Guarded Hybrid Team Logics

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Abstract

Team logics are extensions of first-order logic where formulae are not evaluated over assignments, but over sets ("*teams*") of assignments. In its most basic form, this does not increase the expressiveness of the logic because we can only form statements about the common properties of all assignments ("*flatness*"). Therefore, additional "*team atoms*" are introduced to allow for assertions about interdependencies between the assignments like dependence or inclusion. We propose to consider *binders* known from hybrid logic to increase the expressiveness, where the bound teams may then be referenced as regular relations. We call this *hybrid team logic* (HTL). Additionally, we define the positive and negative fragments of HTL (HTL^+ and HTL^-) by requiring that relations that arise from binding only occur positively or negatively, respectively.

We find that HTL and its positive and negative fragments are equivalent to prominent team logics: HTL^+ is eqivalent to inclusion logic, HTL^- is equivalent to exclusion/dependence logic and HTL itself is equivalent to independence or inclusion/exclusion logic. This classifies HTL as equivalent to existential second order logic and HTL^+ as equivalent to the positive fragment of greatest fixpoint logic.

Binders also enhance the expressiveness of guarded team logics because they enable access to information that normally is obscured by the built-in limitations of these logics. We will take a closer look at guarded hybrid team logics and establish a finite model property for the guarded fragment of HTL using model checking games. More precisely, we encode winning strategies of model checking games as relations, a process that is a natural fit for binders. Further, we notice that the hierarchy of guarded team logics is more complex than the hierarchy of non-guarded team logics, and we establish a hierarchy of prominent union-closed guarded team logics.

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1 Introduction

Team semantics is a generalization of Tarski semantics in which logical formulae are not evaluated for single assignments, but for *sets of assignments* called *teams*. This opens new avenues to reason about interdependencies between assignments, which are relevant e.g. for large sets of data. In fact, most prominent team logics feature notions that have been studied in database theory like dependence [3], independence [14], inclusion [9] or exclusion [10].

Team semantics was originally conceived by Hodges [22] to provide a compositional, model theoretic semantics for independence-friendly logic [21]. Since then, it has been established as a basis for logics of imperfect information. Here, it is prevalent to view the aforementioned interdependencies as atomic properties of teams, an approach that emerged with Väänänen's dependence logic [29] and includes (conditional) independence logic [20] and inclusion/exclusion logic [12]. The expressiveness of these logics is well understood. On sentences, dependence, independence and exclusion logic are all equivalent to existential second order logic Σ^1_1 and inclusion logic is eqivalent to positive greatest fixpoint logic $\nu\mathrm{FO}^+$. On formulae, independence and inclusion/exclusion logic are again equivalent to Σ^1_1, while dependence logic, exclusion logic and inclusion logic are equivalent to specific fragments of

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Σ_1^1 and $\nu\mathrm{FO}^+$, respectively (see Proposition 8 for more details). When including additional propositional connectives like strong negation, even dependence logic reaches the expressive power of full second order logic [24].

However, the expressive power comes at the cost of a comparatively high complexity. Therefore, it seems natural to explore variants of the mentioned team logics that are inspired by logics with desirable algorithmic and model-theoretic properties. One promising direction is the study of guarded logics with team semantics. The basic guarded logic, i.e. the guarded fragment GF of first-order logic, was introduced by Andréka, van Benthem and Németi [1] to explain and generalise the good model theoretic properties of modal logics (for an overview over modal logics, see [7, 8]). GF is defined by restricting first-order quantification in such a way that formulae can only be evaluated with respect to *guarded tuples*, which are tuples of elements that occur together in some atomic fact. This yields a logic that is decidable [1] and where every formula has a finite model [16], amongst other convenient properties. For a more in-depth survey, see e.g. [18]. Many variations and extensions of GF have been studied, for example guarded fixpoint logic, guarded second order logic or the guarded fragment over finite models (see e.g. [4, 5, 6, 17, 27]).

In particular, there exist explorations of guarded team logics by Grädel and Otto [19] and Lück [26]. These focus mostly on the analysis of guarded team logics with additional propositional connectives like strong negation, and establish guarded bisimulation as a suitable tool to analyse the expressiveness of these logics. More specifically, a core feature commonly found with guarded logics is invariance under guarded bisimulation. However, the addition of team atoms will, in general, interfere with bisimulation invariance.

This work aims to provide further insight into guarded team logics that cannot be characterised by guarded bisimulation invariance. For this, a novel team logic called "*hybrid team logic*" (HTL), as well as its positive and negative fragment, are introduced. We show that in their non-guarded version, these are equivalent to independence logic, inclusion logic and dependence logic, respectively. Further, we notice that guarded hybrid team logics are uniquely suited for a type of reduction that encodes winning strategies in model checking games as formulae in the basic guarded fragment, thus reducing the satisfiability problem to the satisfiability problem of GF and thereby providing an easy proof for decidability and a finite model property for guarded fragments of HTL. We then establish a partial expressive hierarchy of guarded team logics, which is more complex than the hierarchy of non-guarded team logics. Due to space limitations, we mainly focus on union closed logics.

2 First-Order Team Logic

In this section, we lay the foundation for the coming sections by providing basic definitions and recalling relevant results from the study of first-order team logic and its extensions.

2.1 Basic definitions

▶ **Definition 1.** *We use the following conventions throughout the paper.*

- *Let A be a set and $\overline{a} = (a_1, \dots, a_n) \in A^n$ be a tuple. We write $[\overline{a}] \coloneqq \{a_1, \dots, a_n\}$ for the set of components, $|\overline{x}| \coloneqq n$ for the length, and $\overline{a} \in A^*$ if n is irrelevant.*
- *An* assignment *to variables $\overline{x}$ into a set $A \neq \emptyset$ is a map $t\colon [\overline{x}] \to A$. We write* $\mathrm{dom}(t) = [\overline{x}]$*. For any tuple $\overline{y} = (y_1, \dots, y_k) \in \mathrm{dom}(t)^k$, we write $t(\overline{y}) \coloneqq (t(y_1), \dots, t(y_k))$, and similarily $t(X) \coloneqq \{t(x) \mid x \in X\}$ for $X \subseteq \mathrm{dom}(t)$.*
- *An* update *$t\left[\frac{a}{x}\right]\colon \mathrm{dom}(t) \cup \{x\} \to A$ of an assignment t is the assignment that maps x to a, and agrees with t everywhere else. Further, $t\left[\frac{\overline{a}}{\overline{x}}\right] = t\left[\frac{a_1}{x_1}\right] \dots \left[\frac{a_n}{x_n}\right]$.*

▶ **Definition 2.** *We use the following conventions with regard to teams.*

- *A* team T *is a set of assignments with a shared domain* $\mathrm{dom}(T)$*. A team may be empty. If* $\mathrm{dom}(T) = \emptyset$*, it may only contain the empty assignment.*
- *For tuples* $\overline{x} \in \mathrm{dom}(T)^*$ *and sets* $X \subseteq \mathrm{dom}(T)$*, we write* $T(\overline{x}) := \{t(\overline{x}) \mid t \in T\}$ *and* $T(X) = \{t(X) \mid t \in T\}$*.*
- *Let* $F\colon T \to \mathcal{P}(A) \setminus \emptyset$*. The* update *of* T *(along* F*) is the team* $T\left[\frac{F}{x}\right] := \{t\left[\frac{a}{x}\right] \mid t \in T, a \in F(t)\}$*. For any set* $B \subseteq A$*, we write* $T\left[\frac{B}{x}\right]$ *for the special case where* $F(t) = B$ *for all* $t \in T$*.*

Now, we can inductively provide team semantics for formulae of first-order logic in negation normal form. We are only using relational signatures σ.

▶ **Definition 3** (Team semantics for FO)**.** *Let* $\mathfrak{A}$ *be a* σ*-structure with universe* A*. Let* T *be a team in* $\mathfrak{A}$ *and* $\varphi \in \mathrm{FO}_\sigma$*. We assume that* $\mathrm{free}(\varphi) \subseteq \mathrm{dom}(T)$*.*

- *If* φ *is a literal then* $\mathfrak{A}, T \models \varphi$ *iff* $\mathfrak{A}, t \models \varphi$ *for all* $t \in T$*.*
- *If* $\varphi = \varphi_1 \wedge \varphi_2$ *then* $\mathfrak{A}, T \models \varphi$ *iff* $\mathfrak{A}, T \models \varphi_1$ *and* $\mathfrak{A}, T \models \varphi_2$*.*
- *If* $\varphi = \varphi_1 \vee \varphi_2$ *then* $\mathfrak{A}, T \models \varphi$ *iff there are* $T_1, T_2 \subseteq T$ *with* $T_1 \cup T_2 = T$ *so that* $\mathfrak{A}, T_1 \models \varphi_1$ *and* $\mathfrak{A}, T_2 \models \varphi_2$*. We call* T_1, T_2 *a* split *of* T*.*
- *If* $\varphi = \exists x \psi$ *then* $\mathfrak{A}, T \models \varphi$ *iff there is an update* $T\left[\frac{F}{x}\right]$ *so that* $\mathfrak{A}, T\left[\frac{F}{x}\right] \models \psi$*.*
- *If If* $\varphi = \forall x \psi$ *then* $\mathfrak{A}, T \models \varphi$ *iff* $\mathfrak{A}, T\left[\frac{A}{x}\right] \models \psi$*. We call* $T\left[\frac{A}{x}\right]$ *the* universal update *of* T *in* $\mathfrak{A}$*.*

In addition to the syntax and semantics presented here, other propositional connectives can be considered. However, most of them (like intuitionistic disjunction, implication or strong negation) do not preserve some of the following convenient properties:

▶ **Proposition 4.** FO *with team semantics satisfies the following properties:*

- ***Flatness****:* $\mathfrak{A}, T \models \varphi$ *if and only if for all* $t \in T, \mathfrak{A}, \{t\} \models \varphi$*.*
- ***Union-closure****:* $\mathfrak{A}, T_1 \models \varphi$ *and* $\mathfrak{A}, T_2 \models \varphi$ *implies* $\mathfrak{A}, T_1 \cup T_2 \models \varphi$*.*
- ***Downward closure****:* $\mathfrak{A}, T \models \varphi$ *and* $T' \subseteq T$ *implies* $\mathfrak{A}, T' \models \varphi$
- ***Locality****:* $\mathfrak{A}, T \models \varphi$ *if and only if* $\mathfrak{A}, T{\restriction_{\mathrm{free}(\varphi)}} \models \varphi$*.*
- ***Empty team property****:* $\mathfrak{A}, \emptyset \models \varphi$*.*

One could argue whether flatness is a desirable property, as it essentially reduces FO with team semantics to classical FO. Still, this version of FO with team semantics provides a clear base for the addition of team atoms.

2.2 Team Atoms

Team atoms are added to first-order team logic to describe atomic team properties that correspond to interdependencies between the assignments in a team. The first and probably best-known resulting logic was dependence logic [29]. Other notable team logics include independence logics [20], inclusion logic and exclusion logic [12] and its combinations, which extend FO with team semantics by one or more of the following team atoms:

▶ **Definition 5** (Team atoms)**.** *Let* $\mathfrak{A}$ *be a structure, let* T *be a team in* $\mathfrak{A}$ *and let* $\overline{x}, \overline{y}, \overline{z} \in \mathrm{dom}(T)^*$ *be tuples of variables, where* $\overline{x}$ *and* $\overline{y}$ *have the same length.*

- ***Dependence****:* $\mathfrak{A}, T \models\ =\!(\overline{x}, \overline{z})$ *if and only if the assignments to* $\overline{z}$ *depend on the assignments to* $\overline{y}$*, in the sense that for all* $t, t' \in T$*,* $t(\overline{x}) = t'(\overline{x})$ *implies* $t(\overline{z}) = t'(\overline{z})$*.*
- ***Independence****:* $\mathfrak{A}, T \models (\overline{x} \perp_{\overline{z}} \overline{y})$ *if and only if for all* $t, t' \in T$ *with* $t(\overline{z}) = t'(\overline{z})$*, there is a* $t'' \in T$ *with* $t(\overline{x}) = t''(\overline{x})$ *and* $t'(\overline{y}) = t''(\overline{y})$*.*
- ***Inclusion****:* $\mathfrak{A}, T \models (\overline{x} \subseteq \overline{y})$ *if and only if for all* $t \in T$ *there is a* $t' \in T$ *with* $t(\overline{x}) = t'(\overline{y})$*.*
- ***Exclusion****:* $\mathfrak{A}, T \models (\overline{x} | \overline{y})$ *if and only if for all* $t, t' \in T$ *we have* $t(\overline{x}) \neq t'(\overline{y})$*.*

$$\begin{tikzcd}
& \Sigma^1_1 \arrow[rr, leftrightarrow] & & \mathrm{FO}(\mathrm{inc},\mathrm{exc}) \arrow[rr, leftrightarrow] & & \mathrm{FO}(\mathrm{indep}) \arrow[dr, dotted] & \\
\nu\mathrm{FO}^+ \arrow[ur] \arrow[rr, leftrightarrow] & & \mathrm{FO}(\mathrm{inc}) \arrow[ur] & & \mathrm{FO}(\mathrm{exc}) \arrow[ul] \arrow[rr, leftrightarrow] & & \mathrm{FO}(\mathrm{dep}) \arrow[ul] \\
& & & \text{FO with team semantics} \arrow[ul] \arrow[ur] & & &
\end{tikzcd}$$

Figure 1 An overview over the hierarchy of team logics. $L \to L'$ means that L' is at least as expressive as L, with dotted arrows for sentences.

All of these are proper extensions of FO with team semantics, which can be seen by analysing which of the properties in Proposition 4 are preserved.

▶ **Proposition 6.** $\mathrm{FO}(\mathrm{dep},\mathrm{indep},\mathrm{inc},\mathrm{exc})$ *satisfies locality and has the empty team property.* $\mathrm{FO}(\mathrm{dep})$ *and* $\mathrm{FO}(\mathrm{exc})$ *are downward closed, but not union closed.* $\mathrm{FO}(\mathrm{inc})$ *is union-closed, but not downward closed.* $\mathrm{FO}(\mathrm{indep})$ *is neither downward nor union-closed. (cf. [12, 20, 29])*

To further describe the expressive power of these logics, we can, on one hand, compare them amongst themselves.

▶ **Proposition 7.**

- $\mathrm{FO}(\mathrm{dep}) \equiv \mathrm{FO}(\mathrm{exc})$ *[12].*
- $\mathrm{FO}(\mathrm{dep}) \equiv \mathrm{FO}(\mathrm{indep})$ *for sentences [29],[20].*
- $\mathrm{FO}(\mathrm{inc},\mathrm{exc}) \equiv \mathrm{FO}(\mathrm{indep})$ *[12].*

On the other hand, it may be desirable to compare a team logic L to another logic L' that is not designed to handle teams. This can be achieved by interpreting the evaluation of a team over given variables as a relation. For example, for any given formula $\varphi(\overline{x}) \in L$, we can then ask whether there is a corresponding $\varphi'(R) \in L'$ with a new relation symbol R so that for every structure $\mathfrak{A}$ and team T, we have that $\mathfrak{A}, T \models \varphi(\overline{x})$ if an only if $(\mathfrak{A}, T(\overline{x})) \models \varphi'(R)$.

In particular, all standard variations of first-order team logic are fragments of existential second order Σ^1_1 in this sense. To be more precise:

▶ **Proposition 8.**

- *For every formula* $\varphi \in \mathrm{FO}(\mathrm{indep})$*, there is a corresponding sentence* $\varphi(R) \in \Sigma^1_1$ *and vice versa [12].*
- *For every formula* $\varphi \in \mathrm{FO}(\mathrm{dep})$*, there is a corresponding sentence* $\varphi(R) \in \Sigma^1_1$ *where* R *only appears negatively, and vice versa [25].*
- *For every formula* $\varphi \in \mathrm{FO}(\mathrm{inc})$*, there is a corresponding sentence* $\forall\overline{x}(R\overline{x} \to \psi(R,\overline{x})) \in \nu\mathrm{FO}^+$ *(positive greatest fixpoint logic) and vice versa [13].*

▶ Note. Positive greatest fixpoint logic $\nu\mathrm{FO}^+$ is an extension of first-order logic by positive occurrences of *greatest fixpoint operators* $[\mathrm{gfp}_{S,\overline{x}}\, \psi(S,\overline{x})]\overline{y}$, thus being a fragment of least fixed point logic. More details can be found e.g. in [23] for fixed point logics in general, and in [13] for $\nu\mathrm{FO}^+$ in particular.

See Figure 1 for a summary of this section.

3 Hybrid Team Logics

We introduce a new team logic called hybrid team logic. The name is inspired by hybrid modal logics, a collection of extensions of modal logic by first-order machinery that was first introduced in 1967 by Prior in [28] to deal with specific issues in temporal logics. For a detailed account of the fundamentals of hybrid logics, see e.g. [2].

One of the main features of modal hybrid logics is the $\downarrow$ *binder*, which was introduced by Goranko in [15] to "bind" the current world as the interpretation of a constant. This concept can be transferred to team logics in the sense that teams can be bound as interpretations of new relational variables.

▶ **Definition 9.** Hybrid team logic (HTL) *is an extension of first-order logic with team semantics by* binders $\downarrow$ *with the following semantics: for all structures* $\mathfrak{A}$*, teams* T*, variables* $\overline{x} \in \mathrm{dom}(T)^*$ *and formulae* $\varphi(X) \in \mathrm{HTL}$ *where* X *is a new relation symbol of arity* $|\overline{x}|$*,*

$$\mathfrak{A}, T \models \downarrow_{\overline{x}} X \varphi(X) \quad \Leftrightarrow \quad (\mathfrak{A}, T(\overline{x})), T \models \varphi(X).$$

The variables in $\overline{x}$ *are considered free variables, i.e.* $\mathrm{free}(\downarrow_{\overline{x}} X\varphi(X)) = \mathrm{free}(\varphi) \cup [\overline{x}]$*.*
The positive (negative) fragment HTL^+ *(*HTL^-*) is the fragment of* HTL *where bound relations may only occur positively (negatively).*

We immediately notice that inclusion and exclusion atoms can be expressed in hybrid team logic on an elementary level.

▶ **Lemma 10.** $\mathrm{FO(inc)} \subseteq \mathrm{HTL}^+$*,* $\mathrm{FO(exc)} \subseteq \mathrm{HTL}^-$ *and* $\mathrm{FO(inc, exc)} \subseteq \mathrm{HTL}$*.*

Proof. We need to show that for every $\varphi \in \mathrm{FO(inc)}$ ($\mathrm{FO(exc)}$, $\mathrm{FO(inc, exc)}$), there is a $\varphi' \in \mathrm{HTL}^+$ (HTL^-, HTL) so that for all structures $\mathfrak{A}$ and teams T,

$$\mathfrak{A}, T \models \varphi \quad \Leftrightarrow \quad \mathfrak{A}, T \models \varphi'.$$

Recall that X is a relational variable, i.e. $\mathfrak{A}, T \models X\overline{x}$ if and only if $T(\overline{x}) \subseteq X^{\mathfrak{A}}$ and $\mathfrak{A}, T \models \neg X\overline{x}$ if and only if $T(\overline{x}) \cap X^{\mathfrak{A}} = \emptyset$. With that, it is straightforward to verify that

$$\mathfrak{A}, T \models (\overline{x} \subseteq \overline{y}) \quad \Leftrightarrow \quad \mathfrak{A}, T \models \downarrow_{\overline{y}} X(X\overline{x})$$

and

$$\mathfrak{A}, T \models (\overline{x} | \overline{y}) \quad \Leftrightarrow \quad \mathfrak{A}, T \models \downarrow_{\overline{y}} X(\neg X\overline{x}).$$

With that, we can replace every occurrence of an inclusion or exclusion atom in φ to get the desired φ'. ◀

Taking a closer look at the positive fragment HTL^+ of hybrid team logic, we shall see that it is equivalent to inclusion logic FO(inc) and positive greatest fixpoint logic $\nu\mathrm{FO}^+$. Considering Lemma 10, it is enough to show that HTL^+ is a fragment of $\nu\mathrm{FO}^+$ in the sense of Proposition 8.

One way to think about this equivalence is that each of the logics in question provides its own tools (independence atoms, binders, teams, fixpoints), which can be *simulated* by the other logics. For example, in the proof of $\mathrm{FO(inc)} \equiv \nu\mathrm{FO}^+$ in [13], fixpoints are simulated in FO(inc) by expanding any given team to a cartesian product, i.e. introducing a second team (over fresh free variables) that represents the fixpoint and can be handled independently.

Another example already occurred in Proposition 8, where $\nu\mathrm{FO}^+$ could be used to simulate teams via an additional relation. In general, this team will be manipulated when evaluating a formula, e.g. by splitting or updating. This can be simulated with the help of fixpoints.

▶ **Theorem 11.** *For every* $\varphi(\overline{x}) \in \mathrm{HTL}^{+}$ *there is a* $\varphi^{*}(R, \overline{x}) \in \nu\mathrm{FO}^{+}$ *so that* R *only appears positively and*

$$\mathfrak{A}, T \models \varphi(\overline{x}) \quad \Leftrightarrow \quad (\mathfrak{A}, T(\overline{x})), t \models \varphi^{*}(R, \overline{x}) \text{ for all } t \in T.$$

Proof. We use syntactic induction and retrace the proof of Theorem 15 in [13] regarding everything except binders. If $\varphi = \downarrow_{\overline{x}} S\psi(S, \overline{xy})$, by induction we have $\psi^{*}(R, S, \overline{xy})$ and use

$$\varphi^{*} = \psi^{*}(R, \exists\overline{y}R\_\overline{y}, \overline{xy}),$$

i.e. $\exists\overline{y}R\_\overline{y}$ is supposed to replace S in the sense that every instance of $S\overline{x}$ is replaced by $\exists\overline{y}R\overline{xy}$. This way, we have

$$\begin{aligned} & \mathfrak{A}, T \models \varphi \\ \Leftrightarrow \quad & (\mathfrak{A}, T(\overline{x})), T \models \psi(S, \overline{xy}) \\ \Leftrightarrow \quad & (\mathfrak{A}, T(\overline{xy}), T(\overline{x}))\,, t \models \psi^{*}(R, S, \overline{xy}) \text{ for all } t \in T \\ \Leftrightarrow \quad & (\mathfrak{A}, T(\overline{xy}))\,, t \models \varphi^{*} \text{ for all } t \in T, \end{aligned}$$

because $(\mathfrak{A}, T(\overline{x})), t \models S\overline{x}$ if and only if $(\mathfrak{A}, T(\overline{xy})), t \models \exists\overline{y}R\overline{xy}$. ◀

▶ **Corollary 12.** $\mathrm{HTL}^{+} \equiv \mathrm{FO}(\mathrm{inc}) \equiv \nu\mathrm{FO}^{+}$

The question of whether the unique features available in some logic can be simulated in another logic will be revisited in Section 5.

For the sake of completeness, we mention that the other team logics are also equivalent to their respective counterparts from Lemma 10. A proof can be found in Appendix A.

▶ **Proposition 13.** $\mathrm{HTL}^{-} \equiv \mathrm{FO}(\mathrm{exc})$ *and* $\mathrm{HTL} \equiv \mathrm{FO}(\mathrm{inc}, \mathrm{exc})$.

4 Guarded Team Logics

In classical first-order logic, there are several equivalent ways to define the guarded fragment. In particular, there are syntactic and semantic definitions. In the former case, we add guards to quantification in the sense that, if quantification appears in a formula, it has to have the form $\exists\overline{x}(\alpha(\overline{xy}) \land \psi(\overline{xy}))$ or $\forall\overline{x}(\alpha(\overline{xy}) \to \psi(\overline{xy}))$ for some guard α. In the latter case, we require the images of all assignments to be guarded.

Here, we work with one of the least restrictive variations of guarded logics, similar to [18].

▶ **Definition 14.** *Let* σ *be a relational signature and* $\mathfrak{A}$ *be a* σ*-structure with universe* A. *The* set of guards $\mathbb{G}(\mathfrak{A})$ *is defined as*

$$\mathbb{G}(\mathfrak{A}) := \{G \subseteq A \mid G \subseteq [\overline{a}], \overline{a} \in R^{\mathfrak{A}}, R \in \sigma \cup \{=\}\}.$$

As we can see, the set of guards in $\mathfrak{A}$ contains all sets that are completely "covered" by a tuple that occurs in a relation. The inclusion of "=" in the selection of guards entails that all singleton sets are guarded.

▶ **Definition 15.** *Let* $\mathfrak{A}$ *be a relational structure.*

- *A tuple* $\overline{a}$ *is guarded in* $\mathfrak{A}$ *if* $[\overline{a}] \in \mathbb{G}(\mathfrak{A})$.
- *An assignment* t *is guarded if* $t(\mathrm{dom}(t)) \in \mathbb{G}(\mathfrak{A})$.
- *A team* T *is guarded if* $T(\mathrm{dom}(T)) \subseteq \mathbb{G}(\mathfrak{A})$.

It is clear that, in general, not all updates of guarded teams are guarded. However, there always is a *universal guarded update*, i.e. a unique maximal guarded update that can take the place of universal updates in guarded semantics. With this, we can define a guarded variant of team logic by replacing the classical quantification of FO with *guarded quantification*.

▶ **Definition 16.** Guarded team logic GTL *is defined analogously to standard* FO *with team semantics, where quantification is replaced by guarded quantification in the following sense:*

- $\mathfrak{A}, T \models \forall_g x\psi$ *iff* $A, T' \models \psi$ *for the universal guarded update* T' *of* $T\restriction_{\mathrm{free}(\psi)\setminus\{x\}}$.
- $\mathfrak{A}, T \models \exists_g x\psi$ *iff* $A, T' \models \psi$ *for some guarded update* T' *of* $T\restriction_{\mathrm{free}(\psi)\setminus\{x\}}$.

For tuples $\overline{x} = (x_1, \dots, x_n)$*, we write* $\exists_g \overline{x}\psi$ *instead of* $\exists_g x_1 \dots \exists_g x_n \psi$ *(similarly for* $\forall$*).*

Extending GTL *by adding binders, inclusion atoms etc. yields guarded hybrid team logic* GHTL*, guarded inclusion logic* GTL(inc) *etc. respectively. Binding teams with the* $\downarrow$*-operator does not change the set of guards.*

One of the features of guarded logics is the fact that quantification can be thought of as moving from guarded patch to guarded patch. This is reflected by the evaluation of guarded quantification through the restriction of T to those variables that are relevant for the inner formula.

▶ Note. In general, there are several options to design guarded team logics. One obvious candidate would be to use team semantics with the standard guarded fragment of FO, which would entail guarding each quantification with a relational atom. However, this would prohibit mixed teams, i.e. teams that contain assignments that are guarded by different relations. In the non-team case, there is no tangible difference between both versions as all assignments are considered independently, anyway. With teams, however, there are cases where extensions of GF with team semantics are strictly weaker than extensions of GTL as defined above (see Appendix B.2 for details).

▶ Note. GTL is technically not a fragment of FO with team semantics. The reason is that guarded quantification implicitly introduces a disjunction over all relations in the signature. For infinite signatures, this is not reproducible in FO. For any finite signature however, we can find a translation between GTL and FO with team semantics (and their extensions). For the rest of this paper, we therefore assume that all signatures are finite if not explicitly stated otherwise (see Appendix B for details).

4.1 Properties of Guarded Team Logics

In Sections 2 and 3, we introduced several team logics and mentioned their properties in Proposition 4. It is straightforward to verify that these properties translate to the respective guarded variants of these logics. (See Appendix B for more details.)

This leaves the question whether we can lift desirable properties of classical guarded logic to the team setting. We take a closer look at two specific properties.

▶ **Proposition 17** ([1, 16])**.** *The guarded fragment of classical first-order logic is* decidable *since it has the* finite model property*, i.e. every satisfiable formula has a finite model.*

▶ Remark 18. All logics with team semantics that are examined here have the empty team property, i.e. all formulae are satisfied by the empty team across all structures. A sensible notion of decidability and finite model property for team logics would therefore only regard satisfaction by non-empty teams.

We can immediately show that guarded dependence logic GTL(dep) cannot have the finite model property:

▶ **Lemma 19.** *Let E be a binary relation symbol. The sentence*

$$\begin{aligned}\varphi \coloneqq\ & \exists_g x(\forall_g y \neg Eyx) \wedge \\ & \quad \forall_g z(\exists_g w(z \neq w \wedge Ezw) \wedge \\ & \qquad \forall_g w(\neg Ezw \vee (Ezw \wedge {=}(z,w) \wedge {=}(w,z))))\end{aligned}$$

is a satisfiable formula in GTL(dep) *that is satisfied by a structure* $\mathfrak{A}$ *only if* $\mathfrak{A}$ *contains at least one infinite simple E-path.*

Proof. If $\mathfrak{A} \models \varphi$, then there is at least one element a without any E-predecessor. We also know that every element has exactly one successor and at most one predecessor. This implies that $\mathfrak{A}$ consists of only cycles or infinite paths with or without starting point. The vertex a cannot lie on a circle, so it has to be the starting point of an infinite path. Clearly, $(\mathbb{N}, \{(n, n+1) \mid n \in \mathbb{N}\})$ satisfies φ. ◀

In contrast, guarded hybrid team logic GHTL has the finite model property and is decidable. Both can be shown by reducing the satisfiability of formulae in GHTL to the satisfiability of the classical guarded fragment.

▶ **Theorem 20.** *Let σ be a relational signature and $\varphi(\overline{x}) \in \mathrm{GHTL}_\sigma$. There is a signature $\tau \supseteq \sigma \cup \{R_\varphi\}$ and a formula $\varphi^*(R_\varphi) \in \mathrm{GF}_\tau$ so that all σ-structures $\mathfrak{A}$ and guarded teams T satisfy φ if and only if there is a expansion of $\mathfrak{A}$ to a τ-structure $\mathfrak{A}^*$ so that $(\mathfrak{A}^*, T(\overline{x})) \models \varphi^*(R_\varphi)$.*

Proof. The general strategy is to expand σ by relation symbols R_ψ for all instances of subformulae ψ of φ. Then, we include clauses in φ^* that are only satisfied if the interpretations of the R_ψ provide a winning strategy for a basic model checking game that is similar to the one presented in [29, section 5.2].

Let $S(\varphi)$ be the set of (instances of) subformulae of φ, including φ itself, and let $\mathcal{X}(\varphi)$ be the set of relational variables that are bound in φ. We define $\tau = \sigma \cup \{R_\psi\}_{\psi \in S(\varphi)} \cup \mathcal{X}(\varphi)$ and formulae $\varphi^*(\psi)$ according to Appendix C.1 and define

$$\varphi^* \coloneqq \bigwedge_{\psi \in S(\varphi)} \varphi^*(\psi).$$

There are a few technical details that have to be considered (see Appendix C.2), but overall, it is straightforward to verify by syntactic induction that this is as required. ◀

We see that guarded hybrid team logic is uniquely suited for this type of translation, because the concept of interchangeability between teams and relations that is at the core of the proof is already included in the logic itself.

Moreover, we get a glimpse of why guarded dependence logic loses the finite model property: the dependence atom is fundamentally non-guarded in the sense that it cannot be defined by a formula in classical guarded logic. Another explanation would be the similarity of GF(dep) to guarded logic with counting quantifiers, which also does not have the finite model property and is undecidable [16].

▶ **Corollary 21.** GHTL *has the finite model property and is decidable in the sense of Remark 18.*

Proof. From Theorem 20, we know that any $\varphi \in \mathrm{GHTL}$ is satisfied by some (possibly infinite) structure $\mathfrak{A}$ and non-empty team T with domain $\overline{x}$ if and only if $\varphi^* \wedge \exists \overline{x} R_\varphi \overline{x}$ is satisfied by an appropriate expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ with $R_\varphi^{\mathfrak{A}^*} = T(\overline{x})$.

- Due to the finite model property of GF, we find a finite model $(\mathfrak{B}^*, T'(\overline{x}))$ satisfying $\varphi^* \wedge \exists \overline{x} R_\varphi \overline{x}$. Using Theorem 20, we get a finite model $\mathfrak{B}, T'$ of φ with non-empty T' which proves the finite model property.
- The translation from φ to $\varphi^* \wedge \exists \overline{x} R_\varphi \overline{x} \in \mathrm{GF}$ is computable in linear time, so decidability of GHTL follows directly from the decidability of GF. ◀

5 A Hierarchy of Union-Closed Team Logics

One takeaway from Lemma 19 and Corollary 21 is that the hierarchy of team logics in the guarded case differs from the hierarchy in the non-guarded case.

▶ **Corollary 22.** $\mathrm{GTL}(\mathrm{dep}) \not\subseteq \mathrm{GHTL}^-$.

Of course, all the obvious inclusions like $\mathrm{GTL}(\mathrm{exc}) \subseteq \mathrm{GTL}(\mathrm{inc}, \mathrm{exc})$ still hold, and the translations of Lemma 10 work on an atomic level and are therefore unaffected by changes to quantification.

▶ **Corollary 23.** $\mathrm{GTL}(\mathrm{inc}) \subseteq \mathrm{GHTL}^+$, $\mathrm{GTL}(\mathrm{exc}) \subseteq \mathrm{GHTL}^-$, $\mathrm{GTL}(\mathrm{inc}, \mathrm{exc}) \subseteq \mathrm{GHTL}$.

This also implies that there are formulae in guarded dependence logic that cannot be expressed in guarded exclusion logic, contrary to the non-guarded case.

In this section, we further investigate the relative expressiveness of union-closed guarded team logics, i.e. guarded inclusion logic GTL(inc), positive guarded hybrid team logic GHTL^+ and guarded positive greatest fixpoint logic $\nu\mathrm{GF}^+$.

An upper bound for the expressiveness of these logics is GHTL. It is clear that $\mathrm{GTL}(\mathrm{inc}) \subseteq \mathrm{GHTL}^+ \subseteq \mathrm{GHTL}$. The proof for $\nu\mathrm{GF}^+ \subseteq \mathrm{GHTL}$ can roughly be outlined as follows:

1. As mentioned towards the end of Section 3, in the non-guarded case, we could simulate fixpoints by introducing them as new teams over fresh variables. This way, we could effectively handle more than one team simultaneously. However, this strategy is not available any more because in general, it would require a shared guard for all teams.
2. Let L be a team logic and R a relation symbol that may occur both positively and negatively. For all tuples $\overline{x}$ of appropriate length and $\psi \in L$ with $\mathrm{free}(\psi) \subseteq [\overline{x}]$, we can define a sentence $\varphi(R, \psi)$ that is satisfied in a structure $\mathfrak{A}$ if and only if $R^{\mathfrak{A}}$, interpreted as a team with domain $[\overline{x}]$, satisfies ψ.
3. We can bind a team, introduce a fixpoint, bind the fixpoint, and then "recover" the team (using the sentence in 2.) and check whether it is in the fixpoint. This circumvents the problem in 1.

However, the same strategy cannot be employed in GHTL^+ or GHTL^-, because the sentence in 2. uses both positive and negative instances of R. Neither $\nu\mathrm{GF}^+$ nor GHTL^- contains the other because the former is union-closed and the latter is downward closed, but not vice versa. By contrast, Theorem 11 can be easily adapted.

▶ **Lemma 24.** *For every $\varphi(\overline{x}) \in \mathrm{GHTL}^+$ there is a formula $\varphi^*(R, \overline{x}) \in \nu\mathrm{GF}^+$ in which R only appears positively and*

$$\mathfrak{A}, T \models \varphi(\overline{x}) \quad \Leftrightarrow \quad (\mathfrak{A}, T(\overline{x})), t \models \varphi^*(R, \overline{x}) \text{ for all } t \in T.$$

Proof. The proof is very similar to the one of Theorem 11. In most steps, quantification does not matter, so we focus on the ones where it does:

- If $\varphi = \exists_g \overline{y}\psi(\overline{xy})$, let

$$\varphi^* = \exists_g \overline{y}[\mathrm{gfp}_{S,\overline{xy}}\, \exists_g \overline{z} R\overline{xz} \wedge \psi^*(S, \overline{xy})]\overline{xy},$$

where the length of $\overline{xz}$ is equal to the arity of R. The tuple $\overline{z}$ is supposed to represent the variables that are dropped from the domain of the team when evaluating the quantification. An identical argument can be made for universal quantification.

- If $\varphi = \downarrow_{\overline{x}} S\psi(S, \overline{xy})$, let

$$\varphi^* = \psi^*(R, \exists_g \overline{y} R\_\overline{y}, \overline{xy}),$$

matching the non-guarded case. ◀

We shall see that we have $\mathrm{GHTL}^+ \equiv \nu\mathrm{GF}^+$ for sentences but $\mathrm{GHTL}^+ \subsetneq \nu\mathrm{GF}^+$ for arbitrary formulae.

5.1 $\mathrm{GHTL}^+ \equiv \nu\mathrm{GF}^+$ on Sentences

To simplify notation, we introduce some abbreviations:

- We write $[\mathrm{gfp}\, \psi]$ instead of $[\mathrm{gfp}_{S,\overline{x}}\, \psi(S, \overline{x})]$ whenever the variables are not in focus.
- For all structures $\mathfrak{A}$ and fixpoints $[\mathrm{gfp}\, \psi]$, we write $[\mathrm{gfp}\, \psi]^{\mathfrak{A}}$ for the interpretation of $[\mathrm{gfp}\, \psi]$ in $\mathfrak{A}$.

One direction of $\mathrm{GHTL}^+ \equiv \nu\mathrm{GF}^+$ is already included in Lemma 24. To show $\nu\mathrm{GF}^+ \subseteq \mathrm{GHTL}^+$, we provide a translation $\varphi^* \in \mathrm{GHTL}^+$ for any sentence $\varphi \in \nu\mathrm{GF}^+$. To do that, we design φ^* so that first, all necessary fixpoints are simulated and bound, and then the sentence is evaluated as usual, with the bound pseudo-fixpoints replacing the actual fixpoints.

In the process of applying the translation, we transition from $\nu\mathrm{GF}^+$ to GHTL^+ fixpoint by fixpoint. This leads to intermediate steps that involve the syntax of both logics. To handle this combined logic, we need team semantics for $\nu\mathrm{GF}^+$, which we get by extending the semantics of guarded team logic GTL by a clause for fixpoints:

▶ **Definition 25.** *For all structures $\mathfrak{A}$, teams T and fixpoints $[\mathrm{gfp}\, \psi]$,*

$$\mathfrak{A}, T \models [\mathrm{gfp}\, \psi]\overline{x} \quad :\Leftrightarrow \quad T(\overline{x}) \subseteq [\mathrm{gfp}\, \psi]^{\mathfrak{A}}.$$

▶ **Lemma 26.** *Let $\psi \in \nu\mathrm{GF}^+$ and let R be a new relation symbol.*

1. $\mathfrak{A}, T \models [\mathrm{gfp}\, \psi]\overline{x}$ *iff* $(\mathfrak{A}, [\mathrm{gfp}\, \psi]^{\mathfrak{A}}), T \models R\overline{x}$.
2. $\nu\mathrm{GF}^+$ *with team semantics has the flatness property.*
3. $\mathfrak{A}, \{t\} \models \psi$ *iff* $\mathfrak{A}, t \models \psi$.
4. *If* $(\mathfrak{A}, T(\overline{x})), T \models \psi(S, \overline{x})$*, then* $\mathfrak{A}, T \models [\mathrm{gfp}\, \psi]\overline{x}$.
5. *If T is maximal so that* $\mathfrak{A}, T \models [\mathrm{gfp}\, \psi]\overline{x}$*, then* $(\mathfrak{A}, T), T \models \psi(S, \overline{x})$.

Proof. 1. follows directly from Definition 25. The proofs for 2. and 3. are identical to the proofs for non-guarded team logic, using 1. The proofs of 4. and 5. can then be reduced to the respective proofs of [13, Lemma 14]. ◀

▶ Note. Lemma 26, part 2, does not imply flatness for GHTL^+, even though it is a fragment in the sense of Lemma 24. For this, the translations φ^* would have to be flat with regard to the relational encoding of the team, i.e.

$$\text{for all } t \in T : (\mathfrak{A}, T(\overline{x})), t \models \varphi^* \Leftrightarrow \text{for all } t' \in T : (\mathfrak{A}, \{t'(\overline{x})\}), t' \models \varphi^*.$$

As is, fixpoints may contain non-guarded tuples because the inner formulae are satisfiable by non-guarded tuples (e.g. $[\mathrm{gfp}_{S,x_1x_2}\top]^{\mathfrak{A}}$ contains every pair in the universe of $\mathfrak{A}$). However, there are ways to replace these fixpoints without changing the guarded parts.

▶ **Lemma 27.** *Let $\varphi \in \nu\mathrm{GF}^+$, let $\mathfrak{A}$ be a structure and $\overline{a}$ be a tuple in $\mathfrak{A}$.*

1. *There is a formula $\varphi^g \in \nu\mathrm{GF}^+$ such that $\mathfrak{A}, \overline{a} \models \varphi^g$ if and only if $\mathfrak{A}, \overline{a} \models \varphi$ and $\overline{a} \in \mathbb{G}(\mathfrak{A})$.*
2. *$[\mathrm{gfp}\,\varphi^g]^{\mathfrak{A}} = [\mathrm{gfp}\,\varphi]^{\mathfrak{A}} \cap \mathbb{G}(\mathfrak{A})$.*

Proof.

1. We can construct φ^g using trivial quantification, e.g. $\varphi^g(\overline{x}) := \exists_g \overline{x}'(\overline{x}' = \overline{x} \wedge \varphi(\overline{x}'))$.
2. To show $[\mathrm{gfp}\,\varphi^g]^{\mathfrak{A}} \subseteq [\mathrm{gfp}\,\varphi]^{\mathfrak{A}} \cap \mathbb{G}(\mathfrak{A})$, we use the first part of this lemma twice. First, we immediately get $[\mathrm{gfp}\,\varphi^g]^{\mathfrak{A}} \subseteq \mathbb{G}(\mathfrak{A})$ because φ^g is only satisfiable by guarded tuples. Second, for all $\overline{a} \in [\mathrm{gfp}\,\varphi^g]^{\mathfrak{A}}$, we have $(\mathfrak{A}, [\mathrm{gfp}\,\varphi^g]^{\mathfrak{A}}), \overline{a} \models \varphi^g(\overline{x}, S)$ and therefore $(\mathfrak{A}, [\mathrm{gfp}\,\varphi^g]^{\mathfrak{A}}), \overline{a} \models \varphi(\overline{x}, S)$ and with an argument similar to [13, Lemma 14] we get $\mathfrak{A}, \overline{a} \models [\mathrm{gfp}\,\varphi]\overline{x}$.
   For the other direction, let $\overline{a} \in [\mathrm{gfp}\,\varphi]^{\mathfrak{A}} \cap \mathbb{G}(\mathfrak{A})$, so we have $(\mathfrak{A}, [\mathrm{gfp}\,\varphi]^{\mathfrak{A}}), \overline{a} \models \varphi(\overline{x}, S)$ and therefore $(\mathfrak{A}, [\mathrm{gfp}\,\varphi]^{\mathfrak{A}}), \overline{a} \models \varphi^g(\overline{x}, S)$ because $\overline{a}$ is guarded. When evaluating $\varphi^g(\overline{x}, S)$, every occurrence of $S\overline{y}$ will be evaluated by a guarded tuple (because $\overline{a}$ and every update of $\overline{a}$ due to quantification is guarded). The non-guarded part of $[\mathrm{gfp}\,\varphi]^{\mathfrak{A}}$ is therefore irrelevant for the evaluation and can be omitted, which yields $(\mathfrak{A}, [\mathrm{gfp}\,\varphi]^{\mathfrak{A}} \cap \mathbb{G}(\mathfrak{A})), \overline{a} \models \varphi^g(\overline{x}, S)$. Again, we refer to [13, Lemma 14] to get $\mathfrak{A}, \overline{a} \models [\mathrm{gfp}\,\varphi^g]\overline{x}$ and are done. ◀

We can now provide a tool to replace fixpoints by bound teams. To this end we allow the creation of formulae that may contain both binders and fixpoints, as long as no fixpoint contains a binder (bound relations are allowed). This way, we can always find a fixpoint with a first-order inner formula, which is key for the translation to work.

▶ **Lemma 28.** *Let $\varphi([\mathrm{gfp}\,\psi])$ be a sentence in $\nu\mathrm{GF}^+$ so that $[\mathrm{gfp}\,\psi]$ appears in φ and ψ itself does not contain fixpoints. Then*

$$\varphi([\mathrm{gfp}\,\psi]) \equiv \exists_g \overline{x} \downarrow_{\overline{x}} X(\psi(X, \overline{x}) \wedge \varphi(X)).$$

Proof. We show the equivalence for an arbitrary structure $\mathfrak{A}$. Without loss of generality, we can assume that fixpoints are guarded because they are always evaluated by guarded teams. As such, only the guarded part of the fixpoint matters, and we can always find a corresponding fixpoint due to Lemma 27.

For one direction, assume $\mathfrak{A} \models \varphi([\mathrm{gfp}\,\psi])$. Because we can assume that fixpoints are guarded, it suffices to show

$$(\mathfrak{A}, [\mathrm{gfp}\,\psi]^{\mathfrak{A}}), [\mathrm{gfp}\,\psi]^{\mathfrak{A}} \models \psi(X, \overline{x}) \wedge \varphi(X).$$

For $(\mathfrak{A}, [\mathrm{gfp}\,\psi]^{\mathfrak{A}}), [\mathrm{gfp}\,\psi]^{\mathfrak{A}} \models \psi(X, \overline{x})$, we use Lemma 26, part 5, as $[\mathrm{gfp}\,\psi]^{\mathfrak{A}}$ obviously is maximal. Also, we can use the assumption and Lemma 26, part 1, to replace every instance of $[\mathrm{gfp}\,\psi]\overline{x}$ in φ with $X\overline{x}$ to get $(\mathfrak{A}, [\mathrm{gfp}\,\psi]^{\mathfrak{A}}), [\mathrm{gfp}\,\psi]^{\mathfrak{A}} \models \varphi(X)$.

For the other direction, let T be a witness for the existential claim. From $(\mathfrak{A}, T(\overline{x})), T \models \psi(X, \overline{x})$, it follows that $T \subseteq [\mathrm{gfp}\,\psi]^{\mathfrak{A}}$ according to Lemma 26, part 4, and Definition 25. We also have $(\mathfrak{A}, T(\overline{x})) \models \varphi(X)$, and because X occurs only positively in φ, satisfaction of φ is preserved whenever the interpretation of X is replaced by a superset. This yields $(\mathfrak{A}, [\mathrm{gfp}\,\psi]^{\mathfrak{A}}) \models \varphi(X)$. We replace all instances of $X\overline{x}$ with $[\mathrm{gfp}\,\psi]\overline{x}$ using Lemma 26, part 1, and are done. ◀

▶ **Theorem 29.** $\nu\mathrm{GF}^+ \equiv \mathrm{GHTL}^+$ *for sentences.*

Proof. $\mathrm{GHTL}^+ \subseteq \nu\mathrm{GF}^+$ is already established for formulae in Lemma 24. Towards $\nu\mathrm{GF}^+ \subseteq \mathrm{GHTL}^+$, let σ be a signature and $\varphi \in \nu\mathrm{GF}^+$ be a sentence. We start by applying Lemma 28 on one of the "innermost" fixpoints $[\mathrm{gfp}\,\psi_1]$ of φ, i.e. one that does not contain other fixpoints, to get

$$\varphi_1 \coloneqq \exists_g \overline{x} \downarrow_{\overline{x}} X_1(\psi_1(X_1, \overline{x}) \wedge \varphi(X_1)).$$

We notice that $\varphi(X_1)$ is again a sentence in $\nu\mathrm{GF}^+$ over the signature $\sigma \cup \{X_1\}$ that now contains one fixpoint less. This allows for repeated application of Lemma 28 until all fixpoints are eliminated and we get a formula of the form

$$\begin{aligned}\varphi_n \coloneqq &\exists_g \overline{x}_1 \downarrow_{\overline{x}} X_1(\psi_1(X_1, \overline{x}_1)\\ &\wedge \exists_g \overline{x}_2 \downarrow_{\overline{x}} X_2(\psi_2(X_2, \overline{x}_2)\\ &\dots\\ &\wedge \exists_g \overline{x}_n \downarrow_{\overline{x}} X_n(\psi_n(X_n, \overline{x}_n) \wedge \varphi(X_1, \dots, X_n)) \dots)),\end{aligned}$$

where $\varphi(X_1, \dots, X_n)$ is a first-order sentence. All instances of bound relations are positive, because they only replace fixpoints or relation variables in fixpoints, which only appear positive in $\nu\mathrm{GF}^+$. Therefore, $\varphi_n \in \mathrm{GHTL}^+$ and we are done. ◀

5.2 $\mathrm{GF(inc)} \subsetneq \mathrm{GHTL}^+ \subsetneq \nu\mathrm{GF}^+$ on Formulae

The aim of this section is to show that on formulae, guarded positive greatest fixpoint logic is more expressive than positive guarded hybrid team logic, which in turn is more expressive than guarded inclusion logic. For this, we take a closer look at the nature of quantification in guarded logics.

Quantification in guarded logics can either be *local* or *global*, which means that the new assignment or team either has to be guarded together with (parts of) the old assignment or team, or the previous team can be "forgotten". More specifically, we say that a subformula $Q_g \overline{x} \psi$ (where $Q \in \{\forall, \exists\}$) corresponds to a global move if $\mathrm{free}(\psi) \subseteq [\overline{x}]$, i.e. $Q_g \overline{x} \psi$ is a sentence. This motivates the following definitions :

► **Definition 30.**

- *For all formulae φ, any subformula of φ that is a sentence on its own is called a* subsentence *of φ. In general, there will be subsentences that do contain other subsentences.*
- *Gaifman-neighbourhoods [11]: For any σ-structure $\mathfrak{A}$ with universe A and subset $B \subseteq A$, we inductively define the l-*neighbourhood *$\mathfrak{n}(B, l)$ according to $\mathfrak{n}(B, 0) = B$ and*

$$\mathfrak{n}(B, l+1) = \left\{ a \in A \;\middle|\; \exists \overline{a} \in \bigcup_{R \in \sigma} R^{\mathfrak{A}} \text{ s.t. } a \in [\overline{a}] \text{ and } [\overline{a}] \cap \mathfrak{n}(B, l) \neq \emptyset \right\}.$$

- *For any assignment t with domain $\mathrm{dom}(t) = X$, we write $\mathfrak{n}(t, l) \coloneqq \mathfrak{n}(t(X), l)$. For any team T with domain $\mathrm{dom}(T) = X$, we define*

$$\mathfrak{n}(T, l) \coloneqq \mathfrak{n}\left(\bigcup_{t \in T} t(X), l\right), \quad \text{or equivalently} \quad \mathfrak{n}(T, l) \coloneqq \bigcup_{t \in T} \mathfrak{n}(t, l).$$

- *Let T be a team in $\mathfrak{A}$ with $\mathrm{dom}(T) = X$ and $T' \subseteq T$. We say that T' is an l-*local cluster *in T if, for all $t \in T$, we have $t \in T'$ or*

$$\mathfrak{n}(t, l) \cap \mathfrak{n}(T', l) = \emptyset.$$

- *Let $\mathfrak{A}, \mathfrak{B}$ be structures, $T_{\mathfrak{A}}, T_{\mathfrak{B}}$ be teams with domain X and $C_{\mathfrak{A}}, C_{\mathfrak{B}}$ be l-local clusters in $T_{\mathfrak{A}}, T_{\mathfrak{B}}$ respectively. We say that $C_{\mathfrak{A}}$ and $C_{\mathfrak{B}}$ are* l-locally isomorphic *($\mathfrak{A}, C_{\mathfrak{A}} \simeq^l \mathfrak{B}, C_{\mathfrak{B}}$) if there is a bijection $\pi\colon C_{\mathfrak{A}} \to C_{\mathfrak{B}}$ that can be extended to an isomorphism $\iota\colon \mathfrak{n}(C_{\mathfrak{A}}, l) \to \mathfrak{n}(C_{\mathfrak{B}}, l)$ on the induced substructures in the sense that for all $x \in X$ and $t_{\mathfrak{A}} \in T_{\mathfrak{A}}$, we have $\iota(t_{\mathfrak{A}}(x)) = \pi(t_{\mathfrak{A}})(x)$.*
- *The local rank $\mathfrak{lr}(\varphi)$ is defined inductively, identically to the standard quantifier rank for guarded logics, with one essential difference: $\mathfrak{lr}(\varphi) = 0$ if φ is a (sub)sentence.*

The concept of local clusters and local isomorphism gives us a strong criterion for local indistinguishability.

5.2.1 $\mathrm{GHTL}^+ \not\subseteq \mathrm{GF(inc)}$

If there are two teams in the same structure that cannot be distinguished locally, both teams satisfy the same GF(inc)-formulae.

▶ **Lemma 31.** *Let $\mathfrak{A}$ be a σ-structure and $l \in \mathbb{N}$. Let T_1 and T_2 be teams in $\mathfrak{A}$ such that for every $(l+1)$-local cluster C_2 in T_2, there is an $(l+1)$-local cluster C_1 in T_1 such that $\mathfrak{A}, C_1 \simeq^{l+1} \mathfrak{A}, C_2$. Let $\varphi \in \mathrm{GF(inc)}$ be a formula with locality rank $\mathfrak{lr}(\varphi) \leq l$. Then $\mathfrak{A}, T_1 \models \varphi$ implies $\mathfrak{A}, T_2 \models \varphi$.*

Proof. The full proof can be found in Appendix D. In short, we use syntactic induction. For (sub)sentences, the current teams are irrelevant and we have $\mathfrak{A}, T_1 \models \varphi$ if and only if $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A}, T_2 \models \varphi$. For everything else, we mainly have to make sure that we can match every local move (or split) in T_1 by a local move (or split) in T_2 that preserves the preconditions. This is always provided by the local isomorphism. ◀

It is clear that the precision of this invariance is not comparable to results like bisimulation invariance of GF or GTL. Still, it is a strong enough tool so show our intended result by providing an example of an inexpressible property using a specific class of structures.

▶ **Definition 32.** *For $n \geq 3$, let $\mathfrak{B}_n$ consist of an n-cycle and an n-line, i.e. $\mathfrak{B}_n$ is an $\{E\}$-structure with universe $B_n := \{b_1, \dots, b_n, c_1, \dots, c_n\}$ and*

$$E^{\mathfrak{B}_n} = \{(b_i, b_{i+1}) \mid 1 \leq i < n)\} \cup \{(c_i, c_{i+1}) \mid 1 \leq i < n\} \cup \{(c_n, c_1)\}.$$

▶ **Lemma 33.** *Let $\ell \in \mathbb{N}$ and $\varphi \in \mathrm{GF(inc)}$ with $\mathfrak{lr}(\varphi) \leq \ell$. Let $\mathfrak{B}_{2\ell+4}$ be as defined in Definition 32 and let $T_C := \{x \mapsto c_{\ell+2}\}$ and $T_L := \{x \mapsto b_{\ell+2}\}$ be two teams with domain $\{x\}$ consisting of just one assignment each. Then*

$$\mathfrak{B}_{2\ell+4}, T_C \models \varphi \qquad \Leftrightarrow \qquad \mathfrak{B}_{2\ell+4}, T_L \models \varphi.$$

Proof. If we can show that the prerequisites of Lemma 31 are fulfilled in both directions, we are done. Obviously, both T_C and T_L consist of only one $(\ell+1)$-local cluster, for which we only have to show that there is an isomorphism $\iota\colon \mathfrak{n}(T_C, \ell+1) \to \mathfrak{n}(T_L, \ell+1)$ so that $\iota(c_{\ell+2}) = b_{\ell+2}$. We notice that $\mathfrak{n}(T_C, \ell+1) = \{c_1, \dots, c_{2\ell+3}\}$ and $\mathfrak{n}(T_L, \ell+1) = \{b_1, \dots, b_{2\ell+3}\}$. It is then straightforward to check that $\iota(c_i) = b_i$ fulfils the requirements. ◀

▶ **Theorem 34.** $\mathrm{GHTL}^+ \not\subseteq \mathrm{GF(inc)}$.

Proof. In Lemma 33, we saw that there cannot be a formula separating $\mathfrak{B}_n, T_L$ from $\mathfrak{B}_n, T_C$ for all $n \in \mathbb{N}$. However, there is such a formula in GHTL^+, namely

$$\varphi(x) := \downarrow_x X(\exists_g yz(Eyz \wedge \downarrow_{yz} Y(\exists_g w(Ezw \wedge Yzw) \wedge \exists_g uv(Euv \wedge Yuv \wedge Xu)))),$$

which is always satisfied by T_C but not by T_L. After binding the starting team to X, we move to a binary team T_{yz} with the property that for each edge in T_{yz}, we can move one step further on the graph and still be in T_{yz}. This can only be the case if T_{yz} consists exactly of all edges of the cycle. We bind this team and then move to another team T_{uv} that may only consist of edges on the cycle where the first node is in X. This can only be possible if the vertex in the original team was on the cycle. ◀

5.2.2 $\nu\mathrm{GF}^+ \not\subseteq \mathrm{GHTL}^+$

This section builds upon the previous one. In particular, we want to provide an inexpressibility result for GHTL^+ similar to Lemma 31. All the observations about local behaviour in guarded logics still apply, but we now have to account for the fact that teams can be bound with $\downarrow$ and then "carried" through global moves. To handle this, we can make use of the fact that these bound teams appear only positively, which means that a subsentence is still satisfied when replacing bound teams by supersets.

▶ **Lemma 35.** *Let $\tau = \sigma \cup \sigma^+$ be a signature and $l \in \mathbb{N}$. Let $\mathfrak{A}$ be a σ-structure and let $\mathfrak{A}_1$, $\mathfrak{A}_2$ be expansions of $\mathfrak{A}$ to τ so that for all $R \in \sigma^+$, we have $R^{\mathfrak{A}_1} \subseteq R^{\mathfrak{A}_2}$. Let $T_1 \subseteq T_2$ be teams in $\mathfrak{A}_2$ such that T_1 is an $(l+1)$-local cluster in T_2 (with respect to $\mathfrak{A}_2$) and for every $(l+1)$-local cluster C_2 in T_2, there is a $(l+1)$-local cluster C_1 in $T_1 \subseteq T_2$ so that $\mathfrak{A}_1, C_1 \simeq^{l+1} \mathfrak{A}_2, C_2$.*
Let $\varphi \in \mathrm{GHTL}^+$ be a formula with locality rank $\mathfrak{lr}(\varphi) \leq l$ such that all relations in σ^+ appear only positively. Then $\mathfrak{A}_1, T_1 \models \varphi$ implies $\mathfrak{A}_2, T_2 \models \varphi$.

▶ Note. Even though local clusters are supposed to capture the intuition of locally isolated partitions, the union of two or more disjoint local clusters is still a local cluster and a union of local isomorphisms is still a local isomorphism.

Proof of Lemma 35. Overall, the proof is very similar to the proof of Lemma 31. Concerning local behaviour, it is identical except for the case $\varphi = \downarrow_{\overline{x}} R\psi$. Here we need to show that $(\mathfrak{A}_1, T_1(\overline{x})), T_1 \models \psi$ implies $(\mathfrak{A}_2, T_2(\overline{x})), T_2 \models \psi$. If we can show that $(\mathfrak{A}_1, T_1(\overline{x})), T_1$ and $(\mathfrak{A}_2, T_2(\overline{x})), T_2$ satisfy the preconditions of the lemma, we are done by induction.

- First, we notice that $T_1(\overline{x}) \subseteq T_2(\overline{x})$ because $T_1 \subseteq T_2$.
- T_1 is still an $(l+1)$-local cluster in T_2: else, the neighbourhoods of T_1 and $T_2 \setminus T_1$ in $(\mathfrak{A}_2, T_2(\overline{x}))$ would overlap. This means there is a short path from T_1 to T_2, i.e. a tuple of guarded sets $(G_1, \ldots, G_n)$ so that $n \leq 2l+2$, $G_1 \in T_1(X)$ and $G_n \in T_2(X) \setminus T_1(X)$ with $\mathrm{dom}(T_2) = \mathrm{dom}(T_1) = X$. We can assume that this path is minimal, which in particular means that no G_i is guarded by $T_2(\overline{x})$ for $1 \leq i < n$. But this means this path would already exist in $\mathfrak{A}_2$ without the expansion by $T_2(\overline{x})$, and T_1 would not have been a cluster in T_2 in the first place.
- The same argument can be extended to all other clusters in T_2 and T_1 in the sense that they still are clusters after the expansion, so we can keep the correspondence of clusters between T_1 and T_2.
- For every pair of corresponding cluster $\mathfrak{A}_1, C_1 \simeq^{l+1} \mathfrak{A}_2, C_2$, with local isomorphism (ι, π), we have to show $(\mathfrak{A}_1, T_1(\overline{x})), C_1 \simeq^{l+1} (\mathfrak{A}_2, T_2(\overline{x})), C_2$. For this, it suffices to show that for all tuples $\overline{a}$ with $[\overline{a}] \subseteq \mathfrak{n}(C_1, l+1)$ we have $\overline{a} \in T_1(\overline{x})$ if and only if $\iota(\overline{a}) \in T_2(\overline{x})$. But $\overline{a} \in T_1(\overline{x})$ if and only if there is a $t_1 \in T_1$ with $\overline{a} = t(\overline{x})$, so $\iota(\overline{a}) = \iota(t_1(\overline{x})) = \pi(t_1)(\overline{x}) \in C_2(\overline{x}) \subseteq T_2(\overline{x})$ (the other direction is similar).

As soon as we reach a subsentence ψ, we can ignore the team, which leaves the interpretations of relations in σ^+ as the only remaining difference between $\mathfrak{A}_1$ and $\mathfrak{A}_2$. But these relations only appear positively, and all interpretations of these relations in $\mathfrak{A}_2$ are supersets of the interpretations in $\mathfrak{A}_1$. So any evaluation that satisfies ψ in $\mathfrak{A}_1$ can be applied to $\mathfrak{A}_2$ as well. ◀

▶ **Lemma 36.** *Let $\ell \in \mathbb{N}$ and $\varphi \in \mathrm{GHTL}^+$ with $\mathfrak{lr}(\varphi) \leq \ell$. Let $\mathfrak{B}_{2\ell+4}$ be as defined in Definition 32 and let $T_B \coloneqq \{x \mapsto b_{\ell+2}, x \mapsto c_{\ell+2}\}$ and $T_C \coloneqq \{x \mapsto c_{\ell+2}\}$ be two teams with domain $\{x\}$. Then*

$$\mathfrak{B}_{2\ell+4}, T_C \models \varphi \qquad \Rightarrow \qquad \mathfrak{B}_{2\ell+4}, T_B \models \varphi.$$

Proof. Again, it is enough to show that Lemma 35 is applicable. We assume $\sigma^+ = \emptyset$. We have $T_C \subseteq T_B$ and each assignment in T_B is its own $(\ell+1)$-local cluster, both of which are locally isomorphic to the cluster in T_C. ◀

▶ **Theorem 37.** $\nu\mathrm{GF}^+ \not\subseteq \mathrm{GHTL}^+$.

Proof. Similar to the proof of Theorem 34, we provide a formula $\varphi \in \nu\mathrm{GF}^+$ in the required format (see Proposition 8) that is satisfied by all $(\mathfrak{B}_n, T_C)$ but not by (B_n, T_B):

$$\varphi \coloneqq \forall_g x(Rx \to [\mathrm{gfp}_{S,x}\, \exists_g y(Exy \wedge Sy)]x)$$

is satisfied in $\mathfrak{B}_n$ if and only if all elements in the teams are part of the cycle, so exactly as required. ◀

6 Closing Remarks

We have seen that guarded team logics bring together many desirable properties of guarded logics and team logics. Hybrid team logics in particular not only offer a new perspective on well-known team logics as seen in Section 3, we could also easily show in Corollary 21 that the guarded variants are decidable and have the finite model property. In Section 5 we also showed that positive guarded hybrid team logic provides an intermediate step in the hierarchy of expressiveness between guarded inclusion logic and guarded positive greatest fixpoint logic. An important ingredient for this was the separation of local and global behaviour of guarded formulae, which we used to establish a strict hierarchy through Lemma 31 and Lemma 35.

It remains an open question whether $\mathrm{GF(inc)} \equiv \mathrm{GHTL}^+$ for sentences. Further, the expressiveness of downward closed guarded team logics like guarded dependence logic, was largely set aside for this paper. This also extends to related questions, e.g. which fragments of existential second order logic correspond to guarded independence logic and guarded hybrid team logic, respectively. It might also be worthwhile to revisit basic design questions for guarded team logics such as which notion of guardedness is appropriate in which contexts, or what effect the addition of further propositional connectives would have.

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A Proposition 13: Negative and Full Hybrid Team Logic

We outline a proof for Proposition 13. In both cases, one part of the equivalence is already shown in Lemma 10. The other part uses syntactic induction, which poses one particular challenge: when we evaluate a formula that contains binders, we may accumulate bound relations over the course of this evaluation. In the proof of Theorem 11, the translation from φ to φ^* "records" each change of the team using fixpoints, which automatically makes the bound teams available if needed. Therefore, we do not have to deal with the afromentioned accumulation in the induction.

In the proofs of Proposition 13 (and also if we wanted to prove $\mathrm{HTL}^+ \subseteq \mathrm{FO(inc)}$ directly), this strategy is not available. Instead, we want to save bound teams over fresh free variables and refer to them using exclusion (or inclusion) atoms. This results in a slightly more general statement.

▶ **Theorem 38.** *Let σ be a signature and Γ be a set of relational variables with $\sigma \cap \Gamma = \emptyset$. Let $(\overline{x}_X)_{X\in\Gamma}$ be a family of pairwise disjoint tuples of variables such that $|\overline{x}_X|$ is equal to the arity of X, and let $\overline{y}$ be a tuple of additional fresh variables.*

For every $\varphi(\overline{x},\Gamma) \in \mathrm{HTL}^-$ so that all $X \in \Gamma$ appear only negatively, there is a $\varphi^+(\overline{x},(\overline{x}_X)_{X\in\Gamma}) \in \mathrm{FO(exc)}$ such that

$$(\mathfrak{A},(X^{\mathfrak{A}})_{X\in\Gamma}),T \models \varphi(\overline{y},\Gamma) \quad\Leftrightarrow\quad \mathfrak{A},T\begin{bmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{bmatrix}_{X\in\Gamma} \models \varphi^+(\overline{y},(\overline{x}_X)_{X\in\Gamma}).$$

Proof. Literals in σ, $\wedge$, and quantification are straightforward. If $\varphi = \neg X\overline{y}$ for some $X \in \Gamma$, we use $\varphi^+ = (\overline{x}_X \bot \overline{y})$ as both formulae are satisfied if and only if $T(\overline{y}) \cap X^{\mathfrak{A}} = \emptyset$. This leaves disjunctions and binders. In both cases, we make use of the fact that we may use dependence atoms because $\mathrm{FO(exc)} \equiv \mathrm{FO(exc, dep)}$.

- If $\varphi = \alpha \vee \beta$, let

$$\begin{aligned}\varphi^+ = \exists zz'ww'(&=\!(\overline{y},zz') \wedge =\!(\overline{y},ww') \wedge (z=z' \vee w=w')\\ &\wedge (z\neq z' \vee (z=z'\wedge\alpha^+)) \wedge (w\neq w' \vee (w=w'\wedge\beta^+))).\end{aligned}$$

  In short, a team S satisfies φ^+ if and only if it can be split into two subteams $S_\alpha \cup S_\beta = S$ so that S_α satisfies α^* and S_β satisfies β^*. In particular, this split only depends on the assignments to $\overline{y}$, so if S has the form $S\restriction_{\overline{y}} \times S'$ with $\mathrm{dom}(S') = \mathrm{dom}(S) \setminus [\overline{y}]$, the subteams have the form $S_\alpha\restriction_{\overline{y}} \times S'$ and $S_\beta\restriction_{\overline{y}} \times S'$.

  Because $\overline{y}$ and all $\overline{x}_X$ are supposed to be disjoint, $T\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma}$ has this form and we have $\mathfrak{A},T\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma} \models \varphi^+$ if and only if there is some split $T_\alpha \cup T_\beta = T$ such that $\mathfrak{A},T_\alpha\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma} \models \alpha^+$ and $\mathfrak{A},T_\beta\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma} \models \beta^+$. By induction, this is equivalent to $\mathfrak{A},T_\alpha \models \alpha$ and $\mathfrak{A},T_\beta \models \beta$, which is equivalent to $\mathfrak{A},T \models \varphi$ as required.
- If $\varphi = \downarrow_{\overline{y}'} Y\psi(\overline{y},\Gamma,Y)$ for some $[\overline{y}'] \subseteq [\overline{y}]$, let

$$\varphi^+ = \forall\overline{x}_Y\exists zz'(=\!(\overline{x}_Y,zz') \wedge ((z=z'\wedge\psi^+(\overline{y},(\overline{x}_X)_{X\in\Gamma},\overline{x}_Y)) \vee (z\neq z'\wedge(\overline{x}_Y|\overline{y}')))).$$

  Similar to the above case, a team S satisfies φ^+ if and only if there is a superset S' of $S(\overline{y}')$ so that $S\left[\begin{smallmatrix}S'\\ \overline{x}_Y\end{smallmatrix}\right]$ satisfies ψ^+. (†)

  Applying (†) in one direction, if $\mathfrak{A},T\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma} \models \varphi^+$, there is a superset T' of $T(\overline{y}')$ so that $\mathfrak{A},T\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma}\left[\begin{smallmatrix}T'\\ \overline{x}_Y\end{smallmatrix}\right] \models \psi^+$. Because exclusion logic is downward closed, this implies $\mathfrak{A},T\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma}\left[\begin{smallmatrix}T(\overline{y}')\\ \overline{x}_Y\end{smallmatrix}\right] \models \psi^+$, which by induction is equivalent to $(\mathfrak{A},(X^{\mathfrak{A}})_{X\in\Gamma},T(\overline{y}')),T \models \psi$. Using the definition of binders, we get $(\mathfrak{A},(X^{\mathfrak{A}})_{X\in\Gamma}),T \models \varphi$ as required.

  For the other direction, we can trace the same steps backwards, except for the one using downward closure. Considering (†), we know that $\mathfrak{A},T\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma}\left[\begin{smallmatrix}T(\overline{y}')\\ \overline{x}_Y\end{smallmatrix}\right] \models \psi^+$ directly implies $\mathfrak{A},T\left[\begin{smallmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{smallmatrix}\right]_{X\in\Gamma} \models \varphi^+$ and we are done. ◀

▶ **Theorem 39.** *Let σ be a signature and Γ be a set of relational variables with $\sigma \cap \Gamma = \emptyset$. Let $(\overline{x}_X)_{X\in\Gamma}$ be a family of pairwise disjoint tuples of variables such that $|\overline{x}_X|$ is equal to the arity of X, and let $\overline{y}$ be a tuple of additional fresh variables.*

For every $\varphi(\overline{x},\Gamma) \in \mathrm{HTL}$ there is a $\varphi^\#(\overline{x},(\overline{y}_X)_{X\in\Gamma}) \in \mathrm{FO(inc, exc)}$ such that

$$(\mathfrak{A},(X^{\mathfrak{A}})_{X\in\Gamma}),T \models \varphi(\overline{y},\Gamma) \quad\Leftrightarrow\quad \mathfrak{A},T\begin{bmatrix}X^{\mathfrak{A}}\\ \overline{x}_X\end{bmatrix}_{X\in\Gamma} \models \varphi^\#(\overline{y},(\overline{x}_X)_{X\in\Gamma}).$$

Proof. Again, ordinary literals, $\wedge$, and quantification is straightforward. Negative literals in Γ and $\vee$ are exactly as in the proof above, which does not rely on any properties that are specific to FO(exc). This leaves binders.

In the proof above, we approximated the bound team with a superset, which was sufficient because of downward closure. Here, we can identify the team directly using both inclusion and exclusion atoms. So if $\varphi = \downarrow_{\overline{y}'} Y \psi(\overline{y}, \Gamma, Y)$ for some $[\overline{y}'] \subseteq [\overline{y}]$, let

$$\varphi^{\#} = \forall \overline{x}_Y \exists z z' (=(\overline{x}_Y, z z') \wedge ((z = z' \wedge (\overline{x}_Y \subseteq \overline{y}') \wedge \psi^{\#}) \vee (z \neq z' \wedge (\overline{x}_Y | \overline{y}')))).$$

A team S satisfies $\varphi^{\#}$ if and only if $S\left[\frac{S(\overline{y}')}{\overline{x}_Y}\right]$ satisfies $\psi^{\#}$. With that, we can adapt the above proof to not require downward closure and are done. ◀

Proposition 13 follows from these theorems for $\Gamma = \emptyset$ and Lemma 10.

B Syntactic Definitions of Guarded Team Logics

Proof that for a given finite signature, every formula in guarded logics is expressible as a formula in non-guarded logics:

▶ **Lemma 40.** *Let σ be a finite signature. Let $\mathcal{L}$ be a first-order team logic and let $\mathcal{GL}$ be the guarded variant in the sense of Definition 16, i.e. quantification is replaced by guarded quantification. Then for all $\varphi \in \mathcal{GL}$ there is a $\varphi^\sigma \in \mathcal{L}$ so that for all σ-structures $\mathfrak{A}$ with team T we have*

$$\mathfrak{A}, T \models \varphi \quad \Leftrightarrow \quad \mathfrak{A}, T \models \varphi^\sigma.$$

Proof. It suffices to show that quantification is replaceable. For this, we use an auxilliary formula $\mathcal{G}^\sigma(\overline{x}) \in \text{FO}$ that is satisfied by a team T if and only if $T(\overline{x})$ is guarded. With this, for all formulae ψ with $\text{free}(\psi) = [\overline{xy}]$, we have

$$\exists_g \overline{x} \psi \equiv \exists \overline{x} (\mathcal{G}^\sigma(\overline{xy}) \wedge \psi^\sigma) \quad \text{and} \quad \forall_g \overline{x} \psi \equiv \forall \overline{x} (\neg \mathcal{G}^\sigma(\overline{xy}) \vee (\mathcal{G}^\sigma(\overline{xy}) \wedge \psi^\sigma),$$

where $\neg \mathcal{G}^\sigma(\overline{xy})$ is the negated formula in negation normal form. As $\mathcal{G}^\sigma$ is in FO and therefore flat, any formula with the same purpose for GF can be used (see for example [17, Section 3]). ◀

In particular, this proves that the properties of Proposition 6 transfer to their guarded variants, which we show for union-closed logics as an example.

▶ **Lemma 41.** *Let $\mathcal{L}$ be a union-closed team logic and $\mathcal{GL}$ be the guarded variant of $\mathcal{L}$. Then $\mathcal{GL}$ is also union-closed.*

Proof. Let $\varphi \in \mathcal{GL}$, and let $\mathfrak{A}$ be a σ-structure with teams T_1 and T_2 such that $\mathfrak{A}, T_1 \models \varphi$ and $\mathfrak{A}, T_2 \models \varphi$. By Lemma 40, there exists a $\varphi^\sigma \in \mathcal{L}$ that is equivalent to φ on σ-structures, so $\mathfrak{A}, T_1 \models \varphi^\sigma$ and $\mathfrak{A}, T_2 \models \varphi^\sigma$. The union-closure of $\mathcal{L}$ yields $\mathfrak{A}, T_1 \cup T_2 \models \varphi^\sigma$ and therefore $\mathfrak{A}, T_1 \cup T_2 \models \varphi$ as required. ◀

We see that this same strategy can be applied to all properties in Proposition 4.

B.1 A Note on Structures with Infinite Signature

Over infinite signatures, there are formulae in GTL that cannot have an equivalent formula in FO with team semantics, which we prove by giving a counterexample:

Let $\sigma = \{E_1, E_2, \ldots\}$ consist of infinitely many binary relations, and let $\mathfrak{A}$ be a σ-structure with universe $\mathbb{N} \times \{0, 1\}$ and $((n, b), (m, c)) \in E_i$ if and only if $n = m = i$ and $b \neq c$, i.e. each relation consist of exactly one (symmetric) edge and each element in the universe has exactly one partner. The sentence $\varphi := \forall_g x \exists_g y (x \neq y)$ is satisfied if and only if each element has a partner, so it is satisfied in $\mathfrak{A}$, but not in any reduct of $\mathfrak{A}$. But as we know, any formula $\psi \in \mathrm{FO}$ can only contain a finite set of relations $\sigma_\psi \subset \sigma$, and is therefore satisfied in $\mathfrak{A}$ if and only if it is satisfied in the reduct of $\mathfrak{A}$ to σ_ψ. Therefore, ψ cannot be equivalent to φ.

B.2 The Standard Guarded Fragment with Team Semantics

We take a closer look at GF with team semantics, i.e. FO with team semantics where each quantification in a formula is accompanied by a guard that only consists of one atom (see [1]). We provide an example for a property that is expressible in GTL(inc), but not in GF(inc). In GTL(inc), the sentence $\varphi := \exists_g xy((Exy \vee Fxy) \wedge (y \subseteq x))$ is satisfied by a (directed) graph with coloured edges if and only if there is an infinite walk along E- *and* F-edges. This is impossible to express in GF(inc), as can be seen by comparing the following two structures.

Let $\mathfrak{A}$ be a structure with universe $A = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid m \leq n\}$ and relations $E^{\mathfrak{A}} := \{((n, i), (n, i + 1)) \mid i \equiv 0 \mod 2\}$ and $F^{\mathfrak{A}} := \{((n, i), (n, i + 1)) \mid i \equiv 1 \mod 2\}$, i.e. $\mathfrak{A}$ consists of finite paths of increasing length that alternate between E- and F-edges. Let $\mathfrak{B}$ be a structure with universe $B := A \uplus \{\infty\} \times \mathbb{N}$ and relations similar to $\mathfrak{A}$ so that we again have an infinite number of alternating paths, but now including a path of infinite length. We have $\mathfrak{B} \models \varphi$, but $\mathfrak{A} \not\models \varphi$ in GTL.

In both $\mathfrak{A}$ and $\mathfrak{B}$, every $E(F)$-edge is disjoint from every other $E(F)$-edge. This means that for every GF-guarded team T in one of the structures, $T \models (\overline{x} \subseteq \overline{y})$ if and only if $T \models \overline{x} = \overline{y}$. Therefore, $\mathfrak{A}$ and $\mathfrak{B}$ can only be distinguished by a sentence in GF(inc) if they can be distinguished by a sentence in GF. Due to flatness, this is only possible if they can be distinguished by a sentence in the standard GF, which is known to be impossible.

C The Proof of Theorem 20 in Detail

C.1 Translations

In Table 1, we provide a detailed account of all translations used in the proof of Theorem 20. For better readability, we use a few abbreviations:

Let $\overline{x}, \overline{y}, \overline{z}$ be tuples. If $[\overline{y}] \subseteq [\overline{x}]$, we use $\alpha(\overline{y}) \subseteq \beta(\overline{x})$ as an abbreviation for $\forall_g \overline{y}(\alpha(\overline{y}) \rightarrow \exists_g \overline{z}(\beta(\overline{x})))$ with $[\overline{x}] = [\overline{y}] \uplus [\overline{z}]$. If $[\overline{x}] \subseteq [\overline{y}]$, we use $\alpha(\overline{y}) \subseteq \beta(\overline{x})$ as an abbreviation for $\forall_g \overline{y}(\alpha(\overline{y}) \rightarrow \beta(\overline{x}))$. Correspondingly $(\alpha(\overline{y}) = \beta(\overline{x})) := (\alpha(\overline{y}) \subseteq \beta(\overline{x})) \wedge (\beta(\overline{x}) \subseteq \alpha(\overline{y}))$.

C.2 Further Details

We provide a proper account of the model checking game that is referenced in the proof.

For all structures $\mathfrak{A}$, team T and $\varphi \in \mathrm{GHTL}$, we can inductively define a model checking game $\mathfrak{G}(\mathfrak{A}, T, \varphi)$ as a two player game with perfect information. Player **II** (*"Verifier"*) wants to show that $\mathfrak{A}, T \models \varphi$, while player **I** (*"Falsifier"*) tries to spoil it for **II**. During the game, both players may move to *positions* $(S, \psi)_{\mathfrak{A}}$ with S being a team in $\mathfrak{A}$ and $\psi \in \mathrm{GHTL}$. The starting position for the game $\mathfrak{G}(\mathfrak{A}, T, \varphi)$ is $(T, \varphi)_{\mathfrak{A}}$.

Table 1 Translations used in the proof of Theorem 20.

$\psi =$	$\varphi^*(\psi) :=$
$\alpha(\overline{x})$ for some literal α in $\sigma \cup \mathcal{X}(\varphi)$	$R_\psi \overline{x} \subseteq \alpha(\overline{x})$
$\psi_1(\overline{y}_1) \wedge \psi_2(\overline{y}_2)$ with $[\overline{x}] = [\overline{y}_1] \cup [\overline{y}_2]$	$(R_\psi \overline{x} = R_{\psi_1} \overline{y}_1) \wedge (R_\psi \overline{x} = R_{\psi_2} \overline{y}_2)$
$\psi_1(\overline{y}_1) \vee \psi_2(\overline{y}_2)$ with $[\overline{x}] = [\overline{y}_1] \cup [\overline{y}_2]$	$(R_\psi \overline{x} \subseteq (R_{\psi_1} \overline{y}_1 \vee R_{\psi_2} \overline{y}_2))$ $\wedge (R_{\psi_1} \overline{y}_1 \subseteq R_\psi \overline{x}) \wedge (R_{\psi_2} \overline{y}_2 \subseteq R_\psi \overline{x})$
$\exists_g \overline{y} \phi(\overline{xy})$ with $[\overline{x}] = \mathrm{free}(\phi) \setminus [\overline{y}]$	$\forall_g \overline{x} (R_\psi \overline{x} \leftrightarrow \exists_g \overline{y} R_\phi \overline{xy})$
$\forall_g \overline{y} \phi(\overline{xy})$ with $[\overline{x}] = \mathrm{free}(\phi) \setminus [\overline{y}]$	$\forall_g \overline{x} (R_\psi \overline{x} \leftrightarrow \forall_g \overline{y} R_\phi \overline{xy})$
$\downarrow_{\overline{x}} X \phi(\overline{xy})$	$(X\overline{x} = R_\psi \overline{xy}) \wedge (R_\psi \overline{xy} = R_\phi \overline{xy})$

In any position $(S, \psi)_{\mathfrak{A}}$, the rules of the game are as follows:

- If ψ is a literal, **II** wins if $\mathfrak{A}, S \models \psi$, else **I** wins.
- If $\psi = \psi_1 \wedge \psi_2$, then **I** decides whether the game proceeds from $(S, \psi_1)_{\mathfrak{A}}$ or $(S, \psi_2)_{\mathfrak{A}}$.
- If $\psi = \psi_1 \vee \psi_2$, then **II** chooses a split $S_1 \cup S_2 = S$ and **I** decides whether the game proceeds from $(S_1, \psi_1)_{\mathfrak{A}}$ or $(S_2, \psi_2)_{\mathfrak{A}}$.
- If $\psi = \exists_g \overline{x} \phi$, then **II** chooses a guarded update S' of S and the game proceeds from $(S', \phi)_{\mathfrak{A}}$.
- If $\psi = \forall_g \overline{x} \phi$, then let S' be the universal guarded update of S and the game proceeds from $(S', \phi)_{\mathfrak{A}}$.
- If $\psi = \downarrow_{\overline{x}} X \phi$, then the game proceeds from $(S, \phi)_{(\mathfrak{A}, S(\overline{x}))}$.

As we can see, the definition of the game mirrors the semantics of GHTL. As such, it is clear that a winning strategy for **II** in the game $\mathfrak{G}(\mathfrak{A}, T, \varphi)$ is equivalent to $\mathfrak{A}, T \models \varphi$.

In the proof, we use the fact that **II** has a winning strategy in the game $\mathfrak{G}(\mathfrak{A}, T, \varphi(\overline{x}))$ if and only if there is an expansion $\mathfrak{A}^*$ of $\mathfrak{A}$ with $R_\varphi^{\mathfrak{A}^*} = T(\overline{x})$ so that $\mathfrak{A}^* \models \varphi^*$. This strategy is provided by the R_ψ in the sense that each of the translations in Appendix C.1 corresponds to a winning strategy in one of the positions of the game. For example, if **II** has a winning strategy in position $(R_\phi^{\mathfrak{A}^*}, \phi)_{\mathfrak{A}}$, then $\mathfrak{A}^* \models \forall_g \overline{x} (R_\psi \overline{x} \leftrightarrow \exists_g \overline{y} R_\phi \overline{xy})$ if and only if **II** has a winning strategy in position $(R_\psi^{\mathfrak{A}^*}, \psi)_{\mathfrak{A}}$ for $\psi = \exists_g \overline{y} \phi(\overline{xy})$.

As already stated at the end of the proof, there are a few technical details that have to be considered:

1. In general, the arity of R_ψ should correspond to the number of free variables in ψ. If ψ is a sentence, then R_ψ would be 0-ary. We circumvent this by choosing R_ψ to be unary. This way, the winning strategy should include the position $(\emptyset, \psi)_{\mathfrak{A}}$ if $R_\psi^{\mathfrak{A}^*}$ is empty, and $(\{\emptyset\}, \psi)_{\mathfrak{A}}$ else. In the latter case, the specific interpretation of R_ψ does not matter, as it only serves as a distinction between the empty assignment $\{\emptyset\}$ and the empty team.
2. This also requires a modification of the translation in the case that ψ is a sentence that starts with a quantification.
   If $\psi = \exists_g \overline{x} \phi(\overline{x})$, then $\varphi^*(\psi) = \exists_g \overline{y} R_\psi \overline{y} \rightarrow \exists_g \overline{x} R_\phi \overline{x}$.
   If $\psi = \forall_g \overline{x} \phi(\overline{x})$, then $\varphi^*(\psi) = \exists_g \overline{y} R_\psi \overline{y} \rightarrow \forall_g \overline{x} R_\phi \overline{x}$.
   This prevents false positives by making sure that the interpretation of R_ϕ is non-empty if the interpretation of R_ψ is non-empty.

D Proof of Lemma 31

Recall Lemma 31:

Let $\mathfrak{A}$ be a σ-structure and $l \in \mathbb{N}$. Let T_1 and T_2 be teams in $\mathfrak{A}$ such that for every $(l+1)$-local cluster C_2 in T_2, there is a $(l+1)$-local cluster C_1 in T_1 such that $\mathfrak{A}, C_1 \simeq^{l+1} \mathfrak{A}, C_2$. Let $\varphi \in \mathrm{GF(inc)}$ be a formula with locality rank $\mathfrak{lr}(\varphi) \leq l$. Then $\mathfrak{A}, T_1 \models \varphi$ implies $\mathfrak{A}, T_2 \models \varphi$.

Proof. We use syntactic induction.

- For classical literals, 1-local isomorphisms imply atomic equivalence.
- Inclusion atoms are also handled by 1-local isomorphism: we assume $\mathfrak{A}, T_1 \models (\overline{x} \subseteq \overline{y})$ and $t \in T_2$. Then there is a 1-local cluster C_2 in T_2 containing t and a corresponding 1-local cluster C_1 in T_1 with bijection π from C_1 to C_2. We then find some $t' \in T_1$ so that $t'(\overline{y}) = \pi^{-1}(t)(\overline{x})$ by assumption. Because C_1 is a 1-local cluster, we also have $t' \in C_1$ and by commutativity of the maps this yields $\pi(t')(\overline{y}) = t(\overline{x})$.
- Conjunctions are straightforward. For disjunctions, we show that every split $T_1^1 \cup T_1^2 = T_1$ can be used to find a split $T_2^1 \cup T_2^2 = T_2$ that preserves the preconditions of the lemma. For this, let C_2 be a cluster in T_2 and C_1 be the corresponding cluster in T_1 with local isomorphism (ι, π). Clearly, $C_1^1 = C_1 \cap T_1^1$ and $C_1^2 = C_1 \cap T_1^2$ form a split of C_1, and thus $C_2^1 = \pi(C_1^1)$ and $C_2^2 = \pi(C_1^2)$ form a split of C_2. Let T_2^1 be the union of all C_2^1 and T_2^2 be the union of all C_2^2. Then $T_2^1 \cup T_2^2 = T_2$. For all $i, j \in \{1, 2\}$, C_i^j is a cluster in T_i^j and $\mathfrak{A}, C_1^j \simeq^{l+1} \mathfrak{A}, C_2^j$. By construction, we find a corresponding C_1^j for all C_2^j and are done.
- For local universal quantification, the guarded universal update of each $(l+1)$-local cluster is still a l-local cluster, and the guarded universal update of two $(l+1)$-locally isomorphic clusters are still l-locally isomorphic.
- For local existential quantification, we can argue similarly to the local universal case. For any cluster C_2 in T_2, there might be several $l+1$-locally isomorphic clusters $C_1^1, C_1^2, \ldots$ in T_1 which might not be l-locally isomorphic in the update of T_1 anymore. However, we just have to update C_2 in a way so that there is *some* corresponding cluster in the update of T_1, for which we can choose any of the possible candidates, e.g. C_1^1, and update C_2 according to the image of C_1^1 under the local isomorphism.
- For global quantification, when evaluating a (sub)sentence, the current teams are irrelevant and we have $\mathfrak{A}, T_1 \models \varphi$ if and only if $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A}, T_2 \models \varphi$. ◀