Graphs and Circuits: Some Further Remarks *

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Abstract

We consider the power of single level circuits in the context of graph complexity. We first prove that the single level conjecture fails for fanin-2 circuits over the basis $\{\oplus, \wedge, 1\}$. This shows that the (surpisingly tight) phenomenon, established by Mirwald and Schnorr (1992) for quadratic functions, has no analogon for graphs. We then show that the single level conjecture fails for unbounded fanin circuits over $\{\vee, \wedge, 1\}$. This partially answers the question of Pudlák, Rödl and Savický (1986). We also prove that $\Sigma_2 \neq \Pi_2$ in a restricted version of the hierarhy of communication complexity classes introduced by Babai, Frankl and Simon (1986). Further, we show that even depth-2 circuits are surprisingly powerful: every bipartite $n \times n$ graph of maximum degree Δ can be represented by a monotone CNF with $O(\Delta \log n)$ clauses. We also discuss a relation between graphs and ACC-circuits.

Keywords: Graph complexity, single level conjecture, Sylvester graphs, communication complexity, ACC-circuits

AMS subject classification: 05C35, 05C60, 68Q17, 68R10, 94C30

1   Introduction

Let $V = \{1, \ldots, n\}$ be a set of n vertices. We identify vertices $u \in V$ with boolean variables x_u, and consider boolean functions $f : \{0, 1\}^V \to \{0, 1\}$ whose set of variables is $X = \{x_u : u \in V\}$. Such a function accepts/rejects a subset of vertices $S \subseteq V$ if it accepts/rejects the incidence vector of S. A *non-edge* is a pair of non-adjacent vertices; if the graph is bipartite then a non-edge is a pair of non-adjacent vertices from different parts (color classes), that is, pairs of vertices in one color class are neither edges nor non-edges.

Following [5], we say that a boolean function *represents* a given graph $G = (V, E)$ if it accepts all edges and rejects all non-edges. That is, the function must behave correctly only on 2-element sets of vertices—on other subsets of vertices the function can take arbitrary values. For example, $f(x_1, x_2, x_3, x_4) = (x_1 \vee x_2) \wedge (x_3 \vee x_4)$ represents a bipartite 2×2 clique $K_{2,2}$, a single variable x_i represents a complete star around i, etc. In particular, the (boolean) quadratic

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function $f_G(X) = \bigvee_{ij \in E} x_i x_j$ as well as its algebraic counterpart $f_G^{\oplus}(X) = \bigoplus_{ij \in E} x_i x_j$ represent the graph G.

The circuit complexity of a graph is the minimum size of a circuit representing this graph. This concept is interesting because monotone lower bounds for graphs imply non-monotone lower bounds for boolean functions. The *adjacency function* of a bipartite $n \times n$ graph with $n = 2^m$, whose vertices are binary vectors of length m, is a boolean function in $2m$ variables which accepts all edges and rejects all non-edges.

Lemma 1.1 (Magnification Lemma ([5])). *In a (non-monotone) circuit computing the adjacency function of a bipartite graph G it is possible to replace the negated inputs with ORs of variables so that the obtained (monotone) circuit represents G. The same holds with Parity gates instead of OR gates.*

This fact is particularly useful in such circuit models where computing an OR (or a Parity) of input literals is "cheap." For example, if the circuit computing f has unbounded fanin OR gates on the bottom (next to the inputs) level, then the obtained (monotone) circuit represents G and has just the same number of gates! Hence, if we could prove that a bipartite $n \times n$ graph G with $n = 2^m$ cannot be represented using, say, fewer than n^ϵ gates, this would immediately imply that the characteristic function f of G requires at least $n^\epsilon = 2^{\epsilon m}$ gates, which is *exponential* in the number $2m$ of variables of f_m (this is where the term "magnification" comes from).

In this paper we present some results concerning the complexity of graphs as well as the single level conjecture for graphs. None of these results solve some big problem nor their proofs are difficult. We hope however that they could be useful when dealing with graph complexity—our understanding of wnat graphs are hard for what kind of circuits is still poor.

2 Mirwald–Schnorr's phenomenon fails for graphs

In this section we consider fanin-2 circuits over the basis $\{\oplus, \wedge, 1\}$. A circuit is a *single level circuit* if it has only one level of AND gates, that is, if every path from an input to the output contains at most one AND gate.

The so-called "single level conjecture" for quadratic functions claimed that single level circuits for quadratic functions are almost optimal, i.e. that the gap is constant.

A strong support for the conjecture was given by Mirwald and Schnorr in [9]: For every graph G, *every* optimal with respect to the number of AND gates circuit over the basis $\{\oplus, \wedge, 1\}$ computing $f_G^{\oplus}$ is a single level circuit. That is, if we count only AND gates then over the basis $\{\oplus, \wedge, 1\}$ we have no gap at all!

Our first result is that the theorem of Mirwald and Schnorr has no analogue for graphs. Let M_n be a bipartite $n \times n$ graph consisting of n mutually disjoint edges, that is, a perfect matching with n edges.

Theorem 2.1. *Over the basis $\{\oplus, \wedge, 1\}$, the graph M_n can be represented by a circuit using only logarithmic in n number of AND gates, but the number of AND gates in single level circuits for M_n is linear.*

Hence, the gap in this case may be as large as $\Omega(n/\log n)$.

Proof. For a graph G, let $L(G)$ be the minimum number of AND gates in a (fanin-2) circuit over the basis $\{\oplus, \wedge, 1\}$ representing G, and let $L_1(G)$ be the single level version of this measure. Our goal is to show that $L(M_n) \leq \log n$ and $L_1(M_n) = \Omega(n)$.

Upper bound: $L(M_n) \leq \log n$. Let $n = 2^r$. We identify vertices of M_n with vectors in $\{0, 1\}^r$ and look at M_n as a bipartite $n \times n$ graph with parts $U = \{0, 1\}^r$ and $W = \{0, 1\}^r$, and where two vertices (vectors) $u \in U$ and $w \in W$ are adjacent if and only if $u = w$. Let $X = \{x_u \mid u \in U\} \cup \{y_w \mid y \in W\}$ be the corresponding set of boolean variables, and consider the functions g_i, $i = 1, \ldots, r$ defined by

$$g_i(X) = \bigoplus_{u \in U, u_i = 0} x_u \oplus \bigoplus_{w \in W, w_i = 1} x_w.$$

It is easy to see, that the function $g_i(X)$ accepts an arc $uv \in U \times W$ (i.e. a vector in $\{0, 1\}^X$ with exactly two 1's in positions u and w) if and only if $u_i = w_i$. Since an arc uv is adjacent in M_n if and only if $u_i = w_i$ for all $i = 1, \ldots, r$, the function $F(X) = \bigwedge_{i=1}^r g_i(X)$ represents the graph M_n, implying that $L(M_n) \leq r = \log n$.

Lower bound: $L_1(M_n) = \Omega(n)$. By a rank of a graph G, rk(G), we will mean the rank over GF(2) of the adjacency matrix of G. Observe that $L_1(G) \leq t$ iff G can be represented by a sum $\bigoplus_{i=1}^t \ell_{i,1} \wedge \ell_{i,2}$ of t products of linear forms over GF(2). The graph represented by a linear form is just a union of two vertex disjoint bipartite cliques, and hence, has rank at most 2. Thus, the graph G itself has rank rk(G) $\leq 4t$, implying that $L_1(G) \geq \frac{1}{4}$rk(G) holds for every bipartite graph G. Since the perfect matching M_n has full rank, the lower bound $L_1(M_n) = \Omega(n)$ follows. $\square$

An interesting question is the status of the single level conjecture for circuits over the basis $\{\oplus, \wedge, 1\}$ in the case of *unbounded fanin* gates. Single level circuits in this case have the form

$$F(X) = \bigoplus_{i=1}^s \bigwedge_{j=1}^r \ell_{ij}(X) \tag{1}$$

where $\ell_{ij}(X) = \bigoplus_{k=1}^n \lambda_{ijk} x_k \oplus \lambda_{ij}$ are linear functions over GF(2). Any product of r linear forms represents a fat matching of size at most 2^r, i.e. a union of at most 2^r vertex-disjoint bipartite cliques (see [6]). Hence, if F represents a graph G, then $s \geq$ rk(G)$/2^r$. Therefore, if r is relatively small, say $r = o(\log n)$, then already a perfect matching M_n requires circuits of size $\Omega(n)$. However, what happens if we do not restrict the middle fanin r? That the question may be interesting follows from a result, due to Razborov [11], that some "combinatorially complicated" graphs—like Ramsey graphs, i.e. n-vertex graphs without a clique or independent set larger than $O(\log n)$—*can* be represented by a circuit of the form (1) with $s = (\log n)^{O(1)}$ and $r = O(\log \log n)$. This is a direct consequence of the following more general result proved in [11]. If $G(s, r)$ denotes a random graph represented by a random circuit of the form (1), obtained by a

random and independent choice of the coefficients λ_{ijk} and λ_{ij}, then for every graph H on k vertices, the graph $G(s,r)$ with $r \geq \log \binom{k}{2} + 1$ contains a copy of H as an induced subgraph with probability at most

$$\binom{n}{k} \left[2^{-\binom{k}{2}} + e^{-s/2^r} \right].$$

3 Depth-3 circuits may be much weaker

We now consider the *unbounded fanin* version of the single level conjecture: does monotone depth-3 circuits (i.e. monotone Σ_3 circuits) for quadratic function and/or graphs are almost optimal? In the context of graphs this question was explicitely raised by Pudlák, Rödl and Savický in [10].

The reason, why the unbounded fanin version of the conjecture for graphs is interesting, is twofold: (i) the presence of unbounded fanin gates may *exponentially* increase the power of single level circuits for quadratic functions, and (ii) a lower bound of the form $n^{\Omega(1)}$ on the size of a monotone single level circuit with unbounded fanin gates would imply a nonlinear lower bound for NC^1 circuits (see [6] for details). Note that every quadratic function in n variables can be computed by a monotone Σ_3-circuit of linear size:

$$f_G(X) = \bigvee_{u \in V} x_u \wedge \left(\bigvee_{v:uv \in E} x_v \right)$$

Below we combine a result of Lokam [8] with the Magnification Lemma to show that also in the case of unbounded fanin circuits the gap may be as large as $\sqrt{\log n}$, and this holds for quadratic functions and for graphs. This (partially) answers the question of [10]. We—like the authors of [10]—conjecture that the actual gap should be much larger.

A *Sylvester graph* is a bipartite $n \times n$ graph H with $n = 2^m$ vertices in each part (color class) identified with subsets of $\{1, \ldots, m\}$; two vertices u and v are adjacent iff $|u \cap v|$ is odd. The *saturated extension* of H is a (non-bipartite) graph $G = (V, E)$ consisting of two cliques with the bipartite graph H in-between. That is, two vertices $u \neq v \in V$ are adjacent in G iff either both these vertices lie in the same color class of H or uv is an edge of H. Since every edge/non-edge of H is also an edge/non-edge of G, every circuit computing G must also represent H.

Theorem 3.1. *Let G be the saturated extension of an $n \times n$ Sylvester graph H. Then the gaps between general monotone and single level circuits for the quadratic function f_G as well as for the graph H are at least $\sqrt{\log n}$.*

Proof. By a "circuit" we will now mean a circuit with unbounded fanin gates. We are going to combine the Magnification Lemma with the following result.

Theorem 3.2 (Lokam [8])**.** *Every monotone depth-3 formula representing an $n \times n$ Hadamard graph has at least $\Omega\left((\log n)^3/(\log \log n)^5\right)$ gates.*

Note that for depth-3 *circuits*, Theorem 3.2 implies the lower bound $\Omega((\log n)^{3/2-\epsilon})$: just take the maximum of the fanins of gates on the top and middle level. By Lemma 1.1, every monotone depth-3 circuit representing the Sylvester graph H, and hence, any monotone depth-3 circuit computing the quadratic function f_G of its saturated extension G, must have at least so many gates.

On the other hand, if we allow larger depth, then the graph H can be represented using much fewer gates. Indeed, the adjacency function of H is the inner product function $IP_m = \sum_{i=1}^{m} x_i y_i \pmod 2$. This function has a trivial (non-monotone) circuit of linear in $m = \log n$ size. By Lemma 1.1, the graph H can be represented by a monotone circuit of size $O(\log n)$. Since G is a saturated extension of H, Lemma 3.8 of [6] implies that also f_G can be computed by a monotone circuit of size $O(\log n)$. $\qquad\square$

4   Upper bounds for depth-3 circuits

Let $\Sigma_3(G)$ denote the minimum size of a monotone depth-3 circuit representing the graph G. It is easy to show that $\Sigma_3(G) = \Omega(\sqrt{n})$ for almost all n-vertex graphs. On the other hand, a lower bound $\Sigma_3(G) = \Omega(n^\epsilon)$ (with an arbitrary small constant $\epsilon > 0$) for an *explicit* graph G, together with a reduction of Valiant [12], would imply that its adjacency function requires non-monotone log-depth circuits of super-linear size (see, e.g., [5]). Unfortunately, our knowledge about the power of depth-3 circuits for graphs is very poor: we cannot even prove large poly-logarithmic lower bounds (the best remains the lower bound of Lokam [8] mentioned above).

In view of these difficulties with proving *lower* bounds, it is natural to try to obtain good *upper* bounds. That is, to understand what graphs are "bad" candidates, i.e. can be represented by small monotone depth-3 circuits.

Let $\mathrm{cnf}(H)$ be the minimum number r of clauses in a monotone CNF

$$F(X) = \Big(\bigvee_{u \in S_1} x_u \Big) \wedge \Big(\bigvee_{u \in S_2} x_u \Big) \wedge \cdots \wedge \Big(\bigvee_{u \in S_r} x_u \Big) \tag{2}$$

representing the graph H. In oder to show that some graph G cannot be represented by a small monotone Σ_3-circuit, it would be enough to show that $\mathrm{cnf}(H)$ is large for every dense enough subgraph H of G.

However, it turns out that already CNFs allow to represent a lot of graphs quite compactly. A CNF (2) represents a graph iff every edge (looked as a 2-element set) intersets all of the sets $S_1, \ldots, S_r$, and every non-edge avoids at least one of these sets. Hence, $\mathrm{cnf}(H)$ equals the minimum number of independent sets covering all non-edges of H. Alon [1] has proved that this number does not exceed $O(\Delta^2 \log n)$, where Δ is the maximum degree of H. For *bipartite* graphs we can prove a somewhat better upper bound.

Theorem 4.1. *Let H be bipartite n-vertex graph, and Δ the minimum over the two color classes of the maximal degree of a vertex in this class. Then $\mathrm{cnf}(H) = O(\Delta \log n)$.*

Proof. Let $H \subseteq U \times W$ and assume w.l.o.g. that Δ is the maximal degree of a vertex in W. Consider the following procedure of choosing a clique $A \times B$ in $\overline{H}$: pick every vertex $u \in U$ independently, with probability $p = 1/(2\Delta)$ to get a subset $A \subseteq U$, and take $B = W \setminus N(A)$, where $N(A)$ is the set of all neighbors of vertices in A. Apply this procedure $t = O(\Delta \ln n)$ times to get k cliques $A_i \times B_i$, $i = 1, \ldots, t$. Let us estimate the probability that some (fixed) edge uv of $\overline{H}$ is not covered by all these cliques. This edge is covered by $A_i \times B_i$, if u was chosen in A_i and no neighbor of v was chosen in A_i. Hence, uv is covered by $A_i \times B_i$ with probability at least $p(1-p)^\Delta \geq pe^{-p\Delta/(1-p)} \geq pe^{-2pd} = p/e$, and the probability that uv is not covered by any of t cliques $A_i \times B_i$ does not exceed $(1 - p/e)^t \leq e^{-tp/e}$. Hence, the probability that some of the non-edges of H remains uncovered does not exceed $n^2 e^{-tp/e} = \exp(2\ln n - t/(2e\Delta))$, which is < 1 for $t = c\Delta \log n$ with a sufficiently large constant c. Hence, the edges of $\overline{H}$ can be covered by $O(\Delta \ln n)$ bipartite cliques, implying that $\mathrm{cnf}(H) = O(\Delta \log n)$. $\qquad\square$

Thus, all bipartite graphs of small degree in at least one color class are "bad" candidates. On the other hand, some graphs of large degree are also bad. Such are, in particular, graphs which can be covered by a small number of bipartite cliques or "fat matchings".

A *fat matching* is a union of vertex-disjoint bipartite cliques (these cliques need not to cover all vertices); the number of such cliques is the *size* of a fat matching. Hence, a matching with k edges is a fat matching of size k, and a bipartite clique is a fat matching of size 1.

Lemma 4.1. *If a bipartite graph G can be covered by t fat matchings, each of size at most 2^r, then $\Sigma_3(G) \leq 2t \max\{1, r\}$.*

Proof. Fix an arbitrary covering of $G \subseteq U \times W$ by t fat matchings of size 2^r. The case $r = 0$ (covering by bipartite graphs) is obvious. So, assume that $r \geq 1$, and let H be any fat matching from that covering with the largest $\mathrm{cnf}(H)$. This graph has the form $H = \bigcup_{i=1}^k A_i \times B_i$ where the sets A_i (as well as the sets B_i) are disjoint.

It is easy to show (see, e.g., [5]) that $\mathrm{cnf}(G)$ is the smallest number d for which each vertex u can be associated with a subset $S_u \subseteq \{1, \ldots, d\}$ such that $S_u \cap S_v = \emptyset$ if uv is an edge, and $S_u \cap S_v \neq \emptyset$ if uv is a non-edge of G.

Let now $S_1, \ldots, S_K$ with $K = \binom{2r}{r} > 2^r = k$ be all distinct r-element subsets of $\{1, \ldots, 2r\}$. Associate all vertices in A_i with the i-th set S_i, and all vertices in B_i with the complement $\overline{S_i}$. Associate all the remaining vertices with the set $S_0 = \{1, \ldots, 2r\}$. It is easy to see that a pair of vertices $u \in U$ and $v \in W$ are adjacent in H iff their associated sets are disjoint. Hence, $\mathrm{cnf}(H) \leq 2r$, implying that $\Sigma_3(G) \leq t \cdot \mathrm{cnf}(H) \leq 2rt$. $\qquad\square$

5 Depth-3 circuits and the hierarhy of communication complexity classes

In 1986 Babai, Frankl and Simon [2] defined a hierarhy of communication complexity classes and asked whether $\Sigma_2^{cc} = \Pi_2^{cc}$ in this hierarchy. The combinato-

rial definition of this hierarchy is the following.

We consider bipartite $n \times n$ graphs. The initial set Π_0^{cc} is defined in [2] as the set of all (bipartite) cliques, and Σ_0^{cc} is the set of their complements.[1] For every $i \geq 0$, a Σ_{i+1}^{cc}-graph is a union of $2^{\text{polylog}(m)}$ Π_i^{cc}-graphs, and a Π_{i+1}^{cc}-graph is a complement of a Σ_i^{cc}-graph.

Problem 5.1 (Babai–Frankl–Simon [2])**.** *Does $\Sigma_2^{cc} \neq \Pi_2^{cc}$?*

Combining the Magnification Lemma and a result from [4] we can answer this question affirmatively when the hierarchy is constructed starting from the set Π_0^c consisting of "canonical" cliques. That is, we look at vertices of a bipartite $n \times n$ graph $H \subseteq U \times W$ with $n = 2^m$ as binary vectors of length m; hence, $U = W = \{0,1\}^m$. Recall that a *subcube* of $\{0,1\}^m$ is a subset $A \subseteq \{0,1\}^m$ of the form

$$A = \{a : a_{i_1} = \sigma_1, \ldots, a_{i_k} = \sigma_k\}$$

for some $1 \leq i_1 < i_2 < \cdots < i_k \leq m$ and $\sigma_1, \ldots, \sigma_k \in \{0,1\}$. A clique $A \times B$ is *canonical* if both A and B are subcubes of $\{0,1\}^m$.

Let now the initial set Π_0^c consist of all canonical cliques, and let Σ_i^c and Π_i^c be the classes of the resulting hierarchy. Note that this is a very restricted version of the original hierarchy since the first class Π_0^c contains much fewer graphs than that of the original hierarchy.

Theorem 5.2. $\Sigma_2^c \neq \Pi_2^c$.

For the proof we need the following fact, which can be easily derived from the Magnification Lemma in [5].

A Σ_3-circuit for $H \subseteq U \times W$ with $U = W = \{0,1\}^m$ is *canonical* if for every OR gate $g = \bigvee_{u \in S} x_u$ on the bottom (next to the inputs) level, both sets $S \cap U$ and $S \cap W$ are complements of subcubes of $\{0,1\}^m$. Let $\Sigma_3^{\text{can}}(H)$ be the minimum size of a monotone cannonical Σ_3-circuit representing H. It is easy to see that $H \in \Sigma_2^c$ iff $\Sigma_3^{\text{can}}(H) \leq 2^{\text{polylog}(m)}$. On the other hand, from the proof of the Magnification Lemma in [5] it is not difficult to derive the following

Lemma 5.1. *If f is the adjacency function of H, then $\Sigma_3(f) = \Sigma_3^{\text{can}}(H)$.*

Proof. (sketch) Let $f(y,z)$ be the adjacency function of H, and let $F(y,z)$ be a Σ_3-circuit computing f. Replace each input literal y_i^σ (resp., z_i^σ) by an OR of variables

$$\bigvee\{x_u : u \in U,\ u_i = \sigma\}$$

(resp., $\bigvee\{x_w : w \in W,\ w_i = \sigma\}$). The obtained monotone Σ_3-circuit $F'(X)$ on the variables $X = \{x_v\ v \in U \cup W\}$ is canonical. Moreover, it is easy to check (see [5]) that $F'(X)$ represents H. Hence, $\Sigma_3(f) \geq \Sigma_3^{\text{can}}(H)$. The other direction $\Sigma_3(f) \leq \Sigma_3^{\text{can}}(H)$ is also easy because in each bottom OR gate $g = \bigvee_{u \in S} x_u$ of a monotone canonical circuit representing H, each set of variables $\{x_u : u \in U,\ u_i = \sigma\}$ corresponds to the literal y_i^σ. $\square$

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[1] A complement of a bipartite graph $H \subseteq U \times W$ is the bipartite graph $\overline{H} = (U \times W) \setminus H$.

Corollary 5.3. *A graph H belongs to Σ_2^c iff its adjacency function can be computed by a Σ_3 circuit of size $2^{\mathrm{polylog}(m)}$.*

Now we turn to the actual proof of the theorem.

Proof. (of Theorem) Let $m = r^2$ and consider the following Kneser-type graph $H_m \subseteq U \times W$. Its vertices are 0-1 matrices $u = \{u_{ij}\}$ of dimension $r \times r$, and two matrices u and v are adjacent iff $\exists i \; \forall j \; u_{ij} \cdot v_{ij} = 0$. Hence,

$$f_m(x, y) = \bigvee_{i=1}^{r} \bigwedge_{j=1}^{r} (\overline{x}_{ij} \vee \overline{y}_{ij})$$

is the adjacency function of H_m. By Corollary 5.3, the graph H_m belongs to Σ_2^c. On the other hand, it is proved in [4] that any Π_3-circuit computing f_m requites $2^{\Omega(m)}$ gates. Since $\neg f_m$ is the adjacency function of the complement $\overline{H}_m$ of H_m, we obtain that $\Sigma_3^{\mathrm{can}}(\overline{H}_m) = \Sigma_3(\neg f_m) = \Pi_3(f_m) \geq 2^{\Omega(m)}$. By Corollary 5.3, the complement of H_m does not belong to Σ_2^c. $\qquad\square$

6 Graphs and ACC-circuits

A SYM-*circuit* of size d is a depth-2 circuit of the form $SYM(C_1, \ldots, C_d)$ where SYM is a symmetric boolean function in d variables and each C_i is an OR of some variables and their negations. If there are no negated variables, then the corresponding circuit in a SYM^+-*circuit*. A circuit is *monotone* if it has no negated variables. The *type* of such a circuit is the subset $K \subseteq \{0, 1, \ldots, d\}$ such that $SYM(x_1, \ldots, x_d) = 1$ iff $\sum_i x_i \in K$. An ACC-circuit is a constant depth circuit with unbounded fanin AND, OR and arbitrary MOD m gates; such a gate outputs 1 precisely when the sum of input bits is divisible by m.

Our interest in SYM-circuits comes from the result of Yao [13] which, together with the Magnification Lemma, implies that if a bipartite $n \times n$ graph G cannot be recognized by a SYM^+-circuit of size $d \leq 2^{(\log \log n)^{O(1)}}$, then its adjacency function cannot be computed by ACC-circuits of polynomial size.

A system $\mathcal{A} = \{A_1, \ldots, A_n\}$ of (not necessarily distict) subsets of $\{1, \ldots, d\}$ *represents* a given n-vertex graph G if there is a subset $L \subseteq \{0, 1, \ldots, d\}$ (called the *type* of the representation) such that

(i) $|A_i \cap A_j| \in L$ if $\{i, j\}$ is an edge, and

(ii) $|A_i \cap A_j| \notin L$ if $\{i, j\}$ is a non-edge of G.

The *intersection dimension* of G, $\dim(G)$, is the smallest size d of the universum for which such a representation exists. If the type L is given, then the corresponding measure is denoted by $\dim_L(G)$; hence, $\dim(G)$ is the minimum of $\dim_L(G)$ over all possible types L.

Proposition 6.1. *For every graph G, $\dim(G)$ is the minimum size of a SYM^+-circuit representing G.*

Proof. Take an arbitrary graph $G = ([n], E)$, and let

$$SYM(\bigvee_{i \in S_1} x_i, \bigvee_{i \in S_2} x_i, \ldots, \bigvee_{i \in S_d} x_i)$$

be a minimal SYM^+-circuit representing this graph, where SYM is a symmetric boolean function of some type $K \subseteq \{0, 1, \ldots, d\}$. We will show only one direction $d \geq \dim(G)$; the other is similar. For each vertex $i \in [n]$, let $A_i = \{j : i \notin S_j\}$. Then $|A_i \cap A_j|$ is the mumber of clauses rejecting the arc $e = \{i, j\}$. Hence, e is an edge in G iff the number of clauses accepting e belongs to K iff $d - |A_i \cap A_j| \in K$ iff $|A_i \cap A_j| \in L$, where $L = \{d - k : k \in K\}$. $\quad\square$

It is easy to see that $\dim(G) \leq n$ for every $n \times n$ bipartite graph G: just associate to each vertex i on the left part the set $A_i = \{i\}$, and to each vertex j on the right part the set B_j of its neighbors. Then $|A_i \cap B_j| \in \{0, 1\}$, and $|A_i \cap B_j| = 1$ precisely when i and j are adjacent; hence, we can take $L = \{1\}$. Thus, even for type $L = \{1\}$ we have that $\dim_L(G) \leq n$ for every $n \times n$ bipartite graph G.

We also have that $\dim(G) \geq \log n$ for every "non-trivial" graph (i.e. graph, no two vertices of which have the same neighborhood): we need different sets for different vertices. However, this trivial upper bound is exponentially far from the lower bound $\dim(G) = \Omega(n)$ which, by an easy counting, is valid for almost all graphs. On the other hand, as mentioned above, the solution of the following problem would give us a super-polynomial lower bound for ACC-circuits.

Problem 6.2. *Exhibit an explicit bipartite $n \times n$ graph G with* $\dim(G) \geq 2^{(\log \log n)^{\omega(1)}}$.

What graphs have large intersection dimension? What about Ramsey graphs, i.e. graphs without a clique or independent set of size $O(\log n)$? A naive approach to show that Ramsey graphs cannot have small intersection dimension would be to show that, if d is small, then for any system $\mathcal{A} = \{A_1, \ldots, A_n\}$ of subsets of $\{1, \ldots, d\}$, the coloring $c_\mathcal{A}(i, j) := |A_i \cap A_j|$ of the edges of K_n must produce a monochromatic clique of size $\omega(\log n)$. However, a result of Kostochka and Rödl [7] on weak Δ-systems shows that this will not work: there exists a family $\mathcal{A}$ of n subsets of $\{1, \ldots, d\}$ such that $d \leq (\log n)^3$ and the coloring $c_\mathcal{A}$ of K_n produces *no* monochromatic clique of size $\omega(\log n)$.

When trying to estimate the intersection dimension of a graph, we are faced with the following problem. We have a system $\mathcal{A} = \{A_1, \ldots, A_n\}$ of subsets of $\{1, \ldots, d\}$ and (if the type L of the intersections is not given) the only knowledge about this system is that the intersection sizes $|A_i \cap A_j|$ must be consistent with a given graph $G = (V, E)$: the pairs A_i, A_j corresponding to edges and to non-edges must have different intersection sizes. Hence, the whole information about the set-system $\mathcal{A}$ we are interested in is given by its intersection matrix

$$I(\mathcal{A}) = \{|A_i \cap A_j| : 1 \leq i, j \leq n\}.$$

Since $I(\mathcal{A})$ is a matrix of scalar products of the corresponding characteristic vectors, the size d of the universum is at least the rank of $I(\mathcal{A})$ over the reals.

Hence, the most direct (and most difficult) way would be to try to prove that the intersection matrix $I(\mathcal{A})$ of every set-system $\mathcal{A}$ representing G must have large rank.

Another, less direct approach could be to try to use the properties of a given graph G to construct a multi-linear polynomial $f(x_1, \ldots, x_d) = \sum_{I \subseteq [d]} a_I \prod_{i \in I} x_i$ and to show that the matrix[2]

$$I_f(\mathcal{A}) = \{f(A_i \cap A_j) : 1 \leq i, j \leq n\}$$

must have large rank. (Note that the intersection matrix $I(\mathcal{A})$ is the matrix $I_f(\mathcal{A})$ with $f = \sum_{i=1}^{d} x_i$.) If the weight $w(f)$ of f—that is, the number of monomials X_I with non-zero coefficients—is not too large with respect to the number d of variables of f then, as shown by Grolmusz in [3], d must be also large enough.

Theorem 6.3 (Grolmusz [3]). *Let R be either a field or a ring $\mathbb{Z}_m$ for some m. Let $\mathcal{A} = \{A_1, \ldots, A_n\}$ be a family of subsets of $\{1, \ldots, d\}$. Let $f(x_1, \ldots, x_d)$ be a multi-linear polynomial with non-negative integer coefficients. Then the matrix $I_f(\mathcal{A})$ has rank at most $w(f)$ over R.*

Proof. (sketch) Let $f(x_1, \ldots, x_d) = \sum_{I \subseteq [d]} a_I X_I$, where $X_I = \prod_{i \in I} x_i$. Take the incidence $d \times n$ matrix M of $\mathcal{A}$. The incidence matrix M' of $f(\mathcal{A})$ is a $N \times n$ matrix with $N = \sum_{I \subseteq [d]} a_I$, whose rows correspond to monomials X_I of f. There are a_I identical rows in M' corresponding to the same monomial $X_I = \prod_{i \in I} x_i$; the row corresponding to such a monomial is just a component-wise AND of the rows $i \in I$ of M. Let $\mathcal{A}' = \{A'_1, \ldots, A'_n\}$, where A'_i is a subset of $\{1, \ldots, N\}$ defined by the i-th column of M'. Note that

$$f(A_i \cap A_j) = \sum_{I : X_I(A_i \cap A_j) = 1} a_I.$$

Using this, it is easy to verify that

$$f(A_i \cap A_j) = |A'_i \cap A'_j|$$

for all i, j. Hence, $I_f(\mathcal{A}) = I(\mathcal{A}')$. Since the rank of $I(\mathcal{A}')$ cannot exceed the number $w(f)$ of its different rows, we are done. $\square$

A general frame to use this result to show that some graph G must have large intersection dimension, could be as follows. Use the properties of the given graph G to construct polynomial $f(x_1, \ldots, x_d)$ of weight $w(f) \leq d^{O(1)}$ such that, for every system $\mathcal{A}$ of subsets of $\{1, \ldots, d\}$ representing the graph G, the matrix $I_f(\mathcal{A})$ has large rank, say, at least r. Then $d \geq r^{\Omega(1)}$.

Bellow we show how this argument can be used to derive non-trivial lower bounds on the intersection dimension if either: (i) the "modular size" of the intersection type L is not too large, or (ii) if the type L is arbitrary but we impose some additional conditions on the representing set systems $\mathcal{A}$.

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[2]$f(A_i \cap A_j)$ denotes the value of f on the incidence vector of $A_i \cap A_j$.

A difficult thing when dealing with SYM-circuits $SYM(C_1, \ldots, C_d)$ is that SYM can be a symmetric boolean function of *arbitrary* type $L \subseteq \{0, 1, \ldots, d\}$. On the other hand, if we know more about the type L, then lower bounds may be much easier to prove. We illustrate this by the followng simple function.

Let $NE_m(\vec{y}, \vec{z})$ be a boolean function in $2m$ variables such that $NE_{2m}(\vec{y}, \vec{z}) = 1$ iff $\vec{y} \neq \vec{z}$ (the non-equality function). Let $SYM(C_1, \ldots, C_d)$ be a SYM-circuit computing NE_{2m}, where g is a symmetric boolean function in d variables of some type L. We now show that, if the "modular size" of L is small, then the size d of the circuit must be exponential in m.

For a subset $L \subseteq \{0, 1, \ldots, d\}$ and an integer $m \geq 1$, let $L \pmod{m}$ denote the set of distinct residues of elements of L modulo m. Let also $\overline{L} = \{0, 1, \ldots, d\} \setminus L$. We say that L has

(i) *modular size a* if there is an integer $m \geq 1$ such that the set $L \pmod{m}$ has at most a elements and is disjoint from $\overline{L} \pmod{m}$;

(ii) *weak modular size b* if for every $\ell \in L$ there is a prime $p_\ell \leq b$ such that the residue of ℓ modulo p_ℓ does not belong to $\overline{L} \pmod{p_\ell}$.

In particular, the modular size of any set L never exceed its size $|L|$ (just take $m = 1$ in (i)). Hence, such simple symmetric functions, like AND, OR or Parity, have modular size $a = 1$.

Since NE_m is the adjacency function of the complement $\overline{M_n}$ of an n to n matching M_n with $n = 2^m$, exponential lower bounds on the SYM-circuit size of NE_m, when the type of a circuit has small (weak) modular size, follows from the following

Proposition 6.4. *If L has modular size a and weak modular size b, then*

$$\dim_L(\overline{M_n}) \geq \frac{1}{e} \max \left\{ n^{1/a}, \left(\frac{n}{|L|} \right)^{1/(b-1)} \right\}.$$

Proof. Let $d = \dim_L(G)$, and let $\mathcal{A} = \{A_1, \ldots, A_n, B_1, \ldots, B_n\}$ be the corresponding systems of subsets of $\{1, \ldots, d\}$ associated with the vertices of $\overline{M_n}$; hence,

$$|A_i \cap B_j| \in L \iff i \neq j.$$

To prove the first estimate $\dim_L(G) \geq n^{1/a}/e$, take an $m \geq 1$ such that the set $K = L \pmod{m}$ has size $|K| \leq a$ and shares no common element with $\overline{L} \pmod{m}$. Consider the following multi-linear polynomial of degree at most a in d variables $z = (z_1, \ldots, z_d)$ over the ring $\mathbb{Z}_m$:

$$f(z) = \prod_{k \in K} \left(\sum_{i=1}^{d} z_i - k \right).$$

Then $f(A_i \cap B_i) \neq 0$ since $|A_i \cap B_i| \in \overline{L}$ and $K = L \pmod{m}$ is disjoint from $\overline{L} \pmod{m}$. Moreover, if $i \neq j$ then $|A_i \cap B_j| \in L$, and hence, $f(A_i \cap B_j) = 0$. This implies that the matrix $I_f(\mathcal{A})$ has rank n over $\mathbb{Z}_m$, and Theorem 6.3

implies that $n \leq w(f)$. Since $w(f) \leq \sum_{i=0}^{a} \binom{d}{i} \leq \left(\frac{ed}{a}\right)^a$, the desired lower bound $d \geq n^{1/s}/e$ follows.

To prove the second estimate $\dim_L(G) \geq (n/|L|)^{1/(b-1)}/e$, take a subset $I \subseteq \{1, \ldots, n\}$ such that $|I| \geq n/|L|$ and all intersections $A_i \cap B_i$ with $i \in I$ have the same size $x \in L$. Let $p \leq b$ be a prime number corresponding to x, i.e. the residue $x \pmod{p}$ does not appear in $\overline{L} \pmod{p}$. Let $\mathcal{A}' = \{A_i, B_i : i \in I\}$ be the corresponding sub-system of $\mathcal{A}$. Consider the following multi-linear polynomial of degree $p - 1$ in d variables $z = (z_1, \ldots, z_d)$ over $GF(p)$:

$$f(z) = 1 - g(z)^{p-1} \quad \text{with} \quad g(z) = \sum_{i=1}^{d} z_i - x.$$

Then for every $i, j \in I$ we have that $f(A_i \cap B_i) = 1$ because $g(A_i \cap B_i) = 0$, and $f(A_i \cap B_j) = 0$ if $i \neq j$ because then $g(A_i \cap B_i) \neq 0$. Hence, again, the matrix $I_f(\mathcal{A}')$ has rank $|I|$ over $GF(p)$, and Theorem 6.3 implies that $|I| \leq w(f)$. Since $|I| \geq n/|L|$ and $w(f) \leq \sum_{i=0}^{p-1} \binom{d}{i} \leq \left(\frac{ed}{p-1}\right)^{p-1}$, the desired lower bound on d follows. $\qquad\square$

Proposition 6.4 implies that, using types of modular size a, the function NE_m cannot be computed by SYM-circuits of size smaller tham $2^{\Omega(m/a)}$. But this function has a very small ACC circuit: $NE_m(\vec{y}, \vec{z}) = \bigvee_{i=1}^{m} y_i \oplus z_i$. This is no contradiction, because there are types L for which $\dim_L(\overline{M_n}) = O(\log n)$. This follows from a simple fact that the edges of $\overline{M_n}$ can be covered by $O(\log n)$ bipartite complete subgraphs of $\overline{M_n}$.

In our next example we show how the linear algebra method can be applied in the situations, where we know something more about the set system $\mathcal{A}$ than that it just is consistent with our graph—the type L in this case can be arbitrary!

Let $\mathcal{A}$ be a family of finite sets. Say that a pair $X \neq Y \in \mathcal{A}$ of its members is a *unifying pair* if

$$|A \cap B \cap X| = |A \cap B \cap Y| \text{ for all } A \neq B \in \mathcal{A} \setminus \{X, Y\}.$$

To ensure this, it would be enough, for example, that $A \cap B \subseteq X \cap Y$ for all $A \neq B \in \mathcal{A} \setminus \{X, Y\}$.

Say that a graph $G = (V, E)$ is *k-separated* if for every pair of distinct vertices $i \neq j$ there exists a subset $S \subseteq V \setminus \{i, j\}$ of $|S| = k$ vertices such that i is connected to all vertices in S and j is connected to none of the vertices in S. For example, an n-vertex Paley graph is k-separated with $k = n/4$.

Proposition 6.5. *Let $\mathcal{A}$ be a system of n subsets of $\{1, \ldots, d\}$, and assume that it contains a unifying pair. If $\mathcal{A}$ represents some k-separated graph with $k \geq 1$, then $d \geq k$.*

Proof. Fix a prime number $p \geq d$ and work over the field $GF(p)$. Let $X, Y \in \mathcal{A}$ be a unifying pair in $\mathcal{A}$, and let x and y be the corresponding to the sets X, Y pair of vertices of G. Take a subset $S \subseteq V \setminus \{x, y\}$ of $|S| = k$ vertices, all of which are joined to x and none of which is joined to y. Then

$$|A_i \cap X| \neq |A_i \cap Y| \quad \text{for all } i \in S \tag{3}$$

just because $|A_i \cap X| \in L$ and $|A_i \cap Y| \notin L$. Consider the following polynomial in d variables (over $GR(p)$)

$$f(z_1, \ldots, z_d) = \sum_{k \in X} z_k - \sum_{l \in Y} z_l.$$

Let $\mathcal{A}' = \{A_i : i \in S\}$. Then the (i, j)-entry of the intersection matrix $I_f(\mathcal{A}')$ is $|A_i \cap A_j \cap X| - |A_i \cap A_j \cap Y|$. By (3), the diagonal entries are nonzero integers. On the other hand, all other entries are zeros because $|A_i \cap A_j \cap X| = |A_i \cap A_j \cap Y|$ for all $i \neq j \in S$. Hence, the matrix $I_f(\mathcal{A}')$ has full rank (over $GF(p)$), implying that $k = |S| \leq w(f) \leq |X \cup Y| \leq d$. $\qquad\square$

7 A lower bound for fanin-2 circuits

In this section we consider standard (fanin-2) circuits over $\{\vee, \wedge, \neg\}$. Let $C_+(G)$ be the minimum size of a *monotone* circuit representing the graph G.

Combining the Magnification Lemma with a result of Pudlák, Rödl and Savický [10] about monotone complexity of boolean sums, it can be shown (see [6]) that a lower bound $C_+(G) \geq 12n + \varphi(n)$ for an explicit bipartite $n \times n$ graph G with $n = 2^m$ would imply a lower bound $\varphi(2^m)$ on the size of *non-monotone* circuits computing an explicit boolean function in $2m$ variables (the adjacency function of G). Hence, proving even linear lower bounds $C_+(G) \geq Cn$ for graphs should be a very difficult task. Still, the following fact shows that at least for $C = 2$ this can be easily done.

Proposition 7.1. *Let $G_n = K_{n-1} + E_1$ be a complete graph on $n - 1$ vertices plus one isolated vertex. Then $C(G_n) \geq 2n - 6$.*

Proof. Let $G_n = (V, E)$ where $V = \{1, \ldots, n\}$ and n is the isolated vertex. Let $F(x_1, \ldots, x_n)$ be an arbitrary monotone circuit representing G_n.

Claim 7.1. *If $n \geq 3$ then every input gate x_i, $i = 1, \ldots, n - 1$ has fanout at least 2.*

If the claim is true, then by setting one variable to a constant 0 at least two gates become redundant. This gives the recurrence $C(G_n) \geq C(G_{n-1}) + 2$, and this holds until $n \geq 3$. Hence, we have that $C(G_n) \geq 2(n - 3) = 2n - 6$.

It remains to prove the claim. For this, assume that some input gate, say, x_1 has fanout 1. Let $g(x_1, x_i)$ be the (unique) gate entered by this input gate. We will show that then the circuit accepts some non-edge of G_n, i.e. some arc $\{j, n\}$ with $j \neq n$, a contradiction.

Case 1: $g = \wedge$ and $i = n$. Then

$$1 = F(1, 1, 0, \ldots, 0) = F(0, 1, 0, \ldots, 0) \leq F(0, 1, 0, \ldots, 1),$$

Case 2: $g = \wedge$ and $i \neq n$, say, $i = 2$. Then

$$1 = F(1, 0, 1, 0 \ldots, 0) = F(0, 0, 1, 0, \ldots, 0) \leq F(0, 0, 1, 0, \ldots, 1),$$

Case 3: $g = \vee$ and $i = n$. Then

$$1 = F(1, 1, 0, \ldots, 0) \leq F(1, 1, 0, \ldots, 1) = F(0, 1, 0, \ldots, 1),$$

Case 4: $g = \vee$ and $i \neq n$, say, $i = 2$. Then

$$1 = F(1, 1, 0, \ldots, 0) = F(0, 1, 0, \ldots, 0) \leq F(0, 1, 0, \ldots, 1),$$

$\square$

8 Structure of optimal formulas for graphs

A *length* $|F|$ of a formula is the number of input literals in it. Fix a minimal monotone formula representing an n-vertex graph $G = ([n], E)$, and let m_i be the number of occurences of the variable x_i in this formula; hence, the formula has length $\sum_{i=1}^{n} m_i$. Let d_i be the degree of vertex i in G. The representation $\bigvee_{ij \in E} x_i x_j$ implies that $\sum_{i=1}^{n} m_i \leq 2|E| = \sum_{i=1}^{n} d_i$. In the representation

$$f_G(X) = \bigvee_{u \in V} x_u \wedge \left(\bigvee_{v:uv \in E} x_v \right)$$

by a single level formula we have that $m_i \leq d_i + 1$ for all $i = 1, \ldots, n$. Interestingly, this last property is shared by *any* minimal formula.

Proposition 8.1. *Let $G = ([n], E)$ be a graph without complete stars. If $F(x_1, \ldots, x_n)$ is a minimal monotone formula representing G, then $m_i \leq d_i + 1$ for all $i = 1, \ldots, n$.*

Proof. Suppose that $m_i > d_i + 1$ for some i. Let $F_{x_i=0}$ be the formula obtained from F by setting to 0 all occurences of the variable x_i. We claim that the formula

$$F' = F_{x_i=0} \vee F_i \quad \text{with} \quad F_i = x_i \wedge (\bigvee_{j:ij \in E} x_j)$$

represents G. Let $a \in \{0, 1\}^n$ be an input with precisely two 1's. Hence, $F(a) = 1$ iff the two positions of these 1's are adjacent in G. We consider two cases.

If $a_i = 0$ then $F_i(a) = 0$ and $F_{x_i=0}(a) = F(a)$, implying that in this case $F'(a) = F(a)$. Assume therefore that $a_i = 1$, and let j be the second position for which $a_j = 1$. Then $F_i(a) = 1$ iff $ij \in E$ iff $F(a) = 1$. Moreover, $F_{x_i:=0}(a) = 0$ because otherwise, F would accept a single vertex $\{j\}$, a contradiction with star-freeness of G. Hence, also in this case we have that $F'(a) = F(a)$.

Thus, the new formula F' represents the graph G and has length

$$|F'| \leq |F_{x_i=0}| + (d+1) \leq |F| - m_i + (d+1) < |F|,$$

a contradiction with the minimality of F. $\square$

Proposition 8.1 could wake an impression that, in the context of graphs, the single level for formulas could be true. Unfortunately, this is not the case. Let $K \subseteq U \times V$ be a bipartite Kneser $n \times n$ graph. In this case U and W consist of all $n = 2^r$ subsets u of $\{1, \ldots, r\}$, and $uv \in K$ iff $u \cap v = \emptyset$. It is easy to see that the formula

$$F(X) = \bigwedge_{i=1}^{r} \bigvee_{w \in S_i} x_w \tag{4}$$

with $S_i = \{w : i \notin w\}$ represents the graph K. This formula has length at most $nr = n \log n$. On the other hand, it is shown in [6] that any monotone single level formula representing K must have length at least n^{1+c} for a constant $c > 0$.

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