Quantization of self–similar probabilities

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Extended Abstract

(A detailed version of the results stated below can be found in [3])

The search for effective methods for the discretization of signals led electrical engineers to the study of quantization for probabilities starting in the 1940s.

Mathematically the problem can be phrased as follows: Given a norm $\| \ \|$ on $\mathbb{R}^d$ and a probability space $(\Omega, \mathcal{A}, \mu)$ let $\mathcal{L}^r$ be the space of all random vectors $X: \Omega \to \mathbb{R}^d$ with $\|X\|_r := E(\|X\|^r)^{1/r} < \infty$. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the subset of all $Y \in \mathcal{L}^r$ with $\mathrm{card}(Y(\Omega)) \leq n$ and, for a given $X \in \mathcal{L}^r$,

$$e_{n,r} = e_{n,r}(X) = \inf\{\|X - Y\|_r : Y \in \mathcal{F}_n\}$$

the **n-th quantization error** of X of order r. Note that $e_{n,r}$ only depends on the distribution P of X.

If the limit exists

$$D_r = D_r(X) = \lim_{n\to\infty} \frac{\log n}{-\log e_{n,r}}$$

is called the **quantization dimension** of X or P.

If $X \in \mathcal{L}^{r+\delta}$ for some $\delta > 0$ and if P has a non–vanishing absolutely continuous part then $D_r = d$ and

$$\lim_{n\to\infty} n^{1/D_r} e_{n,r}^{D_r} \quad \text{exists in } (0, +\infty) \tag{1}$$

(by results of Zador [4] and Bucklew–Wise [1], see Graf–Luschgy [2] for a complete proof.)

The present talk deals with the quantization problem for random vectors with self–similar distributions.

Given contracting similarity maps $S_1, \ldots, S_N: \mathbb{R}^d \to \mathbb{R}^d$ and a probability vector $(p_1, \ldots, p_N)$ with $p_i > 0$ for all $i \in \{1, \ldots, N\}$ there exists a unique probability measure P on $\mathbb{R}^d$ with

$$P = \sum_{i=1}^{N} p_i P \circ S_i^{-1}.$$

P is called the self–similar probability corresponding to $(S_1, \ldots, S_N; p_1, \ldots, p_N)$. The support A of P is the unique non–empty compact subset of $\mathbb{R}^d$ with

$$A = S_1(A) \cup \ldots \cup S_N(A).$$

In the following we will assume that $(S_1, \ldots, S_N)$ satisfies the **open set condition**, i.e. there exists a non–empty open subset U of $\mathbb{R}^d$ with $S_i(U) \subset U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all i, j with $i \neq j$. Moreover let X be a random vector in $\mathbb{R}^d$ which has the self–similar distribution P (as above).

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Recall that $(a_1, \ldots, a_N) \in (\mathbb{R}\backslash\{0\})^N$ is **arithmetic** if there is an $a \in \mathbb{R}$ with $(a_1, \ldots, a_N) \in (a\mathbb{Z})^N$. Let s_i denote the contraction constant of S_i. Then we can formulate the main result.

Main Theorem (see [3], Theorem 4.1)
If $D_r \in (0, +\infty)$ is the unique real number with

$$\sum_{i=1}^{N} (p_i s_i^r)^{\frac{D_r}{r+D_r}} = 1$$

then D_r is the quantization dimension of X. Moreover,

$$0 < \liminf_{n\to\infty} n^{1/D_r} e_{n,r}(X) \leq \limsup_{n\to\infty} n^{1/D_r} e_{n,r}(X) < \infty$$

and the limit exists provided $(\log(p_1 s_1^r), \ldots, \log(p_n s_N^r))$ is not arithmetic.

The main tool for proving the existence of the limit in the above theorem relies on a general notion of subadditivity for sequences of real numbers: Let $t = (t_1, \ldots, t_N) \in (0,1)^N$ be given. A sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is called t–**additive** iff

$$\forall n_1, \ldots, n_N \in \mathbb{N}: v_{n_1+\ldots+n_N} \leq t_1 v_{n_1} + \ldots + t_N v_{n_N}.$$

Then one obtains the following proposition.

Proposition (see [3], Theorem 3.2)
If $\sum_{i=1}^{N} t_i < 1$, if $(\log\ t_1, \ldots, \log\ t_N)$ is not arithmetic and if $(v_n)_{n\in\mathbb{N}}$ is t–subadditive then there exist a unique $\rho \in (0,1)$ with $\sum_{i=1}^{N} t_i^\rho = 1$ and

$$\lim_{n\to\infty} n^{(1-\rho)/\rho} v_n \quad \text{exists in} \quad [0, +\infty).$$

Idea of proof for the main theorem: To prove the existence of the limit in the main theorem set $t_i = p_i\ s_i^r$ and observe that $(e_{n,r}^r)_{n\in\mathbb{N}}$ is t–subadditive. By the definition of D_r in the main theorem we have for $\rho = \frac{D_r}{r+D_r}$ that $\sum_{i=1}^{N} t_i^\rho = 1$. Since $(1-\rho)/\rho = \frac{r}{D_r}$ the proposition implies the existence of $\lim_{n\to\infty} n^{\frac{r}{D_r}} e_{n,r}^r$ in $[0, +\infty)$ provided $(\log(p_1 s_1^r), \ldots, \log(p_N s_N^r))$ is not arithmetic.

Next we will investigate the empirical distribution on asymptotically optimal sets. A sequence of subsets of $\mathbb{R}^d$ with $1 \leq \text{card}(\alpha_n) \leq n$ is called asymptotically optimal for X (or P) provided

$$\lim_{n\to\infty} \frac{1}{e_{n,r}} \left(\int \min_{a\in\alpha_n} \|x-y\|^r dP(x) \right)^{1/r} = 1.$$

Theorem (see [3], Theorem 5.5)
Let $(q_1, \ldots, q_N) = ((p_1 s_1^r)^{\frac{D_r}{r+D_r}}, \ldots, (p_N s_N^r)^{\frac{D_r}{r+D_r}})$ and let P_r be the self–similar probability corresponding to $(S_1, \ldots, S_N; q_1, \ldots, q_N)$. Assume that $\lim_{n\to\infty} n^{1/D_r} e_{n,r}(P)$ exists in $(0, \infty)$ and let $(\alpha_n)_{n\in\mathbb{N}}$ be any asymptotically optimal sequence for P. Then the sequence

$$\left(\frac{1}{card(\alpha_n)} \sum_{a\in\alpha_n} \delta_a \right)_{n\in\mathbb{N}}$$

of empirical measures on α_n weakly converges to P_r.

References

[1] J.A. Bucklew and G.L. Wise, Multidimensional asymptotic quantization theory with r–th power distortion measures, *IEEE Trans. Information Theory* **28**(1982), 239–247

[2] S. Graf and H. Luschgy, Foundations of quantization for probability distributions, *Lect. Notes in Math.* **1730**, Springer 2000

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[4] P.L. Zador, Asymptotic quantization error of continuous signals and the quantization dimension, *IEEE Trans. Information Theory* **28** (1982), 139–149