On the sensitivity of some APN permutations to swapping points

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Abstract

We define a set called the pAPN-spectrum of an (n, n)-function F, which measures how close F is to being an APN function, and investigate how the size of the pAPN-spectrum changes when two of the outputs of a given F are swapped. We completely characterize the behavior of the pAPN-spectrum under swapping outputs when $F(x) = x^{2^n-2}$ is the inverse function over $\mathbb{F}_{2^n}$. We also investigate this behavior for functions from the Gold and Welch monomial APN families, and experimentally determine the size of the pAPN-spectrum after swapping outputs for representatives from all infinite monomial APN families up to dimension $n = 10$.

Keywords: Boolean function, almost perfect nonlinear (APN), partial APN.

1 Introduction

Let $\mathbb{F}_{2^n}$ be the finite field with 2^n elements for some positive integer n. We call a function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2$ a *Boolean function* on n variables. The set of all Boolean functions on n variables will be denoted by $\mathcal{B}_n$.

For a Boolean function $f : \mathbb{F}_{2^n} \to \mathbb{F}_2$, we define the *Walsh-Hadamard transform* to be the integer valued function

$$\mathcal{W}_f(u) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x)+\mathrm{Tr}_1^n(ux)},$$

where $\mathrm{Tr}_1^n : \mathbb{F}_{2^n} \to \mathbb{F}_2$ is the absolute trace function, $\mathrm{Tr}_1^n(x) = \sum_{i=0}^{n-1} x^{2^i}$.

A vectorial Boolean function, or (n, m)-function, is a map $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$, for some positive integers m and n. When $m = n$, it can be uniquely represented as a univariate polynomial over $\mathbb{F}_{2^n}$ (using the natural identification of the finite field $\mathbb{F}_{2^n}$ with the vector space $\mathbb{F}_2^n$) of the form

$$F(x) = \sum_{i=0}^{2^n-1} a_i x^i, \ a_i \in \mathbb{F}_{2^n}.$$

The binary weight $w_2(i)$ of a positive integer i is the number of non-zero bits in its binary expansion, i.e. $w_2(i) = \sum_{j=0}^{K} a_j$, where $i = \sum_{j=0}^{K} a_j 2^j$ for some positive integer K and for $a_j \in \{0, 1\}$, with all sums involved being computed over the integers. The algebraic degree of $F(x)$ is then the largest binary weight of an exponent i with $a_i \neq 0$. For an (n, n)-function F and for $a, b \in \mathbb{F}_{2^n}$, we define the Walsh transform $\mathcal{W}_F(a, b)$ of F to be the Walsh-Hadamard transform of its component function $\mathrm{Tr}_1^n(bF(x))$ at a, that is,

$$\mathcal{W}_F(a, b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\mathrm{Tr}_1^n(bF(x)+ax)}.$$

For an (n, n)-function F, and $a, b \in \mathbb{F}_{2^n}$, we let $\Delta_F(a, b) = |\{x \in \mathbb{F}_{2^n} \mid F(x + a) + F(x) = b\}|$. We call the quantity $\Delta_F = \max\{\Delta_F(a, b) : a, b \in \mathbb{F}_{2^n}, a \neq 0\}$ the *differential uniformity* of F. If $\Delta_F \leq \delta$, then we say that F is differentially δ-uniform. Since $x + a$ is a solution to $F(x + a) + F(x) = b$ whenever x is, the differential uniformity is always even and is thus at least 2 for any F. If $\delta = 2$, then F is an *almost perfect nonlinear* (*APN*) function.

APN functions are of significant interest in cryptography for the construction of block ciphers since they provide optimal resistance to differential cryptanalysis. Furthermore, some classes of APN functions correspond to optimal objects in other areas of mathematics and computer science, such as coding theory, projective geometry, and combinatorial design theory. Nonetheless, being cryptographically strong functions, APN functions are by design unpredictable and difficult to construct and analyze. For the purpose of making their analysis more tractable, a number of characterizations of APN-ness have been derived and can be found in the literature. We give some of them below [3, 6, 7, 16].

Lemma 1. *Let F be an (n, n)-function.*

(*i*) *We always have*

$$\sum_{a,b \in \mathbb{F}_{2^n}} \mathcal{W}_F^4(a, b) \geq 2^{3n+1}(3 \cdot 2^{n-1} - 1),$$

with equality if and only if F is APN.

(*ii*) *If, in addition, F is APN and satisfies $F(0) = 0$, then*

$$\sum_{a,b \in \mathbb{F}_{2^n}} \mathcal{W}_F^3(a, b) = 2^{2n+1}(3 \cdot 2^{n-1} - 1).$$

(*iii*) *(Janwa-Wilson-Rodier Condition*[1]*) F is APN if and only if all the points $x, y, z \in \mathbb{F}_{2^n}$ satisfying*

$$F(x) + F(y) + F(z) + F(x + y + z) = 0$$

belong to the curve $(x + y)(x + z)(y + z) = 0$.

Along with S. Kwon, we introduced in [4] a notion of partial APN-ness in our attempt to resolve a conjecture on the upper bound on the algebraic degree of APN functions [3]. For a fixed $x_0 \in \mathbb{F}_{2^n}$, we call an (n, n)-function a (*partial*) *x_0-APN function* (which we typically refer to as x_0-APN, partially APN, or just pAPN, for short) if all points, x, y satisfying

$$F(x_0) + F(x) + F(y) + F(x_0 + x + y) = 0 \tag{1}$$

belong to the curve

$$(x_0 + x)(x_0 + y)(x + y) = 0. \tag{2}$$

We will refer to the set of points $x_0 \in \mathbb{F}_{2^n}$ for which a function is x_0-APN as the *pAPN-spectrum* of the function. Certainly, an APN function is x_0-APN for any point x_0; that is, its pAPN-spectrum is $\mathbb{F}_{2^n}$.

An alternative way to express the fact that a given function F is x_0-APN is to say that for any $a \neq 0$ the equation $F(x + a) + F(x) = F(x_0 + a) + F(x_0)$ has only two solutions x, namely $x = x_0$ and $x = x_0 + a$.

We shall denote by $\frac{1}{a}$ or $1/a$ the multiplicative inverse of a in $\mathbb{F}_{2^n}$, adopting the usual convention $\frac{1}{0} = 1/0 = 0$.

In this paper we show an intriguing property of the inverse, Gold and Welch functions: swapping two of their output values leads to a reduction in the size of their pAPN-spectra; in some cases, this reduction is quite significant. In the case of the inverse function, we completely characterize the cases in which the resulting function has an empty pAPN-spectrum.

2 Considerations and useful remarks

Throughout, we shall be using the following result [1, 17], which describes the existence of solutions for quadratic and cubic equations over binary finite fields.

Theorem 2. *Let n be a natural number, and consider the finite field $\mathbb{F}_{2^n}$.*

(1) *The equation $x^2 + ax + b = 0$, with $a, b \in \mathbb{F}_{2^n}$, $a \neq 0$, has solutions in $\mathbb{F}_{2^n}$ if and only if $\mathrm{Tr}_1^n\left(\frac{b}{a^2}\right) = 0$.*

(2) *The equation $x^3 + ax + b = 0$, with $a, b \in \mathbb{F}_{2^n}$, $b \neq 0$, has (t_1, t_2 are the roots of $t^2 + bt + a^3 = 0$):*

   (*i*) *three solutions in $\mathbb{F}_{2^n}$ if and only if $\mathrm{Tr}_1^n(a^3/b^2) = \mathrm{Tr}_1^n(1)$ and t_1, t_2 are cubes in $\mathbb{F}_{2^n}$ for n even, and in $\mathbb{F}_{2^{2n}}$ for n odd;*

[1] We have been calling this the "Rodier condition", but we realized that it did occur in the literature prior to Rodier's work, for power monomials in [10], so we will now call it by the three names.

(*ii*) *a unique solution in* $\mathbb{F}_{2^n}$ *if and only if* $\mathrm{Tr}_1^n(a^3/b^2) \neq \mathrm{Tr}_1^n(1)$;

(*iii*) *no solutions in* $\mathbb{F}_{2^n}$ *if and only if* $\mathrm{Tr}_1^n(a^3/b^2) = \mathrm{Tr}_1^n(1)$ *and* t_1, t_2 *are not cubes in* $\mathbb{F}_{2^n}$ *for* n *even, respectively,* $\mathbb{F}_{2^{2n}}$ *for* n *odd.*

A construction proposed in [18] designed to construct differentially 4-uniform permutations that involves swapping two outputs of a given (n, n)-function, has been the subject of many papers since then (see [5, 13, 14, 15, 19], to cite just a few works; a generalization allowing the modification of any two output values, of which swapping is a special case, is investigated in [11]). This naturally leads to the question of how swapping two outputs of a given function F would affect its pAPN-spectrum. We now describe the Janwa-Wilson-Rodier equation for an (n, n)-function F with two output points swapped. More precisely, given two points $x_0 \neq x_1$ in $\mathbb{F}_{2^n}$, we let $G_{x_0x_1}$ be the $\{x_0, x_1\}$-swapping of F defined by

$$G_{x_0x_1}(x) = F(x) + \left((x+x_0)^{2^n-1} + (x+x_1)^{2^n-1}\right)(y_0+y_1), \tag{3}$$

where $y_0 = F(x_0), y_1 = F(x_1)$. We will sometimes denote $G_{x_0x_1}$ simply by G if there is no danger of confusion.

Note that $x^{2^n-1} = 1$ in $\mathbb{F}_{2^n}$ unless $x = 0$, and so for any $x, y \in \mathbb{F}_{2^n}$, the expression $(x+y)^{2^n-1}$ is equal to 1 if $x \neq y$ and is equal to 0 if $x = y$.

The Janwa-Wilson-Rodier equation of $G = G_{x_0x_1}$ at $\zeta \in \mathbb{F}_{2^n}$ becomes

$$\begin{aligned} 0 = G(\zeta) + G(x) + G(y) + G(x+y+\zeta) &= F(\zeta) + F(x) + F(y) + F(x+y+\zeta) \\ &+ \left((\zeta+x_0)^{2^n-1} + (\zeta+x_1)^{2^n-1} + (x+x_0)^{2^n-1} + (x+x_1)^{2^n-1} + (y+x_0)^{2^n-1}\right. \\ &\left.+(y+x_1)^{2^n-1} + (x+y+\zeta+x_0)^{2^n-1} + (x+y+\zeta+x_1)^{2^n-1}\right)(y_0+y_1). \end{aligned} \tag{4}$$

We consider several cases depending on the value of ζ:

- If $\zeta = x_0$, then (4) becomes (for $x \neq \zeta \neq y \neq x$)

$$\begin{aligned} 0 = F(x_0) + F(x) + F(y) + F(x+y+x_0) \\ + \left((x+x_1)^{2^n-1} + (y+x_1)^{2^n-1} + (x+y+x_0+x_1)^{2^n-1}\right)(y_0+y_1). \end{aligned} \tag{5}$$

- If $\zeta = x_1$, then (4) becomes (for $x \neq \zeta \neq y \neq x$)

$$\begin{aligned} 0 = F(x_1) + F(x) + F(y) + F(x+y+x_1) \\ + \left((x+x_0)^{2^n-1} + (y+x_0)^{2^n-1} + (x+y+x_0+x_1)^{2^n-1}\right)(y_0+y_1). \end{aligned} \tag{6}$$

- If $x_0 \neq \zeta \neq x_1$, then (4) becomes (for $x \neq \zeta \neq y \neq x$)

$$\begin{aligned} 0 = F(\zeta) + F(x) + F(y) + F(x+y+\zeta) \\ + \left((x+x_0)^{2^n-1} + (x+x_1)^{2^n-1} + (y+x_0)^{2^n-1}\right. \\ \left.+(y+x_1)^{2^n-1} + (x+y+\zeta+x_0)^{2^n-1} + (x+y+\zeta+x_1)^{2^n-1}\right)(y_0+y_1). \end{aligned} \tag{7}$$

We shall be referring to equations (5)–(7) throughout the paper.

When studying how swapping outputs affects the pAPN-spectrum, we do not restrict ourselves to APN functions and often drop the conditions on the parameters in the definition of the infinite families; for example, in our experimental results for the Gold functions in Table 6, we consider all functions of the form x^{2^i+1} over $\mathbb{F}_{2^n}$ regardless of the value of $\gcd(i,n)$. In a number of cases, the functions in question are not APN, but are still differentially two-valued, i.e., there is a positive integer $s>1$ such that all non-zero derivatives of these functions are 2^s-to-1. While such a function is clearly not ζ-APN for any $\zeta \in \mathbb{F}_{2^n}$, it is also easy to see that swapping two of its outputs will always result in an empty pAPN-spectrum. The following proposition therefore allows us to eliminate some trivial cases.

Proposition 3. *Let $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be such that $\Delta_F(a,b) \geq 4$ whenever $\Delta_F(a,b) \neq 0$. Then F has an empty pAPN-spectrum. Furthermore, for any $x_0, x_1 \in \mathbb{F}_{2^n}$, the pAPN-spectrum of the $\{x_0,x_1\}$-swapping $G_{x_0x_1}$, as defined in (3), is also empty.*

Proof. We use the fact that a function F is ζ-APN if and only if the equation $D_aF(\zeta) = D_aF(x)$ only has the trivial solutions $x=\zeta$ and $x=a+\zeta$ for any $a \in \mathbb{F}_{2^n}^*$. Since $\Delta_F(a, D_aF(\zeta)) \geq 4$ for any $a \in \mathbb{F}_{2^n}^*$ and any $\zeta \in \mathbb{F}_{2^n}$ by the hypothesis, it is clear that F cannot be ζ-APN for any ζ.

Suppose now that $x_0, x_1 \in \mathbb{F}_{2^n}$, and $G = G_{x_0x_1}$ is obtained by swapping the outputs of F at x_0 and x_1. Consider some $\zeta \in \mathbb{F}_{2^n}$. Let $a,b \in \mathbb{F}_{2^n}$ be such that $x_0 = \zeta + a$ and $x_1 = \zeta + b$. First, suppose that $ab = 0$, say $a = 0$. Then

$$D_bG(\zeta) = G(\zeta) + G(\zeta+b) = F(\zeta+b) + F(\zeta) = D_bF(\zeta).$$

Since $\Delta_F(b, D_bF(\zeta)) \geq 4$, there must be some $w \in \mathbb{F}_{2^n}$ such that $D_bF(w) = D_bF(\zeta)$ and $w \neq \zeta, \zeta+b$. Thus $\{x_0,x_1\} \cap \{w, b+w\} = \emptyset$ and hence

$$D_bG(w) = D_bF(w) = D_bF(\zeta) = D_bG(\zeta),$$

showing that G is not ζ-APN.

Suppose now that $ab \neq 0$, and let $c = a+b$. We then have

$$D_cG(\zeta) = G(\zeta) + G(\zeta+a+b) = F(\zeta) + F(\zeta+a+b) = D_cF(\zeta)$$

due to $\{x_0,x_1\} \cap \{\zeta, \zeta+a+b\} = \emptyset$. Since $\Delta_F(c, D_cF(\zeta)) \geq 4$, we can find $w \in \mathbb{F}_{2^n}$ with $D_cF(w) = D_cF(\zeta)$ and $w \neq \zeta, \zeta+a+b$. Suppose now that $x_0 = \zeta + a = w$. Then $x_1 = \zeta + b = w + a + b = w + c$. Thus, $\{x_0,x_1\}$ and $\{w, w+c\}$ are either identical or disjoint. In both cases, we have

$$D_cG(w) = D_cF(w) = D_cF(\zeta) = D_cG(\zeta),$$

witnessing that G is not ζ-APN. □

3 The pAPN property for the inverse function swapped at two outputs

Theorem 4. *Let $F(x) = x^{2^n-2}$ be the inverse function on $\mathbb{F}_{2^n}$ and let $G_{x_0x_1}$ be the $\{x_0, x_1\}$-swapping of F for some $x_0, x_1 \in \mathbb{F}_{2^n}$ with $x_0 \neq x_1$. If n is odd, then:*

(i) If $x_0 = 0$ or $x_1 = 0$, then $G_{x_0x_1}$ is not ζ-APN for any $\zeta \in \mathbb{F}_{2^n}$.

(ii) If $x_0x_1 \neq 0$, then G is not ζ-APN for $\zeta \notin \{0, x_0, x_1\}$. Furthermore:

(a) if G is 0-APN, then $\mathrm{Tr}_1^n\left(\frac{x_0}{x_1}\right) = \mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right) = 1$;

(b) if G is x_0-APN, then $\mathrm{Tr}_1^n\left(\frac{x_0}{x_1}\right) = 1$;

(c) if $\mathrm{Tr}_1^n\left(\frac{x_0}{x_1}\right) = 1$, then G is x_0-APN if and only if there is no $\alpha \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$ such that $\mathrm{Tr}_1^n\left(\frac{t(\alpha+\alpha^{-1})}{t^2+\alpha^2+\alpha^{-2}+1}\right) = 0$, where $t = \frac{x_0}{x_1}$;

(d) if G is x_1-APN, then $\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right) = 1$;

(e) if $\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right) = 1$, then G is x_1-APN if and only if there is no $\alpha \in \mathbb{F}_{2^n} \setminus \mathbb{F}_2$ such that $\mathrm{Tr}_1^n\left(\frac{t(\alpha+\alpha^{-1})}{t^2+\alpha^2+\alpha^{-2}+1}\right) = 0$, where $t = \frac{x_1}{x_0}$.

If n is even (we let ω be a primitive element of $\mathbb{F}_4$), then:

(i) If say $x_0 = 0$, then G_{0x_1} is not ζ-APN if $\zeta \in \{0, x_1\}$, or $\mathrm{Tr}_1^n\left(\frac{x_1}{x_1+\zeta}\right) = 0$, or $\mathrm{Tr}_1^n\left(\frac{\zeta^2+\zeta}{x_1^2}\right) = 0$.

(ii) If $x_0x_1 \neq 0$, then we examine four cases depending on the value of the pair $\left(\mathrm{Tr}_1^n\left(1/x_0^3\right), \mathrm{Tr}_1^n\left(1/x_1^3\right)\right)$:

(a) if $\mathrm{Tr}_1^n\left(1/x_0^3\right) = \mathrm{Tr}_1^n\left(1/x_1^3\right) = 1$, then G is not ζ-APN for $\zeta \notin \{\omega x_0, \omega x_1, \omega^2 x_0, \omega^2 x_1\}$;

(b) if $\mathrm{Tr}_1^n\left(1/x_0^3\right) = 0$ and $\mathrm{Tr}_1^n\left(1/x_1^3\right) = 1$, then G is not ζ-APN for $\zeta \notin \{\omega x_0, \omega^2 x_0\}$;

(c) if $\mathrm{Tr}_1^n\left(1/x_0^3\right) = 1$ and $\mathrm{Tr}_1^n\left(1/x_1^3\right) = 0$, then G is not ζ-APN for $\zeta \notin \{\omega x_1, \omega^2 x_1\}$;

(d) if $\mathrm{Tr}_1^n\left(1/x_0^3\right) = \mathrm{Tr}_1^n\left(1/x_1^3\right) = 0$, then G is not ζ-APN for any $\zeta \in \mathbb{F}_{2^n}$.

Proof. We first examine the case when $x_0 = 0$. Let ζ be an arbitrary element of $\mathbb{F}_{2^n}$, and consider the Janwa-Wilson-Rodier equation for G_{0x_1} at ζ. We distinguish three subcases, namely $\zeta = 0$, $\zeta = x_1$, and $\zeta \neq 0, x_1$, which we treat next.

Suppose first that $\zeta = 0$. We then work under the assumption $xy(x+y) \neq 0$, and obtain from (5)

$$\begin{aligned} 0 =& F(x) + F(y) + F(x+y) + \left((x+x_1)^{2^n-1} + (y+x_1)^{2^n-1} + (x+y+x_1)^{2^n-1}\right) y_1 \\ =& x^{2^n-2} + y^{2^n-2} + (x+y)^{2^n-2} \\ & + \left((x+x_1)^{2^n-1} + (y+x_1)^{2^n-1} + (x+y+x_1)^{2^n-1}\right) y_1. \end{aligned}$$

Taking x such that $x \neq 0, x_1$ and letting $y = x+x_1$, we get $x^{2^n-2}+(x+x_1)^{2^n-2}+x_1^{2^n-2} = 0$. Multiplying both sides by $x_1x(x+x_1)$ renders $x^2+xx_1+x_1^2 = 0$, which, by Theorem 2, has two solutions if and only if $\mathrm{Tr}_1^n(x_1^2/x_1^2) = \mathrm{Tr}_1^n(1) = 0$, and that is true if and only if n is even. Therefore, G_{0x_1} cannot be 0-APN when n is even.

If n is odd, then we take $0 \neq x \neq x_1 \neq x \neq y \neq x + x_1$ and equation (5) becomes

$$F(x_1) + F(x) + F(y) + F(x + y) = 0,$$

that is,

$$x^2y + xy^2 + x_1y^2 + x_1x^2 + xyx_1 = 0,$$

and taking an arbitrary $a \neq 0, 1$, we see that the pair $x = x_1\left(1 + \frac{1}{a^2+a}\right)$, $y = x_1\left(a + \frac{1}{a+1}\right)$ is a solution to the above equation. We now argue that $xy \neq 0$ and $x \neq y$. Both of these conditions are equivalent to the equation $a^2+a+1 = 0$ having no solutions in $\mathbb{F}_{2^n}$, which is true since n is odd and $a^2 + a = 1$ would imply $\mathrm{Tr}_1^n(a^2 + a) = \mathrm{Tr}_1^n(1)$. Next, we verify that $y \neq x+x_1$. Assuming that $y = x+x_1$ leads to $a^3+a^2+a+1 = (a+1)^3 = 0$, which is impossible by the choice of a. Thus, G_{0x_1} is not 0-APN when n is odd.

We now consider the case of $x_0 = 0, \zeta = x_1$. Equation (6) transforms into

$$F(x_1) + F(x) + F(y) + F(x + y + x_1) + \left(x^{2^n-1} + y^{2^n-1} + (x + y + x_1)^{2^n-1}\right) y_1. \quad (8)$$

Let $x, y, a \in \mathbb{F}_{2^n}$ be such that $x \neq y = ax \neq 0$ (thus, $a \neq 0, 1$) and $x \neq x_1(a + 1)^{-1}$ (so that $y \neq x + x_1$). Then (8) becomes

$$\begin{aligned} 0 &= x_1^{2^n-2} + x^{2^n-2} + y^{2^n-2} + (x + y + x_1)^{2^n-2} + y_1 \\ &= x^{2^n-2} + y^{2^n-2} + (x + y + x_1)^{2^n-2}, \end{aligned}$$

which is equivalent to $0 = x^2 + y^2 + xy + x_1(x + y) = x^2(a^2 + a + 1) + x_1x(a + 1)$, rendering the solution $x = x_1(a + 1)(a^2 + a + 1)^{-1}$. It is easy to see that neither x nor ax can be equal to x_1, and so G_{0x_1} is not x_1-APN.
Finally, we consider the case of $\zeta \neq 0, x_1$. For $x_0 = 0$, equation (7) becomes

$$\begin{aligned} 0 = {} & F(\zeta) + F(x) + F(y) + F(x + y + \zeta) \\ & + \left(x^{2^n-1} + (x + x_1)^{2^n-1} + y^{2^n-1} + (y + x_1)^{2^n-1}\right. \\ & \left. +(x + y + \zeta)^{2^n-1} + (x + y + \zeta + x_1)^{2^n-1}\right) y_1. \end{aligned} \quad (9)$$

We now assume that G_{0x_1} is ζ-APN, and so (9) has no nontrivial solutions. Take $y = 0$ and $x_1+\zeta \neq x \neq x_1$ in (9). We get $\zeta^{-1}+x^{-1}+(x+\zeta)^{-1}+y_1 = 0$, which is equivalent to $x^2(1 + y_1\zeta) + x\zeta(1 + y_1\zeta) + \zeta^2 = 0$, and moreover (with $y_1 = 1/x_1$), $x^2 + x\zeta + \frac{\zeta^2x_1}{x_1+\zeta} = 0$. By Theorem 2 this equation has no solution, i.e., G is ζ-APN, if and only if

$$\mathrm{Tr}_1^n\left(\frac{\frac{\zeta^2x_1}{x_1+\zeta}}{\zeta^2}\right) = \mathrm{Tr}_1^n\left(\frac{x_1}{x_1 + \zeta}\right) = 1. \quad (10)$$

Now, take $0 \neq y = x_1 \neq x \neq 0$ in (9), as well as $x \neq x_1 + \zeta$, $x \neq \zeta$. We get $\zeta^{-1} + x^{-1} + (x + x_1 + \zeta)^{-1} = 0$, which is equivalent to $x^2 + x(x_1 + \zeta) + x_1\zeta + \zeta^2 = 0$, which has no solutions if and only if

$$\mathrm{Tr}_1^n\left(\frac{\zeta(x_1+\zeta)}{(x_1+\zeta)^2}\right) = \mathrm{Tr}_1^n\left(\frac{\zeta}{x_1+\zeta}\right) = 1. \tag{11}$$

Now, put together the conditions from equations (10) and (11). We obtain

$$0 = \mathrm{Tr}_1^n\left(\frac{x_1}{x_1+\zeta}\right) + \mathrm{Tr}_1^n\left(\frac{\zeta}{x_1+\zeta}\right) = \mathrm{Tr}_1^n(1).$$

When n is odd, $\mathrm{Tr}_1^n(1) = 1$. We obtain a contradiction, and therefore, G_{0x_1} cannot be ζ-APN for n odd.

We now turn to the case when $x_0x_1 \neq 0$. We first assume that n is odd and $\zeta \neq 0$. We examine two subcases, depending on whether ζ is one of x_0, x_1 or not.

We first assume that $\zeta = x_0$ (the case when $\zeta = x_1$ is treated in a similar manner). Then equation (5) becomes

$$\begin{aligned} 0 = {} & x_0^{2^n-2} + x^{2^n-2} + y^{2^n-2} + (x+y+x_0)^{2^n-2} \\ & + \left((x+x_1)^{2^n-1} + (y+x_1)^{2^n-1} + (x+y+x_0+x_1)^{2^n-1}\right)\left(x_0^{2^n-2} + x_1^{2^n-2}\right). \end{aligned} \tag{12}$$

If the parenthesized expression vanishes, then an even number of its terms must evaluate to 0, which leads to only trivial solutions. At least one of the terms must therefore evaluate to 1, and so we consider the following possibilities:

- $y = x_1$ and $x = x_0$ immediately implies the trivial solution $x = \zeta$.
- $y = x_1$ and $x \neq x_0$. Equation (12) reduces to $0 = x_0^{2^n-2} + x^{2^n-2} + x_1^{2^n-2} + (x + x_1 + x_0)^{2^n-2}$, which is equivalent to $x^2 + (x_0 + x_1)x + x_0x_1 = 0$, leading to the trivial solutions $x = x_0, x_1$.
- $y = x_0$ and $x \neq x_1$. Equivalent to the previous case.
- $y = x_0$ and $x = x_1$ immediately implies the trivial solution $y = \zeta$.
- $x, y \neq x_0, x_1$ but $y = x + x_0 + x_1$. Equation (12) reduces to

  $$\begin{aligned} 0 &= x_0^{2^n-2} + x^{2^n-2} + (x+x_0+x_1)^{2^n-2} + (x_0+x_1+x_0)^{2^n-2} + x_0^{2^n-2} + x_1^{2^n-2} \\ &= x^{2^n-2} + x_1^{2^n-2}. \end{aligned}$$

  This has no solutions, since $x \neq x_1$.
- $x, y \neq x_0, x_1$, $y \neq x + x_0 + x_1$. The equation above is then

  $$\begin{aligned} 0 &= x_0^{2^n-2} + x^{2^n-2} + y^{2^n-2} + (x+y+x_0)^{2^n-2} + x_0^{2^n-2} + x_1^{2^n-2} \\ &= x^{2^n-2} + y^{2^n-2} + (x+y+x_0)^{2^n-2} + x_1^{2^n-2}. \end{aligned} \tag{13}$$

Suppose $\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right)=0$. Taking $x=0$, equation (13) reduces to

$$0=y^{2^n-2}+(y+x_0)^{2^n-2}+x_1^{2^n-2},$$

which is equivalent to $y^2+yx_0+x_0x_1=0$, which has solutions in y if and only if $\mathrm{Tr}_1^n\left(\frac{x_0x_1}{x_0^2}\right)=\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right)=0$. In this case, notice that $y=x_0$ is not a solution, so that we can conclude that, if $\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right)=0$, the function is not x_0-APN.

If $\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right)\neq 0$, we consider the case $x\neq 0$ (similarly, $y\neq 0$). We can then write $y=\alpha x$, with $\alpha\neq 0,1$. Equation (13) then reduces to

$$0=\alpha(1+\alpha)x^2+(x_0\alpha+x_1(1+\alpha+\alpha^2))x+(1+\alpha)x_0x_1,$$

which is equivalent to

$$0=x^2+\frac{x_0\alpha+x_1(1+\alpha+\alpha^2)}{\alpha(1+\alpha)}x+\frac{x_0x_1}{\alpha}.$$

Label $t=x_0/x_1, z=x/x_1$. Dividing both sides by x_1^2, the above equation becomes

$$z^2+\frac{t\alpha+\alpha^2+\alpha+1}{\alpha^2+\alpha}z+\frac{t}{\alpha}=0,$$

which has a solution if and only if

$$\mathrm{Tr}_1^n\left(\frac{\frac{t}{\alpha}}{\left(\frac{t\alpha+\alpha^2+\alpha+1}{\alpha^2+\alpha}\right)^2}\right)=\mathrm{Tr}_1^n\left(\frac{t(\alpha+\alpha^{-1})}{t^2+\alpha^2+\alpha^{-2}+1}\right)=0.$$

Consider now the case of $\zeta\neq x_0,x_1$ with $\zeta\neq 0$. Assume first that $xy\neq 0$. Then, we can write $x=\beta\zeta$, and $y=\alpha\zeta$, with $\alpha,\beta\neq 0,1$ and $\alpha\neq\beta$. Equation (7) then becomes

$$0=\zeta^{2^n-2}(1+\alpha^{2^n-2}+\beta^{2^n-2}+(1+\alpha+\beta)^{2^n-2})+P(x_0^{2^n-2}+x_1^{2n-2}),$$

where $P=(x+x_0)^{2^n-1}+(x+x_1)^{2^n-1}+(y+x_0)^{2^n-1}+(y+x_1)^{2^n-1}+(x+y+\zeta+x_0)^{2^n-1}+(x+y+\zeta+x_1)^{2^n-1}$. Assume that $P=0$ (which can be achieved, for instance, if all the parenthesized expressions in P are different from zero). The equation becomes

$$0=\zeta^{2^n-2}(1+\alpha^{2^n-2}+\beta^{2^n-2}+(1+\alpha+\beta)^{2^n-2}),$$

which, since $\zeta\neq 0$, is equivalent to

$$0=1+\alpha^{2^n-2}+\beta^{2^n-2}+(1+\alpha+\beta)^{2^n-2},$$

which, multiplying both sides by $\alpha\beta(1+\alpha+\beta)$, becomes

$$\begin{aligned}0&=\alpha\beta(1+\alpha+\beta)+\beta(1+\alpha+\beta)+\alpha(1+\alpha+\beta)+\alpha\beta\\&=\alpha\beta+\alpha^2\beta+\alpha\beta^2+\beta+\alpha\beta+\beta^2+\alpha+\alpha^2+\alpha\beta+\alpha\beta\\&=\alpha+\alpha^2+\beta+\beta^2+\alpha^2\beta+\alpha\beta^2.\end{aligned}$$

Writing $\beta = \gamma\alpha$, with $\gamma \neq 0, 1, \frac{1}{\alpha}$, the equation above becomes

$$0 = \alpha + \alpha^2 + \gamma\alpha + \gamma^2\alpha^2 + \gamma\alpha^3 + \gamma^2\alpha^3 = \alpha(1+\gamma)(\gamma\alpha^2 + (1+\gamma)\alpha + 1).$$

Since $\alpha \neq 0, \gamma \neq 1$, we obtain the equivalent equation

$$\alpha^2 + \frac{1+\gamma}{\gamma}\alpha + \frac{1}{\gamma} = 0,$$

which has solutions if and only if $\mathrm{Tr}_1^n\left(\frac{\frac{1}{\gamma}}{\left(\frac{1+\gamma}{\gamma}\right)^2}\right) = \mathrm{Tr}_1^n\left(\frac{\gamma}{1+\gamma^2}\right) = 0$. Since $\frac{\gamma}{1+\gamma^2} = \frac{1}{1+\gamma} + \left(\frac{1}{1+\gamma}\right)^2$, we always have that $\mathrm{Tr}_1^n\left(\frac{\gamma}{1+\gamma^2}\right) = 0$, so this equation always has solutions. We can then always choose an appropriate solution of the Janwa-Wilson-Rodier equation, so the function is not ζ-APN if $\zeta \neq 0, x_0, x_1$.

For n even, the conditions from equations (10) and (11) are equivalent, since $\mathrm{Tr}_1^n(1) = 0$, and $\frac{x_1}{x_1+\zeta} + \frac{\zeta}{x_1+\zeta} = 1$. Therefore, when n is even and $\mathrm{Tr}_1^n\left(\frac{x_1}{x_1+\zeta}\right) = 0$, the function G_{0x_1} is not ζ-APN.

Now, we shall show the claim of our theorem for $x_0x_1 \neq 0$; there are two possibilities for the Janwa-Wilson-Rodier equation (7) at ζ. The parenthesized expression is either equal to 0 or to 1. If its value is 0 and $\zeta = 0$, but $x, y, x+y \neq 0$, then the equation transforms into

$$x^{-1} + y^{-1} + (x+y)^{-1} = 0,$$

which is equivalent to (with $y = ax$, $a \neq 0, 1$)

$$a^2 + a + 1 = 0,$$

which always has solutions for n even since $\mathrm{Tr}_1^n(1) = 0$. We can choose $x \neq 0$ so that the parenthesized expression is 0, and, therefore, the function is never 0-APN for n even. However, for n odd, the equation $a^2 + a + 1 = 0$ does not have solutions. We thus have to consider the case where $\zeta = 0$ and the expression in the parentheses in (7) is equal to 1. In that case, we must have that $x = x_0$, or $x = x_1$, or $y = x_0$, or $y = x_1$, or $x+y = x_0$, or $x + y = x_1$. We take first the case $x = x_0$. Equation (7) becomes:

$$0 = x_0^{-1} + y^{-1} + (y + x_0)^{-1} + x_0^{-1} + x_1^{-1},$$

which is equivalent to

$$0 = y^2 + yx_0 + x_0x_1,$$

which has solutions if and only if $\mathrm{Tr}_1^n\left(\frac{x_0x_1}{x_0^2}\right) = \mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right) = 0$. In that case, the function is not 0-APN. By symmetry, the case $x = x_1$ gives the condition $\mathrm{Tr}_1^n\left(\frac{x_0}{x_1}\right) = 0$. In that case, the function is not 0-APN.

The other five cases lead to the same conditions. We conclude then that the function is not 0-APN, for n odd, when either $\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right) = 0$ or $\mathrm{Tr}_1^n\left(\frac{x_0}{x_1}\right) = 0$, and is 0-APN otherwise.

Consider now $\zeta \neq 0$. If the parenthesized expression in (7) is 0, $\zeta \neq x_0, x_1$, and x, $y, x+y+\zeta \neq 0$, then equation (7) transforms into

$$\zeta^{-1} + x^{-1} + y^{-1} + (x+y+\zeta)^{-1} = 0,$$

which is equivalent to $0 = x^2y + xy^2 + x^2\zeta + y^2\zeta + x\zeta^2 + y\zeta^2 = (x+y)(x+\zeta)(y+\zeta)$, rendering trivial solutions.

If the parenthesized expression in (7) is 0, $\zeta \neq x_0, x_1$ and $x = 0$, but $y, y+\zeta \neq 0$, then equation (7) transforms into

$$\zeta^{-1} + y^{-1} + (y+\zeta)^{-1} = 0,$$

which is equivalent to $0 = y^2 + \zeta y + \zeta^2$, which has solutions if and only if $\mathrm{Tr}_1^n\left(\frac{\zeta^2}{\zeta^2}\right) = \mathrm{Tr}_1^n(1) = 0$, which is always true for n even. These solutions are always nontrivial, since $y = x = 0$ and $y = \zeta$ are never solutions, under $\zeta \neq 0$. These solutions are, of course, only valid if the parenthesised expression evaluates to 0. For $\zeta = x_1 + x_0$, however, this expression is always zero, and so the function cannot be ζ-APN.

Take now $x_1 \neq x_0 + \zeta$. We know that $y^2 + \zeta y + \zeta^2 = 0$ has exactly two different roots, $y_0 = \zeta\omega$ and $y_1 = \zeta\omega^2$, where ω is a primitive element of $\mathbb{F}_4$. When $y_j = x_k$ for $j, k = 0, 1$ or $y_j = x_k + \zeta$, these solutions are not valid. Suppose that $y_0 = x_0$. The equation $x_0^2 + \zeta x_0 + \zeta^2 = 0$ has solutions in ζ if and only if $\mathrm{Tr}_1^n(1/x_0^3) = 0$. The other forbidden roots induce the condition $\mathrm{Tr}_1^n(1/x_1^3) = 0$, or $\mathrm{Tr}_1^n(1/x_0^3) = 0$.

So, for swaps such that $\mathrm{Tr}_1^n(1/x_0^3) = 0$, there will be two values $\zeta_0 \neq \zeta_1$, namely the solutions $\zeta_0 = x_0\omega$ and $\zeta_1 = x_0\omega^2$ of the equation $x_0^2 + \zeta x_0 + \zeta^2 = 0$, for which the function can still be ζ-APN, and similarly for x_1 under the condition $\mathrm{Tr}_1^n(1/x_1^3) = 0$, producing new (though not necessarily distinct) solutions. Summarizing, we have that:

- If $\mathrm{Tr}_1^n(1/x_0^3) = 0 = \mathrm{Tr}_1^n(1/x_1^3)$, then G can be ζ-APN for at most four values of ζ.
- If $\mathrm{Tr}_1^n(1/x_0^3) = 0$, $\mathrm{Tr}_1^n(1/x_1^3) = 1$, then G can be ζ-APN for at most two values of ζ.
- If $\mathrm{Tr}_1^n(1/x_0^3) = 1$, $\mathrm{Tr}_1^n(1/x_1^3) = 0$, then G can be ζ-APN for at most two values of ζ.
- If $\mathrm{Tr}_1^n(1/x_0^3) = 1 = \mathrm{Tr}_1^n(1/x_1^3)$, then G cannot be ζ-APN for any ζ.

By symmetry, we obtain a similar result in the case of $y = 0$. If the expression in the parentheses in (7) is 0 and $y = x + \zeta$, but $x \neq 0, \zeta$, then (7) transforms into $\zeta^{-1} + x^{-1} + (x+\zeta)^{-1} = 0$, which is equivalent to $x^2 + \zeta x + \zeta^2 = 0$. We have already handled this equation in the case $x = 0$ above, and we do not get any new information from this.

If the parenthesized expression in (7) is 1, we cannot possibly have $x_0 = x_1 + \zeta$. We must then have that $x = x_0$, or $x = x_1$, or $y = x_0$, or $y = x_1$, or $x + y = \zeta + x_0$, or $x + y = \zeta + x_1$. We take first the case $\zeta \neq x_0, x_1, x_0 + x_1$. If $x = x_0$, then the equation becomes $\zeta^{-1} + x_0^{-1} + y^{-1} + (x_0 + y + \zeta)^{-1} + x_0^{-1} + x_1^{-1} = 0$, which is equivalent to

$$y^2(x_1 + \zeta) + y(\zeta + x_0)(\zeta + x_1) + \zeta x_1(\zeta + x_1) = 0,$$

and that, since $x_1 \neq \zeta$, is equivalent to $y^2+y(\zeta+x_0)+\zeta x_0 = 0$, that is, $(y+\zeta)(y+x_0) = 0$.

Note that both solutions implied by this equation are invalid, since $y = \zeta$ is one of the trivial solutions, and $y = x_0$ leads to the expression in the parentheses in (7) to evaluate to 0, and hence implies $x = y$, another trivial solution. The other cases also yield trivial solutions.

We now consider $\zeta \in \{x_0, x_1, x_0 + x_1\}$. Suppose that $\zeta = x_0$, and the parenthesized expression in (5) is 1. Then, we have that $x = x_1$, or $y = x_1$, or $x + y = x_0 + x_1$. On inspection, they either yield trivial solutions, or a contradiction. We have then that the function is ζ-APN. □

Remark 5. *In the proof of the last item of the case n odd, $\zeta x_0 x_1 \neq 0$ above, taking $\beta = \alpha + \alpha^{-1}$, we can easily show that $\mathrm{Tr}_1^n\left(\frac{t\,\beta}{t^2+\beta^2+1}\right) = 0$ has solutions. To see this, we look at the equation $\frac{t\,\beta}{t^2+\beta^2+1} = t^2+t$ $(t \neq 0)$, which has solutions β if $\beta^2+\frac{1}{1+t}\beta+(1+t)^2 = 0$, and this last equation, by Theorem 2, has solutions if and only if $\mathrm{Tr}_1^n\left(\frac{(t+1)^2}{\frac{1}{(t+1)^2}}\right) = \mathrm{Tr}_1^n((t+1)^4) = \mathrm{Tr}_1^n(t+1) = 0$, which holds, by our assumption that $\mathrm{Tr}_1^n\left(\frac{x_1}{x_0}\right) \neq 0$ and n is odd. Unfortunately, it is not always true that there exists α such that $\alpha + \alpha^{-1} = \beta$ (this last equation has a solution α if and only if $\mathrm{Tr}_1^n(\beta^{-2}) = \mathrm{Tr}_1^n(\beta^{-1}) = 0$).*

To supplement the above discussion, we perform an exhaustive search by going over all pairs $(x_0, x_1) \in \mathbb{F}_{2^n}^2$ and compute the size of the pAPN-spectrum of the (x_0, x_1)-swapping of the inverse function x^{2^n-2} over $\mathbb{F}_{2^n}$ for $4 \leq n \leq 10$. The results are presented in Table 1 below. The sizes of the pAPN-spectra of all (x_0, x_1)-swaps are given in the last column, with multiplicities given in superscript, e.g., the entry 0^{45} for $n = 4$ indicates that the pAPN-spectrum of the (x_0, x_1)-swapping is empty for 45 pairs $\{x_0, x_1\}$. The remaining columns give the exponent d, the differential uniformity of x^{2^n-2} (which is known to be 2, respectively 4 for odd, respectively, even n), and the size of the pAPN-spectrum of x^{2^n-2}.

n	d	δ_F	Spectrum	Swapped spectrum
4	7	4	0	$0^{45}, 2^{60}, 8^{15}$
5	15	2	32	0^{31}, 6^{155}, 8^{155}, 9^{155}
6	31	4	0	0^{1197}, 2^{567}, 4^{189}, 30^{63}
7	63	2	128	0^{127}, 26^{889}, 28^{889}, 29^{889}, 30^{889}, 32^{2667}, 35^{889}, 36^{889}
8	127	4	0	0^{19125}, 2^{10200}, 4^{3060}, 128^{255}
9	255	2	512	0^{511}, 116^{4599}, 118^{4599}, 119^{4599}, 120^{6132}, 122^{9198}, 124^{22995}, 125^{4599}, 126^{4599}, 127^{4599}, 128^{9198}, 129^{4599}, 130^{13797}, 131^{4599}, 133^{4599}, 134^{13797}, 135^{4599}, 136^{4599}, 138^{4599}
10	511	4	0	0^{277233}, 2^{230175}, 4^{15345}, 510^{1023}

Table 1: pAPN-spectra of two-point swaps of the inverse function

4 The Gold APN case

A natural question arising from the above investigations is, how does swapping output values affect the other infinite families of APN monomials. In this section, we present our results on the Gold functions.

We will need the following theorem from [12], which shows that a trinomial $z^{p^k}-az-b$ in the finite field $\mathbb{F}_{p^n}$ has either zero, one, or p^g roots, where $g=\gcd(n,k)$. This result was made more explicit by [8].

Theorem 6. *Let p be a prime. Let $f(z)=z^{p^k}-az-b$ in $\mathbb{F}_{p^n}$, $g=\gcd(n,k)$, $m=n/\gcd(n,k)$ and Tr_g be the trace function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^g}$. For $0\le i\le m-1$, we define $t_i=\sum_{j=i}^{m-2}p^{n(j+1)}$, $\alpha_0=a,\beta_0=b$. If $m>1$, then, for $1\le r\le m-1$, we set*

$$\alpha_r=a^{1+p^k+\cdots+p^{kr}} \text{ and } \beta_r=\sum_{i=0}^{r}a^{s_i}b^{p^{ki}},$$

where $s_i=\sum_{j=i}^{r-1}p^{k(j+1)}$, for $0\le i\le r-1$ and $s_r=0$. Then:

- *if $\alpha_{m-1}=1$ and $\beta_{m-1}\neq 0$, then f has no roots in $\mathbb{F}_{p^n}$;*
- *if $\alpha_{m-1}\neq 1$, then f has precisely one root in $\mathbb{F}_{p^n}$, namely $x=\beta_{m-1}/(1-\alpha_{m-1})$;*
- *if $\alpha_{m-1}=1$ and $\beta_{m-1}=0$, then f has precisely p^g roots in $\mathbb{F}_{p^n}$ given by $x+\delta\tau$, where $\delta\in\mathbb{F}_{p^g}$, τ is fixed in $\mathbb{F}_{p^n}$ with $\tau^{p^k-1}=a$, and, for any $e\in\mathbb{F}_{p^n}^*$ with $\mathrm{Tr}_g(e)\neq 0$, where $x=\dfrac{1}{\mathrm{Tr}_g(e)}\displaystyle\sum_{i=0}^{m-1}\left(\sum_{j=0}^{i}e^{p^{kj}}\right)a^{t_i}b^{p^{ki}}$.*

Theorem 7. *Let $F(x)=x^{2^k+1}$ be the Gold function on $\mathbb{F}_{2^n}$, where n is odd and $\gcd(k,n)=1$. Let G_{0x_1} be the $\{0,x_1\}$-swapping of F for some $x_1\in\mathbb{F}_{2^n}^*$. Then:*

- *G_{0,x_1} is not 0-APN;*
- *G_{0,x_1} is not x_1-APN for $0\neq x_1\in\mathbb{F}_{2^n}$ if and only if there exists $0\neq t\in\mathbb{F}_{2^n}$ such that $\displaystyle\sum_{i=0}^{n-1}t^{2^{ki}}=0$;*
- *if $0\neq\zeta\neq x_1$, then G_{0x_1} is ζ-APN if and only if there are no solutions to either of $u^{2^k}+u+(x_1/\zeta)^{2^k+1}=0$, and $y^{2^k}+y(x_1+\zeta)^{2^k-1}+x_1^{2^k}+\frac{x_1\zeta^{2^k}}{x_1+\zeta}=0$; equivalently, G_{0x_1} is ζ-APN if and only if $\displaystyle\sum_{i=0}^{n-1}\left(\frac{x_1}{\zeta}\right)^{2^{ki}}\neq 0$ and $\displaystyle\sum_{i=0}^{n-1}\left((x_1+\zeta)^{-2^k}\left(x_1^{2^k}+\frac{x_1\zeta^{2^k}}{x_1+\zeta}\right)\right)^{2^{ki}}\neq 0$.*

Proof. Let $G_{x_0x_1}$ be the $\{x_0, x_1\}$-swapping of F. The Janwa-Wilson-Rodier condition (4) of $G_{x_0x_1}$ at ζ becomes

$$\begin{aligned} &x^{2^k}y + x^{2^k}\zeta + y^{2^k}x + y^{2^k}\zeta + \zeta^{2^k}x + \zeta^{2^k}y \\ &+ \left((\zeta + x_0)^{2^n-1} + (\zeta + x_1)^{2^n-1} + (x + x_0)^{2^n-1} + (x + x_1)^{2^n-1} + (y + x_0)^{2^n-1}\right. \\ &\left.+(y + x_1)^{2^n-1} + (x + y + \zeta + x_0)^{2^n-1} + (x + y + \zeta + x_1)^{2^n-1}\right)(y_0 + y_1) = 0. \end{aligned} \tag{14}$$

We will use below the fact that under $\gcd(k, n) = 1$, the equation $z^{2^k-1} = a$ has a unique solution in $\mathbb{F}_{2^n}$. Let $x_0 = 0$ (hence $y_0 = 0$). We consider three cases depending on the value of ζ.

In the first case, suppose that $\zeta = 0$. If $0 \neq x \neq y \neq 0$, then equation (14) becomes

$$x^{2^k}y + y^{2^k}x + \left((x + x_1)^{2^n-1} + (y + x_1)^{2^n-1} + (x + y + x_1)^{2^n-1}\right)y_1 = 0.$$

If $x = x_1$ (similarly, for $y = x_1$ and $x + y = x_1$), then we get (certainly, $0 \neq y \neq x_1$, respectively, $0 \neq x \neq x_1$), $x_1^{2^k}y + y^{2^k}x_1 = 0$, rendering $(y/x_1)^{2^k-1} = 1$, and since $\gcd(k, n) = 1$, this last equation has only the trivial solution $y = x_1$.

We now assume $x \neq x_1 \neq y$ and $x + y \neq x_1$. Thus, recalling that $y_1 = x_1^{2^k+1}$, equation (14) becomes $x^{2^k}y + y^{2^k}x + x_1^{2^k+1} = 0$. Taking $u = x/x_1, v = y/x_1$, and dividing by $x_1^{2^k+1}$ above, we get $u^{2^k}v + v^{2^k}u + 1 = 0$. Let us take α with $\alpha^{2^k} + \alpha \neq 0, 1$. Such an α certainly exists; we can, for instance, take α to be a primitive element of $\mathbb{F}_{2^n}$. Writing $v = \alpha u$, the above equation becomes

$$u^{2^k+1} = \left(\alpha^{2^k} + \alpha\right)^{-1}.$$

Since n is odd, $\gcd(2^k + 1, 2^n - 1) = 1$, and so, the equation above has a unique solution $u \neq 1$ in $\mathbb{F}_{2^n}$ for every $\alpha \in \mathbb{F}_{2^n}$ satisfying $\alpha^{2^k} + \alpha \neq 0, 1$. Thus, G_{0x_1} cannot be 0-APN.

In the second case, let $\zeta = x_1 \neq 0$. If $x_1 \neq x \neq y \neq x_1$, then equation (14) becomes

$$\begin{aligned} &x^{2^k}y + x^{2^k}x_1 + y^{2^k}x + y^{2^k}x_1 + x_1^{2^k}x + x_1^{2^k}y \\ &\quad + \left(x^{2^n-1} + y^{2^n-1} + (x + y + x_1)^{2^n-1}\right)y_1 = 0. \end{aligned} \tag{15}$$

If $x = 0$, then $y \neq 0, x_1$ and the above equation becomes $y^{2^k}x_1 + x_1^{2^k}y = 0$, which only has the trivial solutions $y = 0$ and $y = x_1$. The cases when $y = 0$ and $y = x + x_1$ are handled similarly.

We next assume that $xy \neq 0, x + y \neq x_1$. Thus, equation (14) becomes

$$x^{2^k}y + x^{2^k}x_1 + y^{2^k}x + y^{2^k}x_1 + x_1^{2^k}x + x_1^{2^k}y + x_1^{2^k+1} = 0. \tag{16}$$

Dividing by $x_1^{2^k+1}$ and labelling $u = x/x_1, v = y/x_1$, we obtain

$$u^{2^k}v + u^{2^k} + v^{2^k}u + v^{2^k} + u + v + 1 = 0. \tag{17}$$

We now let $w = u + v$ and rewrite (17) as $w^{2^k}(u+1) + w(u+1)^{2^k} + 1 = 0$, that is, $w^{2^k} + w(u+1)^{2^k-1} + (u+1)^{-1} = 0$.

We now apply Theorem 6. Here, $p = 2$, $a = (u+1)^{2^k-1}$, $b = (u+1)^{-1}$, and $m = \frac{n}{\gcd(k,n)} = n$. Then,

$$\alpha_{n-1} = \left((u+1)^{2^k-1}\right)^{1+2^k+\cdots+2^{k(n-1)}} = \left((u+1)^{2^k-1}\right)^{\frac{2^{kn}-1}{2^k-1}} = (u+1)^{2^{kn}-1} = 1.$$

Furthermore,

$$\begin{aligned}\beta_{n-1} &= \sum_{i=0}^{n-1} \left((u+1)^{2^k-1}\right)^{\sum_{j=i}^{n-2} 2^{k(j+1)}} \left((u+1)^{-1}\right)^{2^{ki}} \\ &= \sum_{i=0}^{n-1} (u+1)^{2^{k(i+1)}\left(2^{k(n-i-1)}-1\right)-2^{ki}} \\ &= (u+1)^{2^{kn}} \sum_{i=0}^{n-1} (u+1)^{-2^{ki}(2^k+1)}.\end{aligned}$$

Thus, $\beta_{n-1} = 0$ if and only if there exists u such that $\sum_{i=0}^{n-1} (u+1)^{-2^{ki}(2^k+1)} = 0$. We conclude that G_{0x_1} is not x_1-APN if and only if there exists $t \neq 0$ such that $\sum_{i=0}^{n-1} t^{2^{ki}} = 0$.

In the final case, we assume that $0 \neq \zeta \neq x_1 \neq 0$. Equation (14) becomes

$$\begin{aligned}0 &= x^{2^k}y + y^{2^k}x + x^{2^k}\zeta + \zeta^{2^k}x + y^{2^k}\zeta + \zeta^{2^k}y \\ &+ \left(x^{2^n-1} + (x+x_1)^{2^n-1} + y^{2^n-1}\right. \\ &\left.+(y+x_1)^{2^n-1} + (x+y+\zeta)^{2^n-1} + (x+y+\zeta+x_1)^{2^n-1}\right) y_1.\end{aligned} \tag{18}$$

We will show that equation (18) has no nontrivial solutions x, y. All of the resulting subcases are similar, so we will explicitly describe only some of them.

If the expression in the parentheses in (18) is equal to 0, then we need to investigate the equation

$$x^{2^k}y + y^{2^k}x + x^{2^k}\zeta + \zeta^{2^k}x + y^{2^k}\zeta + \zeta^{2^k}y = 0.$$

Writing $y = \alpha x$, we get

$$x^{2^k+1}\alpha + x^{2^k+1}\alpha^{2^k} + x^{2^k}\zeta + x\zeta^{2^k} + \alpha^{2^k}x^{2^k}\zeta + \alpha x\zeta^{2^k} = 0,$$

which becomes

$$x^{2^k+1}(\alpha + \alpha^{2^k}) + x^{2^k}\zeta(1 + \alpha^{2^k}) + x\zeta^{2^k}(\alpha+1) = 0.$$

Dividing by x^{2^k+1}, and labelling $z = \frac{\zeta}{x}$, we get

$$z^{2^k}(\alpha+1) + z(\alpha+1)^{2^k} + \alpha + \alpha^{2^k} = 0.$$

Dividing by $1+\alpha$ and observing that $\frac{\alpha^{2^k}+\alpha}{\alpha+1} = \frac{\alpha^{2^k}+1+\alpha+1}{\alpha+1} = \frac{(\alpha+1)^{2^k}+\alpha+1}{\alpha+1} = (\alpha+1)^{2^k-1}+1$, we obtain

$$z^{2^k} + z(\alpha+1)^{2^k-1} + (\alpha+1)^{2^k-1} + 1 = 0,$$

which can be factored as

$$(z+1)^{2^k} + (z+1)(\alpha+1)^{2^k-1} = 0,$$

that is,

$$(z+1)((z+1)^{2^k-1} + (\alpha+1)^{2^k-1}) = 0,$$

with roots $z = 1$ and $z = \alpha$. Both of these, however, are trivial, since then $x = \zeta$, respectively, $y = \zeta$.

Assume now that the parenthesized expression in (18) does not evaluate to 0 (which can only happen if an odd number of terms vanish). Equation (18) becomes

$$x^{2^k}y + y^{2^k}x + x^{2^k}\zeta + \zeta^{2^k}x + y^{2^k}\zeta + \zeta^{2^k}y + x_1^{2^k+1} = 0.$$

If $x = 0$, then (18) becomes $y^{2^k}\zeta + \zeta^{2^k}y + x_1^{2^k+1} = 0$. Dividing by ζ^{2^k+1} and labelling $u = y/\zeta$, we get $u^{2^k} + u + (x_1/\zeta)^{2^k+1} = 0$. By the same argument as in the previous case, solutions to this equation exist if and only if $\sum_{i=0}^{n-1}(x_1/\zeta)^{(2^k+1)2^{ki}} = 0$. Thus, if there exist solutions to this equation other than $u = \frac{x_1}{\zeta}$ or $u = 1 + \frac{x_1}{\zeta}$ (which would give $y = x_1$ or $y = \zeta + x_1$, making the parenthesized expression in (18) vanish), then G_{0x_1} is not ζ-APN (note that $y = 0, y = \zeta$ cannot be solutions).

We have to ensure that the potential solutions of $u^{2^k}+u+(x_1/\zeta)^{2^k+1} = 0$ are different from $u = \frac{x_1}{\zeta}$ (which would give $y = x_1$) and $u = 1 + \frac{x_1}{\zeta}$ (which would give $y = \zeta + x_1$), since in both cases the expression inside the parentheses in (18) would vanish. If $y = x_1$ or $y = \zeta + x_1$, since $x = 0$, then (18) becomes $x_1^{2^k}\zeta + \zeta^{2^k}x_1 + x_1^{2^k+1} = 0$. Dividing by $x_1^{2^k+1}$ and relabelling $z = \frac{\zeta}{x_1}$, we obtain the equation $z^{2^k} + z + 1 = 0$, which has no solutions by Theorem 6.

If $x = x_1$, then (18) transforms into

$$x_1^{2^k}y + y^{2^k}x_1 + x_1^{2^k}\zeta + \zeta^{2^k}x_1 + y^{2^k}\zeta + \zeta^{2^k}y + x_1^{2^k+1} = 0,$$

which can be rewritten as $y^{2^k} + y(x_1+\zeta)^{2^k-1} + x_1^{2^k} + \frac{x_1\zeta^{2^k}}{x_1+\zeta} = 0$. If a solution y exists to this previous equation (observe that y cannot be equal to x_1), then G_{0x_1} is not ζ-APN. By a similar argument as the one in the second case, by Theorem 6 we get $\alpha_{n-1} = 1$, and

$$\beta_{n-1} = (x_1+\zeta)^{2^{kn}} \sum_{i=0}^{n-1} \left((x_1+\zeta)^{-2^k} \left(x_1^{2^k} + \frac{x_1\zeta^{2^k}}{x_1+\zeta} \right) \right)^{2^{ki}}.$$

Therefore, G_{0x_1} is not ζ-APN if and only if $\sum_{i=0}^{n-1} \left((x_1+\zeta)^{-2^k} \left(x_1^{2^k} + \frac{x_1\zeta^{2^k}}{x_1+\zeta} \right) \right)^{2^{ki}} = 0$.

The remaining cases give the same equations (up to relabelling). □

Remark 8. *Our computations for $4 \leq n \leq 10$ suggest that swapping any outputs in a Gold APN function produce a function with a non-empty pAPN-spectrum, but we do not yet have a theoretical argument explaining this. See Table 6 in the appendix for detailed computational results.*

5 The Welch APN case

Recall that the Welch APN function is defined over $\mathbb{F}_{2^n}$ as $F(x) = x^{2^k+3}$ for $n = 2k+1$. In this section, we generalize this function by allowing k in x^{2^k+3} to be any positive integer.

To simplify notation, we denote

$$\begin{aligned} E(\zeta, x_1, x, y) &= \zeta^{2^n-1} + (\zeta + x_1)^{2^n-1} + x^{2^n-1} + (x + x_1)^{2^n-1} + y^{2^n-1} \\ &\quad + (y + x_1)^{2^n-1} + (x + y + \zeta)^{2^n-1} + (x + y + \zeta + x_1)^{2^n-1}, \\ C(\zeta, x, y) &= \zeta^{2^k+3} + x^{2^k+3} + y^{2^k+3} + (x + y + \zeta)^{2^k+3} \end{aligned}$$

in $\mathbb{F}_{2^n}$. Certainly, $E(\zeta, x_1, x, y) \in \{0, 1\}$.

Theorem 9. *Let $F(x) = x^{2^k+3}$ be the Welch function on $\mathbb{F}_{2^n}$, where n is odd and let G_{0x_1} be the $\{0, x_1\}$-swapping of F for some $0 \neq x_1 \in \mathbb{F}_{2^n}$. Then*:

- *G_{0x_1} is not 0-APN if $\gcd(2^k + 3, 2^n - 1) = 1$ (which always happens if $n = 2k+1$), nor x_1-APN in general*;
- *if $\zeta \neq 0, x_1$, then G_{0x_1} is not ζ-APN if and only if there is a solution (x, y) of the system $C(\zeta, x, y) = 0$ and $E(\zeta, x_1, x, y) = 0$, or $C(\zeta, x, y) = x_1^{2^k+3}$ and $E(\zeta, x_1, x, y) = 1$, where $x_1, \zeta \neq x \neq y \neq x_1, \zeta$.*

Proof. Let G_{0x_1} be the $\{0, x_1\}$-swapping of F. The Janwa-Wilson-Rodier condition (4) of G_{0x_1} at ζ becomes

$$\begin{aligned} &\zeta^{2^k+3} + x^{2^k+3} + y^{2^k+3} + (x + y + \zeta)^{2^k+3} \\ &+ \left(\zeta^{2^n-1} + (\zeta + x_1)^{2^n-1} + x^{2^n-1} + (x + x_1)^{2^n-1} + y^{2^n-1}\right. \\ &\left.+ (y + x_1)^{2^n-1} + (x + y + \zeta)^{2^n-1} + (x + y + \zeta + x_1)^{2^n-1}\right) x_1^{2^k+3} = 0. \end{aligned} \tag{19}$$

First, assume that $\zeta = 0$. Then (19) becomes

$$\begin{aligned} &x^{2^k} y^3 + x^{2^k+2} y + x^{2^k+1} y^2 + y^{2^k} x^3 + y^{2^k+1} x^2 + y^{2^k+2} x \\ &+ \left(x_1^{2^n-1} + x^{2^n-1} + (x + x_1)^{2^n-1} + y^{2^n-1}\right. \\ &\left.+ (y + x_1)^{2^n-1} + (x + y)^{2^n-1} + (x + y + x_1)^{2^n-1}\right) x_1^{2^k+3} = 0. \end{aligned} \tag{20}$$

If the expression inside the parentheses vanishes (with $y = \alpha x \neq 0, x$), the equation becomes

$$\alpha(\alpha^{2^k-1} + 1)(\alpha^2 + \alpha + 1) = 0,$$

which does not have solutions other than $\alpha = 0, 1$ (which contradict $y \neq 0, x$), since $\gcd(k, n) = 1$ and n is odd. Thus, we need to assume that the parenthesized expression in (20) does not vanish, that is, $E(0, x_1, x, y) = 1$. The equation thus becomes

$$x^{2^k}y^3 + x^{2^k+2}y + x^{2^k+1}y^2 + y^{2^k}x^3 + y^{2^k+1}x^2 + y^{2^k+2}x + x_1^{2^k+3} = 0.$$

Taking $y = \alpha x \neq 0, x$ (so, $\alpha \neq 0, 1$) we obtain,

$$x^{2^k+3}\alpha(\alpha^{2^k-1} + 1)(\alpha^2 + \alpha + 1) = x_1^{2^k+3},$$

and since $\alpha(\alpha^{2^k-1} + 1)(\alpha^2 + \alpha + 1) \neq 0$, if $\gcd(2^k + 3, 2^n - 1) = 1$, then there exists a unique solution

$$x^{2^k+3} = \frac{x_1^{2^k+3}}{\alpha(\alpha^{2^k-1} + 1)(\alpha^2 + \alpha + 1)},$$

and so G_{0x_1} is not 0-APN.

We argue now that when $n = 2k + 1$ we have $\gcd(2^k + 3, 2^n - 1) = 1$. Let us denote $d = \gcd(2^k + 3, 2^n - 1)$. We then have

$$\begin{aligned} 2^k &\equiv -3 \pmod{d} \\ 2^{2k+1} &\equiv 1 \pmod{d}, \end{aligned}$$

and so

$$\begin{aligned} 2^{2k+1} &\equiv 2 \cdot 3^2 \pmod{d} \\ 2^{2k+1} &\equiv 1 \pmod{d}, \end{aligned}$$

which, by subtraction, renders

$$17 \equiv 0 \pmod{d}.$$

Thus, $d = 1$ or $d = 17$. However, by [9, Lemma 9] we know that $\gcd(2^s + 1, 2^n - 1) = \frac{2^{\gcd(n,2s)}-1}{2^{\gcd(n,s)}-1}$, which, if $s = 2$ and n is odd becomes $\gcd(2^n - 1, 2^4 + 1) = 1$. Therefore, $\gcd(2^k + 3, 2^n - 1) = 1$, when $n = 2k + 1$.
Now, suppose that $\zeta = x_1$. Then (19) becomes

$$\begin{aligned} &x_1^{2^k+3} + x^{2^k+3} + y^{2^k+3} + (x + y + x_1)^{2^k+3} + \left(x_1^{2^n-1} + x^{2^n-1} + (x + x_1)^{2^n-1}\right. \\ &\left. + y^{2^n-1} + (y + x_1)^{2^n-1} + (x + y + x_1)^{2^n-1} + (x + y)^{2^n-1}\right) x_1^{2^k+3} = 0. \end{aligned} \tag{21}$$

If the parenthesized expression above does not vanish, that is, $E(\zeta, x_1, x, y) = 1$, the equation becomes

$$x^{2^k+3} + y^{2^k+3} + (x + y + x_1)^{2^k+3} = 0,$$

which, dividing by $x_1^{2^k+3}$, and taking $u = x/x_1, v = y/x_1$, becomes

$$u^{2^k+3} + v^{2^k+3} + (u + v + 1)^{2^k+3} = 0.$$

Noting that $u = 0$ can not be a solution, we take $v = \alpha u$ with $\alpha \neq 0, 1$ and divide both sides by u^{2^k+3}. Since $\gcd(2^k + 3, 2^n - 1) = 1$, then a unique $(2^k + 3)$-root exists and this last equation becomes

$$\beta = \frac{1 + u(1 + \alpha)}{u} = (1 + \alpha^{2^k+3})^{1/(2^k+3)}$$

which (taking α such that $\beta + \alpha + 1 \neq 0$) renders the solution $u = (\beta + \alpha + 1)^{-1}$. Surely, one can find many values of α such that $1 \neq u \neq v \neq 1$, and consequently, $x_1 \neq x \neq y \neq x_1$. Therefore, G_{0x_1} is not x_1-APN, either.
Finally, assume that $0 \neq \zeta \neq x_1$. If the expression in the parentheses in (19) is zero, that is, $E(\zeta, x_1, x, y) = 0$, the equation becomes

$$\zeta^{2^k+3} + x^{2^k+3} + y^{2^k+3} + (x + y + \zeta)^{2^k+3} = 0.$$

If the expression in the parentheses in (19) is not zero, that is, $E(\zeta, x_1, x, y) = 1$, the equation is then

$$x_1^{2^k+3} + \zeta^{2^k+3} + x^{2^k+3} + y^{2^k+3} + (x + y + \zeta)^{2^k+3} = 0,$$

which concludes the proof of the theorem. □

Remark 10. *As with the Gold function, our computational results in Table* 5 *suggest that swapping any two points of the Welch APN function leads to a function with a non-empty spectrum. At the moment, we cannot theoretically justify why this happens.*

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A Experimental data on the infinite APN families

For functions from each of the infinite APN monomial families over $\mathbb{F}_{2^n}$ with $n \leq 10$ (except for the inverse family which is characterized by Theorem 4), we have computed the size of the pAPN-spectrum of $G_{x_0x_1}$ for all possible pairs $(x_0, x_1) \in \mathbb{F}_{2^n}^2$. The results are given in Tables 2, 3, 4, 5, 6 below.

In all cases, the results are computed for generalizations of the respective infinite families, with all restrictions on the parameters dropped. This means that we consider the function

- $x^{2^{4i}+2^{3i}+2^{2i}+2^i-1}$ for Dobbertin,
- $x^{2^{2i}-2^i+1}$ for Kasami,
- $x^{2^i+2^{i/2}-1}$ or $x^{2^i+2^{(3i+1)/2}-1}$ for even and odd values of i, respectively, for Niho,
- x^{2^i+3} for Welch, and
- x^{2^i+1} for Gold

over $\mathbb{F}_{2^n}$, with the parameter i being any positive integer in the range $1 \leq i \leq n-1$.

The first two columns of each table specify the degree n of the extension field $\mathbb{F}_{2^n}$ and the value of the parameter i. The third column gives the smallest element from the cyclotomic coset of the resulting exponent d. The fourth and fifth columns give the differential uniformity and size of the pAPN-spectrum of x^d over $\mathbb{F}_{2^n}$, respectively. Finally, the last column describes how the pAPN-spectrum changes after swapping two output values of the function. More precisely, for every pair $\{x_0, x_1\} \subseteq \mathbb{F}_{2^n}$ with $x_0 \neq x_1$, we compute the size of the pAPN-spectrum of $G_{x_0x_1}$; the last column then lists the sizes of all possible spectra obtained in this way. The frequencies with which these sizes occur over all possible pairs $\{x_0, x_1\}$ are given as superscripts. For example, the first row of Table 2 contains $0^{45}, 2^{60}, 8^{15}$ in the last column. This means that, out of the 120 pairs $\{x_0, x_1\} \subseteq \mathbb{F}_{2^4}$, 45 pairs produce a function with an empty pAPN-spectrum, 60 pairs produce a function which is ζ-APN for two values of ζ, and the remaining 15 pairs lead to functions that are ζ-APN for 8 values of ζ.

By Proposition 3, all exponents d such that x^d has 2^s-to-1 derivatives for some fixed $s > 1$ are omitted. All such functions and all two-point swaps of these functions have an empty pAPN-spectrum by the proposition, and are therefore of very limited

interest. These include all Gold functions with $\gcd(i,n) > 1$ and all Kasami functions with $\gcd(i,n) > 1$ and $n/\gcd(i,n)$ odd. They also include the exponents $i = 3, 4$ for $n = 6$ and $i = 5$ for $n = 10$ in the Dobbertin case; $i = 3$ for $n = 6$ in the Kasami case; $i = 1$ for even n, $i = 4$ for $n = 6$ and $i = 8$ for $n = 10$ in the Welch case; $i = 1, 2$ for n even, $i = 3$ for $n = 5$, $i = 4$ for $n = 6$, $i = 5$ for $n = 8$ and $i = 6$ for $n = 9$ in the Niho case.

We note that in some cases, swap operations lead to a full-sized pAPN-spectrum, indicating that the corresponding function is APN. This occurs exclusively in even dimensions for APN functions, and is caused by pairs $\{x_0, x_1\}$ with $x_0 \neq x_1$ but $F(x_0) = F(x_1)$, where F is the function in question. Consider, for example, $F(x) = x^3$ for $n = 6$ and $i = 2$ in Table 2; there are 63 pairs leading to a pAPN-spectrum of size 64. We know that APN power functions over even-degree extensions of $\mathbb{F}_2$ are 3-to-1; in this case, x^3 has 21 non-zero images y, for each of which there are three pre-images x_1, x_2, x_3 such that $F(x_1) = F(x_2) = F(x_3) = y$. Since a pair of elements from among $\{x_1, x_2, x_3\}$ can be selected in three different ways, each of the 21 images contributes three pairs, leading to these 63 pairs which trivially preserve the APN-ness of the initial function.

The only exceptions to this occur for $n = 4$; for example, for $F(x) = x^3$ in Table 2, there are 30 pairs giving a full pAPN-spectrum, while the trivial pairs as described above account for only 15 of these. To the best of our knowledge, $n = 4$ is the highest extension degree for which APN functions at Hamming distance 2 from each other exist; this is reflected in e.g. [11] and agrees with the results presented in the tables.

Conversely, we can observe that the inverse function is the only APN function among the ones considered whose pAPN-spectrum can become empty after a two-point swap. We ran a separate experiment in which we computed the sizes of the pAPN-spectra of all two-points swaps for representatives from all known CCZ-equivalence classes of APN functions, and observed the same phenomenon: the inverse function is the only one for which an empty pAPN-spectrum could be obtained by swapping two points. Based on this, we formulate the following conjecture.

Conjecture 11. *Let F be any APN power function over $\mathbb{F}_{2^n}$, CCZ-inequivalent to the inverse power function x^{2^n-2}, and let $G_{x_0x_1}$ be the (x_0, x_1)-swapping of F for some $(x_0, x_1) \in \mathbb{F}_{2^n}^2$. Then the pAPN-spectrum of G is not empty.*

We note that the multiset of the sizes of the pAPN-spectra of all functions obtained by swapping two points in a given function is not CCZ-invariant. Counterexamples can be found easily, for instance by considering the Kim function and its CCZ-equivalent permutation [2] over $\mathbb{F}_{2^6}$: the pAPN-spectra of all functions obtained by swapping two outputs of the former are of even size, while pAPN-spectra of odd size can be obtained from the latter. Hence, our conjecture relates only to power APN functions and does not include the ones CCZ-equivalent to them.

Some of the functions listed in the table have a singleton pAPN-spectrum, e.g. $F(x) = x^{47}$ for $i = 3$ and $n = 7$ in Table 2. All such functions are 0-APN.

The function $F(x) = x^{15}$ over $\mathbb{F}_{2^8}$, as given in Table 4, is remarkable due to the fact that all possible pairs $\{x_0, x_1\}$ lead to a function with a singleton pAPN-spectrum.

When $x_0 = 0$, the resulting function is x_1-APN, and when $x_0 \neq 0$, the resulting function is 0-APN.

Table 2: pAPN-spectra of two-point swaps of the Dobbertin function

n	i	d	δ_F	Spectrum	Swapped spectrum
4	1,3	7	4	0	0^{45}, 2^{60}, 8^{15}
	2	3	2	16	16^{30}, 4^{90}
5	1,2,3,4	15	2	32	0^{31}, 6^{155}, 8^{155}, 9^{155}
6	1	23	10	0	0^{2016}
	2	3	2	64	10^{189}, 12^{378}, 16^{189}, 22^{378}, 24^{378}, 26^{378}, 64^{63}, 8^{63}
	5	31	4	0	0^{1197}, 2^{567}, 30^{63}, 4^{189}
7	1,5	29	2	128	25^{889}, 28^{889}, 29^{889}, 30^{1778}, 31^{2667}, 32^{889}, 42^{127}
	2,4	43	2	128	22^{889}, 26^{889}, 28^{127}, 30^{889}, 32^{2667}, 36^{1778}, 38^{889}
	3	47	4	1	0^{4572}, 1^{3556}
	6	63	2	128	0^{127}, 26^{889}, 28^{889}, 29^{889}, 30^{889}, 32^{2667}, 35^{889}, 36^{889}
8	1	29	10	0	0^{32640}
	2,6	21	4	1	0^{14025}, 1^{18615}
	3	43	30	0	0^{32640}
	4	9	2	256	256^{255}, 48^{2040}, 52^{2040}, 54^{2040}, 56^{2040}, 58^{6120}, 60^{3060}, 62^{5100}, 70^{510}, 74^{255}, 80^{2040}, 86^{4080}, 88^{3060}
	5	59	12	0	0^{32640}
	7	127	4	0	0^{19125}, 128^{255}, 2^{10200}, 4^{3060}
9	1	29	8	0	0^{130816}
	2	117	6	1	0^{80227}, 1^{50589}
	3	5	2	512	112^{13797}, 114^{1533}, 118^{4599}, 120^{13797}, 122^{13797}, 124^{9198}, 126^{14308}, 128^{18396}, 130^{9198}, 132^{9198}, 134^{9198}, 136^{4599}, 142^{4599}, 144^{4599}
	4	95	8	0	0^{130816}
	5	83	6	1	0^{80227}, 1^{50589}
	6	17	2	512	106^{4599}, 114^{4599}, 118^{9198}, 120^{22995}, 122^{13797}, 124^{18396}, 126^{11242}, 128^{4599}, 132^{9198}, 136^{18396}, 138^{4599}, 142^{9198}
	7	85	8	0	0^{130816}
	8	255	2	512	0^{511}, 116^{4599}, 118^{4599}, 119^{4599}, 120^{6132}, 122^{9198}, 124^{22995}, 125^{4599}, 126^{4599}, 127^{4599}, 128^{9198}, 129^{4599}, 130^{13797}, 131^{4599}, 133^{4599}, 134^{13797}, 135^{4599}, 136^{4599}, 138^{4599}
10	1	29	4	0	0^{523776}
	2,4,6,8	213	2	1024	1024^{1023}, 224^{10230}, 228^{10230}, 230^{15345}, 232^{25575}, 241^{10230}, 243^{10230}, 244^{25575}, 245^{10230}, 246^{20460}, 247^{10230}, 250^{30690}, 251^{20460}, 252^{20460}, 254^{30690}, 255^{10230}, 258^{10230}, 260^{20460}, 261^{5115}, 262^{1023}, 263^{10230}, 264^{25575}, 265^{10230}, 266^{10230}, 267^{10230}, 268^{10230}, 269^{10230}, 270^{10230}, 271^{20460}, 272^{20460}, 274^{5115}, 275^{20460}, 278^{20460}, 279^{20460}, 283^{10230}, 291^{10230}
	3	151	6	0	0^{523776}
	7	89	6	0	0^{523776}
	9	511	4	0	0^{277233}, 2^{230175}, 4^{15345}, 510^{1023}

Table 3: pAPN-spectra of two-point swaps of the Kasami function

n	i	d	δ_F	Spectrum	Swapped spectrum
4	1,3	3	2	16	16^{30}, 4^{90}
	2	7	4	0	0^{45}, 2^{60}, 8^{15}
5	1,4	3	2	32	10^{31}, 6^{155}, 8^{310}
	2,3	11	2	32	10^{31}, 6^{155}, 8^{310}
6	1,5	3	2	64	10^{189}, 12^{378}, 16^{189}, 22^{378}, 24^{378}, 26^{378}, 64^{63}, 8^{63}
7	1,6	3	2	128	22^{889}, 26^{889}, 28^{127}, 30^{889}, 32^{2667}, 36^{1778}, 38^{889}
	2,5	13	2	128	21^{127}, 27^{889}, 28^{889}, 29^{2667}, 30^{889}, 32^{1778}, 38^{889}
	3,4	23	2	128	25^{889}, 28^{889}, 29^{889}, 30^{1778}, 31^{2667}, 32^{889}, 42^{127}
8	1,7	3	2	256	256^{255}, 48^{2040}, 52^{4080}, 54^{4080}, 56^{2040}, 58^{4080}, 62^{3060}, 66^{2040}, 70^{510}, 74^{255}, 76^{1020}, 80^{2040}, 82^{2040}, 88^{3060}, 90^{2040}
	2,6	13	12	0	0^{32640}
	3,5	39	2	256	256^{255}, 53^{2040}, 55^{2040}, 57^{2040}, 60^{4080}, 61^{4080}, 62^{6630}, 65^{2040}, 81^{2040}, 83^{2040}, 85^{4080}, 88^{1020}, 98^{255}
	4	31	16	0	0^{32640}
9	1,8	3	2	512	112^{9198}, 114^{4599}, 116^{13797}, 118^{4599}, 120^{9198}, 122^{4599}, 124^{9198}, 126^{20440}, 128^{4599}, 130^{13797}, 132^{13797}, 136^{4599}, 138^{4599}, 140^{9198}, 142^{4599}
	2,7	13	2	512	108^{1533}, 118^{4599}, 119^{9198}, 120^{4599}, 121^{9198}, 122^{4599}, 123^{13797}, 124^{4599}, 125^{4599}, 126^{4599}, 127^{4599}, 128^{22995}, 129^{13797}, 130^{4599}, 132^{4599}, 133^{13797}, 135^{4599}, 144^{511}
	4,5	47	2	512	116^{4599}, 117^{9198}, 121^{4599}, 123^{4599}, 124^{4599}, 125^{9198}, 126^{9198}, 127^{9198}, 128^{13797}, 129^{4599}, 131^{27594}, 132^{10731}, 133^{4599}, 135^{9198}, 136^{4599}, 99^{511}
10	1,9	3	2	1024	1024^{1023}, 212^{1023}, 216^{10230}, 218^{20460}, 220^{20460}, 222^{10230}, 224^{10230}, 226^{20460}, 230^{30690}, 232^{20460}, 238^{15345}, 240^{10230}, 242^{5115}, 246^{5115}, 252^{10230}, 256^{10230}, 258^{30690}, 262^{30690}, 264^{10230}, 266^{20460}, 268^{35805}, 270^{20460}, 272^{10230}, 276^{20460}, 278^{20460}, 280^{30690}, 284^{30690}, 286^{10230}, 288^{20460}, 290^{10230}, 292^{10230}, 294^{10230}
	3,7	57	2	1024	1024^{1023}, 219^{20460}, 220^{10230}, 227^{10230}, 228^{10230}, 229^{10230}, 231^{10230}, 232^{36828}, 233^{10230}, 234^{10230}, 235^{10230}, 240^{10230}, 242^{20460}, 244^{10230}, 248^{5115}, 255^{10230}, 259^{10230}, 260^{20460}, 263^{10230}, 266^{40920}, 269^{10230}, 270^{10230}, 271^{10230}, 272^{10230}, 273^{20460}, 274^{10230}, 275^{10230}, 276^{10230}, 277^{20460}, 278^{20460}, 279^{20460}, 280^{10230}, 281^{20460}, 282^{10230}, 283^{10230}, 284^{30690}, 290^{10230}
	5	63	32	0	0^{523776}

Table 4: pAPN-spectra of two-point swaps of the Niho function

n	i	d	δ_F	Spectrum	Swapped spectrum
4	3	3	2	16	16^{30}, 4^{90}
5	1,2	5	2	32	10^{31}, 6^{155}, 8^{310}
	4	7	2	32	10^{31}, 6^{155}, 8^{310}
6	3	15	8	0	0^{2016}
	5	7	6	0	0^{2016}
7	1,2,5	5	2	128	20^{889}, 28^{127}, 30^{1778}, 32^{2667}, 34^{889}, 36^{889}, 38^{889}
	3	29	2	128	25^{889}, 28^{889}, 29^{889}, 30^{1778}, 31^{2667}, 32^{889}, 42^{127}
	4	19	4	1	0^{4572}, 1^{3556}
	6	15	2	128	22^{889}, 26^{889}, 28^{1016}, 32^{889}, 34^{1778}, 36^{2667}
8	3	39	2	256	256^{255}, 53^{2040}, 55^{2040}, 57^{2040}, 60^{4080}, 61^{4080}, 62^{6630}, 65^{2040}, 81^{2040}, 83^{2040}, 85^{4080}, 88^{1020}, 98^{255}
	4	19	16	0	0^{32640}
	6	29	10	0	0^{32640}
	7	15	14	1	1^{32640}
9	1,2	5	2	512	112^{13797}, 114^{1533}, 118^{4599}, 120^{13797}, 122^{13797}, 124^{9198}, 126^{14308}, 128^{18396}, 130^{9198}, 132^{9198}, 134^{9198}, 136^{4599}, 142^{4599}, 144^{4599}
	3	39	8	0	0^{130816}
	4	19	2	512	116^{4599}, 117^{511}, 119^{4599}, 121^{4599}, 122^{4599}, 123^{4599}, 124^{9198}, 125^{27594}, 126^{9198}, 127^{13797}, 128^{9198}, 129^{9198}, 130^{4599}, 131^{4599}, 132^{9198}, 133^{4599}, 135^{6132}
	5	63	6	1	0^{129283}, 1^{1533}
	7	13	2	512	108^{1533}, 118^{4599}, 119^{9198}, 120^{4599}, 121^{9198}, 122^{4599}, 123^{13797}, 124^{4599}, 125^{4599}, 126^{4599}, 127^{4599}, 128^{22995}, 129^{13797}, 130^{4599}, 132^{4599}, 133^{13797}, 135^{4599}, 144^{511}
	8	31	2	512	106^{4599}, 114^{4599}, 118^{9198}, 120^{22995}, 122^{13797}, 124^{18396}, 126^{11242}, 128^{4599}, 132^{9198}, 136^{18396}, 138^{4599}, 142^{9198}
10	3	39	32	0	0^{523776}
	4	19	6	0	0^{523776}
	5	125	34	0	0^{523776}
	6	71	6	0	0^{523776}
	7	9	2	1024	1024^{1023}, 206^{20460}, 208^{10230}, 210^{10230}, 212^{11253}, 220^{20460}, 222^{10230}, 230^{10230}, 232^{5115}, 234^{10230}, 236^{15345}, 238^{10230}, 242^{25575}, 248^{15345}, 254^{5115}, 256^{20460}, 258^{10230}, 260^{20460}, 262^{5115}, 264^{30690}, 266^{20460}, 268^{30690}, 270^{40920}, 272^{30690}, 274^{30690}, 278^{20460}, 280^{10230}, 286^{20460}, 288^{10230}, 292^{10230}, 294^{10230}, 300^{10230}, 308^{10230}
	8	61	6	0	0^{523776}
	9	31	30	0	0^{523776}

Table 5: pAPN-spectra of two-point swaps of the Welch function

n	i	d	δ_F	Spectrum	Swapped spectrum
4	2,3	7	4	0	$0^{45}, 2^{60}, 8^{15}$
5	1	5	2	32	10^{31}, 6^{155}, 8^{310}
	2,4	7	2	32	10^{31}, 6^{155}, 8^{310}
	3	11	2	32	10^{31}, 6^{155}, 8^{310}
6	2,5	7	6	0	0^{2016}
	3	11	10	0	0^{2016}
7	1	5	2	128	20^{889}, 28^{127}, 30^{1778}, 32^{2667}, 34^{889}, 36^{889}, 38^{889}
	2,6	7	6	1	0^{5461}, 1^{2667}
	3	11	2	128	21^{127}, 27^{889}, 28^{889}, 29^{2667}, 30^{889}, 32^{1778}, 38^{889}
	4	19	4	1	0^{4572}, 1^{3556}
	5	13	2	128	21^{127}, 27^{889}, 28^{889}, 29^{2667}, 30^{889}, 32^{1778}, 38^{889}
8	2,7	7	6	0	0^{32640}
	3	11	10	0	0^{32640}
	4	19	16	0	0^{32640}
	5	25	6	0	0^{32640}
	6	13	12	0	0^{32640}
9	1	5	2	512	112^{13797}, 114^{1533}, 118^{4599}, 120^{13797}, 122^{13797}, 124^{9198}, 126^{14308}, 128^{18396}, 130^{9198}, 132^{9198}, 134^{9198}, 136^{4599}, 142^{4599}, 144^{4599}
	2,8	7	6	1	0^{129283}, 1^{1533}
	3	11	8	0	0^{130816}
	4	19	2	512	116^{4599}, 117^{511}, 119^{4599}, 121^{4599}, 122^{4599}, 123^{4599}, 124^{9198}, 125^{27594}, 126^{9198}, 127^{13797}, 128^{9198}, 129^{9198}, 130^{4599}, 131^{4599}, 132^{9198}, 133^{4599}, 135^{6132}
	5	35	6	1	0^{129283}, 1^{1533}
	6	25	8	0	0^{130816}
	7	13	2	512	108^{1533}, 118^{4599}, 119^{9198}, 120^{4599}, 121^{9198}, 122^{4599}, 123^{13797}, 124^{4599}, 125^{4599}, 126^{4599}, 127^{4599}, 128^{22995}, 129^{13797}, 130^{4599}, 132^{4599}, 133^{13797}, 135^{4599}, 144^{511}
10	2,9	7	6	0	0^{523776}
	3	11	10	0	0^{523776}
	4	19	6	0	0^{523776}
	5	35	34	0	0^{523776}
	6	49	8	0	0^{523776}
	7	25	8	0	0^{523776}

Table 6: pAPN-spectra of two-point swaps of the Gold function

n	i	d	δ_F	Spectrum	Swapped spectrum
4	1,3	3	2	16	16^{30}, 4^{90}
5	1,4	3	2	32	10^{31}, 6^{155}, 8^{310}
	2,3	5	2	32	10^{31}, 6^{155}, 8^{310}
6	1,5	3	2	64	10^{189}, 12^{378}, 16^{189}, 22^{378}, 24^{378}, 26^{378}, 64^{63}, 8^{63}
7	1,6	3	2	128	22^{889}, 26^{889}, 28^{127}, 30^{889}, 32^{2667}, 36^{1778}, 38^{889}
	2,5	5	2	128	20^{889}, 28^{127}, 30^{1778}, 32^{2667}, 34^{889}, 36^{889}, 38^{889}
	3,4	9	2	128	22^{889}, 26^{889}, 28^{1016}, 32^{889}, 34^{1778}, 36^{2667}
8	1,7	3	2	256	256^{255}, 48^{2040}, 52^{4080}, 54^{4080}, 56^{2040}, 58^{4080}, 62^{3060}, 66^{2040}, 70^{510}, 74^{255}, 76^{1020}, 80^{2040}, 82^{2040}, 88^{3060}, 90^{2040}
	3,5	9	2	256	256^{255}, 48^{2040}, 52^{2040}, 54^{2040}, 56^{2040}, 58^{6120}, 60^{3060}, 62^{5100}, 70^{510}, 74^{255}, 80^{2040}, 86^{4080}, 88^{3060}
9	1,8	3	2	512	112^{9198}, 114^{4599}, 116^{13797}, 118^{4599}, 120^{9198}, 122^{4599}, 124^{9198}, 126^{20440}, 128^{4599}, 130^{13797}, 132^{13797}, 136^{4599}, 138^{4599}, 140^{9198}, 142^{4599}
	2,7	5	2	512	112^{13797}, 114^{1533}, 118^{4599}, 120^{13797}, 122^{13797}, 124^{9198}, 126^{14308}, 128^{18396}, 130^{9198}, 132^{9198}, 134^{9198}, 136^{4599}, 142^{4599}, 144^{4599}
	4,5	17	2	512	106^{4599}, 114^{4599}, 118^{9198}, 120^{22995}, 122^{13797}, 124^{18396}, 126^{11242}, 128^{4599}, 132^{9198}, 136^{18396}, 138^{4599}, 142^{9198}
10	1,9	3	2	1024	1024^{1023}, 212^{1023}, 216^{10230}, 218^{20460}, 220^{20460}, 222^{10230}, 224^{10230}, 226^{20460}, 230^{30690}, 232^{20460}, 238^{15345}, 240^{10230}, 242^{5115}, 246^{5115}, 252^{10230}, 256^{10230}, 258^{30690}, 262^{30690}, 264^{10230}, 266^{20460}, 268^{35805}, 270^{20460}, 272^{10230}, 276^{20460}, 278^{20460}, 280^{30690}, 284^{30690}, 286^{10230}, 288^{20460}, 290^{10230}, 292^{10230}, 294^{10230}
	3,7	9	2	1024	1024^{1023}, 206^{20460}, 208^{10230}, 210^{10230}, 212^{11253}, 220^{20460}, 222^{10230}, 230^{10230}, 232^{5115}, 234^{10230}, 236^{15345}, 238^{10230}, 242^{25575}, 248^{15345}, 254^{5115}, 256^{20460}, 258^{10230}, 260^{20460}, 262^{5115}, 264^{30690}, 266^{20460}, 268^{30690}, 270^{40920}, 272^{30690}, 274^{30690}, 278^{20460}, 280^{10230}, 286^{20460}, 288^{10230}, 292^{10230}, 294^{10230}, 300^{10230}, 308^{10230}