Sparse Graphic Degree Sequences Have Planar Realizations

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Abstract

A sequence $d = (d_1, d_2, \ldots, d_n)$ of positive integers is *graphic* if it is the degree sequence of some simple graph G, and *planaric* if it is the degree sequence of some simple planar graph G. It is known that if $\sum d \le 2n-2$, then d has a realization by a forest, hence it is trivially planaric. In this paper, we seek bounds on $\sum d$ that guarantee that if d is graphic then it is also planaric. We show that this holds true when $\sum d \le 4n-4-2\omega_1$, where ω_1 is the number of 1's in d. Conversely, we show that there are graphic sequences with $\sum d = 4n-2\omega_1$ that are non-planaric. For the case $\omega_1 = 0$, we show that d is planaric when $\sum d \le 4n-4$. Conversely, we show that there is a graphic sequence with $\sum d = 4n-2$ that is non-planaric. In fact, when $\sum d \le 4n-6-2\omega_1$, d can be realized by a graph with a 2-page book embedding.

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1 Introduction

Background. In a graph G with n vertices, the degree of a vertex is the number of edges incident to it. Let $\deg(G)$ denote the sequence of length n of vertex degrees of G. The Degree Realization problem concerns deciding, given a sequence d of n positive integers, whether d has a *realizing graph*, namely, a graph G with n vertices such that $\deg(G) = d$, and finding such a graph if exists. A *graphic* sequence is one admitting a realizing graph. A full characterization of graphic degree sequences was given by Erdös and Gallai [7]. Havel and Hakimi [11, 12] described an algorithm that, given a sequence d, generates a realization, or verifies that d is not graphic.

The realizability characterization of [7] for general graphs takes into account all the elements of the sequence d. In contrast, the realizability of degree sequences by some special graph classes can be characterized more economically. An extreme example is realizability by a *forest* (cycle-free graph). Here, a single parameter suffices, namely, the *volume* $\sum d = \sum_{i=1}^{n} d_i$ of d. Concretely, if $\sum d \le 2n-2$, then d can be realized by a forest, and if $\sum d \ge 2n$ then it cannot [10]. A slightly more involved and less economical characterization applies for realizations by *cacti* graphs. A cactus graph is a connected graph in which every edge occurs on at most one cycle, namely, different cycles do not share edges (but may share one vertex).

The paper [16] gives a full characterization for sequences that can be realized by a cactus graph based only on the volume of the sequence, the number of 2's in the sequence, and the number of odd degrees in the sequence.

In this paper, we are interested in realizability by planar graphs, which turns out to be a challenging task that is still an open problem after more than half a century. A sequence $d = (d_1, d_2, \ldots, d_n)$ of positive integers is *planaric* if it is the degree sequence of some planar graph G. Planaric sequences were studied in, e.g., [13, 1, 18, 8, 9, 17] and more. At the moment, however, a complete characterization for planaric sequences is not yet available.

Contributions. Let d be a sequence with n positive integers and let ω_i be the *multiplicity* of i in d. This paper investigates the impact of the volume $\sum d$ and the multiplicity parameters ω_1 and ω_2 of the sequence d on its realizability by a planar graph. One direction follows from [18] which implies that if $\sum d > 6n - 12 - 2\omega_2 - 4\omega_1$ then d is not planaric. (This is tight in the sense that there are planaric sequences with $\sum d \leq 6n - 12 - 2\omega_2 - 4\omega_1$.) In this paper, we focus on the converse direction, i.e., we seek bounds on these parameters that guarantee that if d is graphic then it is also planaric. A simple bound is obtained by recalling the above-mentioned known fact that if $\sum d \leq 2n - 2$, then d has a a realization by a forest with $(2n - \sum d)/2$ components, hence it is planaric [10]. Here, we give stronger bounds for this problem, depending on $\sum d$ and ω_1. It turns out that most of the technical effort involves handling the *leaf-free* case (the case in which d does not contain 1s). We establish the following.

▶ **Theorem 1.** *Every graphic sequence d with $\omega_1 = 0$ and $\sum d \leq 4n - 4$ is planaric.*

This in turn enables us to prove our more general main result.

▶ **Theorem 2.** *Every graphic sequence d with $\sum d \leq 4n - 4 - 2\omega_1$ is planaric.*

In fact, when $\sum d \leq 4n - 6 - 2\omega_1$, our constructed realizing graphs are not only planar but also enjoy a 2-page book embedding, yielding the following corollary.

▶ **Corollary 3.** *Every graphic sequence d with $\sum d \leq 4n - 6 - 2\omega_1$ can be realized by a graph with a 2-page book embedding.*

This corollary can be interpreted as saying that the family $\mathcal{D}$ of all graphic sequences d such that $\sum d \leq 4n - 6 - 2\omega_1$ enjoy a 2-page book embedding realization. In comparison, the main result of [2] gives a partition of $\mathcal{D}$ into *non-outerplanaric* sequences and sequences enjoying a 2-book embedding realization. Moreover, by [2] if d is outerplanaric and $\sum d \geq 2n$, then $\sum d \leq 4n - 6 - 2\omega_1$. Therefore, this corollary can be seen as an alternative way to obtain the main result of [2].

Conversely, we show that there are graphic sequences with $\sum d \leq 4n - 2\omega_1$ that are non-planaric. For the case of $\omega_1 = 0$, there is a known graphic but non-planaric sequence with $\sum d = 4n - 2$. The gap between the bounds for $\omega_1 > 0$ is left for future study.

Note that the parameters $\sum d$, ω_1 and ω_2 are insufficient for charting the borderline between planaric and non-planaric sequences, and leave a "grey area" in between our upper and lower bounds, in which some sequences are planaric and some are not. This hints that a full characterization may require using additional parameters, and perhaps involve all the degrees, as is the case with realizability by general graphs.

Related work. Planaric sequences for regular planar graphs were classified in [13], and planaric bipartite biregular degree sequences were studied in [1]. In [18], Schmeichel and Hakimi determined which graphic sequences with $d_1 - d_n = 1$ are planaric, and presented similar results for $d_1 - d_n = 2$ with a small number of unsolved cases. Some of the sequences left unsolved in [18] were later resolved in [8, 9]. Some additional studies on special cases of the planaric degree realization problem are discussed in Rao's survey [17].

An economical characterization is given in [5] for the class of 2-trees, which is a subfamily of planar graphs, that is based on $\sum d$, ω_2, and ω_{odd}. For the Outerplanar Degree Realization problem, a full characterization of *forcibly* outerplanar graphic sequences (namely, sequences each of whose realizations is outerplanar) was given in [6]. A characterization of the degree sequence of maximal outerplanar graphs having exactly two 2-degree nodes was provided in [5]. A characterization of the degree sequences of maximal outerplanar graphs with at most four vertices of degree 2 was given in [15]. In [3] it is shown that a nonincreasing n-element graphic sequence d is outer-planaric if either $\omega_1 = 0$ and $\sum d \le 3n - 3$, or $\omega_1 > 0$ and $\sum d \le 3n - \omega_1 - 2$. Conversely, there are graphic sequences that are not outer-planaric with $\omega_1 = 0$ and $\sum d = 3n - 2$, as well as ones with $\omega_1 > 0$ and $\sum d = 3n - \omega_1 - 1$

2 Preliminaries

Given a sequence $d = (d_1, \ldots, d_n)$ of n integers, we assume that it is non-increasing, namely that $d_{i+1} \le d_i$, for every $i \in \{1, \ldots, n-1\}$. Given two sequences d and d', denote by $d \ominus d' = (d_1 - d'_1, \ldots, d_n - d'_n)$ their componentwise difference. For a nonincreasing sequence d of n nonnegative integers, let $\mathsf{pos}(d)$ denote the prefix consisting of the positive integers of d. We use the shorthand a^k to denote a subsequence of k consecutive a's. For any graph G, let $E(G)$ be the edge set of G.

Euler's theorem implies that if d is planaric, where $n \ge 3$, then $\sum d \le 6n - 12$. Call d a *maximal* Euler sequence if $\sum d = 6n - 12$.

Known planaric sequences. A sequence d is called a *k-sequence* if $d_1 - d_n = k$. Schmeichel and Hakimi [18] divided the analysis for 2-sequences into maximal and non-maximal 2-sequences. They left a few open cases, some of which were resolved by Fanelli [8, 9].

▶ **Lemma 4** ([9, 18])**.** *Every graphic non-maximal Euler 2-sequence is planaric except for* $(4^5, 2)$, $(5^5, 3^3)$, $(5^{11}, 3)$, $(5^{13}, 3)$, $(6^{n-7}, 4^7)$ *for* $n > 7$, $(7, 5^{15})$, $(7, 5^{17})$, *and possibly* $(7^3, 5^{17})$, *whose status is unresolved.*

The following lemma describess another (relatively small[1]) class of degree sequences known to be planaric.

▶ **Lemma 5** ([18],Theorem 5(a))**.** *For* $n \ge 3$, *if* d *such that* $d_1 \ge d_2 \ge \cdots \ge d_n$ *is graphic,* $\sum d \le 6n - 12$ *and* $d_3 \le 3$, *then* d *is planaric.*

▶ **Observation 6.** *If* G *is a planar graph, then adding a parallel edge to* $E(G)$ *maintains the planarity of the resulting graph.*

Minimum pivot Havel-Hakimi algorithm. The *minimum pivot* version of the Havel-Hakimi algorithm [12, 11] for realizing a degree sequence $d = (d_1, \ldots, d_n)$ associated with the vertices $v_1, \ldots, v_n$, presented explicitly in [19] is based on repeatedly performing the following operation, hereafter referred to as the *MP-step*, until all the vertices reach their required degrees. Suppose that the current sequence of residual degrees is $\delta = (\delta_1, \cdots, \delta_h)$.

[1] The condition $d_3 \le 3$ implies that the number of sequences in the class is upper bounded by n^4.

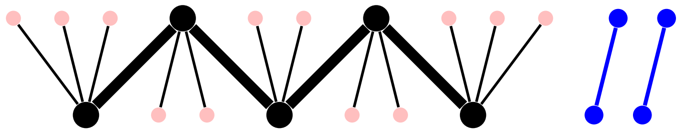


Figure 1 Realization of the forestic sequence $(4^5, 1^{16})$ by an alternating caterpillar and a matching. The spine is depicted by bold black vertices and edges.

The MP-step.

- Pick as a pivot one of the vertices with the minimum non-zero residual degree v_i whose degree is δ_i (break ties arbitrarily).
- Set v_i's neighbors to be the δ_i vertices with the highest residual degrees $v_{i_1}, v_{i_2}, \ldots, v_{i_{\delta_i}}$ (break ties arbitrarily).
- Set $\delta_i \leftarrow 0$ and reduce by 1 the residual degrees of its selected neighbors. That is, set $\delta_{i_j} \leftarrow \delta_{i_j} - 1$ for $j \in \{1, \ldots, \delta_i\}$.

The Minimum pivot Havel-Hakimi algorithm terminates when all the n residual degrees are zero, that is, when $\delta_j = 0$ for $j \in \{1, \ldots, n\}$. The key observation is that, whenever the MP-step transforms the residual degree sequence δ into δ', the following holds: δ is graphic if and only if δ' is graphic.

Caterpillar-based realizations. It is known that if $\sum d \leq 2n - 2$, then d can be realized by a cycle-free graph (forest). In this case, d is called a *forestic* sequence. If $\sum d = 2n - 2$, then d can be realized by a tree and the sequence is called a *treeic* sequence. The following lemma concerns the realization of forestic sequences. We make use of a special type of realizations of forestic and treeic sequences by *caterpillar trees*. In a caterpillar tree $G = (V, E)$, all the non-leaves vertices are arranged on a path, called the *spine*.

▶ **Lemma 7.** *A forestic sequence $d = (d_1, d_2, \ldots, d_n)$ of positive integers can be realized by a union of a caterpillar tree and a matching. Moreover, the order of the vertices on the spine may be chosen arbitrarily.*

For the sake of our later constructions, let us outline the way the realization of Lemma 7 is obtained. Run the *minimum pivot* version of the Havel-Hakimi algorithm while applying the MP-step until all the degrees in the residual sequence are at most 2. Then realize the residual sequence with a path (of arbitrary order) and a matching. The interior of the path is the spine of the caterpillar while the pivots and the two end vertices of the path are the leaves of the caterpillar. Our later constructions make critical use of the “arbitrary ordering” property. It is convenient to illustrate a caterpillar tree with its spine drawn horizontally (in a zigzagged fashion), and its groups of leaves drawn alternately above and below the spine. We refer to this representation as an *alternating caterpillar.* (See Figure 1.)

Outer-planar graphs. An *outer-planar* graph is a graph that has a planar embedding in which all the vertices occur on the outer face. A *maximal outer-planar* (MOP) graph is an outer-planar graph such that adding any new edge to it results in a non-outer-planar graph. Given a planar embedding in which all the vertices occur on the outer face, an *external* edge is an edge residing on the outer face. If $d = (d_1, \ldots, d_n)$ is an outer-planaric degree sequence where $n \geq 2$, then $\sum d \leq 4n - 6$, with equality if and only if d is maximal outer-planaric [20].

A (directed) *circuit* in G is an *ordered set* of vertices $C = \{v_0, v_1, \ldots, v_{k-1}\}$ such that $v_i \neq v_{(i+1) \bmod k}$ and $(v_i, v_{(i+1) \bmod k}) \in E$ for every $i = 0, \ldots, k - 1$. In a directed circuit, vertices may appear more than once while each edge may appear at most once. Note that

$e = (u, w)$ is an external edge if u and w are neighbors on a circuit which is part of the outer face. All other edges of G are *internal*. An *internal triangle* in G is a triangle all of whose edges are internal. Jao and West [14] show that the number of internal triangles in a maximal outer-planar graph (MOP) is related to ω_2.

▶ **Lemma 8** ([14]). *Let G be a MOP on n vertices, let $d = \deg(G)$, and let t be the number of internal triangles. If $n \geq 4$, then $t = \omega_2 - 2$.*

Given a graph $G = (V, E)$, let $E = E_1 \cup \ldots \cup E_p$ be a partition of its edges such that each subgraph $G_i = (V, E_i)$ is outerplanar. For a book embedding of G, think of a book in which the pages (half-planes) are filled by outerplanar embeddings of the G_i's such that the vertices are embedded on the spine of the book and in the same location on each page. This constraint is equivalent to requiring that the vertices appear in the same order along the cyclic order of each of the outerplanar embeddings of the G_i's. The *book thickness* or *pagenumber* [4] is the minimal number of pages for which a graph has a valid book embedding. Note that a graph is outerplanar if and only if it has pagenumber 1, and it is known that the pagenumber of planar graphs is at most 4 [21].

▶ **Lemma 9** ([4]). *A graph G has a 2-book embedding if and only if G is a subgraph of a Hamiltonian planar graph.*

3 Tools and sufficient conditions for OP and MOP realizations

3.1 Leaf-free sequences with $\sum d \leq 4n - 6$ and $\omega_2 = 2$ are outer-planaric

We start with a special class of sequences for which we present a basic construction of an *outerplanar* realization. A number of our later constructions of planar realizations start from this basic construction (typically applied to a sub-sequence) and modify it in various ways in order to derive the required planar realization. In many of these cases, the modification requires adding a few more edges. For each additional edge, this requires finding a pair of vertices u, w such that

(i) the edge (u, w) does not appear in the construction, and
(ii) adding it preserves planarity.

We make use of the following lemma, established in [2].

▶ **Lemma 10** ([2]). *Let d be a graphic sequence such that $d_1 \geq d_2 \geq \cdots \geq d_n$ with $\omega_1 = 0$, $\omega_2 = 2$, and $\sum d \leq 4n - 6$. Then d is outer-planaric.*

We outline the construction of the realizing graph, since it will be instrumental in what follows. Let $d' = \mathsf{pos}(d \ominus (2^n))$. Then $n' = n - 2$ and $\sum d' \leq 2n' - 2$ because

$$\sum d' = \sum d - 2n \leq 2n - 6 = 2(n' + 2) - 6 = 2n' - 2 .$$

By Lemma 7, d' can be realized by a graph $G' = (V', E')$ composed of a union of an alternating caterpillar T' and a matching M'. Next, construct an outer-planar realization G for d based on G' as follows.

Let $S = (x_1, \ldots, x_s)$ be the vertices on the spine of T', and let $X_i = \{\ell_{i,1}, \ldots, \ell_{i,k_i}\} \subseteq V'$ be the leaves adjacent to the spine vertex x_i, for $i \in \{1, \ldots, s\}$. Note that in T', $deg(x_i) = k_i + 1$ for $i \in \{1, s\}$ and $deg(x_i) = k_i + 2$ otherwise. Assume that the matching is $M' = \{(y_1, z_1), (y_2, z_2), \ldots, (y_t, z_t)\}$. To construct an outer-planar realization of d, add to

G' a set of edges that form a Hamiltonian cycle (including two additional new vertices of degree 2). The construction consists of two steps. We describe it for $s = 0$ and for an odd s; an analogous construction applies for a positive even s.

(1) Construct two paths
$P_1 = (x_1, \ell_{2,1}, \dots, \ell_{2,k_2}, x_3, \ell_{4,1}, \dots, \ell_{4,k_4}, \dots, x_{s-2}, \ell_{s-1,1}, \dots, \ell_{s-1,k_{s-1}}, x_s, y_1, y_2, \dots, y_t)$,
$P_2 = (z_t, \dots, z_2, z_1, \ell_{s,k_s}, \dots, \ell_{s,1}, x_{s-1}, \dots, \ell_{3,k_3}, \dots, \ell_{3,1}, x_2, \ell_{1,k_1}, \dots, \ell_{1,1})$,
connecting the spine vertices in odd and even positions, respectively. If $s = 0$, then $P_1 = (y_1, \dots, y_t)$ and $P_2 = (z_t, \dots, z_1)$.

(2) Add two new vertices x_0 and x_{s+1}. If $s > 0$ and $t > 0$, then connect x_0 with x_1 and $\ell_{1,1}$ and connect x_{s+1} with y_t and z_t. If $s = 0$ and $t > 0$, then connect x_0 with y_1 and z_1 and connect x_{s+1} with y_t and z_t. If $t = 0$ and $s > 0$, then connect x_0 with x_1 and $\ell_{1,1}$ and connect x_{s+1} with x_s and ℓ_{s,k_s}. These two vertices form a cycle C together with P_1 and P_2. The new edges added to G' to construct G are the edges of the cycle C ($E(C)$) which are the edges from the two paths P_1 ($E(P_1)$) and P_2 ($E(P_2)$) and the four edges that connect x_0 and x_{s+1} to these paths.

Observe that after adding the cycle to G', the degree of each one of the n' vertices is increased by 2. Together, with the two new vertices of degree 2 the modified graph G is an outer-planar realization of the original sequence d. For an illustration of the resulting outer-planar graph G for $s = 5$ and $t = 2$, see Figure 2.

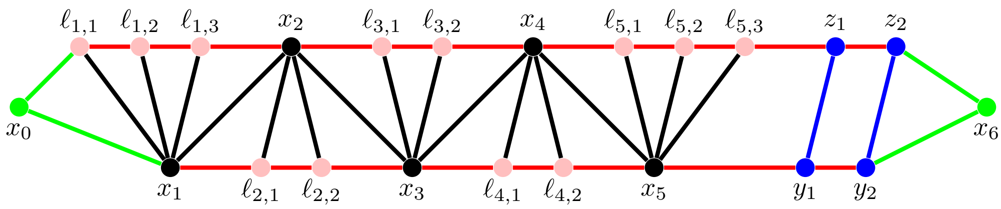


Figure 2 Illustration for the outer-planar construction described in Lemma 10.

3.2 Leaf-free sequences with $\sum d = 4n - 6$ and $\omega_2 = 3$ are outer-planaric

The following lemma is used in our analysis for sequences in which $\sum d = 4n - 4$, $w_1 = 0$ and $w_2 = 3$.

▶ **Lemma 11.** *Let d, such that $d_1 \geq d_2 \geq \cdots \geq d_n$, be a degree sequence such that*
(i) $\sum d = 4n - 6$,
(ii) $d_1 \geq 5$,
(iii) $d_3 \geq 4$,
(iv) $d_n = 2$, *and*
(v) $\omega_2 = 3$.
Then
(a) *d can be realized by a maximal outer-planaric graph G,*
(b) *A vertex $v \in V$ cannot be adjacent to three vertices of degree 2 in G.*

Proof. Let d be as in the lemma. By (iii) and (v), $n \geq 6$. Moreover, if $n = 6$ then $d_4 = 2$ by (iv) and (v), and combining it with (i) we get $\sum d = 18 = d_1 + d_2 + d_3 + 6$, so $d_1 + d_2 + d_3 = 12$, which contradicts (ii) and (iii). Therefore, $n \geq 7$. Also note that if $d_4 = 2$ then $n \leq 6$ by (iv)

and (v), leading to the same contradiction. Hence, $d_4 \geq 3$. Due to Lemma 8, if d has a MOP realization G, then G has exactly one internal triangle because $\omega_2 - 2 = 1$. To prove the lemma, we construct a MOP realization of d in which the internal triangle is formed by the vertices whose degrees are d_1, d_2, and d_3. We divide the sequences satisfying the conditions of the lemma into three (possibly overlapping) families, named $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, a nd show the realizations of sequences that belong to each class separately.

Family $\mathcal{A}$. This family contains all the sequences that satisfy the requirements of the lemma with the additional requirement that $d_1 + d_2 < 10$. Since $d_1 \geq 5$ and $d_3 \geq 4$, it follows that in these sequences $d_1 = 5$ and $d_2 = d_3 = 4$. Therefore, $\mathcal{A}$ contains all the sequences of the type $(5, 4^{\omega_4}, 3^{\omega_3}, 2^3)$ for $\omega_4 \geq 2$. To satisfy the $\sum d = 4n - 6$ requirement of the lemma, it must be the case that $\omega_3 = 1$ as shown below.

$$\begin{aligned} \sum d &= 5 + 4\omega_4 + 3\omega_3 + 6 = 4\omega_4 + 3\omega_3 + 11 \\ 4n - 6 &= 4(\omega_4 + \omega_3 + 4) - 6 = 4\omega_4 + 4\omega_3 + 10 \end{aligned}$$

This implies that $\omega_3 = 1$. To summarize, $\mathcal{A}$ contains all the sequences of the type

$$a(\omega_4) = (5, 4^{\omega_4}, 3, 2^3)$$

of length $n = \omega_4 + 5$ for $\omega_4 \geq 2$.

The following describes how to construct a MOP graph with one inner triangle, denoted by $G(\omega_4)$, realizing the sequence $a(\omega_4)$ for $\omega_4 \geq 3$. Let the $n = \omega_4 + 5$ vertices of $G(\omega_4)$ be

$$(u_1, u_2, u_3, t_0, t_1, \ldots, t_{\omega_4-3}, t_{\omega_4-2}, t_{\omega_4-1}, r_1, r_2)$$

and associate them respectively with the degrees $(5, 4, 4, \overbrace{4, \ldots, 4}^{\omega_4-2}, 3, 2, 2, 2)$. Note that the ω_4 vertices of degree 4 are $u_2, u_3, t_0, t_1, \ldots, t_{\omega_4-3}$ while the degrees of t_{ω_4-2} and t_{ω_4-1} are 3 and 2 respectively. Let the $2\omega_4 + 7$ $(= \sum d/2)$ edges of $G(\omega_4)$ be the three edges of the triangle (u_1, u_2, u_3), the six edges (u_1, t_0), (u_1, r_1), (u_2, r_1), (u_2, r_2), (u_3, r_2), and (u_3, t_0), the $\omega_4 - 1$ edges forming the path $(t_0, t_1, \ldots, t_{\omega_4-2}, t_{\omega_4-1})$, and the $\omega_4 - 1$ edges forming the two paths $(t_0, t_2, t_4 \ldots)$ and $(u_1, t_1, t_3, \ldots)$ of length $\lfloor(\omega_4 - 1)/2\rfloor$ and $\lceil(\omega_4 - 1)/2\rceil$ respectively. For an odd ω_4 the first path is $(t_0, t_2, t_4 \ldots, t_{\omega_4-3}, t_{\omega_4-1})$ and the second path is $(u_1, t_1, t_3, \ldots, t_{\omega_4-4}, t_{\omega_4-2})$ while for an even ω_4 the first path is $(t_0, t_2, t_4 \ldots, t_{\omega_4-4}, t_{\omega_4-2})$ and the second path is $(u_1, t_1, t_3, \ldots, t_{\omega_4-3}, t_{\omega_4-1})$.

Figure 3 illustrates the outer-planar layout of $G(5)$. Observe that for an odd ω_4 the outer Hamiltonian cycle of $G(\omega_4)$ that contains all of its vertices is

$$(u_1, r_1, u_2, r_2, u_3, t_0, t_2, \ldots, t_{\omega_4-3}, t_{\omega_4-1}, t_{\omega_4-2}, t_{\omega_4-4}, \ldots, t_3, t_1, u_1)$$

and for an even ω_4 it is

$$(u_1, r_1, u_2, r_2, u_3, t_0, t_2, \ldots, t_{\omega_4-4}, t_{\omega_4-2}, t_{\omega_4-1}, t_{\omega_4-3}, \ldots, t_3, t_1, u_1)$$

In both cases, (u_1, u_2, u_3) is the only inner triangle.

Family $\mathcal{B}$. This family contains all the sequences that satisfy the requirements of the lemma with an additional requirement that $d_4 = 3$. Hence, $d_1 = 4 + j$ for $j \geq 1$ because $d_1 \geq 5$, $d_2 = 4 + i$ and $d_3 = 4 + h$ for $i, h \geq 0$ because $d_2 \geq d_3 \geq 4$, and $j \geq i \geq h$ because d is a non-increasing sequence. Therefore, $\mathcal{B}$ contains all the sequences of the type

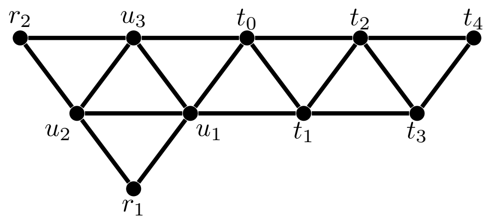


Figure 3 The MOP realization $G(5)$ of the sequence $(5, 4^5, 3, 2^3)$.

$b(j,i,h) = ((4+j), (4+i), (4+h), 3^{\omega_3}, 2^3)$ for $j \geq i \geq h \geq 0$ and $j \geq 1$. To satisfy the $\sum d = 4n - 6$ requirement of the lemma, it must be the case that $\omega_3 = (j+i+h)$ as shown below.

$$\begin{aligned} \sum d &= (4+j)+(4+i)+(4+h)+3\omega_3+6 = (j+i+h)+3\omega_3+18 \\ 4n-6 &= 4(\omega_3+6)-6 = 4\omega_3+18 \end{aligned}$$

This implies $\omega_3 = (j+i+h)$. To summarize, $\mathcal{B}$ contains all the sequences of the type

$$b(j,i,h) = ((4+j), (4+i), (4+h), 3^{j+i+h}, 2^3)$$

of length $n = j+i+h+6$ for $j \geq i \geq h \geq 0$ and $j \geq 1$.

The following describes how to construct a MOP graph with one inner triangle, denoted by $G(j,i,h)$, realizing the sequence $b(j,i,h)$ for $j \geq i \geq h \geq 0$ and $j \geq 1$. Let the $n = j+i+h+6$ vertices of $G(j,i,h)$ be

$$(u_1, u_2, u_3, p_0, \ldots, p_{j-1}, q_0, \ldots, q_{i-1}, r_0, \ldots, r_{h-1}, p_j, q_i, r_h)$$

and associate them respectively with the degrees

$$((4+j), (4+i), (4+h), \overbrace{3,\ldots,3}^{j}, \overbrace{3,\ldots,3}^{i}, \overbrace{3,\ldots,3}^{h}, 2, 2, 2)$$

Let the $2(j+i+h)+9$ $(= \sum d/2)$ edges of $G(j,i,h)$ be the three edges of the triangle (u_1,u_2,u_3), the three edges (u_1,q_0), (u_2,r_0), and (u_3,p_0), the $j+1$ edges (u_1,p_ℓ) for $0 \leq \ell \leq j$, the $i+1$ edges (u_2,q_ℓ) for $0 \leq \ell \leq i$, the $h+1$ edges (u_3,r_ℓ) for $0 \leq \ell \leq h$, the j edges forming the path $(p_0,p_1,\ldots,p_j)$, the i edges forming the path $(q_0,q_1,\ldots,q_i)$, and the h edges forming the path $(r_0,r_1,\ldots,r_h)$,

Figure 4 illustrates the outer-planar layout of $G(4,3,2)$. Observe that the outer Hamiltonian cycle of $G(j,i,h)$ that contains all of its vertices is

$$(u_1, p_j, p_{j-1}, \ldots, p_0, u_3, r_h, r_{h-1}, \ldots, r_0, u_2, q_i, q_{i-1}, \cdots, q_0, u_1)$$

and that (u_1,u_2,u_3) is the only inner triangle.

Family $\mathcal{C}$. This family contains all the sequences that satisfy the requirements of the lemma but do not belong to $\mathcal{A} \cup \mathcal{B}$. That is, the requirement *(iii)* is modified and a new requirement *(vi)* is added as follows,

(iii) $d_4 \geq 4$

(iv) $d_1 + d_2 \geq 10$

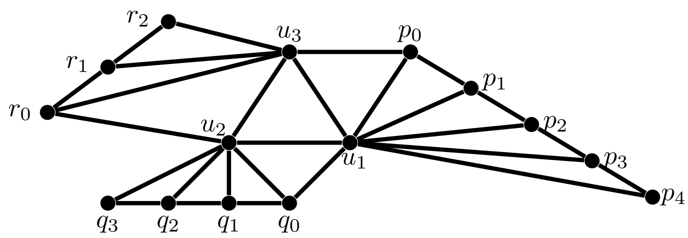


Figure 4 The MOP realization $G(4,3,2)$ of the sequence $(8,7,6,3^9,2^3)$.

The outline of the construction of the MOP realization of d that satisfies the new set of six requirements is as follows. The construction is done in two phases. In the first phase, d is modified to a shorter sequence d' of length $n' = n-3$ by contracting the three largest degrees in d into one degree and eliminating one appearance of 2 in the sequence d. We will show that the new sequence d' satisfies all the requirements of Lemma 10. As a result, this lemma will provide an outer-planar realization G' of d'. In the second phase, the construction of the realization G of the sequence d of length n will be completed by replacing the highest degree vertex in G' with a triangle of vertices and adding another vertex of degree 2 while making sure that all the $n = n' + 3$ vertices in G has the required degrees from the sequence d.

Formally, generate the sequence d' such that $d'_1 \geq d'_2 \geq \cdots \geq d'_n$ of length $n' = n-3$ by removing d_n and merging d_1, d_2 and d_3 as follows.

$$d'_i = \begin{cases} d_1 + d_2 + d_3 - 10, & \text{for } i = 1, \\ d_{i+2}, & \text{for } i = 2, \ldots, n-3. \end{cases}$$

Note that since $d_1 + d_2 \geq 10$, it follows that $d_1 + d_2 + d_3 - 10 \geq d_3 \geq d_4 = d'_2$ and therefore since d is non-increasing it follows that d' is also non-increasing.

Note that $\sum d = 4n - 6$ by the assumptions of the lemma and that $\omega'_2 = 2$ and $d'_{n'} = 2$ by the definition of d'. Consequently,

$$\sum d' = \sum d - 10 - 2 = 4n - 18 = 4(n' + 3) - 18 = 4n' - 6 \ .$$

By Lemma 10, d' is outer-planaric.

By the proof of Lemma 10, $d'' = \mathsf{pos}(d' \ominus (2^{n'}))$ can be realized by a caterpillar T and there exists an outer-planar realization G' of d' that is based on a caterpillar T with vertices $S = (x_1, \ldots, x_s)$ forming its spine and a cycle C composed of two paths P_1 and P_2 and two new vertices x_0 and x_{s+1}. (An illustration of this outer-planar graph can be found in Figure 5.) Recall that in T, $X_i = \{\ell_{i,1}, \ldots, \ell_{i,k_i}\} \subseteq V'$ is the set of leaves adjacent to the spine vertex x_i, for $i \in \{1, \ldots, s\}$. This realization G' does not have an internal triangle. We continue referring to these vertices as the spine vertices and leaves although G' is not a caterpillar.

Transforming G' into a realization G of d involves two steps. In the first step, x_1 whose degree is d'_1 will be replaced by three vertices u_1, u_2, and u_3. Note that by Lemma 7, the order of the vertices on the spine can be chosen arbitrarily. As a result, it can be assumed that the degree of x_1 is d'_1. In the second step, the outer cycle of G' will be modified to cover the new vertices with the addition of a new vertex of degree 2.

By the construction of G' from the caterpillar T and since $d'_2 = d_4 \geq 4$ by the new requirement *(iv)*, it follows that the spine of T has at lease two vertices x_1 and x_2 and x_1 is connected in G' to the two vertices x_0 (one of the two vertices of degree 2 in G') and x_2. In

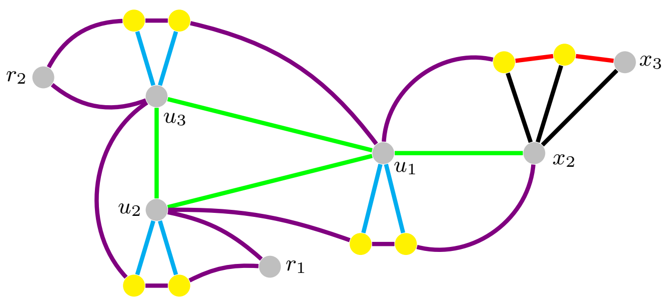


Figure 5 Realization of the sequence $d = (7, 6^2, 5, 3^8, 2^3)$. (Red and black edges are part of the original construction.)

addition, x_1 is connected in G' to the k_1 leaves $\{\ell_{1,1}, \ldots, \ell_{1,k_1}\}$ from the set X_1, as well as to $\ell_{2,1}$ (or to x_3 in case x_2 does not have leaves). Therefore,

$$k_1 = d'_1 - 3 = d_1 + d_2 + d_3 - 13 \ . \tag{1}$$

The expansion of x_1 into a triangle of vertices is done as follows. Remove x_0, x_1, and the leaves of X_1 from G'. Add the three vertices u_1, u_2, and u_3, add a triangle of edges connecting them, namely, the edges (u_1, u_2), (u_1, u_3) and (u_2, u_3), and add the edge (u_1, x_2). (See the green edges in Figure 5.)

Next, split the k_1 leaves in X_1 between the vertices u_1, u_2, and u_3 toward satisfying their degrees d_1, d_2, and d_3, respectively. Specifically, add a set U_1 of $d_1 - 5$ new leaves adjacent to u_1, and a set U_i of $d_i - 4$ new leaves adjacent to u_i, for $i = 2, 3$. (See the blue edges in Figure 5.) The split is perfect since $\sum_{i=1}^{3} |U_i| = k_1$ by Equation 1. In summary,

- u_1 is adjacent to u_2, u_3, x_2, and the $d_1 - 5$ vertices in U_1;
- u_2 is adjacent to u_1, u_3, and the $d_2 - 4$ vertices in U_2; and
- u_3 is adjacent to u_1, u_2, and the $d_3 - 4$ vertices in U_3 .

Therefore, at this point $\deg(u_i) = d_i - 2$, for $i = 1, 2, 3$.

Finally, add two more degree-2 vertices r_1 and r_2 to complete the outer cycle. One of them replaces x_0 while the other is an additional degree 2 vertex, as d_n was removed in the definition of d'. For $i = 1, 2, 3$, connect the leaves in U_i to form a path whose starting and ending vertices (of degree 1) are u_i^s and u_i^e respectively. Next, add the edges (x_2, u_1^s) and (u_1^e, u_2) if $|U_1| > 0$, otherwise add the edge (x_2, u_2). Analogously, add the edges (u_2, r_1), (r_1, u_2^s), (u_2^e, u_3) if $|U_2| > 0$, otherwise add edges (u_2, r_1) and (r_1, u_3). Add edges (u_3, r_2), (r_2, u_3^s), and (u_3^e, u_1) if $|U_3| > 0$, otherwise add edges (u_3, r_2) and (r_2, u_1). Finally, add the edge $(u_1, \ell_{2,1})$, or (u_1, x_3) in case x_2 does not have leaves. (See the violet edges in Figure 5.) At this stage the degrees of u_1, u_2, u_3 are d_1, d_2, and d_3 respectively and the degrees of r_1 and r_2 are 2.

See Figure 5 for an illustration of the realization G of d. One can verify that the construction is a realization of d. ◀

3.3 Degree-Two Removal (Procedure `Deg_2_Remove`)

As it turns out, sequences with a small number of 2 degrees are easier to realize directly. Consequently, when dealing with a sequence d with many 2's, a convenient approach is to first transform it into a "similar" sequence d' with only a few 2's, construct a graph G' realizing d', and then transform G' into a graph G realizing the original d. We next present a procedure called *Degree-Two Removal (*`Deg_2_Remove`*)* that will be used to that end in some of our constructions. The input to this procedure is a graphic sequence for which $d_3 \geq 4$,

$\omega_2 \geq 3$ and $\omega_1 = 0$. The procedure applies repeatedly the MP-step of the Havel-Hakimi algorithm until in the residual sequence either $\omega_2 < 3$ or the second maximum degree is less than 4. As a result, throughout its execution it is always the case that the residual sequence is graphic. For any degree sequence $d = (d_1, d_2, \dots, d_n)$, let $d(\ell)$ be the ℓ'th largest degree in d, for $1 \leq \ell \leq n$. (Note that possibly $d(\ell) = d(\ell + 1)$ for some ℓ values.)

Procedure 1 `Deg_2_Remove`.

Set $\bar{d} \leftarrow d$
Set $\bar{\omega}_2 \leftarrow |\{i \mid d_i = 2\}|$
while $\bar{d}(2) \geq 4$ **and** $\bar{\omega}_2 \geq 3$ **do**
  Let j be such that $\bar{d}_j = \bar{d}(1)$ and let j' be such that $\bar{d}_{j'} = \bar{d}(2)$
  $\bar{d}_j \leftarrow \bar{d}_j - 1$
  $\bar{d}_{j'} \leftarrow \bar{d}_{j'} - 1$
  Change one degree-2 in $\bar{d}$ to 0
  set $\bar{\omega}_2 \leftarrow \bar{\omega}_2 - 1$
Set $\bar{d}^s \leftarrow$ `sort`$(\bar{d})$ `/* The final` $\bar{d}$ `sorted in non-increasing order */`

Assume that Procedure `Deg_2_Remove` executes k iterations of its while-loop. Observe that during the run of the `Deg_2_Remove` procedure, no new degree 2 vertices appear because $\bar{d}(2) \geq 4$ is one of the conditions of the while-loop. As a result, $\bar{\omega}_2$, which is initially the number of degree-2 vertices in d, decreases by 1 after each iteration. Let $\bar{\omega}_2^i$ be the value of $\bar{\omega}_2$ after the i'th iteration, for $0 \leq i \leq k$. For $0 \leq i \leq k$, let $\bar{d}^i$ be the sequence $\bar{d}$ after the i'th iteration (note that $\bar{d}^s$ is the sorted version of $\bar{d}^k$), and let $\bar{n}^i$ be the length of $\mathsf{pos}(\bar{d}^i)$. We make use of the set A indices of high degrees that were reduced by Procedure `Deg_2_Remove` and the set B of indices of 2 degrees that were eliminated by the procedure. Formally,

$$A = \{i \mid d_i > \bar{d}_i,\ d_i \geq 4\}, \qquad B = \{i \mid d_i = 2,\ \bar{d}_i = 0\}. \tag{2}$$

▶ **Observation 12.**
(i) $\bar{n}^{i+1} = \bar{n}^i - 1$,
(ii) $\bar{\omega}_2^{i+1} = \bar{\omega}_2^i - 1$,
(iii) $\sum \bar{d}^{i+1} = \sum \bar{d}^i - 4$,
(iv) $|B| = k$,
(v) $\sum_{i \in A}(d_i - \bar{d}_i^k) = 2k$,
(vi) $\sum \bar{d}^k - 4\bar{n}^k = \sum d - 4n$,
(vii) $\bar{n}^k = n - k$,
(viii) $\bar{d}^i$ *is graphic for* $1 \leq i \leq k$.

▶ **Observation 13.** *When Procedure* `Deg_2_Remove` *terminates, the following properties hold.*
(i) *Either* $\bar{d}^k(2) \geq 3$ *and* $\bar{\omega}_2 = 2$, *or* $\bar{d}^k(2) = 3$ *and* $\bar{\omega}_2 \geq 3$.
(ii) *If* $\bar{d}_2^s \geq 5$, *then* $\bar{d}_i^k \geq 4$ *for every* $i \in A$.

The following lemma (whose proof is omitted) demonstrates the usefulness of Procedure 1 in reducing the number of degree 2 vertices to generate an outer-planar sequence. Later, it will be shown how to add back the removed degree 2 vertices to get a planar realization for the original sequence.

▶ **Lemma 14.** *Applying Procedure* `Deg_2_Remove` *(Procedure 1) on a graphic sequence* d *with* $\sum d \leq 4n - 6$, $\omega_1 = 0$ *and* $\omega_2 \geq 3$, *the output sequence* $\bar{d}^k$ *satisfies that* $\mathsf{pos}(\bar{d}^k)$ *is outer-planaric.*

4 Main Result

This section presents our main result (Theorem 2). Specifically, Subsection 4.1 shows that this theorem is implied directly by Theorem 1. The following two subsections establish Theorem 1, where Subsection 4.2 handles the easy case of sequences with few 2 degrees ($\omega_2 \le 2$) and Subsection 4.3 handles the more elaborate case of sequences with many 2 degrees ($\omega_2 \ge 3$). Finally, Subsection 4.4 provides examples showing that our bounds are almost tight.

4.1 The planarity of low-volume sequences with $\omega_1 > 0$

We first rely on Theorem 1 to prove our main result.

Proof of Theorem 2. If $\sum d \le 2n - 2$, then d can be realized by a forest, hence it is planaric [10]. For the case where $\sum d \ge 2n$, we prove the claim by induction on ω_1. In the base case, $\omega_1 = 0$, the claim follows by Theorem 1. Now assume that the claim holds for $\omega_1 \le i$ and consider $\omega_1 = i + 1$. Construct d' by setting $d'_1 = d_1 - 1$, $d'_n = 0$ and $d'_i = d_i$ for $i \in \{2, \dots, n-1\}$. Since $\sum d \ge 2n$ and $\omega_1 > 0$, we have that $d_1 \ge 3$ and hence $d'_1 \ge 2$. Let $d'' = \mathsf{pos}(d')$ and denote the number of 1-degrees in d'' by ω''_1. Then $\omega''_1 = \omega_1 - 1$ and $n'' = n - 1$, implying that $\sum d'' \le 4n'' - 4 - 2\omega''_1$. Also note that when applying the MP-step of the Havel-Hakimi method on d here, using d_n as pivot, we get d'', and hence d'' is graphic. Therefore, d'' satisfies the conditions of induction hypothesis and hence can be realized by a planar graph G''. To complete the construction, add one leaf to the vertex with degree d'_1 in G''. This yields a planar graph G realizing d. The theorem follows. ◀

The following two subsections are dedicated to the leaf-free case ($\omega_1 = 0$), and prove Theorem 1.

4.2 The planarity of leaf-free sequences with $\sum d \le 4n - 4$ and $\omega_2 \le 2$

The case of "few degrees 2" is relatively easier, and is covered by the following lemma.

▶ **Lemma 15.** *Every graphic sequence d such that $d_1 \ge d_2 \ge \cdots \ge d_n$ with $\omega_1 = 0$, $\omega_2 \le 2$, and $\sum d \le 4n - 4$ is planaric.*

Proof. If $d_3 < 4$, then d is planaric by Lemma 5. From now on assume in addition that $d_3 \ge 4$. We consider three cases depending on ω_2.

Case 1: $\omega_2 = 0$.
In this case, $d_n = 3$ since $d_n \ge 4$ would imply $\sum d \ge 4n$. Let $d' = d \ominus (2^n)$. Then $n' = n$ and $\sum d' = \sum d - 2n \le 2n' - 4$. Therefore, d' is a forestic sequence. By Lemma 7, d' can be realized by a graph $G' = (V', E')$ composed of a union of an alternating caterpillar T' and a matching M'. This matching contains at least one edge because when $\sum d' \le 2n' - 4$, any realization forest must contain at least two connected components.

Define the spine S, the leaf sets X_i in T' and the matching M' analogously to the proof of Lemma 10. Observe that since $d_3 \ge 4$ and the spine contains all the vertices whose degree in G' is at least 2, it follows that the spine S contains at least three vertices.

To construct a planar realization of d, add to G' a set of edges that form two disjoint cycles. The construction consists of two steps. We describe it for odd s; an analogous construction applies for even s.

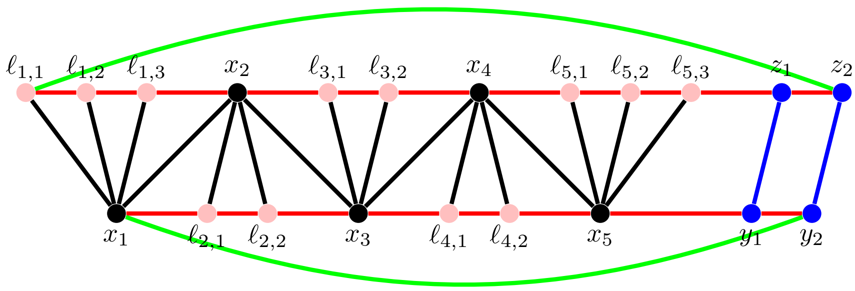


Figure 6 Illustration for a planar construction when $\omega_2 = 0$. This is essentially the outer-planar graph of Figure 2 after omitting the vertices x_0 and x_6 with the addition of the green edges (x_1, y_2) and $(z_2, \ell_{1,1})$.

(1) Construct two paths P_1 and P_2 as in the proof of Lemma 10.

(2) Connect x_1 with y_t and $\ell_{1,1}$ with z_t, thereby transforming the paths P_1 and P_2 into a cycle.

For an illustration of the resulting outer-planar graph G for $s = 5$ and $t = 2$, see Figure 6.

Case 2: $\omega_2 = 1$.

In this case, $d_n = 2$ and $d_{n-1} = 3$, since again $d_{n-1} \geq 4$ would contradict the assumption $\sum d \leq 4n - 4$. If $d_1 \leq 4$, then $d = (4^{\omega_4}, 3^{\omega_3}, 2)$ with even $\omega_3 \geq 2$ is planaric by Lemma 4. Hereafter, we assume that $d_1 \geq 5$.

Define d' by setting $d'_1 = d_1 - 1$, $d'_{n-1} = 2$ (replacing $d_{n-1} = 3$) and $d'_i = d_i$ for all other $i \in \{2, \ldots, n-2, n\}$. Then $\sum d' \leq 4n' - 6$ and $n' = n$. Let $d'' = \mathsf{pos}(d' \ominus (2^n))$. We have $n'' = n' - 2$ and $\sum d'' = \sum d' - 2n \leq 2n' - 6 = 2(n'' + 2) - 6 = 2n'' - 6$. By Lemma 7, d'' can be realized by a graph $G'' = (V'', E'')$ composed of a union of an alternating caterpillar T'' and a matching M''. Notice that $d''_1 \geq 2$ since $d_1 \geq 5$. Therefore, the vertex with degree d''_1 occurs on the spine of T'' and can be identified as x_1 since by Lemma 7 the spine-vertices may appear in any order.

Define the spine S, the leaf sets X_i in T'' and the matching M'' analogously to the proof of Lemma 10. Since $d_3 \geq 4$, by the construction of d' and d'', there are at least two vertices on the spine S and $s \geq 2$. To construct a planar realization of d, we add a set of edges to G'' as done in the proof of Lemma 10 and one extra edge. The construction consists of two steps. Again, we describe it only for odd s.

(1) Construct two paths P_1, P_2 and $E(C)$ as in the proof of Lemma 10

(2) Connect x_{s+1} and x_1 by an edge.

An illustration of the above steps is presented in Figure 7. Note that steps (1) and (2) build an outer-planar graph $G' = (V, E')$ with $E(G') = E(G'') \cup E(C)$ realizing d'. Notice that $(x_{s+1}, x_1) \notin E(G')$ since $s \geq 2$. Therefore, step (3) yields a simple planar graph $G = (V, E)$ with $E(G) = E(G'') \cup E(C) \cup \{(x_{s+1}, x_1)\}$ realizing d.

Case 3: $\omega_2 = 2$ (i.e., $d_n = d_{n-1} = 2$ and $d_{n-2} \geq 3$).

In this case, for $\sum d \leq 4n - 6$, the result follows directly from Lemma 10.

Next consider $\sum d = 4n - 4$. This case is divided into three sub-cases.

Case 3.1: $d_1 = 4$.

In this case, d is a 2-sequence, i.e., it satisfies $d_1 - d_n = 2$, and such d is known to be planaric, see Lemma 4.

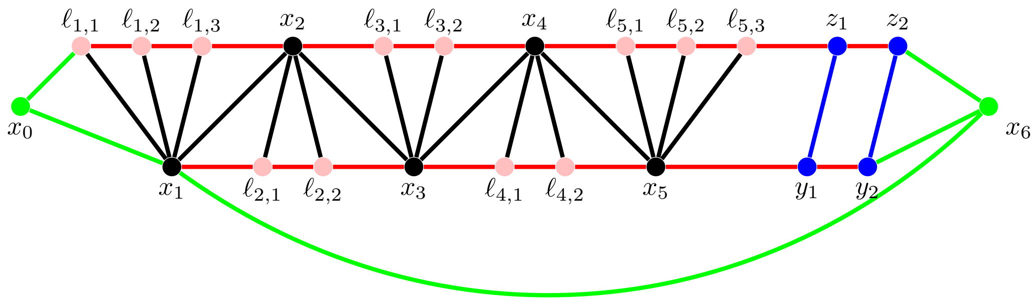


Figure 7 Illustration for planar construction when $\omega_2 = 1$. This is essentially the outer-planar graph of Figure 2 with the addition of the green edge (x_1, x_6).

Case 3.2: $d_1 = 5$ and $d_2 = 4$.
In this case, $\sum d = 4n - 4$ dictates that $d = (5, 4^{\omega_4}, 3, 2^2)$ for $\omega_4 \geq 2$ since $d_3 \geq 4$. The following describes how to construct a planar graph, denoted by $G(\omega_4)$, realizing the sequence d for $\omega_4 \geq 2$. Let the $n = \omega_4 + 4$ vertices of $G(\omega_4)$ be $(u, x_1, x_2, \ldots, x_{\omega_4}, r_1, r_2, r_3)$, and associate them respectively with the degrees $(5, \overbrace{4, \ldots, 4}^{\omega_4}, 3, 2, 2)$. Let the $2\omega_4 + 6$ $(= \sum d/2)$ edges of $G(\omega_4)$ be the five edges (u, r_1), (u, r_2), (u, r_3), (u, x_1), (u, x_2), the four edges $(r_1, x_{\omega_4 - 1})$, (r_1, x_{ω_4}), (r_2, x_{ω_4}), and (r_3, x_1), the $\omega_4 - 1$ edges forming the path $(x_1, x_2, \ldots, x_{\omega_4 - 1}, x_{\omega_4})$, and the $\omega_4 - 2$ edges forming the two paths $(x_1, x_3, \ldots)$ and $(x_2, x_4, \ldots)$ of length $\lceil (\omega_4 - 2)/2 \rceil$ and $\lfloor (\omega_4 - 2)/2 \rfloor$ respectively. For an even ω_4 the first path is $(x_1, x_3, \ldots, x_{\omega_4 - 1})$ and the second path is $(x_2, x_4, \ldots, x_{\omega_4})$ while for an odd ω_4 the first path is $(x_1, x_3, \ldots, x_{\omega_4})$ and the second path is $(x_2, x_4, \ldots, x_{\omega_4 - 1})$. Figure 8 illustrates the planar layout of $G(6)$.

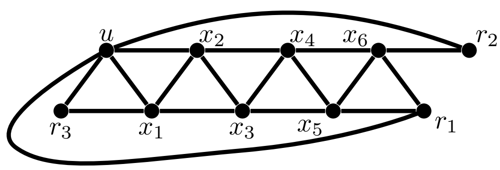


Figure 8 The planar realization of the sequence $G(6) = (5, 4^6, 3, 2^2)$.

Case 3.3: $d_1 \geq 6$ or $d_2 \geq 5$.
If $d_{n-2} \geq 4$, then $\sum d \geq 4n - 2$, which contradicts with $\sum d = 4n - 4$. Therefore, $d_{n-2} = 3$. Construct d' from d by letting $d'_1 = d_1 - 1$, $d'_{n-2} = 2$ and $d'_i = d_i$ for all other $i \in \{2, \ldots, n-3, n-1, n\}$. Then $\mathsf{pos}(d') = d'$, $n' = n$, $\sum d' = 4n' - 6$ and $\omega'_2 = 3$.

Combining $d_3 \geq 4$ and the condition of this case, the maximum degree in d' is at least 5 and the third maximum degree is at least 4. Hence, d' satisfies the conditions of Lemma 11, and therefore it can be realized by an outer-planar graph G'. In the construction of G', there exists a vertex u of degree 2 not adjacent to the vertex v of degree d'_1 by (b) of Lemma 11. Construct the planar graph G realizing d by adding the edge (u, v) to G'.

Summarizing the above three cases, d is planaric. ◀

4.3 The planarity of leaf-free sequences with $\sum d \leq 4n - 4$ and $\omega_2 \geq 3$

This subsection handles the more complex case of "many degrees 2". The analysis is separated into two main parts. First (Lemma 16), we consider sequences of volume $\sum d \leq 4n - 6$. Later (Lemma 17) we analyze the extremal case where $\sum d = 4n - 4$.

▶ **Lemma 16.** *Every graphic sequence* d*, such that* $d_1 \geq d_2 \geq \cdots \geq d_n$*, with* $\omega_1 = 0$*,* $\omega_2 \geq 3$*, and* $\sum d \leq 4n - 6$ *is planaric.*

For lack of space, we present an overview of the construction, deferring the complete proof to the full version. The construction of this lemma involves four phases. Phase (1) transforms d into a sequence $\bar{d}^k$ with just two degrees 2 (or, $\bar{d}^k(2) = 3$). This is done by using Procedure `Deg_2_Remove`. Phase (2) constructs an outer-planar graph G_k realizing $\bar{d}^k$, which is illustrated in Figure 9(a). Phase (3) adds a multi-graph $\bar{G}$ with edges connecting the vertices corresponding to indices in the set A defined in Eq. (2). It is illustrated in Figure 9(b). For Phase (4) we need to use the *vertex insertion* operation, defined as follows. For any multi-graph $\hat{G}$, let $I(\hat{G})$ denote the graph obtained from $\hat{G}$ by inserting one new vertex $z_{u,v}$ into every edge $e = (u, v)$ in $E(\hat{G})$, namely, replacing e by the 2-edge path $(u, z_{u,v}, v)$. Note that this operation cancels all parallel edges, so the resulting graph is simple. Observe that if we transform $\bar{G}$ into $I(\bar{G})$, then $I(\bar{G})$ realizes $\mathsf{pos}(d \ominus \bar{d}^k)$ and the combined graph $I(\bar{G}) \cup G_k$ realizes d. Moreover, as G_k is outer-planar, if there are no cross edges between vertices of $\bar{G}$, then $G_k \cup I(\bar{G})$ is a *planar* realization for d, as shown in Figure 9(c). However, the multi-graph $\bar{G}$ might contain crossing edges, as shown in Figure 9(b). In this case, some preliminary processing is needed. At the beginning of Phase (4), apply procedure `Edge_Swap` and replace $\bar{G}$ by a *modified* multi-graph G^M with no edge crossings. Only then, invoke the vertex insertion operation to insert k new vertices into the edges of G^M and get $I(G^M)$. Since the inserted new vertices in $I(G^M)$ do not exist in G_k, the graph $G_k \cup I(G^M)$ is simple. Consequently, $G_k \cup I(G^M)$ can be shown to be a simple planar graph realizing d, as shown in Figure 9(d).

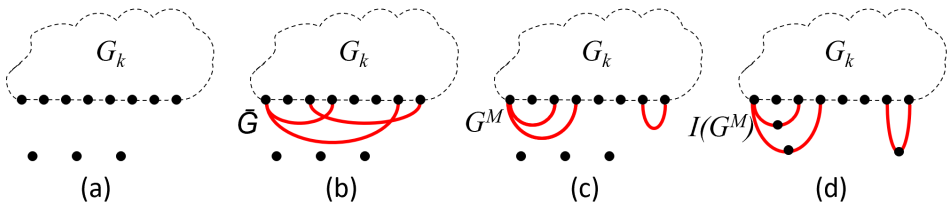


Figure 9 A schematic description of the realization process in Case 4. The preliminary step involves separating the sequence d into a sequence $\bar{d}^k$ with reduced high degrees and only two degrees 2 (depicted by the higher row of vertices in the figures, forming the spine of G_k), and k degrees 2 kept separately (the lower row in the figures).

Careful inspection of the proofs of Lemmas 5, 9 and 16 reveals that when $\sum d \leq 4n - 6 - 2\omega_1$, the constructed realizing graphs are not only planar but also enjoy a 2-page book embedding, yielding Corollary 3.

The next lemma shows the case for $\sum d = 4n - 4$ and $\omega_1 = 0$, $\omega_2 \geq 3$. A schematic description of the construction process in this case is as follows. First apply Procedure `Deg_2_Remove` (Procedure 1 in Section 2) to create a modified degree sequence $\bar{d}^k$ and sets A and B. Next construct a simple graph G_k realizing $\bar{d}^k$, as a combination of an outer-planar graph G' and an edge (u, v), using the method described in the proof of Lemmas 10 and 11. Then, apply procedure `Deg_2_Recover` on G' and output a simple planar graph G. As a final step, construct a graph G'' with $E(G'') = E(G) \cup \{(u, v)\}$. Note that by Procedure `Deg_2_Recover`, $G'' = G_k \cup I(G^M)$. Since $I(G^M)$ realizes $\mathsf{pos}(d \ominus \bar{d}^k)$, G'' realizes d. Since (u, v) does not exist in G, G'' is a simple graph. Combining it with the fact that G is an "almost outer-planaric" graph, if one can show that there are no cross edges between (u, v) and any edges in $E(G^M)$, then G'' is a simple planar graph. The formal proof involves a rather complex case analysis, and is omitted for lack of space.

▶ **Lemma 17.** *Every graphic sequence d with $\omega_1 = 0$, $\omega_2 \geq 3$, and $\sum d = 4n - 4$ is planaric.*

Combining Lemmas 15, 16 and 17 yields the proof for our main Theorem 1.

4.4 Almost tight negative examples

We complement the positive result of Theorem 2 by almost tight negative examples. Consider first the case of $\omega_1 = 0$, for which we present a tight example.

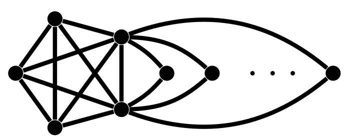
▶ **Lemma 18.** *There exists a graphic sequence of volume $4n - 2$ and $\omega_1 = 0$, which is non-planaric.*

Proof. The sequence $d = (4^5, 2)$ is non-planaric by Lemma 4, and satisfies $\sum d = 4n - 2$. ◀

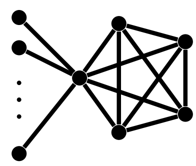
However, we do not know non-planaric sequences for which $\omega_1 = 0$ and $\sum d = 4n - 2$ for $n > 6$. Instead, the next lemma shows that for any $n \geq 5$ there exists a non-planaric sequence with $\omega_1 = 0$ and $\sum d = 4n$.

▶ **Lemma 19.** *For any non-negative integer k, the sequence $d[k] = ((4 + k)^2, 4^3, 2^k)$, for which $\sum d[k] = 4n$, is graphic but not planaric.*

Proof. See Figure 10a for the unique realization of the sequence $d[k] = ((4 + k)^2, 4^3, 2^k)$. This realization is non-planaric because it has a a K_5 subgraph consisting of the two vertices of degree $4 + k$ and the three vertices of degree 4. This is a unique realization becaus the two degree $4 + k$ vertices each must be connected to all other $n - 1 = 4 + k$ vertices. ◀



(a) The unique realization of $((4 + k)^2, 4^3, 2^k)$.



(b) The unique realization of $(4 + k, 4^4, 1^k)$.

Figure 10 Two non-planaric families of sequences.

Turning to sequences with $\omega_1 > 0$, there is again a small gap. The next lemma shows that for any $n \geq 6$ there exists a non-planaric sequence with $\omega_1 > 0$ and $\sum d = 4n - 2\omega_1$.

▶ **Lemma 20.** *For any non-negative integer k, the sequence $d'[k] = (4 + k, 4^4, 1^k)$, for which $\sum d'[k] = 4n - 2\omega_1$, is graphic but not planaric.*

Proof. See Figure 10b for the unique realization of the sequence $d'[k] = ((4 + k, 4^4, 1^k)$. This realization is non-planaric because it has a K_5 subgraph consisting of the vertex of degree $4 + k$ and the four vertices of degree 4. This is a unique realization becaus the degree $4 + k$ vertex must be connected to all other $n - 1 = 4 + k$ vertices. ◀

Recall that Theorem 2 states that every graphic sequence d such that $\sum_d \leq 4n - 4 - 2\omega_1$ is planaric. In light of Lemma 19 and Lemma 20 it follows that the remaining gap for the case $\omega_1 = 0$ involves sequences d with $\sum d = 4n - 2$ while the remaining gap for the case $\omega_1 > 0$ involves sequences d with $\sum d = 4n - 2 - 2\omega_1$.

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