Self-Orthogonal Minimal Codes From (Vectorial) p-ary Plateaued Functions

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Abstract

In this article, we derive the weight distribution of linear codes stemming from a subclass of (vectorial) p-ary plateaued functions $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ for a prime p, which includes all the explicitly known examples of weakly and non-weakly regular plateaued functions. In the literature, this construction of linear codes is referred to as the first generic construction. First, we partition the class of p-ary plateaued functions into three classes $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$, according to the behavior of their dual function f^*. Using these classes, we refine the results presented in a series of articles [11, 13, 17, 19, 22]. Namely, we derive the full weight distributions of codes stemming from all s-plateaued functions for $n+s$ odd (parametrized by the weight of the dual $wt(f^*)$), whereas for $n+s$ even, the weight distributions are derived from the class of s-plateaued functions in $\mathscr{C}_1$ parametrized using two parameters (including $wt(f^*)$ and a related parameter Z_0). Additionally, we provide more results on the different weight distributions of codes stemming from functions in subclasses of these three different kinds. The exact derivation of such distributions is achieved by using some well-known equations over finite fields to count certain dual preimages. In order to improve the dimension of these codes, we then study the vectorial case, thus providing the weight distributions of a few codes associated to known vectorial plateaued functions and obtaining codes with parameters $[p^n-1, 2n, p^n-p^{n-1}-p^{(n+s-2)/2}(p-1)]$. For the first time, we provide the full weight distributions of codes from non-weakly regular vectorial p-ary plateaued functions for every p. This class includes all known explicit examples in the literature. The obtained codes are minimal and self-orthogonal virtually in all cases. Notably, we show that this is the best one can achieve—there are no q-ary self-dual minimal codes for any prime power q, except for the ternary tetracode and the binary repetition code.

Keywords—linear codes, vectorial functions, plateaued functions, minimal codes, self-orthogonal codes, finite fields.

1 Introduction

There are a vast number of methods for constructing linear codes—constructions based on p-ary functions are among the most renowned methods. In their pioneering work, Carlet, Charpin and

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Zinoviev [3] showed the first explicit connection between AB (and APN) functions and linear codes. Soon after, Carlet and Ding [4] constructed codes based on perfect nonlinear mappings. Since then, many authors have addressed the construction of linear codes using p-ary functions [2, 8, 21, 9, 12, 13, 14, 15, 20, 23, 24].

In this work, we address the construction of p-ary codes from (vectorial) plateaued functions. There has been much work on linear codes stemming from perfect nonlinear functions, however, less is known for the plateaued case. Elaborating on the results of [11, 13, 17, 19, 22], we present the full weight distribution of subclasses of weakly and non-weakly regular s-plateaued functions $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ yielding three-weight codes and five-weight codes, for $n+s$ odd and $n+s$ even, respectively. Depending on the behavior of the dual function, we observe that there exists a partition of the class of s-plateaued functions into three subclasses that are called $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$. Some general properties of these classes are then deduced (see Lemma 3). These results are obtained by using well-known solutions of equations over cyclotomic fields, which are field extensions of the rational numbers by adding the complex p-th root of unity. To derive the exact Walsh distributions (thus, the weight distributions of associated codes), functions from each class $\mathscr{C}_i$ are taken into account to compute the cardinalities of preimages of suitable dual functions. The parameters of the obtained codes are $[p^n-1, n+1, (p-1)p^{n-1}-p^{\frac{n+s-1}{2}}]$ and $[p^n-1, n+1, p^n-p^{n-1}-p^{(n+s-2)/2}(p-1)]$, for $n+s$ odd and $n+s$ even, respectively. In order to obtain linear codes with a larger dimension, we study the vectorial case. Little is known about infinite families of vectorial plateaued functions outside of the quadratic, planar or bent realm. However, some examples have been given in the literature. Based on such examples, we extract general properties of these functions to obtain the weight distribution of codes stemming from a class of vectorial plateaued functions yielding three-weight codes with parameters $[p^n-1, 2n, p^n-p^{n-1}-p^{(n+s-2)/2}(p-1)]$. We then prove that essentially all obtained codes are minimal and self-orthogonal, which makes these codes quite interesting also from a practical point of view. Surprisingly, we prove that this is the best we can expect since there are no q-ary self-dual minimal codes for any prime power q, except for trivial examples (the ternary tetracode and the binary repetition code).

2 Preliminaries

Let $\mathbb{F}_{p^n}$ denote the finite field with p^n elements, where $n>0$ and p is prime. Let $\mathbb{F}_p^n$ be an n-dimensional vector space over $\mathbb{F}_p$. A function F from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^m}$ is called a vectorial p-ary function. When $p=2$, F is simply referred to as a vectorial Boolean function. The adjective vectorial is dropped when we refer to functions mapping to the prime field $\mathbb{F}_p$ (thus $m=1$). Such functions will be usually denoted with lowercase letters. We treat a function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ and its truth table as the same object whenever there is no ambiguity. The component functions $F_a : \mathbb{F}_{p^n} \to \mathbb{F}_p$ of a vectorial function $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ are the mappings $x \mapsto \mathrm{Tr}_1^m(aF(x))$ for $a \in \mathbb{F}_{p^m}^\star$, where $\mathbb{F}_{p^m}^\star = \mathbb{F}_{p^m} \setminus \{0\}$ and the function Tr_1^m denotes the usual trace function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$, i.e., $\mathrm{Tr}_1^m(x) = x + x^p + x^{p^2} + \cdots + x^{p^{(m-1)}}$.

The Walsh transform of $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ at a point $b \in \mathbb{F}_{p^n}$ is the sum of characters given by

$$W_f(b) = \sum_{x \in \mathbb{F}_{p^n}} \xi_p^{f(x)+\mathrm{Tr}_1^n(bx)}, \tag{1}$$

where $\xi_p = e^{2\pi i/p}$ is the complex primitive p-th root of unity. The *inverse Walsh transform* of f is then defined by

$$p^n \xi_p^{f(x)} = \sum_{b \in \mathbb{F}_{p^m}} W_f(b) \xi_p^{-\mathrm{Tr}_1^n(bx)}. \tag{2}$$

The Walsh spectrum of f is the multi-set of values $\{* \; W_f(b) \; : \; b \in \mathbb{F}_{p^n} \; *\}$. For a vectorial function F, its Walsh spectrum is given by $\{* \; W_{F_a}(b) \; : \; (a,b) \in \mathbb{F}_{p^m}^\star \times \mathbb{F}_{p^n} \; *\}$. The set of linear functions from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^m}$ will be denoted by $\mathcal{L}_{n,m}$, whereas the set of affine functions will be denoted by $\mathcal{A}_{n,m}$. Oftentimes, we omit the double subindex when $m = 1$. A p-ary s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ is characterized by the property $|W_f(b)|^2 = 0$ or p^{n+s} for each $b \in \mathbb{F}_{p^n}$. The number $p^{\frac{n+s}{2}}$ is called the amplitude of f. If $s = 0$, then there are no zero spectral values and we call such a function a p-ary bent function. It can be shown [13] that the non-zero Walsh values of a p-ary s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ can be expressed as $u_b p^{-(n+s)/2} W_f(b) = \xi_p^{f^*(b)}$ for a complex number u_b with $|u_b| = 1$ and a p-ary function f^*, where $f^*\colon \mathsf{supp}(W_f) \to \mathbb{F}_p$, where $\mathsf{supp}(W_f) = \left\{b \in \mathbb{F}_{p^n} : |W_f(b)|^2 = p^{n+s}\right\}$. If $u_b = 1$ for every $b \in \mathbb{F}_{p^n}$, then we say that f is a p-ary regular s-plateaued function. More generally, if the value of u_b does not depend on b, then the function f is called p-ary weakly regular s-plateaued, and non-weakly regular s-plateaued otherwise. The function f^* is called the dual of f. Furthermore, it was shown [13] that a weakly regular s-plateaued function f satisfies $W_f(b) = \epsilon_f \sqrt{p^*}^{n+s} \xi_p^{f^*(b)}$, where $\epsilon_f = \pm 1$ is called the sign of the Walsh transform of f and $p^* = \eta(-1)p$, where the function $\eta : \mathbb{F}_p \to \{-1, 0, 1\}$ denotes the Legendre symbol on $\mathbb{F}_p$ defined by

$$\eta(j) = \begin{cases} 0, & j = 0 \\ 1, & \exists i \in \mathbb{F}_p^\star, i^2 = j \\ -1, & otherwise. \end{cases} \tag{3}$$

Throughout, the notation $\left(\frac{j}{p}\right)$ will be used interchangeably with $\eta(j)$. In a similar fashion, one can easily show that a non-weakly regular s-plateaued function f satisfies $W_f(b) = \epsilon_f(b) \sqrt{p^*}^{n+s} \xi_p^{f^*(b)}$, where $\epsilon_f(b) = \pm 1$ will be called the sign of the Walsh transform of f at $b \in \mathbb{F}_{p^n}$.

2.1 Linear codes from functions

A linear $[n, k, d]$ code $\mathcal{C}$ over the alphabet $\mathbb{F}_p$ is a k-dimensional linear subspace of $\mathbb{F}_p^n$, whose minimum Hamming distance (equivalently, the minimum weight of its non-zero codewords) is d. Every code considered in this paper is a linear code, thus we will not distinguish between the terms linear code and code. The code $\mathcal{S}_n$ spanned by all linear functionals over $\mathbb{F}_{p^n}^\star$ is a $[p^n - 1, n, p^n - p^{n-1}]$ code, called the n-affine simplex code, i.e., $\mathcal{S}_n = \{(L(x))_{x \in \mathbb{F}_{p^n}^\star} : L \in \mathcal{L}_n\}$ (a pruning of the first order Reed-Muller code).

Let a_j be the number of codewords in $\mathcal{C}$ with Hamming weight j, $0 \leq j \leq n$. The weight distribution of a code $\mathcal{C}$ is the vector $(1, a_1, \ldots, a_n)$ and it is fully specified by its weight enumerator polynomial, which is the polynomial $1 + a_1 z + \cdots + a_n z^n$. We say that a code with parameters $[n, k, d]$ is distance-optimal, or simply optimal, provided that there does not exist an $[n, k, d']$ linear code with $d < d'$. A generic method to specify linear codes from a mapping $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ with

$F(0) = 0$ is described as follows. For positive integers n and m, the linear code $\mathcal{C}_F \subset \mathbb{F}_p^{p^n-1}$ is defined by

$$\mathcal{C}_F = \{\mathbf{c}_{a,u} : a \in \mathbb{F}_{p^m}, b \in \mathbb{F}_{p^n}\}, \tag{4}$$

where $\mathbf{c}_{a,u} := \left(\mathrm{Tr}_1^m(aF(x)) + \mathrm{Tr}_1^n(ux)\right)_{x\in\mathbb{F}_{p^n}^\star}$. The dimension of $\mathcal{C}_F$ is at most $n+m$ and its length is $p^n - 1$. If $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ has no linear components, the linear code $\mathcal{C}_F$ derived from the generic construction in (4) has dimension exactly $n + m$. Moreover, its weights can be expressed by the Walsh transform of absolute trace functions of the map $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ as shown in [11].

3 Cyclotomic relations relevant for plateaued functions

Let QR denote the set of quadratic residues modulo p and let NQR be the set of quadratic non-residues modulo p. Denote by $\mathfrak{i}$ the complex imaginary unit, i.e., $\mathfrak{i} = \sqrt{-1}$.

Lemma 1. *(Folklore) The following relations are true for the Legendre symbol, defined in (3), and* ξ_p*:*

1. $\sum_{j\in\mathbb{F}_p^\star} \eta(j) = \sum_{j\in QR^\star} 1 + \sum_{j\in NQR}(-1) = 0;$
2. $\sum_{j\in\mathbb{F}_p^\star} \xi_p^j = -1;$
3. *For any* $a \in \mathbb{Z}$*, the integral equation*

$$\sum_{j\in\mathbb{F}_p^\star} a_j\xi_p^j = \begin{cases} a\sqrt{p}, & p \equiv 1 \pmod 4; \\ \mathfrak{i}a\sqrt{p}, & p \equiv 3 \pmod 4; \end{cases}$$

has a unique solution $a_j = a\eta(j) \in \mathbb{Z}$.

Note that $\mathfrak{i} \notin \mathbb{Z}(\xi)$ since it is not a root of unity for $p \not\equiv 0 \pmod 4$ [18]. Therefore, $\sum_{i=1}^{p-1} a_i\xi_p^i = p^{\frac{\theta}{2}}\nu$ for $a_i \in \mathbb{Z}$, $\theta \in \mathbb{N}$ and $\nu \in \{1, \mathfrak{i}\}$ implies that either θ is odd or $\nu \neq \mathfrak{i}$. Therefore, we have the following.

Lemma 2. *[10] Let* $(a_1, \ldots, a_n) \in \mathbb{Z}^p$*,* $\theta \in \mathbb{N}$ *and* $\nu \in \{1, \mathfrak{i}\}$*. Suppose that* $\sum_{i=1}^{p-1} a_i\xi_p^i = p^{\frac{\theta}{2}}\nu$*.*

1. *If* $\theta \equiv 0 \pmod 2$*, then* $\nu = 1$*;*
2. *If* $\theta \equiv 1 \pmod 2$*, then*

$$\nu = \begin{cases} 1, & p \equiv 1 \pmod 4; \\ \mathfrak{i}, & p \equiv 3 \pmod 4. \end{cases}$$

4 Dual value distributions of plateaued functions

Using a similar notation as in [13], given $f_1\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ and any function $f_2\colon \mathsf{supp}(W_{f_1}) \to \mathbb{F}_p$, we define the sets $N_{f_2}(j) = \{x \in \mathsf{supp}(W_{f_1}) : f_2(x) = j\}$ and the numbers $n_{f_2}(j) = \#N_{f_2}(j)$, for $j \in \mathbb{F}_p$. Following the terminology introduced in [16, 17], for a given set $S \subseteq \mathbb{F}_{p^n}$, we say that a function $f\colon S \to \mathbb{F}_p$ is *bent relative to* S if $|W_f(b)| = \#S^{1/2}$ for all $b \in \mathbb{F}_{p^n}$, where $W_f(b)$ is considered as the restriction to S of the Walsh transform of f, i.e., $W_f(b) = \sum_{x\in S} \xi_p^{f(x)+\mathrm{Tr}_1^n(bx)}$. For weakly regular plateaued functions, the dual function f^* is bent relative to $\mathsf{supp}(W_f)$. For non-weakly regular plateaued functions, the dual may or may not be bent relative to $\mathsf{supp}(W_f)$. There are infinitely many examples of both cases.

In general, let $S \subseteq \mathbb{F}_{p^m}$ and let $f: S \to \mathbb{F}_p$ be a function such that $W_f(0) = \sum_{x\in S} \xi_p^{f(x)} = t(f)\nu p^{\frac{\mu}{2}}\xi_p^j$, where $t(f) = \pm 1$ or 0, $\nu \in \{1, \mathfrak{i}\}$, $j \in \mathbb{F}_p$ for some $\mu \in \mathbb{N}$, $\mu > 0$. The number $t(f)$ will be called the *type of* f. For an s-plateaued function $f: \mathbb{F}_{p^n} \to \mathbb{F}_p$ with $0 \le s \le n$, let $\Gamma^+(f)$ and $\Gamma^-(f)$ be sets that partition $S = \mathsf{supp}(W_f)$ and are given by

$$\Gamma^+(f) = \left\{b \in S : W_f(b) = \nu p^{\frac{n+s}{2}}\xi_p^{f^*(b)}\right\},\ \Gamma^-(f) = \left\{b \in S : W_f(b) = -\nu p^{\frac{n+s}{2}}\xi_p^{f^*(b)}\right\},$$

where $\nu \in \{1, \mathfrak{i}\}$. Note that in this case $t(f) = \epsilon_f(0)\eta(-1)^{n+s}$, where $\epsilon_f(0)$ denotes the sign of W_f at 0. For an s-plateaued function $f: \mathbb{F}_{p^n} \to \mathbb{F}_p$, define the numbers

$$A_j := \#(N_{f^*}(j) \cap \Gamma^+(f)) \text{ and } B_j := \#(N_{f^*}(j) \cap \Gamma^-(f))$$

for $j \in \mathbb{F}_p$. We also define $Z_j := A_j - B_j$.

Lemma 3. *Let $f: \mathbb{F}_{p^n} \to \mathbb{F}_p$ be any s-plateaued function. Let $f(0) = j_0$. Then $A_{j_0} \neq B_{j_0}$ (i.e. $Z_{j_0} \neq 0$). The distribution values A_j, B_j associated to f satisfy exactly one of the following.*

i) *$A_j \neq B_j$ for every j;*

ii) *The number $n - s$ is even and $A_j = B_j$ for each $j \neq j_0$. In this case, $A_{j_0} = p^{\frac{n-s}{2}} + B_{j_0}$ and, $\sum_{j\neq j_0} A_j = \sum_{j\neq j_0} B_j = \frac{p^{n-s}+p^{\frac{n-s}{2}}}{2} - A_{j_0} = \frac{p^{n-s}-p^{\frac{n-s}{2}}}{2} - B_{j_0}$;*

iii) *The number $n - s$ is odd and $A_{j+j_0} = B_{j+j_0}$ for $j \in \mathcal{I}$ and $A_{j+j_0} - B_{j+j_0} = 2\sigma \left(\frac{j}{p}\right) p^{\frac{n-s-1}{2}}$ for $j \notin \mathcal{I}$, where*

$$\sigma = \begin{cases} 1, & p \equiv 1 \pmod 4; \\ -1, & p \equiv 3 \pmod 4; \end{cases}$$

and

$$\mathcal{I} = \begin{cases} QR^\star, & \frac{Z_{j_0}}{|Z_{j_0}|} = -\sigma; \\ NQR, & \textit{otherwise}. \end{cases}$$

In this case, $Z_{j_0} = -\sigma \left(\frac{j}{p}\right) p^{\frac{n-s-1}{2}}$ for (any) $j \in \mathcal{I}$. Moreover, if $A_{j_0} \neq 0$, then $\sum_{i\neq j_0} A_i = \frac{p^{n-s}+\sigma\left(\frac{j}{p}\right)p^{\frac{n-s+1}{2}}}{2} - A_{j_0}$, and, if $B_{j_0} \neq 0$, then $\sum_{i\neq j_0} B_i = \frac{p^{n-s}-\sigma\left(\frac{j}{p}\right)p^{\frac{n-s+1}{2}}}{2} - B_{j_0}$, for (any) $j \notin \mathcal{I}$.

Proof. Consider the inverse Walsh transform (2) of $f(x)$ at $x = 0$,

$$p^n\xi_p^{j_0} = \sum_{b\in\mathbb{F}_{p^n}} W_f(b) = \sum_{j\in\mathbb{F}_p} (A_j - B_j)\xi_p^j \nu p^{\frac{n+s}{2}}.$$

Using 2) of Lemma 1, this can be rewritten as

$$\sum_{j\neq j_0}(A_j - B_j - Z_{j_0})\xi_p^j \nu p^{\frac{n+s}{2}} = p^{\frac{n+s}{2}}\xi_p^{j_0}, \tag{5}$$

which can be rearranged as

$$\sum_{j\neq j_0}(A_j - B_j - Z_{j_0})\xi_p^{j-j_0} = p^{\frac{n-s}{2}}\nu^{-1}. \tag{6}$$

Suppose that $n-s$ is even. Thus, $\nu = 1$ by Lemma 2. We first show that $Z_{j_0} \neq 0$. On the contrary, suppose that $Z_{j_0} = 0$. Then (6) implies that $A_j - B_j = -p^{\frac{n-s}{2}}$ by Lemma 1. Since f is plateaued, $\sum_{j\in\mathbb{F}_p}(A_j + B_j) = p^{n-s}$. Then $2\sum_{j\in\mathbb{F}_p} A_j = p^{n-s} - p^{\frac{n-s}{2}}(p-1)$, which is a contradiction since $p^{n-s} - p^{\frac{n-s}{2}}(p-1)$ is an odd number. Therefore $Z_{j_0} \neq 0$. To prove *ii*), let us suppose that *i*) is not true, i.e., suppose that there is an index $j' \neq j_0$ such that $A_{j'} = B_{j'}$. From (6), we get $A_j - B_j = Z_{j_0} - p^{\frac{n-s}{2}}$ for each j. In particular, $0 = A_{j'} - B_{j'} = Z_{j_0} - p^{\frac{n-s}{2}}$, so that $Z_{j_0} = p^{\frac{n-s}{2}}$ and $A_j = B_j$ for every $j \neq j_0$. The second part of *ii*) comes from this and the fact that $\sum_{j\in\mathbb{F}_p}(A_j + B_j) = p^{n-s}$.

Suppose that $n-s$ is odd. To show that $Z_{j_0} \neq 0$, suppose the opposite. Equation (6) implies that

$$A_j - B_j = \sigma\left(\frac{j-j_0}{p}\right)p^{\frac{n-s-1}{2}},$$

where $\sigma = 1$ if $p \equiv 1 \pmod 4$ and $\sigma = -1$ if $p \equiv 3 \pmod 4$, by Lemma 1. Since f is plateaued, $\sum_{j\in\mathbb{F}_p}(A_j + B_j) = p^{n-s}$. Then $2\sum_{j\in\mathbb{F}_p} A_j = p^{n-s}$, which is a contradiction since p^{n-s} is odd. Hence $Z_{j_0} \neq 0$. Again, suppose that *i*) is not true, i.e. suppose that there is an index $j' \neq j_0$ such that $A_{j'} = B_{j'}$. We will prove *iii*). From (6), we get $A_j - B_j = Z_{j_0} + \sigma\left(\frac{j-j_0}{p}\right)p^{\frac{n-s-1}{2}}$ for each j. In particular, $0 = A_{j'} - B_{j'} = Z_{j_0} + \sigma\left(\frac{j'-j_0}{p}\right)p^{\frac{n-s-1}{2}}$, so that $Z_{j_0} = -\sigma\left(\frac{j'-j_0}{p}\right)p^{\frac{n-s-1}{2}}$. This tells us that for every j such that $\left(\frac{j-j_0}{p}\right) = \left(\frac{j'-j_0}{p}\right)$, we have $A_j = B_j$. Defining $\mathcal{I}$ as in the statement, this is equivalent to $A_{j+j_0} = B_{j+j_0}$ for every $j \in \mathcal{I}$. Additionally, $A_j - B_j = 2\sigma\left(\frac{j-j_0}{p}\right)p^{\frac{n-s-1}{2}}$ for each $j - j_0 \notin \mathcal{I}$. The second part of *iii*) comes from the above and the fact that $\sum_{j\in\mathbb{F}_p}(A_j + B_j) = p^{n-s}$. □

Using the previous lemma, we can partition the set of s-plateaued functions into three classes. These classes will be denoted by $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$, respectively. Thus, $\mathscr{C}_1$ corresponds to the functions specified in *i*) of Lemma 3, $\mathscr{C}_2$ corresponds to the functions specified in *ii*) and $\mathscr{C}_3$ corresponds to the functions specified in *iii*). First we show that $\mathscr{C}_i$, $1 \leq i \leq 3$, is non-empty (Example 1). Afterwards, we will determine the exact values of A_j, B_j for certain s-plateaued functions in each family $\mathscr{C}_i$.

Example 1. *Any weakly regular plateaued function whose dual is surjective belongs to* $\mathscr{C}_1$ *for which there are several infinite families of functions.*

To construct an infinite family inside $\mathscr{C}_2$*, consider the function* $f(x) = \mathrm{Tr}_1^3(x^7)$ *over* $\mathbb{F}_{3^3}$*. This function is a non-weakly* 1*-plateaued function with zero dual, namely,* $\{W_f(b) : b \in \mathbb{F}_{3^3}\} = \{0, 9, -9\}$

with distribution $\{*0^{18}, 9^6, -9^3*\}$*, so* $f^*(x) = 0$ *for each* $x \in \mathsf{supp}(W_f)$*. For any* $l \in \mathbb{N}$*, we consider the* l*-th iteration of the direct sum of* f *with itself,* $f^l : \mathbb{F}_{3^{3(l-1)}} \times \mathbb{F}_{3^3} \to \mathbb{F}_3$*, defined recursively by*

$$f^1 = f;$$

$$f^l(x, y) = f^{l-1}(x) + f(y) \text{ for } l \geq 2,$$

which is an l*-plateaued function defined on* $\mathbb{F}_{3^{3l}}$ *(upon identifying* $\mathbb{F}_{3^{3l}} \cong \mathbb{F}_{3^{3(l-1)}} \times \mathbb{F}_{3^3}$*) with constant zero dual (recall that the Walsh values of the direct sum of functions are directly related to the Walsh spectra of the summands).*

For $\mathscr{C}_3$*, consider the function* $g(x) = \mathrm{Tr}_1^3(2x^4 + x^2)$ *in* $\mathbb{F}_{3^3}$*. This function is a weakly regular* 2*-plateaued function with* $\{W_f(b) : b \in \mathbb{F}_{3^3}\} = \{0, \mathrm{i}3^{5/2}, \mathrm{i}3^{5/2}\xi_3^2\}$ *with distribution*

$$\{*0^{24}, (\mathrm{i}3^{5/2})^1, (\mathrm{i}3^{5/2}\xi_3^2)^2*\}$$

For any $l \in \mathbb{N}$*, consider* f^l *as before. The direct sum of* f^l *with* g *gives a non-weakly regular* $(l+2)$*-plateaued function in* $\mathbb{F}_{3^{3(l+1)}}$ *with*

$$\{W_f(b) : b \in \mathbb{F}_{3^{3(l+1)}}\} = \{0, \mathrm{i}3^{\frac{5+4l}{2}}, -\mathrm{i}3^{\frac{5+4l}{2}}, \mathrm{i}3^{\frac{5+4l}{2}}\xi_3^2, -\mathrm{i}3^{\frac{5+4l}{2}}\xi_3^2\},$$

which belongs to $\mathscr{C}_3$*.*

Although the previous example (Example 1) shows that the classes $\mathscr{C}_1, \mathscr{C}_2$ and $\mathscr{C}_3$ are non-empty, it also gives rise to some existence problems. Namely, the following questions arise naturally.

Question 1. Are the classes $\mathscr{C}_2$ and $\mathscr{C}_3$ non-empty for $p > 3$ and for every n?

Note also that every function in $\mathscr{C}_2$ is non-weakly regular. Indeed, any function in $\mathscr{C}_2$ is an s-plateaued function that satisfies $A_j = B_j$ for each $j \neq j_0$. Obviously, any such function is non-weakly regular if $A_j = B_j \neq 0$ for some $j \neq j_0$. Then, we only need to show that a zero dual plateaued function is non-weakly regular: it must hold that

$$(A_{j_0} - B_{j_0})p^{\frac{n+s}{2}} = p^n \xi_p^{j_0},$$

in this case, $\nu = 1$ and $n + s$ is even. Hence, $j_0 = 0$ as the left-hand side of this equation is an integer. By plateauedness, $A_0 + B_0 = p^{n-s}$. Therefore, a zero dual plateaued function satisfies the system of equations

$$A_0 - B_0 = p^{\frac{n-s}{2}} \text{ and } A_0 + B_0 = p^{n-s},$$

which implies $A_0 \neq 0$ and $B_0 \neq 0$.

Question 2. Are there infinite classes of functions in $\mathscr{C}_2$ whose dual is non-zero?

Question 3. Are there infinite classes of functions in $\mathscr{C}_3$ whose dual is surjective (necessarily non-weakly regular plateaued)?

In the following (Propositions 1-6), we determine the exact values of A_j, B_j for certain subfamilies of p-ary plateaued functions which carry enough information about the dual to derive these values.

Proposition 1. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an s-plateaued function in $\mathscr{C}_1$ with $f(0) = f^*(0) = 0$. Suppose that $W_f(0) = t(f)\nu p^{\frac{n+s}{2}}$ and $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu, \nu' \in \{1, \mathrm{i}\}$ and $\theta \in \mathbb{N}$, $\theta > 1$. For $j \in \mathbb{F}_p^\star$, the numbers A_j, B_j are either zero or depend on A_0 and B_0, respectively. Moreover, $A_0 + B_0 = p^{n-s-1}$ when θ is odd and $A_0 + B_0 = p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$ for θ even. The values of A_j, B_j are displayed in Table 1 for different parities of $n+s$ and θ.*

Proof. Suppose that $n+s$ is even. Suppose that θ is even, too. By Lemma 1,

$$A_j - B_j = A_0 - B_0 - p^{\frac{n-s}{2}} \tag{7}$$

for each $j \in \mathbb{F}_p^\star$. On the other hand, $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$. By Lemma 2, $\nu' = 1$. Since $t(f^*)p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$, we have

$$A_j + B_j = A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}} \tag{8}$$

for each j. From (7) and (8), one can obtain the values of A_j, B_j in terms of A_0, B_0 respectively. Lastly,

$$p^{n-s} = \sum_{j=0}^{p-1}(A_j + B_j) = (p-1)(A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}}) + A_0 + B_0.$$

Thus, $A_0 + B_0 = p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$. Assume now that θ is odd. Since $t(f^*)\nu' p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$, we have

$$A_j + B_j = A_0 + B_0 + \left(\frac{j}{p}\right) t(f^*)p^{\frac{\theta}{2}} \tag{9}$$

for each j. From (7) and (9), one can obtain the values of A_j, B_j in terms of A_0, B_0 respectively. Lastly,

$$p^{n-s} = \sum_{j=0}^{p-1}(A_j + B_j) = p(A_0 + B_0)$$

Thus, $A_0 + B_0 = p^{n-s-1}$.

For the case when $n+s$ is odd, use the fact that (by Lemma 1)

$$A_j - B_j = A_0 - B_0 + \left(\frac{j}{p}\right) p^{\frac{n-s}{2}} \tag{10}$$

for each $j \in \mathbb{F}_p^\star$. Combining this with (8) and (9), we obtain the desired result. □

Remark 1. *Proposition 1 extends the results of [11, 13, 17, 19, 22]. Namely, in [11], the value distribution of the dual of a weakly regular bent function f was studied. Then the extension to weakly regular plateaued functions was given in [13]. In [17], the case of f being a non-weakly regular bent function whose dual is bent with respect to $\mathsf{supp}(f)$ was investigated. Later, in [19], the authors presented the case of non-weakly regular s-plateaued functions f whose dual is bent with respect to $\mathsf{supp}(f)$, which was further analyzed in [22]. Therefore Proposition 1 is the most general result of this kind.*

Table 1: Values of A_j, B_j and $A_0 + B_0$ in Proposition 1 for different parities of $n+s$ and θ, where the pairs stand for $(n+s \pmod 2), \theta \pmod 2)$.

	A_j	B_j	$A_0 + B_0$
$(0,0)$	0, or, $A_0 - p^{\frac{n-s}{2}-1}\frac{(1+t(f^*))}{2}$	0, or, $B_0 - p^{\frac{n-s}{2}-1}\frac{(1-t(f^*))}{2}$	$p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$
$(0,1)$	0, or, $A_0 + \frac{\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta}{2}} - p^{\frac{n-s}{2}-1}}{2}$	0, or, $B_0 + \frac{\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta}{2}} + p^{\frac{n-s}{2}-1}}{2}$	p^{n-s-1}
$(1,0)$	0, or, $A_0 + \frac{\left(\frac{j}{p}\right)p^{\frac{n-s}{2}} - t(f^*)p^{\frac{\theta}{2}}}{2}$	0, or, $B_0 + \frac{-\left(\frac{j}{p}\right)p^{\frac{n-s}{2}} - t(f^*)p^{\frac{\theta}{2}}}{2}$	$p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}$
$(1,1)$	0, or, $A_0 + \frac{\left(\frac{j}{p}\right)(t(f^*)p^{\frac{\theta}{2}} + p^{\frac{n-s}{2}})}{2}$	0, or, $B_0 + \frac{\left(\frac{j}{p}\right)(t(f^*)p^{\frac{\theta}{2}} - p^{\frac{n-s}{2}})}{2}$	p^{n-s-1}

Remark 2. *Proposition 1 covers the value distributions of all known instances of weakly and non-weakly bent functions.*

Next, we analyze three subclasses of functions in $\mathscr{C}_2$ and determine the values of A_j, B_j. The necessary conditions to derive these values are imposed on the $W_{f^*}(0)$, namely, we study the cases $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$ for some $\theta \in \mathbb{N}$. The case of θ even is treated in Proposition 2, whereas Proposition 4 deals with the case θ odd. The subclass of zero dual functions is further analyzed in Proposition 3.

Proposition 2. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an unbalanced s-plateaued function in $\mathscr{C}_2$ such that $f(0) = f^*(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$ for some $\theta \in \mathbb{N}$, $\theta > 1$, θ even. Then, $A_0 = \frac{p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}$, $B_0 = \frac{p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}$. Moreover, for $j \in \mathbb{F}_p^\star$, $A_j = B_j = A_0 - \frac{p^{\frac{n-s}{2}} + t(f^*)p^{\frac{\theta}{2}}}{2}$.*

Proof. Since $t(f^*)p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(2A_j - 2A_0 + p^{\frac{n-s}{2}})\xi_p^j$, we have

$$A_j = A_0 - \frac{p^{\frac{n-s}{2}} + t(f^*)p^{\frac{\theta}{2}}}{2} \tag{11}$$

for each j. From Lemma 3, $\frac{p^{n-s} + p^{\frac{n-s}{2}}}{2} = \sum_{j=0}^{p-1} A_j = (p-1)(A_0 - \frac{p^{\frac{n-s}{2}} + t(f^*)p^{\frac{\theta}{2}}}{2}) + A_0$. Thus,

$$A_0 = \frac{p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}.$$

So that

$$B_0 = \frac{p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}.$$

The values of A_j are then obtained via (11). □

Proposition 3. *Let $f \in \mathscr{C}_2$ be a plateaued function with $f(0) = f^*(0) = 0$ such that $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}\nu$, $\theta \in \mathbb{N}$ and $\nu \in \{1, \mathrm{i}\}$. Then f^* is the constant zero function if and only if $\nu = 1$, $\theta = 2(n-s)$ and $t(f^*) = 1$.*

Proof. Suppose that f^* is the constant zero function. Compute

$$t(f^*)p^{\frac{\theta}{2}}\nu = W_{f^*}(0) = \sum_{x\in \mathsf{supp}(W_f)} 1 = p^{n-s}.$$

Hence, $\nu = 1$, $\theta = 2(n-s)$ and $t(f^*) = 1$. Conversely, suppose that $W_{f^*}(0) = p^{n-s}$. By Proposition 2, $A_0 = \frac{p^{n-s}+p^{\frac{n-s}{2}}}{2}$, $B_0 = \frac{p^{n-s}-p^{\frac{n-s}{2}}}{2}$ and, for each $j \in \mathbb{F}_p^\star$, $A_j = B_j = \frac{p^{n-s}+p^{\frac{n-s}{2}}}{2} - \frac{p^{\frac{n-s}{2}}+p^{n-s}}{2} = 0$. Then, the function f^* is the constant zero function. □

Proposition 4. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an unbalanced s-plateaued function in $\mathscr{C}_2$ such that $f(0) = f^*(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathfrak{i}\}$ and $\theta \in \mathbb{N}$, $\theta > 1$, θ odd. Then, we have $A_0 = \frac{p^{n-s-1}+p^{\frac{n-s}{2}}}{2}$, $B_0 = \frac{p^{n-s-1}-p^{\frac{n-s}{2}}}{2}$. Moreover, for $j \in \mathbb{F}_p^\star$, the value of $A_j(= B_j)$ is equal to $A_j = B_j = A_0 - \frac{p^{\frac{n-s}{2}}+\left(\frac{j}{p}\right)t(f^*)p^{\frac{\theta-1}{2}}}{2}$.*

Proof. Since $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta-1}{2}}\sqrt{p}$, we have

$$2A_j = 2A_0 - p^{\frac{n-s}{2}} + \left(\frac{j}{p}\right) t(f^*)p^{\frac{\theta-1}{2}}$$

for each j. Summing these terms up, we get

$$p^{n-s} = \sum_{j=0}^{p-1} 2A_j - p^{\frac{n-s}{2}} = 2pA_0 - p^{\frac{n-s}{2}+1}.$$

Thus $2A_0 = p^{n-s-1} + p^{\frac{n-s}{2}}$ and the result follows. □

Similarly as for $\mathscr{C}_2$, we analyze two subclasses of functions in $\mathscr{C}_3$, depending on whether θ is even or odd, where $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$.

Proposition 5. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an unbalanced s-plateaued function in $\mathscr{C}_3$ such that $f(0) = f^*(0) = 0$. Let the set $\mathcal{I}$ be defined as in Lemma 3. Suppose that $W_{f^*}(0) = t(f^*)p^{\frac{\theta}{2}}$ for some $\theta \in \mathbb{N}$, $\theta > 1$, θ even. Then,*

$$A_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} - \sigma\left(\frac{i}{p}\right)p^{\frac{n-s-1}{2}}}{2}$$

and

$$B_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} + \sigma\left(\frac{i}{p}\right)p^{\frac{n-s-1}{2}}}{2}$$

for any $i \in \mathcal{I}$. Moreover, for $j \in \mathcal{I}$, $A_j = B_j = \frac{p^{n-s-1}-t(f^*)p^{\frac{\theta}{2}-1}}{2}$ and, for $j \notin \mathcal{I}$, we have

$$A_j = \frac{p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1}}{2} - \sigma\left(\frac{j}{p}\right)p^{\frac{n-s-1}{2}}$$

and $B_j = \frac{p^{n-s-1}-t(f^*)p^{\frac{\theta}{2}-1}}{2} + \sigma\left(\frac{j}{p}\right)p^{\frac{n-s-1}{2}}$, where $\sigma = 1$ if $p \equiv 1 \pmod 4$ and $\sigma = -1$ if $p \equiv 3 \pmod 4$.

Proof. Since $t(f^*)p^{\frac{\theta}{2}} = W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$, we have

$$A_j + B_j = A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}} \tag{12}$$

for each j. Summing up, we get

$$p^{n-s} - A_0 - B_0 = \sum_{j=1}^{p-1} A_j + B_j = (p-1)(A_0 + B_0 - t(f^*)p^{\frac{\theta}{2}}).$$

Thus, $A_0 + B_0 = p^{n-s-1} + t(f^*)p^{\frac{\theta}{2}-1}(p-1)$. By Lemma 3, we know that $A_0 - B_0 = -\sigma \left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}$ for any $i \in \mathcal{I}$. Combining these two equations, we have

$$A_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} - \sigma \left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}}{2}$$

and

$$B_0 = \frac{p^{n-s-1} + t(f^*)(p-1)p^{\frac{\theta}{2}-1} + \sigma \left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}}{2}.$$

The result follows at once from *iii)* of Lemma 3. □

Proposition 6. *Let $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ be an unbalanced s-plateaued function in $\mathscr{C}_3$ such that $f^*(0) = 0$. Let the set $\mathcal{I}$ be defined as in Lemma 3. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathrm{i}\}$ and $\theta \in \mathbb{N}$, $\theta > 1$, θ odd. Then*

$$A_0 = \frac{p^{n-s-1} - \left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}}{2} \quad \text{and } B_0 = \frac{p^{n-s-1} + \left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}}{2}$$

for any $i \in \mathcal{I}$. Moreover, for $j \in \mathcal{I}$, we have $A_j = B_j = \frac{p^{n-s-1} + \left(\frac{j}{p}\right) t(f^*) p^{\frac{\theta-1}{2}}}{2}$ and, for $j \notin \mathcal{I}$, we have

$$A_j = \frac{p^{n-s-1} + \left(\frac{j}{p}\right) t(f^*) p^{\frac{\theta-1}{2}}}{2} - \sigma \left(\frac{j}{p}\right) p^{\frac{n-s-1}{2}} \quad \text{and } B_j = \frac{p^{n-s-1} + \left(\frac{j}{p}\right) t(f^*) p^{\frac{\theta-1}{2}}}{2} + \sigma \left(\frac{j}{p}\right) p^{\frac{n-s-1}{2}},$$

where $\sigma = 1$ if $p \equiv 1 \pmod 4$ and $\sigma = -1$ if $p \equiv 3 \pmod 4$.

Proof. Since $t(f^*)\nu' p^{\frac{\theta-1}{2}} \sqrt{p} = W_{f^*}(0) = \sum_{j=1}^{p-1}(A_j + B_j - A_0 - B_0)\xi_p^j$, we have

$$A_j + B_j = A_0 + B_0 + \left(\frac{j}{p}\right) t(f^*) p^{\frac{\theta-1}{2}} \tag{13}$$

for each j. Summing up, we get $A_0 + B_0 = p^{n-s-1}$. By Lemma 3, we know that $A_0 - B_0 = -\sigma \left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}$ for any $i \in \mathcal{I}$. Combining these two equations, we have $A_0 = \frac{p^{n-s-1} - \sigma\left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}}{2}$ and $B_0 = \frac{p^{n-s-1} + \sigma\left(\frac{i}{p}\right) p^{\frac{n-s-1}{2}}}{2}$. Combining these values with *iii)* of Lemma 3, we obtain the desired conclusion. □

Remark 3. *Propositions 1, 2 and 6 cover all known examples of plateaued functions (up to now).*

5 Codes from plateaued functions

In this section, we will use plateaued functions $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ to construct linear codes using (4). This approach extends the results in [11, 13, 17, 19, 22]. In order to explicitly compute the weights of the derived codes $\mathcal{C}_f$, where f is an s-plateaued function, we must count the number of elements in the preimage of a given function. We will do so by considering the possible dual value distributions studied in Section 4. In the following sections, we derive the full weight distributions of codes stemming from plateaued functions such that $f(0) = 0$, where the distributions are parametrized by $wt(f^*)$ when $n + s$ is odd and by $wt(f^*)$ and Z_0 when $n + s$ is even, in the latter it is also required that $f^*(0) = 0$.

5.1 The weight distribution of $\mathcal{C}_f$ for $n + s$ odd

Throughout this section, $n + s$ will be odd. Define the following three subclasses of plateaued functions:

$$\widehat{\mathcal{P}_2} = \{f \in \mathscr{C}_3 \mid f \text{ is weakly regular}\},$$

$$\widetilde{\mathcal{P}_2} = \{f \in \mathscr{C}_1 \mid \forall i \in QR^\star A_i = 0, B_i \neq 0 \text{ and } \forall i \in NQR\ A_i \neq 0, B_i = 0\},$$

and

$$\overline{\mathcal{P}_2} = \{f \in \mathscr{C}_1 \mid \forall i \in NQR\ A_i = 0, B_i = 0 \text{ and } \forall i \in NQR\ A_i = 0, B_i \neq 0\}.$$

Define $\mathcal{P}_2 = \widehat{\mathcal{P}_2} \cup \widetilde{\mathcal{P}_2} \cup \overline{\mathcal{P}_2}$. These classes yield codes with two weights (see Remark 4), thus they can be regarded as exceptions since every other plateaued function gives rise to a 3-valued code, as shown in the following theorem, which is quite general and it does not necessarily follow from Lemma 1.

Theorem 1. *Let $n > 0$ and $0 \leq s < n$ be integers such that $n + s$ is odd. Let f be any s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = 0$ such that $f \notin \mathcal{P}_2$. The code $\mathcal{C}_f$ in (4) ($m = 1$) is a three-valued code with parameters $[p^n - 1, n + 1, (p - 1)p^{n-1} - p^{\frac{n+s-1}{2}}]$, whose weight distribution is displayed in Table 2.*

Proof. The weights are easily derived from the results in [11], which are $w_1 := p^n - p^{n-1} - p^{(n+s-1)/2}, w_2 := p^n - p^{n-1}$ and $w_3 := p^n - p^{n-1} + p^{(n+s-1)/2}$. Note that there are exactly three weights since $f \notin \mathcal{P}_2$. Indeed, when $n - s$ is odd, counting the number of different weights in $\mathcal{C}_f$ comes down to counting the number of possible values taken by $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u)) \in \{-1, 0, 1\}$ for each $(a, u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n}$ (see [11] for more insight on this derivation). It is easy to see that the number $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u))$ equals 1 if and only if $a^{-1}u \in A_j$ for some $j \in QR^\star$ or $a^{-1}u \in B_j$ for some $j \in NQR$. Similarly, $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u))$ equals -1 if and only if $a^{-1}u \in A_j$ for some $j \in NQR$ or $a^{-1}u \in B_j$ for some $j \in QR^\star$. Therefore, there are exactly two weights if and only if $f \in \mathcal{P}_2$.

Denote by X, Y and Z the number of codewords attaining the weight $p^n - p^{n-1} - p^{(n+s-1)/2}, p^n - p^{n-1}$ and $p^n - p^{n-1} + p^{(n+s-1)/2}$, respectively. Using the first two Pless power moments, we get the system of equations

$$X + Y + Z = p^{n+1} - 1 \tag{14}$$

$$w_1X + w_2Y + w_3Z = p^n(p - 1)(p^n - 1). \tag{15}$$

Since the number of balanced codewords can be counted as

$$Y = p^n - 1 + (p-1)(p^n - p^{n-s}) + (p-1)(p^{n-s} - wt(f^*)) = p^{n+1} - (p-1)wt(f^*) - 1,$$

we can solve the above system in terms of $wt(f^*)$. Namely,

$$X = \frac{(p-1)}{2}(wt(f^*) - (p-1)p^{(n-s-1)/2}) \tag{16}$$

$$Z = \frac{(p-1)}{2}(wt(f^*) + (p-1)p^{(n-s-1)/2}) \tag{17}$$

□

Table 2: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 1, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ with $f(0) = 0$ and $f \notin \mathcal{P}_2$, when $n+s$ is odd.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-1)/2}$	$\frac{(p-1)}{2}(wt(f^*) + (p-1)p^{\frac{n-s-1}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)wt(f^*) - 1$
$p^n - p^{n-1} + p^{(n+s-1)/2}$	$\frac{(p-1)}{2}(wt(f^*) - (p-1)p^{\frac{n-s-1}{2}})$

When $f \in \mathcal{P}_2$, one can show that the code $\mathcal{C}_f$ is two-valued, thus the frequencies corresponding to the same weight must be added up. From Theorem 1, we can easily derive the weight distribution of $\mathcal{C}_f$ for s-plateaued functions with $f(0) = 0$ such that $wt(f^*) = p^n - p^{n-s-1}$. The corresponding values are displayed in Table 3.

Remark 4. *As noted in the proof of Theorem 1, when $n-s$ is odd, the number of different weights in $\mathcal{C}_f$ equals the number of possible values taken by $\epsilon_f(a^{-1}u)\eta(f^*(a^{-1}u))$ for each $(a,u) \in \mathbb{F}_p^\star \times \mathbb{F}_{p^n}$ (which takes values in $\{-1,0,1\}$). Therefore, it can be shown that the code $\mathcal{C}_f$ has two weights if and only if $f \in \mathcal{P}_2$.*

Table 3: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 1, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ with $f(0) = 0$, $f \notin \mathcal{P}_2$, and $wt(f^*) = p^{n-s} - p^{n-s-1}$, when $n+s$ is odd.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-1)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} + p^{\frac{n-s-1}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)^2 p^{n-s-1} - 1$
$p^n - p^{n-1} + p^{(n+s-1)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - p^{\frac{n-s-1}{2}})$

5.2 The weight distribution of $\mathcal{C}_f$ for $n+s$ even

We now demonstrate that there is an essential difference when $n+s$ is even since the derived codes are 5-valued. This is a non-trivial consequence of the behavior of some exponential sums of the quadratic automorphism of $\mathbb{Q}(\xi_p)$, which depends on the parity of $n+s$.

Theorem 2. *Let $n > 0$ and $0 \le s \le n$ be integers such that $n + s$ is even. Let $f \in \mathscr{C}_1$ be an s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = f^*(0) = 0$. The code $\mathcal{C}_f$ in (4) ($m = 1$) is a five-valued code with parameters $[p^n - 1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is summarized in Table 4.*

Proof. The weights are easily obtained from the results of [11]. We have the weights $w_1 = p^n - p^{n-1} - t(f^*)p^{(n+s-2)/2}(p-1)$, $w_2 = p^n - p^{n-1} - t(f^*)p^{(n+s-2)/2}$, $w_3 = p^n - p^{n-1}$, $w_4 = p^n - p^{n-1} + t(f^*)p^{(n+s-2)/2}$ and $w_5 = p^n - p^{n-1} + t(f^*)p^{(n+s-2)/2}(p-1)$. Using the relation between weights of the code and the Walsh transform, the number of codewords with weight w_1 is equal to $|\{(a,u) \in \mathbb{F}_p^* \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = 1\}|$, which equals $(p-1)A_0$. Moreover,

$$(p-1)A_0 = \frac{(p-1)}{2}(p^{n-s} - wt(f^*) - t(f^*)Z_0).$$

Similarly, the number of codewords with weight w_5 is $|\{(a,u) \in \mathbb{F}_p^* \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = -1\}|$, that is,

$$(p-1)B_0 = \frac{(p-1)}{2}(p^{n-s} - wt(f^*) + t(f^*)Z_0).$$

The number of codewords of weight w_2 and w_4 is equal to $|\{(a,u) \in \mathbb{F}_p^* \times \mathbb{F}_{p^n} : f^*(a^{-1}u) \neq 0, \epsilon_f(a^{-1}u) = 1\}|$ and $|\{(a,u) \in \mathbb{F}_p^* \times \mathbb{F}_{p^n} : f^*(a^{-1}u) \neq 0, \epsilon_f(a^{-1}u) = -1\}|$, respectively. These values are

$$(p-1)\sum_{j=1}^{p-1} A_j = \frac{(p-1)}{2}(wt(f^*) + t(f^*)(p-1)(Z_0 - t(f^*)p^{\frac{n-s}{2}}))$$

and

$$(p-1)\sum_{j=1}^{p-1} B_j = \frac{(p-1)}{2}(wt(f^*) - t(f^*)(p-1)(Z_0 - t(f^*)p^{\frac{n-s}{2}})),$$

respectively. Finally, there are $p^n - 1 + (p-1)(p^n - p^{n-s})$ balanced codewords (i.e. corresponding to the weight w_3). □

Table 4: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 2, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_1$ such that $f(0) = f^*(0) = 0$, when $n+s$ even, and $Z_0 = A_0 - B_0$.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} - wt(f^*) + Z_0)$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)}{2}(wt(f^*) - (p-1)Z_0 + (p-1)p^{\frac{n-s}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)}{2}(wt(f^*) + (p-1)Z_0 - (p-1)p^{\frac{n-s}{2}})$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} - wt(f^*) - Z_0)$

Corollary 1. *Let $n > 0$ and $0 \le s \le n$ be integers such that $n + s$ is even. Let $f \in \mathscr{C}_1$ be any s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = f^*(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{n-s}{2}}$ for some $\nu' \in \{1, \mathfrak{i}\}$. Let $Z_0 := A_0 - B_0$. The code $\mathcal{C}_f$ in (4) ($m = 1$) is a five-valued code with parameters $[p^n - 1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distributions are displayed in Table 5.*

Proof. By Proposition 1,

$$wt(f^*) = p^{n-s} - (A_0 + B_0) = p^{n-s} - p^{n-s-1} - t(f^*)(p-1)p^{\frac{n-s}{2}-1}$$

since $n-s$ is even. Using Theorem 2, we get the desired result by plugging the obtained value for $wt(f^*)$. □

Table 5: Weight distribution of the code $\mathcal{C}_f$, derived in Corollary 1, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_1$ such that $f(0) = f^*(0) = 0$ and $W_{f^*}(0) = t(f^*)\nu' p^{\frac{n-s}{2}}$, when $n+s$ is even.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + t(f^*)p^{\frac{n-s}{2}} - t(f^*)p^{\frac{n-s}{2}-1} + Z_0)$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} + p^{\frac{n-s}{2}} - t(f^*)p^{\frac{n-s}{2}-1} - Z_0)$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - p^{\frac{n-s}{2}} - t(f^*)p^{\frac{n-s}{2}-1} + Z_0)$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + t(f^*)p^{\frac{n-s}{2}} - t(f^*)p^{\frac{n-s}{2}-1} - Z_0)$

It is not too hard to see that when $f \in \mathscr{C}_2$ and f^* is the constant zero function the code $\mathcal{C}_f$ is a three-weight code. The possible values for weights of unbalanced codewords are

$$w_1 := p^n - p^{n-1} - p^{(n+s-2)/2}(p-1) \text{ and } w_3 := p^n - p^{n-1} + p^{(n+s-2)/2}(p-1),$$

whose frequencies come from the cardinalities

$$|\{(a,u) \in \mathbb{F}_p^* \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = 1\}|$$

and

$$|\{(a,u) \in \mathbb{F}_p^* \times \mathbb{F}_{p^n} : f^*(a^{-1}u) = 0, \epsilon_f(a^{-1}u) = -1\}|,$$

which are equal to $(p-1)A_0$ and $(p-1)B_0$, respectively. By Lemma 3, we know that $A_0 = \frac{p^{n-s}+p^{\frac{n-s}{2}}}{2}$, $B_0 = \frac{p^{n-s}-p^{\frac{n-s}{2}}}{2}$. Thus, the weight enumerator polynomial is given by

$$1 + \frac{(p-1)}{2}(p^{n-s} + p^{\frac{n-s}{2}})z^{w_1} + (p^{n+1} - (p-1)p^{n-s} - 1)z^{w_2} + \frac{(p-1)}{2}(p^{n-s} - p^{\frac{n-s}{2}})z^{w_3},$$

where $w_2 = p^n - p^{n-1}$. For further reference, we record this discussion in Table 6.

Table 6: Weight distribution of the code $\mathcal{C}_f$ for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_2$ such that $f(0) = 0$, whose dual f^* is constant zero (so $n+s$ is even).

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} + p^{\frac{n-s}{2}})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s} - p^{\frac{n-s}{2}})$

Now we turn to analyzing the remaining cases of functions in $\mathscr{C}_2$. First, we consider the case $f \in \mathscr{C}_2$ and θ even, where $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$.

Theorem 3. *Let $n > 0$ and $0 \le s \le n$ be integers such that $n+s$ is even. Let $f \in \mathscr{C}_2$ be an s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = f^*(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathfrak{i}\}$ and $\theta \in \mathbb{N}, \theta > 0$ even. Suppose that f^* is not constant zero. The code $\mathcal{C}_f$ in (4) ($m = 1$) is a five-valued code with parameters $[p^n - 1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is summarized in Table 7.*

Proof. Again, the weights are seen to be $w_1 = p^n - p^{n-1} - t(f^*)p^{(n+s-2)/2}(p-1)$, $w_2 = p^n - p^{n-1} - t(f^*)p^{(n+s-2)/2}$, $w_3 = p^n - p^{n-1}$, $w_4 = p^n - p^{n-1} + t(f^*)p^{(n+s-2)/2}$ and $w_5 = p^n - p^{n-1} + t(f^*)p^{(n+s-2)/2}(p-1)$. Following a similar reasoning as in Theorem 2 and using Proposition 2 to obtain the values of A_j, B_j for each $j \in \mathbb{F}_p$, the number of codewords with weights w_1 and w_5 are, respectively,

$$(p-1)A_0 = (p-1)\left(\frac{p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}\right),$$

and

$$(p-1)B_0 = (p-1)\left(\frac{p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1}}{2}\right).$$

Since $A_j = B_j$ for each $j \in \mathbb{F}_p^*$ in this case, the number of codewords with weights w_2 and w_4 equals

$$(p-1)\sum_{j=1}^{p-1} A_j = \frac{(p-1)}{2}(p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1}).$$

Finally, there are $p^n - 1 + (p-1)(p^n - p^{n-s})$ balanced codewords.

□

Table 7: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 3, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_2$ such that $f(0) = f^*(0) = 0$ and $W_{f^*}(0) = t(f^*)\nu p^{\frac{\theta}{2}}$, when $n+s$ and θ are even.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1})$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1})$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}(p^{n-s-1} - t(f^*)p^{\frac{\theta}{2}-1})$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} - p^{\frac{n-s}{2}} + t(f^*)(p-1)p^{\frac{\theta}{2}-1})$

The last case remaining is $f \in \mathscr{C}_2$ and θ odd, where $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$. The main difference with the previous case is the use of Proposition 4 instead of Proposition 2.

Theorem 4. *Let $n > 0$ and $0 \le s \le n$ be integers such that $n+s$ is even. Let $f \in \mathscr{C}_2$ be an s-plateaued function defined over $\mathbb{F}_{p^n}$ with $f(0) = f^*(0) = 0$. Suppose that $W_{f^*}(0) = t(f^*)\nu' p^{\frac{\theta}{2}}$ for some $\nu' \in \{1, \mathfrak{i}\}$ and $\theta \in \mathbb{N}$, $\theta > 0$ odd. The code $\mathcal{C}_f$ in (4) ($m = 1$) is a five-valued code with parameters $[p^n - 1, n+1, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, whose weight distribution is displayed in Table 8.*

Proof. As before, the weights are $w_1 = p^n - p^{n-1} - t(f^*)p^{(n+s-2)/2}(p-1)$, $w_2 = p^n - p^{n-1} - t(f^*)p^{(n+s-2)/2}$, $w_3 = p^n - p^{n-1}$, $w_4 = p^n - p^{n-1} + t(f^*)p^{(n+s-2)/2}$ and $w_5 = p^n - p^{n-1} + t(f^*)p^{(n+s-2)/2}(p-1)$. Using Proposition 4, we compute the frequencies of codewords. The number of codewords with weight w_1 is

$$(p-1)A_0 = \frac{(p-1)}{2}(p^{n-s-1} + p^{\frac{n-s}{2}}).$$

The number of codewords with weight w_5 is

$$(p-1)B_0 = \frac{(p-1)}{2}(p^{n-s-1} - p^{\frac{n-s}{2}}).$$

The number of codewords of weight w_2 and w_4 is

$$(p-1)\sum_{j=1}^{p-1} A_j = \frac{(p-1)^2}{2} p^{n-s-1}.$$

Finally, there are $p^n - 1 + (p-1)(p^n - p^{n-s})$ balanced codewords. □

Table 8: Weight distribution of the code $\mathcal{C}_f$, derived in Theorem 4, for an s-plateaued function $f\colon \mathbb{F}_{p^n} \to \mathbb{F}_p$ in $\mathscr{C}_2$ such that $f(0) = f^*(0) = 0$ and $W_{f^*}(0) = t(f^*)\nu p^{\frac{\theta}{2}}$, when $n+s$ is even and θ is odd.

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} + p^{\frac{n-s}{2}})$
$p^n - p^{n-1} - p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}p^{n-s-1}$
$p^n - p^{n-1}$	$p^{n+1} - (p-1)p^{n-s} - 1$
$p^n - p^{n-1} + p^{(n+s-2)/2}$	$\frac{(p-1)^2}{2}p^{n-s-1}$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{(p-1)}{2}(p^{n-s-1} - p^{\frac{n-s}{2}})$

5.3 The weight distribution of $\mathcal{C}_F$

In this section, we extend the results in the previous sections to the case of vectorial plateaued functions. Little is known about infinite families of vectorial plateaued functions that are not bent, planar or quadratic. Namely, the only known examples are some power functions [6].

A vectorial plateaued function whose all components have the same amplitude $p^{\frac{n+s}{2}}$ is simply called a vectorial s-plateaued function.

Example 2. *For an integer k with $n/\gcd(n,k)$ odd, consider the functions $F\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $F(x) = x^{(p^{2k}+1)/2}$ and $F'\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $F'(x) = x^{p^{2k}-p^k+1}$ (Kasami). Then, both $F(x)$ and $F'(x)$ are s-plateaued, whose all components have zero duals.*

Example 3. *Working in $\mathbb{F}_{3^5}$, consider the 1-plateaued function $F\colon \mathbb{F}_{3^5} \to \mathbb{F}_{3^5}$ defined by $F(x) = x^{\frac{3^2+1}{2}} = x^5$. Using MAGMA, we have verified that the code $\mathcal{C}_F$ is a minimal self-orthogonal code with parameters $[242, 10, 144]$, dual distance $d^\perp = 2$ and weight enumerator polynomial $1 + 10890z^{144} + 39446z^{162} + 8712z^{180}$.*

The weight distributions for these two vectorial examples are easily derived in general.

Theorem 5. *Let $n > 0$ and $0 \leq s \leq n$ be integers such that $n + s$ is even. Let $F: \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be a vectorial s-plateaued function such that $F(0) = 0$, whose all components have zero duals. The code $\mathcal{C}_F$ in (4) ($m = n$) is a three-valued code with parameters $[p^n - 1, 2n, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$. Its weight distribution is given in Table 9.*

Proof. The code $\mathcal{C}_F$ can be seen as the union of the codes corresponding to the components of F, i.e., $\mathcal{C}_F = \bigcup_{a \in \mathbb{F}_{p^n}^\star} \mathcal{C}_{F_a}$. Since the components have zero duals, the weight distributions of the codes $\mathcal{C}_{F_a}$ are deduced from Table 6. Each unbalanced codeword in a single component contributes the numbers displayed in Table 6 divided by $p - 1$ since multiples of the component in question have to be excluded from this counting. Similarly, the number of balanced codewords not coming from linear functions is $p^n - p^{n-s}$ (subtract $p^n - 1$ from the second row in Table 6 and divide by $p - 1$). To count the total frequencies, each of the above frequencies has to be multiplied by $p^n - 1$, due to the number of components. This way we count frequencies from each component, except the frequency corresponding to balanced codewords stemming from linear functions, which is counted only once. The weight distribution is then easily obtained. □

Table 9: Weight distribution of $\mathcal{C}_F$ in Theorem 5, where $F: \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ is an s-plateaued function such that $F(0) = 0$, whose all components have zero dual ($n + s$ is even).

Weight w	Number of codewords
$p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)$	$\frac{1}{2}(p^n - 1)(p^{n-s} + p^{\frac{n-s}{2}})$
$p^n - p^{n-1}$	$p^n - 1 + (p^n - 1)(p^n - p^{n-s})$
$p^n - p^{n-1} + p^{(n+s-2)/2}(p-1)$	$\frac{1}{2}(p^n - 1)(p^{n-s} - p^{\frac{n-s}{2}})$

To build an infinite family of vectorial non-weakly regular s-plateaued functions whose components belong to $\mathscr{C}_1$, consider the following simple construction.

Construction A.

- Let $F_1(x)$ be any planar function (i.e., its components are bent) over $\mathbb{F}_{p^n}$, whose components are weakly regular (e.g. x^2).
- Let $F_2(y)$ be any of the functions over $\mathbb{F}_{p^n}$ provided in Example 2.
- Consider the direct sum $F : \mathbb{F}_{p^n} \times \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ given by $F(x, y) = F_1(x) + F_2(y)$. This is a vectorial 1-plateaued function, whose components are non-weakly regular and they belong to $\mathscr{C}_1$.
- Construct the code $\mathcal{C}_F$, which is a $[p^{2n} - 1, 3n]$-code.

Example 4. *Let $F_1, F_2 : \mathbb{F}_{3^3} \to \mathbb{F}_{3^3}$ be given by $F_1(x) = x^2$ and $F_2(y) = y^5$. The components of the function F_1 have Walsh spectra $\{* \ \epsilon \mathrm{i} 3^{\frac{3}{2}}, \epsilon \mathrm{i} 3^{\frac{3}{2}} \xi_3, \epsilon \mathrm{i} 3^{\frac{3}{2}} \xi_3^2 \ *\}$ with multiplicities $(9, 12, 6)$, where the sign ϵ depends on the component. This yields that the weight of the dual of each component is $18 = 3^3 - 3^{3-1}$. The components of the function F given in Construction A are non-weakly* *regular plateaued and have amplitude* $3^{\frac{6+1}{2}}$*. Moreover, they belong to* $\mathscr{C}_1$ *and satisfy the conditions in Theorem 1. Consider the code* $\mathcal{C}_F$*. This is a* 3*-weight code with parameters* $[728, 9, 477]$*. From Table 2, we see that the number of balanced codewords (without counting the ones coming from linear functions) is*

$$(3^3 - 1)(3^6 - 3^5 + 3^4) = 14742,$$

accounting a total of $3^6 - 1 + 14742 = 15470$ *balanced codewords. There are* $(3^3 - 1)(3^4 + 3^2) = 2340$ *codewords of weight* $3^6 - 3^5 - 3^3 = 459$*, whereas there are* $(3^3 - 1)(3^4 - 3^2) = 1872$ *of weight* $3^6 - 3^5 + 3^3 = 513$*. Hence, the weight enumerator polynomial of* $\mathcal{C}_F$ *is*

$$1 + 2340z^{459} + 15470z^{486} + 1872z^{513}.$$

The minimum distance of a linear code with parameters $[728, 9]$ *is at most* 483*. Hence, the minimum distance of the code* $\mathcal{C}_F$ *is rather close to it* ($\frac{459}{483} = 0.9503\ldots$)*. Besides having few weights and a large minimum distance,* $\mathcal{C}_F$ *enjoys some other desirable properties that will be described in Section 6.*

Remark 5. *In general, the weight distributions of codes from vectorial plateaued functions are easy to derive when we know the distribution of amplitudes of components and the classes they belong to (under the assumption that we can compute the dual weights and the values of* Z_0*). The challenge then arises when there is a mix of types and possible amplitudes, so that there is no clue on how many of each type there are or the amplitude distribution. In contrast, there are several instances when one can indeed compute the weight distributions of the codes, e.g., for known planar functions, subclasses of quadratic forms, etc.*

6 Properties of the obtained codes

A linear code $\mathcal{C}$ is said to be minimal if the supports of any two linearly independent codewords are not included in each other. It is easily seen that the obtained codes are minimal (almost always) by Ashikhmin-Barg's condition [1], which states that if the ratio between the minimum weight and the maximum weight of a p-ary code is strictly larger than $\frac{p-1}{p}$ then the code is minimal.

To illustrate this fact, consider the weight distribution of the code $\mathcal{C}_F$ given in Table 9. The minimum weight is $(p-1)(p^{n-1} - p^{(n+s-2)/2})$ and the maximum weight is $(p-1)(p^{n-1} + p^{(n+s-2)/2})$. The ratio equals

$$\frac{(p-1)(p^{n-1} - p^{(n+s-2)/2})}{(p-1)(p^{n-1} + p^{(n+s-2)/2})} = \frac{p^{n-1} - p^{(n+s-2)/2}}{p^{n-1} + p^{(n+s-2)/2}},$$

which is larger than $\frac{p-1}{p}$ if and only if

$$p^n - p^{(n+s)/2} > p^n + p^{(n+s)/2} - p^{n-1} - p^{(n+s-2)/2}.$$

Equivalently, $p^{(n-s-2)/2} + p^{-1} > 2$. The last inequality is true whenever $n - 2 > s$.

Furthermore, codes stemming from plateaued functions are always self-orthogonal. In general, self-orthogonality in the case of odd characteristic p can be characterized by the following property: a code $\mathcal{C}$ is self-orthogonal if and only if $\mathbf{c} \cdot \mathbf{c} = 0$ for all $\mathbf{c} \in \mathcal{C}$. For characteristic two, a code is self-orthogonal if and only if all weights are divisible by 2.

Theorem 6. *Let $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be any plateaued function such that $f(0) = 0$. The code $\mathcal{C}_f$ is included in its dual $\mathcal{C}_f^{\perp}$, i.e., $\mathcal{C}_f$ is self-orthogonal.*

Proof. By looking at the form of codewords, it suffices to prove that

$$\sum_{x\in\mathbb{F}_{p^n}} f(x)^2 + (\mathrm{Tr}_1^n((v_1+v_2)x))f(x) + \mathrm{Tr}_1^n(v_1x)\mathrm{Tr}_1^n(v_2x)$$

is divisible by p. Since $(\mathrm{Tr}_1^n((v_1+v_2)x))$ is balanced, the sum $\sum_{x\in\mathbb{F}_{p^n}}(\mathrm{Tr}_1^n((v_1+v_2)x))$ is divisible by p. Moreover, so is $\mathrm{Tr}_1^n(v_1x)\mathrm{Tr}_1^n(v_2x)$ by a similar reason. The value of $\sum_{x\in\mathbb{F}_{p^n}} f(x)^2$ is determined through the sums $\sum_{j=1}^{\frac{p-1}{2}} j^2(|f^{-1}(j)|+|f^{-1}(-j)|)$. It is a well-known result that $f^{-1}(j)$ is congruent to 0 modulo p for each j [25]. □

Remark 6. *A consequence of the previous approach is that codes stemming from vectorial plateaued functions $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^m}$ are also self-orthogonal. The code $\mathcal{C}_F$ can be regarded as the union $\bigcup_{a\in\mathbb{F}_{p^m}^{\star}} \mathcal{C}_{F_a}$. Thus, any codeword $\mathbf{c}$ (not induced by a linear function) belongs to at least one, in fact, $p-1$ codes corresponding to components, so that $\mathbf{c}\cdot\mathbf{c} = 0$ by Theorem 6.*

A code that is simultaneously minimal and self-orthogonal is *the best* we can expect, namely, there are no minimal self-dual codes besides two exceptions, as shown in the following.

Self-dual codes with $q = 2, 3$ are classified in three types [5]: a self-dual code with some codeword of weight not divisible by 4 is called singly-even or Type I; a self-dual code whose all codewords are divisible by 4 is called doubly-even or Type II; and self-dual codes over $\mathbb{F}_3$ are Type III.

Proposition 7. *There are no self-dual minimal linear codes for $q > 3$. The only self-dual minimal ternary code is the tetracode $[4, 2, 3]_3$, whereas the only self-dual minimal binary code is the repetition code $[2, 1, 2]_2$.*

Proof. Let $\mathcal{C}$ be a linear code with parameters $[n, k, d]_q$ with n even. If $\mathcal{C}$ is minimal then $k+q-2 \le d_{min} \le d_{max} \le n-k+1$, where d_{min} and d_{max} denote the minimum and the maximum distance in $\mathcal{C}$, respectively. For a reference of these results, see [7]. Thus if $\mathcal{C}$ is self-dual and minimal, we have

$$\frac{n}{2} + q - 2 \le d_{min} \le d_{max} \le \frac{n}{2} + 1. \tag{18}$$

Hence, for $q > 3$, the result follows. Let $q = 3$, i.e., $\mathcal{C}$ is a Type III code. In this case, the only possibility is that $d_{min} = d_{max} = \frac{n}{2} + 1$, so that $\mathcal{C}$ is also a one-weight code with parameters $[n, n/2, n/2 + 1]$. It's known that n is divisible by 4. Since $\mathcal{C}$ is self-dual, it is self-orthogonal, so $n/2 + 1 \equiv 0 \pmod 3$, which implies $n \equiv 1 \pmod 3$. Then $n = 4(3r + 1)$ for some $r \ge 0$. For a Type III code [5], $d_{min} \le 3\lfloor \frac{n}{12} \rfloor + 3$. Hence, $d_{min} \le 3\lfloor r + \frac{1}{3} \rfloor + 3 = 3r + 3$. On the other hand, $d_{min} = 6r + 3$, which yields $r = 0$. Thus, $\mathcal{C}$ must be the tetracode $[4, 2, 3]_3$.

Let $q = 2$. By (18), we get: $d_{min} = d_{max} = \frac{n}{2}$, $d_{min} = d_{max} = \frac{n}{2} + 1$ or $d_{min} = \frac{n}{2}$ and $d_{max} = \frac{n}{2} + 1$. Since a self-dual binary code is even, it must be that $d_{min} = d_{max}$. First suppose that $\mathcal{C}$ is of Type II, i.e. all codewords are divisible by four. In this case, $n \equiv 0 \pmod 8$, say, $n = 8r$ for some $r \ge 1$. It's well-known [5] that for self-dual binary codes it holds $d_{min} \le 2\lfloor \frac{n}{8} \rfloor + 2$. This yields $d_{min} \le 2r + 2$. Since $d_{min} = 4r$, the only possibility is that $\mathcal{C}$ has parameters $[8, 4, 4]_2$, that is, the extended Hamming code, which contains the all one vector $\mathbf{1}$. Now suppose that $\mathcal{C}$ is of

Type I (there are some codewords which are not divisible by four). For Type I codes, it holds [5] $d_{min} \leq 2\lfloor \frac{n+6}{10} \rfloor$ for $n \notin E := \{2, 12, 22, 32\}$. Assume that $n \notin E$. Suppose that $n \equiv 0 \pmod 4$, say, $n = 4r$ for some $r \geq 1$. It follows that $d_{min} = \frac{n}{2}$. This yields $2r = d_{min} \leq 2\lfloor \frac{2r+3}{5} \rfloor$. So $r \leq \lfloor \frac{2r+3}{5} \rfloor$, which is true only for $r = 1$, in other words, the code $\mathcal{C}$ has parameters $[4, 2, 2]_2$, which can be seen to contain **1**. Suppose that $n \equiv 2 \pmod 4$, say, $n = 4r + 2$ for some $r \geq 1$. It follows that $d_{min} = \frac{n}{2} + 1$. This yields $2r + 2 = d_{min} \leq 2\lfloor \frac{2r+3}{5} \rfloor$. So $r + 1 \leq \lfloor \frac{2r+3}{5} \rfloor$, which cannot happen. Using again the bound $d_{min} \leq 2\lfloor \frac{n}{8} \rfloor + 2$, we can rule out all the values of $n \in E$ except for $n = 2$. This finishes the proof. □

7 Conclusions

In this work, we have derived the weight distributions of linear codes associated with a subclass of p-ary plateaued functions, focusing on both single-output and vectorial cases. By partitioning p-ary plateaued functions into the classes $\mathscr{C}_1, \mathscr{C}_2$, and $\mathscr{C}_3$, based on the properties of their dual function f^*, we have extended and refined the existing results from the literature. Specifically, for s-plateaued functions with $n + s$ odd, we obtained complete weight distributions parametrized by the dual weight $wt(f^*)$. For the case when $n + s$ is even, we derived weight distributions using both $wt(f^*)$ and an additional parameter Z_0 for functions in the class $\mathscr{C}_1$. Our study extends previous research by presenting new results on weight distributions from functions belonging to subclasses of the plateaued function classes. Furthermore, by transitioning to the vectorial case, we have provided the weight distributions for codes associated with known vectorial plateaued functions. This approach has led to the construction of codes with parameters $[p^n - 1, 2n, p^n - p^{n-1} - p^{(n+s-2)/2}(p-1)]$, thus expanding the repertoire of linear codes with the properties of minimality and self-orthogonality. The results in this paper represent an advancement in the study of linear codes derived from p-ary plateaued functions, particularly vectorial ones, and offer valuable insights for future research into the properties and applications of these codes. Future work could explore more generalized constructions and investigate other subclasses of plateaued functions, potentially broadening the range of parameters and improving the properties of the resulting codes.

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