LWE with Quantum Amplitudes: Algorithm, Hardness, and Oblivious Sampling

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Abstract

The learning with errors problem (LWE) is one of the most important building blocks for post-quantum cryptography. To better understand the quantum hardness of LWE, it is crucial to explore quantum variants of LWE. To this end, Chen, Liu, and Zhandry [Eurocrypt 2022] defined $\mathsf{S}|\mathsf{LWE}\rangle$ and $\mathsf{C}|\mathsf{LWE}\rangle$ problems by encoding the error of LWE samples into quantum amplitudes, and showed efficient quantum algorithms for a few interesting amplitudes. However, algorithms or hardness results of the most interesting amplitude, Gaussian, were not addressed before.

In this paper, we show new algorithms, hardness results and applications for $\mathsf{S}|\mathsf{LWE}\rangle$ and $\mathsf{C}|\mathsf{LWE}\rangle$ with real Gaussian, Gaussian with linear or quadratic phase terms, and other related amplitudes. Let n be the dimension of LWE samples. Our main results are

1. There is a $2^{\widetilde{O}(\sqrt{n})}$-time algorithm for $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude with *known* phase, given $2^{\widetilde{O}(\sqrt{n})}$ many quantum samples. The algorithm is modified from Kuperberg's sieve, and in fact works for more general amplitudes as long as the amplitudes and phases are completely *known*.
2. There is a polynomial time quantum algorithm for solving $\mathsf{S}|\mathsf{LWE}\rangle$ and $\mathsf{C}|\mathsf{LWE}\rangle$ for Gaussian with quadratic phase amplitudes, where the sample complexity is as small as $\widetilde{O}(n)$. As an application, we give a quantum oblivious LWE sampler where the core quantum sampler requires only quasi-linear sample complexity. This improves upon the previous oblivious LWE sampler given by Debris-Alazard, Fallahpour, Stehlé [STOC 2024], whose core quantum sampler requires $\widetilde{O}(nr)$ sample complexity, where r is the standard deviation of the error.
3. There exist polynomial time quantum reductions from standard LWE or worst-case GapSVP to $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude with small *unknown* phase, and arbitrarily many samples. Compared to the first two items, the appearance of the unknown phase term places a barrier in designing efficient quantum algorithm for solving standard LWE via $\mathsf{S}|\mathsf{LWE}\rangle$.

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1 Introduction

The learning with errors problem asks to learn a secret vector given many noisy linear samples.

Definition 1 (Learning with errors (LWE) [Reg09])**.** *Let n, m, q be positive integers. Let $\mathbf{s} \in \mathbb{Z}_q^n$ be a secret vector. The search LWE problem $\mathsf{LWE}_{n,m,q,\alpha}$ asks to find the secret $\mathbf{s}$ given access to an oracle that outputs $\mathbf{a}_i$, $\langle \mathbf{s}, \mathbf{a}_i \rangle + e_i \pmod q$ on its i^{th} query, for $i = 1, \cdots, m$. Here each $\mathbf{a}_i$ is a uniformly random vector in $\mathbb{Z}_q^n$, and each error term e_i is sampled from the Gaussian distribution over $\mathbb{Z}$ with standard deviation $\alpha q/\sqrt{2\pi}$.*

The LWE problem is extremely versatile, leading to advanced encryption schemes such as fully homomorphic encryptions [Gen09, BV11, Mah18]. It is also shown by Regev to be quantumly as hard as the approximate short vector problems for all lattices [Reg09]. LWE and lattice problems in general (e.g. [HPS98, Reg09]) are also popular candidates for the NIST post-quantum cryptography standardization, due to their conjectured hardness against quantum computers. In fact, the fastest quantum and classical algorithms for LWE all run in $2^{\Omega(n)}$ time. However, the conjectured quantum hardness of lattice problem is still lacking solid evidences. Finding quantum algorithms for lattice problems has therefore been a major open problem in the area of quantum computation and cryptography in the past decade.

One way of exploring the quantum power for solving LWE is to consider the quantum variants of LWE, by encoding quantum states into the LWE problem (henceforth, we refer to the original LWE problem as "classical LWE" or "standard LWE" to distinct them from the quantum variant mentioned below). To this end, Chen, Liu, and Zhandry [CLZ22] define the following quantum variants of LWE:

Definition 2 (Solve $|\mathsf{LWE}\rangle$, $\mathsf{S}|\mathsf{LWE}\rangle$)**.** *Let n, m, q be positive integers. Let f be a function from $\mathbb{Z}_q$ to $\mathbb{C}$. Let $\mathbf{s} \in \mathbb{Z}_q^n$ be a secret vector. The problem $\mathsf{S}|\mathsf{LWE}\rangle_{n,m,q,f}$ asks to find $\mathbf{s}$ given access to an oracle that outputs independent samples $\mathbf{a}_i$, $\sum_{e_i \in \mathbb{Z}_q} f(e_i) | \langle \mathbf{s}, \mathbf{a}_i \rangle + e_i \bmod q\rangle$ on its i^{th} query, for $i = 1, \cdots, m$. Here each $\mathbf{a}_i$ is a uniformly random vector in $\mathbb{Z}_q^n$.*

Definition 3 (Construct $|\mathsf{LWE}\rangle$ states, $\mathsf{C}|\mathsf{LWE}\rangle$)**.** *Let n, m, q be positive integers. Let f be a function from $\mathbb{Z}_q$ to $\mathbb{C}$. The problem of constructing LWE states $\mathsf{C}|\mathsf{LWE}\rangle_{n,m,q,f}$ asks to construct a quantum state of the form*

$$\sum_{\mathbf{s} \in \mathbb{Z}_q^n} \bigotimes_{i=1}^{m} \left(\sum_{e_i \in \mathbb{Z}_q} f(e_i) | \langle \mathbf{s}, \mathbf{a}_i \rangle + e_i \bmod q\rangle \right),$$

given the input $\{\mathbf{a}_i\}_{i=1,\ldots,m}$ where each $\mathbf{a}_i$ is a uniformly random vector in $\mathbb{Z}_q^n$.

Note that if there is a quantum algorithm that solves $\mathsf{S}|\mathsf{LWE}\rangle_{n,m,q,f}$ without collapsing the input state, then there is a quantum algorithm that solves $\mathsf{C}|\mathsf{LWE}\rangle_{n,m,q,f}$ for the same n, m, q, f. So in the introduction we will say more about $\mathsf{S}|\mathsf{LWE}\rangle$ since it is more fundamental than $\mathsf{C}|\mathsf{LWE}\rangle$. We will mention $\mathsf{C}|\mathsf{LWE}\rangle$ when $\mathsf{C}|\mathsf{LWE}\rangle$ is necessary for the application.

Chen, Liu, and Zhandry [CLZ22] then show a quantum filtering technique to solve $\mathsf{S}|\mathsf{LWE}\rangle$ when the DFT of the error amplitude is non-negligible everywhere over $\mathbb{Z}_q$. Two interesting error amplitudes covered by their result are the bounded uniform amplitude and Laplacian amplitude.

In particular, the classical LWE problem with bounded uniform error distribution is proven to be as hard as worst-case lattice problems [DM13, MP13]. Thus, their work gives a strong indication that $\mathsf{S}|\mathsf{LWE}\rangle$ is easier than classical LWE for certain error distributions.

However, the most interesting amplitude, Gaussian, was not addressed in [CLZ22]. For classical LWE, Gaussian distribution is the default error distribution since Regev [Reg09] shows a quantum reduction from worst-case lattice problems to LWE with Gaussian error, given *arbitrarily* many LWE samples. Interestingly, it was implicitly shown in [SSTX09] and [BKSW18] that $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude is as hard as standard LWE when the number of samples is *very small.* But if we are given *arbitrarily* many samples, is the $\mathsf{S}|\mathsf{LWE}\rangle$ problem with Gaussian amplitude still hard?

Not only did understanding $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude shed light on our knowledge of the standard LWE, but it was also considered a bedrock on which interesting quantum protocols can be based, especially unclonable cryptography. The idea was first shown by Zhandry [Zha19] for a potential approach to construct a very powerful cryptographic primitive called quantum lightning, while its concrete and secure instantiation still remains unknown. After that, Khesin, Lu, and Shor proposed another lightning construction [KLS22] based on $\mathsf{S}|\mathsf{LWE}\rangle$ but later was broken by Liu, Montgomery, and Zhandry [LMZ23]. Poremba [Por23], and Ananth, Poremba, and Vaikuntanathan [APV23] build certifiable deletion based on $\mathsf{S}|\mathsf{LWE}\rangle$, basing on certain conjectured security. If $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude was not secure, then all schemes mentioned before may not even have semantic security, let alone its unclonability.

In addition, recently, Debris-Alazard, Fallahpour, and Stehlé [DFS24] showed that solving the $\mathsf{C}|\mathsf{LWE}\rangle$ problem for proper amplitudes implies a quantum *oblivious LWE sampler.* Roughly speaking, an oblivious LWE sampler is able to take as input the public matrix $\mathbf{A}$, output an LWE sample $\mathbf{A}^T\mathbf{s}+\mathbf{e} \bmod q$ without knowing the secret $\mathbf{s}$. If an oblivious LWE sampler exists, it refutes the typical knowledge assumption for LWE used in several lattice-based SNARK constructions (e.g. [GMNO18], see [DFS24] for more discussions) which assume that anyone who generates an LWE sample must know the LWE secret. While we don't know of any classical oblivious LWE sampler, Debris-Alazard et al. showed that solving $\mathsf{C}|\mathsf{LWE}\rangle$ is perfect for the task of oblivious LWE sampling, since if we are able to generate a state like $\sum_{\mathbf{s}\in\mathbb{Z}_q^n}\bigotimes_{i=1}^m\left(\sum_{e_i\in\mathbb{Z}_q} f(e_i)|\langle\mathbf{s},\mathbf{a}_i\rangle+e_i \bmod q\rangle\right)$, and then measure it, we will get an LWE sample $\mathbf{A}^T\mathbf{s}+\mathbf{e} \bmod q$ with a random and unknown secret $\mathbf{s}$. Debris-Alazard et al. then design an amplitude f whose norm is Gaussian, and whose phase is cleverly chosen, so that the algorithm of quantum unambiguous measurement [CB98] is able to solve $\mathsf{C}|\mathsf{LWE}\rangle$ for such an amplitude with relatively few samples.

In sum, the motivations of this work are addressing the following questions:

What amplitudes are useful for $\mathsf{S}|\mathsf{LWE}\rangle$ *or* $\mathsf{C}|\mathsf{LWE}\rangle$ *towards solving standard LWE or improving quantum applications like oblivious LWE sampling?*

Can we show algorithms or hardness for solving $\mathsf{S}|\mathsf{LWE}\rangle$ *or* $\mathsf{C}|\mathsf{LWE}\rangle$ *for Gaussian and other related amplitudes?*

1.1 Main results

In this paper, we show new quantum algorithms, hardness results, and applications for $\mathsf{S}|\mathsf{LWE}\rangle$ and $\mathsf{C}|\mathsf{LWE}\rangle$ with Gaussian and other related amplitudes. A summary of the results is given in Table 1.

Error Amplitude	# Samples	Algorithm or Hardness	Reference
Gaussian	$\tilde{O}(n)$	As hard as LWE or approx-GapSVP	[SSTX09, BKSW18]
Gaussian	$2^{\tilde{O}(\sqrt{n})}$	$2^{\tilde{O}(\sqrt{n})}$-time quantum algorithm	This work §3
Complex Gaussian	$\tilde{O}(n)$	$\mathsf{poly}(n)$-time quantum algorithm	This work §4
Gaussian with unknown phase	Arbitrary	As hard as LWE or approx-GapSVP	This work §5,6
Known with non-negl DFT	$\mathsf{poly}(n)$	$\mathsf{poly}(n)$-time quantum algorithm	[CLZ22, DFS24]

Table 1: Algorithm and hardness of $\mathsf{S}|\mathsf{LWE}\rangle$ for various error amplitudes

Before introducing our new algorithms for solving $\mathsf{S}|\mathsf{LWE}\rangle$, let us take a step back and review the existing methods of solving $\mathsf{S}|\mathsf{LWE}\rangle$ for a general amplitude f. Fix some amplitude $f : \mathbb{Z}_q \to \mathbb{C}$ and suppose for now that f is centered at 0. Then each sample of $\mathsf{S}|\mathsf{LWE}\rangle_{n,m,q,f}$ is given by

$$\mathbf{a} \in \mathbb{Z}_q^n, \quad |\phi_{f,\langle\mathbf{a},\mathbf{s}\rangle}\rangle := \sum_{x\in\mathbb{Z}_q} f(x)|x + \langle\mathbf{a},\mathbf{s}\rangle \bmod q\rangle. \tag{1}$$

The state $|\phi_{f,\langle\mathbf{a},\mathbf{s}\rangle}\rangle$ can be seen as a shift of $|\phi_{f,0}\rangle$ where the center of the state is shifted from 0 to $\langle\mathbf{a},\mathbf{s}\rangle$. A general approach of solving $\mathsf{S}|\mathsf{LWE}\rangle$ is then: try to design a quantum algorithm that predicts the center $\langle\mathbf{a},\mathbf{s}\rangle$ of each state $|\phi_{f,\langle\mathbf{a},\mathbf{s}\rangle}\rangle$ correctly, and if we predict the centers of n states correctly, use Gaussian elimination to find out $\mathbf{s}$.

However, a difficulty faced by the general approach is: for a "typical" error amplitude, like Gaussian of some width $r \in (\sqrt{n}, q/\sqrt{n})$, i.e., for $\rho_r(x) := \exp\left(-\pi\frac{x^2}{r^2}\right)$, the overlap between $|\phi_{\rho_r,0}\rangle$ and $|\phi_{\rho_r,1}\rangle$ is quite large, which means it is inherently hard to distinguish states with adjacent shifts with high confidence. The quantum algorithms used in [CLZ22, DFS24] have to use sophisticated measurements depending on f to guarantee that once the center c of $|\phi_{f,c}\rangle$ is predicted correctly for some samples, we know it is correct. But the probability of predicting correctly in the algorithms used in [CLZ22, DFS24] is proportional to the square of the minimum of $|\mathsf{DFT}_q(f)|$ and is in general very low (the success probability of [DFS24] is better than [CLZ22] in its dependency on q). Although the success probability is non-negligible for certain f, it is exponentially small for the typical Gaussian amplitude. For example, in [DFS24], the required sample complexity is $\frac{n\cdot\omega(logn)}{q\cdot\min_{x\in\mathbb{Z}_q}|\mathsf{DFT}_q(f)(x)|^2}$. By taking $\min_{x\in\mathbb{Z}_q}|\mathsf{DFT}_q(\rho_r)(x)|^2 \approx \frac{1}{\frac{q}{\sqrt{2}r}}\cdot\rho_{\frac{q}{\sqrt{2}r}}\left(\frac{q}{2}\right) \approx \frac{\sqrt{2}r}{q}\cdot e^{-\pi\frac{r^2}{2}}$, the sample complexity for solving $\mathsf{S}|\mathsf{LWE}\rangle$ for Gaussian amplitude ρ_r is greater than $O(n/r)\cdot e^{\pi\frac{r^2}{2}}$, which is exponential in n when $r > \sqrt{n}$ (when r is in $O(n^{0.5-c})$ for any $c > 0$, the classical Arora-Ge algorithm [AG11] has solved classical LWE with error distribution ρ_r in subexponential time and sample complexity already). Therefore [CLZ22, DFS24] are not able to provide subexponential time algorithms for solving $\mathsf{S}|\mathsf{LWE}\rangle$ for the Gaussian amplitude ρ_r when $r > \sqrt{n}$.

Subexponential time algorithm for general amplitudes. Our first result is a sub-exponential time quantum algorithm for solving $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude, given sub-exponentially many $\mathsf{S}|\mathsf{LWE}\rangle$ samples. In fact, it works for a more general amplitude f as long as two points in the discrete Fourier transform of f are more than subexponentially small (the DFT of Gaussian certainly has two non-negligible points). The algorithm combines Kuperberg's sieve [Kup05] and quantum rejection sampling [ORR13].

Theorem 4 (Theorem 32, informal)**.** *Let $f : \mathbb{Z} \to \mathbb{C}$ be a known, normalized error amplitude function for $\mathsf{S}|\mathsf{LWE}\rangle$ such that for the DFT_q of f, denoted by g, there exists two distinct values $j_1, j_2 \in \mathbb{Z}_q$ such that $\gcd(j_1 - j_2, q) = 1$ and $|g(j_1)|, |g(j_2)| \geq 2^{-\sqrt{n}\log q}$, and $g(j_1), g(j_2)$ are computable in time $2^{\Theta(\sqrt{n}\log q)}$. Then there exists a quantum algorithm that, given $m = 2^{\Theta(\sqrt{n}\log q)}$ samples of $\mathsf{S}|\mathsf{LWE}\rangle$ with amplitude f, finds the secret within a time complexity of $2^{\Theta(\sqrt{n}\log q)}$.*

Polynomial time algorithm for complex Gaussian amplitudes. Our second result shows when the amplitude f is Gaussian with quadratic imaginary phase (henceforth complex Gaussian), i.e., $f_{r,t}(x) = \exp\left(-\pi\left(\frac{1}{r^2} + \frac{i}{t}\right)x^2\right)$ for some $r \in \mathbb{R}$ and $t \in \mathbb{N}^+$, there are $\mathsf{poly}(n, \log q)$ time quantum algorithms for solving $\mathsf{S}|\mathsf{LWE}\rangle$ and $\mathsf{C}|\mathsf{LWE}\rangle$.

Our algorithm makes use of an interesting property of the complex Gaussian function $f_{r,t}(x) = \exp\left(-\pi\left(\frac{1}{r^2} + \frac{i}{t}\right)x^2\right)$: when $r > t \cdot \Omega(\sqrt{n})$, the overlap between $|\phi_{f_{r,t},c_1}\rangle$ and $|\phi_{f_{r,t},c_2}\rangle$ (following the notation in Eqn. (1)) is negligible whenever $c_1 \not\equiv c_2 \pmod{t}$. In other words, given a state $|\phi_{f_{r,t},c}\rangle$, it should in principle be easy to extract the value of $c \bmod t$. Indeed, we show when $r > \tilde{O}(n) \cdot t$, if we simply measure the least significant bits of $|\phi_{f_{r,t},c}\rangle$ by the purely imaginary Gaussian basis $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$, where

$$|\psi_d\rangle = \sum_{x\in\mathbb{Z}_t} e^{-\frac{\pi i(x-d)^2}{t}}|x\rangle,$$

and output the measurement result $d \in \mathbb{Z}_t$, then $d = c \bmod t$ with probability $1 - O(1/n)$, which finds the center $c \bmod t$ of $|\phi_{f_{r,t},c}\rangle$ with high probability. Utilizing this center finding procedure, we can solve a variant of $\mathsf{S}|\mathsf{LWE}\rangle$ where $q = q_1q_2$ with coprime factors q_1, q_2, such that the first half of samples can be viewed as from $\mathsf{S}|\mathsf{LWE}\rangle_{n,\tilde{O}(n),q_1,f_{r,q_1}}$, while the second half can be viewed as samples from $\mathsf{S}|\mathsf{LWE}\rangle_{n,\tilde{O}(n),q_2,f_{r,q_2}}$ with the same secret $\mathbf{s} \bmod q$. Then we employ Guess-then-Gaussian-Elimination on the two halves to recover $\mathbf{s} \bmod q_1$ and $\mathbf{s} \bmod q_2$, and finally use the Chinese Remainder Theorem to recover the whole secret in $\mathbb{Z}_q$.

The idea can be generalized to the case where q has more than two factors, and it directly works for solving $\mathsf{C}|\mathsf{LWE}\rangle$. Our main theorem is

Theorem 5. *Let n, m, q, ℓ be positive integers and r be a real number. Suppose that $m = 2\ell n \cdot \omega(\log n)$, q is a composite number satisfying $q = q_1q_2\cdots q_\ell$ where $q_1, q_2, \cdots, q_\ell$ are coprime, r satisfies $\frac{q}{\sqrt{n}} > r > 30n\log n \cdot \max\{q_1, q_2, \cdots, q_\ell\}$.*

There exists a quantum algorithm running in time $\mathsf{poly}(n, \ell, \log q)$ that, takes input $\mathbf{A} \leftarrow \mathcal{U}(\mathbb{Z}_q^{n\times m})$, outputs a state ρ such that the trace distance between ρ and $\phi := |\phi\rangle\langle\phi|$ is negligible, where

$$|\phi\rangle = \sum_{\mathbf{s}\in\mathbb{Z}_q^n}\sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_r(\mathbf{x})e^{-\pi i\sum_{j=1}^{\ell}\|\mathbf{x}_j\|^2/q_j}|(\mathbf{A}^T\mathbf{s} + \mathbf{x}) \bmod q\rangle$$

is a quantum LWE state with Gaussian amplitude and quadratic phase terms, here we write $\mathbf{x}$ into ℓ blocks $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_\ell)$ with $\mathbf{x}_j \in \mathbb{Z}^{m/\ell}, j = 1, 2, \cdots, \ell$.

Application to oblivious LWE sampling. As mentioned in [DFS24], to build a quantum oblivious LWE sampler via solving $\mathsf{C}|\mathsf{LWE}\rangle$, the key is to choose the amplitude f where the real part is Gaussian, and the imaginary part is suitable so that an efficient quantum algorithm can find the center c of $|\phi_{f,c}\rangle$ with non-negligible probability. In [DFS24], f is chosen to be *Gaussian with*

Error Amplitude	# Samples	Algorithm	Reference
Bounded uniform	$O(nrq^4)$	Quantum filtering	[CLZ22]
Gaussian with half phase	$\tilde{O}(nr)$	Unambiguous measurement	[DFS24]
Complex Gaussian	$\tilde{O}(n)$	Imaginary Measurement and CRT	This work §4

Table 2: Algorithm and sample complexity for $\mathsf{C}|\mathsf{LWE}\rangle$ and Oblivious LWE sampling. Here n is the dimension of the LWE secret, r is the standard deviation of the error distribution, q is the modulus. All quantum algorithms in this table run in $\mathsf{poly}(n)$ time.

half phase: $f(x) = \exp\left(-\pi \frac{x^2}{r^2}\right) \cdot \mathrm{sgn}^+(x)$, where $\mathrm{sgn}^+(x)$ outputs 1 when $x \geq 0$, -1 when $x < 0$. They are able to solve $\mathsf{C}|\mathsf{LWE}\rangle_{n,m,q,f}$ for such f with sample complexity $m \in \tilde{O}(nr)$ and with prime modulus q. Once they get an oblivious LWE sample with parameter $(n, m, q, r/q)$ (here $r/\sqrt{2\pi}$ is the standard deviation of the LWE error term), they can throw away additional samples, and apply the classical modulus switching transformation [BLP+13] to get oblivious LWE samples with parameter $(n, m', q', r'/q')$ for certain $m' \leq m$, $q' < q$, $r' > r$ (we omit more detailed restrictions in the introduction).

Using the complex Gaussian amplitude, we can directly use Theorem 5 with $\ell \in O(1)$ to reduce the sample complexity of $\mathsf{C}|\mathsf{LWE}\rangle$ from $\tilde{O}(nr)$ to $\tilde{O}(n)$. Here is the main theorem for the core quantum component of our oblivious LWE sampler:

Theorem 6 (Informal version of Corollary 42)**.** *Let n, m, q, ℓ, r satisfy the same conditions as in Theorem 5. Assume the quantum hardness of $\mathsf{LWE}_{n,m,q,r/(\sqrt{2}q)}$, there exists a witness-oblivious quantum sampler for $\mathsf{LWE}_{n,m,q,r/(\sqrt{2}q)}$.*

Once we get an oblivious LWE sampler for some composite modulus q, we can also use modulus switching to get an oblivious LWE sampler for certain prime modulus $q' < q$. The detailed parameters are given in Corollary 44 in the main body.

In Table 2 we summarize the known poly-time algorithms for solving $\mathsf{C}|\mathsf{LWE}\rangle$ and oblivious LWE sampling. In Figure 1 we provide some examples of interesting amplitudes addressed in our paper or previous papers.

Hardness results. Given our new quantum algorithms for solving $\mathsf{S}|\mathsf{LWE}\rangle$ for Gaussian and complex Gaussian amplitudes, readers may wonder whether they lead to algorithmic advantages for standard LWE. In fact, the complex Gaussian function was introduced by Chen [Che24] in a failed attempt of solving standard LWE. We haven't been able to fix the problem in [Che24]. Instead, we extract the intuition from [Che24] and show that complex Gaussian is at least suitable for solving $\mathsf{S}|\mathsf{LWE}\rangle$, $\mathsf{C}|\mathsf{LWE}\rangle$, and reduce the quantum sample complexity for oblivious LWE sampling. We are not able to show that solving $\mathsf{S}|\mathsf{LWE}\rangle$, $\mathsf{C}|\mathsf{LWE}\rangle$ with complex Gaussian amplitude implies solving standard LWE.

How about the real Gaussian amplitude? Whether Theorem 4 leads to sub-exponential time quantum algorithms for standard LWE? To address this question, let us recall the quantum reduction from GapSVP and SIVP to classical LWE with Gaussian error distribution due to Regev [Reg09]. This reduction works even given *arbitrarily many* classical LWE samples. At first glance, one would conjecture that the reduction can be modified to a quantum reduction from worst-case lattice problems to $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude.

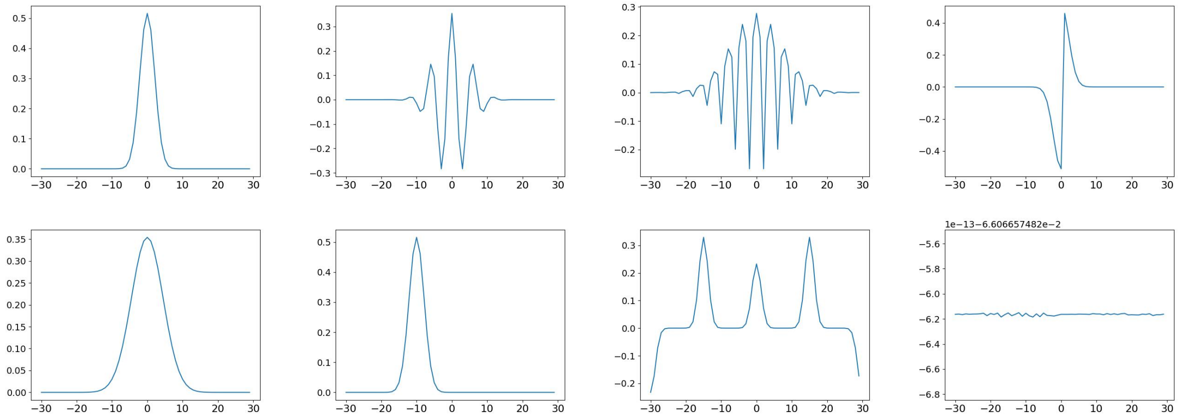


Figure 1: Interesting $\mathsf{S}|\mathsf{LWE}\rangle$ error amplitudes (top) and their DFTs (bottom). All pictures are depicting the real parts of the functions. The x-axis is the input (from -30 to 29, all examples are given over $\mathbb{Z}_{60}$). The y-axis is the amplitude. Four pictures on the top from left to right are: (1) Gaussian, where our sub-exponential algorithm applies; (2) Gaussian with imaginary linear phase, where our reductions apply when the phase (or the center of the DFT) is unknown; (3) Gaussian with imaginary quadratic phase, where our oblivious LWE sampler uses; (4) Gaussian where the phase changes in the middle, where the oblivious LWE sampler in [DFS24] uses.

However, we are only able to modify Regev's reduction into a quantum reduction from worst-case lattice problem to $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude with some small but *unknown* phase, with arbitrarily many quantum samples. Before stating our main theorem, let us first introduce the definition of $\mathsf{S}|\mathsf{LWE}\rangle$ with phase.

Definition 7 ($\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$)**.** *Let n, m, q be LWE parameters. Define the following components: (1) an amplitude function $f : \mathsf{supp}(f) \to \mathbb{R}_{\geq 0}$ where $\mathsf{supp}(f) = \{x/Q : -q/2 < x/Q \leq q/2, x \in \mathbb{Z}\}$ for some integer Q; (2) a mapping $\theta : \mathsf{supp}(\theta) \to \mathbb{R}$ where $\mathsf{supp}(\theta)$ is a subset of high-dimensional integer vectors; (3) a distribution D_θ over the set $\mathsf{supp}(\theta)$.*

We say a quantum algorithm is capable of solving $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}_{n,m,q,f,\theta,D_\theta}$, if for any hidden vector $\mathbf{s} \in \mathbb{Z}_q^n$, when provided with m samples of

$$\mathbf{a} \leftarrow \mathcal{U}(\mathbb{Z}_q^n), \quad \mathbf{y} \leftarrow D_\theta, \quad \sum_{e \in \mathsf{supp}(f)} f(e) \exp(2\pi \mathrm{i} \cdot e\theta(\mathbf{y})) |(\langle \mathbf{a}, \mathbf{s} \rangle + e) \bmod q\rangle,$$

the algorithm outputs $\mathbf{s}$ with probability at least $1 - 2^{-\Omega(n)}$.

On the first pass of the definition, readers can think of the integer vector $\mathbf{y}$ as some auxiliary information. The phase term $\theta(\mathbf{y})$ is a function of $\mathbf{y}$. As it is defined, the function θ may or may not be efficiently computable (it is not efficiently computable in our result). So we can think of $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ as a variant of $\mathsf{S}|\mathsf{LWE}\rangle$ with a phase term in the amplitude.

Theorem 8 (Theorem 56, informal)**.** *Let $q = q(n) > 10n$ be an integer of at most $\mathsf{poly}(n)$ bits, $\alpha \in (0, \frac{1}{5\sqrt{n}})$ such that $\alpha q > 2\sqrt{n}$. Suppose there exists a quantum algorithm that solves $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ where the amplitude $f(e) := \exp\left(-\pi \frac{e^2}{(\alpha q)^2}\right)$ and the phase θ is not efficiently computable, with*

$m = 2^{o(n)}$ samples and in time complexity T. Then there exists a quantum algorithm that solves GapSVP_γ and SIVP_γ, where $\gamma \in \tilde{O}(n/\alpha)$, in time $\mathsf{poly}(n, m, T)$.

The informal statement above omits the distribution of the unknown phase term $\theta(\mathbf{y})$ and some other details. All those details can be found in the statement of Theorem 56. Morally, Theorem 56 says there is a quantum reduction from worst-case lattice problems to $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude with unknown phase, with arbitrarily many samples. Let us remark that the distribution of the unknown phase is *known* to be Gaussian with *small* width.

We also provide a quantum reduction directly from classical LWE to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ in Theorem 48. This reduction goes through the (extrapolated) dihedral coset problem, originally used in [Reg02, BKSW18]. It achieves slightly worse parameters compared to Theorem 56, but is much simpler to describe. For more details we refer the readers to Section 5. Although the result appears to be qualitatively similar to Theorem 56, as it also says $\mathsf{S}|\mathsf{LWE}\rangle$ with Gaussian amplitude with small unknown phase is as hard as classical LWE; the algorithm used in the reduction is very different, therefore it might offer a different approach for potential improvements.

Compared to the reduction of [BKSW18], while their reduction can be seen as converting classical LWE to $\mathsf{S}|\mathsf{LWE}\rangle$ samples with non-negligible probability of failure (meaning that the amplitude of $\mathsf{S}|\mathsf{LWE}\rangle$ samples does not follow an expected shape; the event of failure is not efficiently detectable), our reduction converts classical LWE to $\mathsf{S}|\mathsf{LWE}\rangle$ samples which always have a Gaussian amplitude, but the phase has a small unknown term. Strictly speaking, our result is incomparable with the result in [BKSW18]. But we hope that maybe some quantum algorithms in future can handle $\mathsf{S}|\mathsf{LWE}\rangle$ samples with small unknown phase better than $\mathsf{S}|\mathsf{LWE}\rangle$ samples with failure.

Is the unknown phase an inherent barrier? Overall, we show subexponential time quantum algorithm for $\mathsf{S}|\mathsf{LWE}\rangle$ with known amplitudes. We also reduce worst-case lattice problems, or classical LWE, to $\mathsf{S}|\mathsf{LWE}\rangle$ with unknown phase. Readers may wonder whether the *unknown phase* would be an inherent barrier for getting a subexponential quantum algorithm for solving standard LWE. We think it is a wonderful question for future work. On one hand, the unknown phase appeared in our reduction is not completely random (if the unknown phase were completely random then the quantum LWE samples are as useless as classical LWE samples) – we know they are small and follow Gaussian distributions, so there are some hope of further utilizing the unknown phases (such as trying to use the filtering technique in [CLZ22] to guess the phase). On the other hand, all our attempts of utilizing the unknown phase failed. The reasons of failures are rather technical and entangled with the details of our algorithms, so we refer readers to the main body for discussions therein. We think it is worth to understand whether the unknown phase is an inherent obstacle towards solving LWE in quantum subexponential time, or there is a hope of finding innovative methods in handling the unknown phase appeared in our reductions.

Organization. The rest of this paper is organized as follows. In Section 2 we provide the background of quantum computation and lattice problems. In Section 3 we provide our sub-exponential time quantum algorithm for $\mathsf{S}|\mathsf{LWE}\rangle$ with completely known amplitude. In Section 4 we provide our polynomial time quantum algorithm for $\mathsf{S}|\mathsf{LWE}\rangle$ with complex Gaussian amplitudes, and the application to oblivious LWE sampling. In Section 5 we provide the reduction from classical LWE to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ via extrapolated DCP. In Section 6 we provide the reduction from worst-case lattice problem to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ via quantizing Regev's reduction. We choose to present the reduction from classical LWE to $\mathsf{S|LWE\rangle^{phase}}$ via extrapolated DCP first, since this reduction is relatively easier to follow. Section 3, Section 4, Section 5, Section 6 are all in fact self-contained and independent, containing their own overview if necessary, so readers can start from any section without reading the others.

2 Preliminaries

Notations and terminology. Let $\mathbb{C}, \mathbb{R}, \mathbb{R}_{\geq 0}, \mathbb{Z}, \mathbb{N}, \mathbb{N}^+$ be the set of complex numbers, real numbers, non-negative real numbers, integers, natural numbers (non-negative integers), and positive integers. Denote $\mathbb{Z}/q\mathbb{Z}$ by $\mathbb{Z}_q$. By default we represent the elements of $\mathbb{Z}_q$ by elements in $(-q/2, q/2] \cap \mathbb{Z}$. For any integer $q \geq 2$, let $\omega_q = e^{2\pi \mathrm{i}/q}$ denote the primitive q-th root of unity. The rounding operation $\lfloor a \rceil : \mathbb{R} \to \mathbb{Z}$ rounds a real number a to its nearest integer (if $a \in \mathbb{Z} + 0.5$, we round it to $a + 0.5$). For positive integer q the rounding operation $\lfloor a \rceil_q : \mathbb{R} \to q\mathbb{Z}$ rounds a real number a to its nearest integer which is a multiple of q. For $n \in \mathbb{N}^+$, let $[n] := \{1, 2, \cdots, n\}$.

A vector in $\mathbb{R}^n$ (represented in column form by default) is written as a bold lower-case letter, e.g. $\mathbf{v}$. For a vector $\mathbf{v}$, the i^{th} component of $\mathbf{v}$ will be denoted by v_i. A matrix is written as a bold capital letter, e.g. $\mathbf{A}$. The i^{th} column vector of $\mathbf{A}$ is denoted $\mathbf{a}_i$.

For $x \in \mathbb{R}$ and $q \in \mathbb{N}^+$, let $x \bmod q$ be the unique real number $z \in (-q/2, q/2]$ such that $x - z$ is a multiple of q. For a vector $\mathbf{v} \in \mathbb{R}^n$, we do mod coordinate-wise, i.e. the i^{th} coordinate of $\mathbf{v} \bmod q$ is given by $v_i \bmod q$. To avoid ambiguity, we give mod lower precedence than addition/subtraction. For example, $\mathbf{a} + \mathbf{b} \bmod q$ means $(\mathbf{a} + \mathbf{b}) \bmod q$.

The length of a vector is the ℓ_p-norm $\|\mathbf{v}\|_p := (\sum v_i^p)^{1/p}$, or the infinity norm given by its largest entry $\|\mathbf{v}\|_\infty := \max_i\{|v_i|\}$. The ℓ_p norm of a matrix is the norm of its longest column: $\|\mathbf{A}\|_p := \max_i \|\mathbf{a}_i\|_p$. By default we use ℓ_2-norm unless explicitly mentioned. Let $\mathbf{x} \in \mathbb{C}^n$, then $\|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$. Let B_p^n denote the open unit ball in $\mathbb{R}^n$ in the ℓ_p norm.

When a variable v is drawn uniformly random from the set S we denote as $v \leftarrow \mathcal{U}(S)$. When a function f is applied on a set S, it means $f(S) := \sum_{x \in S} f(x)$.

Definition 9 (Statistical distance)**.** *For two distributions over $\mathbb{R}^n$ with probability density functions f_1 and f_2, we define the statistical distance between them as*

$$D(f_1, f_2) = \frac{1}{2} \int_{\mathbb{R}^n} |f_1(\mathbf{x}) - f_2(\mathbf{x})| d\mathbf{x}.$$

We say two distributions (respectively, quantum states) are ϵ-close to each other if their statistical distance (respectively, trace distance by default) is at most ϵ. We say two pure (unnormalized) states $|\phi\rangle$ and $|\psi\rangle$ are ϵ-close in ℓ_2 distance if $\||\phi\rangle - |\psi\rangle\| \leq \epsilon \max(\||\phi\rangle\|, \||\psi\rangle\|)$.

Fourier transform. The Fourier transform of a function $h : \mathbb{R}^n \to \mathbb{C}$ is defined to be

$$\hat{h}(\mathbf{w}) = \int_{\mathbb{R}^n} h(\mathbf{x}) \exp(-2\pi \mathrm{i} \langle \mathbf{x}, \mathbf{w} \rangle) d\mathbf{x}.$$

Define the convolution of two functions as $f * g(\mathbf{y}) = \int_{\mathbb{R}^n} f(\mathbf{x}) g(\mathbf{y} - \mathbf{x}) d\mathbf{x}$. Then $\widehat{f * g} = \hat{f} \cdot \hat{g}$ and $\widehat{f \cdot g} = \hat{f} * \hat{g}$.

We recall some formulas about Fourier transform (cf. [Gra08, P.100, Proposition 2.2.11]). If h is defined by $h(\mathbf{x}) = g(\mathbf{x}+\mathbf{v})$ for some function $g : \mathbb{R}^n \to \mathbb{C}$ and vector $\mathbf{v} \in \mathbb{R}^n$, then

$$\hat{h}(\mathbf{w}) = \hat{g}(\mathbf{w}) \cdot \exp(2\pi i \langle \mathbf{v}, \mathbf{w} \rangle). \tag{2}$$

If h is defined by $h(\mathbf{x}) = g(\mathbf{x}) \exp(2\pi i \langle \mathbf{x}, \mathbf{v} \rangle)$ for some function $g : \mathbb{R}^n \to \mathbb{C}$ and vector $\mathbf{v} \in \mathbb{R}^n$, then

$$\hat{h}(\mathbf{w}) = \hat{g}(\mathbf{w} - \mathbf{v}). \tag{3}$$

As a corollary of Eqns. (2) and (3), if h is defined by $h(\mathbf{x}) = f(\mathbf{x}+\mathbf{v}) \exp(2\pi i \langle \mathbf{x}, \mathbf{z} \rangle)$ for some function $f : \mathbb{R}^n \to \mathbb{C}$ and vectors $\mathbf{v}$, $\mathbf{z} \in \mathbb{R}^n$, then we define $g(\mathbf{x}) := f(\mathbf{x}+\mathbf{v})$, so $h(\mathbf{x}) = g(\mathbf{x}) \exp(2\pi i \langle \mathbf{x}, \mathbf{z} \rangle)$. Therefore $\hat{g}(\mathbf{w}) = \hat{f}(\mathbf{w}) \cdot \exp(2\pi i \langle \mathbf{v}, \mathbf{w} \rangle)$, and

$$\hat{h}(\mathbf{w}) = \hat{g}(\mathbf{w} - \mathbf{z}) = \hat{f}(\mathbf{w} - \mathbf{z}) \cdot \exp(2\pi i \langle \mathbf{v}, \mathbf{w} - \mathbf{z} \rangle).$$

2.1 Lattices

An n-dimensional lattice $\mathcal{L}$ of rank $k \le n$ is a discrete additive subgroup of $\mathbb{R}^n$. Given k linearly independent basis vectors $\mathbf{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_k \in \mathbb{R}^n\}$, the lattice generated by $\mathbf{B}$ is

$$\mathcal{L}(\mathbf{B}) = \mathcal{L}(\mathbf{b}_1, \cdots, \mathbf{b}_k) = \left\{ \sum_{i=1}^{k} x_i \cdot \mathbf{b}_i, x_i \in \mathbb{Z} \right\}.$$

By default we work with full-rank lattices unless explicitly mentioned.

The minimum distance $\lambda_1(\mathcal{L})$ of a lattice $\mathcal{L}$ is the length (in the ℓ_2 norm by default) of its shortest nonzero vector: $\lambda_1(\mathcal{L}) = \min_{\mathbf{x} \in \mathcal{L} \setminus \{\mathbf{0}\}} \|\mathbf{x}\|$. More generally, the i^{th} successive minimum $\lambda_i(\mathcal{L})$ is the smallest radius r such that $\mathcal{L}$ contains i linearly independent vectors of norm at most r. We write λ_1^p as the minimum distance in the ℓ_p norm.

For a point $\mathbf{y} \in \mathbb{R}^n$, its distance to $\mathcal{L}$ is given by $\mathrm{dist}(\mathbf{y}, \mathcal{L}) = \min_{\mathbf{x} \in \mathcal{L}} \{\|\mathbf{y} - \mathbf{x}\|\}$. Define "the ball around lattice" as $B_{\mathcal{L}}(r) = \{\mathbf{x} \in \mathbb{R}^n : \mathrm{dist}(\mathbf{x}, \mathcal{L}) < r\}$. For $\mathbf{y} \in B_{\mathcal{L}}(\lambda_1(\mathcal{L})/2)$, the (unique) closest vector to $\mathbf{y}$ in $\mathcal{L}$ is given by $\kappa_{\mathcal{L}}(\mathbf{y}) = \mathrm{argmin}_{\mathbf{x} \in \mathcal{L}} \{\|\mathbf{y} - \mathbf{x}\|\}$. For convenience, we omit the $\lambda_1(\mathcal{L})/2$ term and define $B_{\mathcal{L}}$ as $B_{\mathcal{L}}(\lambda_1(\mathcal{L})/2)$, over which $\kappa_{\mathcal{L}}$ is uniquely defined.

The dual of a lattice $\mathcal{L} \in \mathbb{R}^n$ is defined as

$$\mathcal{L}^* := \{\mathbf{y} \in \mathbb{R}^n : \langle \mathbf{y}, \mathbf{x} \rangle \in \mathbb{Z} \text{ for all } \mathbf{x} \in \mathcal{L}\}.$$

If $\mathbf{B}$ is a basis of a full-rank lattice $\mathcal{L}$, then $\mathbf{B}^{-T}$ is a basis of $\mathcal{L}^*$. The determinant of a full-rank lattice $\mathcal{L}(\mathbf{B})$ is $\det(\mathcal{L}(\mathbf{B})) = |\det(\mathbf{B})|$.

Lemma 10 (Poisson Summation Formula)**.** *For any lattice $\mathcal{L}$ and any Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$, we have $f(\mathcal{L}) = \det(\mathcal{L}^*) \hat{f}(\mathcal{L}^*)$.*

Gaussians and lattices. For any $s > 0$, define the Gaussian function on $\mathbb{R}^n$ with parameter s following the convention in [MR07]

$$\forall \mathbf{x} \in \mathbb{R}^n, \ \rho_s(\mathbf{x}) = \exp(-\pi \|\mathbf{x}\|^2 / s^2).$$

For any $\mathbf{c} \in \mathbb{R}^n$, define $\rho_{s,\mathbf{c}}(\mathbf{x}) := \rho_s(\mathbf{x} - \mathbf{c})$. The subscripts s and $\mathbf{c}$ are taken to be 1 and $\mathbf{0}$ (respectively) when omitted. Note that the standard deviation of ρ_s is $s/\sqrt{2\pi}$. The Fourier transform for Gaussian satisfies $\hat{\rho}_s = s^n \rho_{1/s}$. From Poisson summation formula we have $\rho_s(\mathcal{L}) = s^n \cdot \det(\mathcal{L}^*) \cdot \rho_{1/s}(\mathcal{L}^*)$.

For a full-rank, symmetric, positive definite $n \times n$ matrix $\mathbf{\Sigma}$, define the Gaussian function on $\mathbb{R}^n$ with parameter $\sqrt{\mathbf{\Sigma}}$ following the convention in [MP12]

$$\forall \mathbf{x} \in \mathbb{R}^n, \ \rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{x}) = \exp(-\pi \cdot \mathbf{x}^T \mathbf{\Sigma}^{-1} \mathbf{x}).$$

For any $\mathbf{c} \in \mathbb{R}^n$, real $s > 0$, and n-dimensional lattice $\mathcal{L}$, define the discrete Gaussian distribution $D_{\mathcal{L}+\mathbf{c},s}$ as

$$\forall \mathbf{x} \in \mathcal{L} + \mathbf{c}, \ D_{\mathcal{L}+\mathbf{c},s}(\mathbf{x}) = \frac{\rho_s(\mathbf{x})}{\rho_s(\mathcal{L} + \mathbf{c})}.$$

Similarly, for a full-rank, symmetric, positive definite $n \times n$ matrix $\mathbf{\Sigma}$, define the discrete Gaussian distribution $D_{\mathcal{L}+\mathbf{c},\sqrt{\mathbf{\Sigma}}}$ as

$$\forall \mathbf{x} \in \mathcal{L} + \mathbf{c}, \ D_{\mathcal{L}+\mathbf{c},\sqrt{\mathbf{\Sigma}}}(\mathbf{x}) = \frac{\rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{x})}{\rho_{\sqrt{\mathbf{\Sigma}}}(\mathcal{L} + \mathbf{c})}.$$

The following Gaussian tail bound over lattices is due to Banaszczyk.

Lemma 11 (Lemma 1.5 of [Ban93])**.** *For any n-dimensional lattice $\mathcal{L}$, and $r \geq \frac{1}{\sqrt{2\pi}}$, $\mathbf{c} \in \mathbb{R}^n$,*

$$\begin{aligned} \rho(\mathcal{L} \setminus B^n(r\sqrt{n})) &< \left(r\sqrt{2\pi e} \cdot e^{-\pi r^2}\right)^n \rho(\mathcal{L}), \\ \rho((\mathcal{L} - \mathbf{c}) \setminus B^n(r\sqrt{n})) &< 2\left(r\sqrt{2\pi e} \cdot e^{-\pi r^2}\right)^n \rho(\mathcal{L}). \end{aligned} \tag{4}$$

As a direct corollary, for $r > \frac{C}{\sqrt{2\pi}}\alpha\sqrt{n}$ with $C > 1$ be a constant, we have that

$$\rho_\alpha((\mathcal{L} - \mathbf{c}) \setminus B^n(r)) < 2^{-\Omega(n)} \rho_\alpha(\mathcal{L}).$$

Lemma 12 (Lemma 2.10 [Ban95])**.** *For any n-dimensional lattice L, $\mathbf{c} \in \mathbb{R}^n$, $r > 0$, one has*

$$\rho((L - \mathbf{c}) \setminus r\mathcal{B}^n_\infty) < \left(2n \cdot e^{-\pi r^2}\right) \rho(L).$$

Lemma 13 (Claim 8.1 [RS17])**.** *For any $n \geq 1$, $s > 0$,*

$$s^n(1 + 2e^{-\pi s^2})^n \leq \rho_s(\mathbb{Z}^n) \leq s^n(1 + (2 + 1/s)e^{-\pi s^2})^n.$$

Smoothing parameter. We recall the definition of smoothing parameter for Gaussian over lattices and some useful facts.

Definition 14 (Smoothing parameter [MR07])**.** *For any lattice $\mathcal{L}$ and positive real $\epsilon > 0$, the smoothing parameter $\eta_\epsilon(\mathcal{L})$ is the smallest real $s > 0$ such that $\rho_{1/s}(\mathcal{L}^* \setminus \{\mathbf{0}\}) \leq \epsilon$.*

Lemma 15 (Lemma 2.12 [Reg09])**.** *For any n-dimensional lattice $\mathcal{L}$, and any real $\epsilon > 0$,*

$$\eta_\epsilon(\mathcal{L}) \leq \lambda_n(\mathcal{L}) \cdot \sqrt{\ln(2n(1 + 1/\epsilon))/\pi}.$$

Lemma 16 (Claim 2.13 [Reg09])**.** *For any n-dimensional lattice $\mathcal{L}$, and any real $\epsilon > 0$,*

$$\eta_\epsilon(\mathcal{L}) \geq \sqrt{\frac{\ln 1/\epsilon}{\pi}} \frac{1}{\lambda_1(\mathcal{L}^*)}.$$

Lemma 17 (Claim 3.8 of [Reg09])**.** *For any n-dimensional lattice $\mathcal{L}$, $\mathbf{c} \in \mathbb{R}^n$, $\epsilon > 0$, and $r \geq \eta_\epsilon(\mathcal{L})$*

$$\rho_r(\mathcal{L} + \mathbf{c}) \in r^n \det(\mathcal{L}^*)(1 \pm \epsilon).$$

q-ary lattices. Given $n < m \in \mathbb{N}$ and a modulus $q \geq 2$, for $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, define q-ary lattices as

$$\begin{aligned} \mathcal{L}_q(\mathbf{A}) &= \left\{ \mathbf{x} \in \mathbb{Z}^m : \exists \mathbf{s} \in \mathbb{Z}^n \text{ such that } \mathbf{x} = \mathbf{A}^T \cdot \mathbf{s} + q\mathbb{Z}^m \right\}; \\ \mathcal{L}_q^\perp(\mathbf{A}) &= \left\{ \mathbf{x} \in \mathbb{Z}^m : \mathbf{A} \cdot \mathbf{x} = \mathbf{0} \pmod{q} \right\}. \end{aligned}$$

Those two lattices are dual of each other up to a factor of q, i.e., $\mathcal{L}_q(\mathbf{A}) = q \cdot \mathcal{L}_q^\perp(\mathbf{A})^*$.

Lemma 18. *Let $q \geq 2, m \geq 2n \log_2 q$, then for all but at most $q^{-0.16n}$ fraction of $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, we have*

$$\lambda_1^\infty(\mathcal{L}_q(\mathbf{A})) \geq \frac{q}{4}.$$

Proof. The lemma is proven when q is a prime in [GPV08, Lemma 5.3]. Here we extend the proof to a general q.

For any fixed non-zero $\mathbf{s} \in \mathbb{Z}_q^n$, wlog assuming s_1 is a non-zero entry of $\mathbf{s}$. Then for any $\mathbf{a} \in \mathbb{Z}_q^n$, $y := \langle \mathbf{a}, \mathbf{s} \rangle \mod q$ can be written as $y = s_1 a_1 + v \mod q$ for some $v \in \mathbb{Z}_q$. We observe that for any $q \in \mathbb{N}$, for any $v \in \mathbb{Z}_q$, for any non-zero $s_1 \in \mathbb{Z}_q$,

$$\Pr_{a_1 \in \mathbb{Z}_q} [\ s_1 a_1 + v \bmod q \in (-q/4, q/4) \cap \mathbb{Z}\ ] \leq 2/3,$$

where the equality holds when $q \in 3^k \cdot \mathbb{N}$ for some $k \geq 1$, $s_1 \in (q/3) \cdot \mathbb{Z}/q\mathbb{Z}$, $s_1 \neq 0$, and for some $v \in \mathbb{Z}_q$ (for example, when $q = 15$, $s_1 = 5$, and $v = 2$).

Therefore, over the randomness of $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$, the probability that $\mathbf{A}^T \mathbf{s} = \mathbf{y} \bmod q$ for some $\mathbf{y} \in \mathbb{Z}^m$ such that $\|\mathbf{y}\|_\infty < q/4$ is at most $(2/3)^m \leq (3/2)^{-2n \log_2 q} \leq q^{-1.16n}$. Applying a union bound over all $\mathbf{s} \in \mathbb{Z}_q^n$ completes the proof of Lemma 18. □

Lattice problems. We have formally defined the LWE, $\mathsf{S|LWE\rangle}$, $\mathsf{C|LWE\rangle}$, and $\mathsf{S|LWE\rangle^{phase}}$ problems in the introduction. Now let us recall the definitions for other lattice problems.

The shortest vector problem (SVP) asks to find a lattice vector of length λ_1. More generally, let $\gamma(n) \geq 1$ be an approximation factor, we consider the approximation version of SVP and its close variants.

Definition 19 (Approximate SVP)**.** *Given a basis $\mathbf{B}$ of an n-dimensional lattice $\mathcal{L}$, the SVP_γ problem asks to output a non-zero lattice vector $\mathbf{Bx}$, $\mathbf{x} \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, such that $\|\mathbf{Bx}\| \leq \gamma(n) \cdot \lambda_1(\mathcal{L})$.*

Definition 20 (GapSVP)**.** *Given a basis $\mathbf{B}$ of an n-dimensional lattice $\mathcal{L}$ and a number $d > 0$, the GapSVP_γ problem asks to decide whether $\lambda_1(\mathcal{L}) \leq d$ or $\lambda_1(\mathcal{L}) > d \cdot \gamma(n)$.*

Definition 21 (Shortest independent vector problem (SIVP))**.** *Given a basis* $\mathbf{B}$ *of an* n*-dimensional lattice* $\mathcal{L}$*, the* SIVP_γ *problem asks to output a set of* n *linearly independent vectors of length at most* $\gamma(n) \cdot \lambda_n(\mathcal{L})$*.*

Definition 22 (Discrete Gaussian Sampling Problem (DGS))**.** *Given a basis* $\mathbf{B}$ *of an* n*-dimensional lattice* $\mathcal{L}$ *and a parameter* $s > 0$*, the* DGS_s *problem asks to output a vector whose distribution is statistically close to* $D_{\mathcal{L},s}$*.*

Definition 23 (Quantum Discrete Gaussian Sampling Problem ($|\mathsf{DGS}\rangle$))**.** *Given a basis* $\mathbf{B}$ *of an* n*-dimensional lattice* $\mathcal{L}$ *and a parameter* $s > 0$*, the* $|\mathsf{DGS}\rangle_s$ *problem asks to output a quantum state that is* $2^{-\Omega(n)}$*-close to* $\sum_{\mathbf{v}\in\mathcal{L}} \rho_s(\mathbf{v})|\mathbf{v}\rangle$ *in trace distance.*

If a quantum algorithm solves $|\mathsf{DGS}\rangle_s$, then it immediately solves $\mathsf{DGS}_{s/\sqrt{2}}$ by simply measuring the quantum state. Let us also recall the relationships among DGS, GapSVP, and SIVP.

Lemma 24 (Lemma 3.20 of [Reg09])**.** *For any* $\gamma = \gamma(n) \geq 1$*, there exists a polynomial time reduction from* $\mathsf{GapSVP}_{100\sqrt{n}\gamma}$ *for* $\mathcal{L}$ *to* $\mathsf{DGS}_{\sqrt{n}\gamma/\lambda_1(\mathcal{L}^*)}$ *for* $\mathcal{L}^*$*.*

Lemma 25 (Lemma 3.17 of [Reg09])**.** *For any* $\gamma > \omega(\sqrt{\log n})$*, there exists a polynomial time reduction from* $\mathsf{SIVP}_{2\sqrt{n}\gamma}$ *for* $\mathcal{L}$ *to* $\mathsf{DGS}_{\gamma\lambda_n(\mathcal{L})}$ *for* $\mathcal{L}$*.*

2.2 Quantum computation

We assume readers are familiar with the basic concepts of quantum computation. All the quantum background we need in this paper are available in standard textbooks of quantum computation, e.g., [NC16]. When writing a quantum state such as $\sum_{x\in S} f(x)|x\rangle$, we typically omit the normalization factor except when needed.

The trace distance between two quantum states ρ and σ is defined as $\delta(\rho,\sigma) := \frac{1}{2}\operatorname{tr}|\rho-\sigma|$. Note that when ρ and σ commute they are diagonal in the same basis,

$$\rho = \sum_i r_i |i\rangle\langle i|, \quad \sigma = \sum_i s_i |i\rangle\langle i|,$$

for some orthonormal basis $|i\rangle$, then $\delta(\rho,\sigma) = \frac{1}{2}\operatorname{tr}|\sum_i (r_i - s_i)|i\rangle\langle i|| = \frac{1}{2}\sum_i |r_i - s_i|$.

The trace distance is preserved under unitary transformations, and is contractive under trace-preserving operations.

Lemma 26. *Let* $|\phi\rangle$*,* $|\psi\rangle$ *be un-normalized vectors s.t.* $\||\phi\rangle\| \geq \mu$ *and* $\||\phi\rangle - |\psi\rangle\| \leq \epsilon$*. Then*

$$\delta\left(\frac{1}{\||\phi\rangle\|}|\phi\rangle, \frac{1}{\||\psi\rangle\|}|\psi\rangle\right) = \sqrt{1 - \left(\frac{|\langle\phi|\psi\rangle|}{\||\phi\rangle\|\||\psi\rangle\|}\right)^2} \leq O\left(\sqrt{\frac{\epsilon}{\mu}}\right).$$

Lemma 27 (Gentle measurement [Win99])**.** *Let* ρ *be a quantum state and let* $(\mathbf{\Pi}, \mathbf{I} - \mathbf{\Pi})$ *be a two-outcome projective measurement such that* $\operatorname{tr}(\mathbf{\Pi}\rho) \geq 1-\epsilon$*. Let* $\rho' = \frac{\mathbf{\Pi}\rho\mathbf{\Pi}}{\operatorname{tr}(\mathbf{\Pi}\rho)}$ *be the state after applying the measurement and post-selecting on getting the first outcome. Then* $\delta(\rho,\rho') \leq 2\sqrt{\epsilon}$*.*

We need the following lemma about the trace distance between discrete Gaussian states.

Lemma 28. *When $q > 2\sqrt{n}\max(\beta_1,\beta_2)$ and $R \geq \frac{2\sqrt{n}}{\min(\beta_1,\beta_2)}$, the trace distance between*

$$|\phi_1\rangle = \sum_{e\in\mathbb{Z}_{qR}/R} \rho_{\beta_1}(e)|e\rangle \qquad \text{and} \qquad |\phi_2\rangle = \sum_{e\in\mathbb{Z}_{qR}/R} \rho_{\beta_2}(e)|e\rangle$$

is at most $\sqrt{\frac{(\beta_1-\beta_2)^2}{\beta_1^2+\beta_2^2}}(1+2^{-\Omega(n)})$.

Proof.

$$\begin{aligned}\frac{\langle\phi_1|\phi_2\rangle}{\||\phi_1\rangle\|\||\phi_2\rangle\|} &= \sum_{e\in\mathbb{Z}_{qR}/R} \rho_{\frac{\beta_1\beta_2}{\sqrt{\beta_1^2+\beta_2^2}}}(e) \Bigg/ \sqrt{\sum_{e\in\mathbb{Z}_{qR}/R}\rho_{\beta_1/\sqrt{2}}(e)\sum_{e\in\mathbb{Z}_{qR}/R}\rho_{\beta_2/\sqrt{2}}(e)} \\ &= \sum_{e\in\mathbb{Z}/R} \rho_{\frac{\beta_1\beta_2}{\sqrt{\beta_1^2+\beta_2^2}}}(e) \Bigg/ \sqrt{\sum_{e\in\mathbb{Z}/R}\rho_{\beta_1/\sqrt{2}}(e)\sum_{e\in\mathbb{Z}/R}\rho_{\beta_2/\sqrt{2}}(e)}\,(1+2^{-\Omega(n)}) \\ &= \frac{\beta_1\beta_2}{\sqrt{\beta_1^2+\beta_2^2}}\cdot\frac{\sqrt{2}}{\sqrt{\beta_1\beta_2}}\sum_{e\in R\mathbb{Z}}\rho_{\frac{\sqrt{\beta_1^2+\beta_2^2}}{\beta_1\beta_2}}(e) \Bigg/ \sqrt{\sum_{e\in R\mathbb{Z}}\rho_{\sqrt{2}/\beta_1}(e)\sum_{e\in R\mathbb{Z}}\rho_{\sqrt{2}/\beta_2}(e)}\,(1+2^{-\Omega(n)}) \\ &= \frac{\sqrt{2\beta_1\beta_2}}{\sqrt{\beta_1^2+\beta_2^2}}(1+2^{-\Omega(n)}),\end{aligned}$$

where we use the Poisson summation formula and Banaszczyk's tail bound.

So their trace distance is at most

$$\sqrt{1-\left(\frac{|\langle\phi_1|\phi_2\rangle|}{\||\phi_1\rangle\|\||\phi_2\rangle\|}\right)^2} \leq \sqrt{\frac{(\beta_1-\beta_2)^2}{\beta_1^2+\beta_2^2}}(1+2^{-\Omega(n)}).$$

□

We use the following quantum algorithms:

Lemma 29 (Quantum Fourier Transform (QFT) [Kit95])**.** *Let $q \geq 2$ be an integer. The following unitary operator QFT_q can be implemented by $\mathsf{poly}(\log q)$ elementary quantum gates. When QFT_q is applied on a quantum state $|\phi\rangle := \sum_{x\in\mathbb{Z}_q} f(x)|x\rangle$, we have*

$$\mathsf{QFT}_q|\phi\rangle = \sum_{y\in\mathbb{Z}_q}\sum_{x\in\mathbb{Z}_q}\frac{1}{\sqrt{q}}\cdot\omega_q^{xy}\cdot f(x)|y\rangle.$$

We use the following lemma [ORR13] to change the amplitude of a state.

Lemma 30 (Quantum rejection sampling)**.** *Let $f : D \to \mathbb{C}$ be a normalized amplitude of some quantum state $|\phi_f\rangle := \sum_{x\in D} f(x)|x\rangle$. Let $\gamma : D \to [0,1]$ be a polynomial time computable function. There is a quantum algorithm that takes as input $|\phi_f\rangle$, outputs a state $\sum_{x\in D}\frac{1}{\sqrt{M}}\gamma(x)f(x)|x\rangle$ with probability M where $M = \sum_{x\in D}\gamma^2(x)|f(x)|^2$.*

In this paper we are interested in preparing the Gaussian state $|\sigma_{n,R}\rangle := \sum_{\mathbf{x}\in\mathbb{Z}^n\cap B^n(R\sqrt{n})}\rho_R(\mathbf{x})|\mathbf{x}\rangle$ for some radius $R \leq 2^{\mathsf{poly}(n)}$. Given Lemma 11, there is a $2^{-\Omega(n)}$ mass in the tail of $\rho_R(\mathbf{x})$ outside $B^n(R\sqrt{n})$. This means for any integer $P \in (2R\sqrt{n}, 2^{\mathsf{poly}(n)})$, $|\sigma_{n,R}\rangle$ is $2^{-\Omega(n)}$ close to $\sum_{\mathbf{x}\in\mathbb{Z}_P^n}\rho_R(\mathbf{x})|\mathbf{x}\rangle$. This also means we can prepare $|\sigma_{n,R}\rangle$ by generating n independent samples of one-dimensional Gaussian state $|\sigma_{1,R}\rangle$, which can be done efficiently using [GR02].

Lemma 31 (Gaussian state preparation)**.** *Let $n, R \in \mathbb{N}$ such that $1 \le R \le 2^{n^c}$ for some constant $c \ge 0$. Then we can create a $\mathsf{poly}(n)$ size unitary U that maps $|\mathbf{0}\rangle$ to a state within trace distance $2^{-\Omega(n)}$ from $|\sigma_{n,R}\rangle$ with $2n \lceil n^c \cdot \log n \rceil$ qubits.*

3 Quantum Sub-exponential Time Algorithm for $\mathsf{S}|\mathsf{LWE}\rangle$

In this section, we provide a quantum sub-exponential time algorithm designed to solve $\mathsf{S}|\mathsf{LWE}\rangle$ instances with specific amplitudes. More precisely, we consider scenarios where the discrete Fourier transform of these amplitudes has at least $2^{-\sqrt{n}\log q}$ mass on two distinct points. Our approach is built upon two key steps: (1) generate a DCP state for each quantum $\mathsf{S}|\mathsf{LWE}\rangle$ sample, resulting in a sub-exponential collection of DCP states; (2) employ the Kuperberg sieve technique [Kup05] on the DCP states to successfully recover the secret vector. Formally, we state the main theorem of this section as follows:

Theorem 32 (Main theorem)**.** *For any efficiently computable normalized amplitude function $f : \mathbb{Z} \to \mathbb{C}$ such that there exists two distinct points j_1 and j_2 from $\mathbb{Z}_q$ with $\gcd(j_1 - j_2, q) = 1$ and $|\mathsf{DFT}_q(f)(j_1)|$ and $|\mathsf{DFT}_q(f)(j_2)|$ are both greater than $2^{-\sqrt{n}\log q}$ ($\mathsf{DFT}_q(f)$ is the discrete Fourier transform of f, defined as $\mathsf{DFT}_q(f)(j) = \frac{1}{\sqrt{q}} \sum_{e\in\mathbb{Z}} f(e)\omega_q^{je}$ for $j \in \mathbb{Z}_q$), there exists a quantum algorithm that, given $\ell = 2^{\Theta(\sqrt{n}\log q)}$ samples of vector $\mathbf{a} \leftarrow \mathcal{U}(\mathbb{Z}_q^n)$ and $\mathsf{S}|\mathsf{LWE}\rangle$ state of form*

$$\mathsf{S}|\mathsf{LWE}\rangle = \sum_{e\in\mathbb{Z}} f(e) |\langle \mathbf{a}, \mathbf{s}\rangle + e \bmod q\rangle,$$

finds the secret vector $\mathbf{s} \in \mathbb{Z}_q^n$ within a time complexity of $2^{\Theta(\sqrt{n}\log q)}$.

Note that when q is prime, then $\gcd(j_1 - j_2, q) = 1$ is always satisfied. So the condition $\gcd(j_1 - j_2, q) = 1$ is only needed to be taken care of when q is composite.

To prove Theorem 32, we begin by introducing the Kuperberg sieve algorithm [Kup05].

Lemma 33 (Kuperberg sieve)**.** *Let $\mathbf{s} \in \mathbb{Z}_q^n$ be a secret vector. There exists a quantum algorithm that given $\ell^* = 2^{\Theta(\sqrt{n}\log q)}$ samples of*

$$\mathbf{a} \leftarrow \mathcal{U}(\mathbb{Z}_q^n), \quad |\psi_{\mathbf{a}}\rangle = |0\rangle + \omega_q^{\langle \mathbf{a}, \mathbf{s}\rangle} |1\rangle,$$

finds out the secret vector $\mathbf{s}$ in time $2^{\Theta(\sqrt{n}\log q)}$.

Now we present the proof of Theorem 32 here.

Proof of Theorem 32. Suppose we have $\ell = \ell^* \cdot 2^{4\sqrt{n}\log q} = 2^{\Theta(\sqrt{n}\log q)}$ instances of $\mathsf{S}|\mathsf{LWE}\rangle$ states. Our quantum algorithm proceeds as follows:

1. Compute $\mathsf{DFT}_q(f)(j)$ for each $j \in \mathbb{Z}_q$ and find two distinct points j_1 and j_2 from $\mathbb{Z}_q$ with $\gcd(j_1 - j_2, q) = 1$ and $|\mathsf{DFT}_q(f)(j_1)|$ and $|\mathsf{DFT}_q(f)(j_2)|$ are both greater than $2^{-\sqrt{n}\log q}$. If it fails, abort.

2. Apply QFT to any $\mathsf{S|LWE}\rangle$ state, resulting in the state

$$\mathsf{QFT}_q \cdot \mathsf{S|LWE}\rangle = \frac{1}{\sqrt{q}} \sum_{j \in \mathbb{Z}_q} \sum_{e \in \mathbb{Z}} f(e) \omega_q^{j(\langle \mathbf{a}, \mathbf{s} \rangle + e)} |j\rangle = \sum_{j \in \mathbb{Z}_q} \omega_q^{\langle j \cdot \mathbf{a}, \mathbf{s} \rangle} \mathsf{DFT}_q(f)(j) |j\rangle,$$

3. Define $\gamma(j) : \mathbb{Z}_q \to [0, 1]$ as

$$\gamma(j) = \frac{\min\{|\mathsf{DFT}_q(f)(j_1)|, |\mathsf{DFT}_q(f)(j_2)|\}}{|\mathsf{DFT}_q(f)(j)|} \text{ for } j = j_1, j_2$$

and $\gamma(j) = 0$ otherwise, apply quantum rejection sampling (Lemma 30) to obtain the state

$$\frac{\mathsf{DFT}_q(f)(j_1)}{|\mathsf{DFT}_q(f)(j_1)|} \omega_q^{\langle j_1 \cdot \mathbf{a}, \mathbf{s} \rangle} |j_1\rangle + \frac{\mathsf{DFT}_q(f)(j_2)}{|\mathsf{DFT}_q(f)(j_2)|} \omega_q^{\langle j_2 \cdot \mathbf{a}, \mathbf{s} \rangle} |j_2\rangle, \tag{5}$$

with probability

$$M = \sum_{j \in \mathbb{Z}_q} \gamma^2(j) |\mathsf{DFT}(f)(j)|^2 = 2(\min\{|\mathsf{DFT}_q(f)(j_1)|, |\mathsf{DFT}_q(f)(j_2)|\})^2 > 2^{-2\sqrt{n}\log q}.$$

4. For the states that have been successfully transformed to Equation (5), apply a unitary operation

$$U : |j_1\rangle \to \frac{\overline{\mathsf{DFT}_q(f)(j_1)}}{|\mathsf{DFT}_q(f)(j_1)|} |0\rangle, |j_2\rangle \to \frac{\overline{\mathsf{DFT}_q(f)(j_2)}}{|\mathsf{DFT}_q(f)(j_2)|} |1\rangle,$$

this results in the state

$$|\psi_{(j_2 - j_1)\mathbf{a}}\rangle = |0\rangle + \omega_q^{\langle (j_2 - j_1)\mathbf{a}, \mathbf{s} \rangle} |1\rangle,$$

where $(j_2 - j_1)\mathbf{a}$ is a known vector in $\mathbb{Z}_q^n$ that is uniformly random by the assumption that $\gcd(j_1 - j_2, q) = 1$.

5. Select ℓ^* such states obtained in step 3 and apply the Kuperberg sieve algorithm to recover the secret vector $\mathbf{s} \in \mathbb{Z}_q^n$.

It is evident that step 1 runs in time $O(\mathsf{poly}(n) q \log q) \leq 2^{O(\sqrt{n}\log q)}$ and transforming any $\mathsf{S|LWE}\rangle$ state to a DCP-like state requires a time complexity of $2^{O(\sqrt{n}\log q)}$. Therefore, the run time of our quantum algorithm is constrained by both the quantity of $\mathsf{S|LWE}\rangle$ states and the application of the Kuperberg sieve, which both exhibit a complexity of $2^{\Theta(\sqrt{n}\log q)}$. In step 2, the count of states successfully transformed to Equation (5) will be at least $M^2 \ell = \ell^*$ with a probability exponentially close to 1. This concludes the proof. □

Corollary 34. *Suppose m, n, q are LWE parameters. There exists a quantum algorithm that, given $2^{\Theta(\sqrt{n}\log q)}$ samples of vector $\mathbf{a} \leftarrow \mathcal{U}(\mathbb{Z}_q^n)$ and $\mathsf{S|LWE}\rangle$ state of form*

$$\mathsf{S|LWE}\rangle = \sum_{e \in \mathbb{Z}} \rho_\sigma(e) \exp(2\pi \mathrm{i} \cdot ce/q) | \langle \mathbf{a}, \mathbf{s} \rangle + e \bmod q\rangle,$$

where the Gaussian width σ satisfies $\sigma = \Omega(\sqrt{n}), \sigma \leq q$, and c is an arbitrary known number that can be different for different samples, solves the secret vector $\mathbf{s} \in \mathbb{Z}_q^n$ within a time complexity of $2^{\Theta(\sqrt{n}\log q)}$.

Proof. Let us define

$$N = \sum_{e\in\mathbb{Z}} \rho_\sigma^2(e) = \sum_{e\in\mathbb{Z}} \frac{\sigma}{\sqrt{2}} \rho_{\sqrt{2}/\sigma}(e) \approx \frac{\sigma}{\sqrt{2}},$$

where the final approximation holds under the assumption that $\sigma = \Omega(n)$. In this case, the summation of $\rho_{\sqrt{2}/\sigma}$ is concentrated at $\rho_{\sqrt{2}/\sigma}(0)$, with exponentially small weight elsewhere.

In the given problem scenario,

$$f(e) = \frac{1}{\sqrt{N}} \rho_\sigma(e) \exp(2\pi \mathrm{i} \cdot ce/q),$$

thus

$$\begin{aligned}
\mathsf{DFT}_q(f)(j) &= \frac{1}{\sqrt{qN}} \sum_{e\in\mathbb{Z}} \rho_\sigma(e) \exp(2\pi\mathrm{i} \cdot (j+c)e/q) \\
&=_{(*)} \frac{1}{\sqrt{qN}} \sum_{e\in\mathbb{Z}} \sigma \rho_{1/\sigma}\left(e - \frac{j+c}{q}\right) \\
&\approx_{(**)} \frac{1}{\sqrt{qN}} \sigma \rho_{1/\sigma}\left(\left\lfloor \frac{j+c}{q} \right\rceil - \frac{j+c}{q}\right) \\
&= \frac{\sigma}{\sqrt{qN}} \rho_{q/\sigma}(j + c - \lfloor j+c \rceil_q),
\end{aligned}$$

here $(*)$ is from the Poisson summation formula, $(**)$ holds due to the assumption that $\sigma = \Omega(\sqrt{n})$. In this case, the summation $\sum_{j\in\mathbb{Z}_q} \rho_{1/\sigma}\left(e - \frac{j+c}{q}\right)$ is concentrated at $\rho_{1/\sigma}\left(\left\lfloor \frac{j+c}{q} \right\rceil - \frac{j+c}{q}\right)$, with exponentially small weight elsewhere.

Define $j_1 = \lfloor -c \rceil \bmod q, j_2 = (\lfloor -c \rceil + 1) \bmod q$, we can establish that $|j + c - \lfloor j+c \rceil_q| \leq 1$ holds for both $j = j_1$ and $j = j_2$. This implies that

$$|\mathsf{DFT}_q(f)(j)| \geq \sqrt{\frac{\sqrt{2\sigma}}{q} \rho_{q/\sigma}(1)(1 - 2^{-\Omega(n)})} \geq \sqrt{\frac{\sqrt{2}}{q} e^{-\pi}(1 - 2^{-\Omega(n)})} \gg 2^{-\sqrt{n}\log q},$$

for both $j = j_1, j_2$.

As a result, we can deduce the validity of the original statement by straightforwardly applying Theorem 32. □

Remark 35. *Readers may be curious about why our sub-exponential algorithm cannot handle* $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ *instances with an unknown phase. Informally speaking, when the phase term of* $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ *samples is unknown, we can only obtain a DCP-like state with varying weights in the superposition. Although the ratio of weights on* $|0\rangle$ *and* $|1\rangle$ *can be bounded by an inverse polynomial, this ratio tends to become extremely large during the sieving step of Kuperberg's algorithm. As a result, the final state collapses into either* $|0\rangle$ *or* $|1\rangle$*, and the information about* $\mathbf{s}$ *is entirely lost.*

4 Complex Gaussian and Oblivious LWE Sampling

In this section, we provide a quantum polynomial time algorithm that can obliviously sample LWE instances. Our approach follows the reduction from oblivious LWE sampling to the $\mathsf{C}|\mathsf{LWE}\rangle$ problem introduced in [DFS24]. However, our method for solving the $\mathsf{C|LWE}\rangle$ problem is fundamentally different from the approach in [DFS24]; it employs complex Gaussian amplitudes, a technique first introduced in [Che24], along with a center-finding trick. Not only does our algorithm improve upon the sample complexity of the core quantum algorithm in [DFS24], but it is also relatively straightforward and easy to comprehend.

4.1 Center finding

We first show an efficient quantum algorithm for finding the center $c \bmod t$ of a complex Gaussian state $|\phi_c\rangle = \sum_{x\in\mathbb{Z}} \rho_r(x)e^{-\frac{\pi i x^2}{t}}|x+c\rangle$. Note that although the state $|\phi_c\rangle$ appears to be supported over $\mathbb{Z}$, it is within $2^{-\Omega(n)}$ trace distance from $\sum_{x\in[c-r\sqrt{n},c+r\sqrt{n}]\cap\mathbb{Z}} \rho_r(x)e^{-\frac{\pi i x^2}{t}}|x+c\rangle$, so it can be stored within $O(\log|c| + \log(r\sqrt{n}))$ many qubits.

Theorem 36 (Center finding)**.** *Let n, t be positive integers, r be a positive real with $r \geq 30tn\log n$. Then there exists a polynomial time quantum algorithm that, given a complex Gaussian state*

$$|\phi_c\rangle = \sum_{x\in\mathbb{Z}} \rho_r(x)e^{-\frac{\pi i x^2}{t}}|x+c\rangle$$

with unknown center $c \in \mathbb{Z}$, finds $c \bmod t$ with probability $1 - 1/n$.

Before proving the theorem, we first observe that for any $c_1, c_2 \in \mathbb{Z}$ with $c_1 \not\equiv c_2 \bmod t$, the complex Gaussian states $|\phi_{c_1}\rangle$, $|\phi_{c_2}\rangle$ are nearly orthogonal and, therefore, almost perfectly distinguishable. We prove this for the special case of $|\phi_0\rangle$ v.s. $|\phi_c\rangle$, and the proof generalizes to any pair of distinct centers $|\phi_{c_1}\rangle$, $|\phi_{c_2}\rangle$:

$$\begin{aligned}
\frac{|\langle\phi_0|\phi_c\rangle|}{\||\phi_0\rangle\|\||\phi_c\rangle\|} &= \left|\sum_{x\in\mathbb{Z}} \rho_r(x)\rho_r(x-c)e^{-\frac{\pi i(x-c)^2-\pi i x^2}{t}}\right| \Big/ \sum_{x\in\mathbb{Z}} \rho_r(x)^2 \\
&= \rho_{r/\sqrt{2}}(c/2)\left|\sum_{x\in\mathbb{Z}} \rho_{r/\sqrt{2}}(x-c/2)e^{\frac{2\pi i c x}{t}}\right| \Big/ \sum_{x\in\mathbb{Z}} \rho_{r/\sqrt{2}}(x) \\
&=_{(*)} \rho_{r/\sqrt{2}}(c/2)\left|\sum_{y\in\mathbb{Z}} \rho_{\sqrt{2}/r}(y-c/t)e^{2\pi i(y-c/t)\cdot c/2}\right| \Big/ \sum_{y\in\mathbb{Z}} \rho_{\sqrt{2}/r}(y) \\
&\approx_{(**)} \rho_{r/\sqrt{2}}(c/2)\rho_{\sqrt{2}/r}(\lfloor c/t\rceil - c/t),
\end{aligned} \tag{6}$$

where $(*)$ is from the Poisson summation formula, $(**)$ holds due to the assumption that $r/t \in \Omega(\sqrt{n})$. In this case, the summation $\sum_{y\in\mathbb{Z}} \rho_{\sqrt{2}/r}(y-c/t)$ (with some phase term that does not affect the norm) is concentrated at $\rho_{\sqrt{2}/r}(\lfloor c/t\rceil - c/t)$, and the summation $\sum_{y\in\mathbb{Z}} \rho_{\sqrt{2}/r}(y)$ is concentrated at $\rho_{\sqrt{2}/r}(0) = 1$, with exponentially small weight elsewhere.

Therefore, when $c \not\equiv 0 \bmod t$, the overlap of $|\phi_0\rangle$ and $|\phi_c\rangle$ can be bounded by

$$\left|\frac{\langle\phi_0|\phi_c\rangle}{\||\phi_0\rangle\|\||\phi_c\rangle\|}\right| \leq \rho_{r/\sqrt{2}}(c/2)\rho_{\sqrt{2}/r}(1/t)\left(1 + 2^{-\Omega(n)}\right) = 2^{-\Omega(n)}.$$

Using a similar argument, for $c_1 \not\equiv c_2 \bmod t$, the overlap of $|\phi_{c_1}\rangle$ and $|\phi_{c_2}\rangle$ is bounded by $2^{-\Omega(n)}$. Keeping the above observation in mind, the algorithm of center finding is straightforward:

measure the lower order bits (for modulus t) of $|\phi_c\rangle$ under the basis $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$, and output the measurement result $d \in \mathbb{Z}_t$, where

$$|\psi_d\rangle = \sum_{x\in\mathbb{Z}_t} e^{-\frac{\pi i(x-d)^2}{t}}|x\rangle.$$

We'll first show that $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$ is a basis that can be efficiently prepared, and then show the measurement result d equals the center c with high probability.

For $d, d' \in \mathbb{Z}_t$, the overlap of $|\psi_d\rangle$ and $|\psi_{d'}\rangle$ is

$$\begin{aligned}\frac{\langle\psi_{d'}|\psi_d\rangle}{\||\psi_{d'}\rangle\|\||\psi_d\rangle\|} &= \frac{1}{t}\sum_{x\in\mathbb{Z}_t} e^{-\left(\frac{\pi i(x-d)^2}{t}-\frac{\pi i(x-d')^2}{t}\right)} \\ &= \frac{1}{t}e^{\frac{\pi i(d'^2-d^2)}{t}}\sum_{x\in\mathbb{Z}_t} e^{\frac{2\pi i(d-d')x}{t}} \\ &= \begin{cases} 1 & d = d' \\ 0 & d \neq d' \end{cases}.\end{aligned}$$

Therefore $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$ is indeed a basis. To show $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$ can be efficiently prepared, we use the following lemma:

Lemma 37. *There is a* $\mathsf{poly}(\log t)$ *size quantum circuit that implements the unitary transformation* $\sum_{d\in\mathbb{Z}_t}|d\rangle\langle\psi_d|$.

Proof. For any $d \in \mathbb{Z}_t$, we transform $|d\rangle \to |\psi_d\rangle = \sum_{x\in\mathbb{Z}_t} e^{-\frac{\pi i(x-d)^2}{t}}|x\rangle$ as follows:

$$|d\rangle \mapsto_{(1)} e^{-\frac{\pi i d^2}{t}}|d\rangle \mapsto_{QFT_{\mathbb{Z}_t}} \sum_{x\in\mathbb{Z}_t} e^{\frac{2\pi i x d}{t}} e^{-\frac{\pi i d^2}{t}}|x\rangle \mapsto_{(2)} \sum_{x\in\mathbb{Z}_t} e^{-\frac{\pi i x^2}{t}} e^{\frac{2\pi i x d}{t}} e^{-\frac{\pi i d^2}{t}}|x\rangle = \sum_{x\in\mathbb{Z}_t} e^{-\frac{\pi i(x-d)^2}{t}}|x\rangle,$$

where (1), (2) use the phase kick-back trick. □

We then calculate the probability of obtaining $c_1 := c \bmod t$ when measuring the lower order bits of $|\phi_c\rangle$ under the basis $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$. We write the term $x + c$ in the summation of $|\phi_c\rangle$ by $kt + y$, with summation on $k \in \mathbb{Z}$ and $y \in \mathbb{Z}_t$:

$$|\phi_c\rangle = \sum_{k\in\mathbb{Z}}\sum_{y\in\mathbb{Z}_t} \rho_r(kt + y - c)e^{-\frac{\pi i(kt+y-c)^2}{t}}|k\rangle_{\mathsf{A}}|y\rangle_{\mathsf{B}}$$

where registers A and B store the higher order bits and the lower order bits of $x + c$, respectively.

We observe that $|\phi_c\rangle$ can also be written in the following form:

$$\begin{aligned}
|\phi_c\rangle &= \sum_{k\in\mathbb{Z}}\sum_{y\in\mathbb{Z}_t} \rho_r(kt+y-c) e^{-\frac{\pi i(kt+y-c)^2}{t}+\frac{\pi i(y-c_1)^2}{t}} \cdot e^{-\frac{\pi i(y-c_1)^2}{t}} |k\rangle_{\mathsf{A}}|y\rangle_{\mathsf{B}} \\
&= \sum_{k\in\mathbb{Z}}\sum_{y\in\mathbb{Z}_t} \rho_r(kt+y-c) e^{-\frac{\pi i\left(k^2t^2+(y-c)^2\right)}{t}+\frac{\pi i(y-c_1)^2}{t}} \cdot e^{-\frac{\pi i(y-c_1)^2}{t}} |k\rangle_{\mathsf{A}}|y\rangle_{\mathsf{B}} \\
&= \sum_{k\in\mathbb{Z}} e^{-\pi i k^2 t} \sum_{y\in\mathbb{Z}_t} \rho_r(kt+y-c) e^{2\pi i\frac{(c-c_1)y}{t}+\frac{\pi i(c_1^2-c^2)}{t}} \cdot e^{-\frac{\pi i(y-c_1)^2}{t}} |k\rangle_{\mathsf{A}}|y\rangle_{\mathsf{B}} \\
&= \sum_{k\in\mathbb{Z}} e^{-\pi i k^2 t+\frac{\pi i(c_1^2-c^2)}{t}} |k\rangle_{\mathsf{A}} \sum_{y\in\mathbb{Z}_t} \rho_r(kt+y-c) e^{-\frac{\pi i(y-c_1)^2}{t}} |y\rangle_{\mathsf{B}}
\end{aligned}$$

The probability of obtaining c_1 when measuring B in basis $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$ can be computed as first measuring A in computational basis and getting some outcome k, and then obtaining c_1 when measuring B in basis $\{|\psi_d\rangle\}_{d\in\mathbb{Z}_t}$:

$$\begin{aligned}
\Pr(d=c_1) &= \sum_{k\in\mathbb{Z}} \left(\sum_{y\in\mathbb{Z}_t} \rho_r(kt+y-c)\right)^2 \Big/ t\sum_{x\in\mathbb{Z}} \rho_r(x)^2 \\
&= \sum_{k\in\mathbb{Z}} \frac{\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c)^2}{\sum_{x\in\mathbb{Z}}\rho_r(x)^2} \cdot \frac{\left(\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c)\right)^2}{t\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c)^2} \\
&=_{(1)} \sum_{k\in\mathbb{Z}} \frac{\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2}{\sum_{x\in\mathbb{Z}}\rho_r(x)^2} \cdot \frac{\left(\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)\right)^2}{t\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2} \\
&\geq \sum_{-\frac{r\log n}{t}\leq k\leq\frac{r\log n}{t},k\in\mathbb{Z}} \frac{\left(\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)\right)^2}{t\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2} \cdot \frac{\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2}{\sum_{x\in\mathbb{Z}}\rho_r(x)^2} \\
&\geq \sum_{-\frac{r\log n}{t}\leq k\leq\frac{r\log n}{t},k\in\mathbb{Z}} \frac{\min_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2}{\max_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2} \cdot \frac{\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2}{\sum_{x\in\mathbb{Z}}\rho_r(x)^2} \\
&\geq_{(2)} e^{-\frac{8\pi t\log n}{r}} \sum_{-\frac{r\log n}{t}\leq k\leq\frac{r\log n}{t},k\in\mathbb{Z}} \cdot \frac{\sum_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2}{\sum_{x\in\mathbb{Z}}\rho_r(x)^2} \\
&\geq_{(3)} 1-\frac{8\pi t\log n}{r}-\mathsf{negl}(n)
\end{aligned}$$

where (1) uses $c\equiv c_1 \pmod t$, (2) uses for all $-\frac{r\log n}{t}\leq k\leq\frac{r\log n}{t}$, $g(k):=\frac{\min_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2}{\max_{y\in\mathbb{Z}_t}\rho_r(kt+y-c_1)^2}\geq e^{-\frac{8\pi t\log n}{r}}$ (to see why, observe that when $k\geq 3$, $g(k)\geq\frac{\min_{y\in\mathbb{Z}_t}\rho_r(kt+t/2-c_1)^2}{\rho_r(kt-t/2-c_1)^2}\geq e^{-\frac{8\pi kt^2}{r^2}}\geq e^{-\frac{8\pi t\log n}{r}}$, same when $k\leq -3$, when $|k|\leq 2$, $g(k)\geq e^{-\frac{32\pi t^2}{r^2}}\geq e^{-\frac{8\pi t\log n}{r}}$); (3) is due to Lemma 12.

By our choice of parameters, $1-\frac{8\pi t\log n}{r}-\mathsf{negl}(n)\geq 1-\frac{1}{n}$, which ends the proof of Theorem 36.

4.2 Constructing quantum LWE states with Gaussian amplitudes

In this subsection, we show polynomial time quantum algorithms for solving $\mathsf{S|LWE\rangle}$, $\mathsf{C|LWE\rangle}$ problems where the amplitude is Gaussian with a specific choice of phase terms. Here we will state the theorem and algorithm for $\mathsf{C|LWE\rangle}$, the corresponding ones for $\mathsf{S|LWE\rangle}$ are analogous.

Theorem 38. *Let n, m, q, ℓ be positive integers and r be a real number. Suppose that $m = 2\ell n \cdot \omega(\log n)$, q is a composite number satisfying $q = q_1 q_2 \cdots q_\ell$ where $q_1, q_2, \cdots, q_\ell$ are coprime, r satisfies $\frac{q}{\sqrt{n}} > r > 30 n \log n \cdot \max\{q_1, q_2, \cdots, q_\ell\}$.*

There exists a quantum algorithm running in time $\mathsf{poly}(n, \ell, \log q)$ that, takes input $\mathbf{A} \leftarrow \mathcal{U}(\mathbb{Z}_q^{n\times m})$, outputs a state ρ such that the trace distance between ρ and $\phi := |\phi\rangle\langle\phi|$ is negligible, where

$$|\phi\rangle = \sum_{\mathbf{s}\in\mathbb{Z}_q^n} \sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_r(\mathbf{x}) e^{-\pi i \sum_{j=1}^{\ell} \|\mathbf{x}_j\|^2/q_j} |(\mathbf{A}^T\mathbf{s} + \mathbf{x}) \bmod q\rangle$$

is a quantum LWE state with Gaussian amplitude and quadratic phase terms, with probability $1 - \mathsf{negl}(n)$ over the randomness of $\mathbf{A}$. Here we write $\mathbf{x}$ into ℓ blocks $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_\ell)$ with $\mathbf{x}_j \in \mathbb{Z}^{m/\ell}, j = 1, 2, \cdots, \ell$.

The algorithm begin by constructing the state (within trace distance $2^{-\Omega(n)}$)

$$|\phi_0\rangle = \sum_{\mathbf{s}\in\mathbb{Z}_q^n} |\mathbf{s}\rangle \sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_r(\mathbf{x}) e^{-\pi i \sum_{j=1}^{\ell} \|\mathbf{x}_j\|^2/q_j} |(\mathbf{A}^T\mathbf{s} + \mathbf{x}) \bmod q\rangle,$$

by combining the complex Gaussian state preparation procedure [Che24, Lemma 2.15] with the tail bound in Lemma 11, then recovering $\mathbf{s} \bmod q$ via the state in the second register

$$|\psi_{\mathbf{s}}\rangle = \sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_r(\mathbf{x}) e^{-\pi i \sum_{j=1}^{\ell} \|\mathbf{x}_j\|^2/q_j} |(\mathbf{A}^T\mathbf{s} + \mathbf{x}) \bmod q\rangle.$$

and subtract it in the first register to obtain $|\phi\rangle$. At a high level, the strategy of recovering $\mathbf{s} \bmod q$ is to divide $\mathbf{A}$ into ℓ groups $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_\ell \in \mathbb{Z}_q^{n\times m/\ell}$. We use the j^{th} block of $m/\ell = 2n \cdot \omega(\log n)$ samples $(\mathbf{A}_j^T\mathbf{s} + \mathbf{x}_j) \bmod q$ to (coherently) compute $\mathbf{s} \bmod q_j$. Then by Chinese Remainder Theorem (CRT), one can (coherently) recover $\mathbf{s} \bmod q$ via $\mathbf{s} \bmod q_j, j = 1, 2, \cdots, \ell$ and subtract it in the first register. It suffices to prove the following lemma:

Lemma 39. *For any $j = 1, 2, \cdots, \ell$, given $\mathbf{A}_j \leftarrow \mathcal{U}(\mathbb{Z}_q^{n\times m/\ell})$ and the state*

$$|\psi_{\mathbf{s},j}\rangle = \sum_{\mathbf{x}_j\in\mathbb{Z}^{m/\ell}} \rho_r(\mathbf{x}_j) e^{-\pi i \|\mathbf{x}_j\|^2/q_j} |(\mathbf{A}_j^T\mathbf{s} + \mathbf{x}_j) \bmod q\rangle,$$

one can recover $\mathbf{s} \bmod q_j$ in time $\mathsf{poly}(n)$ with probability $1 - \mathsf{negl}(n)$ over the randomness of $\mathbf{A}_j$ and measurements.

Proof. First observe that, when looking at the lower order bits (for modulus q_j) of $|\psi_{\mathbf{s},j}\rangle$, it shares the same density matrix with the state

$$|\psi'_{\mathbf{s},j}\rangle = \sum_{\mathbf{x}_j\in\mathbb{Z}^{m/\ell}} \rho_r(\mathbf{x}_j) e^{-\pi i \|\mathbf{x}_j\|^2/q_j} |(\mathbf{A}_j^T\mathbf{s} \bmod q_j + \mathbf{x}_j) \bmod q\rangle.$$

Therefore, when applying the center-finding technique (Theorem 36, which only works on the lower order bits), the measurement result of $|\psi_{\mathbf{s},j}\rangle$ and $|\psi'_{\mathbf{s},j}\rangle$ shares the same distribution.

Notice that the absolute value of each entry of $\mathbf{A}_j^T\mathbf{s} \bmod q_j + \mathbf{x}_j$ is at most $q_j/2 + rn^{1/3} \le q/2$ with overwhelming probability (by Lemma 12), when $\mathbf{x}_j$ follows the distribution of ρ_r^2. Thus $|\psi'_{\mathbf{s},j}\rangle$ is $\mathsf{negl}(n)$-close to the state

$$|\psi''_{\mathbf{s},j}\rangle = \sum_{\mathbf{x}_j \in \mathbb{Z}^{m/\ell}} \rho_r(\mathbf{x}_j) e^{-\pi i \|\mathbf{x}_j\|^2/q_j} |\mathbf{A}_j^T\mathbf{s} \bmod q_j + \mathbf{x}_j\rangle.$$

Observe that $|\psi''_{\mathbf{s},j}\rangle$ is exactly a tensor product of complex Gaussian state $|\phi_c\rangle$ defined in Theorem 36 for c being entries of $\mathbf{A}_j^T\mathbf{s} \bmod q_j$. Applying the center-finding technique (Theorem 36) coordinate by coordinate on the state $|\psi''_{\mathbf{s},j}\rangle$, we can obtain a measurement outcome $\tilde{\mathbf{y}} \in \mathbb{Z}_{q_j}^{m/\ell}$, which is an approximation of $\mathbf{A}_j^T\mathbf{s} \bmod q_j$ such that each coordinate is independently correct with probability at least $1 - \frac{1}{n}$ over the measurements (we will call it $(1-\frac{1}{n})$-approximation of $\mathbf{A}_j^T\mathbf{s} \bmod q_j$).

Now let's construct a (classical) polynomial time probabilistic algorithm $\mathcal{B}$, which uses Gaussian elimination along with a verification process to obtain $\mathbf{s} \bmod q_j$ given $\mathbf{A}_j \leftarrow \mathcal{U}(\mathbb{Z}_q^{n\times m/\ell})$ and $\tilde{\mathbf{y}}$, which is supposed to be an $(1-\frac{1}{n})$-approximation of $\mathbf{A}_j^T\mathbf{s} \bmod q_j$, with overwhelming probability over the randomness of $\mathbf{A}_j$ and $\tilde{y}$. Concretely, the algorithm runs

1. Divide $\mathbf{A}_j$ into groups of n by $2n$ matrices $\mathbf{A}_{j,1}, \mathbf{A}_{j,2}, \cdots, \mathbf{A}_{j,m/(2\ell n)} \in \mathbb{Z}_q^{n\times 2n}$, and write $\tilde{\mathbf{y}}$ into blocks $\tilde{\mathbf{y}} = (\tilde{\mathbf{y}}_1, \tilde{\mathbf{y}}_2, \cdots, \tilde{\mathbf{y}}_{m/(2\ell n)})$ with each block belonging to $\mathbb{Z}_{q_j}^{2n}$;
2. For each $i = 1, 2, \cdots, m/(4\ell n)$:
   (a) If $\mathbf{A}_{j,i}$ is not full-rank (with rank n), go to the next iteration;
   (b) Solve the linear equations $\mathbf{A}_{j,i}^T\mathbf{s} = \tilde{\mathbf{y}}_i \bmod q_j$ using Gaussian elimination. If the linear equations are not solvable, go to the next iteration; otherwise obtain a solution $\mathbf{s}' \in \mathbb{Z}_{q_j}^n$;
   (c) Compare $\mathbf{y}' = \mathbf{A}_{j,i+m/(4\ell n)}^T\mathbf{s}' \bmod q_j$ with $\tilde{\mathbf{y}}_{i+m/(4\ell n)}$. If there are at least 0.9 fraction of entries are matched, then output $\mathbf{s} = \mathbf{s}'$; otherwise, go to the next iteration.

By construction, $\mathcal{B}$ runs in polynomial time. Now let's show its correctness.

First, the verification process in step 2(c) ensures that the probability of outputting a wrong answer is negligible. Namely, when a wrong answer $\mathbf{s}' \neq \mathbf{s} \bmod q_j$ is obtained, each entry of $\tilde{\mathbf{y}}_{i+m/(4\ell n)}$ equals to the corresponding row of $\mathbf{A}_{j,i+m/(4\ell n)}^T\mathbf{s} \bmod q_j$ with probability at least $1-1/n$. Since the matrix $\mathbf{A}_j$ is chosen uniformly at random, each row of $\mathbf{A}_{j,i+m/(4\ell n)}^T\mathbf{s} \bmod q_j$ and $\mathbf{y}' = \mathbf{A}_{j,i+m/(4\ell n)}^T\mathbf{s}' \bmod q_j$ are different with probability at least $\frac{1}{2}$. Therefore, each entry of $\mathbf{y}'$ and $\tilde{\mathbf{y}}_{i+m/(4\ell n)}$ are equal with probability at most $\frac{1}{2} + \frac{1}{n} < 0.6$. By Chernoff bound, this $\mathbf{s}' \neq \mathbf{s} \bmod q_j$ only passes 2(c) with negligible probability.

Second, the probability that the algorithm can output the correct secret $\mathbf{s} \bmod q_j$ is overwhelming. Notice that a random n by $2n$ matrix is full-rank in $\mathbb{Z}_{q_j}$ with overwhelming probability (see [Che24, Claim 3.3]), so the failure probability in step 2(a) is negligible. Observe that none of the entries in $\tilde{\mathbf{y}}_i$ is wrong with probability at least $\left(1-\frac{1}{n}\right)^{2n} > 0.1$, over the randomness of measurement in the center-finding process. Thus, among the $m/(4\ell n) = \omega(\log n)$ linear equations $\mathbf{A}_{j,i}^T\mathbf{s} = \tilde{\mathbf{y}}_i \bmod q_j$ for $i = 1, 2, \cdots, m/(4\ell n)$, except with negligible probability, at least one of them consists of rows without any error, which leads to the correct $\mathbf{s}$ in step 2(b). This correct $\mathbf{s}$ will pass the verification process in step 2(c) with overwhelming probability because each entry of $\mathbf{y}'$ and $\tilde{\mathbf{y}}_{i+m/(4\ell n)}$ are equal with probability at least $1 - \frac{1}{n}$ in this case.

Lemma 39 follows from the fact that when applying the center-finding technique, the measurement result of $|\psi_{\mathbf{s},j}\rangle$ and $|\psi''_{\mathbf{s},j}\rangle$ are indistinguishable. □

4.3 Oblivious LWE sampling

While constructing a quantum LWE state with a specific choice of phase terms may not seem interesting at first glance, the problem is actually closely related to a problem called oblivious LWE sampling for a vast range of LWE parametrizations.

Roughly speaking, in oblivious LWE sampling, the sampler is asked to sample from the LWE distribution $(\mathbf{A}, \mathbf{A}^T\mathbf{s} + \mathbf{e})$, while any extractor, observing the sampler (but not disturbing the sampler's state), cannot extract the secret $\mathbf{s}$.

Here we abuse the notation $\mathsf{LWE}_{n,m,q,\alpha}$ to represent the distribution of valid LWE samples $\mathbf{A}, \mathbf{b} = \mathbf{A}^T\mathbf{s} + \mathbf{e} \bmod q$, where $\mathbf{A} \leftarrow U(\mathbb{Z}_q^{n\times m})$, $\mathbf{s} \leftarrow U(\mathbb{Z}_q^n)$, and $\mathbf{e} \leftarrow D_{\mathbb{Z},\alpha q}^m$. Later we will use two variants of $\mathsf{LWE}_{n,m,q,\alpha}$. Let $\mathsf{LWE}_{n,m,q,\leq\alpha}$ denote the case where the error is sampled from $D_{\mathbb{Z},\beta q}$ for some $\beta \leq \alpha$. Let $\mathsf{LWE}_{n,m,q,\alpha}(D_{\mathbb{Z}^n,s})$ denote the distribution that produces LWE samples where the secret is sampled from $D_{\mathbb{Z}^n,s}$ for some $s > 0$.

Definition 40 (Witness-Oblivious Quantum Samplers, [DFS24])**.** *Let n, m, q, α be LWE parameters following the same definition as in Definition 1. A quantum polynomial time algorithm $\mathcal{S}$ is called a* witness-oblivious quantum sampler *for $\mathsf{LWE}_{n,m,q,\alpha}$ if it has the following properties:*

1. *Given as input the security parameter 1^λ, a matrix $\mathbf{A} \in \mathbb{Z}_q^{n\times m}$, and polynomial ancillas initialized to $|0\rangle$, $\mathcal{S}$ outputs a pair $(\mathbf{A}, \mathbf{b})$ where $\mathbf{b} \in \mathbb{Z}_q^m$ such that for a uniformly distributed $\mathbf{A}$, the distribution of $\mathcal{S}(1^\lambda, \mathbf{A}, |0\rangle^{\mathsf{poly}(\lambda)})$ is statistically indistinguishable with the distribution $\mathsf{LWE}_{n,m,q,\alpha}$;*

2. *For any valid quantum polynomial-time extractor $\mathcal{E}$ for $\mathcal{S}$, we have that*

$$\Pr\left[\mathbf{s} = \mathbf{s}' \text{ and } \mathbf{e} = \mathbf{e}' : \substack{\mathbf{A}\leftarrow\mathbb{Z}_q^{n\times m} \\ ((\mathbf{A},\mathbf{b}=\mathbf{A}^T\mathbf{s}+\mathbf{e}),(\mathbf{s}',\mathbf{e}'))\leftarrow\langle\mathcal{S},\mathcal{E}\rangle(\tau_{\mathcal{S}}=(1^\lambda,\mathbf{A},|0\rangle^{\mathsf{poly}(\lambda)}),\tau_{\mathcal{E}}=|0\rangle^{\mathsf{poly}(\lambda)})}\right] \leq \mathsf{negl}(\lambda)$$

   where $\langle\mathcal{S}, \mathcal{E}\rangle(\tau_{\mathcal{S}}, \tau_{\mathcal{E}})$ denote the joint output of the sampler $\mathcal{S}$ and the extractor $\mathcal{E}$ on the input $\tau_{\mathcal{S}}$ and $\tau_{\mathcal{E}}$, and an extractor $\mathcal{E}$ is called valid *for the sampler $\mathcal{S}$ if and only if $\mathcal{E}$ does not change the state of the sampler $\mathcal{S}$ up to negligible trace distance when $\mathcal{S}$ and $\mathcal{E}$ are running together.*

Theorem 41 (Theorem 2, [DFS24])**.** *Let n, m, q, α be LWE parameters. Assume the quantum hardness of $\mathsf{LWE}_{n,m,q,\alpha}$. If a quantum polynomial algorithm $\mathcal{S}$ solves $\mathsf{C}|\mathsf{LWE}\rangle_{n,m,q,f}$, where the amplitude f satisfies $|f|^2 \propto D_{\mathbb{Z},\alpha q}$, then $\mathcal{S}$ followed by computational measurements is a witness-oblivious quantum sampler for $\mathsf{LWE}_{n,m,q,\alpha}$.*

Corollary 42. *Let n, m, q, ℓ be positive integers and r be a real number. Suppose that $m \geq 2\ell n \cdot \omega(\log n)$, q is a composite number satisfying $q = q_1 q_2 \cdots q_\ell$ where $q_1, q_2, \cdots, q_\ell$ are coprime, r satisfies $\frac{q}{\sqrt{n}} > r > 30n \log n \cdot \max\{q_1, q_2, \cdots, q_\ell\}$. Assume the quantum hardness of $\mathsf{LWE}_{n,m,q,r/(\sqrt{2}q)}$, there exists a witness-oblivious quantum sampler for $\mathsf{LWE}_{n,m,q,r/(\sqrt{2}q)}$.*

Proof. This is a direct corollary of Theorem 38 and Theorem 41. □

As noted in [DFS24], once we get an oblivious $\mathsf{LWE}_{n,m,q,\alpha}$ sampler for certain composite modulus q, we can throw away additional samples to get an oblivious $\mathsf{LWE}_{n,m',q,\alpha}$ sampler for $m' \leq m$. We can also use the modulus switching technique in [BLP$^+$13] to get an oblivious LWE sampler for $\mathsf{LWE}_{n,m,q',\alpha'}$ where $q' < q$ is not necessarily composite.

Here we spell out the details of the modulus switching part. To use modulus switching, we need to start from LWE samples where the length of the secret is shorter than q (it is easy to modify the quantum algorithm for $\mathsf{C}|\mathsf{LWE}\rangle$ so that the secret is a superposition of a general distribution). For consistency let us fix the secret to be sampled from $D_{\mathbb{Z}^n,s}$ and adapt from [BLP$^+$13, Corollary 3.2]:

Lemma 43. *For any positive integers n, m, $q \geq q' \geq 1$, $\alpha, \alpha' \in (0,1)$ such that $m, \log q \in \mathsf{poly}(n)$,*

$$(\alpha')^2 \geq \alpha^2 + \left(\frac{s\sqrt{n}}{q'}\right)^2 \cdot \omega(\log n),$$

there is a $\mathsf{poly}(n)$ time classical algorithm that takes as input a sample from $\mathsf{LWE}_{n,m,q,\leq\alpha}(D_{\mathbb{Z}^n,s})$, outputs a sample within negligible statistical distance from $\mathsf{LWE}_{n,m,q',\leq\alpha'}(D_{\mathbb{Z}^n,s})$.

As a corollary of Corollary 42 and Lemma 43:

Corollary 44. *Let n, m, q, ℓ be positive integers and r be a real number. Suppose that $m \geq 2\ell n \cdot \omega(\log n)$, q is a composite number satisfying $q = q_1 q_2 \cdots q_\ell$ where $q_1, q_2, \cdots, q_\ell$ are coprime, r satisfies $\frac{q}{\sqrt{n}} > r > 30n \log n \cdot \max\{q_1, q_2, \cdots, q_\ell\}$. Suppose positive integers $m' \leq m, q' \leq q$ and real numbers s, $r' > 0$ satisfies*

$$\left(\frac{r'}{\sqrt{2}q'}\right)^2 \geq \left(\frac{r}{\sqrt{2}q}\right)^2 + \left(\frac{s\sqrt{n}}{q'}\right)^2 \cdot \omega(\log n),$$

and assume the quantum hardness of $\mathsf{LWE}_{n,m,q,r/(\sqrt{2}q)}(D_{\mathbb{Z}^n,s})$, there exists a witness-oblivious quantum sampler for $\mathsf{LWE}_{n,m',q',r'/(\sqrt{2}q')}(D_{\mathbb{Z}^n,s})$.

For example, to produce an oblivious LWE sample from $\mathsf{LWE}_{n,m',q',r'/(\sqrt{2}q')}(D_{\mathbb{Z}^n,s})$ where $q' \in \tilde{O}(n^2)$ is prime, $r' \in \tilde{O}(n^{1.5})$, $s \in O(\sqrt{n})$, we can start from an oblivious LWE sample from $\mathsf{LWE}_{n,m,q,r/(\sqrt{2}q)}(D_{\mathbb{Z}^n,s})$ where $r \in \tilde{O}(n^{1.5})$, $q \in \tilde{O}(n^2)$ such that $q' < q < 2q'$, $q = q_1 q_2 q_3 q_4$ where $q_1, q_2, q_3, q_4 \in \tilde{O}(\sqrt{n})$ are coprime. The sample complexity required for producing $\mathsf{LWE}_{n,m,q,r/(\sqrt{2}q)}(D_{\mathbb{Z}^n,s})$ from our quantum algorithm in Corollary 42 is in $\tilde{O}(n)$. If we use the quantum oblivious LWE sampler in [DFS24], the sample complexity required for producing $\mathsf{LWE}_{n,m',q',r'/(\sqrt{2}q')}(D_{\mathbb{Z}^n,s})$ is $\tilde{O}(n^{2.5})$.

5 Hardness of $\mathsf{S|LWE\rangle}$ with Unknown Phase via Extrapolated DCP

In this section, we show how to obtain a quantum reduction from classical LWE to $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ with Gaussian amplitude. Our reduction goes through the Extrapolated Dihedral Coset problem (EDCP), derived from a modification of the reduction by Regev [Reg04] and Brakerski et al. [BKSW18]. Our reduction consists of two steps:

Step 1 Given a classical LWE instance, our quantum reduction first generates an Extrapolated DCP state with amplitudes following a Gaussian distribution centered at an unknown value. More precisely, we establish the following theorem:

Theorem 45. *Let $n, m, q \in \mathbb{N}^+$, $\alpha = \Omega(\sqrt{n}), \beta, \gamma \in (0,1)$ satisfy $m \geq n \log q$, $\alpha\gamma\sqrt{m} < \beta < \frac{1}{16\sqrt{m\log(\beta q)}}$. There exists a $\mathsf{poly}(n)$ time quantum algorithm that, given a classical LWE instance $\mathsf{LWE}_{n,m,q,\gamma} = (\mathbf{A}, \mathbf{A}^T\mathbf{s} + \mathbf{e})$, generates, with probability $1 - 2^{-\Omega(n)}$, a vector $\mathbf{y} \in \mathbb{Z}_q^m \cap B_{\mathcal{L}_q(\mathbf{A})}$ and an Extrapolated DCP state of form*

$$|\mathsf{EDCP}\rangle = \sum_{j \in \mathbb{Z}_q} \rho_\sigma(j - c)|j\rangle|(\mathbf{v} + j \cdot \mathbf{s}) \bmod q\rangle, \tag{7}$$

where

1. *the vector $\mathbf{v}$ is chosen uniformly at random from $\mathbb{Z}_q^n$ and is unknown,*
2. *the Gaussian width $\sigma = \frac{\alpha\beta q}{\sqrt{\alpha^2\|\mathbf{e}\|^2 + \beta^2 q^2}}$,*
3. *the vector $\mathbf{y}$ is sampled by first sampling $\mathbf{x} \in \mathbb{Z}^m \cap B^m(\lambda_1(\mathcal{L}_q(\mathbf{A}))/2)$ with probability proportional to $\Pr(\mathbf{x}) \propto \rho_{\beta q\sqrt{\mathbf{\Sigma}/2}}(\mathbf{x})$ where $\mathbf{\Sigma} = \mathbf{I}_m + \frac{\alpha^2}{\beta^2 q^2}\mathbf{e}\mathbf{e}^T$, then outputting $\mathbf{y} = (\mathbf{A}^T\mathbf{v} + \mathbf{x}) \bmod q$,*
4. *the center $c = -\frac{\alpha^2\langle \mathbf{x}, \mathbf{e}\rangle}{\alpha^2\|\mathbf{e}\|^2 + \beta^2 q^2}$ (we don't know how to efficiently compute c with success probability $1 - 2^{-\Omega(n)}$ since we don't know $\mathbf{x}$ and $\mathbf{e}$; we can guess c correctly with non-negligible probability, but the event of guessing correctly is not efficiently checkable).*

Step 2 Given an Extrapolated DCP state with amplitudes following a Gaussian distribution centered at an unknown value, we adapt the quantum reduction proposed by Brakerski et al. [BKSW18] to transform it to an $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ instance. The resulting $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ instance will have amplitudes represented as a Gaussian distribution multiplied by an unknown phase term. More precisely, we establish the following theorem:

Theorem 46. *There exists an efficient quantum algorithm that, given an Extrapolated DCP state of form as Equation (7), generates an $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ state of form*

$$\sum_{e \in \mathbb{Z}} \rho_{q/\sigma}(e) \exp(2\pi \mathrm{i} \cdot ce/q)|(\langle \mathbf{a}, \mathbf{s}\rangle + e) \bmod q\rangle \tag{8}$$

along with a known vector $\mathbf{a} \leftarrow \mathcal{U}(\mathbb{Z}_q^n)$. Here the parameters σ, c correspond to those mentioned in Theorem 45.

In the remaining part of this section, we will provide the detailed proofs for Theorem 45 and Theorem 46. By combining these two theorems, we achieve a quantum reduction from classical LWE to $\mathsf{S|LWE\rangle^{phase}}$ with Gaussian amplitude. This establishes that solving the classical LWE problem is as hard as solving the $\mathsf{S|LWE\rangle^{phase}}$ with Gaussian amplitude for the phase function defined below. Formally, we define our special parameters and functions, and propose the main theorem for this section here:

Definition 47. *Let $n, m, q \in \mathbb{N}^+$, $\alpha = \Omega(\sqrt{n}), \beta, \gamma \in (0,1)$ satisfy $m \geq n \log q$, $\alpha\gamma\sqrt{m} < \beta < \frac{1}{16\sqrt{m \log(\beta q)}}$. Given a matrix $\mathbf{A} \in \mathbb{Z}_q^{n \times m}$ and a vector $\mathbf{e}$ such that $\|\mathbf{e}\| \leq \sqrt{m}\gamma q$, we define:*

1. *A family of functions $\{f_E : \mathbb{Z} \to \mathbb{R}\}_{E \in [m\gamma^2 q^2]}$ with $f_E(e) = \rho_{q/\sigma(E)}(e)$ where $\sigma(E) = \frac{\alpha\beta q}{\sqrt{\alpha^2 E + \beta^2 q^2}}$.*

2. *A distribution $D_{\theta,\mathbf{e},\mathbf{A}}(\mathbf{y})$ over $\mathbb{Z}_q^m \cap B_{\mathcal{L}_q(\mathbf{A})}$ given by $\Pr(\mathbf{y}) \propto \rho_{\beta q\sqrt{\mathbf{\Sigma}/2}}(\mathbf{y}')$ where $\mathbf{\Sigma} = \mathbf{I}_m + \frac{\alpha^2}{\beta^2 q^2}\mathbf{e}\mathbf{e}^T$ and $\mathbf{y}' = \mathbf{y} - \kappa_{\mathcal{L}_q(\mathbf{A})}(\mathbf{y})$.*

3. *A phase function $\theta_{\mathbf{e},\mathbf{A}} : \mathbb{Z}_q^m \cap B_{\mathcal{L}_q(\mathbf{A})} \to \mathbb{R}$ with $\theta_{\mathbf{e}}(\mathbf{y}) = -\frac{\alpha^2 \langle \mathbf{y}', \mathbf{e} \rangle}{q(\alpha^2\|\mathbf{e}\|^2 + \beta^2 q^2)}$ (It is not efficiently computable when assuming classical LWE is hard).*

Theorem 48 (Main theorem, from LWE to $\mathsf{S|LWE\rangle^{phase}}$)**.** *Following the parameters defined in Definition 47. Assume for any $E \in [m\gamma^2 q^2]$, there exists a quantum algorithm that takes $\mathbf{A} \leftarrow \mathcal{U}(\mathbb{Z}_q^{n \times m})$ as input, with $1 - 2^{-\Omega(n)}$ probability over $\mathbf{A}$, solves (see Definition 7) $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,\ell,q,f_E,\theta_{\mathbf{e},\mathbf{A}},D_{\theta,\mathbf{e},\mathbf{A}}}$ for every $\mathbf{e} \in \mathbb{Z}^m$ such that $\|\mathbf{e}\|_2^2 = E$, with $\ell = 2^{o(n)}$ and time complexity $T = 2^{o(n)}$, then there exists a quantum algorithm that takes a classical LWE sample $(\mathbf{A}, \mathbf{A}^T\mathbf{s} + \mathbf{e})$ as input, and outputs the secret vector $\mathbf{s} \in \mathbb{Z}_q^n$ with success probability $1 - 2^{-\Omega(n)}$ and time complexity $O((T + \ell \cdot \mathsf{poly}(n,q)) \cdot m\gamma^2 q^2)$.*

Remark 49. *For an example of parameters, let $q \in \tilde{O}(n^2)$, $m \in \Omega(n \log q)$, $\|\mathbf{e}\|_2^2 \in \tilde{O}(n)$, $\alpha \in \tilde{O}(n^{0.5})$, $\beta \in \tilde{O}(n^{-0.5})$, $\gamma \in \tilde{O}(n^{-1.5})$.*

Proof. We proceed to address the classical LWE instance $(\mathbf{A}, \mathbf{A}^T\mathbf{s} + \mathbf{e})$ as follows:

1. Enumerate $E \in \{1, 2, \cdots, m\gamma^2 q^2\}$ to make a guess for $\|\mathbf{e}\|^2$.

2. Apply Theorem 45 and Theorem 46 ℓ times to generate ℓ instances of $\mathsf{S|LWE\rangle^{phase}}$ in the form of Equation (8).

3. Utilize the quantum algorithm in the assumption for $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,\ell,q,f_E,\theta_{\mathbf{e},\mathbf{A}},D_{\theta,\mathbf{e},\mathbf{A}}}$, with those ℓ $\mathsf{S|LWE\rangle^{phase}}$ instances as input, to derive a solution $\mathbf{s}'$.

4. Employ any verification algorithm (e.g., as proposed by Regev [Reg09, Lemma 3.6]) to ascertain whether $\mathbf{s}' = \mathbf{s}$. If this condition holds, output $\mathbf{s}'$ and conclude the process.

It can be easily verified that this algorithm operates with a runtime of $O((T + \ell \cdot \mathsf{poly}(q,n)) \cdot m\gamma^2 q^2)$. Furthermore, as indicated in Theorem 45, the probability of successfully generating ℓ $\mathsf{S|LWE\rangle^{phase}}$ states in step 2 is exponentially close to 1. Thus, when $E = \|\mathbf{e}\|^2$ (i.e., when we guess E correctly), the probability that the solution $\mathbf{s}' = \mathbf{s}$ is exponentially close to 1. Consequently, the aforementioned algorithm achieves success probability exponentially close to 1. $\square$

5.1 Reduce classical LWE to Extrapolated DCP

Our quantum reduction from classical LWE to Extrapolated DCP follows the general design of Regev's reduction [Reg04] and the reduction proposed by Brakerski et al. [BKSW18]. In these reductions, the Euclidean space $\mathbb{R}^n$ is divided into grids, with each grid cell having a width that lies between the length of the error vector $\|\mathbf{e}\|$ and the length of the shortest vector in the lattice $\lambda_1(\mathcal{L}_q(\mathbf{A}))$. The key observation is that when randomly selecting a vector $\mathbf{x} \in \mathbb{R}^n$, the vectors $\mathbf{x}, \mathbf{x}+\mathbf{e}, \cdots, \mathbf{x}+k\cdot\mathbf{e}$ will be in the same grid cell with high probability, creating a superposition in the quantum world.

We modify the reductions in [Reg04,BKSW18] by introducing Gaussian balls around all lattice points in $\mathcal{L}_q(\mathbf{A})$, where the radius of each ball is a quantity smaller then the length of shortest vector in the lattice $\lambda_1(\mathcal{L}_q(\mathbf{A}))$. Note that the reductions in [Reg04,BKSW18] use Euclidean balls or cubes.

Here we give the detailed proof of Theorem 45.

Proof of Theorem 45. Following the parameters defined in Theorem 45. Relying on both Lemma 18 and Banaszczyk's tail bound from Lemma 11, we make the assumptions that $\lambda_1(\mathcal{L}_q(\mathbf{A})) \geq \frac{q}{4}$ and $\|\mathbf{e}\| < \sqrt{m}\gamma q$ for the remainder of this proof; these assumptions hold with probability $1 - 2^{-\Omega(n)}$. Our quantum reduction from classical LWE to Extrapolated DCP works as follows (for simplicity, we omit the normalization factors):

1. We start by preparing the superposition using the Gaussian state sampler (see Lemma 31)

$$\sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)|j\rangle \sum_{\mathbf{v}\in\mathbb{Z}_q^n} |\mathbf{v}\rangle \sum_{\mathbf{x}\in\mathbb{Z}_q^m} \rho_{\beta q}(\mathbf{x})|\mathbf{x}\rangle.$$

2. We apply a unitary to compute $(j, \mathbf{v}, \mathbf{x}) \to (\mathbf{A}^T\mathbf{v} - j\cdot(\mathbf{A}^T\mathbf{s}+\mathbf{e}) + \mathbf{x}) \bmod q$ on the third register, obtaining the state

$$\sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)|j\rangle \sum_{\mathbf{v}\in\mathbb{Z}_q^n} |\mathbf{v}\rangle \sum_{\mathbf{x}\in\mathbb{Z}_q^m} \rho_{\beta q}(\mathbf{x})|(\mathbf{A}^T\mathbf{v} - j\cdot(\mathbf{A}^T\mathbf{s}+\mathbf{e}) + \mathbf{x}) \bmod q\rangle. \tag{9}$$

   This state approximates, with an error of $1 - 2^{-\Omega(n)}$, the same state structure with the only difference being the range of $\mathbf{x}$ in the summation, which is now $\mathbb{Z}^m$ rather than $\mathbb{Z}_q^m$. By expressing $\mathbf{A}^T\mathbf{v} - j\cdot(\mathbf{A}^T\mathbf{s}+\mathbf{e}) + \mathbf{x}$ as $\mathbf{A}^T(\mathbf{v} - j\cdot\mathbf{s}) + (\mathbf{x} - j\cdot\mathbf{e})$, we perform a change of variables $\mathbf{v} \leftarrow (\mathbf{v} + j\cdot\mathbf{s}) \bmod q$ and $\mathbf{x} \leftarrow \mathbf{x} + j\cdot\mathbf{e}$, yields that the state of form Equation (9) is $2^{-\Omega(n)}$-close to the state

$$\sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)|j\rangle \sum_{\mathbf{v}\in\mathbb{Z}_q^n} |(\mathbf{v}+j\cdot\mathbf{s}) \bmod q\rangle \sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e})|(\mathbf{A}^T\mathbf{v}+\mathbf{x}) \bmod q\rangle. \tag{10}$$

   To proceed, we need the following lemma to guarantee that each vector in the support of the third register $\mathbf{y} := \mathbf{A}^T\mathbf{v} + \mathbf{x} \bmod q$ corresponds to a unique $\mathbf{x}$, in order to match with the target of Theorem 45 and to simplify later analyses. The proof of this lemma is deferred to Appendix A.2.

Lemma 50. *Assume that* $(\beta q\sqrt{m}+\alpha\gamma qm)\sqrt{\log(\beta q)} < \lambda_1(\mathcal{L}_q(\mathbf{A}))/2$ *and* $\beta q > \sqrt{m}$*, then the state in Equation (10) is* $2^{-\Omega(n)}$*-close to the state*

$$\sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)|j\rangle \sum_{\mathbf{v}\in\mathbb{Z}_q^n} |(\mathbf{v}+j\cdot\mathbf{s}) \bmod q\rangle \sum_{\substack{\mathbf{x}\in\mathbb{Z}^m,\\ \|\mathbf{x}\|<\lambda_1(\mathcal{L}_q(\mathbf{A}))/2}} \rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e})|(\mathbf{A}^T\mathbf{v}+\mathbf{x}) \bmod q\rangle. \tag{11}$$

Remark 51. *The condition of this lemma can be relaxed if we instead use the proof technique from Claim A.5 in [Reg23]. To be more specific, the above lemma holds as long as* $\beta q\sqrt{m}+\alpha\gamma qm < \lambda_1(\mathcal{L}_q(\mathbf{A}))/2$ *and* $q/2 > \alpha\sqrt{m}$*. However, the relaxed condition provides similar parameters in the overall reduction, and therefore, we do not emphasize it here.*

Under the condition of Theorem 45, $(\beta q\sqrt{m}+\alpha\gamma qm)\sqrt{\log(\beta q)} < 2\beta q\sqrt{m\log(\beta q)} < q/8 < \lambda_1(\mathcal{L}_q(\mathbf{A}))/2$ and $\beta q > \alpha\gamma q\sqrt{m} > \sqrt{m}$, so the state in Equation (10) is $2^{-\Omega(n)}$-close to the state in Equation (11), which can be rewritten as follows

$$\sum_{\substack{\mathbf{v}\in\mathbb{Z}_q^n,\mathbf{x}\in\mathbb{Z}^m,\\ \|\mathbf{x}\|<\lambda_1(\mathcal{L}_q(\mathbf{A}))/2}} \left(\sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)\rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e})|j\rangle|(\mathbf{v}+j\cdot\mathbf{s}) \bmod q\rangle\right)|(\mathbf{A}^T\mathbf{v}+\mathbf{x}) \bmod q\rangle \tag{12}$$

3. We measure the state in Equation (12) on the third register and denote the result as $\mathbf{y} = (\mathbf{A}^T\mathbf{v}+\mathbf{x}) \bmod q$ (note that we have $\|\mathbf{x}\| < \lambda_1(\mathcal{L}_q(\mathbf{A}))/2$, so the vectors $\mathbf{v}$ and $\mathbf{x}$ are both unique), then the remaining state on the first two registers is

   $$\sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)\rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e})|j\rangle|(\mathbf{v}+j\cdot\mathbf{s}) \bmod q\rangle.$$

   The amplitude of this state is computed as follows

   $$\begin{aligned}\rho_\alpha(j)\rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e}) &= \exp\left[-\pi\left(\frac{j^2}{\alpha^2}+\frac{j^2\|\mathbf{e}\|^2+2j\langle\mathbf{x},\mathbf{e}\rangle+\|\mathbf{x}\|^2}{\beta^2q^2}\right)\right]\\ &\propto \exp\left[-\pi\left(\frac{(\alpha^2\|\mathbf{e}\|^2+\beta^2q^2)j^2+2j\alpha^2\langle\mathbf{x},\mathbf{e}\rangle}{\alpha^2\beta^2q^2}\right)\right]\\ &\propto \rho_\sigma(j-c),\end{aligned}$$

   where the Gaussian width $\sigma = \frac{\alpha\beta q}{\sqrt{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}} \in (\alpha/\sqrt{2},\alpha)$ and the center $c = -\frac{\alpha^2\langle\mathbf{x},\mathbf{e}\rangle}{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}$. Unfortunately, the center c remains unknown because we have no knowledge of $\mathbf{x}$ other than that it is the error term in the LWE sample $\mathbf{y} = (\mathbf{A}^T\mathbf{v}+\mathbf{x}) \bmod q$. The analysis of the distribution of $\mathbf{y}$ and c is deferred to Section 5.3.

It's evident that in the state of Equation (11), the amplitude for every $\mathbf{v}\in\mathbb{Z}_q^n$ is the same, which implies that the distribution of the vector $\mathbf{v}$ is uniformly random. This completes the proof. □

Remark 52. *It seems that our reduction bears similarities to the reduction from classical LWE to G-EDCP (Extrapolated DCP with amplitudes following a Gaussian distribution) proposed by Brakerski et al. [BKSW18]. However, our reduction, compared to both Regev's reduction and Brakerski* *et al.'s reduction, exhibits superior success probability. In the previous reductions, the failure probability is inverse-polynomial, leading to the reduction of classical LWE to only polynomially many (Extrapolated) DCP states. In contrast, our reduction achieves $1 - 2^{-\Omega(n)}$ success probability, allowing for the construction of sub-exponentially many Extrapolated DCP states without failure, the cost is introducing an unknown center in the Gaussian distribution of the amplitude. This novel approach offers the potential for creating sub-exponential time quantum algorithms for the standard LWE problem.*

5.2 Reduce Extrapolated DCP to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$

The second step of our quantum reduction from classical LWE to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ involves reducing the obtained Extrapolated DCP states to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$. This step is an adaptation of the reduction from G-EDCP to LWE proposed by Brakerski et al. [BKSW18]. We give the detailed proof of Theorem 46 here.

Proof of Theorem 46. Suppose we are given an Extrapolated DCP state with the form as Equation (7). Our quantum reduction works as follows (for simplicity, we omit the normalization factors):

1. Apply Quantum Fourier Transformation on $\mathbb{Z}_q^n$ for the second register, obtaining the state
$$\sum_{\mathbf{a}\in\mathbb{Z}_q^n}\sum_{j\in\mathbb{Z}_q}\omega_q^{\langle\mathbf{a},\mathbf{v}+j\cdot\mathbf{s}\rangle}\rho_\sigma(j-c)|j\rangle|\mathbf{a}\rangle.$$

2. Measure the second register to get a particular measurement result $\hat{\mathbf{a}}$, which is randomly chosen from $\mathbb{Z}_q^n$ with a uniform distribution. By omitting the global phase term $\omega_q^{\langle\hat{\mathbf{a}},\mathbf{v}\rangle}$, the remaining state is
$$\sum_{j\in\mathbb{Z}_q}\omega_q^{\langle\hat{\mathbf{a}},j\cdot\mathbf{s}\rangle}\rho_\sigma(j-c)|j\rangle.$$

3. Apply another Quantum Fourier Transformation on $\mathbb{Z}_q$ and incorporate Gaussian tails of j again, obtaining a state $2^{-\Omega(n)}$-close to the state
$$\sum_{b\in\mathbb{Z}_q}\sum_{j\in\mathbb{Z}}\omega_q^{j(\langle\hat{\mathbf{a}},\mathbf{s}\rangle+b)}\rho_\sigma(j-c)|b\rangle.$$

4. Use the Poisson summation formula on the amplitude and change the summation variable to $e \leftarrow \langle\hat{\mathbf{a}},\mathbf{s}\rangle + b - q\cdot j$, this state can be rewritten as
$$\begin{aligned}&\sum_{b\in\mathbb{Z}_q}\sum_{j\in\mathbb{Z}}\omega_q^{j(\langle\hat{\mathbf{a}},\mathbf{s}\rangle+b)}\rho_\sigma(j-c)|b\rangle\\ =&\sum_{b\in\mathbb{Z}_q}\sum_{j\in\mathbb{Z}}\sigma\rho_{1/\sigma}\left(j-\frac{\langle\hat{\mathbf{a}},\mathbf{s}\rangle+b}{q}\right)\cdot\exp\left(-2\pi\mathrm{i}\cdot c\left(j-\frac{\langle\hat{\mathbf{a}},\mathbf{s}\rangle+b}{q}\right)\right)|b\rangle\\ \propto&\sum_{e\in\mathbb{Z}}\rho_{q/\sigma}(e)\cdot\exp(2\pi\mathrm{i}\cdot ce/q)|(\langle-\hat{\mathbf{a}},\mathbf{s}\rangle+e)\bmod q\rangle\end{aligned}$$

Finally, this state along with the classical vector $-\hat{\mathbf{a}}$ will be the output $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ instance of our quantum reduction. □

5.3 The distribution of unknown center

For additional technical insights, we present a more detailed analysis of the distribution of center c in the Extrapolated DCP states (see Equation (7)) we get. To achieve this, we begin by examining the distribution of $\mathbf{x}$ after measurement on the third register of the state given in Equation (12). It is evident that the probability of obtaining a specific vector $\mathbf{x}$ is proportional to

$$\begin{aligned}\Pr(\mathbf{x}) &\propto \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2 \rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e})^2 \\ &\approx \sum_{j\in\mathbb{Z}} \rho_\alpha(j)^2 \rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e})^2 \\ &= \sum_{j\in\mathbb{Z}} \exp\left[-2\pi\left(\frac{j^2}{\alpha^2}+\frac{j^2\|\mathbf{e}\|^2+2j\langle\mathbf{x},\mathbf{e}\rangle+\|\mathbf{x}\|^2}{\beta^2q^2}\right)\right] \\ &= \sum_{j\in\mathbb{Z}} \rho_{\sigma/\sqrt{2}}(j-c)\cdot\exp\left[-2\pi\left(\frac{\|\mathbf{x}\|^2}{\beta^2q^2}-\frac{\alpha^2\langle\mathbf{x},\mathbf{e}\rangle^2}{\beta^2q^2(\alpha^2\|\mathbf{e}\|^2+\beta^2q^2)}\right)\right].\end{aligned}$$

We observe that $\sigma = \frac{\alpha\beta q}{\sqrt{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}} > \frac{\alpha}{\sqrt{2}} = \Omega(\sqrt{n})$, which implies that almost all of the weight of $\rho_{\sqrt{2}/\sigma}$ is concentrated at $\rho_{\sqrt{2}/\sigma}(0)$ with exponentially small weight elsewhere. Using the Poisson summation formula, we get that

$$\sum_{j\in\mathbb{Z}} \rho_{\sigma/\sqrt{2}}(j-c) = \sum_{j\in\mathbb{Z}} \frac{\sigma}{\sqrt{2}}\rho_{\sqrt{2}/\sigma}(j)\cdot\exp(-2\pi\mathrm{i}\cdot cj) \in \frac{\sigma}{\sqrt{2}}(1\pm 2^{-\Omega(n)}).$$

Let us denote $\mathbf{\Sigma} = \left(\mathbf{I}_m - \frac{\alpha^2}{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}\mathbf{e}\mathbf{e}^T\right)^{-1} = \mathbf{I}_m + \frac{\alpha^2}{\beta^2q^2}\mathbf{e}\mathbf{e}^T$, the remaining term can be written as

$$\begin{aligned}\exp\left[-2\pi\left(\frac{\|\mathbf{x}\|^2}{\beta^2q^2}-\frac{\alpha^2\langle\mathbf{x},\mathbf{e}\rangle^2}{\beta^2q^2(\alpha^2\|\mathbf{e}\|^2+\beta^2q^2)}\right)\right] &= \exp\left[-2\pi\cdot\frac{1}{\beta^2q^2}(\mathbf{x})^T\left(\mathbf{I}_m-\frac{\alpha^2}{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}\mathbf{e}\mathbf{e}^T\right)\mathbf{x}\right] \\ &= \exp\left[-2\pi\cdot\frac{1}{\beta^2q^2}(\mathbf{x})^T\mathbf{\Sigma}^{-1}\mathbf{x}\right] \\ &= \rho_{\beta q\sqrt{\mathbf{\Sigma}/2}}(\mathbf{x}).\end{aligned}$$

This means that the distribution of $\mathbf{x}$ follows the discrete Gaussian distribution with center $\mathbf{0}$ and covariance matrix $\beta^2q^2\mathbf{\Sigma}/2$. Correspondingly, the distribution of $\mathbf{y}\in\mathbb{Z}_q^m\cap B_{\mathcal{L}_q(\mathbf{A})}$ is given by $\Pr(\mathbf{y}) = \rho_{\beta q\sqrt{\mathbf{\Sigma}/2}}(\mathbf{x})$.

To derive the distribution of the unknown center $c = \frac{\alpha^2\langle\mathbf{x},\mathbf{e}\rangle}{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}$, we observe that the distribution of $\mathbf{x}$ is smooth enough to be treated as a continuous Gaussian distribution since the eigenvalues of $\beta^2q^2\mathbf{\Sigma}/2$ are $\beta^2q^2/2$ and $(\beta^2q^2+\alpha^2\|\mathbf{e}\|^2)/2$. So the distribution of the unknown center c can be approximated by the discrete Gaussian distribution with minimum step $\frac{\alpha^2}{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}$, center 0 and variance

$$\sigma_c^2 = \left(\frac{\alpha^2}{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}\right)^2\cdot\mathbf{e}^T\left(\beta^2q^2\mathbf{\Sigma}/2\right)\mathbf{e} = \frac{\alpha^4\|\mathbf{e}\|^2}{2(\alpha^2\|\mathbf{e}\|^2+\beta^2q^2)}.$$

In conclusion, we propose the following statement for the distribution of the unknown center in the Extrapolated DCP states:

Theorem 53. *The distribution of c in Equation (7) of Theorem 45 approximately follows the discrete Gaussian distribution $D_{\frac{\alpha^2}{\alpha^2\|\mathbf{e}\|^2+\beta^2q^2}\mathbb{Z},\sigma_c}$ where $\sigma_c = \frac{\alpha^2\|\mathbf{e}\|}{\sqrt{2(\alpha^2\|\mathbf{e}\|^2+\beta^2q^2)}}$.*

Remark 54. *As readers may notice, the Gaussian width σ of j and the Gaussian width σ_c of c (the center of the distribution of j) satisfy $\sigma_c = \frac{\alpha\|\mathbf{e}\|}{\sqrt{2}\beta q}\sigma$. In our settings, if we assume $\beta q \gg \alpha \cdot \|\mathbf{e}\|$, then the distribution of j is a discrete Gaussian distribution with a small shift. However, this shift is non-negligible, preventing our $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ state from being exponentially close to a $\mathsf{S}|\mathsf{LWE}\rangle$ state without unknown phase.*

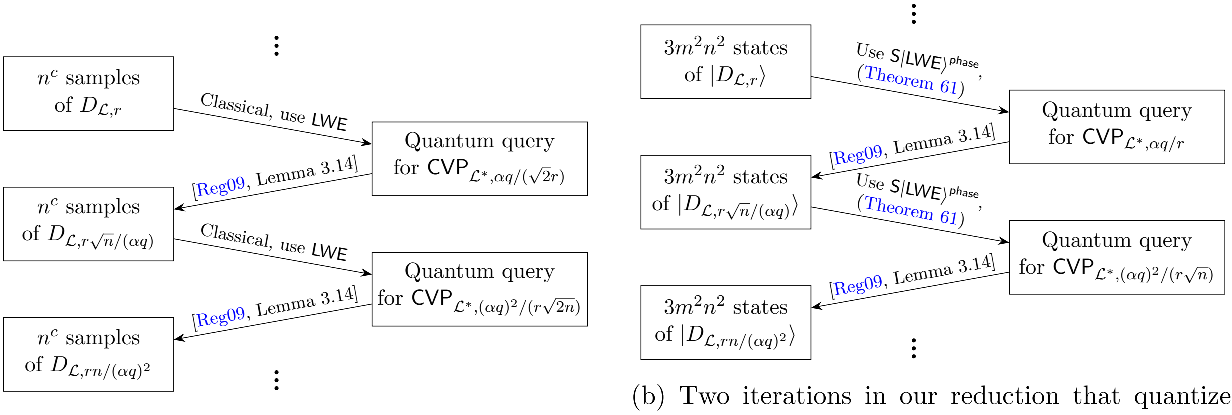
6 Hardness of $\mathsf{S}|\mathsf{LWE}\rangle$ with Unknown Phase via Quantizing Regev's Iterative Reduction

In this section, we show how to reduce from the problem of generating discrete Gaussian states ($|\mathsf{DGS}\rangle$, Definition 23) to a variant of $\mathsf{S}|\mathsf{LWE}\rangle$ with an unknown phase term on the amplitude ($\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$), by modifying Regev's iterative reduction [Reg09] from the problem of generating discrete Gaussian samples (DGS, Definition 22) to LWE. Combined with the known reductions from GapSVP and SIVP to DGS in Lemma 24 and Lemma 25, it gives a quantum reduction from $\mathsf{GapSVP}_{\tilde{O}(n^{1.5})}$ and $\mathsf{SIVP}_{\tilde{O}(n^{1.5})}$ to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$.

6.1 Overview of our reduction

As in [Reg09], our proof is iterative. We start from generating discrete Gaussian states with exponentially large widths (in Regev's reduction, it is classical discrete Gaussian sample with exponentially large widths; both can be done efficiently). Then, each iteration produces discrete Gaussian states (samples) with smaller widths. Repeating the iteration for polynomial number of times gives the discrete Gaussian states (samples) for the $|\mathsf{DGS}\rangle$ (DGS) problem. We illustrate Regev's reduction in Figure 2a, aligned with our reduction in Figure 2b, and then explain with more details.

In order to quantize Regev's iterative reduction, we focus on quantizing the only classical step in the reduction – solving CVP. Roughly speaking, given a CVP instance $\mathbf{x}$, Regev [Reg09] utilizes LWE oracle to solve CVP by feeding it with samples $\mathbf{a} := \mathcal{L}^{-1}\mathbf{v} \bmod q$ and $\langle \mathbf{x}, \mathbf{v}\rangle \bmod q$ where $\mathbf{v} \leftarrow D_{\mathcal{L},r}$, which are close to the LWE sample $\mathbf{a}$ and $\langle \mathbf{a}, \mathbf{s}\rangle + e \bmod q$ where $\mathbf{s} = (\mathcal{L}^*)^{-1}\kappa_{\mathcal{L}^*}(\mathbf{x}) \bmod q$ and e is sampled from the Gaussian distribution. To quantize this step, a natural idea is to replace the classical $\mathbf{v}$ with a superposition state of Gaussian samples $\sum_{\mathbf{v}\in\mathcal{L}} \rho_r(\mathbf{v})|\mathbf{v}\rangle$, measure $\mathbf{a} = \mathcal{L}^{-1}\mathbf{v} \bmod q$, and compute $\langle \mathbf{x}, \mathbf{v}\rangle \bmod q$ in another register, hoping that the register contains an $\mathsf{S}|\mathsf{LWE}\rangle$ state. However, we should be careful to make sure that the $\mathbf{v}$ register does not collapse to a classical $\mathbf{v}$. Our solution is to measure the $\mathbf{v}$ register in Fourier basis, which can ensure that each $\mathbf{v} \in q\mathcal{L} + \mathcal{L}\mathbf{a}$ appears in the amplitude of the $\langle \mathbf{x}, \mathbf{v}\rangle$ register. But it also inevitably introduces a phase term that we are unable to compute efficiently from the measurement results. The above discussion ignores the Gaussian distribution to smooth the error distribution. More details can be found in Section 6.2.



n^c samples of $D_{\mathcal{L},r}$
Classical, use LWE
Quantum query for $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/(\sqrt{2}r)}$
[Reg09, Lemma 3.14]
n^c samples of $D_{\mathcal{L},r\sqrt{n}/(\alpha q)}$
Classical, use LWE
Quantum query for $\mathsf{CVP}_{\mathcal{L}^*,(\alpha q)^2/(r\sqrt{2n})}$
[Reg09, Lemma 3.14]
n^c samples of $D_{\mathcal{L},rn/(\alpha q)^2}$
$3m^2n^2$ states of $|D_{\mathcal{L},r}\rangle$
Use $\mathsf{S|LWE\rangle^{phase}}$, (Theorem 61)
Quantum query for $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$
[Reg09, Lemma 3.14]
$3m^2n^2$ states of $|D_{\mathcal{L},r\sqrt{n}/(\alpha q)}\rangle$
Use $\mathsf{S|LWE\rangle^{phase}}$, (Theorem 61)
Quantum query for $\mathsf{CVP}_{\mathcal{L}^*,(\alpha q)^2/(r\sqrt{n})}$
[Reg09, Lemma 3.14]
$3m^2n^2$ states of $|D_{\mathcal{L},rn/(\alpha q)^2}\rangle$

(a) Two iterations in Regev's reduction [Reg09].

(b) Two iterations in our reduction that quantizes Regev's reduction.

Figure 2: The correspondence between Regev's reduction (from DGS to LWE) and our reduction (from $|\mathsf{DGS}\rangle$ to $\mathsf{S|LWE\rangle^{phase}}$).

The above described reduction leads us to requiring an $\mathsf{S|LWE\rangle^{phase}}$ oracle with specific parameter. We first formally define our special parameters and functions, and propose the main theorem for this section here:

Definition 55. *Let $\mathcal{L}$ be an n-dimensional integer lattice. Given parameters $q, R \in \mathbb{N}^+$ such that $R \in 2^{\mathsf{poly}(n)}$, $\alpha, r \in \mathbb{R}^+$ and a vector $\mathbf{x} \in \mathbb{R}^n$ such that $\operatorname{dist}(\mathbf{x}, \mathcal{L}^*) \le \lambda_1(\mathcal{L}^*)/2$, we define*

1. *An amplitude function $f : \mathbb{Z}_{qR}/R \to \mathbb{R}$ with $f(e) = \rho_{\sqrt{2}\alpha q}(e)$ which is completely known.*
2. *A family of distribution $D_\theta^{(r,\mathbf{x})}$ over $\mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}$ parameterized by $(r, \mathbf{x})$ and given by $\Pr(\mathbf{y}) \propto \rho_{\sqrt{\mathbf{\Sigma}/2}}(\mathbf{z}(\mathbf{y}))$ where $\mathbf{\Sigma} := \frac{\mathbf{I}}{r^2} + \frac{\mathbf{x}'\mathbf{x}'^T}{2\alpha^2 q^2 - r^2\|\mathbf{x}'\|^2}$, $\mathbf{x}' := \mathbf{x} - \kappa_{\mathcal{L}^*}(\mathbf{x})$ and $\mathbf{z}(\mathbf{y}) := \mathbf{y}/R - \kappa_{(q\mathcal{L})^*}(\mathbf{y}/R)$.*
3. *A family of phase function $\theta^{(r,\mathbf{x})} : \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*} \to \mathbb{R}$ parameterized by $(r, \mathbf{x})$ with $\theta^{(r,\mathbf{x})}(\mathbf{y}) = \frac{r^2\langle \mathbf{x}', \mathbf{z}(\mathbf{y})\rangle}{2\alpha^2 q^2}$, where $\mathbf{x}' = \mathbf{x} - \kappa_{\mathcal{L}^*}(\mathbf{x})$ and $\mathbf{z}(\mathbf{y}) = \mathbf{y}/R - \kappa_{(q\mathcal{L})^*}(\mathbf{y}/R)$. This function is not known to be efficiently computable since it requires to solve approximate CVP.*

Theorem 56 (Main theorem, from $|\mathsf{DGS}\rangle$ to $\mathsf{S|LWE\rangle^{phase}}$)**.** *Let $\mathcal{L}$ be an n-dimensional integer lattice. Let $\epsilon = \epsilon(n)$ be a negligible function such that $\epsilon(n) < 2^{-n}$, $q = q(n) > 10n$ be an integer of at most $\mathsf{poly}(n)$ bits, $\alpha \in (0, \frac{1}{5\sqrt{n}})$ such that $\alpha q > 2\sqrt{n}$, $R = R(n)$ be an exponentially large integer such that $R > \max\{2^{2n+2} n\lambda_n(\mathcal{L})^2, \frac{2^{4n+1}\sqrt{2}n\lambda_n(\mathcal{L}^*)\lambda_n(\mathcal{L})}{\alpha q}, 2^{3n}\lambda_n(\mathcal{L}^*)\}$. Let $r_0 > 4\sqrt{n}\eta_\epsilon(\mathcal{L})/\alpha$ be the width parameter of the $|\mathsf{DGS}\rangle$ problem.*

Assume there exists quantum algorithms that can solve $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,m,q,f,\theta^{(r,\mathbf{x})},D_\theta^{(r,\mathbf{x})}}$ for any choice of pair $(r, \mathbf{x})$ such that $\alpha q r_0/\sqrt{n} < r < 2^{2n}\sqrt{2}\lambda_n(\mathcal{L})$, $\mathbf{x} \in \mathcal{L}^*/R$ and $\operatorname{dist}(\mathbf{x}, \mathcal{L}^*) \le \alpha q/r$, with $m = 2^{o(n)}$ samples and in time complexity T. Then there exists a quantum algorithm that can generate a state that is $2^{-\Omega(n)}$-close to the discrete Gaussian state $|D_{\mathcal{L},r_0}\rangle = \sum_{\mathbf{v}\in\mathcal{L}} \rho_{r_0}(\mathbf{v})|\mathbf{v}\rangle$ in time complexity $O((m^4 + m^3 T)\mathsf{poly}(n))$.

Then the $|\mathsf{DGS}\rangle$ problem is easily reduced to either GapSVP or SIVP. The connection to GapSVP and SIVP is a Corollary of Theorem 56 and Lemmas 15, 16, 24, and 25.

Corollary 57. *Under the same assumption used in Theorem 56, there exists quantum algorithms for solving* GapSVP_γ *and* SIVP_γ *for* $\gamma \in \tilde{O}(n/\alpha)$ *in time complexity* $\mathsf{poly}(n, m, T)$.

Remark 58. *Readers may think the assumption of Theorem 56 looks too strong because the family of phase functions* $\{\theta^{(r,\mathbf{x})}\}$ *is a very large family. However, we know the absolute value of* $\theta^{(r,\mathbf{x})}(\mathbf{y})$ *is small with high probability, when* $(r, \mathbf{x})$ *follows the setting in Theorem 56 and* $\mathbf{y}$ *is sampled from the corresponding distribution* $D_\theta^{(r,\mathbf{x})}$. *This is because when* $\mathbf{y} \leftarrow D_\theta^{(r,\mathbf{x})}$, $\mathbf{z}(\mathbf{y}) = \mathbf{y}/R - \kappa_{(q\mathcal{L})^*}(\mathbf{y}/R)$ *follows the distribution* $\rho_{\sqrt{\mathbf{\Sigma}/2}}(\mathbf{z}(\mathbf{y}))$ *over support* $\mathbb{Z}_R^n/R \cap B_{(q\mathcal{L})^*}$ *and thus has* ℓ_2 *norm at most* $\frac{\sqrt{n}\alpha q}{r\sqrt{2\alpha^2 q^2 - r^2\|\mathbf{x}'\|^2}}$ *with* $1 - 2^{-\Omega(n)}$ *probability. Then* $\left|\theta^{(r,\mathbf{x})}(\mathbf{y})\right| \leq \frac{\sqrt{n}}{2\alpha q}$ *with* $1 - 2^{-\Omega(n)}$ *probability. As our algorithm in Section 3 can solve the problem in sub-exponential time if* $\theta^{(r,\mathbf{x})}(\mathbf{y})$ *is always zero, there might be a way to solve the problem when* $\theta^{(r,\mathbf{x})}(\mathbf{y})$ *is close to zero.*

In what follows, we will display the idea of our proof for Theorem 56, which will be an expansion of the proof idea described earlier with Figure 2. As is described before, our proof is iterative. We start from generating discrete Gaussian states with exponentially large widths. Then, equipped with an $\mathsf{S|LWE\rangle^{phase}}$ solver, we iteratively generate discrete Gaussian states with smaller widths in each step. Repeating the iterative step for polynomial number of times gives the discrete Gaussian states we desired. Formally, our initialization and iterative steps are:

Theorem 59 (The initialization step, [Reg09, Lemma 3.12])**.** *There exists an efficient quantum algorithm that given any* n*-dimensional integer lattice* $\mathcal{L}$ *and width* $r > 2^{2n}\sqrt{2}\lambda_n(\mathcal{L})$, *output a state that is* $2^{-\Omega(n)}$*-close to the state* $|D_{\mathcal{L},r}\rangle = \sum_{\mathbf{v}\in\mathcal{L}} \rho_r(\mathbf{v})$.

Theorem 60 (The iterative step)**.** *Let* $\mathcal{L}$ *be an* n*-dimensional integer lattice,* $q > 2$ *be an integer. Define the parameters* $\epsilon \in (0, 2^{-n})$, $\alpha \in (0, \frac{1}{5\sqrt{n}})$, $r > 4q\eta_\epsilon(\mathcal{L})$, *and a precision parameter* $R > \max\{2\sqrt{n}r\sqrt{\log r}, \frac{2\sqrt{n}}{\alpha q}, \frac{2^{2n+1}nr\lambda_n(\mathcal{L}^*)}{\alpha q}, 2^{3n}\lambda_n(\mathcal{L}^*)\}$ *as an integer.*

Assume that there exists a quantum algorithm that solves $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,m,q,f,\theta^{(r,\mathbf{x})},D_\theta^{(r,\mathbf{x})}}$ *for any* $\mathbf{x} \in \mathcal{L}^*/R$ *with* $\mathrm{dist}(\mathbf{x}, \mathcal{L}^*) < \alpha q/r$ *in time complexity* T. *Then there exists a quantum algorithm that, given* $3m^2n^2$ *discrete Gaussian states* $|D_{\mathcal{L},r}\rangle = \sum_{\mathbf{v}\in\mathcal{L}} \rho_r(\mathbf{v})|\mathbf{v}\rangle$, *produces* $3m^2n^2$ *discrete Gaussian states that are* $2^{-\Omega(n)}$*-close to* $|D_{\mathcal{L},r\sqrt{n}/\alpha q}\rangle$, *in time complexity* $O((m^4 + m^3T)\mathsf{poly}(n))$.

The iterative step consists of two steps:

Step 1 Given a CVP instance, we can use a collection of discrete Gaussian states $|D_{\mathcal{L},r}\rangle$ to construct an $\mathsf{S|LWE\rangle^{phase}}$ instance. Solving the $\mathsf{S|LWE\rangle^{phase}}$ instance will in return solve the CVP problem. More precisely, we show the following theorem:

Theorem 61. *Let* $\mathcal{L}$ *be an* n*-dimensional integer lattice, define the parameters* $\epsilon \in (0, 2^{-n})$, $\alpha \in (0, \frac{1}{5\sqrt{n}})$, $r > 4q\eta_\epsilon(\mathcal{L})$, *and a precision parameter* $R > \max\{2\sqrt{n}r\sqrt{\log r}, \frac{2\sqrt{n}}{\alpha q}, \frac{2^{2n+1}nr\lambda_n(\mathcal{L}^*)}{\alpha q}\}$ *as an integer.*

Assume that there exists an quantum algorithm that solves $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,m,q,f,\theta^{(r,\mathbf{x})},D_\theta^{(r,\mathbf{x})}}$ *for any* $\mathbf{x} \in \mathcal{L}^*/R$ *with* $\mathrm{dist}(\mathbf{x}, \mathcal{L}^*) < \alpha q/r$ *in time complexity* T. *Then there exists a quantum algorithm that, given* $3m^2n^2$ *discrete Gaussian states* $|D_{\mathcal{L},r}\rangle = \sum_{\mathbf{v}\in\mathcal{L}} \rho_r(\mathbf{v})|\mathbf{v}\rangle$, *answers quantum query to* $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$ *on the support* $\mathcal{L}^*/R$ *(denoted by* $|\mathbf{x}, \mathbf{y}\rangle \to |\mathbf{x}, \mathbf{y} + \kappa_{\mathcal{L}^*}(\mathbf{x})\rangle$ *with* $\mathbf{x} \in \mathcal{L}^*/R$ *such that* $\mathrm{dist}(\mathbf{x}, \mathcal{L}^*) \leq \alpha q/r$*) up to exponentially small error, with exponentially small disturbance to the states* $|D_{\mathcal{L},r}\rangle$, *and in time* $O((m^2 + mT)\mathsf{poly}(n))$.

Step 2 (Same as the quantum step in Regev's reduction) A query to the CVP oracle can help to generate a discrete Gaussian state with a smaller width. More precisely:

Theorem 62 ([Reg09, Lemma 3.14])**.** *There exists an efficient quantum algorithm that, given any n-dimensional lattice $\mathcal{L}$, a number $d < \lambda_1(\mathcal{L}^*)/2$ and an integer $R > 2^{3n}\lambda_n(\mathcal{L}^*)$, outputs $|D_{\mathcal{L},\sqrt{n}/d}\rangle = \sum_{\mathbf{v}\in\mathcal{L}} \rho_{\sqrt{n}/d}(\mathbf{v})|\mathbf{v}\rangle$, with only one quantum query on the second register of state*

$$\sum_{\mathbf{x}\in\mathcal{L}^*/R, \|\mathbf{x}\|\le d} \rho_{d/\sqrt{n}}(\mathbf{x})|\mathbf{x}, \mathbf{x} \bmod \mathcal{P}(\mathcal{L}^*)\rangle,$$

to the $\mathsf{CVP}_{\mathcal{L}^*,d}$ oracle, which is on the support $\mathcal{L}^*/R$.

The full picture of the proof for the main reduction Theorem 56 is illustrated in Figure 3. The proof starts with the initial step (Theorem 59) then applies the iterative step (Theorem 60) for $\mathsf{poly}(n)$ times. The iterative step consists of two parts, first the construction of CVP oracle (Theorem 61) with the help of discrete Gaussian states from the previous iteration and the help of an $\mathsf{S|LWE\rangle^{phase}}$ solver, and then the generation of a narrower discrete Gaussian state through one query to the CVP oracle (Theorem 62). Since the discrete Gaussian states generated in the previous iteration are only disturbed by an exponentially small amount upon each query of the CVP oracle, we can reuse them for $3m^2n^2$ times to construct $3m^2n^2$ narrower discrete Gaussian states for use of the next iteration.

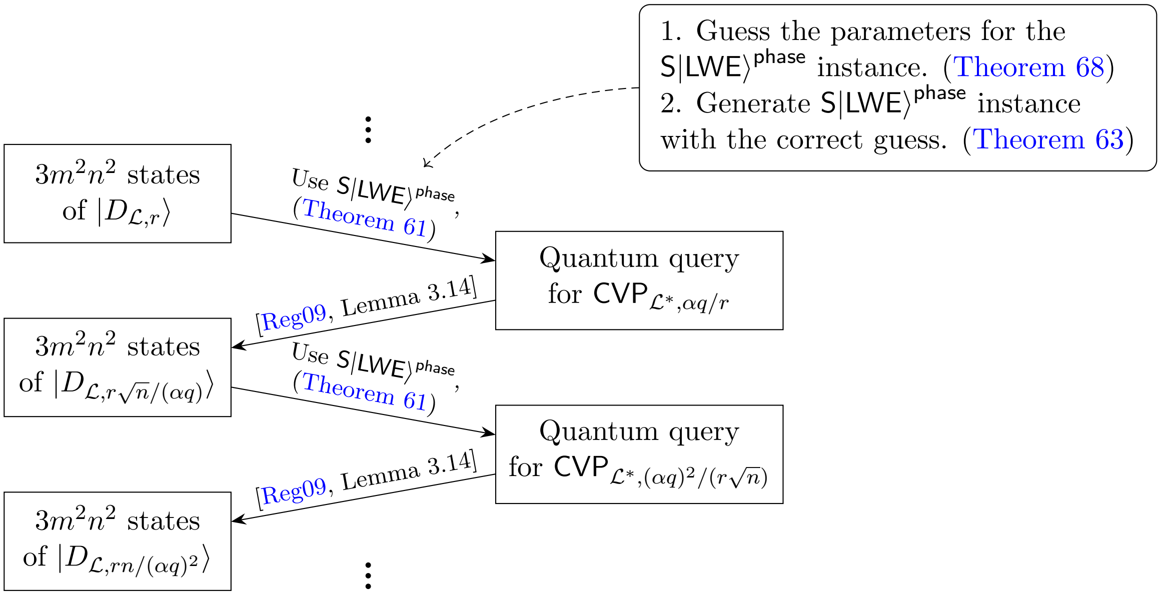

1. Guess the parameters for the $\mathsf{S|LWE\rangle^{phase}}$ instance. (Theorem 68)
2. Generate $\mathsf{S|LWE\rangle^{phase}}$ instance with the correct guess. (Theorem 63)
$3m^2n^2$ states of $|D_{\mathcal{L},r}\rangle$
Use $\mathsf{S|LWE\rangle^{phase}}$, (Theorem 61)
Quantum query for $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$
[Reg09, Lemma 3.14]
$3m^2n^2$ states of $|D_{\mathcal{L},r\sqrt{n}/(\alpha q)}\rangle$
Use $\mathsf{S|LWE\rangle^{phase}}$, (Theorem 61)
Quantum query for $\mathsf{CVP}_{\mathcal{L}^*,(\alpha q)^2/(r\sqrt{n})}$
[Reg09, Lemma 3.14]
$3m^2n^2$ states of $|D_{\mathcal{L},rn/(\alpha q)^2}\rangle$

Figure 3: Illustration of two iterations of the reduction algorithm.

It is left to prove Theorem 61, to which the remainder of this section will be devoted. To prove it, we start by generating an $\mathsf{S|LWE\rangle^{phase}}$ instance but with an unknown Gaussian width, instead of the fixed and known width $\sqrt{2}\alpha q$ in the $\mathsf{S|LWE\rangle^{phase}}$ solver, as displayed in Theorem 63 in Section 6.2. We then address and resolve the issue of the unknown width in order to solve the CVP instance using the $\mathsf{S|LWE\rangle^{phase}}$ oracle, in the proof of Theorem 68 in Section 6.3. Finally, we note that the procedure in Theorem 68 answers the CVP quantum query with $1 - 2^{-\Omega(n)}$ probability.

Therefore using the idea of gentle measurement, we can answer each CVP quantum query with exponentially small disturbance to the states $|D_{\mathcal{L},r}\rangle$, as discussed in Section 6.4.

6.2 Generating the $\mathsf{S|LWE}\rangle^{\mathsf{phase}}$ samples

In this subsection, we show how to create the $\mathsf{S|LWE}\rangle^{\mathsf{phase}}$ instance for Theorem 61 but with an unknown Gaussian width. Given a $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$ instance, the idea is to replace the classical Gaussian samples from $D_{\mathcal{L},r}$ in [Reg09] (that helps to produce the LWE instance) with a superposition state of Gaussian samples $|D_{\mathcal{L},r}\rangle := \sum_{\mathbf{v}\in\mathcal{L}} \rho_r(\mathbf{v})|\mathbf{v}\rangle$ that helps to produce the $\mathsf{S|LWE}\rangle^{\mathsf{phase}}$ instance.

Theorem 63. *Let $\mathcal{L}$ be an n-dimensional integer lattice, define the parameters $\epsilon \in (0, 2^{-n})$, $\alpha \in (0, \frac{1}{5\sqrt{n}})$, $\sigma \in [\alpha q, \sqrt{2}\alpha q]$, $r > 4q\eta_\epsilon(\mathcal{L})$, and a precision parameter $R > 2\sqrt{n}r\sqrt{\log r}$ as an integer. Given a $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$ instance $\mathbf{x} \in \mathcal{L}^*/R$ and a state $|D_{\mathcal{L},r}\rangle$ as input, there exists an efficient quantum algorithm that generates a random vector $\mathbf{a} \leftarrow \mathcal{U}(\mathbb{Z}_q^n)$ and a state $2^{-\Omega(n)}$-close to the following state*

$$\gamma_t^{\mathbf{a}} = \sum_{\mathbf{y}\in\mathbb{Z}_R^n \cap R\cdot B_{(q\mathcal{L})^*}} \rho_{\sqrt{\mathbf{\Sigma}/2}}(\mathbf{z}(\mathbf{y}))|\mathbf{y}\rangle\langle\mathbf{y}| \otimes |\psi_t^{\mathbf{a},\mathbf{y}}\rangle\langle\psi_t^{\mathbf{a},\mathbf{y}}|$$

where $t = \sqrt{\sigma^2 + r^2\|\mathbf{x}'\|^2}$, $\mathbf{s} = (\mathcal{L}^*)^{-1}\kappa_{\mathcal{L}^*}(\mathbf{x}) \bmod q$, $\mathbf{x}' = \mathbf{x} - \kappa_{\mathcal{L}^*}(\mathbf{x})$, $\mathbf{z}(\mathbf{y}) = \mathbf{y}/R - \kappa_{(q\mathcal{L})^*}(\mathbf{y}/R)$, $\mathbf{\Sigma} = \frac{\mathbf{I}_n}{r^2} + \frac{\mathbf{x}'\mathbf{x}'^T}{\sigma^2}$, and the state $|\psi_t^{\mathbf{a},\mathbf{y}}\rangle$ is an $\mathsf{S|LWE}\rangle^{\mathsf{phase}}$ state

$$|\psi_t^{\mathbf{a},\mathbf{y}}\rangle := \sum_{u\in\mathbb{Z}_{qR}/R} \rho_t(u) \exp\left(2\pi \mathrm{i} \cdot u \frac{r^2\langle\mathbf{x}', \mathbf{z}(\mathbf{y})\rangle}{t^2}\right) |\langle\mathbf{s}, \mathbf{a}\rangle + u \bmod q\rangle. \tag{13}$$

Proof. From now on, when it is clear from the context, we use $\mathbf{z}$ to denote the value $\mathbf{z}(\mathbf{y})$, which actually depends on $\mathbf{y}$.

Here is the procedure of our quantum algorithm. For simplicity, we ignore the normalization factors when writing down superposition states.

1. Prepare the initial state

$$|D_{\mathcal{L},r}\rangle \otimes \sum_{e\in\mathbb{Z}_{qR}/R} \rho_\sigma(e)|e\rangle,$$

   which is $2^{-\Omega(n)}$-close to $\sum_{\mathbf{v}\in\mathcal{L},\|\mathbf{v}\|\le\sqrt{n}r} \rho_r(\mathbf{v})|\mathbf{v}\rangle \otimes \sum_{e\in\mathbb{Z}_{qR}/R} \rho_\sigma(e)|e\rangle$ by Banaszczyk's Gaussian tail bound.

2. Measure $\mathbf{a} := \mathcal{L}^{-1}\mathbf{v} \bmod q$ to get an outcome $\mathbf{a}$ and a result state $2^{-\Omega(n)}$-close to

$$\sum_{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\|\mathbf{v}\|\le\sqrt{n}r} \rho_r(\mathbf{v})|\mathbf{v}\rangle \otimes \sum_{e\in\mathbb{Z}_{qR}/R} \rho_\sigma(e)|e\rangle$$

   According to Lemma 17, when $r/\sqrt{2} > q\eta_\epsilon(\mathcal{L})$, the distribution of $\mathbf{a}$ is $2^{-\Omega(n)}$-close to uniform.

3. Apply a unitary to add the inner product $\langle\mathbf{x}, \mathbf{v}\rangle \bmod q$ to the last register and get

$$\sum_{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\|\mathbf{v}\|\le\sqrt{n}r} \rho_r(\mathbf{v})|\mathbf{v}\rangle \otimes \sum_{e\in\mathbb{Z}_{qR}/R} \rho_\sigma(e)|\langle\mathbf{s}, \mathbf{a}\rangle + \langle\mathbf{x}', \mathbf{v}\rangle + e \bmod q\rangle. \tag{14}$$

Note that since we assumed $\mathbf{x} \in \mathcal{L}^*/R$, the second register always has its value in $\mathbb{Z}_{qR}/R$.

Intuitively, since $u := \langle \mathbf{x}', \mathbf{v} \rangle + e \leq \alpha q\sqrt{n} + \sigma\sqrt{n} < q/2$ with high probability and $R > 2\sqrt{n}r$, Equation (14) should be $2^{-\Omega(n)}$-close to the following state displayed in Equation (15). The proof is deferred to Appendix A.3.

Lemma 64. *Suppose that $q/2 > \alpha q\sqrt{n} + \sigma\sqrt{n}$, $R > 2\sqrt{n}r\sqrt{\log r}$ and $r > \sqrt{2}q\eta_\epsilon(\mathcal{L})$ for $\epsilon < 2^{-n}$, then the state in Equation (14) is $2^{-\Omega(n)}$-close to the state*

$$\sum_{\mathbf{v} \in q\mathcal{L}+\mathcal{L}\mathbf{a}} \rho_r(\mathbf{v}) |\mathbf{v} \bmod R\rangle \otimes \sum_{u \in \mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v} \rangle) |\langle \mathbf{s}, \mathbf{a} \rangle + u \bmod q\rangle. \tag{15}$$

Remark 65. *Similar to Lemma 50, the condition of this lemma can be relaxed to $q/2 > \alpha q\sqrt{n} + \sigma\sqrt{n}$ and $R > 2\sqrt{n}r$ if we instead use the proof technique from Claim A.5 in [Reg23], with a slight modification on the lemma statement. Either way, we have the same asymptotic values for the parameters in the end.*

By the assumption that $\sigma \leq \sqrt{2}\alpha q$, we have that $\alpha q\sqrt{n} + \sigma\sqrt{n} \leq (1+\sqrt{2})\alpha q\sqrt{n} < q/2$. Therefore, according to the lemma mentioned above, we can obtain a state that is $2^{-\Omega(n)}$-close to the state displayed in Equation (15).

4. Recall that $\omega_R = e^{2\pi \mathrm{i}/R}$. Applying QFT_R to the first register, we can get a state $2^{-\Omega(n)}$-close to

$$\sum_{\mathbf{y} \in \mathbb{Z}_R^n} \sum_{\mathbf{v} \in q\mathcal{L}+\mathcal{L}\mathbf{a}} \rho_r(\mathbf{v}) \cdot \omega_R^{\langle \mathbf{v}, \mathbf{y} \rangle} |\mathbf{y}\rangle \otimes \sum_{u \in \mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v} \rangle) |\langle \mathbf{s}, \mathbf{a} \rangle + u \bmod q\rangle. \tag{16}$$

We show that the state in Equation (16) is $2^{-\Omega(n)}$-close to the following state displayed in Equation (17). The proof is deferred to Appendix A.4.

Lemma 66. *Suppose that $\epsilon < 2^{-n}, R > 2\sqrt{n}r\sqrt{\log r}, r > 4q\eta_\epsilon(\mathcal{L})$ and $\sigma \geq \alpha q$, then the state in Equation (16) is $2^{-\Omega(n)}$-close to the state*

$$\begin{aligned} &\sum_{\mathbf{y} \in \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}} \rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{z}) \exp\left(2\pi \mathrm{i} \left\langle \mathcal{L}\mathbf{a}, \kappa_{(q\mathcal{L})^*}(\mathbf{y}/R) \right\rangle\right) |\mathbf{y}\rangle \\ &\qquad \otimes \sum_{u \in \mathbb{Z}_{qR}/R} \rho_t(u) \exp\left(2\pi \mathrm{i} \cdot u \frac{r^2 \langle \mathbf{x}', \mathbf{z} \rangle}{t^2}\right) |\langle \mathbf{s}, \mathbf{a} \rangle + u \bmod q\rangle, \end{aligned} \tag{17}$$

where $t, \mathbf{s}, \mathbf{z}, \mathbf{\Sigma}$ are specified in Theorem 63.

By the assumption that $\sigma \in [\alpha q, \sqrt{2}\alpha q]$, we have that $\alpha q\sqrt{n} + \sigma\sqrt{n} < (1+\sqrt{2})\alpha q\sqrt{n} < (1+\sqrt{2})q/5 < q/2$. Therefore, according to Lemma 66, we can obtain a state that is $2^{-\Omega(n)}$-close to the state (recall the definition of $|\psi_t^{\mathbf{a},\mathbf{y}}\rangle$ in Equation (13))

$$\sum_{\mathbf{y} \in \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}} \rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{z}) \exp\left(2\pi \mathrm{i} \left\langle \mathcal{L}\mathbf{a}, \kappa_{(q\mathcal{L})^*}(\mathbf{y}/R) \right\rangle\right) |\mathbf{y}\rangle \otimes |\psi_t^{\mathbf{a},\mathbf{y}}\rangle$$

5. Measure the first register to get a state $2^{-\Omega(n)}$-close to the state $\gamma_t^{\mathbf{a}}$.[1]

[1] We measure the first register only for a better illustration of the residual state. In fact, we'll use the gentle measurement principle to defer all measurements, as described later in Section 6.4.

Finally, combining the measurement result $\mathbf{a}$ from step 2 and the state $\gamma_t^{\mathbf{a}}$ from step 5 gives the desired sample. □

Remark 67. *An important caveat is that we should not discard the* $\mathbf{y}$ *register and hope we can solve the problem given only* $\mathbf{a}$ *and the* $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ *state* $|\psi^{\mathbf{a},\mathbf{y}}_{\sqrt{2}\alpha q}\rangle$ *whose error amplitude is Gaussian with a small phase, because such a solver is so strong that it solves* LWE *directly. This is because such a solver utilizes no information about* $\mathbf{y}$ *(the seed that generates* θ*). So it should output* $\mathbf{s}$ *given samples* $\mathbf{a}$ *and the second register after step 3 (Note that in the actual procedure the solver is given the second register after step 5, but step 4 and step 5 are both local operations acting on the first register, which should not influence the state of the second register). This leads to an algorithm that outputs* $\mathbf{s}$ *given samples* $(\mathbf{a}, \langle \mathbf{s}, \mathbf{a}\rangle + e \bmod q)$ *where* $\mathbf{a} \leftarrow \mathcal{U}_{\mathbb{Z}_q^n}$ *and* $e = \langle \mathbf{x}', \mathbf{v}\rangle$ *for* $\mathbf{v}$ *distributed according to* $D_{q\mathcal{L}+\mathcal{L}\mathbf{a},r/\sqrt{2}}$ *and a fixed vector* $\|\mathbf{x}'\| \leq \frac{\alpha q}{r}$*. As the distribution of* e *is close to a Gaussian distribution, this will give a surprising method to solve* LWE*.*

This also explains why we describe the auxiliary information $\mathbf{y}$ *carefully in Definition 7 and Definition 55 instead of discarding* $\mathbf{y}$ *and strengthening the solver in Theorem 56 to solve* $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,m,q,f,\theta,D}$ *for any unknown* θ *such that* $|\theta| \leq \frac{\sqrt{n}}{2\alpha q}$ *(see Remark 58) and an arbitrary distribution* D*.*

Given that we must use the information of $\mathbf{y}$*, one may hope to compute* $\mathbf{z}(\mathbf{y})$ *to learn something about the phase* $\theta^{(r,\mathbf{x})}(\mathbf{y}) = \frac{r^2\langle \mathbf{x}', \mathbf{z}(\mathbf{y})\rangle}{2\alpha^2 q^2}$*. However,* $\mathbf{z}(\mathbf{y})$ *has* ℓ_2 *norm roughly* $\frac{\sqrt{n}}{r}$ *(see Remark 58). So computing* $\mathbf{z}(\mathbf{y})$ *from* $\mathbf{y}$ *is a* $\mathsf{CVP}_{\mathcal{L}^*,\frac{q\sqrt{n}}{r}}$ *instance, which is even harder than the goal of this iteration (a quantum query to* $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$*). How to utilize* $\mathbf{y}$ *requires more attempts and is an important step towards our ultimate goal of solving standard* LWE *efficiently.*

6.3 Dealing with the unknown Gaussian width

Now that we know how to generate $(\mathbf{a}, \gamma_t^{\mathbf{a}})$ which resembles an $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ instance. However, the $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ solver requires instances with an error distribution of the fixed width $\sqrt{2}\alpha q$. To bridge this gap, we experiment with different values of σ to obtain a suitable width that is sufficiently close to $\sqrt{2}\alpha q$. Equipped with the $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ solver, we can in turn solve the CVP problem. We realize this idea in the proof of the following theorem:

Theorem 68. *Let* $\mathcal{L}$ *be an* n*-dimensional integer lattice, define the parameters* $\epsilon \in (0, 2^{-n})$*,* $\alpha \in (0, \frac{1}{5\sqrt{n}})$*,* $r > 4q\eta_\epsilon(\mathcal{L})$*, and a precision parameter* $R > \max\{2\sqrt{n}r\sqrt{\log r}, \frac{2\sqrt{n}}{\alpha q}\}$ *as an integer.*

Assume that there exists a quantum algorithm $\mathcal{A}$ *that, given* m *samples of independently uniformly random vector* $\mathbf{a} \in \mathbb{Z}_q^n$ *and state* $\gamma^{\mathbf{a}}_{\sqrt{2}\alpha q}$*, solves the secret vector* $\mathbf{s}$ *in time complexity* T *with probability* $1 - 2^{-\Omega(n)}$*. Then there exists a quantum algorithm that, given a* $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$ *instance* $\mathbf{x} \in \mathcal{L}^*/R$ *and* $3m^2n$ *states* $|D_{\mathcal{L},r}\rangle$ *as input, outputs* $\mathbf{s} = (\mathcal{L}^*)^{-1}\kappa_{\mathcal{L}^*}(\mathbf{x}) \bmod q$ *with probability* $1 - 2^{-\Omega(n)}$ *in time* $O((m^2 + mT)\mathsf{poly}(n))$*.*

Proof. Let $\sigma' = \sqrt{2\alpha^2 q^2 - r^2\|\mathbf{x}'\|^2}$ and $\sigma_i = \alpha q\left(1 + (\sqrt{2}-1)\frac{i}{2m}\right), t_i = \sqrt{\sigma_i^2 + r^2\|\mathbf{x}'\|^2}$ for $i = 0, 1, \cdots, 2m$. In this condition, we have $\sigma_i \in [\alpha q, \sqrt{2}\alpha q]$. Since $r\|\mathbf{x}'\| < \alpha q$, there must exist an index $0 \leq j < 2m$ such that $\sigma_j < \sigma' \leq \sigma_{j+1}$.

Then σ_j is a suitable value of σ to generate samples for the quantum algorithm $\mathcal{A}$. Formally,

Lemma 69. *The quantum algorithm $\mathcal{A}$ can solve $\mathbf{s}$ with probability at least $1/2$, when given m independent samples of vector $\mathbf{a} \in \mathbb{Z}_q^n$ and state $\gamma_{t_j}^{\mathbf{a}}$.*

Before the proof of the lemma, let's first show how to construct a CVP algorithm based on the lemma. Here is the procedure of our quantum algorithm.

1. (Generate classical LWE samples for verification of the solution) Apply Theorem 63 to n states $|D_{\mathcal{L},r}\rangle$ with $\sigma = \sigma_0$ to obtain n samples of vectors $\mathbf{a} \in \mathbb{Z}_q^n$ and states $\gamma_{t_0}^{\mathbf{a}}$. Then, measure the second register of $\gamma_{t_0}^{\mathbf{a}}$ to obtain n classical LWE samples of the form $\langle \mathbf{a}, \mathbf{s}\rangle + u \bmod q$.
2. Enumerate σ from the set $\{\sigma_i : i \in \{0, 1, \cdots, 2m-1\}\}$.
3. For each $\sigma = \sigma_i$, apply Theorem 63 to mn states $|D_{\mathcal{L},r}\rangle$ to obtain mn samples of vectors $\mathbf{a} \in \mathbb{Z}_q^n$ and states $\gamma_{t_i}^{\mathbf{a}}$ with a precision of $1 - 2^{-\Omega(n)}$.
4. Utilize the quantum algorithm $\mathcal{A}$ on a group of m samples of vectors $\mathbf{a} \in \mathbb{Z}_q^n$ and states $\gamma_{t_i}^{\mathbf{a}}$ to derive a solution $\mathbf{s}'$.
5. Employ any verification process (e.g., as proposed by Regev in [Reg09, Lemma 3.6]) with the assistance of the n classical LWE samples obtained in step 1 to check whether $\mathbf{s}' = \mathbf{s}$. If this condition holds, output $\mathbf{s}'$ and conclude the process.

This procedure will use a maximum of $2m^2n + n < 3m^2n$ samples of $|D_{\mathcal{L},r}\rangle$ and operates with a time complexity of $O((m^2 + mT)\mathsf{poly}(n))$. Moreover, it will output the correct $\mathbf{s}$ with a probability of at least $1 - 2^{-\Omega(n)}$ when $\sigma = \sigma_j$, by Lemma 69. □

Lemma 69 follows from the fact that $\gamma_{t_j}^{\mathbf{a}}$ is close to $\gamma_{\sqrt{2}\alpha q}^{\mathbf{a}}$ as t_j is close to $\sqrt{2}\alpha q$. For completeness, let's provide the formal proof here.

Proof of Lemma 69. We label the m samples to be $\{(\mathbf{a}_i, \gamma_{t_j}^{\mathbf{a}_i})\}_{i \in [m]}$. By assumption, the quantum algorithm $\mathcal{A}$ can solve $\mathbf{s}$ with probability at least $1 - 2^{-\Omega(n)}$, when given m samples $\{(\mathbf{a}_i, \gamma_{\sqrt{2}\alpha q}^{\mathbf{a}_i})\}_{i \in [m]}$. The trace distance between the states in these two different types of samples is given by

$$\begin{aligned}
\delta\left(\bigotimes_{i=1}^{m} \gamma_{t_j}^{\mathbf{a}_i}, \bigotimes_{i=1}^{m} \gamma_{\sqrt{2}\alpha q}^{\mathbf{a}_i}\right) &\le \sum_{i=1}^{m} \delta\left(\gamma_{t_j}^{\mathbf{a}_i}, \gamma_{\sqrt{2}\alpha q}^{\mathbf{a}_i}\right) \\
&\le \sum_{i=1}^{m} \delta\left(|D_{q\mathcal{L}+\mathcal{L}\mathbf{a}_i,r}\rangle \sum_{e \in \mathbb{Z}_{qR}/R} \rho_{\sigma_j}(e)|e\rangle, |D_{q\mathcal{L}+\mathcal{L}\mathbf{a}_i,r}\rangle \sum_{e \in \mathbb{Z}_{qR}/R} \rho_{\sigma'}(e)|e\rangle\right) + 2^{-\Omega(n)} \\
&= \sum_{i=1}^{m} \delta\left(\sum_{e \in \mathbb{Z}_{qR}/R} \rho_{\sigma_j}(e)|e\rangle, \sum_{e \in \mathbb{Z}_{qR}/R} \rho_{\sigma'}(e)|e\rangle\right) + 2^{-\Omega(n)} \\
&\le_{(*)} m\sqrt{\frac{(\sigma' - \sigma_j)^2}{\sigma_j^2 + \sigma'^2}}(1 + 2^{-\Omega(n)}) + 2^{-\Omega(n)} \\
&\le m \cdot \sqrt{\frac{(\alpha q/(2m))^2}{2(\alpha q)^2}}(1 + 2^{-\Omega(n)}) + 2^{-\Omega(n)} \\
&\le \frac{1}{2\sqrt{2}} + 2^{-\Omega(n)},
\end{aligned}$$

where $(*)$ is according to Lemma 28 and $\sigma_j, \sigma' \in [\alpha q, \sqrt{2}\alpha q), R > \frac{2\sqrt{n}}{\alpha q}$.

Therefore, the quantum algorithm $\mathcal{A}$ will output $\mathbf{s}$ when given $\{(\mathbf{a}_i, \gamma_{t_j}^{\mathbf{a}_i})\}_{i\in[m]}$ with probability at least

$$1 - \frac{1}{2\sqrt{2}} - 2^{-\Omega(n)} > \frac{1}{2}. \qquad \square$$

One last gap between Theorem 68 and the theorem we need to prove (Theorem 61) is that, in Theorem 68 the algorithm only outputs $\mathbf{s} = (\mathcal{L}^*)^{-1}\kappa_{\mathcal{L}^*}(\mathbf{x}) \bmod q$, which answers a modulo version of $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$ for instance $\mathbf{x} \in \mathcal{L}^*/R$. However, as [Reg09] shows, CVP is efficiently reducible to its modulo version, which closes this final gap. Formally,

Theorem 70 ([Reg09, a slight modification of Lemma 3.5])**.** *Let $\mathcal{L}$ be an n-dimensional integer lattice, define distance parameter $d \in (0, \lambda_1(\mathcal{L})/2)$, and integers $q > 2, R > \frac{2^{2n+1}n\lambda_n(\mathcal{L})}{d}$. Assume there exists an algorithm $\mathcal{A}$ that, on input $\mathbf{x} \in \mathcal{L}/R$ with the guarantee $\mathrm{dist}(\mathbf{x}, \mathcal{L}) \le d$, outputs $\mathbf{s} = \mathcal{L}^{-1}\kappa_{\mathcal{L}}(\mathbf{x}) \bmod q$ with probability $1 - 2^{-\Omega(n)}$, then there exists a polynomial-time algorithm that, on input $\mathbf{x} \in \mathcal{L}/R$ with guarantee $\mathrm{dist}(\mathbf{x}, \mathcal{L}) \le d$, outputs $\kappa_{\mathcal{L}}(\mathbf{x})$ with probability $1 - 2^{-\Omega(n)}$ using at most n calls to $\mathcal{A}$.*

Proof. It is merely the same as the proof of Lemma 3.5 in [Reg09], but with a slight modification. Compute a sequence $\mathbf{x}_1, \mathbf{x}_2, \cdots$ where $\mathbf{x}_1$ is the input $\mathbf{x}$ and $\mathbf{x}_{i+1}$ is given by the following: for $\mathbf{x}_i \in \mathcal{L}/R$, call $\mathcal{A}$ to compute $\mathbf{s}_i = \mathcal{L}^{-1}\kappa_{\mathcal{L}}(\mathbf{x}_i) \bmod q$, and then compute $\mathbf{y}_i := (\mathbf{x}_i - \mathcal{L}\mathbf{s}_i)/q$. Now that $\mathbf{y}_i \in \mathcal{L}/(Rq)$ and $\mathrm{dist}(\mathbf{y}_i, \mathcal{L}) = \mathrm{dist}(\mathbf{x}_i, \mathcal{L})/q$, we can apply Babai's nearest plane algorithm [Bab86] to find a point $\mathbf{x}_{i+1} \in \mathcal{L}/R$ such that $\mathrm{dist}(\mathbf{x}_{i+1}, \mathbf{y}_i) \le 2^n \mathrm{dist}(\mathbf{y}_i, \mathcal{L}/R) \le 2^{n-1}n\lambda_n(\mathcal{L}/R) < d/2^{n+2}$. Then $\mathrm{dist}(\mathbf{x}_{i+1}, \mathcal{L}) \le \mathrm{dist}(\mathbf{y}_i, \mathcal{L}) + \mathrm{dist}(\mathbf{x}_{i+1}, \mathbf{y}_i) < \mathrm{dist}(\mathbf{x}_i, \mathcal{L})/q + d/2^{n+2}$.

Thus $\mathrm{dist}(\mathbf{x}_i, \mathcal{L}) \le \frac{1}{q^{i-1}}(\mathrm{dist}(\mathbf{x}_1, \mathcal{L}) - dq/(2^{n+2}(q-1))) + dq/(2^{n+2}(q-1)) \le \frac{d}{q^{i-1}} + \frac{d}{2^{n+1}}$. Then $\kappa_{\mathcal{L}}(\mathbf{x}_{i+1}) = \kappa_{\mathcal{L}}(\mathbf{y}_i)$ because $\mathrm{dist}(\mathbf{y}_i, \mathcal{L}) + \mathrm{dist}(\mathbf{x}_{i+1}, \mathbf{y}_i) < \mathrm{dist}(\mathbf{x}_i, \mathcal{L})/q + d/2^{n+2} < d < \lambda_1(\mathcal{L})/2$.

After n steps, we have a point $\mathbf{x}_{n+1}$ such that $\mathrm{dist}(\mathbf{x}_{n+1}, \mathcal{L}) \le \frac{d}{q^n} + \frac{d}{2^{n+1}} < \frac{d}{2^n}$. Hence we can apply Babai's nearest plane algorithm [Bab86] to recover $\kappa_{\mathcal{L}}(\mathbf{x}_{n+1})$.

Note that $\kappa_{\mathcal{L}}(\mathbf{x}_i) = q\kappa_{\mathcal{L}}(\mathbf{y}_i) + \mathcal{L}\mathbf{s}_i = q\kappa_{\mathcal{L}}(\mathbf{x}_{i+1}) + \mathcal{L}\mathbf{s}_i$. We can recover $\kappa_{\mathcal{L}}(\mathbf{x}_1)$ step by step. Notice that each call to $\mathcal{A}$ has failure probability $2^{-\Omega(n)}$ and we use n calls to $\mathcal{A}$, so our algorithm has failure probability at most $2^{-\Omega(n)}$, which completes the proof. $\square$

As a direct corollary of Theorem 68 and Theorem 70, we can reduce CVP to $\mathsf{S}|\mathsf{LWE}\rangle^{\mathsf{phase}}$ given a sufficient number of states $|D_{\mathcal{L},r}\rangle$. Formally,

Corollary 71. *Let $\mathcal{L}$ be an n-dimensional integer lattice, define the parameters $\epsilon \in (0, 2^{-n})$, $\alpha \in (0, \frac{1}{5\sqrt{n}})$, $r > 4q\eta_\epsilon(\mathcal{L})$, and a precision parameter $R > \max\{2\sqrt{n}r\sqrt{\log r}, \frac{2\sqrt{n}}{\alpha q}, \frac{2^{2n+1}nr\lambda_n(\mathcal{L}^*)}{\alpha q}\}$ as an integer. Assume that there exists a quantum algorithm $\mathcal{A}$ that, given m samples of uniformly random vector $\mathbf{a} \in \mathbb{Z}_q^n$ and state $\gamma^{\mathbf{a}}_{\sqrt{2}\alpha q}$, solves the secret vector $\mathbf{s}$ in time complexity T. Then there exists an algorithm that given a $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$ instance $\mathbf{x} \in \mathcal{L}^*/R$ and $3m^2n^2$ states $|D_{\mathcal{L},r}\rangle$ as input, outputs $\kappa_{\mathcal{L}^*}(\mathbf{x})$ with probability $1 - 2^{-\Omega(n)}$ in time $O((m^2 + mT)\mathsf{poly}(n))$.*

6.4 Answering the quantum CVP query with small disturbance on $|D_{\mathcal{L},r}\rangle$

We are now ready to conclude the proof of Theorem 61. Corollary 71 provides a method to answer classical queries for $\mathsf{CVP}_{\mathcal{L}^*,\alpha q/r}$ on instance $\mathbf{x} \in \mathcal{L}^*/R$ with $1 - 2^{-\Omega(n)}$ probability. By deferred measurement principle and gentle measurement principle, we expect that it can help us to answer quantum query $|\mathbf{x}, \mathbf{y}\rangle \to |\mathbf{x}, \mathbf{y} + \kappa_{\mathcal{L}^*}(\mathbf{x})\rangle$ with $\mathrm{dist}(\mathbf{x}, \mathcal{L}^*) \leq \alpha q/r$, using $3m^2n^2$ states $|D_{\mathcal{L},r}\rangle$ while introducing only an exponentially small disturbance on them. However, it is important to note that the gentle measurement principle is primarily suitable for producing measurement results (which is classical) rather than answering quantum query. Therefore, we shall provide a formal proof for using gentle measurement principle to answer quantum query:

Proof of Theorem 61. The existence of the quantum algorithm $\mathcal{A}$ in Corollary 71 is a direct consequence of the assumption that a quantum algorithm can solve the $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,m,q,f,\theta^{(r,\mathbf{x})},D_\theta^{(r,\mathbf{x})}}$ problem. Specifically, let's recall the expression

$$\gamma^{\mathbf{a}}_{\sqrt{2}\alpha q} = \sum_{\mathbf{y}\in\mathbb{Z}_R^n\cap R\cdot B_{(q\mathcal{L})^*}} \rho_{\sqrt{\mathbf{\Sigma}/2}}(\mathbf{z}(\mathbf{y}))|\mathbf{y}\rangle\langle\mathbf{y}| \otimes |\psi^{\mathbf{a},\mathbf{y}}_{\sqrt{2}\alpha q}\rangle\langle\psi^{\mathbf{a},\mathbf{y}}_{\sqrt{2}\alpha q}|.$$

The quantum algorithm $\mathcal{A}$ proceeds by first measuring the result of $|\mathbf{y}\rangle$ to obtain a specific $\mathbf{y}$ and an $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ state $|\psi^{\mathbf{a},\mathbf{y}}_{\sqrt{2}\alpha q}\rangle$. The parameters and functions of this state corresponds to the parameters and functions defined in Definition 55. Subsequently, $\mathcal{A}$ applies the $\mathsf{S|LWE\rangle}^{\mathsf{phase}}_{n,m,q,f,\theta^{(r,\mathbf{x})},D_\theta^{(r,\mathbf{x})}}$ solver to compute $\mathbf{s}$.

We defer all the measurements in the algorithm in Corollary 71 including those in the $\mathsf{S|LWE\rangle}^{\mathsf{phase}}$ solver $\mathcal{A}$ to obtain a unitary $U : |\mathbf{x}\rangle|D_{\mathcal{L},r}\rangle^{\otimes(3m^2n^2)}|0^{\mathsf{Aux}}\rangle \to |\mathbf{x}\rangle|\phi_{\mathbf{x}}\rangle$, where the first register of $|\phi_{\mathbf{x}}\rangle$ contains the solution $\kappa_{\mathcal{L}^*}(\mathbf{x})$, and the size of U is $O((m^2 + mT)\mathsf{poly}(n))$. As the algorithm in Corollary 71 outputs $\kappa_{\mathcal{L}^*}(\mathbf{x})$ with probability at least $1 - 2^{-\Omega(n)}$ whenever $\mathrm{dist}(\mathbf{x}, \mathcal{L}^*) \leq \alpha q/r$ and $\mathbf{x} \in \mathcal{L}^*/R$, by gentle measurement principle [Win99], there exists state $|\phi^{\mathsf{Aux}}_{\mathbf{x}}\rangle$ such that the state $|\phi_{\mathbf{x}}\rangle$ is $2^{-\Omega(n)}$-close to the state $|\kappa_{\mathcal{L}^*}(\mathbf{x})\rangle|\phi^{\mathsf{Aux}}_{\mathbf{x}}\rangle$ in ℓ_2-norm. Our quantum algorithm answers the query $|\mathbf{x}, \mathbf{y}\rangle \to |\mathbf{x}, \mathbf{y} + \kappa_{\mathcal{L}^*}(\mathbf{x})\rangle$ with $\mathrm{dist}(\mathbf{x}, \mathcal{L}^*) \leq \alpha q/r$ using the following procedure:

1. Prepare the initial state
$$|\mathbf{x}, \mathbf{y}\rangle|D_{\mathcal{L},r}\rangle^{\otimes(3m^2n^2)}|0^{\mathsf{Aux}}\rangle.$$
2. Apply U to the registers containing $|\mathbf{x}\rangle|D_{\mathcal{L},r}\rangle^{\otimes(3m^2n^2)}|0^{\mathsf{Aux}}\rangle$ to get a state $2^{-\Omega(n)}$-close to the state
$$|\mathbf{x}, \mathbf{y}\rangle|\kappa_{\mathcal{L}^*}(\mathbf{x})\rangle|\phi^{\mathsf{Aux}}_{\mathbf{x}}\rangle$$
in ℓ_2-norm.
3. Apply a unitary to add the value of $\kappa_{\mathcal{L}^*}(\mathbf{x})$ to $\mathbf{y}$ to get a state $2^{-\Omega(n)}$-close to the state
$$|\mathbf{x}, \mathbf{y} + \kappa_{\mathcal{L}^*}(\mathbf{x})\rangle|\kappa_{\mathcal{L}^*}(\mathbf{x})\rangle|\phi^{\mathsf{Aux}}_{\mathbf{x}}\rangle$$
in ℓ_2-norm.
4. Apply $U^\dagger$ to the registers containing $|\mathbf{x}\rangle|\kappa_{\mathcal{L}^*}(\mathbf{x})\rangle|\phi^{\mathsf{Aux}}_{\mathbf{x}}\rangle$ to get a state $2^{-\Omega(n)}$-close to the state
$$|\mathbf{x}, \mathbf{y} + \kappa_{\mathcal{L}^*}(\mathbf{x})\rangle|D_{\mathcal{L},r}\rangle^{\otimes(3m^2n^2)}|0^{\mathsf{Aux}}\rangle$$
in ℓ_2-norm.

Therefore, we can answer the quantum query up to exponentially small error with exponentially small disturbance on the states $|D_{\mathcal{L},r}\rangle$ in time $O((m^2 + mT)\mathsf{poly}(n))$, which ends up the proof of Theorem 61. $\square$

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A Appendix

A.1 Upper bounds on Gaussian tails

Lemma 72. *Let $\mathcal{L} \subseteq \mathbb{R}^n$ be a lattice, $\mathbf{u} \in \mathbb{R}^n$ be a fixed vector, $\epsilon \in (0,1)$ be a small error parameter, σ be a positive real number with $\sigma > 2\eta_\epsilon(\mathcal{L})$. Then we have*

$$\sum_{\mathbf{x} \in \mathcal{L}^*} \rho_{1/\sigma}(\mathbf{x} - \mathbf{u}) < \rho_{1/\sigma}(\mathbf{u} - \kappa_{\mathcal{L}^*}(\mathbf{u})) + \epsilon.$$

If $\mathbf{u}$'s closest vectors in $\mathcal{L}^*$ are not unique, then $\kappa_{\mathcal{L}^*}(\mathbf{u})$ can be an arbitrary one of the closest vectors.

Proof. It suffices to prove that

$$\sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\kappa_{\mathcal{L}^*}(\mathbf{u})\}} \rho_{1/\sigma}(\mathbf{x} - \mathbf{u}) < \epsilon.$$

For any $\mathbf{x} \in \mathcal{L}^*$, we have that

$$\begin{aligned} \|\mathbf{x} - \mathbf{u}\|^2 &\geq \frac{1}{2}\left(\|\mathbf{x} - \mathbf{u}\|^2 + \|\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}\|^2\right) \qquad \text{(by the definition of } \kappa_{\mathcal{L}^*}(\mathbf{u})\text{)} \\ &\geq \frac{1}{4}\left(\|\mathbf{x} - \mathbf{u}\| + \|\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}\|\right)^2 \\ &\geq \frac{1}{4}\|\mathbf{x} - \mathbf{u} - (\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u})\|^2 \qquad \text{(triangle inequality)} \\ &= \frac{1}{4}\|\mathbf{x} - \kappa_{\mathcal{L}^*}(\mathbf{u})\|^2, \end{aligned}$$

so

$$\begin{aligned} \sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\kappa_{\mathcal{L}^*}(\mathbf{u})\}} \rho_{1/\sigma}(\mathbf{x} - \mathbf{u}) &\leq \sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\kappa_{\mathcal{L}^*}(\mathbf{u})\}} \rho_{2/\sigma}(\mathbf{x} - \kappa_{\mathcal{L}^*}(\mathbf{u})) \\ &= \sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\mathbf{0}\}} \rho_{2/\sigma}(\mathbf{x}) \\ &< \epsilon, \end{aligned}$$

as desired. □

Lemma 73. *Let $\mathcal{L} \subseteq \mathbb{R}^n$ be a lattice, $\mathbf{u} \in \mathbb{R}^n$ be a fixed vector, $\epsilon \in (0,1)$ be a small error parameter, σ be a positive real number with $\sigma > 2\sqrt{2}\eta_\epsilon(\mathcal{L})$. Then we have*

$$\sum_{\mathbf{x} \in \mathcal{L}^*} \rho_{1/\sigma}(\mathbf{x} - \mathbf{u}) < \rho_{1/\sigma}(\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}) + \epsilon \cdot \rho_{\sqrt{2}/\sigma}(\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}).$$

If $\mathbf{u}$'s closest vectors in $\mathcal{L}^*$ are not unique, then $\kappa_{\mathcal{L}^*}(\mathbf{u})$ can be an arbitrary one of the closest vectors.

Proof. It suffices to prove that

$$\sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\kappa_{\mathcal{L}^*}(\mathbf{u})\}} \rho_{1/\sigma}(\mathbf{x} - \mathbf{u}) < \epsilon \cdot \rho_{\sqrt{2}/\sigma}(\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}).$$

For any $\mathbf{x} \in \mathcal{L}^*$, we have that

$$\begin{aligned} \|\mathbf{x} - \mathbf{u}\|^2 - \frac{1}{2}\|\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}\|^2 &\geq \frac{1}{4}\left(\|\mathbf{x} - \mathbf{u}\|^2 + \|\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}\|^2\right) \qquad \text{(by the definition of } \kappa_{\mathcal{L}^*}(\mathbf{u})\text{)} \\ &\geq \frac{1}{8}\left(\|\mathbf{x} - \mathbf{u}\| + \|\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}\|\right)^2 \\ &\geq \frac{1}{8}\|\mathbf{x} - \mathbf{u} - (\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u})\|^2 \qquad \text{(triangle inequality)} \\ &= \frac{1}{8}\|\mathbf{x} - \kappa_{\mathcal{L}^*}(\mathbf{u})\|^2, \end{aligned}$$

so

$$\begin{aligned} \sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\kappa_{\mathcal{L}^*}(\mathbf{u})\}} \rho_{1/\sigma}(\mathbf{x} - \mathbf{u}) &\leq \rho_{\sqrt{2}/\sigma}(\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}) \sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\kappa_{\mathcal{L}^*}(\mathbf{u})\}} \rho_{2\sqrt{2}/\sigma}(\mathbf{x} - \kappa_{\mathcal{L}^*}(\mathbf{u})) \\ &= \rho_{\sqrt{2}/\sigma}(\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}) \sum_{\mathbf{x} \in \mathcal{L}^* \setminus \{\mathbf{0}\}} \rho_{2\sqrt{2}/\sigma}(\mathbf{x}) \\ &< \epsilon \cdot \rho_{\sqrt{2}/\sigma}(\kappa_{\mathcal{L}^*}(\mathbf{u}) - \mathbf{u}), \end{aligned}$$

as desired. □

A.2 Proof of Lemma 50

Proof of Lemma 50. Denote the (unnormalized) state in Equation (10) as $|\Phi\rangle$, and the (unnormailized) state in Equation (11) as $|\Phi'\rangle$. Then we have

$$
\begin{aligned}
&\| |\Phi\rangle - |\Phi'\rangle \|^2 \\
&= \sum_{\mathbf{v}\in\mathbb{Z}_q^n} \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2 \left\| \sum_{\substack{\mathbf{x}\in\mathbb{Z}^m,\\ \|\mathbf{x}\|\geq\lambda_1(\mathcal{L}_q(\mathbf{A}))/2}} \rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e}) |(\mathbf{A}^T\mathbf{v}+\mathbf{x}) \bmod q\rangle \right\|^2 \\
&\leq_{(1)} q^n \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2 \left(\sum_{\substack{\mathbf{x}\in\mathbb{Z}^m,\\ \|\mathbf{x}\|\geq\lambda_1(\mathcal{L}_q(\mathbf{A}))/2}} \rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e}) \right)^2 \\
&\leq_{(2)} q^n \sum_{\substack{j\in\mathbb{Z}_q,\\ |j|<\alpha\sqrt{m\log(\beta q)}}} \rho_\alpha(j)^2 \left(\sum_{\substack{\mathbf{x}\in\mathbb{Z}^m,\\ \|\mathbf{x}\|\geq\beta q\sqrt{m\log(\beta q)}}} \rho_{\beta q}(\mathbf{x}) \right)^2 + q^n \sum_{\substack{j\in\mathbb{Z}_q,\\ |j|\geq\alpha\sqrt{m\log(\beta q)}}} \rho_\alpha(j)^2 \left(\sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_{\beta q}(\mathbf{x}) \right)^2,
\end{aligned}
$$

where in (1) we absorb the summation over $\mathbf{v}\in\mathbb{Z}_q^n$ in q^n, and use the fact that ρ is non-negative; in (2), the first term uses the following: when $\|\mathbf{x}\|\geq\lambda_1(\mathcal{L}_q(\mathbf{A}))/2$ and $|j|<\alpha\sqrt{m\log(\beta q)}$, we have $\|\mathbf{x}+j\cdot\mathbf{e}\| > \lambda_1(\mathcal{L}_q(\mathbf{A}))/2 - \alpha\sqrt{m\log(\beta q)}\cdot\gamma q\sqrt{m} > \beta q\sqrt{m\log(\beta q)}$, the second term uses for any $\mathbf{c}\in\mathbb{R}^n$, $\sum_{\mathbf{x}\in\mathbb{Z}^m}\rho_{\beta q}(\mathbf{x}+\mathbf{c}) \leq \sum_{\mathbf{x}\in\mathbb{Z}^m}\rho_{\beta q}(\mathbf{x})$.

From Banaszczyk's tail bound (see Lemma 11), we have that

$$
\sum_{\substack{\mathbf{x}\in\mathbb{Z}^m,\\ \|\mathbf{x}\|\geq\beta q\sqrt{m\log(\beta q)}}} \rho_{\beta q}(\mathbf{x}) < \beta^{-3m}q^{-3m} \sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_{\beta q}(\mathbf{x}), \qquad \sum_{\substack{j\in\mathbb{Z}_q,\\ |j|\geq\alpha\sqrt{m\log(\beta q)}}} \rho_{\alpha/\sqrt{2}}(j) < \beta^{-6m}q^{-6m} \sum_{j\in\mathbb{Z}_q} \rho_{\alpha/\sqrt{2}}(j).
$$

Notice that $\beta q > \sqrt{m}$, so from Lemma 13, $\sum_{\mathbf{x}\in\mathbb{Z}^m}\rho_{\beta q}(\mathbf{x}) < (1+2^{-\Omega(m)})\beta^m q^m$. Therefore,

$$
\begin{aligned}
\| |\Phi\rangle - |\Phi'\rangle \|^2 &< 2q^n(\beta q)^{-6m} \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2 \left(\sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_{\beta q}(\mathbf{x}) \right)^2 \\
&< 2(1+2^{-\Omega(n)})q^n(\beta q)^{-4m} \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2
\end{aligned}
$$

On the other hand, notice that from Lemma 13, $\sum_{\mathbf{x}\in\mathbb{Z}^m}\rho_{\beta q/\sqrt{2}}(\mathbf{x}) > (\beta q/\sqrt{2})^m$, we have that

$$
\begin{aligned}
\| |\Phi\rangle \|^2 &\geq \sum_{\mathbf{v}\in\mathbb{Z}_q^n} \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2 \sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_{\beta q}(\mathbf{x}+j\cdot\mathbf{e})^2 \\
&= q^n \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2 \sum_{\mathbf{x}\in\mathbb{Z}^m} \rho_{\beta q}(\mathbf{x})^2 \\
&> q^n \left(\frac{\beta q}{\sqrt{2}}\right)^m \sum_{j\in\mathbb{Z}_q} \rho_\alpha(j)^2
\end{aligned}
$$

Hence, we get that $\frac{\||\Phi\rangle - |\Phi'\rangle\|^2}{\||\Phi\rangle\|^2} < 2^{-\Omega(n)}$, this implies that $|\Phi\rangle$ and $|\Phi'\rangle$ are $2^{-\Omega(n)}$-close to each other, by Lemma 26, as desired. □

A.3 Proof of Lemma 64

Proof of Lemma 64. For any vector $\mathbf{v} \in q\mathcal{L} + \mathcal{L}\mathbf{a}$ such that $\|\mathbf{v}\| \leq \sqrt{n}r$, as $q/2 > \sigma\sqrt{n}$, the state

$$\sum_{e \in \mathbb{Z}_{qR}/R} \rho_\sigma(e) |\langle \mathbf{s}, \mathbf{a} \rangle + \langle \mathbf{x}', \mathbf{v} \rangle + e \bmod q\rangle \tag{18}$$

is $2^{-\Omega(n)}$-close to the state

$$\sum_{e \in [-\sigma\sqrt{n}, \sigma\sqrt{n}] \cap (\mathbb{Z}/R)} \rho_\sigma(e) |\langle \mathbf{s}, \mathbf{a} \rangle + \langle \mathbf{x}', \mathbf{v} \rangle + e \bmod q\rangle$$

in ℓ_2 norm by Banaszczyk's tail bound (see Lemma 11), which is $2^{-\Omega(n)}$-close to the state

$$\sum_{e \in \mathbb{Z}_{qR}/R - \langle \mathbf{x}', \mathbf{v} \rangle} \rho_\sigma(e) |\langle \mathbf{s}, \mathbf{a} \rangle + \langle \mathbf{x}', \mathbf{v} \rangle + e \bmod q\rangle$$

in ℓ_2 norm due to Banaszczyk's tail bound and the fact that $|\langle \mathbf{x}', \mathbf{v} \rangle| \leq \frac{\alpha q}{r} \cdot \sqrt{n}r < q/2 - \sigma\sqrt{n}$, which implies $[-\sigma\sqrt{n}, \sigma\sqrt{n}] \cap (\mathbb{Z}/R) \subseteq \mathbb{Z}_{qR}/R - \langle \mathbf{x}', \mathbf{v} \rangle$.

We can perform a change of variable $u \leftarrow \langle \mathbf{x}', \mathbf{v} \rangle + e$ to write the last state as

$$\sum_{u \in \mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v} \rangle) |\langle \mathbf{s}, \mathbf{a} \rangle + u \bmod q\rangle. \tag{19}$$

Observe that Equation (18) and Equation (19) are $2^{-\Omega(n)}$-close to each other in ℓ_2 norm for every $\mathbf{v} \in q\mathcal{L} + \mathcal{L}\mathbf{a}$ such that $\|\mathbf{v}\| \leq \sqrt{n}r$. Therefore, the given state

$$\sum_{\substack{\mathbf{v} \in q\mathcal{L} + \mathcal{L}\mathbf{a}, \\ \|\mathbf{v}\| \leq \sqrt{n}r}} \rho_r(\mathbf{v}) |\mathbf{v}\rangle \otimes \sum_{e \in \mathbb{Z}_{qR}/R} \rho_\sigma(e) |\langle \mathbf{s}, \mathbf{a} \rangle + \langle \mathbf{x}', \mathbf{v} \rangle + e \bmod q\rangle$$

is $2^{-\Omega(n)}$-close to the state

$$|\Psi\rangle := \sum_{\substack{\mathbf{v} \in q\mathcal{L} + \mathcal{L}\mathbf{a}, \\ \|\mathbf{v}\| \leq \sqrt{n}r}} \rho_r(\mathbf{v}) |\mathbf{v}\rangle \otimes \sum_{u \in \mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v} \rangle) |\langle \mathbf{s}, \mathbf{a} \rangle + u \bmod q\rangle.$$

We will show $|\Psi\rangle$ is $2^{-\Omega(n)}$-close to the state

$$|\Phi\rangle := \sum_{\substack{\mathbf{v} \in q\mathcal{L} + \mathcal{L}\mathbf{a}, \\ \|\mathbf{v}\| < R/2}} \rho_r(\mathbf{v}) |\mathbf{v}\rangle \otimes \sum_{u \in \mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v} \rangle) |\langle \mathbf{s}, \mathbf{a} \rangle + u \bmod q\rangle.$$

The proof is similar to the proof of Lemma 50. We first give an upper bound for $\||\Psi\rangle - |\Phi\rangle\|^2$:

$$\begin{aligned}
\||\Psi\rangle - |\Phi\rangle\|^2 &= \sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \sqrt{n}r<\|\mathbf{v}\|<R/2}} \rho_{r/\sqrt{2}}(\mathbf{v}) \sum_{u\in\mathbb{Z}_{qR}/R} \rho_{\sigma/\sqrt{2}}(u-\langle\mathbf{x}',\mathbf{v}\rangle)\\
&\le \sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \|\mathbf{v}\|>\sqrt{n}r}} \rho_{r/\sqrt{2}}(\mathbf{v}) \sum_{u\in\mathbb{Z}/R} \rho_{\sigma/\sqrt{2}}(u)\\
&\le 2^{-\Omega(n)} \sum_{\mathbf{v}\in q\mathcal{L}} \rho_{r/\sqrt{2}}(\mathbf{v}) \sum_{u\in\mathbb{Z}/R} \rho_{\sigma/\sqrt{2}}(u)
\end{aligned}$$

by Banaszczyk's tail bound.

On the other hand, notice that

$$\begin{aligned}
\||\Psi\rangle\|^2 &= \sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \|\mathbf{v}\|\le\sqrt{n}r}} \rho_{r/\sqrt{2}}(\mathbf{v}) \sum_{u\in\mathbb{Z}_{qR}/R} \rho_{\sigma/\sqrt{2}}(u-\langle\mathbf{x}',\mathbf{v}\rangle)\\
&\ge \sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \|\mathbf{v}\|\le\sqrt{n}r}} \rho_{r/\sqrt{2}}(\mathbf{v}) \sum_{u\in[-\sigma\sqrt{n},\sigma\sqrt{n}]\cap\mathbb{Z}/R} \rho_{\sigma/\sqrt{2}}(u)\\
&\ge (1-2^{-\Omega(n)}) \sum_{\mathbf{v}\in q\mathcal{L}} \rho_{r/\sqrt{2}}(\mathbf{v}) \sum_{u\in\mathbb{Z}/R} \rho_{\sigma/\sqrt{2}}(u)
\end{aligned}$$

by Banaszczyk's tail bound and Lemma 17 together with the fact that $r/\sqrt{2} > q\eta_\epsilon(\mathcal{L})$ for $\epsilon < 2^{-n}$.

So $\frac{\||\Phi\rangle-|\Psi\rangle\|^2}{\||\Phi\rangle\|^2} < 2^{-\Omega(n)}$, which implies that $|\Phi\rangle$ and $|\Psi\rangle$ are $2^{-\Omega(n)}$-close to each other by Lemma 26.

It remains to prove that $|\Phi\rangle$ is $2^{-\Omega(n)}$-close to

$$|\Phi'\rangle := \sum_{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a}} \rho_r(\mathbf{v})|\mathbf{v} \bmod R\rangle \otimes \sum_{u\in\mathbb{Z}_{qR}/R} \rho_\sigma(u-\langle\mathbf{x}',\mathbf{v}\rangle)|\langle\mathbf{s},\mathbf{a}\rangle + u \bmod q\rangle.$$

We again use a similar argument as the proof of Lemma 50.

We first give an upper bound for $\||\Phi\rangle - |\Phi'\rangle\|^2$:

$$\begin{aligned}
\||\Phi\rangle - |\Phi'\rangle\|^2 &\leq_{(1)} \left(\sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \|\mathbf{v}\|\geq R/2}} \rho_r(\mathbf{v}) \left\| |\mathbf{v} \bmod R\rangle \otimes \sum_{u\in\mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v}\rangle) |\langle \mathbf{s}, \mathbf{a}\rangle + u \bmod q\rangle \right\| \right)^2 \\
&\leq_{(2)} \left(\sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \|\mathbf{v}\|>\sqrt{n}r\sqrt{\log r}}} \rho_r(\mathbf{v}) \sqrt{\sum_{u\in\mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v}\rangle)^2} \right)^2 \\
&\leq_{(3)} r^{-6n} \left(\sum_{u\in\mathbb{Z}/R} \rho_\sigma(u)^2 \right) \left(\sum_{\mathbf{v}\in q\mathcal{L}} \rho_r(\mathbf{v}) \right)^2 \\
&\leq_{(4)} r^{-4n} \left(\sum_{u\in\mathbb{Z}/R} \rho_\sigma(u)^2 \right) \det((q\mathcal{L})^*)^2 (1 + 2^{-\Omega(n)})
\end{aligned}$$

where (1) is due to triangle inequality, (2) uses that $R/2 > \sqrt{n}r\sqrt{\log r}$, (3) is due to Banaszczyk's tail bound, and (4) is due to Lemma 17 and $r \geq \eta_\epsilon(q\mathcal{L})$ for $\epsilon < 2^{-n}$.

On the other hand, notice that

$$\begin{aligned}
\||\Phi\rangle\|^2 &\geq \sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \|\mathbf{v}\|\leq\sqrt{n}r}} \rho_r(\mathbf{v})^2 \sum_{u\in\mathbb{Z}_{qR}/R} \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v}\rangle)^2 \\
&\geq_{(1)} (1 - 2^{-\Omega(n)}) \sum_{\substack{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a},\\ \|\mathbf{v}\|\leq\sqrt{n}r}} \rho_r(\mathbf{v})^2 \sum_{u\in\mathbb{Z}/R} \rho_\sigma(u)^2 \\
&\geq_{(2)} (1 - 2^{-\Omega(n)}) (r/\sqrt{2})^n \det((q\mathcal{L})^*) \sum_{u\in\mathbb{Z}/R} \rho_\sigma(u)^2
\end{aligned}$$

where (1) is due to Banaszczyk's tail bound and the fact that $\sigma\sqrt{n} < q/2 - |\langle \mathbf{x}', \mathbf{v}\rangle|$ when $\|\mathbf{v}\| \leq \sqrt{n}r$, and (2) is due to Banaszczyk's tail bound, Lemma 17 and $r/\sqrt{2} \geq \eta_\epsilon(q\mathcal{L})$ for $\epsilon < 2^{-n}$.

Hence, we have that $\frac{\||\Phi\rangle - |\Phi'\rangle\|^2}{\||\Phi\rangle\|^2} < 2^{-\Omega(n)}$, this implies that $|\Phi\rangle$ and $|\Phi'\rangle$ are $2^{-\Omega(n)}$-close to each other, by Lemma 26, as desired.

□

A.4 Proof of Lemma 66

Proof of Lemma 66. The amplitude of $|\mathbf{y}\rangle |\langle \mathbf{s}, \mathbf{a}\rangle + u \bmod q\rangle$ in the given state (Equation (16)) is

$$\begin{aligned}
\sum_{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a}} \rho_r(\mathbf{v}) \rho_\sigma(u - \langle \mathbf{x}', \mathbf{v}\rangle) \omega_R^{\langle \mathbf{v}, \mathbf{y}\rangle} &= \rho_t(u) \sum_{\mathbf{v}\in q\mathcal{L}+\mathcal{L}\mathbf{a}} \rho_{\sqrt{\mathbf{\Sigma}^{-1}}} \left(\mathbf{v} - \frac{r^2 u}{t^2} \mathbf{x}' \right) \omega_R^{\langle \mathbf{v}, \mathbf{y}\rangle} \\
&\propto \rho_t(u) \sum_{\mathbf{w}\in(q\mathcal{L})^* - \mathbf{y}/R} \rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{w}) \exp\left(2\pi \mathrm{i} \left\langle \mathcal{L}\mathbf{a} - \frac{r^2 u}{t^2} \mathbf{x}', \mathbf{w} \right\rangle \right) \omega_R^{\langle \mathcal{L}\mathbf{a}, \mathbf{y}\rangle},
\end{aligned}$$

where $t = \sqrt{\sigma^2 + r^2\|\mathbf{x}'\|^2} \in [\sigma, \sqrt{2}\sigma)$, matrix $\mathbf{\Sigma} = \frac{\mathbf{I}_n}{r^2} + \frac{\mathbf{x}'\mathbf{x}'^T}{\sigma^2}$ with eigenvalues $1/r^2, (t/r\sigma)^2$, and we use the Poisson Summation Formula to compute the last equality. We define the amplitude as a function

$$\phi(\mathbf{y}, u) := \rho_t(u) \sum_{\mathbf{w} \in (q\mathcal{L})^* - \mathbf{y}/R} \rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{w}) \exp\left(2\pi \mathrm{i} \left\langle \mathcal{L}\mathbf{a} - \frac{r^2 u}{t^2}\mathbf{x}', \mathbf{w} \right\rangle\right) \omega_R^{\langle \mathcal{L}\mathbf{a}, \mathbf{y}\rangle}.$$

Meanwhile, for $\mathbf{y}/R \in B_{(q\mathcal{L})^*}$, we define another amplitude function $\phi'(\mathbf{y}, u)$ as

$$\phi'(\mathbf{y}, u) := \rho_t(u)\rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{z}) \exp\left(2\pi \mathrm{i} \left\langle \mathcal{L}\mathbf{a} - \frac{r^2 u}{t^2}\mathbf{x}', -\mathbf{z} \right\rangle\right) \omega_R^{\langle \mathcal{L}\mathbf{a}, \mathbf{y}\rangle},$$

where we recall that $\mathbf{z} = \mathbf{z}(\mathbf{y}) = \mathbf{y}/R - \kappa_{(q\mathcal{L})^*}(\mathbf{y}/R)$. $\phi'(\mathbf{y}, u)$ is the leading term in $\phi(\mathbf{y}, u)$, as we will implicitly show below.

We prove that the following (unnormalized) states

$$|\Phi\rangle := \sum_{\mathbf{y} \in \mathbb{Z}_R^n} \sum_{u \in \mathbb{Z}_{qR}/R} \phi(\mathbf{y}, u) |\mathbf{y}\rangle | \langle \mathbf{s}, \mathbf{a}\rangle + u \bmod q\rangle \quad \text{(the given state)}$$

$$|\Phi'\rangle := \sum_{\mathbf{y} \in \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}} \sum_{u \in \mathbb{Z}_{qR}/R} \phi'(\mathbf{y}, u) |\mathbf{y}\rangle | \langle \mathbf{s}, \mathbf{a}\rangle + u \bmod q\rangle \quad \text{(the target state in Equation (17))}$$

are $2^{-\Omega(n)}$-close to each other. To prove it, we need in addition the following (unnormalized) state

$$|\Phi''\rangle := \sum_{\mathbf{y} \in \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}} \sum_{u \in \mathbb{Z}_{qR}/R} \phi(\mathbf{y}, u) |\mathbf{y}\rangle | \langle \mathbf{s}, \mathbf{a}\rangle + u \bmod q\rangle.$$

Let's begin by establishing upper bounds for both $\||\Phi''\rangle - |\Phi'\rangle\|^2$ and $\||\Phi\rangle - |\Phi''\rangle\|^2$. The first term is relatively simpler to bound: for any $\epsilon < 2^{-n}$, according to Lemma 73 and the assumption that $\frac{r\sigma}{t} > \frac{r}{\sqrt{2}} > 2\sqrt{2}\eta_\epsilon(q\mathcal{L})$, we get that

$$\begin{aligned} \||\Phi''\rangle - |\Phi'\rangle\|^2 &= \sum_{\mathbf{y} \in \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}} \sum_{u \in \mathbb{Z}_{qR}/R} \left|\phi(\mathbf{y}, u) - \phi'(\mathbf{y}, u)\right|^2 \\ &\le \sum_{\mathbf{y} \in \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}} \sum_{u \in \mathbb{Z}_{qR}/R} \left(\rho_t(u) \sum_{\mathbf{w} \in (q\mathcal{L})^* \setminus \{\mathbf{0}\}} \rho_{t/r\sigma}(\mathbf{w} - \mathbf{z})\right)^2 \\ &< \epsilon^2 \sum_{u \in \mathbb{Z}_{qR}/R} \rho_t(u)^2 \sum_{\mathbf{y} \in \mathbb{Z}_R^n \cap R \cdot B_{(q\mathcal{L})^*}} \rho_{t/r\sigma}(\mathbf{z}). \end{aligned} \tag{20}$$

To establish an upper bound for the latter term $\||\Phi\rangle - |\Phi''\rangle\|^2$, according to Lemma 73 and the fact that $\frac{r\sigma}{t} > 2\sqrt{2}\eta_\epsilon(q\mathcal{L})$, we get that for $\mathbf{y} \in \mathbb{Z}_R^n \setminus R \cdot B_{(q\mathcal{L})^*}$,

$$\begin{aligned}\sum_{\mathbf{w}\in(q\mathcal{L})^*-\mathbf{y}/R} \rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{w}) &\leq \sum_{\mathbf{w}\in(q\mathcal{L})^*-\mathbf{y}/R} \rho_{t/r\sigma}(\mathbf{w}) \\ &\leq \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(t/r\sigma)^2}\right) + \epsilon \cdot \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{2(t/r\sigma)^2}\right) \\ &\leq \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{2(t/r\sigma)^2}\right)\left(\exp\left(-\pi\frac{(\lambda_1((q\mathcal{L})^*)/2)^2}{4/r^2}\right) + \epsilon\right) \\ &\leq 2^{-n+1}\exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{2(t/r\sigma)^2}\right)\end{aligned}$$

where the last inequality holds since $\lambda_1((q\mathcal{L})^*) \geq \sqrt{\frac{n\ln 2}{\pi}} \cdot \frac{1}{q\eta_\epsilon(\mathcal{L})} \geq \sqrt{\frac{n\ln 2}{\pi}} \cdot \frac{4}{r}$ by Lemma 16. Therefore,

$$\begin{aligned}\||\Phi\rangle - |\Phi''\rangle\|^2 &= \sum_{\mathbf{y}\in\mathbb{Z}_R^n\setminus R\cdot B_{(q\mathcal{L})^*}} \sum_{u\in\mathbb{Z}_{qR}/R} |\phi(\mathbf{y},u)|^2 \\ &\leq \sum_{u\in\mathbb{Z}_{qR}/R} \rho_t(u)^2 \sum_{\mathbf{y}\in\mathbb{Z}_R^n\setminus R\cdot B_{(q\mathcal{L})^*}} \left(\sum_{\mathbf{w}\in(q\mathcal{L})^*-\mathbf{y}/R} \rho_{\sqrt{\mathbf{\Sigma}}}(\mathbf{w})\right)^2 \\ &\leq 2^{-2n+2} \sum_{u\in\mathbb{Z}_{qR}/R} \rho_t(u)^2 \sum_{\mathbf{y}\in\mathbb{Z}_R^n\setminus R\cdot B_{(q\mathcal{L})^*}} \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(t/r\sigma)^2}\right).\end{aligned} \tag{21}$$

Combine the two upper bounds in Equation (20) and Equation (21), we get that

$$\begin{aligned}\||\Phi\rangle - |\Phi'\rangle\|^2 &\leq 2\||\Phi\rangle - |\Phi''\rangle\|^2 + 2\||\Phi''\rangle - |\Phi'\rangle\|^2 \\ &\leq 2^{-2n+3} \sum_{u\in\mathbb{Z}_{qR}/R} \rho_t(u)^2 \sum_{\mathbf{y}\in\mathbb{Z}_R^n} \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(t/r\sigma)^2}\right).\end{aligned} \tag{22}$$

On the other hand, we can compute that

$$\begin{aligned}\||\Phi'\rangle\|^2 &= \sum_{\mathbf{y}\in\mathbb{Z}_R^n\cap R\cdot B_{(q\mathcal{L})^*}} \sum_{u\in\mathbb{Z}_{qR}/R} \left|\phi'(\mathbf{y},u)\right|^2 \\ &\geq \sum_{u\in\mathbb{Z}_{qR}/R} \rho_t(u)^2 \sum_{\mathbf{y}\in\mathbb{Z}_R^n\cap R\cdot B_{(q\mathcal{L})^*}} \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(1/\sqrt{2}r)^2}\right).\end{aligned} \tag{23}$$

Combining with the bounds in Equation (22), we get that

$$\frac{\||\Phi\rangle - |\Phi'\rangle\|^2}{\||\Phi'\rangle\|^2} \leq 2^{-2n+3} \frac{\sum_{\mathbf{y}\in\mathbb{Z}_R^n} \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(t/r\sigma)^2}\right)}{\sum_{\mathbf{y}\in\mathbb{Z}_R^n\cap R\cdot B_{(q\mathcal{L})^*}} \exp\left(-\pi\frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(1/\sqrt{2}r)^2}\right)}. \tag{24}$$

Now let's establish an upper bound on the ratio between the two summations on the right-hand side of the above inequality. We accomplish this through the following steps:

1. Expanding the support of $\mathbf{y}$. Given our assumption that $\mathcal{L}$ is an integer lattice, we have $(q\mathcal{L})^* + \mathbf{k} = (q\mathcal{L})^*$ for any $\mathbf{k} \in \mathbb{Z}^n$. Consequently, $\mathrm{dist}((\mathbf{y} - R\mathbf{k})/R, (q\mathcal{L})^*) = \mathrm{dist}(\mathbf{y}/R, (q\mathcal{L})^* + \mathbf{k}) = \mathrm{dist}(\mathbf{y}/R, (q\mathcal{L})^*)$. Therefore, we can expand the support of $\mathbf{y}$ from $\mathbb{Z}^n_R$ to $\mathbb{Z}^n_{RN}$ for any $N \in \mathbb{N}_+$, which is a combination of N^n hypercubes, each in the form $\mathbb{Z}^n_R + R\mathbf{k}$, i.e.

$$\frac{\||\Phi\rangle - |\Phi'\rangle\|^2}{\||\Phi'\rangle\|^2} \le 2^{-2n+3} \lim_{N \to +\infty} \frac{\sum_{\mathbf{y}\in\mathbb{Z}^n_{RN}} \exp\left(-\pi \frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(t/r\sigma)^2}\right)}{\sum_{\mathbf{y}\in\mathbb{Z}^n_{RN}\cap R\cdot B_{(q\mathcal{L})^*}} \exp\left(-\pi \frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(1/\sqrt{2}r)^2}\right)}.$$

2. Bound the numerator and denominator as N approaches infinity. Assume that N is a sufficiently large integer, and denote $\ell_{\max} = \max_{\mathbf{y}\in\mathbb{R}^n} \mathrm{dist}(\mathbf{y}, (q\mathcal{L})^*) < +\infty$. For the numerator, when considering $\mathbf{y} \in \mathbb{Z}^n_{RN}$, the closest vector from $\mathbf{y}/R$ to the lattice $(q\mathcal{L})^*$ will have ℓ_∞ norm at most $RN/2 + \ell_{\max}$. Thus

$$\sum_{\mathbf{y}\in\mathbb{Z}^n_{RN}} \exp\left(-\pi \frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(t/r\sigma)^2}\right) \le \sum_{\substack{\mathbf{u}\in(q\mathcal{L})^*,\\ \|\mathbf{u}\|_\infty \le RN/2+\ell_{\max}}} \sum_{\mathbf{y}\in\mathbb{Z}^n} \rho_{t/r\sigma}(\mathbf{y}/R - \mathbf{u}).$$

Similarly, for the denominator, when considering $\mathbf{u} \in (q\mathcal{L})^*$ with ℓ_∞ norm at most $RN/2 - \ell_{\max}$, the vectors $\mathbf{y} \in \mathbb{Z}^n \cap R \cdot B_{(q\mathcal{L})^*}$ for which the closest vector to $(q\mathcal{L})^*$ is $\mathbf{u}$ will certainly have an ℓ_∞ norm at most $RN/2$. Thus

$$\begin{aligned}
&\sum_{\mathbf{y}\in\mathbb{Z}^n_{RN}\cap R\cdot B_{(q\mathcal{L})^*}} \exp\left(-\pi \frac{\mathrm{dist}(\mathbf{y}/R,(q\mathcal{L})^*)^2}{(1/\sqrt{2}r)^2}\right) \\
\ge &\sum_{\substack{\mathbf{u}\in(q\mathcal{L})^*,\\ \|\mathbf{u}\|_\infty \le RN/2-\ell_{\max}}} \sum_{\substack{\mathbf{y}\in\mathbb{Z}^n,\\ \|\mathbf{y}/R-\mathbf{u}\|<\lambda_1((q\mathcal{L})^*)/2}} \rho_{1/\sqrt{2}r}(\mathbf{y}/R - \mathbf{u}) \\
\ge &\sum_{\substack{\mathbf{u}\in(q\mathcal{L})^*,\\ \|\mathbf{u}\|_\infty \le RN/2-\ell_{\max}}} \left(\sum_{\mathbf{y}\in\mathbb{Z}^n} \rho_{1/\sqrt{2}r}(\mathbf{y}/R - \mathbf{u}) - 2^{-\Omega(n)} \rho_{1/\sqrt{2}r}(\mathbb{Z}^n/R) \right),
\end{aligned}$$

where the final inequality is based on Banaszczyk's tail bound (Lemma 11), with the guarantee that $\lambda_1((q\mathcal{L})^*)/2 > \sqrt{\frac{n \ln 2}{\pi}} \cdot \frac{2}{r} > \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}r}\sqrt{n}$.

According to Lemma 15, it follows that $\eta_{2^{-n}}(\mathbb{Z}^n/R) \le \frac{1}{R} \cdot \sqrt{\frac{2n}{\pi}} < \frac{1}{\sqrt{2}r} < \frac{t}{r\sigma}$. Therefore, by using Lemma 17, we can conclude that

$$\sum_{\mathbf{y}\in\mathbb{Z}^n} \rho_{t/r\sigma}(\mathbf{y}/R - \mathbf{u}) \le (1 + 2^{-\Omega(n)}) \cdot R^n \cdot (t/r\sigma)^n,$$

$$\sum_{\mathbf{y}\in\mathbb{Z}^n} \rho_{1/\sqrt{2}r}(\mathbf{y}/R - \mathbf{u}) - 2^{-\Omega(n)} \rho_{1/\sqrt{2}r}(\mathbb{Z}^n/R) \ge (1 - 2^{-\Omega(n)}) \cdot R^n \cdot (1/\sqrt{2}r)^n.$$

3. Establish an upper bound for the right hand side of Equation (24). By combining the in-

equalities in the previous step, we obtain that

$$\begin{aligned}
\frac{\||\Phi\rangle - |\Phi'\rangle\|^2}{\||\Phi'\rangle\|^2} &\leq 2^{-2n+3}(1+2^{-\Omega(n)}) \cdot (\sqrt{2}t/\sigma)^n \cdot \lim_{N\to+\infty} \frac{\#\{\mathbf{u} \in (q\mathcal{L})^* : \|\mathbf{u}\|_\infty \leq RN/2 + \ell_{\max}\}}{\#\{\mathbf{u} \in (q\mathcal{L})^* : \|\mathbf{u}\|_\infty \leq RN/2 - \ell_{\max}\}} \\
&\leq 2^{-n+3}(1+2^{-\Omega(n)}) \cdot \lim_{N\to+\infty} \frac{\#\{\mathbf{u} \in (q\mathcal{L})^* : \|\mathbf{u}\|_\infty \leq RN/2 + \ell_{\max}\}}{\#\{\mathbf{u} \in (q\mathcal{L})^* : \|\mathbf{u}\|_\infty \leq RN/2 - \ell_{\max}\}} \\
&= 2^{-n+3}(1+2^{-\Omega(n)}) \cdot \lim_{N\to+\infty} \frac{(RN/2 + \ell_{\max})^n}{(RN/2 - \ell_{\max})^n} \\
&= 2^{-n+3}(1+2^{-\Omega(n)}).
\end{aligned}$$

Consequently, we can infer that the provided state $|\Phi\rangle$ is $2^{-\Omega(n)}$-close to $|\Phi'\rangle$, according to Lemma 26, as desired. □