Questions in Logic

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To Sofia, who is a master
of wondering and questioning

Contents

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Ivano A. Ciardelli
Utrecht, January 2016.

Preface

Overview

This thesis pursues two tightly interwoven goals: to bring out the relevance of questions for the field of logic, and to establish a solid theory of the logic of questions within a classical logical setting. These enterprises feed into each other: on the one hand, the development of our formal systems will be motivated by our considerations concerning the role to be played by questions; on the other hand, it is via the development of concrete, workable logical systems that the potential of questions in logic will become clear and tangible. In the systems that we will develop, questions will play four different roles, which is worth previewing here.

Questions as information types

Imagine that a hospital has a protocol for dealing with a certain disease, which determines what treatment should be prescribed depending on a patient's age. We will see that, once logic is construed in such a way as to encompass questions, this situation emerges as one in which a question, 'what the patient's age is' entails another question, 'what treatment should be prescribed'. In general, while statements may be viewed as denoting specific *pieces* of information (say, the information that the patient is 42), questions may be viewed as denoting information *types* (say, the type consisting of all pieces of information which specify the patient's age). By extending the notion of entailment to questions, it becomes possible to capture not only the standard relation of consequence, which holds between specific pieces of information, but also the relation illustrated by our example, which we will call *dependency*, and which holds between different information types.

Questions as placeholders in proofs

Just as instances of dependency can be captured as question entailments, they can be proved to hold by making inferences with questions. As we will see, the role of a question in a logical proof is to stand for a generic piece of information of the corresponding type. For instance, in our proof systems we will be able to assume the question *what the patient's age is*; in doing so, we are not assuming any specific information about the patient's age; rather, we are assuming an arbitrary specification of the patient's age, and we are making inferences as to what other information we are ensured to have on that basis. Thus, in our proofs, a question essentially serves as a placeholder for generic information of a certain type, in the same way in which an individual constant may be used in first-order proofs as a placeholder for an arbitrary individual.

Questions as issues

Our approach to the semantics of statements and questions gives rise to a new, more encompassing notion of modal operators, which allows for a modal account not only of attitudes like *knowledge* and *belief*, which may be viewed as relations between agents and classical propositions, but also of attitudes like *wondering*, which are naturally viewed as relations between agents and issues. Similarly, we can use modalities to express the fact that an agent has complete control of a certain issue, i.e., that the issue's resolution is fully determined by the agent's choices. Again, questions play a crucial role: by embedding questions under suitable 'inquisitive' modal operators, our logics allow us to reason about relations between agents and issues, such as *wondering about* or *being in control of*.

Questions as actions

In recent years, much attention has been paid to logics of information change, i.e., logics for reasoning about how an informational scenario evolves as a result of certain communicative actions. These are known as *dynamic epistemic logics*. However, the process of information exchange cannot be understood purely in terms of information. Rather, information exchange is best viewed as a systematic process of raising and resolving issues. By working with models which explicitly represent issues, and with a language which includes questions, we can use the tools of dynamic epistemic logic to define logics of information *ex*change, in which agents may not only provide new information by making statements, but may also raise new issues by asking questions. From this perspective, the role of questions is to provide the means for issue-raising actions.

Structure of the thesis

The thesis is divided into eight chapters. The content of these chapters is summarized below. The dependency diagram of the dissertation is given in Figure 1.

Chapter 1. On the role of questions in logic

We start by showing that, if we move from the standard relation of truth at a possible world to a relation of *support at an information state*, we obtain a semantic foundation which allows us to interpret statements and questions in a uniform way. This move leads to a substantial generalization of the classical notion of entailment, which encompasses not only the standard relation of logical consequence, but also the relation of logical *dependency*. We lay out the formal foundations of our approach, which is then put to work in the subsequent chapters to obtain specific logical systems.

Chapter 2. Questions in propositional logic

In this chapter, the approach laid out in Chapter 1 is made concrete in the simplest possible setting, that of propositional logic. We describe how classical propositional logic can be enriched with questions, and discuss the features of this logic. The resulting system coincides with the propositional inquisitive logic of Ciardelli and Roelofsen (2011), but it is regarded here from a new perspective—as a conservative extension of classical logic, rather than as a non-classical logic.

Chapter 3. Reasoning with questions

This chapter is concerned with the role of questions in logical proofs. The chapter has three aims. First, we describe a natural deduction system for the propositional logic developed in the previous chapter. Second, we show that proofs involving questions in this system have an interesting computational interpretation, reminiscent of the proofs-as-programs interpretation of intuitionistic logic: namely, whenever a proof witnesses a certain dependency, it actually encodes a *method* for computing this dependency. Finally, we abstract away from the details of the given proof system, and focus on the role played by questions in inferences. Essentially, we will see that a question can be used as a placeholder for a generic piece of information of a certain type, much like individual constants are sometimes used in first-order logic proofs as placeholders for generic individuals. By manipulating such placeholders it is then possible to provide formal proofs of the validity of certain dependencies.

Chapter 4. Questions in first-order logic

In this chapter, our approach is taken to the setting of first-order logic. We describe a conservative extension of classical first-order logic with questions, and discuss how a broad range of interesting questions becomes expressible in this system, touching upon issues such as reference and identity. We identify two fragments of the language which jointly cover the most salient classes of first-order questions, and for each of them we provide a simple axiomatization.

Chapter 5. Relations with dependence logic

In this chapter, we discuss in detail the similarities and differences between our approach to dependency and the one adopted in the framework of dependence logic (Väänänen, 2007), which has seen considerable development in recent years. We will find that the fundamental difference is that in dependence logic, dependency is construed as a relation between variables, while in our approach, it is a relation between questions—in fact, none other than the relation of entailment.

We will argue that the approach based on questions has some important assets to it: first, it allows us to recognize and handle a broader range of dependencies than those considered in dependence logic. Moreover, by uncovering dependency as a facet of entailment, the question-based approach does not only lead to a neat conceptual picture, but also allows us to handle this relation by means of familiar logical tools, both semantically and proof-theoretically.

Chapter 6. Kripke modalities

This chapter is concerned with extending Kripke modalities to the context of a logic which includes questions. As an application, this makes it possible to generalize the *knowledge* modality of epistemic logic to embed both statements and questions, allowing for a uniform analysis of sentences like "A knows that B is home", "A knows whether B is home" and, in the first-order case, "A knows where B is". Technically, our main result is a uniform axiomatization result for the inquisitive counterpart of any given canonical modal logic.

Moreover, we propose and defend a modal account of dependence statements, i.e., statements such as "whether Alice will come depends on whether she finishes her homework". According to this account, a dependence statement can be formalized in our modal logic as a modal conditional among questions. Finally, we provide an axiomatization for the logic that arises from taking this conditional as our primitive modal operator.

Chapter 7. Inquisitive modalities

This chapter explores a generalization of the notion of modal operator, suggested by our enriched semantics. We will replace Kripke models with *inquisitive modal* *models*, which equip each world not just with an information state—a set of successors—but with an inquisitive state, encoding both information and issues.

As an application, we investigate *inquisitive epistemic logic*, an enrichment of epistemic logic in which we can reason about agents who do not only have certain information, but also entertain certain issues, both individually and as a group.

We will provide completeness results both for the basic inquisitive modal logic, and for a range of modal logics that result from imposing various interesting frame conditions. We will see that, while inquisitive modalities are more expressive than standard Kripke modalities, they retain a very well-behaved theory, and they are characterized by simple logical features.

Chapter 8. Dynamics

In this final chapter we dynamify inquisitive epistemic logic, by generalizing the standard account of public announcements in epistemic logic. The resulting logic allows us to reason about the way in which an inquisitive-epistemic situation changes not only when new information is provided by announcing a statement, but also when a new issue is raised by asking a question. A remarkable feature of this logic is that we do not need two separate actions for announcing and asking: one action of *public utterance* suffices; it is the meaning of the sentence being uttered that determines whether the effect of the utterance is to provide new information, or to raise new issues. We will establish a complete axiomatization of this dynamic logic by means of reduction rules that allow us to transform every formula of this logic into a formula of inquisitive epistemic logic.

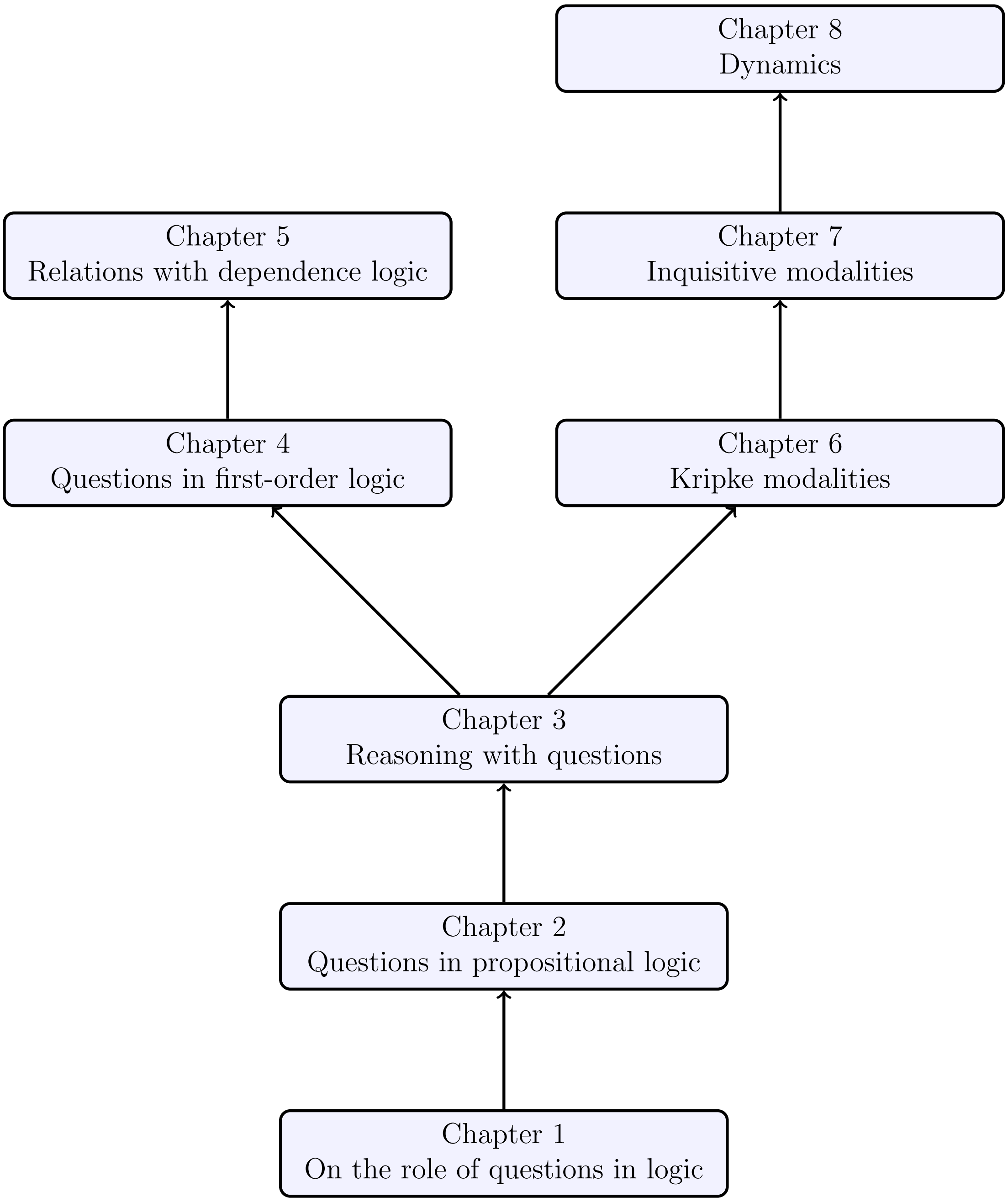

Figure 1: Dependency diagram of the dissertation.

Chapter 1

On the role of questions in logic

Traditionally, questions have been assumed to have no role to play in logic. This chapter is devoted to showing that the opposite is true: questions can be brought within the scope of logic in a natural way, and this results in an exciting generalization of the notion of logical entailment. Traditionally, logical entailment captures patterns such as the one exemplified by (1): the information that Alice and Bob live in the same city, combined with the information that Alice lives in Amsterdam, yields the information that Bob lives in Amsterdam.

(1) Alice and Bob live in the same city
Alice lives in Amsterdam

Bob lives in Amsterdam

By bringing questions into the picture, the notion of entailment is extended so as to capture patterns which we might write as in (2): the information that Alice and Bob live in the same city, combined with the information on where Alice lives, yields the information on where Bob lives.

(2) Alice and Bob live in the same city
Where Alice lives

Where Bob lives

Notice the crucial difference between the two examples: in (1) we are concerned with a relation holding between three specific pieces of information. The situation is different in (2): given the information that Alice and Bob live in the same city, *any* given piece of information on Alice's city of residence yields some corresponding information on Bob's city of residence. We may say that what is at play in (2) are two *types* of information, which we may see as labeled by the questions *where Alice lives* and *where Bob lives*. Entailment captures the fact that, given the assumption that Alice and Bob live in the same city, information of the first type yields information of the second type.

Relations of this kind, which we will refer to as *dependencies*, play a role in a multitude of contexts, ranging from everyday situations to specific fields such as physics, linguistics, and computer science, as discussed in Section 1.6.1. Thus, bringing questions into play broadens the scope of classical logic in a substantial way, allowing it to deal with new, interesting logical relations.[1]

This initial chapter, based on Ciardelli (2015a,b), has a twofold purpose: first, it provides an introduction to the framework of inquisitive semantics (Ciardelli et al., 2013a, 2015a), and offers a novel perspective on it, one which is particularly suitable for the enterprise undertaken in this thesis.[2] Second, it provides a new, independent source of motivation for this framework: while previous work on inquisitive semantics was driven by motivations stemming from linguistics and philosophy of language, in this chapter—and then in the thesis at large—we will argue that the move from a classical semantics to an inquisitive semantics also has solid, independent motivations stemming from within the field of logic.

In order to make our points as simply and generally as possible, we will abstract away for the time being from the details of a specific logical system. We will, e.g., leave notions such as *sentence*, *statement*, *question* and *possible worlds* underspecified, focusing instead on the fundamental ideas underlying the approach, and on the main formal features that follow from them. In other words, the aim of this chapter is to set up the general template for a uniform logic of statements and questions: in the following chapters, this template will then be instantiated in a variety of ways, yielding concrete logical systems where the notions and the facts discussed in this section can be, respectively, rigorously defined and proved.

The chapter is structured as follows. In Section 1.1 we show that, by moving from the standard truth-conditional conception of meaning to an information-oriented conception, we obtain a semantic framework which can interpret both statements and questions in a uniform way. We then describe how the relation of dependency emerges as a facet of the relation of entailment, once the latter is generalized to questions. In Section 1.2 we show that questions can be thought of as denoting *types* of information, and that dependency may be viewed as generalizing entailment from pieces of information to information types. In Section 1.3 we discuss how the notion of *truth* extends to questions, and how this notion is related to the notion of a question's *presupposition*. In Section 1.4 we show that a logic based on a semantics of the kind described here can be equipped

[1] Within logic, dependencies are the subject of much recent work within the framework of *Dependence Logic*. This line of work is an important source of inspiration for this thesis. The relations between this framework and the approach developed here are the subject of Chapter 5.

[2] This perspective is inspired by work on so-called *dichotomous* inquisitive systems, explored in Groenendijk (2011); Ciardelli et al. (2015b). Such systems assume a syntactic distinction of sentences into declaratives and interrogatives. While the systems investigated in this thesis are not dichotomous in this sense, our perspective is close to that of §6 in Ciardelli et al. (2015b).

in a canonical way with an operation of implication that internalizes the meta-language relation of entailment. Section 1.5 summarizes our findings. Finally, in Section 1.6, we describe the relevance of the dependence relation in several domains, motivate some of our setup choices, and compare the approach proposed here to some previous logical approaches to questions.

1.1 Dependency as question entailment

1.1.1 A motivating example

Suppose a certain disease may give rise to two symptoms, S_1 and S_2, the latter much more distressing than the former. Suppose the disease may be countered by means of a certain treatment, which however carries some associated risk. A hospital has the following protocol for dealing with the disease: if a patient presents symptom S_2, then the treatment is always prescribed. However, if the patient only presents symptom S_1, then the treatment is only prescribed in case the patient is in good overall physical condition; if not, the risks associated with the treatment outweigh the benefits, and the treatment is not prescribed.

Given the hospital's protocol, then, whether or not the treatment is prescribed is determined by two things: (i) what symptoms the patient presents and (ii) whether the patient is in good physical condition. In other words, in the context we described, a certain relation holds between the following questions:

μ_1 What symptoms the patient presents

μ_2 Whether the patient is in good physical condition

ν Whether the treatment is prescribed

This relation amounts to the following: as soon as both questions μ_1 and μ_2 are settled, the question ν is bound to be settled as well; that is, as soon as we settle what the patient's symptoms and physical conditions are, it is also settled whether or not the treatment is prescribed. We say that in the given situation, the questions μ_1 and μ_2 *determine* the question ν, and we refer to this relation as a *dependency*. In this section, we look in detail at the logical status of the relation of dependency. We will find out that it is nothing but a manifestation of the familiar notion of *entailment*, once this notion is generalized so that it applies not only to statements, but also to questions.[3]

[3] A terminological remark is in order here: this technical usage of the word *dependency* is not quite in line with the ordinary sense of the term, being weaker in one respect, and stronger in another. On the one hand, if ν is already settled in a certain way in a context, independently of the answer to another question μ, then in our technical sense, ν *does* depend on μ, although in ordinary language, we would say that it does not. In this sense, the technical notion of dependency is weaker than the ordinary notion, since dependencies are not required to be non-trivial. On the other hand, if in the given context the answer to ν is partly determined by the

1.1.2 From states of affairs to states of information

Traditionally, logic has been concerned with relations between a particular kind of sentences, namely *statements*, or *declarative sentences*, which may be regarded as descriptions of a state of affairs. In particular, classical logic arises from the default assumption that the meaning of a statement lies in its truth-conditions, that is, in the conditions that a (complete) state of affairs must satisfy in order for the statement to be true.

We will refer to the formal representation of a complete state of affairs as a *possible world*. The exact nature of possible worlds depends on the specific logical framework. Often, a possible world may be identified simply with a model for the language at stake. However, in so-called *intensional* logics, which aim at representing a whole variety of states of affairs in a single model, possible worlds are internalized as particular entities within the model, which are then equipped with a description of a complete state of affairs with respect to the language at hand. The models that we will use in this thesis are of the latter kind. Therefore, we will henceforth think of possible worlds as being drawn from a given set W_M, which is part of a specific model M.

In the truth-conditional approach, then, semantics consists in the specification of a relation $w \models \alpha$ between possible worlds w and statements α, which holds in case α is true in the possible world w. The central notion of logic, the relation of *entailment*, can then be defined as preservation of truth: α entails β if the truth of α implies the truth of β.

$$\alpha \models \beta \iff \text{for all models } M \text{ and worlds } w \in W_M,\ w \models \alpha \ \text{ implies } \ w \models \beta$$

This is, in a nutshell, the standard way to make sense of the fundamental notion of entailment in classical logic. This perspective really does lead to the expectation that questions have no role to play in logic: after all, it is not even clear a priori what it would mean for a question to be *true* or *false* in a state of affairs. Since entailment is defined in terms of truth, it is also not clear what it would mean for a question to occur as an assumption or as a conclusion of an entailment relation.

However, it is possible to give an alternative semantic foundation for classical logic, which starts out from a more information-oriented perspective. Rather than taking the meaning of a statement α to be given by laying out in which circumstances α is true, we may take it to be given by laying out *what information it takes to settle that* α. In this approach, α is evaluated not with respect to states

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answer to μ, and partly by other factors, then in our technical sense, ν does *not* depend on μ, although in the ordinary sense, it does: in our example, for instance, we would ordinarily say that the treatment depends on the symptoms, even though it is not fully *determined* by the symptoms. In this sense, the technical notion of dependency is stronger than the usual notion. A better name for the relation that we are going to investigate would probably be *determinacy*; I will stick to the term *dependency* for the sake of consistency with the existing literature.

of affairs, but instead with respect to bodies of information, which we will refer to as *information states*.[4]

A simple and perspicuous way of modeling an information state, which goes back at least to Hintikka (1962) and which is widely adopted both in logic and in linguistics, is to identify it with a set s of possible worlds, namely, those worlds that match the available information—that is, that are compatible with it. In other words, if s is a set of possible worlds, then s encodes the information that the actual state of affairs corresponds to one of the possible worlds in s. If $t \subseteq s$, this means that in t, at least as much information as in s is available, and possibly more. We will say that t is an *enhancement* of s or also that t *entails* s.

In the informational approach that we will explore, semantics will thus be given by a relation $s \models \alpha$, called *support*, between information states $s \subseteq W_M$ and statements α, which holds in case α is *settled* in the information state s.

This semantic perspective brings along a corresponding notion of entailment as preservation of support: α entails β if settling that α implies settling that β.

$$\alpha \models \beta \iff \text{for all models } M \text{ and info states } s \subseteq W_M,\ s \models \alpha \text{ implies } s \models \beta$$

Now, we regard a sentence α as being settled in s just in case it follows from the information in s that α is true, i.e., in case s is only compatible with worlds in which α is true. But this means that s settles α iff all the worlds in s are worlds where α is true. Let us write $|\alpha|_M$ for the set of worlds in M where α is true, that is, $|\alpha|_M = \{w \in W_M \mid w \models \alpha\}$. Then, for all models M and information states $s \subseteq W_M$ we have:

$$s \models \alpha \iff s \subseteq |\alpha|_M \tag{1.1}$$

Thus, the support conditions for a statement are completely determined by its truth-conditions. On the other hand, if we consider this connection in the special case that s is a singleton $\{w\}$, we find that it also implies that, conversely, truth-conditions are determined by support conditions.

$$\{w\} \models \alpha \iff \{w\} \subseteq |\alpha|_M \iff w \in |\alpha|_M \iff w \models \alpha \tag{1.2}$$

Intuitively, what this says is that α is true at a world w just in case the information that w is the actual world establishes that α is true.

These connections show that, for statements, the truth-conditional approach and the support-conditional approach are two sides of the same coin: support

[4]Information-oriented semantics have been considered quite often in the literature, especially as a starting point for non-classical logics (e.g., Beth, 1956; Kripke, 1965; Veltman, 1981), but sometimes also as alternative foundations for classical logics (e.g., Fine, 1975b; Humberstone, 1981; van Benthem, 1986; Holliday, 2014). As far as the treatment of classical logic is concerned, our system will be very similar to the ones in the latter tradition, though with one difference in setup, to be discussed in Section 1.6.2. To the best of my knowledge, however, no previous attempt has been made to use such a semantic foundation to bring questions into play in logic.

conditions and truth conditions are interdefinable. What is more, the truth-conditional notion of entailment and the support-conditional one coincide.

To see this, suppose α truth-conditionally entails β, and consider an information state $s \models \alpha$: this means that α is true everywhere in s. Since α truth-conditionally entails β, β must be true everywhere in s, too. But then, β must be supported in s. This shows that α entails β in the support-conditional sense.

Conversely, suppose α entails β in the support-conditional sense, and consider a world $w \models \alpha$. Then, $\{w\}$ is a state which supports α. Since α entails β in the support-conditional sense, $\{w\}$ must also support β, which means that we must have $w \models \beta$. This shows that α entails β in the truth-conditional sense.

What all this means, in short, is that support semantics does not give rise to a new logic of its own, but instead provides an alternative, informational semantic foundation for classical logic.

1.1.3 Questions enter the stage

Given the equivalence between the truth-conditional and the support-conditional view of the semantics of statements, it is not surprising that the former perspective, based on simpler semantic objects, is taken as the standard one. However, support semantics has an advantage which is perhaps not immediately obvious: unlike truth-conditional semantics, it naturally accommodates not only statements, but also *questions*. For, while it is not clear what it means for a question to be true or false given a certain state of affairs, there is a natural sense in which a question can be said to be *settled* in an information state: namely, when the question is completely resolved by the information available in the state.

For a concrete illustration, consider one of the questions in our example, the question μ_1 of what symptoms, out of S_1 and S_2, the patient presents. An information states settles this question in case either (i) it settles that the patient presents neither symptom, or (ii) it settles that the patient presents only S_1, or (iii) it settles that the patient presents only S_2, or finally, (iv) it settles that the patient presents both symptoms. This means that μ_1 is settled in $s \subseteq W_M$ just in case s is included in one of the following four states:

- $a_\emptyset = \{w \in W_M \mid \text{patient has no symptoms in } w\}$
- $a_1 = \{w \in W_M \mid \text{patient has only symptom } S_1 \text{ in } w\}$
- $a_2 = \{w \in W_M \mid \text{patient has only symptom } S_2 \text{ in } w\}$
- $a_{12} = \{w \in W_M \mid \text{patient has both symptoms in } w\}$

Not only are support conditions defined for questions: there are also good reasons to regard them as a good candidate for the role of question meaning. For, questions are used primarily, though not uniquely, in order to specify requests for information: it is therefore natural to expect that to know the meaning of a question is to know what information is requested by asking it, that is, what information state has to be brought about in order for the question to be settled. That is precisely what is encapsulated into the question's support conditions.

1.1.4 Generalizing entailment

So far, we saw that the move from truth to support leads us to a semantic foundation that allows us to interpret questions on a par with statements. As a consequence of this, the notion of entailment defined as preservation of support is just as meaningful for questions as it is for statements. Let us look at this notion in some more detail: in full generality, entailment will be defined as a relation $\Phi \models \psi$ between a set Φ of sentences, which may include questions as well as statements, and a sentence ψ, which may be either a statement or a question.

$$\Phi \models \psi \iff \text{for all models } M \text{ and info states } s \subseteq W_M,\ s \models \Phi \text{ implies } s \models \psi$$

where $s \models \Phi$ is shorthand for '$s \models \varphi$ for all $\varphi \in \Phi$'. Now, focusing on the case of a single premise, we have four possible entailment patterns. Let us examine briefly the significance of each case.

- Statement-to-statement. If α and β are statements, then $\alpha \models \beta$ expresses the fact that settling that α implies settling that β. As we have already discussed, this coincides with the familiar, truth-conditional notion of entailment: $\alpha \models \beta$ holds in case β is true whenever α is.

- Statement-to-question. If α is a statement and μ is a question, then $\alpha \models \mu$ expresses the fact that settling that α implies settling μ. Thus, we may regard $\alpha \models \mu$ as expressing the fact that α *logically resolves* μ.

  E.g., the statement 'Galileo discovered Jupiter's moons' entails the question 'Did Galileo discover anything?', for this questions is resolved in any information state where the truth of the statement is established.

- Question-to-statement. If μ is a question and α is a statement, then $\mu \models \alpha$ means that settling the question μ, in any way, implies settling that α. We may thus regard $\mu \models \alpha$ as expressing the fact that μ *logically presupposes* α.

  E.g., the question 'In what year did Galileo discover Jupiter's moons?' entails the statement 'Galileo discovered Jupiter's moons', for it is impossible to resolve the question without also establishing the truth of the statement.

- Question-to-question. If μ and ν are both questions, then $\mu \models \nu$ expresses the fact that settling μ implies settling ν. This is precisely the relation of *dependency* that we set out to examine, but now in its purely logical version, since *all* possible worlds in all models—not just those worlds that are relevant in a given context, as the protocol of our example—are taken into account. We may thus read $\mu \models \nu$ as expressing that the question μ *logically determines* the question ν.[5]

E.g., the question 'When and where did Galileo discover Jupiter's moons?' entails the question 'When did Galileo discover Jupiter's moons?', since any information resolving the former question also resolves the latter question.

Thus, support-conditional semantics gives rise to an interesting general notion of entailment, which concerns questions as well as statements, and which unifies four natural logical notions: (i) a statement being a logical consequence of another; (ii) a statement logically resolving a question; (iii) a question logically presupposing a statement; and, finally, (iv) a question logically determining another.

1.1.5 Entailment in context

In ordinary situations, it is rarely the purely *logical* notion of consequence that we are concerned with. Rather, we typically take many facts about the world for granted, and then assess whether *on that basis*, the truth of one statement implies the truth of the other. We say, for instance, that "Galileo discovered some celestial bodies" is a consequence of "Galileo discovered Jupiter's moons"; in doing so, we take for granted the fact that Jupiter's moons are celestial bodies; worlds in which Jupiter's moons are not celestial bodies are simply disregarded.

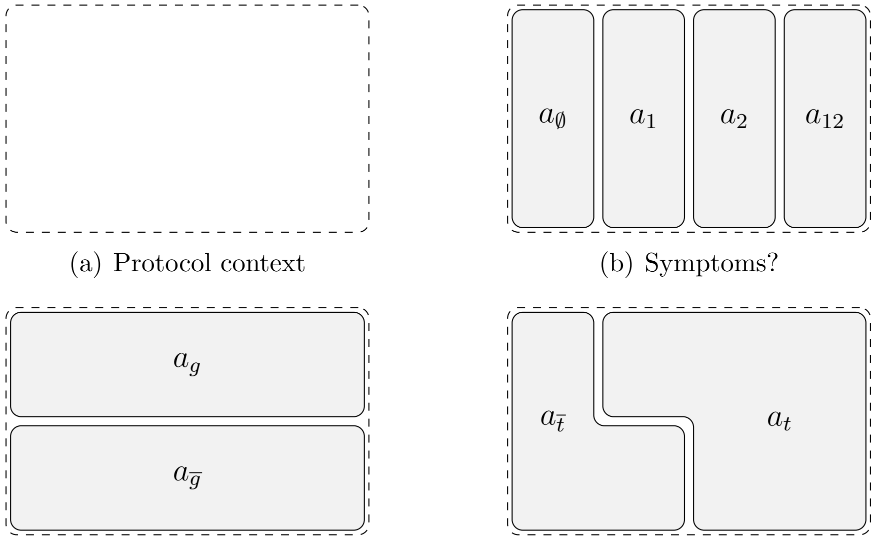
The same holds for questions: when we are concerned with dependency, it is rarely purely *logical* dependency that is at stake. Rather, we are usually concerned with the relations that one question bears to another, given certain background facts about the world. In our initial example, for instance, it is the hospital's protocol that provides the context relative to which the dependency holds. It is only against the background of this specific context that the three questions in the example are linked by any interesting relations.

In order to capture these relations, besides the absolute notion of logical entailment that we discussed, we will also introduce notions of entailment relative to a given model M, and relative to a given *context*. We will model a context simply as an information state s. In assessing entailment relative to s, we take the information embodied by s for granted. This means that, to decide whether an entailment holds or not, only worlds in s, and states consisting of such worlds, are taken into account. Formally, we make the following definition.

1.1.1. Definition. [Contextual entailment]
Let M be a model and let $s \subseteq W_M$ be an information state. We let:

$$\Phi \models_s \psi \iff \text{for all info states } t \subseteq s,\ t \models \Phi \text{ implies } t \models \psi$$

[5]If the reader finds this way of phrasing the relation $\mu \models \nu$ ungrammatical, she should feel free to use the longer phrasing "any answer to μ determines some corresponding answer to ν". However, notice that no notion of answers is needed to formulate the relation. Also, notice that when μ and ν are concrete questions in English, the phrasing "μ determines ν" is perfectly fine, e.g.: "What symptoms the patient has determines what treatment the patient should get."



(a) Protocol context (b) Symptoms?

(c) Condition? (d) Treatment?

Figure 1.1: The meanings of the three questions involved in our initial example, within the context provided by the hospital's protocol. To avoid clutter, only the *maximal* sub-states of the context in which a question is resolved are displayed.

Moreover, we write $\Phi \models_M \psi$ instead of $\Phi \models_{W_M} \psi$ for entailment relative to the set of all the worlds in M.

Contextual entailment captures relations of consequence, resolution, presupposition, and dependency which hold not purely logically, but against the background of a specific context.

Focusing on dependency, let us see how our hospital protocol example is captured as an instance of entailment in context. Consider a model M, and let s denote our hospital protocol context, which consists of the set of worlds which are compatible with the hospital's protocol. Thus, e.g., s may contain worlds where the patient has both symptoms and the treatment is prescribed, but not worlds where the patient has both symptoms and the treatment is *not* prescribed, since such worlds are incompatible with the hospital's protocol.

Now, a state $t \subseteq s$ settles the question μ_1 of what symptoms the patient has in case it settles whether the patient has symptom S_1 and whether she has symptom S_2. This holds just in case t is included in one of the following four states, depicted in figure 1.1(b):

- $a_\emptyset \;=\; \{w \in s \,|\, \text{patient has no symptoms in } w\}$

- $a_1 \;=\; \{w \in s \,|\, \text{patient has only symptom } S_1 \text{ in } w\}$

- $a_2 \;=\; \{w \in s \,|\, \text{patient has only symptom } S_2 \text{ in } w\}$

- $a_{12} = \{w \in s \,|\, \text{patient has both symptoms in } w\}$

Moreover, a state $t \subseteq s$ settles the question μ_2 of whether the patient is in good condition just in case it settles that the patient *is* in good condition, or it settles that the patient is *not* in good condition. This holds just in case t is included in one of the following two states, depicted in Figure 1.1(c):

- $a_g = \{w \in s \,|\, \text{patient is in good condition in } w\}$
- $a_{\overline{g}} = \{w \in s \,|\, \text{patient is not in good condition in } w\}$

Finally, a state $t \subseteq s$ settles the question ν of whether the treatment is prescribed just in case it settles that the treatment is prescribed, or it settles that the treatment is not prescribed. That is, in case t is included in one of the following two states, depicted in Figure 1.1(d):

- $a_t = \{w \in s \,|\, \text{treatment is prescribed in } w\}$
- $a_{\overline{t}} = \{w \in s \,|\, \text{treatment is not prescribed in } w\}$

Now, clearly, relative to the context s, neither μ_1 nor μ_2 by itself entails ν. For instance, μ_1 is settled in the state a_1, but ν is not. This corresponds to the fact that the information that the patient has only symptom S_1 is not sufficient to determine whether the treatment is prescribed or not. Similarly, μ_2 is settled in each of the states a_g and $a_{\overline{g}}$, but ν is not. This corresponds to the fact that the information whether the patient is in good condition is not sufficient to determine whether the treatment is prescribed. Hence, we have $\mu_1 \not\models_s \nu$ and $\mu_2 \not\models_s \nu$, which captures the fact that ν is not determined by either μ_1 or μ_2 in the given context.

At the same time, μ_1 and μ_2 together *do* entail ν relative to s. For, consider a state $t \subseteq s$ which settles both μ_1 and μ_2: since t settles μ_1, t must be included in one of the sets $a_\emptyset$, a_1, a_2, a_{12}; and since t settles μ_2, t must be included in one among a_g and $a_{\overline{g}}$. It is clear by inspecting the figure that any such state must be included in one among a_t and $a_{\overline{t}}$, which means that it also settles ν. Thus, we have $\mu_1, \mu_2 \models_s \nu$, which captures the fact that ν is jointly determined by μ_1 and μ_2 in the given context. As we had anticipated, the dependence relation of our initial example emerges as an instance of entailment—more precisely, as a case of question entailment in context.

1.1.6 Contextual entailment and logical entailment

Contextual entailments can be turned into *logical* entailments by making the relevant features of the context into explicit premises. If Γ is a set of statements, let $|\Gamma|_M$ be the set of worlds in M where all the formulas in Γ are true. We can regard the information state $|\Gamma|_M$ as the context described by Γ in M. We will write $\Phi \models_\Gamma \psi$ in case Φ is guaranteed to entail ψ in the context described by Γ.

$$\Phi \models_\Gamma \psi \overset{def}{\iff} \Phi \models_{|\Gamma|_M} \psi \text{ for all models } M$$

Then, we have the following connection.

1.1.2. Proposition (Decontextualization). $\Phi \models_\Gamma \psi \iff \Gamma, \Phi \models \psi$

Proof. First, it is easy to see that relation 1.1 extends from single statements to sets of statements, so we have $s \models \Gamma \iff s \subseteq |\Gamma|_M$. Using this fact, we have:

$$\begin{array}{rcl} \Gamma, \Phi \models \psi & \iff & \text{for all } M \text{ and all } s \subseteq W_M,\ s \models \Gamma \text{ and } s \models \Phi \text{ implies } s \models \psi \\ & \iff & \text{for all } M \text{ and all } s \subseteq W_M,\ s \subseteq |\Gamma|_M \text{ and } s \models \Phi \text{ implies } s \models \psi \\ & \iff & \text{for all } M \text{ and all } s \subseteq |\Gamma|_M,\ s \models \Phi \text{ implies } s \models \psi \\ & \iff & \text{for all } M,\ \Phi \models_{|\Gamma|_M} \psi \\ & \iff & \Phi \models_\Gamma \psi \end{array}$$

□

This means that, if the relevant contextual assumptions are describable by a set Γ of statements, contextual entailment amounts to logical entailment with the statements in Γ as additional premises. In our hospital protocol example, the context may be described by a statement such as the following:

(γ) The treatment is prescribed if and only if the patient presents symptom S_2, or the patient presents symptom S_1 and is in good physical condition.

In establishing the validity of the contextual entailment $\mu_1, \mu_2 \models_s \nu$ above, all we used was the fact that $s = |\gamma|_M$, not any specific assumption about the model M. This means that we have $\mu_1, \mu_2 \models_\gamma \nu$, which by what we have just seen amounts to $\gamma, \mu_1, \mu_2 \models \nu$.

This illustrates how the context for a dependency can be captured not only semantically, as a restrictor parameter for the entailment relation, but also syntactically, by having a set of statements as additional assumptions.

1.2 Questions as information types

In this section we show that both statements and questions may be regarded as describing information types: statements describe singleton types, which may be identified with specific pieces of information, while questions describe proper, non-singleton information types. This illustrates how the support approach may be viewed as generalizing the classical notion of entailment from pieces of information to arbitrary information types.

1.2.1 Inquisitive propositions

In truth-conditional semantics, the meaning of a sentence α in a model M is encoded by its *truth-set*, that is, by the set of all worlds in M where α is true.

$$|\alpha|_M = \{w \in W_M \mid w \models \alpha\}$$

Similarly, in support-conditional semantics, the meaning of a sentence φ in a model M is encoded by its *support-set*, that is, the set of all states in M where φ is supported:

$$[\varphi]_M = \{s \subseteq W_M \mid s \models \varphi\}$$

The support-set of a formula is a set of information states of a special form. For, suppose an information state s settles a sentence φ: then, any information state t that enhances s will also settle φ. That is, the relation of support is *persistent*:[6,7]

Persistency: if $t \subseteq s$, $s \models \varphi$ implies $t \models \varphi$

This implies that the support-set of a sentence $[\varphi]$ is always *downward closed*, that is, if it contains a state s, it also contains all stronger states $t \subseteq s$.

Downward closure: if $t \subseteq s$, $s \in [\varphi]_M$ implies $t \in [\varphi]_M$

Another way to state downward closure, which will turn out useful later on, uses the notion of *downward closure* of a set of states. In words, the downward closure of a set S of info states is the set of those states which entail some element of S.

1.2.1. Definition. [Downward closure]
If $S \subseteq \wp(W_M)$, the *downward closure* of S is the set $S^{\downarrow}$ defined as follows:

$$S^{\downarrow} = \{s \subseteq W_M \mid s \subseteq t \text{ for some } t \in B\}$$

It is easy to see that $S^{\downarrow}$ is always a downward closed set of states, and moreover, it is the *smallest* downward closed set of states which contains S. The fact that the support-set $[\varphi]_M$ of a formula is downward closed may then be expressed succinctly as follows:

[6] At this point, this may be seen as a stipulation on what it means for a sentence to be *settled* in a state—though in the case of statements this actually follows from the way in which support-conditions are linked to truth-conditions (relation 1.1). Later on in the thesis, persistency will be a fact, that can be formally proved for each of the languages with which we will be concerned.

[7] It is worth pointing out that not all information-based semantics are persistent. E.g., the data semantics of Veltman (1981) contains sentences of the form "may φ" that are supported by a body of information s not if some information is available in s, but rather if some information is compatible with s. The semantics of such sentences (as treated in Veltman's account) cannot be given in terms of what information it takes to *settle* the sentence; thus, such sentences fall outside the immediate scope of the conceptual picture developed here. However, the systems of inquisitive logic described in the following chapters can in principle be extended in a conservative way with operations which yield non-persistent meanings: for the propositional inquisitive logic of the next chapter, one such extension has been investigated by Punčochář (2015b), who also provided a complete axiomatization of the resulting logic.

Downward closure, restated: $[\varphi]_M = ([\varphi]_M)^{\downarrow}$

Downward closure is not the only general feature of the support-set of a sentence. For, consider the empty information state, $\emptyset$. This represents an inconsistent body of information, which is not compatible with any possible world. Notice that relation 1.1 between support and truth implies that $\emptyset$ supports any statement. This may be seen as a natural semantic version of the *ex-falso quodlibet* principle of classical logic. Similarly, it is natural to assume that $\emptyset$ also trivially supports any question, so that for all φ we have the following.[8]

Semantic ex-falso: $\emptyset \in [\varphi]_M$

We will refer to a set of states which contains $\emptyset$ and satisfies downward closure as an *inquisitive proposition.* It is easy to see that, due to downward closure, this definition is equivalent to the following one.

1.2.2. Definition. [Inquisitive propositions]
An inquisitive proposition in a model M is a non-empty and downward closed set $P \subseteq \wp(W_M)$ of information states.

The previous observations ensure that the support-set $[\varphi]_M$ of a sentence in a model is always an inquisitive proposition, which we refer to as the *proposition expressed by* φ in M.

As we will see, a special role is played by the maximal elements in an inquisitive proposition P. We will refer to these elements as the *alternatives* in P.

1.2.3. Definition. [Alternatives]
$\textsc{Alt}(P) = \{s \in P \mid \text{there is no } t \supset s \text{ such that } t \in P\}$

If φ is a sentence, we refer to the alternatives in $[\varphi]_M$ as the *alternatives for* φ. Thus, the alternatives in φ are the minimally informed states that support φ, i.e., those states that contain *just enough* information to settle φ. We will write $\textsc{Alt}_M(\varphi)$ instead of $\textsc{Alt}([\varphi]_M)$.

1.2.2 Pieces and types of information

Consider the proposition expressed by a statement α. In addition to downward closure, this proposition has another crucial feature. Given the relation in 1.1 between the support-conditions for a statement and its truth-conditions, we have:

$$s \in [\alpha]_M \iff s \subseteq |\alpha|_M$$

This connection brings out two interesting facts: first, that $[\alpha]_M$ has a unique maximal supporting state—a unique alternative—namely $|\alpha|_M$.

[8]As for downward closure, once we will turn to concrete logical systems, this assumption will become a provable feature of the system.

- $\textsc{Alt}_M(\alpha) = \{|\alpha|_M\}$

Second, that an information state settles α iff if it entails this unique alternative. By means of downward closure, we can express this succinctly as follows:

- $[\alpha]_M = \{|\alpha|_M\}^{\downarrow}$

Now, the unique alternative for α is naturally regarded as a piece of information—the information that α is true. Thus, a statement α may be regarded as describing a specific piece of information; to say that the statement is settled in a state s is simply to say that this specific piece of information is available in s.

This is not the case for questions. For instance, consider the question μ_1 of which symptoms the patient presents. This question has *four* alternatives, namely, the states $a_\emptyset$, a_1, a_2, a_{12}, where the state $a_\emptyset$ consists of the worlds in which the patient has no symptoms, the state a_1 consists of the worlds in which the patient has only symptom S_1, etc.[9]

- $\textsc{Alt}_M(\varphi) = \{a_\emptyset, a_1, a_2, a_{12}\}$

We also saw that μ_1 is settled in a state s if and only if s entails one of these alternatives. Again, we can express this using the downward closure operation.

- $[\mu_1]_M = \{a_\emptyset, a_1, a_2, a_{12}\}^{\downarrow}$

So, in this case μ_1 cannot be regarded as describing a specific piece of information. Instead, μ_1 can be regarded as describing a non-singleton *type* of information. We can take the pieces of information of this type to be the alternatives for μ_1, which we may regard as pieces of information:

- $a_\emptyset$, the information that the patient has no symptom;
- a_1, the information that the patient has only symptom S_1;
- a_2, the information that the patient has only symptom S_2;
- a_{12}, the information that the patient has both symptoms.

To say that the question μ_1 is settled in a state s is to say that *some* piece of information of this type—that is, some complete specification of the patient's symptoms—is available in s.

Similarly, consider the question μ_2 of whether a given patient is in good physical condition. Provided M contains both good and bad condition worlds, such a question has two alternatives, a_g and $a_{\overline{g}}$, where a_g is the set of good condition worlds, and $a_{\overline{g}}$ is the set of bad condition worlds. That is, we have:

[9] That is, provided each of the sets $a_\emptyset$, a_1, a_2, a_{12} is non-empty in the model M. Throughout this discussion I will assume for simplicity that this is the case.

- $\textsc{Alt}_M(\mu_2) = \{a_g, a_{\overline{g}}\}$

Again, to settle the question is to establish either of these alternatives.

- $[\mu_2]_M = \{a_g, a_{\overline{g}}\}^{\downarrow}$

Thus, μ_2 may be regarded as describing a type of information, whose elements are the two alternatives, regarded as two pieces of information:

- a_g, the information that the patient is in good conditions;
- $a_{\overline{g}}$, the information that the patient is not in good conditions.

The question is settled in a state iff some information of this type is available. To make these observations more general, let us say that a sentence φ is *normal* in case φ is settled in s if s entails some alternative for φ.

1.2.4. Definition. [Normality]
We say that an inquisitive proposition P is normal in case $P = \textsc{Alt}(P)^{\downarrow}$.
We say that a sentence φ is normal in a model M, in case $[\varphi]_M$ is normal.

We have seen above that statements are always normal. The questions in our example are also normal, as are most other natural classes of questions. However, there is no reason to assume that questions in general are normal. We will encounter some non-normal questions in the first-order inquisitive semantics of Chapter 4. For a question μ to be non-normal, it must be the case that μ can be settled in a state s, yet there is no way to weaken s to a maximal information state that settles μ. The example we will discuss in Chapter 4 has the following form, where Z is a set of natural numbers:

(3) What is an example of an upper bound for Z?

We refer to Chapter 4.2 for the details, but the idea is that there is no minimal way to resolve this question: if n is an upper bound for Z, then $n+1$ must be an upper bound as well; as a consequence, providing any number n as an upper bound is less minimal than to provide $n+1$ as an upper bound.[10]

If a question μ is normal, then we can naturally think of it as describing a type of information, where the pieces of information of this type are the alternatives $a \in \textsc{Alt}_M(\mu)$. To settle the question is to establish one of these pieces of information.

Intuitively, another way to think of a question μ is as a non-deterministic description of a piece of information. This intuition will be useful in Chapter 3, where we investigate the role of questions in logical proofs: we will see that,

[10] Ciardelli et al. (2013b) propose a system aimed at making any first-order sentence normal by making the notion of meaning more fine-grained. While the non-normal formulas discussed in this thesis do indeed become normal relative to this refined semantics, it is an open problem (in spite of some investigations: see Herbstritt, 2013) whether this is true in general.

in a proof, a question μ can be regarded as standing for some unspecified piece of information of the corresponding type; all the inferences we can make from μ must then hold no matter what piece of information μ turns out to denote. For instance, if we take the question μ_1 above as an assumption, this means that we are assuming some complete information about the patient's symptoms, without however assuming anything specific about what these symptoms are. Similarly, when we conclude the question ν, this means that under the given assumptions, we are assured to have *some* information about whether or not the treatment is prescribed, though *what* information that is will in general depend upon what information the question assumptions of our proof turn out to denote.

1.2.3 Generators and alternatives

We have seen that, for many examples of sentences φ, we have $[\varphi]_M = \textsc{Alt}_M(\varphi)^{\downarrow}$, and that in this case, we can think of φ as describing a type of information $\textsc{Alt}_M(\varphi)$. However, there are many other sets of information states B such that $[\varphi]_M = B^{\downarrow}$. For instance, since $[\varphi]_M$ is downward closed we have $[\varphi]_M = ([\varphi]_M)^{\downarrow}$.

So, what is so special about alternatives? In this section we give an answer to this question having to do with the way in which an inquisitive proposition may be regarded as being *generated* from a given set of information states.

Let us say that a set of states T is a *generator* for an inquisitive proposition P in case P can be characterized as the set of states which entail some state in T.

1.2.5. Definition. [Generators for an inquisitive proposition]
A set T of info states is a *generator* for an inquisitive proposition P if $P = T^{\downarrow}$.

If T is a generator for $[\varphi]_M$, then we can regard φ as standing for the type of information T: for, φ is settled in a state just in case some piece of information $a \in T$ is available.

Notice that any inquisitive proposition P admits a trivial generator, namely, P itself. However, the examples discussed in the previous subsection show that many inquisitive propositions admit much smaller generators. In the case of a statement α, we saw that the proposition $[\alpha]_M$ admits a singleton generator, namely, $\textsc{Alt}_M(\alpha) = \{|\alpha|_M\}$. In the case of our question μ_1 above, we saw that the proposition $[\mu_1]_M$ admits a generator consisting of only four elements, namely, $\textsc{Alt}_M(\mu_1) = \{a_\emptyset, a_1, a_2, a_{12}\}$.

The generators $\textsc{Alt}_M(\alpha)$ and $\textsc{Alt}_M(\mu_1)$ are very different from the trivial generators $[\alpha]_M$ and $[\mu_1]_M$. First, it is easy to check that any element of $\textsc{Alt}_M(\alpha)$ and $\textsc{Alt}_M(\mu_1)$ is essential to the representation of the corresponding proposition: if we were to remove it, the resulting set would no longer be a generator for the proposition. We say that these generators are *minimal.* Moreover, the elements of these generators are *pairwise logically independent*, in the sense that one element of the generator never entails another. We will say that these generators are *independent.*

1.2.6. Definition. [Minimal and independent generators]
Let T be a generator for an inquisitive proposition P. We say that T is:

- *minimal* if no proper subset $T' \subset T$ is a generator for P;
- *independent* if there are no $t, t' \in T$ such that $t \subset t'$.

The following proposition says that minimal and independent generators coincide.

1.2.7. Proposition (Minimality and independence are equivalent).
Let T be a generator for a proposition P. T is minimal iff it is independent.

Proof. Suppose T is independent and consider a proper subset $T' \subset T$. Let $s \in T - T'$. Since $s \in T$ and T is a generator for P, we have $s \in T^{\downarrow} = P$. However, since $T' \subseteq T$, s cannot be a subset of any element of T': otherwise, it would be a subset of some element of T other than itself, contrary to the independence of T. This means that $s \notin T'^{\downarrow}$. Since $s \in P$, this shows that $T'^{\downarrow} \neq P$, i.e., that T' is not a generator for P. Since $T' \subset T$ was arbitrary, this shows that T is minimal.

For the converse, suppose T is not independent. Then, there are two states $s, t \in T$ such that $s \subset t$. Now consider $T' := T - \{s\}$. We claim that T' is still a generator for P, which shows that T is not minimal.

We have to prove that $T'^{\downarrow} = P$. Since $T' \subset T$, obviously $T'^{\downarrow} \subseteq T^{\downarrow} = P$. To establish the converse inclusion, consider any $u \in P$. Since T is a generator for P, u must be included in some $v \in T$. Now, if $v \neq s$, then $v \in T'$, which implies $u \in T'^{\downarrow}$. On the other hand, if $v = s$, then $u \subseteq s \subseteq t$, and since $t \in T'$, again we have $u \in T'^{\downarrow}$. Thus, in any case we have $u \in T'^{\downarrow}$, and since u was an arbitrary element of P, we conclude $P \subseteq T'^{\downarrow}$. □

We will refer to a minimal generator for a proposition as a *basis*.[11]

1.2.8. Definition. [Basis for an inquisitive proposition]
A *basis* for an inquisitive proposition P is a minimal generator for P.

The next proposition shows that there *is* something special about the alternatives for an inquisitive proposition. Namely, whenever a proposition P admits a basis, the unique basis for P is $\textsc{Alt}(P)$.

1.2.9. Proposition.

- *An inquisitive proposition P has a basis iff P is normal;*

[11] For readers acquainted with linear algebra, there is a close similarity here with the notion of a *basis* for a sub-space X: this can be characterized as a set T of vectors that (i) generates the space X, in the sense that $span(T) = X$ and (ii) is minimal, in the sense that no proper subset $T' \subset T$ generates X. Moreover, instead of (ii) we can equivalently require that (ii') T be *independent*, meaning that any two vectors in T are linearly independent.

- *If P has a basis, then the unique basis for P is* $\textsc{Alt}(P)$.

Proof. First suppose P is normal, that is, $P = \textsc{Alt}(P)^{\downarrow}$. This means that $\textsc{Alt}(P)$ is a generator for P. Moreover, notice that by definition, $\textsc{Alt}(P)$ is independent. By the previous proposition, it follows that $\textsc{Alt}(P)$ is a basis for P.

Moreover, suppose T is another basis for P. Consider any $s \in \textsc{Alt}(P)$: since $s \in P = T^{\downarrow}$, s must be a subset of some element $t \in T$. But now, since t is in T, t must also be in $T^{\downarrow} = P$. Since $s \in \textsc{Alt}(P)$ is by definition a maximal element in P; so, we must have $s = t$, which implies $s \in T$. Since this holds for any $s \in \textsc{Alt}(P)$, we have $\textsc{Alt}(P) \subseteq T$. By the minimality of T, this implies $T = \textsc{Alt}(P)$. This shows that, if P admits a basis, the unique basis is $\textsc{Alt}(P)$.

Conversely, suppose P is not normal, i.e., $P \neq \textsc{Alt}(P)^{\downarrow}$. Since $\textsc{Alt}(P)^{\downarrow} \subseteq P$ by the downward closure of P, we must have $P \not\subseteq \textsc{Alt}(P)^{\downarrow}$. This means that there must be a state $s \in P$ which is not included in any *maximal* state $t \in P$.

Now consider any generator T for P. Since T is a generator and $s \in P$, we must have $s \subseteq t$ for some $t \in T$. Now, we know that t cannot be a maximal element of P. So, let $u \in P$ be such that $t \subset u$. Since $u \in P$ and T is a generator, u must be a subset of some $t' \in T$. But then we have $t \subset u \subseteq t'$, that is, t is a proper subset of t', and both t and t' are in T. This shows that T is not independent, which by the previous proposition implies that T is not a basis. Since T was arbitrary, this shows that there is no basis for P. □

This result shows that, if P is normal, we can regard it as being generated in a canonical way by the set of its alternatives, while if P is non-normal, there is no minimal way of picking a generator for P.

Some previous work in inquisitive semantics (in particular, Ciardelli, 2009, 2010), may be regarded from the present perspective as being concerned with the task of equipping a formula φ not only with an inquisitive proposition $[\varphi]_M$, but also with a designated generator $T_M(\varphi)$ for this proposition. This will not be needed for the enterprise to which this thesis is devoted, namely, the investigation of logics equipped with questions. Here, the focus will be on the semantic relation of support and on the notion of entailment that arises from it.

1.2.4 Entailment as a relation among information types

Consider again our initial example: given the protocol we described, whether or not the treatment is prescribed is determined by the patient's symptoms and by whether the patient is in good physical condition. We have construed this as a relation among questions, that we called *dependency*. If we think of the questions in the examples as names for information types, we may phrase this relation as follows: information of type μ_1 (symptoms?), combined with information of type μ_2 (condition?) yields information of type ν (treatment?).

In this section, we will see that bringing questions into play may be viewed as taking us from a logic of pieces of information to a logic of information types. To see this, let us fix a specific model M, and let us fix for any sentence φ a generator $T\varphi$ of $[\varphi]_M$, so that we can regard φ as describing the type of information $T\varphi$. If our sentences are normal, we saw that the canonical and most economical choice is $T\varphi = \textsc{Alt}_M(\varphi)$, but this assumption will not be needed here.

The following proposition shows that indeed, entailment holds if any piece of information of type $T\varphi$ yields some information of type $T\psi$.

1.2.10. Proposition.
$\varphi \models_M \psi \iff$ *for any* $a \in T\varphi$ *there is some* $a' \in T\psi$ *such that* $a \subseteq a'$

In case a statement is involved in the entailment, this relation can be simplified. First, suppose the formulas α and β at stake are both statements. Then, it is easy to see that the above boils down to the fact that the information that α is true yields the information that β is true.

1.2.11. Proposition (Statement-to-statement entailment).
$\alpha \models_M \beta \iff |\alpha|_M \subseteq |\beta|_M$

So, in this case, entailment is indeed a relation between specific pieces of information. Moreover, suppose α is a statement and μ a question. Then, it is easy to see that the entailment $\alpha \models \mu$ holds in case the information that α is true yields some piece of information of type $T\mu$.

1.2.12. Proposition (Statement-to-question entailment).
$\alpha \models_M \mu \iff |\alpha|_M \subseteq a$ *for some* $a \in T\mu$

For an example, let α and μ be the following statement and question.

(α) Galileo discovered Jupiter's moons.

(μ) Did Galileo discover anything?

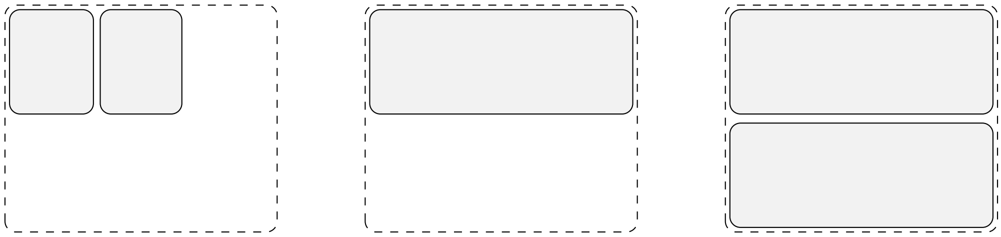
Let $T\mu = \{a_s, a_n\}$, where a_s consists of the worlds where Galileo discovered something, while a_n consists of the worlds where Galileo discovered nothing. We have $|\alpha|_M \subseteq a_s$, which witnesses that α entails μ in M.

Conversely, if μ is a question and α a statement, the entailment $\mu \models \alpha$ holds in case any information of type $T\mu$ yields the information that α is true.

1.2.13. Proposition (Question-to-statement entailment).
$\mu \models_M \alpha \iff a \subseteq |\alpha|_M$ *for all* $a \in T\mu$

For an example, let α be as above, and let μ be following the question:

(μ) In what year did Galileo discover Jupiter's moons?



(a) μ (b) α (c) ν

Figure 1.2: In the given model, the statement α is entailed by the question μ and entails the question ν.

Let $T\mu = \{a_n \,|\, n \in \mathbb{N}\}$, where a_n consists of all the worlds in which Galileo discovered Jupiter's moons in year n. For any $a_n \in T\mu$, we have $a_n \subseteq |\alpha|_M$, which shows that μ entails α in M.

So, the relations $\alpha \models \mu$ and $\mu \models \alpha$ boil down to relations between a piece of information, $|\alpha|_M$, and a type of information $T\mu$: the former relation holds if $|\alpha|_M$ is included in some element of $T\mu$, while the latter holds if $|\alpha|_M$ includes every element of $T\mu$.

Let us now consider the general case in which we have multiple assumptions; for simplicity, we stick to the case in which we have finitely many of them, though this is not necessary. The following proposition states that an entailment $\varphi_1, \ldots, \varphi_n \models_M \psi$ holds in case, whenever we are given a piece of information for each type $T\varphi_i$, these pieces of information jointly yield information of type $T\psi$.

1.2.14. Proposition.

$\varphi_1, \ldots, \varphi_n \models_M \psi \iff$ *for every* $a_1 \in T\varphi_1, \ldots, a_n \in T\varphi_n$ *there is an* $a' \in T\psi$ *such that* $a_1 \cap \cdots \cap a_n \subseteq a'$

As above, for those assumptions that are statements, the universal quantification over $T\alpha$ can be avoided: all we have to consider is one piece of information, $|\alpha|_M$.

Now, as an illustration of the relation of entailment involving both statements and questions as assumptions, let us look back at our initial example. In this case, the sentences that are at stake are the statement γ describing the hospital's protocol, and the three questions μ_1, μ_2, ν, which we can associate with the following information types:

- $T\mu_1 = \{a_\emptyset, a_1, a_2, a_{12}\}$
- $T\mu_2 = \{a_g, a_{\overline{g}}\}$
- $T\nu = \{a_t, a_{\overline{t}}\}$

In view of the above proposition and of the remark on statements, the entailment amounts to the following:

$$\gamma,\, \mu_1,\, \mu_2 \models_M \nu \quad \Longleftrightarrow \quad \begin{array}{l}\text{for any } a \in T\mu_1 \text{ and } a' \in T\mu_2 \text{ there is an } a'' \in T\nu \\ \text{such that } |\gamma|_M \cap a \cap a' \subseteq a''\end{array}$$

That is, the entailment holds if, on the basis of the information $|\gamma|_M$ provided by the protocol, combining a piece of information of type *symptoms* ($a \in T\mu_1$) with one of type *condition* ($a' \in T\mu_2$) is bound to yield some piece of information of type *treatment* ($a'' \in T\nu$). This is indeed the case, as a case-by-case verification shows. We have:

$$\begin{array}{ll}
|\gamma|_M \cap a_\emptyset \cap a_g \subseteq a_{\overline{t}} & \qquad |\gamma|_M \cap a_2 \cap a_g \subseteq a_t \\[1ex]
|\gamma|_M \cap a_\emptyset \cap a_{\overline{g}} \subseteq a_{\overline{t}} & \qquad |\gamma|_M \cap a_2 \cap a_{\overline{g}} \subseteq a_t \\[1ex]
|\gamma|_M \cap a_1 \cap a_g \subseteq a_t & \qquad |\gamma|_M \cap a_{12} \cap a_g \subseteq a_t \\[1ex]
|\gamma|_M \cap a_1 \cap a_{\overline{g}} \subseteq a_{\overline{t}} & \qquad |\gamma|_M \cap a_{12} \cap a_{\overline{g}} \subseteq a_t
\end{array}$$

This shows how the entailment $\gamma,\, \mu_1,\, \mu_2 \models \nu$, involving both statements and questions, captures precisely the relation that we observed to exist between the types of information $T\mu_1$, $T\mu_2$, and $T\nu$ within the context provided by $|\gamma|_M$.

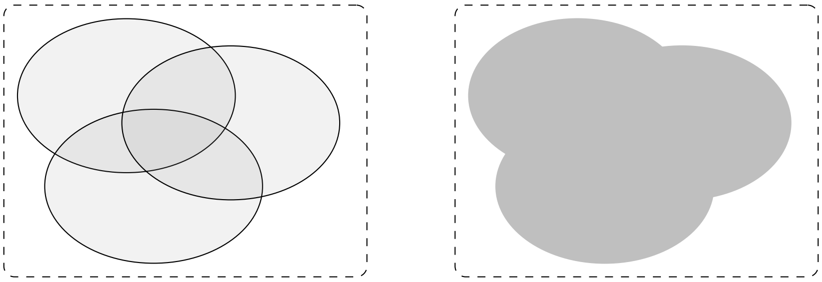
1.3 Truth and question presupposition

Above, we claimed that truth-conditional semantics is not suitable for questions, as it is not even clear what it should mean for a question, say the question *where Madrid is*, to be true or false at a world. However, in this section we will see that, in fact, support semantics does give us a natural way to extend the notion of truth to questions. Only, unlike in the case of a statement, the truth-conditions of a question do not completely determine its semantics.

In Section 1.1.2, we saw that for statements, truth-conditions and support-conditions are inter-definable. In particular, if we are given the support conditions for a given statement α, we can recover the statement's truth-conditions by means of the following connection:

$$w \models \alpha \iff \{w\} \models \alpha$$

That is, α is true at a world if and only if it is supported at the corresponding singleton state. Now, we can take this relation to provide the *definition* of truth in a support-based semantics. This allows us to conservatively extend the notion of truth to questions.



(a) $[\mu]_M$ (b) $|\mu|_M$

Figure 1.3: The proposition expressed by a question, and the question's truth-set.

1.3.1. Definition. [Truth]
We say that a sentence φ is *true* at a world w of a model M, in symbols $w \models \varphi$, in case φ is supported at the state $\{w\}$.

The *truth-set* of φ in M is $|\varphi|_M = \{w \in W_M \mid w \models \varphi\}$.

Now, intuitively, what does it mean for a question μ to be *true* at a world w? The definition says that μ is true at w in case μ would be settled if we were to know that w is the actual world. Now, the singleton $\{w\}$ is a state of complete information: thus, if the question is not settled in this state, this cannot be because not enough information is available: it must be because the question does not admit a truthful resolution at w. This interpretation is backed by the following proposition, which follows immediately from the persistency of support.

1.3.2. Proposition.
For any sentence φ and any world w: $w \models \varphi \iff w \in s$ *for some* $s \models \varphi$

In the particular case of a question μ, this proposition states that μ is true at w if there is some way of resolving μ which is compatible with w. Thus, we can read the relation $w \models \mu$ as expressing the fact that μ is *soluble* at w. As an example, consider the following two questions:

(4) Did Galileo or Kepler discover Jupiter's moons?

(5) In what year did Galileo discover Jupiter's moons?

Suppose we knew that the actual world is a certain world w. Question (4) would be settled for us if and only if Jupiter's moons have in fact been discovered by either Galileo or Kepler in w. If Jupiter's moons have been discovered by some other astronomer in w, or have not been discovered at all, or do not exist, etc., then even if we have complete information, the question is still not settled. In other words, it is natural to regard (4) as having the same truth-conditions as (6).

(6) Either Galileo or Kepler discovered Jupiter's moons.

Now consider question (5). This question is settled in the state $\{w\}$ in case there is indeed some year n such that, in w, Galileo discovered Jupiter's moons in n. If in w Galileo did *not* discover Jupiter's moons, or if the discovery stretched over a period of several years, or again if Jupiter's moons do not exist, etc., then (5) is not settled at $\{w\}$, in spite of having complete information. Thus, it is natural to regard (5) as having the same truth-conditions as (7).

(7) There is a year in which Galileo discovered Jupiter's moons.

Now, there is widespread agreement in the literature that (i) a question always *presupposes* that the world is one where the question is soluble and (ii) this is *all* that the question presupposes. I will refer to this claim as *Belnap's principle*, since as far as I know, the earliest statement of it is found in Belnap (1966), who puts it succinctly as follows:

Belnap's principle
Every question presupposes *precisely* that one of its direct answers is true.

In Belnap's view, the *direct answers* to a question μ correspond to all the ways in which μ may be completely resolved: the question is settled in a state s iff s entails some direct answer to μ. This means that, in our terminology, Belnap's direct answers are supposed to be a generator for $[\mu]_M$.[12] Now, the following fact follows immediately from the definitions of truth and generators.

1.3.3. Proposition.
Let φ be a sentence, and M a model. If $T\varphi$ is a generator for $[\varphi]_M$, then:

$$w \models \varphi \iff w \in a \text{ for some } a \in T\varphi$$

Since Belnap's set of direct answers is a generator, it follows that μ is true at w if and only if some direct answer to μ is true at w. This shows that, in our terminology, Belnap's principle can be re-stated as follows:

Belnap's principle, rephrased
Every question presupposes *precisely* that it is true.

Interestingly, this is exactly how Belnap himself might have originally stated the principle in his 1966 paper. For, the paper ends with the following sentence:

> I should like in conclusion to propose the following linguistic reform: that we all start calling a question "true" just when some direct answer thereto is true.

[12] Strictly speaking, we should say that the set of *truth-sets* of the direct answers are a generator, since Belnap takes direct answers to be statements. The difference is immaterial for our purposes.

Thus, our restatement of Belnap's principle is also in accordance with Belnap's own notion of truth for questions. This is no accident: due to the previous proposition, we can see that the notion of truth that we gave for questions in fact coincides with the one proposed by Belnap.

At the same time, it is noteworthy that in our framework, this notion of truth for question is not obtained by a specific stipulation that links the truth of a question to a more primitive notion of truth for statements; rather, there is just one, uniform definition of truth, which applies to statements and questions alike; the desired link between the truth-conditions of a question and the truth conditions of the answers—intended as statements whose truth-sets give a generator for the question—simply comes out as a fact.

Belnap's principle tells us that we may regard the truth-set $|\mu|_M$ of a question μ as capturing the question's *presupposition*, that is, the body of information that the question presupposes. Following again Belnap (1966), we will say that a statement α *expresses the presupposition* of a question μ in case for any model M, $|\alpha|_M = |\mu|_M$, that is, in case α has the same truth-conditions as μ. Thus, for instance, the statement (6) expresses the presupposition of (4), and the statement (7) expresses the presupposition of (5).

In general, if α expresses the presupposition of μ, then α is settled in a state s iff s entails the presupposition of μ, that is, iff the information available in s ensures that μ may be truthfully resolved.

1.3.4. Proposition.
If π_μ expresses the presupposition of μ, then for any model M and state $s \subseteq W_M$:

$$s \models \pi_\mu \iff s \subseteq |\mu|_M$$

In the logical systems we explore in the following chapters, it will often be convenient to associate with any question μ in the system a specific statement π_μ which expresses the question's presupposition; for brevity, we will allow ourselves to be sloppy and also refer to π_μ as the presupposition of μ.[13]

When we discussed the significance of the various entailment patterns above, we said that, if μ is a question and α is a statement, we can view the entailment $\mu \models \alpha$ as capturing the fact that μ *presupposes* α. Now let π_μ be an arbitrary statement expressing the question's presupposition. The following proposition says that μ presupposes α just in case α follows from the presupposition of μ.

1.3.5. Proposition.
Let μ be a question, α a statement, and let π_μ express the presupposition of μ. Then:

$$\mu \models \alpha \iff \pi_\mu \models \alpha$$

[13]This syntactic notion of presupposition is also in line with the one adopted in some logical frameworks concerned with questions, such as the *interrogative model of inquiry* of Hintikka (1981, 1999) and the *inferential erotetic logic* of Wiśniewski (1994, 1996, 2001).

Thus, if α is a statement and μ a question, we have the following difference between (i) α being presupposed by μ, and (ii) α expressing the presupposition of μ:

- α is presupposed by $\mu \iff |\alpha|_M \supseteq |\mu|_M$: i.e., the truth of α is necessary in order for μ to be truthfully soluble.

- α expresses the presupposition of $\mu \iff |\alpha|_M = |\mu|_M$: i.e., the truth of α is both necessary and sufficient for μ to be truthfully soluble.

For instance, the statement (8-b) is presupposed by (8-a), but unlike (8-c), it does not express the question's presupposition, since its truth is not sufficient to ensure the solubility of the question.

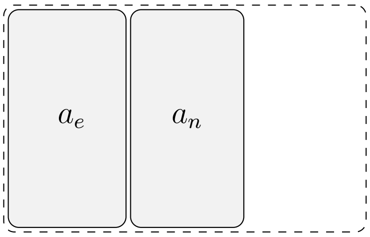
(8) a. Did Galileo, or Kepler discover Jupiter's moons?
    b. Someone discovered Jupiter's moons.
    c. Either Galileo or Kepler discovered Jupiter's moons.

To conclude this section, let us tackle a possible source of confusion. We started by claiming that, in order to bring questions within the scope of logic, we should move beyond a purely truth-conditional conception of semantics. Are we now retracting that claim, and saying that questions are truth-conditional after all? No, we are not: although we can meaningfully extend the notion of truth to questions, that does not mean that the meaning of a question is captured by its truth-conditions. To see this, consider the following two questions:

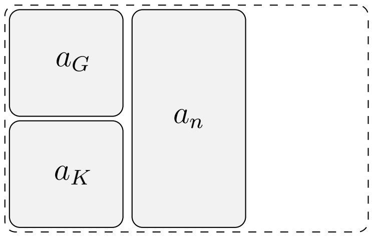
(9) a. Did either Galileo or Kepler discover Jupiter's moons?
    b. Did Galileo, Kepler, or neither of them discover Jupiter's moons?

These questions have identical truth-conditions, since they are soluble at exactly the same worlds. However, they do not have the same support conditions: (9-a) can be settled by establishing that either astronomer made the discovery, or that neither did; (9-b) is more demanding: in order to settle it, one must establish *which* of the two made the discovery, if either of them did. The difference is illustrated in Figure 1.4, where a_G, a_K, a_e and a_n are, respectively, the set of worlds where the discovery was made by Galileo, Kepler, either, or neither. As this example suffices to show, a question's support conditions are not determined by its truth-conditions.

We can take this to be the fundamental semantic difference existing between statements and questions. For statements, support at a state just amounts to truth at each world in the state. For questions, truth at all worlds is not enough. To settle a statement, one only needs to establish that the statement is true. To settle a question, it is not sufficient to establish that the question is true—that is, it is not sufficient to establish the question's presupposition. More is needed. We can make this precise by defining the following notion of *truth-conditionality.*



(a) Either or neither?



(b) Galileo, Kepler, or neither?

Figure 1.4: The alternatives for (9-a) and (9-b). The dashed line represents the logical space W_M. The two questions have the same truth-conditions, namely, they are true at those worlds which are included in some alternative.

1.3.6. Definition. [Truth-conditionality]
We call a sentence φ *truth-conditional* if for all models M and states $s \subseteq W_M$:

$$s \models \varphi \iff w \models \varphi \text{ for all } w \in s$$

In the formal systems developed in this thesis, we will take truth-conditionality to be the fundamental semantic difference between statements and questions. Our languages will not incorporate a fundamental syntactic distinction between statements and questions (though such a distinction can be imposed if desired; see Groenendijk, 2011; Ciardelli et al., 2015b). Rather, we will simply regard truth-conditional formulas as statements, and non truth-conditional formulas as questions.[14] The following proposition suggests another perspective on this difference: statements may be seen as describing specific pieces of information, whereas questions need to be regarded as describing non-singleton information types.

1.3.7. Proposition.
φ is truth-conditional $\iff$ $[\varphi]_M$ admits a singleton generator in any model M.

Proof. If φ is truth-conditional, it is easy to see that $[\varphi]_M$ admits the singleton generator $\{|\varphi|_M\}$ in any model. Conversely, suppose φ always admits a singleton generator and consider a model M. Let $\{a_\varphi\}$ be a singleton generator for $[\varphi]_M$. Now, for the truth-conditions of φ in M we have:

$$w \models \varphi \iff \{w\} \in [\varphi]_M = \{a_\varphi\}^\downarrow \iff w \in a_\varphi$$

[14] A minor problem with this choice is that we cannot recognize tautological questions (e.g., *whether John is John*) as such. For, such questions are trivially resolved in all states, and so, they are trivially truth-conditional. I regard this as a manifestation of the well-known problems that possible-world semantics has in dealing with sentences that are tautological or contradictory. For instance, if our semantics also included *im*possible worlds, such questions would have two alternatives like any other polar question; thus, they would not be truth-conditional, and would be correctly classified as questions.

This shows that $a_\varphi = |\varphi|_M$, that is, the unique element of the generator must be precisely the truth-set of φ. Finally, using this fact we have:

$$s \models \varphi \iff s \in [\varphi]_M = \{a_\varphi\}^{\downarrow} = \{|\varphi|_M\}^{\downarrow} \iff s \subseteq |\varphi|_M$$

that is, φ is supported at a state in M in case it is true everywhere in the state. Since this is true for any M, φ is truth-conditional. □

1.4 Internalizing entailment

1.4.1 Implication in support semantics

In a support semantics, the contexts to which entailment can be relativized are the same kind of object at which formulas are evaluated, namely, information states. This ensures that a support-based logic can always be enriched with an operation of *implication* which internalizes the meta-language relation of entailment. In other words, any logic whose semantics is given in terms of support may be equipped with a connective $\to$ such that, for any sentences φ and ψ, the sentence $\varphi \to \psi$ is settled in an information state s iff φ entails ψ relative to s. In symbols:

$$s \models \varphi \to \psi \iff \varphi \models_s \psi \tag{1.3}$$

Simply by making explicit what the condition $\varphi \models_s \psi$ amounts to, we get the support clause governing this operation:

$$s \models \varphi \to \psi \iff \text{for all } t \subseteq s,\ t \models \varphi \text{ implies } t \models \psi$$

That is, an implication is supported in s in case enhancing s so as to support the antecedent is bound to lead to an information state that supports the consequent. Interestingly, this is, *mutatis mutandis*, precisely the interpretation of implication that we indeed find adopted in most information-based systems.

If we apply this clause to statements, we obtain precisely the usual material conditional of classical logic. To see this, suppose α and β are statements. In view of the connection in 1.1 between support and truth, we get:

$$\begin{aligned} s \models \alpha \to \beta &\iff \forall t \subseteq s,\ t \models \alpha \text{ implies } t \models \beta \\ &\iff \forall t \subseteq s,\ t \subseteq |\alpha|_M \text{ implies } t \subseteq |\beta|_M \\ &\iff s \cap |\alpha|_M \subseteq |\beta|_M \\ &\iff s \subseteq \overline{|\alpha|}_M \cup |\beta|_M \end{aligned}$$

where $\overline{|\alpha|}_M$ denotes the set-theoretic complement of $|\alpha|_M$, that is, the set of worlds in M where α is false. The truth-conditions that this clause delivers for $\alpha \to \beta$ are the standard ones:

$$\begin{aligned} w \models \alpha \to \beta &\iff \{w\} \models \alpha \to \beta \\ &\iff w \in \overline{|\alpha|}_M \cup |\beta|_M \iff w \not\models \alpha \text{ or } w \models \beta \end{aligned}$$

Thus, the conditional $\alpha \to \beta$ is supported in a state s just in case the conditional is true everywhere in s in the ordinary, material interpretation. This shows that the standard material conditional may be seen as arising precisely by internalizing within the language the relation of contextual entailment between statements.

On the other hand, the clause above defines an operation which *generalizes* the material conditional. For, we have seen that support semantics is suitable for interpreting questions, besides statements. If our language does indeed contain questions, implication between them is naturally defined: for any two questions μ and ν, this gives us a corresponding conditional $\mu \to \nu$ with the property of being supported in a state s just in case $\mu \models_s \nu$, that is, just in case μ determines ν relative to s. Thus, the conditional $\mu \to \nu$ allows us to express the dependency of ν on μ within the object language.

Thus, like the classical entailment relation generalizes to questions, allowing us to capture dependencies, so the classical operation of implication generalizes to questions, allowing us to express these dependencies within the language.

1.4.2 Conditional dependencies

In our hospital protocol example, a patient's symptoms do not determine whether or not the treatment is prescribed. However, suppose we know that the patient is in good physical condition: then, her symptoms *do* determine whether the treatment is prescribed. If g is the following statement,

(g) The patient is in good physical condition.

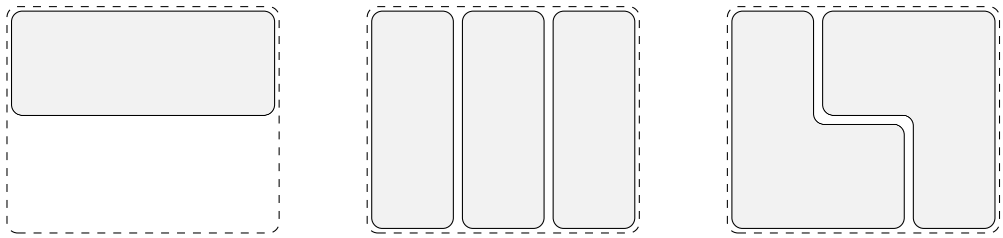
then we may say that the question μ_1 (symptoms?) determines the question ν (treatment?) *conditionally on g*. In general, we will say that a question μ determines another question ν *conditionally on α* in a context s if the dependency holds relative to the set of α-worlds in the context, that is, if we have:

$$\mu \models_{s \cap |\alpha|_M} \nu$$

We will refer to such a relation as a *conditional dependency* in s. Now, given relation 1.3 between entailment and support for implication, the persistency of support, and relation 1.1 between support and truth for a statement, we have:

$$\begin{aligned}
\mu \models_{s\cap|\alpha|_M} \nu \quad &\iff \quad s \cap |\alpha|_M \models \mu \to \nu \\
&\iff \quad \text{for all } t \subseteq s \cap |\alpha|_M,\ t \models \mu \to \nu \\
&\iff \quad \text{for all } t \subseteq s,\ t \subseteq |\alpha|_M \text{ implies } t \models \mu \to \nu \\
&\iff \quad \text{for all } t \subseteq s,\ t \models \alpha \text{ implies } t \models \mu \to \nu \\
&\iff \quad s \models \alpha \to (\mu \to \nu)
\end{aligned}$$

This brings out another remarkable feature of our conditional operator. So far, we saw that when applied to two statements, this operator gives us the standard



(a) α (b) μ (c) ν

Figure 1.5: In the context of the dashed area, μ determines ν conditionally on α.

material conditional, while when applied to two questions, it yields a formula which expresses a dependency. We now see that, in addition, the same operator provides a simple way of expressing *conditional* dependencies as conditionals, having the condition α as antecedent, and the formula $\mu \to \nu$ expressing the dependency as consequent.

1.5 Summing up

We have seen that classical logic can be given an alternative, informational semantics in terms of *support conditions*, which determines when a sentence is *settled* by a body of information, rather than when it is true at a world. Unlike truth-conditional semantics, this semantics can be extended to interpret questions in a natural way. By adopting this more comprehensive semantic foundation, the classical notion of entailment can be generalized to questions, and this makes it possible to capture the relation of dependency as a particular facet of entailment.

We also saw that, in support semantics, a formula may be regarded as describing a type of information: statements describe singleton types, which may be identified with a specific piece of information; questions describe non-singleton types, which are instantiated by several different pieces of information. We may then describe the generalized notion of entailment as capturing not only relations between specific, determinate pieces of information—as in classical logic—but also relations between proper information types: an entailment relation holds if information of the type described by the assumptions is guaranteed to yield information of the type described by the conclusion.

We saw that support semantics suggests a natural way to extend the notion of truth to questions, and that the truth-conditions of a question may be viewed as capturing the question's presupposition.

Finally, we saw that a logic formulated within this framework can be equipped in a canonical way with a conditional operator which reflects the meta-language relation of entailment within the language. While this operator boils down to the standard material conditional when it applies to statements, it can also be applied to questions, producing formulas which are capable of expressing dependencies.

One important part of the conceptual picture is still missing. This has to do with the role of questions in logical proofs. We postpone the discussion of this issue until Chapter 3, because it is a point that it is best illustrated by means of a concrete proof system. However, we may already anticipate the central idea: using questions in proofs allows us to reason with *generic* information—i.e., information which is not completely specified. For instance, by assuming the question *what the symptoms are* we can assume a complete specification of the patient's symptoms, without however assuming anything specific about the symptoms. As we will see, by reasoning with such generic information we can formally prove the existence of certain dependencies; what is more, from such a proof we can effectively extract an algorithm to compute the dependency. That is, when it comes to questions, proofs admit an interesting computational interpretation.

1.6 Discussion

1.6.1 The relevance of dependencies

Once one starts thinking about it, one quickly realizes that the sort of dependency relation that we have been concerned with is quite ubiquitous, both in ordinary contexts and in specific scientific domains. In this section I mention three areas where this notion plays a role, although probably many more can be found.

Natural sciences

Much of the enterprise of natural sciences such as physics and chemistry, consists in finding out what dependencies hold in nature: what are those factors that determine the trajectory of a planet, the temperature of a gas, or the speed of a certain chemical reaction?

One of the earliest achievements of modern science was the discovery that, absent air resistance, the time that a body dropped near the Earth surface employs to reach the ground is completely determined by the height from which it is dropped; another similar discovery of classical mechanics is that, given a flat surface, the distance at which a cannonball will land is completely determined by its initial velocity. Such relations are all cases of dependency in our sense: the answer to one question (say, what the initial velocity is) determines the answer to another question (say, how far the cannonball will land).

Indeed, the epistemic value of a scientific theory, such as classical mechanics or thermodynamics, lies precisely in its ability to establish such dependencies, which is often referred to as the theory's *predictive power*. Our perspective allows us to make this very precise: we can say that a theory Γ, construed as a set of statements, is predictive of a question ν given questions $\mu_1, \dots, \mu_n$ in case the entailment $\Gamma, \mu_1, \dots, \mu_n \models \nu$ holds. Thus, e.g., classical mechanics can be characterized as predictive of a body's position at a time t given (i) the body's position and velocity at a different time t_0, (ii) the body's mass and (iii) the force field in which the body moves.

Linguistics

One of the aims of the theory of pragmatics is to understand when a certain sequence of conversational moves forms a coherent dialogue, and why. A crucial part of this task is to characterize what sentences count as acceptable replies to a question in a certain context. Now consider the following exchange:

(10) Alice: Where can I find you tomorrow?
Bob: If it is sunny I'll be in the park; if not, I'll be at home.

In this dialogue, Bob's reply sounds as informative a response as Alice can possibly hope for. However, strictly speaking, it does not *resolve* Alice's question, since it does not provide a specific place where Bob can be found. Rather, what Bob's reply does is to establish a *dependency* of Alice's question on another question, the question whether it will be sunny. In this case, this might be the best Bob can offer in response to Alice's question. This illustrates the fact that in some cases, the optimal response to a given question may in fact take the form of a dependency on another question.

Databases

A database is a relation, i.e., a collection of vectors of a given size. A vector in a database is called an *entry*, and the various coordinates are called the *attributes*. For instance, the database of a university may contain one entry for each student, and the attributes may be *student ID*, *last name*, *program*, etc.

The traditional role of dependency in database theory is in the specification of constraints that a database should satisfy. Such constraints often take the form of dependencies. E.g., as a university we want an ID number to completely identify a student: this means that the attribute *student ID* should completely determine the value of all other attributes in the database, such as *first name*, *program*, etc. That is, a certain dependency relation should hold.

A related domain where dependency plays a role is *query answering*. Queries are essentially just questions in a specific formal language. When a query is asked, a program accesses the database, computes an answer, and returns it to the user. However, databases are typically large, and consulting them is costly: thus, it is often useful to store the answers to particular queries after these have been answered. Such stored answers are called *views*. Ideally, when a query is asked, one would like to compute an answer just based on the available views, without having to reconsult the database. However, this is only possible if the new query is in fact *determined* by the views, i.e., if a certain dependency relation holds.

1.6.2 Setup choices

Concrete vs. abstract view of information states

The main purpose of this chapter was to show that, by moving from a semantics based on possible worlds to a semantics based on information states, we can interpret both statements and questions in a uniform way, and thereby we obtain an interesting generalization of the classical notion of entailment.

In the literature, two different ways of representing information states have been explored. The first option, which we have taken here, is to regard them as sets of possible worlds—identifying a body of information with the worlds that match that information. We will call this the *concrete* view of information states. The second option is to take information states to be primitive objects, ordered by a given relation $\sqsubseteq$ which holds between two states when the first is an enhancement of the second; so, in this approach information states are just elements of a partially ordered set—which is typically assumed to satisfy some additional properties. I will call this the *abstract* view of information states.

Most of the existing information-based systems adopt the abstract approach. This is true both for works that use information-based semantics as a foundation for non-classical logics, such as Kripke and Beth semantics for intuitionistic logic or Veltman (1981)'s data semantics, and for works that provide information-based semantics for classical logics, such as Fine (1975b); Humberstone (1981); van Benthem (1981, 1986); Holliday (2014).[15] Only more recently, logical semantics based on sets of possible worlds have been explored, e.g., by Cresswell (2004); Väänänen (2008); Ciardelli and Roelofsen (2011); Yang (2014); Punčochář (2015a).

The approach to question semantics suggested in this chapter is compatible with both views on information states, and in principle both views provide a suitable basis for the enterprise of investigating how some familiar systems of classical logic fare when they are enriched with questions, and when the logical operators are allowed to manipulate these questions.

In this thesis, I adopt the concrete view of information states. Part of the reason for this choice is historical. The support-based approach to questions, which originates with Ciardelli (2009) and Groenendijk and Roelofsen (2009), arises out of a line of work in formal semantics, going back to Montague (1973), in which the use of possible world structures is the default approach.

However, the concrete view of information states also has some real advantages for our purposes. First and foremost, this perspective allows us to operate with the very same semantic structures that are used to interpret a given system of classical logic in a possible-world fashion: these structures consist of a set of

[15] I should remark that, although I classify these semantics as information-based, this is not always the perspective adopted in these works. The crucial feature of these systems is that they are based on partial objects, rather than total ones. Although in this thesis I stick to an informational interpretation of such objects, which I find insightful for the present purposes, it is perfectly possible to abstract away from it.

possible worlds, each equipped with a complete specification of all the features that constitute a complete state of affairs from the perspective of the language at hand. While the semantics now takes place at the level of *sets of worlds*, this semantics can be specialized to single worlds in an obvious way, and for formulas in the standard language, this yields the familiar truth-conditions. This allows us to highlight the connections with the standard semantics for the given system, and to make it very clear that what is taking place is a conservative extension of that system with questions.

To give one concrete example, in Chapter 6 we will investigate Kripke modalities in the inquisitive setting. The models on which we will interpret our logic are ordinary Kripke models. Moreover, for any formula, our semantics determines some truth-conditions with respect to a possible world: for standard modal formulas, these truth-conditions are the standard ones, and moreover, they completely determine the formula's semantics; this makes it clear that the standard part of the language receives its usual interpretation. If we wanted to implement the same system based on the abstract view of information states, we should start from a non-world based semantics for classical modal logic, such as given, e.g., in Humberstone (1981) and Holliday (2014). However, the reader may not be familiar with these systems, and therefore it may not be clear how much of the novelties encountered stem from the generalization to questions, and how much from the shift to a non-world based semantic setting. In other words, if we cast the semantics directly in an abstract setting, we would be making at once two moves which are probably better understood one at a time.

A second advantage of the concrete view of information states is that the view of information that it provides, while rather coarse-grained, is also quite explicit. It provides us with a clear way of understanding what information a given state s encodes: it encodes the information that the actual state of affairs corresponds to one of the worlds in s. That is, the information available in s characterizes the actual world as having those features that are common to all the elements $w \in s$. Moreover, if a state t is a proper enhancement of s, then it is always clear *how* the information available in t improves on the information available in s—namely, by providing a more precise description of the actual state of affairs.

In the abstract view, it is often difficult to see what information a certain information state s is supposed to encode. For instance, consider Kripke semantics for intuitionistic logic: a model may consist of an infinite chain $\{s_i \mid i \in \mathbb{N}\}$ of information states, where $s_i \sqsupseteq s_j$ for $i \leq j$, and where each s_i is characterized by the same valuation. Now, the model says that s_1 is a proper enhancement of s_0, but it is difficult to see how s_1 can encode more information than s_0, given that the "situation" at s_1, in terms of both valuation and successors, is precisely the same as in s_0 (more formally: the two generated sub-models are isomorphic).

On the other hand, the abstract view of information also has some interesting advantages over the concrete one. First, once we have identified a suitable logic, the abstract perspective allows us to bring out just *what* features of the space of information states are needed to provide a semantics for this logic. For instance, in the concrete view, the space of information states has the structure of a complete atomic Boolean algebra, but this very special structure is typically not needed in order to have a sound semantics for a given logic. Allowing for a broader class of structures makes it easier to use the semantics to provide countermodels, and, as exemplified by Holliday (2014), it often allows for simpler canonical model constructions which have a nice finitary flavor.

In sum, each of the two views on information has some advantages to it. And, while I think the concrete setup is a better starting point for the present investigation, due to the tighter connection with truth-conditional semantics, I am convinced that the investigation of abstract semantics for the logics introduced here is bound to lead to interesting insights into the structure of these logics, as well as to practical advantages, such as smaller and more manageable models.

"Accessible" information states?

An objection that may be raised against the setup I outlined in this chapter has to do with the fact that any set of worlds can be regarded as an information state. The objection would be that, plausibly, not every set of worlds corresponds to a possible state of information for an agent, even in principle: some bodies of information may, so to speak, be inaccessible.

At the basis of this objection lies a misunderstanding, which is partly to be blamed on our use of the term *information state*—which does suggest the presence of an agent—and partly on the custom, widespread in intuitionistic Kripke semantics, of thinking of an information state and its enhancements as the state of an agent at a given time, and her possible future states.

However, the best way to regard the relation of support is not as involving agents and possible futures at all, but merely as relating some abstract body of information s and a sentence α. Even if s is not something that can actually be known to an agent, it still makes good sense to consider what is settled in s, i.e., to consider what follows from the fact that the actual world is one of those in s.

For instance, it is presumably impossible for an agent to find herself in a state that settles the question (11-a), but this should not prevent us from seeing that (11-a) entails (11-b), since any body of information settling the former question—no matter whether attainable or not—is bound to also resolve the latter question.

(11) a. What sentences of $\mathcal{L}_{\mathsf{PA}}$ are true in $\mathbb{N}$?
b. Is $\varphi_{\mathsf{Goldbach}}$ true in $\mathbb{N}$?

In this respect, one may perhaps compare inaccessible information states to non-computable functions. Even if these states cannot be "realized", for instance because they do not admit a finite description, this need not prevent us from considering these objects mathematically, and investigating their properties.

Furthermore, even when the information state that we are considering *is* the information state of an agent, this does not mean that the only enhancements that matter are those that are possible future states for the agent. In fact, it is not clear why "possible futures" should have anything to do with what is settled by the agent's *current* information.

This is perhaps best seen by considering an example. In the next chapter, we will give the following clause for negation:

$$s \models \neg\varphi \iff \text{for all } t \subseteq s : \ t \models \varphi \text{ implies } t = \emptyset$$

That is, $\neg\varphi$ is settled in s if s is *incompatible* with φ, in the sense that enhancing s so as to support φ is bound to lead to inconsistency. Now suppose the state s of a given agent is compatible with a statement α, i.e., s contains some α-worlds. Then, s does not settle that $\neg\alpha$. This is correctly predicted by the clause, since if s contains some α-worlds, $s \cap |\alpha|_M$ is a consistent enhancement of s that supports α.

Now imagine that, for some reason, it is impossible for the agent to learn that α, so that no state $t \subseteq s$ which supports α is a possible future state for the agent. If we were to restrict the enhancements of s to just those states that *are* possible future states for the agent, the clause would imply, incorrectly, that the agent's information settles that $\neg\alpha$.

The main point here is that what a state does or does not settle depends exclusively on the information available in the state, which in turn is determined by the worlds that it contains. To determine exactly what a state s settles, we may want to check what results from extending the state in certain ways. For instance, in order to determine whether the state s settles $\neg\alpha$, we have to check whether extending the state to support α leads to inconsistency. But this internal exploration of the content of the state has nothing to do with the temporal development of the agent's information. This is why I think that the usual perspective on intuitionistic Kripke models is misleading here: agents and times do not play a role in our picture at this very fundamental level; support is best regarded as a relation between bodies of information s and sentences φ, and whether this relation holds depends exclusively on the content of s and on the meaning of φ. Given this, there seems to be no reason why one should not be allowed to consider what sentences an arbitrary set of worlds s, regarded as a body of information, does or does not support.

Question meanings as support conditions

Our semantic approach is based on a certain view of question meaning, which forms the cornerstone of the framework of inquisitive semantics (Ciardelli et al., 2013a): the meaning of a question is taken to be captured by the question's *support conditions*, i.e., by laying out what information is needed in order to settle the question. In this section, we briefly compare this approach to other classical approaches to questions, explaining why it is more suitable than these alternatives for the enterprise we are embarking on. We will restrict the discussion to those approaches that are most directly related to our own, namely, the *answer-set approach* of Hamblin (1973); Karttunen (1977); Belnap (1982), and the *partition approach* of Groenendijk and Stokhof (1984). Our comparison will be a minimal one: for more detailed discussion, the reader is referred to Ciardelli et al. (2015a).

In the answer-set approach, the meaning of a question μ may be identified with a set $\mathsf{Ans}(\mu)$ of classical propositions—i.e., of sets of worlds—which are called the *basic semantic answers* to the question. For instance, the meaning of (12) is taken to be the set $\{a_F, a_{\overline{F}}\}$, where a_F is the set of worlds where Alice speaks French, and $a_{\overline{F}}$ is the set of worlds where she doesn't.[16]

(12) Does Alice speak French?

From our perspective, the main problem with the answer set approach is that it does not allow us to construe a suitable notion of question entailment, nor to define a satisfactory operation of question conjunction. As Groenendijk and Stokhof (1984) pointed out, combining the answer-set approach with the standard type-theoretic notion of generalized entailment, which amounts to meaning inclusion, does not provide a good notion of entailment for questions. E.g., since no non-trivial question can have a set of basic semantic answers included in $\{a_F, a_{\overline{F}}\}$, we predict that no (non-trivial) question entails (12). However, there is of course a natural sense in which (12) is entailed by (13). This is the notion of question entailment that we, as well as Groenendijk and Stokhof, are interested in.

(13) Which European languages does Alice speak?

Similarly, combining the answer-set approach with the standard type-theoretic notion of conjunction as meaning intersection would predict that most conjunctive questions, such as (14), have an empty set of basic semantic answers.

(14) Who went to the party and who went to the cinema?

One might think that these problems could be overcome by re-defining entailment and conjunction in a point-wise fashion, taking them to apply to each answer rather than to the question meaning as a whole. However, it has been argued in Ciardelli and Roelofsen (2015c) that this solution is not satisfactory either, at least if we aim at a well-behaved logic of questions—which is our priority here.

In our support-based approach, the meaning of a question is a set of sets of worlds—just like in answer-set approaches. However, the elements of this set are not the *basic semantic answers* to the question—whatever these may be exactly—but rather, they are the information states that support the question, i.e., the

[16]More precisely, the meaning of a question is taken to be a function from possible worlds to such sets. This function is defined slightly differently in the various approaches, but these differences are immaterial to the problem raised here.

information states in which the question is settled.

This shift allows us to avoid the problems that we pointed out for the answer-set approach. Indeed, given our approach, the standard type-theoretic notions of entailment and conjunction are perfectly adequate. Consider entailment: we have characterized a question μ as entailing another question ν if ν is supported whenever μ is supported. This simply means that in any model M we have $[\mu]_M \subseteq [\nu]_M$. Thus, question entailment is captured as meaning inclusion. Moreover, we will see in Chapter 2 that the conjunctive questions are unproblematic, too, and that the conjunction of questions can be interpreted by means of intersection. Thus, while assigning to a question the same *type* of object as in answer-set approaches—a set of sets of worlds—our approach does provide us with a suitable semantic foundation for a logic of questions. In fact, we saw that it provides even more, namely, a foundation for an integrated logic of statements and questions.

Now let us turn to the partition approach of Groenendijk and Stokhof (1984). In this approach, the meaning of a question μ is taken to be an equivalence relation $\equiv^\mu$ on the space of possible worlds. The idea is that the relation $w \equiv^\mu w'$ holds in case the true complete answer to μ is the same in w as in w'. For instance, the meaning of (12) is the relation that makes two worlds equivalent in case they assign the same truth-value to the statement "Alice speaks French".

Unlike the answer-set approach, the partition approach does allow us to capture question entailment and conjunction in a suitable way. Thus, this approach provides a good foundation for a logic of questions; and indeed, it has been taken as a basis for the most important precursor of our logics, namely, the Logic of Interrogation of Groenendijk (1999). In fact, this system shows that the partition approach also allows for a uniform notion of entailment in which both questions and statements can participate, like the one we have been developing.

The Logic of Interrogation will be discussed in some detail in the next section, where we also make a precise comparison with our support approach.[17] Through this comparison, the main advantage of the support approach over the partition approach will also become clear. Essentially, we will see that the two approaches are equivalent whenever they are both defined, but the support approach is strictly more general. The reason is that the partition approach is committed to the assumption that there exists such a thing as *the true complete answer* to a question at any given world. As a consequence, it is only suitable to capture those questions for which this assumption is satisfied—call them *partition questions*; however, we will see that some important classes of questions are not of this kind. Summing up, then, we can say that the support approach shares the formal advantages of the partition approach, while not being confined to the restricted class of partition questions.

[17]This logic is also discussed more technically in Section 4.8.2, where it is shown to correspond to a fragment of our first-order system.

1.6.3 Relation with previous work

In this section, the present proposal is situated within the landscape of previous approaches to questions in logic. At this point, we only compare the general structure of our logical framework with alternative options which have been proposed to incorporate questions in logic. More specific connections which exist between the specific systems introduced in this thesis and other related logics will be discussed within the sections devoted to these systems.

Non-entailment directed approaches

Throughout most of history, logicians have paid little attention to questions, mostly confining their investigations to statements. It is not until the second half of the 20th century that logical works devoted to questions have started to appear. In most of these works (e.g. Åqvist, 1965; Harrah, 1961, 1963; Belnap and Steel, 1976; Tichy, 1978) the emphasis has been on providing a logical language for questions, and on characterizing the relation of *answerhood* between statements and questions. Other approaches have focused instead on the role of questions in processes of inquiry, either modeling inquiry itself as a sequence of questioning moves and inference moves, as in the *interrogative model of inquiry* of Hintikka (1999), or characterizing how questions are arrived at in an inquiry scenario, as in the *inferential erotetic logic* of Wiśniewski (1994, 1996, 2001).[18]

However, all these theories share the assumption that dealing with questions requires turning to relations other than logical entailment. Thus, they all pursue enterprises which are rather different from the one that we are concerned with here: bringing questions into play on a par with statements in the very relation of logical entailment, and characterizing how they can be manipulated in entailment-tracking logical proofs.[19]

The Logic of Interrogation

To the best of my knowledge, the first approach that allows for a generalization of the classical notion of entailment to questions is the Logic of Interrogation (LoI) of Groenendijk (1999), based on the partition theory of questions of Groenendijk and Stokhof (1984). The original presentation of the semantics is a dynamic one, in which the meaning of a sentence is identified with its context-change potential. However, as pointed out also by ten Cate and Shan (2007), the dynamic coating is not essential. In its essence, the system may be described as follows: both

[18]For some discussion of the relations between inferential erotetic logic and inquisitive semantics, see Wiśniewski and Leszczyńska-Jasion (2015) and Ciardelli et al. (2015b).

[19]The latter enterprise, making inferences with questions, has even been deemed as nonsensical in the introduction of Belnap and Steel's book *The Logic of Questions and Answers*. On the contrary, we will see in Chapter 3 that questions have an important role to play in logical proofs, and that proofs involving questions have an interesting computational interpretation.

statements and questions are interpreted with respect to pairs $\langle w, w' \rangle$ of possible worlds: a statement is satisfied by such a pair if it is true at both worlds, while a question is satisfied if the true answer to the question is the same in both worlds. In this approach, the meaning of a sentence φ is captured by the set of pairs $\langle w, w' \rangle$ satisfying φ; for any φ, this set is an equivalence relation over a subset of W_M, which we will denote as $\sim_\varphi$. Such an equivalence relation may be equivalently regarded as a partition Π_φ of a subset of the logical space, where the blocks of the partitions are the equivalence classes $[w]^{\sim_\varphi}$ of worlds modulo $\sim_\varphi$.

$$\Pi_M^\varphi = \{[w]^{\sim_\varphi} \mid w \in W_M\}$$

For a statement α, the partition Π_M^α always consists of a unique block, namely, the truth-set $|\alpha|_M$ of the statement. For a question μ, Π_M^μ typically consists of several blocks, which are regarded as the various complete answers to the question.

Since statements and questions are interpreted by means of a uniform semantics, LoI allows for the definition of a notion of entailment in which both statements and questions can take part:

$$\varphi \models_{\mathsf{LoI}} \psi \iff \text{for all } M \text{ and all } w, w' \in W_M : \langle w, w' \rangle \models \varphi \text{ implies } \langle w, w' \rangle \models \psi$$

In terms of partitions, this notion of entailment may be cast as follows:

$$\varphi \models_{\mathsf{LoI}} \psi \iff \text{for all } M, \text{ for all } a \in \Pi_M^\varphi \text{ there is an } a' \in \Pi_M^\psi \text{ such that } a \subseteq a'$$

This is clearly reminiscent of Proposition 1.2.10: here, too, we can think of a sentence as denoting a (possibly singleton) information type; φ entails ψ if any information of type φ always yields some corresponding information of type ψ.

In Groenendijk (1999), this approach is applied to a particular logical language, which is an extension of first-order predicate logic with questions. This gives rise to an interesting combined logic of statements and questions, which was investigated and axiomatized by ten Cate and Shan (2007). We will make use of their axiomatization result in Chapter 4, where we will identify and axiomatize a fragment of first-order inquisitive logic which is essentially equivalent to LoI.

Now, what is the relation between the LoI framework and the approach presented here? Consider a question μ. Given the LoI perspective, it is natural to assume that μ is settled in an information state s in case s entails some complete answer to μ, that is, in case, $s \subseteq a$ for some $a \in \Pi_M^\mu$. We thus have the following relation, where $[\mu]_M$ is the proposition expressed by μ in our approach:

$$[\mu]_M = (\Pi_M^\mu)^\downarrow$$

Moreover, an analogous relation holds for a statement α. For, α is settled in an information state s in case $s \subseteq |\alpha|_M$; given that $\Pi_M^\alpha = \{|\alpha|_M\}$, we have that $s \models \alpha \iff s \subseteq a$ for some $a \in \Pi_M^\alpha$. Thus, for all sentences φ that can be interpreted in LoI, we have:

$$[\varphi]_M = (\Pi_M^\varphi)^\downarrow$$

That is, the set of blocks of the partition induced by φ is always a *generator* for the inquisitive proposition expressed by φ. This allows us to move from the LoI-representation of a sentence, to its support-based representation.

Conversely, notice that the elements of the partition Π^{φ}_{M} can always be characterized as the maximal states included in some $a \in \Pi^{\varphi}_{M}$. This means that the LoI-representation of a sentence φ can be recovered from its support-based representation by taking the *alternatives* for the proposition $[\varphi]_M$:

$$\Pi^{\varphi}_{M} = \textsc{Alt}([\varphi]_M)$$

Thus, for sentences that can be interpreted in LoI, we can go back and forth between the two semantics. Moreover, given that Π^{φ}_{M} is a generator for $[\varphi]_M$, it follows from Proposition 1.2.10 that the notion of entailment that the two frameworks characterize is the same.

$$\varphi \models \psi \iff \varphi \models_{\mathsf{LoI}} \psi$$

Thus, while this has not been highlighted much in the literature, the unified view of entailment that we discussed in Section 1.1 already emerges in LoI.

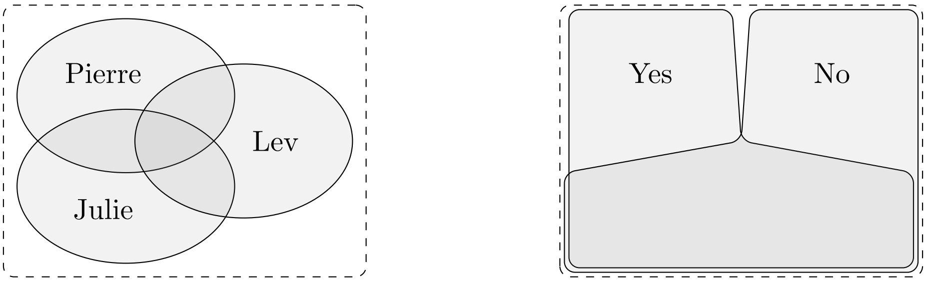
At this point, the reader may be wondering whether what we have done so far is just to provide an alternative semantic foundation for LoI, based on information states rather than pairs of worlds. The answer is *no*. The reason is that the support approach that we discussed in this section is strictly more general than the LoI approach based on pairs of worlds. To see why, consider again the way in which a question μ is interpreted in LoI: a pair of worlds $\langle w, w' \rangle$ satisfies μ in case *the complete answer* to μ is the same in w as in w'. Clearly, this interpretation only makes sense provided that for any world w, there is such a thing as *the complete answer* to μ at w. Now, our analysis of the relation between complete answers and support conditions makes clear what this assumption amounts to: μ must be a *partition question*, in the following sense.

1.6.1. Definition. [Partition questions]
μ is a partition question if any $w \in W_M$ is contained in a unique alternative for μ.

Now, while the class of partition questions does include many natural kinds of questions, there are also important types of questions that fall outside of this class. First of all, we have already seen that a given question may be impossible to resolve at certain worlds. For instance, we saw that the question:

(15) Did Galileo, or Kepler discover Jupiter's moons?

is not soluble at worlds in which neither Galileo nor Kepler discovered Jupiter's moons. Such worlds are not contained in any state where the question is resolved, and a fortiori they are not contained in an alternative for the question, which



(a) A typical French name? (b) If invited, will you go?

Figure 1.6: Illustration of two non-partition questions, (16) and (20). Clearly, in Figure (a), we restricted to a tiny domain of candidates, $\{\text{Julie}, \text{Pierre}, \text{Lev}\}$.

shows that these questions are not partition questions. In general, questions with a presupposition are not representable in LoI.[20]

Besides questions that are not soluble at some worlds, there are also many questions that have multiple minimal ways of being resolved at a given world—multiple "true complete answers" in the LoI terminology. As an example, consider the question (16) below.

(16) What is a typical French name?

On the most salient interpretation, (16) is settled in s in case s establishes that x is a typical French name for some x. In other words, (16) asks for an instance of the property of being a typical French name. Questions that ask for an instance of a given property are known in the literature as *mention-some* questions. Other examples of mention-some questions are the following:

(17) a. Where can I buy an Italian newspaper?
b. How can I get to the station from here?
c. What is an example of a continuous function?

Consider the state a_x consisting of those worlds where x is a typical French name. Thus, for instance, a_{Pierre} is the truth-set of the statement (18):

(18) 'Pierre' is a typical French name.

Clearly, a_x supports the question (16). Moreover, it is natural to assume that our model M is such that, for $x \neq y$, the states a_x and a_y are logically independent, that is, neither is included in the other. It is easy to see that this implies that each a_x is a *maximal* supporting state, i.e., an alternative.

[20] As a matter of fact, questions μ such that any world is included in *at most* one alternative for μ can be made to fit within the LoI framework with a relatively small adjustment: take $\langle w, w' \rangle \models \mu$ to hold in case in both w and w' there exists a complete true answer to μ, and this answer is the same. This strategy does not seem to have been pursued in the literature.

These alternatives do not form a partition of the logical space: rather, they overlap, since a world may well be included in more than one of these alternatives. E.g., the actual world is included in multiple alternatives a_x for the question, each corresponding to one instance x for the property *being a typical French name*.

As this example illustrates, mention-some questions are typically not partition questions, and therefore, they are not representable in LoI. Since such questions constitute a broad and interesting class, the fact that they cannot be represented and reasoned about is a significant limitation of LoI, and of the partition theory of questions on which LoI is based.

Another class of non-partition questions is given by what we might call *approximate value questions*. Consider the following example, due to Yablo (2014).

(19) How many stars are there, give or take ten?

This question is settled in a state s in case s locates the number of stars in some range $[n-10, n+10]$. Let us denote by a_n the set of worlds in which the number of existing stars is within the range $[n-10, n+10]$. Then, (19) is settled in s if and only if $s \subseteq a_n$ for some n. Now, in a model which includes worlds with any given number of stars, each a_n with $n \geq 10$ emerges as an alternative for (19). However, these alternatives are not mutually exclusive. For instance, a world with 15 stars is included both in a_{10} and in a_{20}. Thus, (19) is another example of a non-partition question, and the same is true generally of questions which ask to locate a value within a range of a given size. This is interesting, since experiments in a science like physics typically answer questions of this kind.

Yet another important class of non-partition questions is given by *conditional questions* such as (20), which will be discussed in more detail in the next chapter.

(20) If Mary invites you to the party, will you go?

Such questions, too, cannot be adequately represented within the LoI framework. Summing up, then, the approach described in this chapter coincides with LoI insofar as the treatment of partition questions is concerned, but it is not confined to such questions: in particular, mention-some questions, approximate value questions, and conditional questions, which are out of the reach of LoI, can be handled in a natural way based on the notion of support; and indeed, these classes of questions will all be expressible in the systems discussed in this thesis. Moreover, an important advantage of the support-based approach over LoI is the possibility of internalizing any entailment relation by means of implication, making it possible for our language to include formulas that express dependencies.

Inquisitive pair semantics

Starting with the work of Velissaratou (2000) on conditional questions, the pursuit of greater generality led to the development of a system in which formulas are still evaluated relative to pairs of worlds, but the set of pairs satisfying a given formula is not necessarily an equivalence relation. In this setting, the natural way to read the relation $\langle w, w' \rangle \models \mu$, where μ is a question, is no longer "*the* complete answer to μ is the same in w as in w'", but rather "*some* complete answer to μ is true at both w and w'". This system, laid out in Groenendijk (2009) and Mascarenhas (2009), is now referred to as *inquisitive pair semantics*.

While Groenendijk (2011) showed that this sort of semantics can indeed deal adequately with conditional questions, Ciardelli (2008, 2009), and later Ciardelli et al. (2015b) argued that no pair semantics can provide a satisfactory general framework for question semantics. To get an idea of the problem, consider again our mention-some question (16), and consider a model M where the set of possible worlds is $W = \{w_1, w_2, w_3\}$, where:

- 'Julie' is a typical French name at w_1 and w_2, but not at w_3;
- 'Pierre' is a typical French name at w_2 and w_3, but not at w_1;
- 'Lev' is a typical French name at w_1 and w_3, but not at w_2.

In the information state W, the question (16) is not settled, since there is no single entity of which W implies that it is a typical French name. But this cannot be detected from pairs of worlds alone, since any two worlds *do* share a witness for the property of being a typical French name. If our semantics is based on *pairs* of worlds, we are bound to wrongly predict (16) to be tautological in our model. Examples of this kind suggested a shift from pairs of worlds to information states as points of evaluation, leading to the support-based approach adopted here.

Nelken and Shan's modal approach

After the Logic of Interrogation, a different uniform approach to statements and questions was proposed by Nelken and Shan (2006). In this approach, questions are translated as modal sentences, and they are interpreted by means of truth-conditions: a question is true at a world w in case it is settled by an information state $R[w]$ associated with the world (i.e., the set of successors given by an accessibility relation R). Thus, for instance, Nelken and Shan render the question *whether* p by the modal formula $?p := \Box p \vee \Box\neg p$.

In one respect, this approach is similar to the approach proposed here, since the meaning of a question is essentially taken to be encoded by the conditions under which the question is settled. And indeed, if we consider entailments which involve only questions, the approach of Nelken and Shan makes the expected predictions. However, an asymmetry between statements and questions is maintained: for questions, what matters is whether they are settled by a relevant information state, while for statements, what matters is whether they are true at the world of evaluation. This asymmetry creates problems the moment we start considering cases of entailment involving both statements and questions, such as the one corresponding to our protocol example. It is easy to see that, if such entailments are to be meaningful at all, entailment cannot just amount to preservation of truth. Nelken and Shan propose to fix this by re-defining entailment as *modal consequence*: $\varphi \models \psi$ if, whenever φ is true at *every* possible world in a model, so is ψ. However, this move has the odd consequence of changing the consequence relation for statements in an undesirable way. For instance, if our declarative language indeed contains a Kripke modality, say a knowledge modality K, then if our notion of entailment is redefined as modal consequence, we make undesirable predictions, such as $p \models Kp$. Thus, this approach does not really allow us to extend classical logic with questions in a conservative way.[21]

Incidentally, notice that the asymmetry from which the problem originates can be eliminated by letting statements, too, be interpreted in terms of when they are settled by the state $R[w]$, rather in terms of when they are true at w: that is, we may render a basic statement not as a propositional formula α, but as a modal formula $\Box\alpha$. If we made this move, we would arrive at a framework with a sensible logic, but with some unnecessary complexity: while sentences are interpreted with respect to a world w equipped with an information state $R[w]$, it is only the state $R[w]$ which matters for the interpretation of both statements and questions. We could thus get rid of the worlds, and interpret formulas directly relative to states. This would also allow us to work with simpler models and with a simpler syntax, essentially leading to the approach explored in this thesis.

Other connections

Besides these connections to other logical approaches to questions, the ideas discussed in this chapter are also deeply related to the investigation of the notion of dependency undertaken in the framework of *Dependence Logic* (Väänänen, 2007), and with the treatment of information in modal logics (Hintikka, 1962). We do not discuss these connections here, since each of them is discussed in detail elsewhere in the thesis: the connections with Dependence Logic are the subject of Chapter 5, while the link with modal logic is discussed in Section 6.6.

[21] Besides this, there are other difficulties, too. First, it is hard to make sense of the truth-conditions for questions. E.g., is the question *what is the capital of Spain* actually true? The truth-conditions of a question depend on a given information state, but it is not clear what particular information state we should consider in assessing the truth-conditions of the question at a world. Second, while in Nelken and Shan's system the logical connectives can apply to questions, the results are not always the expected ones; we discuss this in Section 2.6.

Chapter 2

Questions in Propositional Logic

In the previous chapter we discussed how, by moving from the notion of *truth* with respect to a world to the notion of *support* with respect to an information state, we obtain a semantic foundation which is equivalent to the truth-conditional one on statements, but which also allows us to interpret questions. We saw that this gives rise to an interesting generalization of the classical notion of entailment, comprising not only the standard relation of logical consequence, but also the relation of logical dependency.

So far, we have kept our discussion at a very general level, abstracting away from a specific language and from a specific notion of models. In this chapter, we will make our theory concrete in the simplest possible setting, that of propositional logic. In other words, this chapter is devoted to the enterprise of enriching classical propositional logic with questions. The system that results from this enrichment, InqB, will form the basis for all the more expressive systems considered in the subsequent chapters of this thesis.

Formally, InqB coincides with the propositional inquisitive logic of Ciardelli (2009), Groenendijk and Roelofsen (2009) and Ciardelli and Roelofsen (2011). However, we will take a new perspective on this system. In previous work, InqB was thought of as arising from associating standard propositional formulas with meanings that are more fine-grained than plain truth-conditions. As a consequence, InqB emerged as a non-classical logic. Here, we will take a different perspective, suggested by the ideas developed in the previous chapter: we will first provide a support-based semantics for classical propositional logic, equivalent to the standard truth-conditional semantics; we will then exploit the greater generality of support semantics to extend this classical core by means of a question-forming disjunction. As a consequence, InqB will now emerge as a *conservative extension* of classical propositional logic with a new connective. Technically, the difference between the two perspectives is only one of notation. Conceptually, however, the difference is an important one, and it seems that many features of the logic take on a clearer significance from the present perspective.

Unlike for the other chapters in this thesis, the technical results in this chapter are not new, but are due to Ciardelli (2009) and to Ciardelli and Roelofsen (2011). For this reason, the results in this chapter will mostly be presented without proofs. Rather, the focus will be on clarifying the conceptual significance of these results from the perspective of the theory we are constructing.

The chapter is structured as follows. In Section 1, we will see how classical propositional logic can be given a semantics based on the relation of support. In Section 2, this classical core will be enriched with a new connective that allows us to form questions. In Section 3, we will look at the special properties of formulas which are "classical", in the sense that their semantics is completely determined by their truth-conditions. In Section 4 we provide an important normal form result for this logic. In Section 5 we examine the features of the relation of entailment arising from our system. Finally, in Section 6 we discuss the relevance of the inquisitive treatment of logical connectives for the analysis of natural language.

2.1 Support for classical propositional logic

Let $\mathcal{P}$ be a given set of propositional atoms. From the perspective of a propositional language, a state of affairs is characterized completely by a specification of the truth-values of the atoms in $\mathcal{P}$. Thus, a model M for propositional logic will consist simply of a set W, which we regard as a space of possible worlds, together with a propositional valuation $V : W \times \mathcal{P} \to \{0, 1\}$ which specifies for any possible world $w \in W$ and for any atom $p \in \mathcal{P}$ the truth-value of p at w.[1]

2.1.1. Definition. [Propositional information models]
A *propositional information model* for $\mathcal{P}$ is a pair $M = \langle W, V \rangle$, where:

- W is a set, whose elements we refer to as *possible worlds*;
- $V : W \times \mathcal{P} \to \{0, 1\}$ is map that we refer to as the *valuation function*.

An *information state* in an information model $\langle W, V \rangle$ is a set $s \subseteq W$ of possible worlds in M. The state W, which is compatible with all possible worlds, represents the state of complete ignorance, while the set $\emptyset$, which is compatible with no possible worlds, represents the state of inconsistent information. We will refer to W as the *trivial state*, to $\emptyset$ as the *inconsistent state*, and to non-empty states, which are compatible with at least one possible world, as *consistent states*.[2]

[1] We will assume the set $\mathcal{P}$ of propositional atoms to be countable, as customary in propositional logic. This is not strictly necessary, but it simplifies some of the proofs below.

[2] This semantic set-up is slightly different from the one assumed in previous work on InqB. The standard is to assume a fixed model ω, having the propositional valuations themselves as possible worlds. Since this model contains a copy of each possible state of affairs, the

Let us start by providing a support semantics for classical propositional logic. The set $\mathcal{L}_c^{\mathsf{P}}$ of *classical propositional formulas* is given by the following definition:

$$\alpha \ ::= \ p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha$$

That is, we take classical formulas to be built up from atoms and the falsum constant by means of conjunction and implication. We take negation and disjunction to be defined in terms of these primitive connectives, as follows.

2.1.2. Definition. [Defined connectives]

- $\neg\varphi := \varphi \to \bot$
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$

Thus, the language we are assuming is just a standard propositional language. However, we will give the semantics of this language not via a recursive definition of the notion of *truth* with respect to a world, but instead via a recursive definition of the notion of *support* with respect to an information state.

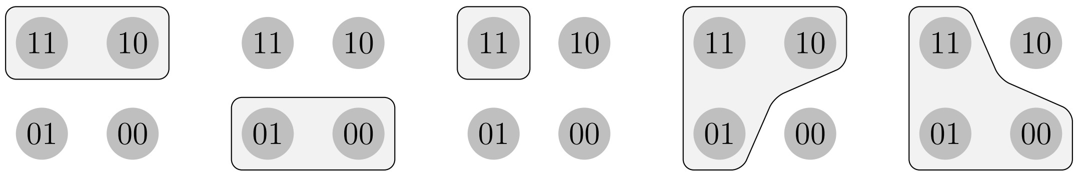
2.1.3. Definition. [Support]
Let M be a propositional information model. The relation of support between states s in M and formulas $\varphi \in \mathcal{L}_c^{\mathsf{P}}$ is defined as follows:

- $s \models p \iff V(w,p) = 1$ for all $w \in s$
- $s \models \bot \iff s = \emptyset$
- $s \models \varphi \wedge \varphi \iff s \models \varphi$ and $s \models \psi$
- $s \models \varphi \to \psi \iff$ for all $t \subseteq s,\ t \models \varphi$ implies $t \models \psi$

Keeping in mind that we read support as capturing the fact that a formula is *settled* in an information state, we can read the clauses as follows. An atom p is settled in s in case the information available in s implies that p is true. The falsum constant $\bot$ is settled in a state only if this is the inconsistent state $\emptyset$. A conjunction is settled in s in case both conjuncts are settled. Finally, implication internalizes contextual entailment in the way described in the previous chapter: $\varphi \to \psi$ is settled in s iff φ entails ψ relative to s, that is, if enhancing s so as to settle φ is bound to lead to a state which also settles ψ.

In order to spell out the clauses for the defined connectives $\neg$ and $\vee$ in a simple way, it will be useful to introduce a derived relation of *compatibility* between information states and formulas: we will say that s is compatible with φ if s can be enhanced consistently to a state that supports φ.

difference is immaterial to the logic. The advantage of the present set-up is that it allows for a smooth transition to modal and first-order logic: we will just need to add to our model other components that contribute to describe the state of affairs at each world, and that allow us to interpret additional operators in the language.



(a) p (b) $\neg p$ (c) $p \wedge q$ (d) $p \vee q$ (e) $p \to q$

Figure 2.1: The alternatives for some classical formulas in a four-worlds model. 11 represents a world where p and q are both true, 10 a world where p is true and q is false, etc. Proposition 2.1.8 implies that a classical formula φ always has a unique alternative, which coincides with the set $|\varphi|_M$ worlds where it is true.

2.1.4. Definition. [Compatibility]
A state s is *compatible* with a formula φ, notation $s \between \varphi$, in case there is a consistent $t \subseteq s$ such that $t \models \varphi$

Using this notion, the derived semantic clauses for negation and disjunction may be expressed as follows.

2.1.5. Proposition (Support conditions for defined connectives).

- $s \models \neg\varphi \iff$ *it is not the case that* $s \between \varphi$
- $s \models \varphi \vee \psi \iff$ *for all consistent* $t \subseteq s$*, either* $t \between \varphi$ *or* $t \between \psi$

That is, a negation $\neg\varphi$ is settled in a state if the state is *in*compatible with φ, i.e., it cannot be consistently enhanced to support φ. As for classical disjunction, $\varphi \vee \psi$ is settled in a state if any consistent enhancement of it is bound to be compatible with either φ or ψ, that is, if s cannot be consistently enhanced to a state that rules out both φ and ψ.

It will be useful to recall the following notions from the previous chapter: the *support-set* of a formula φ is the set of states that support φ, while the *alternatives* for φ are the *maximal* states supporting φ.

2.1.6. Definition. [Support-set, alternatives]

- The support set of φ in a model M is the set: $[\varphi]_M = \{s \subseteq W \mid s \models \varphi\}$
- The set of alternatives for φ in a model M is the set:
  $\textsc{Alt}_M(\varphi) = \{s \subseteq W \mid s \models \varphi$ and there is no $t \supset s$ such that $t \models \varphi\}$

Also, recall that, in a support-based semantics, the notion of truth can be recovered by defining truth at a world w as support with respect to the corresponding singleton state:

$$M, w \models \varphi \overset{def}{\iff} M, \{w\} \models \varphi$$

The following proposition shows that the notion of truth that we obtain in this way coincides with the notion of truth in classical propositional logic.

2.1.7. Proposition (Truth-conditions for classical formulas).
For any model $M = \langle W, V\rangle$ and any world $w \in W$ we have:

- $w \models p \iff V(w, p) = 1$
- $w \not\models \bot$
- $w \models \varphi \wedge \psi \iff w \models \varphi$ *and* $w \models \psi$
- $w \models \varphi \to \psi \iff w \not\models \varphi$ *or* $w \models \psi$
- $w \models \neg\varphi \iff w \not\models \varphi$
- $w \models \varphi \vee \psi \iff w \models \varphi$ *or* $w \models \psi$

Thus, the standard truth-conditions for all classical formulas can be obtained from our support semantics. Moreover, the converse is also true: the support-conditions of a classical formula can be always be derived from its truth-conditions: for, support at a state simply amounts to truth at each world in the state. In the terminology of Definition 1.3.6, all classical formulas are *truth-conditional.*

2.1.8. Proposition (Classical formulas are truth-conditional).
For any model M, any state s, and any formula $\varphi \in \mathcal{L}_c^{\mathsf{P}}$:

$$s \models \varphi \iff w \models \varphi \text{ for all } w \in s$$

Thus, the support-semantics given above and the standard truth-conditional semantics are interdefinable. Moreover, we have seen in Section 1.1.2 that the connection given by Proposition 2.1.8 is sufficient to ensure that the two semantics give rise to the same notion of entailment. So, what we have given so far is a support semantics for classical propositional logic.

2.2 Adding questions to propositional logic

Now that we have re-implemented classical propositional logic in terms of support, we can exploit the extra richness of the support framework over the truth-conditional framework to extend classical propositional logic with questions. We will do this by enriching our classical language with a new connective $\mathbin{\vee\mkern-9mu\vee}$, called *inquisitive disjunction.* Thus, the full language of our system is generated from propositional atoms and $\bot$ by means of the connectives $\wedge, \to$, and $\mathbin{\vee\mkern-9mu\vee}$.

2.2.1. Definition. [Language $\mathcal{L}^{\mathsf{P}}$]
The language $\mathcal{L}^{\mathsf{P}}$ of propositional inquisitive logic is defined as follows:

$$\varphi \ ::= \ p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\mkern-9mu\vee} \varphi$$

Intuitively, we may regard $\varphi \vee\!\!\!\vee \psi$ as standing for the question *whether* φ *or* ψ, which is settled in case one among φ and ψ is settled. Thus, the support conditions for $\vee\!\!\!\vee$ will be stricter than those of the classical disjunction $\vee$: in order to settle $\varphi \vee\!\!\!\vee \psi$, it is not sufficient to establish that one of φ and ψ must be true; we really must be able to settle one of the two.[3]

2.2.2. Definition. [Support for InqB]
The relation of support for InqB is obtained by supplementing Definition 2.1.3 with the following inductive clause:

- $s \models \varphi \vee\!\!\!\vee \psi \iff s \models \varphi$ or $s \models \psi$

It is easy to see that the support relation defined in this way satisfies the *persistency property* discussed in the previous section: that is, if a formula φ is supported in a state s, it is also supported in any enhancement $t \subseteq s$. Moreover, the support relation also satisfies the condition that we called *semantic ex-falso*, stating that the inconsistent state $\emptyset$ supports every formula.[4]

2.2.3. Proposition (Properties of the support relation).
For any model M and any formula $\varphi \in \mathcal{L}^{\mathsf{P}}$, the following properties hold.

Persistence property: *if $s \models \varphi$ and $t \subseteq s$, then $t \models \varphi$*

Empty state property: $\emptyset \models \varphi$

This ensures that the support-set $[\varphi]_M$ of a formula in a model is always an *inquisitive proposition*, in the sense of Definition 1.2.2.

All the truth-conditional clauses given in Proposition 2.1.7 for the classical connectives still hold for the full language $\mathcal{L}^{\mathsf{P}}$. Moreover, the support condition given for $\vee\!\!\!\vee$ gives rise to the following truth-conditions.

2.2.4. Proposition (Truth-conditions for $\vee\!\!\!\vee$).
For any model M and any possible world w: $w \models \varphi \vee\!\!\!\vee \psi \iff w \models \varphi$ or $w \models \psi$

[3]It seems quite possible that an alternative question like *whether φ or ψ* in English is only settled by establishing that one of φ and ψ holds, *to the exclusion of the other* (as assumed by Biezma and Rawlins, 2012; Aloni et al., 2013, among others). If so, an alternative question should be translated in our formal language not as $\varphi \vee\!\!\!\vee \psi$, but as an *exclusive* inquisitive disjunction $\varphi \underline{\vee\!\!\!\vee} \psi := (\varphi \wedge \neg\psi) \vee\!\!\!\vee (\psi \wedge \neg\varphi)$. Nothing important in this thesis hinges on this empirical issue. What matters is that, whichever way we want to construe them, such questions can be represented and reasoned about in the system.

[4]The reader familiar with intuitionistic logic might have noticed that support amounts to satisfaction in an intuitionistic Kripke model having consistent information states as points, and $\supseteq$ as accessibility relation. This connection is explored in detail in Ciardelli and Roelofsen (2009) and Ciardelli (2009).

This immediately shows that, unlike classical formulas, formulas containing $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ are in general not truth-conditional; that is, for formulas containing $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, support at a state does not boil down to truth at each world in the state. A simple way to see this is to see that, by the previous proposition, the formula $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$ is true at every world in every model. If this formula were truth-conditional, it should then be supported at any information state. But this is not the case, since $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$ is not supported at s as soon as s contains both p-worlds and $\neg p$—worlds.

Indeed, the formula $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg p$ captures the polar question *whether p*, which is settled in case the information available in s implies that p, or it implies that $\neg p$. Throughout the thesis, it will be very useful to have a defined operator that allows us to turn a statement into the corresponding polar question.

2.2.5. Definition. [Question mark operator] $?\varphi := \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \neg\varphi$

As discussed in Section 1.3, we will take truth-conditional formulas to represent *statements*, and non truth-conditional formulas to represent *questions*.[5,6]

2.2.6. Definition. [Statements and questions]
We call a formula φ a *statement* if it is truth-conditional, and a *question* otherwise. Henceforth, we will use the letters α and β to refer to statements, μ and ν to refer to questions, and φ and ψ for formulas which may belong to either category.

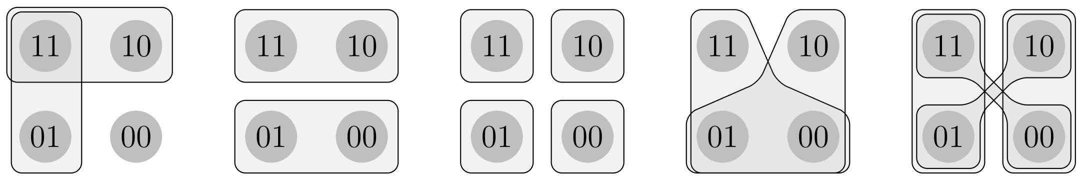
In our propositional logic, all formulas are *normal*, which means that the meaning of φ in a model M is always fully captured by the set of alternatives $\textsc{Alt}_M(\varphi)$.

[5]This is not the perspective usually taken in inquisitive semantics. In most previous work (e.g., Ciardelli et al., 2012), only sentences φ whose support set $[\varphi]_M$ is guaranteed to cover the entire logical space are regarded as questions. Here, I opt for a more liberal view of questions, since I also want to consider questions that can only be resolved in *some* worlds. Indeed, here I take sentences with multiple alternatives which do not cover the whole logical space to correspond to questions like (i-a) in English, which can only be truthfully resolved in worlds in which Mary is coming. In previous work, the same sentences were taken to correspond to hybrid English sentences such as (i-b).

(i) a. Is Mary coming with or without John?
b. Mary is coming, but is John coming as well?

From the present perspective, hybrids like (i-b) cannot be modeled in the system, while from the perspective of previous work, questions like (i-a) cannot be modeled. Ultimately, both kinds of sentences can be represented in a propositional system which explicitly captures presuppositions as a meaning component (for a concrete proposal, see Ciardelli et al., 2012, §7.1). Then, while (i-a) and (i-b) have identical support conditions, they will still be distinguishable in terms of presuppositions: (i-a) presupposes that Mary is coming, while (i-b) presupposes nothing.

[6]Notice that our perspective partitions the formulas of our language into a set of statements and a set of questions. In this respect, it brings InqB closer to the *dichotomous inquisitive semantics with presuppositions* InqD_π of Ciardelli et al. (2015b). The difference is that, here, the dichotomy is not built into the syntax of the language, and no restrictions are placed on the applicability of propositional connectives, which makes the logic somewhat more natural.



(a) $p \mathbin{\vee\!\!\!\vee} q$ (b) $?p$ (c) $?p \wedge ?q$ (d) $p \to ?q$ (e) $?p \to ?q$

Figure 2.2: The alternatives for some questions in InqB.

2.2.7. Proposition (Normality).
For any model M, state s and $\varphi \in \mathcal{L}^P$: $s \models \varphi$ implies $s \subseteq a$ for some $a \in \textsc{Alt}_M(\varphi)$.

This makes it possible to characterize truth-conditionality in terms of alternatives: a formula φ is truth-conditional if it has a unique alternative in any model, which must then coincide with the set $|\varphi|_M$ of worlds in which φ is true.

2.2.8. Proposition.

$$\begin{aligned} \varphi \textit{ is truth-conditional } &\iff \textit{for any model } M,\ \textsc{Alt}_M(\varphi) \textit{ is a singleton} \\ &\iff \textit{for any model } M,\ \textsc{Alt}_M(\varphi) = \{|\varphi|_M\} \end{aligned}$$

Thus, statements are formulas that, in any model, can be settled in only one way—by establishing that they are true—while questions are formulas which, in at least in some models, can be settled in multiple alternative ways.

Let us now take a look at some examples of questions in InqB, illustrated in Figure 2.2. The simplest example is the inquisitive disjunction $p \mathbin{\vee\!\!\!\vee} q$, which is settled when either of p and q is established. This means that in any model M, the set $\{|p|_M, |q|_M\}$ consisting of the truth-set of p and the truth-set of q is a generator for the proposition $[p \mathbin{\vee\!\!\!\vee} q]_M$: that is, we have $[p \mathbin{\vee\!\!\!\vee} q]_M = \{|p|_M, |q|_M\}^{\downarrow}$.

The situation is similar for the question $?p := p \mathbin{\vee\!\!\!\vee} \neg p$, which as we saw captures the polar question *whether* p: $?p$ is settled in a state s when either of p and $\neg p$ is established, i.e., when the state s determines the truth-value of p. Thus, in any model M, the set $\{|p|_M, |\neg p|_M\}$ is a generator for the proposition $[?p]_M$.

Let us now consider questions that are not obtained directly by applying inquisitive disjunction to statements, but that result from combining other questions by means of conjunction and implication.

Consider conjunction first. We will see shortly that, when α and β are statements, $\alpha \wedge \beta$ is itself a statement, with the usual, conjunctive truth-conditions. However, conjunction can now also be applied to questions. For an example, consider the formula $?p \wedge ?q$: as shown by Figure 2.2(c), this is a question that is settled in a state s just when both questions $?p$ and $?q$ are settled, that is, just when the state s determines the truth-value of both p and q. In general, conjunction allows us to combine two questions μ and ν, resulting in a formula $\mu \wedge \nu$ which is settled precisely when both μ and ν are settled.

Next, consider implication. We will see that, when α and β are statements, the formula $\alpha \to \beta$ is itself a statement with the usual truth-conditions. However, as for conjunction, now we can also consider the effect of applying this connective to questions. Let us first consider the case in which the antecedent is a statement. In this case, the clause for $\to$ can be simplified.

2.2.9. Proposition.
Let α be a truth-conditional formula. Then for any M, s, and $\varphi \in \mathcal{L}^P$:

$$s \models \alpha \to \varphi \iff s \cap |\alpha|_M \models \varphi$$

That is, $\alpha \to \varphi$ is supported in a state s just in case φ is supported in the state which results from enhancing s with the information that α. Thus, settling the conditional $\alpha \to \varphi$ amounts to settling φ *under the assumption that* α.

As an example, consider the formula $p \to ?q$. This formula is a question, which is settled in a state s in case the truth-value of q is the same in all the p-worlds in s, that is to say, in case $s \cap |p|_M \subseteq |q|_M$, or $s \cap |p|_M \subseteq |\neg q|_M$. It is easy to see that this is the case if and only if $s \subseteq |p \to q|_M$ or $s \subseteq |p \to \neg q|_M$. Thus, to settle the conditional question $p \to ?q$, one needs to establish either of the conditional statements $p \to q$ and $p \to \neg q$, which correspond to the two alternatives displayed in Figure 2.2(d).

Now consider the case in which both the antecedent and the consequent of our conditional are questions. As we saw in Section 1.4, such a conditional $\mu \to \nu$ captures a dependency, in the sense that $\mu \to \nu$ is settled in a state s iff $\mu \models_s \nu$, that is, iff μ determines ν in the context of the state s.

For an example, consider the formula $?p \to ?q$: this formula is supported in a state s in case within s, whether p is true determines whether q is true. For instance, the implication $?p \to ?q$ is supported by the state $\{10, 01\}$ in the model of Figure 2.2, but not by the state $\{10, 11\}$. In fact, it is easy to see that a state s supports $?p \to ?q$ iff it supports at least one of the following statements:

- $(p \to q) \wedge (\neg p \to q)$
- $(p \to q) \wedge (\neg p \to \neg q)$
- $(p \to \neg q) \wedge (\neg p \to q)$
- $(p \to \neg q) \wedge (\neg p \to \neg q)$

In Figure 2.2(e), each of these statements corresponds to one alternative for the formula $?p \to ?q$. Notice that, having multiple alternatives in at least one model, this formula is a question. We can think of $?p \to ?q$ as a question asking for a *method* of turning information as to whether p into information as to whether q, and to the above statements as encoding the various possible methods. This idea will be made precise in a moment by means of the notion of a *resolution function*.

Notice that, even though an implication like $?p \to ?q$ captures a dependency, this formula should not be taken to be a formalization of a dependency statement like "whether p determines whether q". In Section 6.5 we will give arguments for this claim, and we will propose an analysis of dependence statements that, while taking them to express dependence relations, also treats them as statements in our formal sense. For now, let us just point out that, according to our definition, $?p \to ?q$ is a question, which can be settled in multiple ways, one for each way of turning information as to whether p into information as to whether q.

2.3 Truth-conditional formulas

Proposition 2.1.8 guarantees that any classical formula is truth-conditional. We are now going to see that, conversely, any truth-conditional formula in InqB is equivalent to a classical formula. Thus, adding inquisitive disjunction to our language enables our logic to express questions, but not to express new statements.[7] The key to this result is to show that with any $\varphi \in \mathcal{L}^{\mathsf{P}}$ we can associate a classical formula φ^{cl} which has the same truth-conditions as φ.

2.3.1. Definition. [Classical variant of a formula]
The *classical variant* of a formula $\varphi \in \mathcal{L}^{\mathsf{P}}$, denoted φ^{cl}, is obtained from φ by replacing all occurrences of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ by $\vee$.

It is immediate to verify by induction on φ that, indeed, φ and φ^{cl} always have the same truth-conditions.

2.3.2. Proposition. *For any formula φ and any model M, $|\varphi^{cl}|_M = |\varphi|_M$*

If a formula $\varphi \in \mathcal{L}^{\mathsf{P}}$ is truth-conditional, then φ and φ^{cl} are both truth-conditional formulas with the same truth-conditions. But then, given any model M and state s, we have: $s \models \varphi \iff s \subseteq |\varphi|_M \iff s \subseteq |\varphi^{cl}|_M \iff s \models \varphi^{cl}$, which shows that φ and φ^{cl} are logically equivalent.

2.3.3. Proposition.
If φ is truth-conditional, then $\varphi \equiv \varphi^{cl}$.

Thus, any truth-conditional formula is equivalent to a classical formula. Conversely, if a formula φ is equivalent to a classical formula, then since classical formulas are truth-conditional, φ must be truth-conditional as well. So, truth-conditional formulas in InqB can be characterized as being precisely those formulas which are equivalent to some classical propositional formula.

[7] This is not always the case for richer languages: in Chapter 7, we will see that questions may be embedded under specific *inquisitive modalities*, resulting in new truth-conditional formulas expressing, for instance, that an agent *wonders about* a certain question. In such a system, the presence of questions also enables the language to express statements that have no counterpart in a purely classical language.

2.3.4. Corollary.
For any $\varphi \in \mathcal{L}^P$*,* φ *is truth-conditional* $\iff \varphi \equiv \alpha$ *for some* $\alpha \in \mathcal{L}^P_c$*.*

If μ is a question, then μ is not equivalent to its classical variant μ^{cl}. However, the classical variant μ^{cl} is a statement which is true precisely in those worlds where the question μ is *soluble*, that is, in those worlds where μ can be truthfully resolved. As discussed in Section 1.3, we can see the statement μ^{cl} as expressing the presupposition of the question μ. With some abuse of terminology, we will simply refer to μ^{cl} as *the presupposition of* μ, and we will denote it π_μ.

2.3.5. Definition. [Presupposition of a question]
If μ is a question, its presupposition is the statement $\pi_\mu := \mu^{cl}$.

Another important observation related to truth-conditionality is that, by the semantic clause for $\neg$, negations are always truth-conditional.

2.3.6. Proposition. *For any* $\varphi \in \mathcal{L}^P$*,* $\neg\varphi$ *is truth-conditional.*

In particular, then, the double negation of a formula is always truth-conditional, and equivalent to the classical variant of the formula.

2.3.7. Proposition. *For any* $\varphi \in \mathcal{L}^P$*,* $\neg\neg\varphi \equiv \varphi^{cl}$

Proof. Since $\neg\neg\varphi$ is a truth-conditional, we have $\neg\neg\varphi \equiv (\neg\neg\varphi)^{cl}$. But notice that $(\neg\neg\varphi)^{cl} = \neg\neg(\varphi^{cl})$, and $\neg\neg(\varphi^{cl}) \equiv \varphi^{cl}$ since φ^{cl} is a classical formula. □

As a consequence, truth-conditional formulas may also be characterized as being precisely those formulas which are equivalent to their own double negation. In other words, the double negation law is the hallmark of truth-conditionality.

2.3.8. Proposition. *For any* $\varphi \in \mathcal{L}^P$*,* $\varphi \equiv \neg\neg\varphi \iff \varphi$ *is truth-conditional.*

Finally, one last thing that it be useful to note is that the classical connectives preserve truth-conditionality. In fact, in the case of implication, the truth-conditionality of the consequent is sufficient to ensure the truth-conditionality of the implication, regardless of whether the antecedent is truth-conditional.

2.3.9. Proposition ($\wedge$ and $\to$ preserve truth-conditionality).

- *If* α *and* β *are truth-conditional, so is* $\alpha \wedge \beta$*.*
- *If* α *is truth-conditional, so is* $\varphi \to \alpha$ *for any* φ*.*

2.4 Resolutions and normal form

An important feature of the system InqB is that we can compute, recursively on the structure of a formula φ, a set of classical formulas which can be taken to name the different pieces of information of type φ. We refer to these formulas as the *resolutions* of φ.

2.4.1. Definition. [Resolutions]

- $\mathcal{R}(p) = \{p\}$
- $\mathcal{R}(\bot) = \{\bot\}$
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \mathbin{\mathbb{V}} \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$

Notice that resolutions are by definition classical formulas. Moreover, it is easy to show by induction that any classical formula is the only resolution of itself.

2.4.2. Proposition. *If $\alpha \in \mathcal{L}_c^{\mathsf{P}}$, then $\mathcal{R}(\alpha) = \{\alpha\}$.*

The crucial property of resolutions is stated by the following Proposition: to settle the formula φ is to establish of some resolution α of φ that it is true.

2.4.3. Proposition. *For any formula $\varphi \in \mathcal{L}^{\mathsf{P}}$, any model M and state s:*

$$\begin{aligned} s \models \varphi \quad &\iff \quad s \models \alpha \textit{ for some } \alpha \in \mathcal{R}(\varphi) \\ &\iff \quad s \subseteq |\alpha|_M \textit{ for some } \alpha \in \mathcal{R}(\varphi) \end{aligned}$$

This proposition can be restated as saying that, for any model M, we have:

$$[\varphi]_M = \{|\alpha|_M \mid \alpha \in \mathcal{R}(\varphi)\}^{\downarrow}$$

This means that the truth-sets of the resolutions of φ provide a *generator* for the proposition expressed by φ, in the sense of Definition 1.2.5. Thus, a formula $\varphi \in \mathcal{L}^{\mathsf{P}}$ can always be thought of as describing a type of information whose elements are named by its resolutions.[8]

As a corollary of Proposition 2.4.3, we have the following normal form result, which shows that any formula in InqB is equivalent to an inquisitive disjunction of classical formulas.

[8]Notice that, in general, resolutions do not correspond exactly to the alternatives for the formula. This is because one resolution may be strictly stronger than another, and thus it may fail to give rise to an alternative: for instance, $\bot$ is a resolution of $p \mathbin{\mathbb{V}} \bot$, but $|\bot| = \emptyset$ is not an alternative for $p \mathbin{\mathbb{V}} \bot$. If we wanted to obtain a perfect correspondence between resolutions and alternatives, we could filter out from $\mathcal{R}(\varphi)$ those resolutions that strictly entail another resolution. This is unproblematic, but it is not necessary: for our purposes, what really matters is that the resolutions correspond to elements of a generator for the proposition expressed by φ.

2.4.4. Proposition (Inquisitive normal form).
Let $\varphi \in \mathcal{L}^P$ *and let* $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$*. Then,* $\varphi \equiv \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$

It is interesting to remark that there is a close similarity between the inductive definition of resolutions that we gave, and the inductive definition of *proofs* given in the Brouwer-Heyting-Kolmogorov (BHK) interpretation of intuitionistic logic. In this interpretation, a proof of a conjunction is a pair of two proofs, one for each conjunct; a proof of a disjunction is a proof of either disjunct; and a proof of an implication is a function that turns any proof of the antecedent into a proof of the consequent. Similarly, a resolution of a conjunction is a conjunction of two resolutions, one for each conjunct; a resolution of an inquisitive disjunction is a resolution of either disjunct; and a resolution of an implication corresponds to a function from resolutions of the antecedent to resolutions of the consequent. The main difference between the two notions is that, unlike proofs in the BHK interpretation, resolutions are in turn formulas, that is, objects within the same language in which the original formula lives. Thus, we can look at the definition of resolutions as a language-internal analogue of the BHK interpretation.

By means of the notion of resolutions we can also re-state the support conditions for an implication in an interesting way. To spell this out, we will introduce the notion of a *dependence function.*

2.4.5. Definition. [Dependence function]
A function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$ is said to be a dependence function from φ to ψ in a state s of a model M, notation $f : \varphi \rightsquigarrow_s \psi$, in case for any $\alpha \in \mathcal{R}(\varphi)$, $\alpha \models_s f(\alpha)$. A function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$ is said to be a *logical* dependence function from φ to ψ, notation $f : \varphi \rightsquigarrow \psi$, if it is a dependence function in any state of any model.

Thus, f is a dependence function from φ to ψ in s if it is settled in s that, if a resolution $\alpha \in \mathcal{R}(\varphi)$ is true, then the resolution $f(\alpha) \in \mathcal{R}(\psi)$ is true as well. A *logical* dependence function from φ to ψ is a map that is guaranteed to be a dependence function in any possible state.

2.4.6. Example. Consider three propositional atoms p, q, r, and a model M having one possible world for each combination of truth-values for these atoms. Let us write 101 for a world in which p is true, q is false, and r true, and similarly for the other worlds. Now consider the following function $f : \mathcal{R}(?p) \to \mathcal{R}(q \vee\!\!\!\vee r)$ and the following two states s_1 and s_2:

$$f = \left\{ \begin{array}{c} p \mapsto q \\ \neg p \mapsto r \end{array} \right. \qquad s_1 = \left\{ \begin{array}{cc} 111 & 011 \\ 110 & 001 \end{array} \right\} \qquad s_2 = \left\{ \begin{array}{cc} 101 & 011 \\ 110 & 001 \end{array} \right\}$$

In s_1, the truth of a resolution α of $?p$ implies the truth of the corresponding resolution $f(\alpha)$ of $q \vee\!\!\!\vee r$: that is, we have $p \models_{s_1} q$ and $\neg p \models_{s_1} r$. In s_2, this is not the case, since $p \not\models_{s_2} q$. Thus, f is a resolution function in s_1, but not in s_2:

$$f : ?p \rightsquigarrow_{s_1} q \vee\!\!\!\vee r \qquad\qquad f : ?p \not\rightsquigarrow_{s_2} q \vee\!\!\!\vee r$$

Now, the resolutions of an implication $\varphi \to \psi$ are statements of the form:

$$\gamma_f = \bigwedge_{\alpha \in \mathcal{R}(\varphi)} (\alpha \to f(\alpha))$$

for a function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$. The next proposition states that a function $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$ is a dependence function in a state s if and only if the corresponding statement γ_f is supported in s.

2.4.7. Proposition.
Let $f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)$*. For any model* M *and state* s*,* $s \models \gamma_f \iff f : \varphi \leadsto_s \psi$

Proof. The claim follows immediately from the definitions and the equivalence $\alpha \models_s f(\alpha) \iff s \models \alpha \to f(\alpha)$. □

Now, Proposition 2.4.4 tells us that $\varphi \to \psi$ is supported in a state s in case some formula $\gamma_f \in \mathcal{R}(\varphi \to \psi)$ is supported. But by the previous proposition, this holds if and only if there exists a dependence function $f : \varphi \leadsto_s \psi$. We have thus obtained the following result about the support conditions for an implication.

2.4.8. Proposition (Support for implication, restated).
$M, s \models \varphi \to \psi \iff$ *there exists some* $f : \varphi \leadsto_s \psi$

This shows that a state supports an implication $\varphi \to \psi$ iff it admits a dependence function, i.e., if the information available in s allows us to turn any resolution of φ into some corresponding resolution of ψ.

To conclude this section, let us remark that the notion of resolutions, and the results that we have shown about it, can be extended straightforwardly from single formulas to sets of formulas. The idea is that a resolution of a set Φ of formulas is a set Γ of classical formulas obtained by replacing each formula in Φ by a resolution of it.

2.4.9. Definition. [Resolutions for sets]
If Φ is a set of formulas, a *resolution function* for Φ is a map $f : \Phi \to \mathcal{L}_c^{\mathsf{P}}$ such that for each $\varphi \in \Phi$ we have $f(\varphi) \in \mathcal{R}(\varphi)$. We say that a set Γ of formulas is a *resolution* of Φ if it is the image of Φ under some resolution function:

$$\mathcal{R}(\Phi) = \{f[\Phi] \mid f \text{ is a resolution function for } \Phi\}$$

Thus for instance, the set $\Phi = \{p, ?q, ?r\}$ has the following four resolutions:

- $\{p, q, r\}$
- $\{p, \neg q, r\}$
- $\{p, q, \neg r\}$
- $\{p, \neg q, \neg r\}$

It is clear from the definition that any resolution of a set of formulas is a set of *classical* formulas. Moreover, consider a set Γ of classical formulas: since any $\alpha \in \Gamma$ has itself as unique resolution, there is only one resolution function for Γ, namely, the identity function. As a consequence, Γ is the only resolution of itself.

2.4.10. PROPOSITION. *If* $\Gamma \subseteq \mathcal{L}_c^{\mathsf{P}}$, *then* $\mathcal{R}(\Gamma) = \{\Gamma\}$.

Just like a state supports a formula iff it supports some resolution of it, so a state supports a set of formulas iff it supports some resolution of it.[9] Thus, the resolutions of a set capture the different ways in which all the formulas in the set may be settled.

2.4.11. PROPOSITION. *For any set of formulas* Φ *and any state* s*:*

$$s \models \Phi \iff s \models \Gamma \textit{ for some } \Gamma \in \mathcal{R}(\Phi)$$

Proof. Suppose $s \models \Phi$. For any $\varphi \in \Phi$, $s \models \varphi$, so by Proposition 2.4.3 we have some resolution $\alpha_\varphi \in \mathcal{R}(\varphi)$ such that $s \models \alpha_\varphi$. Now let f be the function which maps each $\varphi \in \Phi$ to the corresponding α_φ: by definition, $f[\Phi] \in \mathcal{R}(\Phi)$ and $s \models f[\Phi]$.[10] The converse direction is immediate, again using Proposition 2.4.3 and the definition of resolutions for sets. $\Box$

Similarly, the notion of a *dependence function* can be extended straightforwardly to the case in which we have a set of determining formulas, as follows.

2.4.12. DEFINITION.
A function $f : \mathcal{R}(\Phi) \to \mathcal{R}(\psi)$ is a *dependence function* from Φ to ψ in a state s, notation $f : \Phi \rightsquigarrow_s \psi$, in case for all $\Gamma \in \mathcal{R}(\Phi)$ we have $\Gamma \models_s f(\Gamma)$. A function $f : \mathcal{R}(\Phi) \to \mathcal{R}(\psi)$ is a *logical* dependence function, notation $f : \Phi \rightsquigarrow \psi$, if it is a dependence function in any state of any model.

2.5 Entailment in InqB

Now that we have enriched classical propositional logic with questions, let us take a look at the features of the resulting logic. Let us start out by considering some special cases of entailment.

[9] Recall from the previous chapter that we say that a state supports a set Φ of formulas in case it supports all formulas in the set: $s \models \Phi \iff s \models \varphi$ for all $\varphi \in \Phi$.

[10] The definition of f does not require the axiom of choice, since we may fix an enumeration of the formulas in $\mathcal{L}^{\mathsf{P}}$, and we may define α_φ as the least resolution of φ supported by s.

2.5.1 Entailments with truth-conditional conclusions

First, in case the conclusion of an entailment relation is truth-conditional, entailment boils down to preservation of truth.

2.5.1. Proposition.
Let $\Phi \cup \{\alpha\} \subseteq \mathcal{L}^P$, where α is truth-conditional. Then:

$$\Phi \models \alpha \iff \textit{for any model } M \textit{ and world } w,\ w \models \Phi \ \textit{ implies } \ w \models \alpha$$

Proof. The left-to-right direction is immediate, since truth is a special case of support. For the converse, suppose for any model M and any world w in M, $w \models \Phi$ implies $w \models \alpha$. We want to show that $\Phi \models \alpha$. So, consider a model M and an arbitrary state $s \models \Phi$ in M. By persistency, this implies that for all $w \in s$ we have $w \models \Phi$. By our assumption, it follows that for all $w \in s$ we have $w \models \alpha$. Since α is truth-conditional, this implies that $s \models \alpha$. Hence, $\Phi \models \psi$. □

Since classical formulas are truth-conditional, this implies that entailment among classical formulas just amounts to the standard truth-conditional notion. Since the truth-conditions of each classical formula are the standard ones, it follows that entailment restricted to the classical fragment of InqB is simply entailment in classical propositional logic.

2.5.2. Proposition (Conservativity over classical logic).
Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^P_c$. Then $\Gamma \models \alpha \iff \Gamma$ entails α in classical propositional logic.

In this precise sense, InqB is a conservative extension of classical propositional logic. This is interesting, since simply by taking $\vee\!\!\!\vee$ to be the "official" disjunction of the system, rather than a new connective, InqB can also be regarded as a non-standard intermediate logic (see Ciardelli, 2009).

Another immediate consequence of Proposition 2.5.1 is that, when the conclusion is truth-conditional, any assumption may be replaced by its classical variant.

2.5.3. Proposition.
Let $\Phi \cup \{\alpha\} \subseteq \mathcal{L}^P$ where α is truth-conditional, and let $\Phi^{cl} = \{\varphi^{cl} \mid \varphi \in \Phi\}$.

$$\Phi \models \alpha \iff \Phi^{cl} \models \alpha$$

In particular, this tells us that a statement is entailed by a question if and only if it is entailed by the question's presupposition: $\mu \models \alpha \iff \pi_\mu \models \alpha$. This tells us that the presupposition of a question can be characterized as the strongest statement, modulo logical equivalence, which is entailed by the question.

Thus, for instance, the question $p \vee\!\!\!\vee q$ entails its presupposition $p \vee q$, all of its consequences, and no other statements. For another example, notice that the presupposition of a polar question $?\alpha$ is the tautology $\alpha \vee \neg\alpha$; thus, tautologies are the only statements that are entailed by a polar question.

2.5.2 Entailments with truth-conditional assumptions

Let us now consider the case in which the *assumptions* of our entailment relation are truth-conditional. In this case, too, the conditions for the entailment to hold can be simplified. The following proposition says that, in this case, to check whether an entailment $\Gamma \models \varphi$ holds we do not have to check all states in which Γ is supported: it suffices to check whether φ is supported in the state which embodies the information carried by Γ, namely, the state $|\Gamma|_M = \{w \mid w \models \alpha \text{ for all } \alpha \in \Gamma\}$.

2.5.4. Proposition.
Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^P$, where all formulas in Γ are truth-conditional. We have:

$$\Gamma \models \varphi \iff \text{for any model } M,\ |\Gamma|_M \models \varphi$$

As a consequence of this proposition, we have the following important property: a set of statementss entails an inquisitive disjunction if and only if it entails a specific disjunct of it.

2.5.5. Proposition (Split Property).
Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}^P$, where Γ is a set of truth-conditional formulas. We have:

$$\Gamma \models \varphi \vee\!\!\!\vee \psi \iff \Gamma \models \varphi \text{ or } \Gamma \models \psi$$

Proof. If $\Gamma \models \varphi$ or $\Gamma \models \psi$, then obviously $\Gamma \models \varphi \vee\!\!\!\vee \psi$. For the converse, suppose $\Gamma \not\models \varphi$ and $\Gamma \not\models \psi$. By the previous proposition, this means that there are two models $M = \langle W, V\rangle$ and $M' = \langle W', V'\rangle$ such that $|\Gamma|_M \not\models \varphi$ and $|\Gamma|_{M'} \not\models \psi$. Now define a new information model $M'' = \langle W'', V''\rangle$, whose set of worlds is the disjoint union $W \uplus W'$, and whose valuation V'' coincides with V on W and with V' on W'. It is easy to prove that, for states $s \subseteq W$, the support relation is the same in M as in M'', while for states $s \subseteq W'$, it is the same in M as in M''. Notice that this implies that $|\Gamma|_{M''} = |\Gamma|_M \cup |\Gamma|_{M'}$. Now, by persistency, since $|\Gamma|_M \not\models \varphi$ we must also have $|\Gamma|_{M''} \not\models \varphi$. Similarly, since $|\Gamma|_{M'} \not\models \psi$ we must also have $|\Gamma|_{M''} \not\models \psi$. Now, since $|\Gamma|_{M''}$ supports neither φ nor ψ, we have $|\Gamma|_{M''} \not\models \varphi \vee\!\!\!\vee \psi$. By the previous proposition, this shows that $\Gamma \not\models \varphi \vee\!\!\!\vee \psi$. □

In particular, if we take $\Gamma = \emptyset$ we obtain the Disjunction Property.

2.5.6. Corollary (Disjunction Property).
For any $\varphi, \psi \in \mathcal{L}^P$, $\models \varphi \vee\!\!\!\vee \psi \iff \models \varphi$ or $\models \psi$.

Now, notice that it follows from the normal form result given by Proposition 2.4.3 that a formula is always entailed by any of its resolutions.

2.5.7. Proposition. *If $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \models \varphi$.*

Thus, anything that entails a resolution of a formula also entails the formula itself. Conversely, the following proposition ensures that, if some statements entail a formula φ, then they entail a particular resolution of φ. In particular, this means that, if some statements logically resolve a question, this must be because they entail a particular resolution of it.

2.5.8. Proposition (Resolution Property).
Let Γ be a set of truth-conditional formulas and let φ be an arbitrary formula:

$$\Gamma \models \varphi \iff \Gamma \models \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

Proof. The left-to-right direction follows immediately form the previous proposition. For the converse, let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. By Proposition 2.4.4, we have $\varphi \equiv \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$. So, $\Gamma \models \varphi$ implies $\Gamma \models \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$. By the Split Property we must then have $\Gamma \models \alpha_i$ for some i. $\square$

By considering the special case in which $\Gamma = \emptyset$, we also get that a formula is logically valid if and only if some resolution of it is logically valid.

2.5.9. Corollary. *For any $\varphi \in \mathcal{L}^P$, $\models \varphi \iff \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$*

In fact, all these properties hold not only for the absolute notion of entailment, but also for entailment in context.[11] First consider the Split Property.

2.5.10. Proposition (Local Split Property).
Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}^P$, where Γ is a set of truth-conditional formulas. For any model M and state s:

$$\Gamma \models_s \varphi \vee\!\!\!\vee \psi \iff \Gamma \models_s \varphi \text{ or } \Gamma \models_s \psi$$

Proof. Suppose $\Gamma \models_s \varphi \vee\!\!\!\vee \psi$. It is easy to see that, since the formulas in Γ are truth-conditional, there is a unique largest substate of s that supports all formulas in Γ, namely, $s \cap |\Gamma|_M$. Now, since $s \cap |\Gamma|_M \models \Gamma$ and $\Gamma \models_s \varphi \vee\!\!\!\vee \psi$, it follows that $s \cap |\Gamma|_M \models \varphi \vee\!\!\!\vee \psi$, which by the support clause for $\vee\!\!\!\vee$ means that either $s \cap |\Gamma|_M \models \varphi$, or $s \cap |\Gamma|_M \models \psi$. Since any state $t \subseteq s$ which supports Γ is included in $s \cap |\Gamma|_M$, Persistency ensures that in the first case we have $\Gamma \models_s \varphi$, while in the second case we have $\Gamma \models_s \psi$. The converse direction is immediate. $\square$

As a consequence of this proposition, we obtain that the Disjunction Property and the Resolution Property also hold for entailment in context. Moreover, since entailment in context is internalized as support of an implication, the Local Split Property actually amounts to the validity of a certain entailment pattern, which will play an important role in our proof system.

[11] Recall that Φ entails ψ in the context of an information state s in case any subset of s which support Φ also supports ψ (cf. Definition 1.1.1).

2.5.11. Corollary.
Let $\alpha, \varphi, \psi \in \mathcal{L}^P$, where α is truth-conditional. We have:

$$\alpha \to (\varphi \vee\!\!\!\vee \psi) \models (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi)$$

Proof. Suppose $s \models \alpha \to (\varphi \vee\!\!\!\vee \psi)$ for an arbitrary state s. We know that this amounts to the validity of the contextual entailment $\alpha \models_s \varphi \vee\!\!\!\vee \psi$. By the previous proposition, this implies that we must have either $\alpha \models_s \varphi$ or $\alpha \models_s \psi$. In the former case, we have $s \models \alpha \to \varphi$, and in the latter case, we have $s \models \alpha \to \psi$. In either case, it follows that $s \models (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi)$. □

2.5.3 The general case

We are now left to consider the general case, where both assumptions and conclusions are allowed to be questions. In this case, too, an interesting characterization of entailment in terms of resolutions can be given: a set of formulas Φ entails a formula ψ iff any resolution of Φ entails some corresponding resolution of ψ.

2.5.12. Theorem (Resolution Theorem).
For any set Φ of formulas and any formula ψ:

$$\Phi \models \psi \iff \textit{for every } \Gamma \in \mathcal{R}(\Phi) \textit{ there is some } \alpha \in \mathcal{R}(\psi) \textit{ such that } \Gamma \models \alpha$$

Proof. For the left-to-right direction, suppose $\Phi \models \psi$ and take any $\Gamma \in \mathcal{R}(\Phi)$. By Proposition 2.5.7, Γ entails any formula in Φ, and so $\Gamma \models \psi$. Then, by the Resolution Property (Proposition 2.5.8) we have $\Gamma \models \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

For the converse, suppose any resolution of Φ entails some resolution of ψ. Consider any model M and state s which supports Φ. By Proposition 2.4.11, $s \models \Phi$ implies $s \models \Gamma$ for some $\Gamma \in \mathcal{R}(\Phi)$. By assumption, $\Gamma \models \alpha$ for some $\alpha \in \mathcal{R}(\psi)$, so we have $s \models \alpha$. Finally, by Proposition 2.4.3 we can conclude $s \models \psi$. Hence, we can conclude $\Phi \models \psi$. □

Notice that, since resolutions are classical formulas, and entailment among classical formulas is classical, this result shows that InqB-entailment is grounded in classical entailment in an interesting way.

Making use of the notion of a dependence function, the Resolution Theorem can be stated as follows: an entailment $\Phi \models \psi$ holds iff there exists a logical dependence function from Φ to ψ.

2.5.13. Corollary.
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^P$, $\Phi \models \psi \iff$ there exists some f such that $f : \Phi \rightsquigarrow \psi$

Thus, when an entailment holds in InqB, this is always witnessed by the existence of a logical dependence function. In the previous chapter, we saw that an entailment involving questions captures a logical dependency. We may regard a dependence function witnessing this entailment as capturing exactly *how* the dependency is realized. In the next chapter, we are going to see that we can always regard a proof of an entailment $\Phi \models \psi$ in inquisitive logic as encoding a logical dependence function $f : \Phi \rightsquigarrow \psi$.

It is worth remarking that both the Resolution Theorem and its formulation in terms of dependence functions hold not just for logical entailment, but also for entailment in context. The proofs are analogous to those of the logical case.

2.5.14. THEOREM (LOCAL RESOLUTION THEOREM).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^P$, for any model M and state s:

$$\Phi \models_s \psi \iff \textit{for every } \Gamma \in \mathcal{R}(\Phi) \textit{ there is } \alpha \in \mathcal{R}(\psi) \textit{ such that } \Gamma \models_s \alpha$$

2.5.15. COROLLARY.
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^P$, $\Phi \models_s \psi \iff$ there exists f such that $f : \Phi \rightsquigarrow_s \psi$

To familiarize with entailments involving questions in InqB, let us examine how the hospital protocol example that we discussed in Section 1.1.1 can be captured as a case of entailment in InqB.

We will assume that our language contains four propositional atoms: s_1 and s_2 stand for the statement that the patient presents the corresponding symptom; g stands for the statement that the patient is in good physical conditions; finally, t stands for the statement that the patient should get the treatment. Then, the protocol of our hospital is encoded by the following classical formula:

$$\gamma \ := \ t \ \leftrightarrow \ s_2 \vee (s_1 \wedge g)$$

The questions involved in the example are captured by the following formulas:

- $?s_1 \wedge ?s_2$: what symptoms the patient has, out of S_1 and S_2;
- $?g$: whether the patient is in good conditions;
- $?t$: whether the treatment is prescribed.

The fact that, in the context of the protocol, the third question is determined by the first two is captured by the validity of the following entailment:

$$\gamma\,, \ \ ?s_1 \wedge ?s_2\,, \ \ ?g \ \ \models \ \ ?t$$

The Resolution Theorem gives us a simple way to see that the entailment is valid. It suffices to check that any resolution of the assumptions entails, in classical logic, a corresponding resolution of the conclusion. Our set of assumptions is:

$$\Phi = \{\gamma, \ ?s_1 \wedge ?s_2, \ ?g\}$$

The resolutions of these assumptions are as follows:

- $\mathcal{R}(\gamma) = \{\gamma\}$
- $\mathcal{R}(?s_1 \wedge ?s_2) = \{s_1 \wedge s_2,\ s_1 \wedge \neg s_2,\ \neg s_1 \wedge s_2,\ \neg s_1 \wedge \neg s_2\}$
- $\mathcal{R}(?g) = \{g, \neg g\}$

Now, a resolution of our set of assumptions Φ consists of a resolution for each assumption. Since $\mathcal{R}(\gamma)$ has one element, $\mathcal{R}(?s_1 \wedge ?s_2)$ has two, and $\mathcal{R}(?g)$ has four, we have in total $1 \times 4 \times 2 = 8$ resolutions for Φ. As the following list shows, each of these sets entails some corresponding resolution of $?t$.

- $\gamma,\ s_1 \wedge s_2,\ g \models t$
- $\gamma,\ \neg s_1 \wedge s_2,\ g \models t$
- $\gamma,\ s_1 \wedge \neg s_2,\ g \models t$
- $\gamma,\ \neg s_1 \wedge \neg s_2,\ g \models \neg t$
- $\gamma,\ s_1 \wedge s_2,\ \neg g \models t$
- $\gamma,\ \neg s_1 \wedge s_2,\ \neg g \models t$
- $\gamma,\ s_1 \wedge \neg s_2,\ \neg g \models \neg t$
- $\gamma,\ \neg s_1 \wedge \neg s_2,\ \neg g \models \neg t$

Notice that what this means is that the following is a logical dependence function from Φ to $?t$, where in the input set, α denotes a resolution of $?s_1 \wedge ?s_2$ and β a resolution of $?g$:

$$f(\{\gamma, \alpha, \beta\}) = \begin{cases} \neg t & \text{if } \alpha = \neg s_1 \wedge \neg s_2 \text{ or } (\alpha = s_1 \wedge \neg s_2 \text{ and } \beta = \neg g) \\ t & \text{otherwise} \end{cases}$$

2.5.4 Classical and intuitionistic features of InqB

One important feature of InqB which it is worthwhile to remark is the validity of the deduction theorem, which is intimately connected to the relation between implication and contextual entailment.

2.5.16. Proposition (Deduction Theorem).
For any set of formulas Φ, and formulas ψ and χ: $\Phi, \psi \models \chi \iff \Phi \models \psi \to \chi$

Now, we have seen above (Proposition 2.3.8) that a formula is truth-conditional iff it is equivalent to its double negation. Since a formula always entails its own double negation, this condition amounts to $\neg\neg\varphi \models \varphi$. By the deduction theorem, we can then characterize truth-conditional formulas as those formulas for which the double negation law $\neg\neg\varphi \to \varphi$ is a logical validity.

2.5.17. Proposition (Double negation and truth-conditionality).
For any $\varphi \in \mathcal{L}^P$: φ is truth-conditional $\iff \models \neg\neg\varphi \to \varphi$

In fact, truth-conditional formulas obey not only the double negation law, but *all* the laws of classical logic.

2.5.18. Proposition (Truth-cond. formulas obey Classical Logic).
Let α be truth-conditional, and let γ be a valid classical formula. Then $\models \gamma[\alpha/p]$.

Proof. Since α is truth-conditional, we have $\alpha \equiv \alpha^{cl}$, and thus $\gamma[\alpha/p] \equiv \gamma[\alpha^{cl}/p]$. Since γ and α^{cl} are classical formulas, and since classical logic is closed under substitution, from the assumption $\models \gamma$ it follows $\models \gamma[\alpha^{cl}/p]$. Finally, since $\gamma[\alpha/p] \equiv \gamma[\alpha^{cl}/p]$ we have $\models \gamma[\alpha/p]$. ☐

Questions, on the other hand, do not in general obey the laws of classical logic; for instance, it follows from Proposition 2.5.17 that $\neg\neg\mu \to \mu$ is always invalid when μ is a question. However, the following fact—which follows from results in Ciardelli and Roelofsen (2011)—shows that all formulas, statements and questions alike, obey the weaker laws of *intuitionistic* logic.

2.5.19. Proposition.
Let $\varphi \in \mathcal{L}^{\mathsf{P}}$, and let $\gamma \in \mathcal{L}^{\mathsf{P}}_c$ be valid in intuitionistic logic. Then, $\models \gamma[\varphi/p]$.

In fact, the statement of this proposition is not as strong as it should be. The reason is that the formula γ is restricted to be in $\mathcal{L}^{\mathsf{P}}_c$, and thus, it does not contain disjunction as a primitive connective. This is not a restriction in classical logic, where disjunction can be defined, but it *is* in intuitionistic logic, where disjunction is not definable from the other connectives. Thus, the previous proposition only guarantees that all inquisitive formulas obey the disjunction-free validities of intuitionistic logic. The following, stronger version of the proposition ensures that the result also holds for intuitionistic validities that do contain disjunction, when this is identified with $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

2.5.20. Proposition (All formulas obey Intuitionistic Logic).
Let $\varphi \in \mathcal{L}^{\mathsf{P}}$, and let $\gamma \in \mathcal{L}^{\mathsf{P}}$ be valid in intuitionistic logic when $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is interpreted as intuitionistic disjunction. Then, $\models \gamma[\varphi/p]$.

Thus, the logic of questions has a constructive flavor. This resonates with what we will see in the next chapter about proofs in InqB: while proofs involving only statements are essentially just proofs in classical logic, proofs involving questions have an interesting constructive interpretation, reminiscent of the proofs-as-programs interpretation of intuitionistic logic: namely, a proof of an entailment $\Phi \models \psi$ may be seen as encoding a logical resolution function $f : \Phi \rightsquigarrow \psi$.

2.5.5 Atoms, substitution, and definability

An interesting and non-standard aspect of our logic is that, while our language contains both statements and questions, *atomic* formulas are always statements. Questions are not part of our logic from the start; rather, they have to be built up syntactically out of statements by means of inquisitive disjunction.

This is not an accidental feature of InqB, but a deliberate architectural choice: we could have allowed atoms to be questions by interpreting them by means of an inquisitive valuation function V which assigns to each atom an inquisitive proposition $V(p) \subseteq \wp(W)$. However, restricting the interpretation of atoms in this way has three important advantages.

First, as we saw, it allows us to regard our system InqB as a conservative extension of classical propositional logic with a new connective, thus preserving a classical fragment in our logic.

Second, it allows us to associate each question with a recursively defined set of statements—its resolutions—which capture the different ways in which the question may be resolved. This plays a crucial role in some of our most interesting results, such as the fact, shown in the next chapter, that from an inquisitive proof of a dependency we can always extract a corresponding dependence function.

Finally, the results of Ciardelli and Roelofsen (2011) show that, if we allowed atoms to be questions, our logic would coincide with the logic of finite problems of Medvedev (1962, 1966).[12] This logic is not finitely axiomatizable, and it is a long standing issue whether it is axiomatizable at all. By assuming atoms to be statements, on the other hand, our logic allows for a simple and finite axiomatization, which provides a natural environment for inferences involving both statements and questions.

Technically, the immediate consequence of taking atoms to be statements is that validity in InqB is not in general preserved under uniform substitution. For, we have seen that there are logical principles that are valid for all statements, but that fail for questions. One example is the double negation law, as shown by Proposition 2.5.17. Thus, the formula $\neg\neg p \to p$ is valid in InqB, while its substitution instance $\neg\neg(p \vee\!\!\!\vee\, q) \to (p \vee\!\!\!\vee\, q)$ is invalid.[13]

On the other hand, since atoms are intended to stand for *arbitrary* statements, we *do* expect validity to be preserved when atoms are substituted by statements. The following proposition ensures that this is indeed the case.

2.5.21. Proposition (Truth-conditional substitution).
For any formula φ and any truth-conditional α, if $\models \varphi$, then $\models \varphi[\alpha/p]$.

[12] The result can also be found in Miglioli et al. (1989), with a slightly different characterization of the logics involved.

[13] Inquisitive logic is not the only logic which is not closed under uniform substitution. Other logics with this property include the modal logic of Carnap (1946), data logic (Veltman, 1981), public announcement logic (Plaza, 1989), and dependence logic (Väänänen, 2007). In each of these cases, the failure of uniform substitution can be traced to the fact that atoms are assumed to stand for a specific kind of sentence: in Carnap's modal logic, atoms stand for contingent sentences; in data logic and dynamic epistemic logic, they stand for sentences which are, in a relevant sense, non-epistemic; in dependence logic, they stand for formulas that are *flat*, i.e., truth-conditional in our sense. For relevant discussion of the lack of closure under substitution, see Holliday et al. (2011, 2012) in the context of public announcement logic, Galliani (2013) and Yang (2015) in the context of dependence logic.

Proof. Since truth-conditional formulas may be characterized as being precisely those formulas which are equivalent to their own double negation, this fact follows from Proposition 3.2.15 and Corollary 3.2.40 of Ciardelli (2009). ☐

A consequence of the fact that validity in InqB is not closed under substitution is that there is a difference between the *definability* and the *uniform definability* of a connective. On the one hand, it is shown in Ciardelli (2009) that, if we take $\neg$ as a primitive connective, the set of connectives $\{\vee\!\!\!\vee, \neg\}$ is expressively complete in a natural sense: this implies that any occurrence of $\wedge$ and $\to$ in a sentence can be paraphrased away by using $\vee\!\!\!\vee$ and $\neg$. Since $\neg$ is definable in terms of $\to$ and $\bot$, the set of connectives $\{\vee\!\!\!\vee, \to, \bot\}$ is also expressively complete.

At the same time, however, it is also shown that each primitive connective is not *uniformly* definable in terms of the others: that is, for $\circ \in \{\wedge, \to, \bot, \vee\!\!\!\vee\}$, there is no $\circ-$free formula $\chi(p, q)$ s.t. for all φ and ψ we have $\varphi \circ \psi \equiv \chi(\varphi, \psi)$. Thus while, for instance, each formula $\varphi \wedge \psi$ may be rewritten as a $\wedge-$free formula, the rewriting is necessarily dependent on the specific formulas φ and ψ.

Conceptually, what this means is that each primitive connective expresses an operation on inquisitive propositions which is not reducible to the operations expressed by the other connectives. As an example, consider conjunction. The support-clause for $\wedge$ can be re-phrased as follows: $[\varphi \wedge \psi]_M = [\varphi]_M \cap [\psi]_M$. Thus, conjunction expresses the operation of intersection on inquisitive propositions. The independence of conjunction from the remaining connectives means that there is no way to define the operation of intersection by composing the operations expressed by the other connectives, $\bot, \to$, and $\vee\!\!\!\vee$. On the other hand, the fact that $\wedge$ is definable from $\bot, \to$, and $\vee\!\!\!\vee$ means that any proposition obtained by means of intersection may also be obtained *without* the use of intersection.

2.6 Discussion

In this section we discuss the relevance of the inquisitive treatment of the connectives for natural language semantics, and we briefly compare the operations available in InqB to those considered in other logics of questions. For an investigation of the deep relations existing between InqB, intuitionistic logic, and various intermediate logics, see Ciardelli (2009) and Ciardelli and Roelofsen (2011). For a discussion of the relations with propositional dependence logic, see Chapter 5.

2.6.1 Relevance for natural language

Conjunction

In natural languages, the word *and* and its counterparts in different languages can be used not only to coordinate two declarative sentences, as in (1-a), but also to coordinate two interrogative sentences, as in (1-b).

(1) a. Alice is going to Paris, and Bob is with her.
b. Where is Alice going, and who is with her?

The truth-conditional account of conjunction allows us to interpret the occurrence of *and* in (1-a), but clearly not the one in (1-b). By contrast, inquisitive semantics generalizes this account so that we can regard (1-a) and (1-b) as being obtained by means of one and the same operation. Indeed, we saw that, when the inquisitive conjunction connective combines two statements like the ones in (1-a), the outcome is a statement having the usual, conjunctive truth-conditions. On the other hand, applying this connective to two questions like the ones in (1-b) we obtain a new question, which is settled precisely when both the original questions are. This is the correct prediction: indeed, in order to settle (1-b), one needs to settle both the question of where Alice is going, and the question of who is with her. In this way, the classical conjunction connective is generalized in a simple way to an operator that is also capable of combining two questions into one, and which allows us to derive the expected semantics for conjunctive questions.[14]

Disjunction

As for the conjunction *and*, so also the disjunction *or* and its cross-linguistic counterparts can be used to coordinate both declarative sentences, as in (2-a), and interrogative sentences, as in (2-b).[15] Furthermore, natural language *or* also has another crucial function: it can be used to *form* questions, as shown in (2-c,d).

(2) a. Either Alice is home, or she is in the garden.
b. Where can I rent a van, or who has a large car that I could borrow?
c. Is Alice home, or is she in the garden?
d. Is Alice home, or not?

The truth-conditional account of disjunction allows us to make sense of the occurrence of *or* in (2-a), but not the occurrences in (2-b-d).

In inquisitive semantics, we do have a disjunction operator, $\mathbin{\vee\!\!\!\vee}$, that allows us to form questions by listing the possible ways in which the question may be settled. This is precisely the role that disjunction plays in (2-c,d): (2-c) can be settled by establishing that Alice is home, or by establishing that she is in the garden; similarly, (2-d) can be settled by establishing that Alice is home, or by

[14] It is worth remarking here that, for this generalization to work, it is crucial that we take the proposition $[\mu]_M$ expressed by a question to consist of all states that settle the question. If we assume an analysis of questions à la Hamblin (1973); Karttunen (1977) in terms of "basic semantic answers", extending conjunction to questions proves problematic. For discussion of this point, see Groenendijk and Stokhof (1984); Ciardelli and Roelofsen (2015c).

[15] The fact that interrogatives can be directly disjoined has been questioned in the literature (Szabolcsi, 1997; Krifka, 2001). However, examples like (2-b) seem to witness that they can. For more serious discussion of this point, see Ciardelli et al. (2015a).

establishing that she is not. Thus, we can derive the meaning of questions (2-c,d) if we analyze the occurrence of *or* in (2-c) and (2-d) as contributing a $\mathbin{\vee\!\!\!\vee}$ operator.

The occurrence of *or* in the coordinated interrogative (2-b) may also be analyzed in terms of inquisitive disjunction. For, (2-b) can be settled either by specifying a place where the speaker can rent a van, thereby settling the first disjunct, or by specifying a person who could lend a large car, thereby settling the second disjunct. This is precisely what is predicted by interpreting *or* as $\mathbin{\vee\!\!\!\vee}$.

Now, it may seem that inquisitive semantics leads us to view natural language *or* as being ambiguous between two different operators: classical disjunction, which is at play in (2-a), and inquisitive disjunction, which is at play in (2-b-d). But this is not a very attractive position. First, the fact that not only in English, but in many other languages as well, disjunctions can fulfill all the roles exemplified above requires an explanation, which an ambiguity account of *or* does not provide.[16] Second, by postulating an ambiguity we would still be left with the task of explaining why one could not interpret the disjunction in (2-a) as $\mathbin{\vee\!\!\!\vee}$, and the ones in (2-b-d) as $\vee$.

Fortunately, inquisitive semantics offers a better alternative to this position. Namely, we can assume that *or* always contributes an inquisitive disjunction, even in (2-a). To account for the fact that (2-a) is not a question, we can then assume that the logical form of declarative sentences always involves a closure operator ! which ensures that the sentence is truth-conditional. In fact, such an operator is definable in InqB, as follows:

$$!\varphi := \neg\neg\varphi$$

We know from Proposition 2.3.7 that $!\varphi$ is always a statement having the same truth-conditions as φ. Thus, we may view ! as a projection operator that turns a formula into a statement, preserving its truth-conditional content (see Ciardelli et al., 2015a). Now, recall that we have $!\varphi \equiv \varphi^{cl}$, where φ^{cl} is the formula obtained by replacing all occurrences of inquisitive disjunction by classical disjunction. This means that, if we analyze natural language *or* as $\mathbin{\vee\!\!\!\vee}$, and if we assume that declarative sentences always involve ! as their main operator, the translation of a declarative sentence in InqB will be equivalent to the one we would have obtained by analyzing *or* as $\vee$. For instance, the logical form of (2-a) would not be $p \vee q$, but the equivalent formula $!(p \mathbin{\vee\!\!\!\vee} q)$.[17]

[16] Although there are languages in which different disjunction words are used for (2-a) and (2-c,d): see Alonso-Ovalle (2006) and Winans (2012).

[17] It is important to remark, however, that while the equivalence $!\varphi \equiv \varphi^{cl}$ holds in the setting of InqB, richer languages may well contain operators O such that the formulas $!O(p \mathbin{\vee\!\!\!\vee} q)$ and $O(p \vee q)$ come apart in truth-conditions. If these operators are related to linguistic constructions, this may help to decide between the two hypotheses we are considering.

In fact, a body of recent work (Simons, 2005; Aloni, 2007; Alonso-Ovalle, 2009; Fine, 2012), while not cast in the framework of inquisitive semantics, can be seen as pointing at modals and conditional antecedents as linguistic environments where an analysis of *or* along the lines of $\mathbin{\vee\!\!\!\vee}$

In conclusion, then, it seems attractive to take the view that the word *or* never directly contributes a classical disjunction; rather, the effect of classical disjunction actually results from a combination of inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ with the closure operator !. In this way, a uniform analysis of the meaning of *or* becomes available, which allows us to account for the different roles of disjunction in (2-a-d) without stipulating an ambiguity. For more on the relevance of inquisitive disjunction to natural language, see Pruitt and Roelofsen (2011); AnderBois (2012); Roelofsen (2015); Ciardelli and Roelofsen (2015a).

Negation

Let us now consider negation. Unlike for conjunction and disjunction, attempts to embed questions under negation result in sentences which are ungrammatical.

(3) a. *It is not the case whether Alice is home.
b. *It is not the case where Alice is.

Thus, while negation can still be associated with the operation $\neg$, leading to the standard predictions for statements, it is not obvious that there are occurrences of negation in natural language that can benefit from the inquisitive account of this connective. However, one may hope that the inquisitive account of negation would provide a semantic explanation for the ungrammaticality of negating questions.

While InqB as such does not provide us with such an explanation, I think that a natural one becomes available as soon as presuppositions are explicitly introduced in our semantic picture. Let me sketch the argument here.

In a presuppositional version of inquisitive semantics (Ciardelli et al., 2012, 2015b), a sentence φ is associated with a pair $[\![\varphi]\!] = \langle P(\varphi), [\varphi]\rangle$, where $P(\varphi)$, the *presupposition* of φ, is a set of worlds, and $[\varphi]$, the *proposition expressed* by φ, is a downward closed set of subsets of $P(\varphi)$.

The most natural treatment of negation in a presuppositional semantics is one where $\neg\varphi$ is assigned the same presupposition as φ, $P(\neg\varphi) = P(\varphi)$.[18] The support clause for $\neg$ remains the same as in InqB, except that now, only states $s \subseteq P(\neg\varphi)$ are allowed to support $\neg\varphi$. Some calculation shows that we obtain:

$$[\![\neg\varphi]\!] = \langle P(\varphi), \wp(P(\varphi) - |\varphi|)\rangle$$

That is, $\neg\varphi$ is always a statement which has the same presupposition as φ, and which is true at a world w in case $w \in P(\varphi) - |\varphi|$, that is, in case the presupposition of φ is satisfied, while φ is false.

is advantageous, and perhaps needed, in order to predict the correct truth-conditions.

The crucial feature here is that in the proposition expressed by $p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q$, the contribution of each disjunct surfaces as a separate alternative, and can thus be manipulated separately by an operator embedding the disjunction; this is not the case for $p \vee q$, where the contributions of the two disjuncts are conflated, and are no longer recoverable from the disjunctive proposition.

[18]See, e.g., Beaver (2001); in fact, the preservation under negation is sometimes taken to be the key test to distinguish presuppositions from plain consequences.

But now suppose μ is a question. We saw in Section 1.3 that in this case, the presupposition of μ is always captured by the set of worlds where μ is true, i.e., where μ is soluble; this means that we should have $P(\mu) = |\mu|$. But then, the negation of μ will receive the following meaning:

$$[\![\neg\mu]\!] = \langle P(\mu), \wp(P(\mu) - |\mu|)\rangle = \langle P(\mu), \{\emptyset\}\rangle$$

Thus, we predict that the negating a question always results in a contradiction, which is only supported by the empty state. Assuming that natural languages have no use for constructs that *never* result in a consistent proposition, this gives us a simple explanation for the ungrammaticality of negating questions.

Conditionals

A huge amount of research in linguistics and philosophy of language is devoted to conditionals. However, one important fact that has not received much attention in this literature is that natural languages contain not just conditional declaratives, like (4-a), but also conditional interrogatives, like (4-b,c).

(4) a. If Alice invites Bob to the party, he will go.
 b. If Alice invites Bob to the party, will he go?
 c. If Alice invites Bob to the party, what will Charlie say?

While accounts of (4-a) abound in the literature, these are typically designed to yield a single classical proposition as the semantic value of the conditional, and thus cannot be used directly to account for conditional questions such as (4-b,c).

Now suppose we analyze the conditional construction in terms of our inquisitive implication operator. Then, a conditional declarative like (4-a) is interpreted in a standard way: it is a statement which is settled in a state s if all worlds in s where Alice invites Bob are worlds where Bob goes to the party. At the same time, the same operation can now be used to interpret conditional questions like (4-b,c). The prediction is that a conditional question of the form *if A then Q* is settled in a state s just in case Q is settled in the state $s \cap |A|_M$ that results from enriching s with the assumption that A. In other words, a conditional question asks for information that settles the consequent conditionally on the antecedent. For instance, we correctly predict that (4-a) can be settled in two ways: either (i) by establishing that Bob will go to the party if invited by Alice or (ii) by establishing that Bob will not go if invited by Alice. For, settling either of these statements is necessary and sufficient to ensure that the question whether Bob will go is settled relative to the set of worlds in s where he is invited by Alice.

In spite of these satisfactory predictions for conditional questions, one may hesitate to interpret the indicative conditionals by means of inquisitive implication. For, in the case of declaratives, this coincides with the material conditional of classical logic, which is notoriously unsatisfactory in embedded contexts. Fortunately, the way in which inquisitive conditional extends the material conditional can be generalized: given any conditional operation $\blacktriangleright$ that takes two propositions and returns a proposition, we may lift this to an inquisitive account which coincides with $\blacktriangleright$ on statements, but that also allows us to make sense of conditional questions. The recipe for this lifting is the following.

2.6.1. Definition. [Inquisitive conditional]
$s \models \varphi > \psi \iff$ for all $a \in \textsc{Alt}_M(\varphi)$ there is an $a' \in \textsc{Alt}_M(\psi)$ s.t. $s \subseteq a \blacktriangleright a'$

It is easy to see that inquisitive implication arises in this way if $\blacktriangleright$ is taken to be the material conditional. If α and β are statements, $>$ coincides with $\blacktriangleright$:

$$s \models \alpha > \beta \iff s \subseteq \alpha \blacktriangleright \beta$$

However, $>$ also allows us to interpret conditional questions. For instance, if we model (4-b) by means of a formula $p > ?q$, then we get the expected prediction that, in order to settle this question, it is necessary and sufficient to establish either of $p > q$ and $p > \neg q$:

$$p > ?q \;\equiv\; (p > q) \mathbin{\vee\!\!\!\vee} (p > \neg q)$$

Besides enabling us to extend an account of conditional statements to a uniform account of conditional statements and questions, there is another important benefit to this inquisitive lifting strategy. For, the following logical equivalence is valid on this account, regardless of how the operator $\blacktriangleright$ behaves.

$$(p \mathbin{\vee\!\!\!\vee} q) > r \equiv (p > r) \wedge (q > r)$$

Thus, if we analyze the conditional as $>$ and disjunction as $\mathbin{\vee\!\!\!\vee}$, as suggested above, we can predict the validity of the inference from (5-a) to (5-b), known as *simplification of disjunctive antecedents*.

(5) a. If you come in the morning or after 3pm, you will find me in my office.
b. If you come after 3pm, you will find me in my office.

This logical principle is intuitively valid (see Nute, 1975; Fine, 1975a; Alonso-Ovalle, 2009), but it is predicted to be invalid by the most standard semantic accounts of conditionals (Stalnaker, 1968; Lewis, 1973; Kratzer, 1986).

At the same time, like these accounts, our conditional operator $>$ need not validate the more general principle known as *strengthening of the antecedent*, which would allow us to infer (6) from (5-b).

(6) If you come after 3pm and after I have left, you will find me in my office.

This is a very interesting result, since the desirable principle of simplification of disjunctive antecedents and the undesirable principle of strengthening of the antecedent are known to be inter-derivable in a standard truth-conditional semantic framework (Ellis et al., 1977), preventing a semantics based on plain truth-conditions from validating the first without validating the second.

By moving to a support-based semantic framework and adopting an inquisitive account of disjunction, these two principles can be disentangled, and we can predict the validity of simplification while at the same time rejecting the validity of strengthening of the antecedent. For more discussion of this issue and for a presentation of the proposal sketched here, we refer to Champollion et al. (2015). Accounts along similar lines—but not extending to conditional questions—have been advanced by van Rooij (2006), Alonso-Ovalle (2009) and Fine (2012).

2.6.2 Connectives in other logics of questions

In this chapter, we have looked at how propositional logic can be extended with questions; an important feature of the system we have achieved is that questions are not just added as a surface layer *on top* of classical propositional logic. Rather, questions are part of the recursive definition of the semantics, and thus they can in turn be manipulated by the logical connectives. In the next chapter we will see that the fundamental logical features of these connectives are preserved by this generalization to questions (see also Roelofsen, 2013, for an algebraic perspective). Throughout the thesis, the fact that questions can be manipulated on a par with statements in inferences will be a key asset of our logics.

This is a feature that sets InqB apart from other logics that include questions. Most of these logics (e.g., Wiśniewski, 1996, 2001; Groenendijk, 1999) simply do not allow questions to be combined. As a matter of fact, the *logic of interrogation* of Groenendijk (1999), discussed in Section 1.6.3, *could* be equipped with a conjunction operation that applies to statements and questions in a uniform way. However, this framework does not allow for an analogous generalization of implication. The reason is that in this system, questions are modeled as partitions, while implication would in general deliver non-partition questions, as witnessed by figures 2.2(d) and 2.2(e). This does not only mean that conditional questions like (4-b) are out of the scope of the system, but also that the relation of dependency cannot be internalized in the language in the way discussed in Section 1.4.

Some other approaches, notably that of Belnap and Steel (1976), do allow questions to be combined by operations such as conjunction and conditionalization to a declarative antecedent. However, these operations are not generalization of the operations that are used to combine statements. Rather, they are new operations, whose effect on questions is obtained by means of ad-hoc stipulations. This leaves the relation between these question-embedding operations and their statement-embedding counterpart unaccounted for. Besides, these operations do not appear to have any sort of proof-theoretic or algebraic characterization that makes them particularly interesting from a logical perspective.

To my knowledge, the only logic coming with a set of connectives that can apply to both statements and questions is that of Nelken and Shan (2006), which we discussed in 1.6.3. In this system, questions are assigned truth-conditions, and can therefore be manipulated directly by means of the classical connectives. This gives the desired results for conjunctive questions, but not for conditional questions. For, using the material conditional to analyze a conditional question like (4-b) would lead to the prediction that (4-b) is settled in case the antecedent is false, or the question in the consequent is settled. This is not the right prediction. On the one hand, the clause requires too little: if the antecedent is false but we don't know it, then (4-b) is *not* settled in our information state, contrary to the prediction. On the other hand, it requires too much: if we know that Bob will go to the party if Alice invites him, then (4-b) is settled in our information state, again contrary to Nelken and Shan's predictions. The main point here is that, in asking a conditional question like (4-b), one is asking to resolve the consequent not *if* the antecedent happens to be true, but rather, *under the assumption* that the antecedent is true. For similar reasons, using Nelken and Shan's material implication does not allow us to express dependencies in the way that is allowed by the InqB conditional.

Overall, it seems that InqB allows us to generalize the connectives of classical logic to questions in a way which is simpler and more satisfactory than in previous logics of questions. In the next chapter, we turn to a systematic investigation of the logical features of these generalized connectives, as well as of our new disjunction operator, and more broadly to the issue of the role of questions in inferences.

Chapter 3

Reasoning with Questions

This chapter is concerned with logical proofs involving questions. The purpose of this chapter is threefold. First, we will introduce a natural deduction system for the propositional logic InqB, and prove the completeness of this system. We will see that all the connectives, including inquisitive disjunction, are governed by simple inference rules. While our system is closely related to the Hilbert-style proof system given in Ciardelli and Roelofsen (2009, 2011), the natural deduction setup makes inquisitive proofs much more transparent, allowing us to understand the role played by questions in these proofs.

Second, we will see that inquisitive proofs have an interesting kind of constructive content. Namely, a proof involving questions as assumptions and conclusion does not just *witness* the existence of a dependency: it actually encodes a function that computes this dependency, i.e., a method for turning any resolution of the assumptions into a corresponding resolution of the conclusion. Thus, we will establish a computational interpretation of our proofs, reminiscent of the proofs-as-programs interpretation of intuitionistic logic.

Finally, an important aim of this chapter is to make a conceptual point about the role of questions in logical reasoning, which complements the discussion in Chapter 1. Contrary to what has traditionally been assumed, questions have a very interesting role to play in inferences: they make it possible to formalize arguments involving generic information of a certain type, such as *where Alice lives*, *what symptoms the patient has*, or *whether the treatment is prescribed.* In other words, questions may be used as placeholders for arbitrary information of the corresponding type. By manipulating such placeholders, we may then provide formal proofs of the existence of certain dependence relations.

The chapter is structured as follows. In Section 3.1 we provide a proof system for InqB, and discuss its features. In Section 3.2 we look at examples of proofs involving questions, and describe a general recipe for extracting a logical dependence function from the proof of a dependency. In Section 3.3 we prove the completeness of the system by means of a strategy which will be adapted in the

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \vee\!\!\!\vee \psi} \qquad \frac{\psi}{\varphi \vee\!\!\!\vee \psi} \qquad \frac{\varphi \vee\!\!\!\vee \psi \quad \begin{array}{c}[\varphi] \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \chi\end{array}}{\chi}$$

Falsum

$$\frac{\bot}{\varphi}$$

$\vee\!\!\!\vee$ split

$$\frac{\alpha \to \psi \vee\!\!\!\vee \chi}{(\alpha \to \psi) \vee\!\!\!\vee (\alpha \to \chi)}$$

$\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Figure 3.1: A sound and complete natural-deduction system for InqB. In these rules, the variables φ, ψ, and χ range over arbitrary formulas, while the variable α is restricted to range over classical formulas.

following chapters to richer logical settings. Finally, in Section 3.4 we abstract away from the specific setting of InqB, and discuss the role of questions in logical proofs. This chapter is based on Ciardelli (2015b); the core ideas have also been presented in a preliminary form in Ciardelli (2014b).

3.1 A natural deduction system for InqB

Let us start out by presenting a natural deduction system for InqB. Such a system is described in Figure 3.1, where the variables φ, ψ, and χ range over all formulas, while α is restricted to *classical* formulas. As customary, we refer to the introduction rule for a connective $\circ$ as ($\circ$i), and to the elimination rule as ($\circ$e). We write $P : \Phi \vdash \psi$ to mean that P is a proof whose set of undischarged assumptions is included in Φ and whose conclusion is ψ, and we write $\Phi \vdash \psi$ to mean that a proof $P : \Phi \vdash \psi$ exists. Finally, we say that two formulas φ and ψ are *provably equivalent*, notation $\varphi \dashv\vdash \psi$, in case $\varphi \vdash \psi$ and $\psi \vdash \varphi$. Let us comment briefly on each of the rules of this system.

Conjunction Conjunction is handled by the standard inference rules. The soundness of these rules corresponds to the following standard feature of conjunction in InqB: a set of assumptions entails a conjunction iff it entails both conjuncts.

3.1.1. Proposition. $\Phi \models \varphi \wedge \psi \iff \Phi \models \varphi$ *and* $\Phi \models \psi$

Notice that these rules are not restricted to classical formulas: conjunctive questions like $?p \wedge ?q$ can be manipulated in inferences just like standard conjunctions.

Implication Implication is also handled by the standard inference rules. The soundness of these rules corresponds essentially to the semantic deduction theorem (Proposition 2.5.16, repeated here), which in turn captures the tight relation existing between implication and entailment.

3.1.2. Proposition. $\Phi \models \varphi \to \psi \iff \Phi, \varphi \models \psi$

Again, these rules are not restricted to classical formulas: implications involving questions (including implications which capture dependencies, like $?p \to ?q$) can also be handled by means of the standard implication rules.

Falsum As usual, $\bot$ has no introduction rule, and can be eliminated to infer any formula. This corresponds to the fact that we have $\bot \models \varphi$ for all formulas φ, which in turn is a consequence of the fact that the inconsistent state $\emptyset$ always supports every formula.

Negation Since $\neg\varphi$ is defined as $\varphi \to \bot$, the usual intuitionistic rules for negation, given in Figure 3.2, follow as particular cases of the rules for implication.

Inquisitive disjunction Inquisitive disjunction is handled by the standard natural deduction rules for disjunction. The soundness of these rules corresponds to the following fact, which follows from the support-clause for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

3.1.3. Proposition. $\Phi, \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \models \chi \iff \Phi, \varphi \models \chi$ *and* $\Phi, \psi \models \chi$.

Classical disjunction Figure 3.2 shows the derived rules for $\vee$. While the introduction rule is the standard one, the elimination rule is restricted to conclusions that are classical formulas. Without this restriction, the rule would not be sound. E.g., we have $p \models ?p$ and $\neg p \models ?p$, but $p \vee \neg p \not\models ?p$: the question $?p$ is logically resolved by both statements p and $\neg p$, but not by the tautology $p \vee \neg p$.

Negation

$$\dfrac{\begin{matrix}[\varphi]\\ \vdots\\ \bot\end{matrix}}{\neg\varphi} \qquad \dfrac{\varphi \quad \neg\varphi}{\bot}$$

Classical disjunction

$$\dfrac{\varphi}{\varphi\vee\psi} \qquad \dfrac{\psi}{\varphi\vee\psi} \qquad \dfrac{\varphi\vee\psi \quad \begin{matrix}[\varphi]\\ \vdots\\ \alpha\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots\\ \alpha\end{matrix}}{\alpha}$$

Figure 3.2: Derived rules for $\vee$ and $\neg$, where α is restricted to classical formulas.

Double negation elimination From the double negation $\neg\neg\alpha$ of a classical formula, our system allows us to infer α. We saw in the previous chapter (Proposition 2.5.17) that the double negation law characterizes truth-conditionality. Thus, by adding this law for all classical formulas, we are capturing the truth-conditionality of the classical fragment of our logic.

In fact, in order to obtain a complete proof system it would be sufficient to allow double negation for *atoms*, as in Ciardelli and Roelofsen (2011). However, allowing it for all classical formulas is handy in practice and, moreover, it makes our system an extension of a standard natural deduction system for classical logic (as given, e.g., in Gamut, 1991). This ensures that any standard natural deduction proof in classical logic is also a proof in our system.

Split The only non-standard ingredient of our proof system is the $\mathbin{\vee\mkern-12mu\vee}$ split rule, which allows us to distribute a classical antecedent over an inquisitive-disjunctive consequent. The soundness of this rule is guaranteed by Corollary 2.5.11, which states that the corresponding entailment is valid whenever the antecedent is truth-conditional. In Section 2.5.2, we saw that this entailment captures the following property, that we referred to as *Local Split*, where α is truth-conditional:

$$\alpha \models_s \varphi \mathbin{\vee\mkern-12mu\vee} \psi \text{ implies } \alpha \models_s \varphi \text{ or } \alpha \models_s \psi$$

Intuitively, this says that if a statement resolves a question in a given context, it must do so by entailing a specific resolution of it. Thus, we may regard the split rule as capturing this fundamental feature of our logic.[1]

[1] The split rule is the counterpart in our system of the Kreisel-Putnam axiom adopted in Ciardelli and Roelofsen (2011):

$$(\neg\varphi \to \psi \mathbin{\vee\mkern-12mu\vee} \chi) \to (\neg\varphi \to \psi) \mathbin{\vee\mkern-12mu\vee} (\neg\varphi \to \chi)$$

Like classical formulas, negations are representative of all and only the truth-conditional formulas in InqB (see Proposition 2.3.6 and Proposition 2.3.7). Thus, taking negative antecedents

3.2 Inquisitive proofs and dependence functions

Now that we have discussed the individual inference rules, let us turn to examine the significance of inquisitive proofs as a whole. To see what proofs involving questions look like, let us consider once again our initial example of a dependency, corresponding to the following entailment:[2]

$$\gamma,\ \ ?s_1,\ \ ?s_2,\ \ ?g\ \ \models\ \ ?t$$

where γ stands for the protocol description, $t \leftrightarrow s_2 \vee (s_1 \wedge g)$. The following is a proof of this entailment in our system, where steps involving only inferences in classical logic have been omitted and denoted by $(\mathsf{C}_1), \ldots, (\mathsf{C}_4)$.

$$\dfrac{?s_2 \qquad \dfrac{\dfrac{\gamma \quad [s_2]}{t}\,(\mathsf{C}_1)}{?t}\,(\mathbb{V}\mathsf{i}) \qquad \dfrac{?s_1 \qquad \dfrac{\dfrac{\gamma \quad [\neg s_1] \quad [\neg s_2]}{\neg t}\,(\mathsf{C}_2)}{?t}\,(\mathbb{V}\mathsf{i}) \qquad \dfrac{?g \qquad \dfrac{\dfrac{\gamma \quad [s_1] \quad [g]}{t}\,(\mathsf{C}_3)}{?t}\,(\mathbb{V}\mathsf{i}) \qquad \dfrac{\dfrac{\gamma \quad [\neg s_2] \quad [\neg g]}{\neg t}\,(\mathsf{C}_4)}{?t}\,(\mathbb{V}\mathsf{i})}{?t}\,(\mathbb{V}\mathsf{e})}{?t}\,(\mathbb{V}\mathsf{e})}{?t}\,(\mathbb{V}\mathsf{e})$$

It is interesting to consider what intuitive argument is encoded by this proof. In words, this may be phrased roughly as follows. We are assuming the information corresponding to the protocol γ, as well as information of types $?s_1, ?s_2$, and $?g$. Now, to have information of type $?s_2$ is to have either the information that s_2, or the information that $\neg s_2$. If we have the information that s_2, then by combining this information with the protocol γ we can infer t, and so we have some information of type $?t$. On the other hand, if we have the information that $\neg s_2$, we have to rely on the information of type $?s_1$. Again, to have information of type $?s_1$ is to have either the information that s_1, or the information that $\neg s_1$. If the information we have is $\neg s_1$, then by combining this with $\neg s_2$ and γ we can infer $\neg t$, and thus we have some information of type t. If the information we have is that s_1, then we have to rely on the information of type $?g$. Again, there are two possibilities: if the information we have is g, then by combining this with s_1 and γ we can infer t, and so we have information of type $?t$; if the information we have is $\neg g$, then by combining this with $\neg s_2$ and γ we can infer $\neg t$, and thus again we have information of type $?t$. So, in any case, under the given assumptions we are assured to have information of type $?t$.[3]

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or taking classical antecedents are just two ways of restricting to antecedents whose truth-conditionality is visible from syntactic form. Thus, the Kreisel-Putnam rule, too, which looked like a fundamental but mysterious ingredient in Ciardelli and Roelofsen (2011), takes on an intuitive significance from the present perspective.

[2] In order to reduce clutter in the proofs, the conjunctive question $?s_1 \wedge ?s_2$ is replaced here by two distinct polar questions, $?s_1$ and $?s_2$. This change is merely cosmetic, and dispensable.

[3] The proof of a dependency does not always have to proceed, as in this case, by splitting

The attentive reader may have noticed an interesting feature of this proof: the proof is constructive, in the sense that it does not just *witness* that, given γ, information of type $?s_1, ?s_2$, and $?g$ yields information of type $?t$; rather, it actually describes how to go about obtaining a piece of information of type $?t$ from any given information of types $?s_1, ?s_2$, and $?g$. This means that, if we replace each of the question assumptions $?s_1, ?s_2, ?g$ in the proof by a specific resolution of it, say $s_1, \neg s_2$, and g respectively, we can read off from the proof a resolution of $?t$ determined by these assumptions.

In other words, our proof does not just witness a dependency $\Phi \models ?t$, where Φ is our set of assumptions: it actually describes a logical dependence function $f : \Phi \rightsquigarrow ?t$, which can be used to turn any data about a given patient into the deliberation about the treatment which logically follows from them.

This is not just a feature of this particular proof, but a manifestation of a general fact: any inquisitive proof encodes a dependence function.

To see how this works, let us write $\overline{\varphi}$ for a sequence $\varphi_1, \dots, \varphi_n$ of formulas, and $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ to mean that $\overline{\alpha}$ is a sequence $\alpha_1, \dots, \alpha_n$ such that $\alpha_i \in \mathcal{R}(\varphi_i)$.

3.2.1. Theorem (Existence of a resolution algorithm).
If $P : \overline{\varphi} \vdash \psi$, we can define inductively on P a procedure F_P which maps each $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ to a proof $F_P(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ having as conclusion a resolution $\beta \in \mathcal{R}(\psi)$.

Proof. Let us describe how to construct the procedure F_P inductively on P. We distinguish a number of cases depending on the last rule applied in P.

- ψ is an undischarged assumption φ_i. In this case, any resolution $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ contains a resolution α_i of φ_i by definition. So, we can just let F_P map $\overline{\alpha}$ to the trivial proof $Q : \overline{\alpha} \vdash \alpha_i$ which consists only of the assumption α_i.

- $\psi = \chi \wedge \xi$ was obtained by ($\wedge$i) from χ and ξ. Then the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash \chi$ and a proof $P'' : \overline{\varphi} \vdash \xi$, for which the induction hypothesis gives two procedures $F_{P'}, F_{P''}$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. We have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ and $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \gamma$, where $\beta \in \mathcal{R}(\chi)$ and $\gamma \in \mathcal{R}(\xi)$. By extending these proofs with an application of ($\wedge$i), we get a proof $Q : \overline{\alpha} \vdash \beta \wedge \gamma$. Since $\beta \wedge \gamma \in \mathcal{R}(\chi \wedge \xi)$, we can let $F_P(\overline{\alpha}) := Q$.

- $\psi = \chi \to \xi$ was obtained by ($\to$i). Then the immediate subproof of P is a proof $P' : \overline{\varphi}, \chi \vdash \xi$, for which the induction hypothesis gives a procedure $F_{P'}$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. Suppose $\beta_1, \dots, \beta_m$ are the resolutions of χ. For $1 \leq i \leq m$, the sequence $\overline{\alpha}, \beta_i$ is a resolution of $\overline{\varphi}, \chi$, and so we have $F_{P'}(\overline{\alpha}, \beta_i) : \overline{\alpha}, \beta_i \vdash \gamma_i$ for some resolution γ_i of ξ. Extending this proof with an application of ($\to$i), have a proof $Q_i : \overline{\alpha} \vdash \beta_i \to \gamma_i$. Since

all possible cases. E.g., from μ and $\mu \to \nu$, we can immediately infer ν by modus ponens. We do not need to look at the resolutions of μ and $\mu \to \nu$. This is convenient, since the number of resolutions for $\mu \to \nu$ is exponential in the number of resolutions for μ, and typically large.

this is the case for $1 \leq i \leq n$, by several applications of the rule ($\wedge$i) we obtain a proof $Q : \overline{\alpha} \vdash (\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_m \to \gamma_m)$. By construction, $(\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_m \to \gamma_m)$ is a resolution of $\chi \to \xi$, and so we can put $F_P(\overline{\alpha}) := Q$.

- $\psi = \chi \vee\!\!\!\vee \xi$ was obtained by ($\vee\!\!\!\vee$i) from one of the disjuncts. Without loss of generality, let us assume it is χ. Thus, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \chi$, for which the induction hypothesis gives a procedure $F_{P'}$. Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. The induction hypothesis gives us a proof $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\chi)$. Since β is also a resolution of $\chi \vee\!\!\!\vee \xi$, we can simply let $F_P(\overline{\alpha}) := F_{P'}(\overline{\alpha})$.

- ψ was obtained by ($\wedge$e) from $\psi \wedge \chi$. Then the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \psi \wedge \chi$, and the induction hypothesis gives a procedure $F_{P'}$. For any resolution $\overline{\alpha}$ of $\overline{\varphi}$, we have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$, where $\beta \in \mathcal{R}(\psi \wedge \chi)$. By definition of resolutions for a conjunction, β is of the form $\gamma \wedge \gamma'$ where $\gamma \in \mathcal{R}(\psi)$ and $\gamma' \in \mathcal{R}(\chi)$. Extending $F_{P'}(\overline{\alpha})$ with an application of ($\wedge$e) we have a proof $Q : \overline{\alpha} \vdash \gamma$. Since $\gamma \in \mathcal{R}(\psi)$, we can just let $F_P(\overline{\alpha}) := Q$.

- ψ was obtained by ($\to$e) from χ and $\chi \to \psi$. Then the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash \chi$, and a proof $P'' : \overline{\varphi} \vdash \chi \to \psi$, for which the induction hypothesis gives procedures $F_{P'}$ and $F_{P''}$. Now consider a resolution $\overline{\alpha}$ of $\overline{\varphi}$. We have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ where $\beta \in \mathcal{R}(\chi)$, and a proof $F_{P''}(\overline{\alpha}) : \overline{\alpha} \vdash \gamma$, where $\gamma \in \mathcal{R}(\chi \to \psi)$. Now, if $\mathcal{R}(\chi) = \{\beta_1, \ldots, \beta_m\}$, then $\beta = \beta_i$ for some i, and by definition of the resolutions of an implication, $\gamma = (\beta_1 \to \gamma_1) \wedge \cdots \wedge (\beta_n \to \gamma_m)$ where $\{\gamma_1, \ldots, \gamma_n\} \subseteq \mathcal{R}(\psi)$. Now, extending $F_{P''}(\overline{\alpha})$ with an application of ($\wedge$e) we obtain a proof $Q'' : \overline{\alpha} \vdash \beta_i \to \gamma_i$. Finally, putting together this proof with $F_{P'}(\overline{\alpha})$ and applying ($\to$e), we obtain a proof $Q : \overline{\alpha} \vdash \gamma_i$. Since the conclusion of this proof is a resolution of ψ, we can let $F_P(\overline{\alpha}) := Q$.

- ψ was obtained by ($\vee\!\!\!\vee$e) from $\chi \vee\!\!\!\vee \xi$. Then the immediate subproofs of P are: a proof $P' : \overline{\varphi} \vdash \chi \vee\!\!\!\vee \xi$; a proof $P'' : \overline{\varphi}, \chi \vdash \psi$; and a proof $P''' : \overline{\varphi}, \xi \vdash \psi$, for which the induction hypothesis gives procedures $F_{P'}$, $F_{P''}$, and $F_{P'''}$. Now take a resolution $\overline{\alpha}$ of $\overline{\varphi}$. We have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\chi \vee\!\!\!\vee \xi) = \mathcal{R}(\chi) \cup \mathcal{R}(\xi)$. Without loss of generality, assume that $\beta \in \mathcal{R}(\chi)$. Then the sequence $\overline{\alpha}, \beta$ is a resolution of $\overline{\varphi}, \chi$. Thus, we have $F_{P''}(\overline{\alpha}, \beta) : \overline{\alpha}, \beta \vdash \gamma$ for some $\gamma \in \mathcal{R}(\psi)$. Now, by substituting any undischarged assumption of β in the proof $F_{P''}(\overline{\alpha}, \beta)$ by an occurrence of the proof $F_{P'}(\overline{\alpha})$, we obtain a proof $Q : \overline{\alpha} \vdash \gamma$ having a resolution of ψ as its conclusion, and we can let $F_P(\overline{\alpha}) := Q$.

- ψ was obtained by ($\bot$e). This means that the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \bot$, for which the induction hypothesis gives a method $F_{P'}$.

Now take any resolution $\overline{\alpha}$ of $\overline{\varphi}$. Since $\mathcal{R}(\bot) = \{\bot\}$, we have $F_{P'}(\overline{\alpha}) : \overline{\alpha} \vdash \bot$. Now take any $\beta \in \mathcal{R}(\psi)$ (notice that, by definition, the set of resolutions of a formula is always non-empty): by extending the proof $F_{P'}(\overline{\alpha})$ with an application of ($\bot$e), we obtain a proof $Q : \overline{\alpha} \vdash \beta$. Since $\beta \in \mathcal{R}(\psi)$, we can let $F_P(\overline{\alpha}) := P$.

- $\psi = (\alpha \to \chi) \mathbin{\vee\!\!\!\vee} (\alpha \to \xi)$ was obtained by an application of the $\mathbin{\vee\!\!\!\vee}$-split rule from $\alpha \to \chi \mathbin{\vee\!\!\!\vee} \xi$, where $\alpha \in \mathcal{L}_c^{\mathsf{P}}$. Then, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \alpha \to \chi \mathbin{\vee\!\!\!\vee} \xi$, for which the induction hypothesis gives a method $F_{P'}$. Now, we can simply let $F_P := F_{P'}$: for, using the fact that $\mathcal{R}(\alpha) = \{\alpha\}$, since α is a classical formula, it is easy to verify that we have $\mathcal{R}(\alpha \to \chi \mathbin{\vee\!\!\!\vee} \xi) = \mathcal{R}((\alpha \to \chi) \mathbin{\vee\!\!\!\vee} (\alpha \to \xi))$.

- $\alpha \in \mathcal{L}_c^{\mathsf{P}}$ was obtained by double negation elimination from $\neg\neg\alpha$. In this case, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \neg\neg\alpha$, for which the induction hypothesis gives a method $F_{P'}$. Since $\neg\neg\alpha$ is a classical formula, we have $\mathcal{R}(\neg\neg\alpha) = \{\neg\neg\alpha\}$. Thus, for any resolution $\overline{\beta}$ of $\overline{\varphi}$ we have $F_{P'}(\overline{\beta}) : \overline{\beta} \vdash \neg\neg\alpha$. Extending this proof with an application of double negation elimination we obtain a proof $Q : \overline{\beta} \vdash \alpha$. Since α is a classical formula and thus a resolution of itself, we can then let $F_P(\overline{\beta}) := Q$. □

Thus, a proof in our system may be seen as a *template* for classical proofs, where questions serve as placeholders for generic information of the corresponding type. As soon as the assumptions of the proof are instantiated to particular resolutions—say, as soon as we input the data relative to a specific patient—this template can be instantiated to a classical proof which infers some specific resolution of the conclusion—in our example, a deliberation about the treatment.

We will refer to the proof $F_P(\overline{\alpha})$ as the *resolution* of the proof P on the input $\overline{\alpha}$. Now, let us denote by f_P the function that maps a resolution $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ to the conclusion of the proof $F_P(\overline{\alpha})$. By definition, we have $f : \mathcal{R}(\overline{\varphi}) \to \mathcal{R}(\psi)$. Moreover, for any $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$, we have $F_P(\overline{\alpha}) : \overline{\alpha} \vdash f_P(\overline{\alpha})$, and thus, by the soundness of our proof system, we have $\overline{\alpha} \models f_P(\overline{\alpha})$. This shows that the function f_P determined by the proof P is a logical dependence function from $\overline{\varphi}$ to ψ.

3.2.2. Corollary (Inquisitive proofs encode dependence functions). *If $P : \Phi \vdash \psi$, then inductively on P we can define a logical dependence function $f_P : \Phi \rightsquigarrow \psi$.*

This connection is reminiscent of the proofs-as-programs correspondence known for intuitionistic logic. As discovered by Curry (1934) and Howard (1980), in intuitionistic logic formulas may be regarded as *types* of a certain type-theory, extending the simply typed lambda-calculus. A proof $P : \varphi \vdash \psi$ in intuitionistic logic may be identified with a term t_P of this type theory which describes a function from objects of type φ to objects of type ψ.

The situation is similar for InqB, except that now, formulas play double duty. On the one hand, formulas may be still be regarded as types. On the other hand, the elements of a type φ may in turn be identified with certain formulas, namely, the resolutions of φ. As in intuitionistic logic, a proof $P : \varphi \vdash \psi$ determines a function f_P from objects of type φ to objects of type ψ; but since these objects may now be identified with classical formulas, the function f_P is now defined within the language of classical propositional logic: $f_P : \mathcal{L}_c^{\mathsf{P}} \to \mathcal{L}_c^{\mathsf{P}}$.

3.3 Completeness proof

By showing that each inference rule of our proof system is sound, the discussion in Section 3.1 implies the soundness of our proof system as a whole.

3.3.1. Proposition (Soundness). *If $\Phi \vdash \psi$, then $\Phi \models \psi$.*

In this section, we will be concerned with establishing that our proof system is also complete, i.e., with proving the following theorem.

3.3.2. Theorem (Completeness). *If $\Phi \models \psi$, then $\Phi \vdash \psi$.*

In previous work (Ciardelli and Roelofsen, 2009, 2011) two different proofs of the completeness of a Hilbert-style relative of our proof system have been given. While these proofs could be adapted straightforwardly to our natural deduction system, I will present yet another completeness proof which explicitly constructs a canonical information model. The main motivation for providing a new proof is that the strategy that we will follow, unlike those used in previous work, can be extended in a natural way to provide completeness proofs for richer logical languages. This will be important for completeness results in the following chapters.

The strategy that we will pursue can be summarized as follows. First, we will define a canonical model having complete theories of classical formulas as its possible worlds. Second, we will prove an analogue of the truth-lemma, the *support lemma*, which connects provability to support in the canonical model: a formula is supported by a state in the canonical model if and only if it is derivable from the intersection of the theories in the state. Finally, we will show that when a formula ψ cannot be derived from a set Φ, we can define a corresponding information state in the canonical model that supports Φ but not ψ.

3.3.1 Preliminary results

Let us start out by establishing a few important facts about our proof system. First, notice that, if we leave out the rules of $\neg\neg$-elimination and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split, what we have is a complete system for intuitionistic propositional logic, with $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ in the role of intuitionistic disjunction. Thus, we have the following fact.

3.3.3. Lemma (Intuitionistic entailments are provable).
If Φ entails ψ in intuitionistic propositional logic, when $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is identified with intuitionistic disjunction, then $\Phi \vdash \psi$.

Second, our proof system allows us to prove the equivalence between a formula and its normal form.

3.3.4. Lemma (Provability of normal form). *For any φ, $\varphi \dashv\vdash \bigvee\mathcal{R}(\varphi)$*

Proof. The proof is by induction on φ. The basic cases for atoms and $\bot$ are trivial, and so is the inductive case for $\vee$. So, only the inductive cases for conjunction and implication remain to be proved. Consider two formulas φ and ψ, with $\mathcal{R}(\varphi) = \{\alpha_1, \ldots, \alpha_n\}$ and $\mathcal{R}(\psi) = \{\beta_1, \ldots, \beta_m\}$. Let us make the induction hypothesis that $\varphi \dashv\vdash \alpha_1 \vee \ldots \vee \alpha_n$ and $\psi \dashv\vdash \beta_1 \vee \ldots \vee \beta_m$, and let us consider the conjunction $\varphi \wedge \psi$ and the implication $\varphi \to \psi$.

- *Conjunction.* Using the induction hypothesis and the rules for $\wedge$, we obtain:

  $$\varphi \wedge \psi \dashv\vdash (\alpha_1 \vee \ldots \vee \alpha_n) \wedge (\beta_1 \vee \ldots \vee \beta_m)$$

  Since the distributivity of conjunction over disjunction is provable in intuitionistic logic, by Lemma 3.3.3 we have:

  $$(\alpha_1 \vee \ldots \vee \alpha_n) \wedge (\beta_1 \vee \ldots \vee \beta_m) \dashv\vdash \bigvee\{\alpha_i \wedge \beta_j \mid i \leq n, j \leq m\}$$

  And we are done, since by definition $\mathcal{R}(\varphi \wedge \psi) = \{\alpha_i \wedge \beta_j \mid i \leq n, j \leq m\}$.

- *Implication.* Using the induction hypothesis and the rules for $\to$, we obtain:

  $$\varphi \to \psi \dashv\vdash (\alpha_1 \vee \ldots \vee \alpha_n) \to (\beta_1 \vee \ldots \vee \beta_m)$$

  By intuitionistic reasoning, we obtain the following:

  $$(\alpha_1 \vee \ldots \vee \alpha_n) \to (\beta_1 \vee \ldots \vee \beta_m) \dashv\vdash \bigwedge_{i \leq n}(\alpha_i \to \beta_1 \vee \ldots \vee \beta_m)$$

  Now, since any resolution is a classical formula, by the $\vee$-split rule we have:

  $$\alpha \to \beta_1 \vee \ldots \vee \beta_m \dashv\vdash (\alpha_i \to \beta_1) \vee \ldots \vee (\alpha_i \to \beta_m)$$

  Since this is the case for for $1 \leq i \leq n$, the rules for $\wedge$ yield:

  $$\bigwedge_{i \leq n}(\alpha_i \to \beta_1 \vee \ldots \vee \beta_m) \dashv\vdash \bigwedge_{i \leq n}((\alpha_i \to \beta_1) \vee \ldots \vee (\alpha_i \to \beta_m))$$

  Finally, using again the provable distributivity of $\wedge$ over $\vee$, we get:

  $$\bigwedge_{i \leq n} \bigvee_{j \leq m} (\alpha_i \to \beta_j) \dashv\vdash \bigvee\{\bigwedge_{i \leq n}(\alpha_i \to f(\alpha_i)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$$

  By definition of resolutions for an implication, the formula on the right is precisely $\bigvee\mathcal{R}(\varphi \to \psi)$. This completes the inductive proof. $\Box$

As a corollary, a formula may always be derived from each of its resolutions.

3.3.5. Corollary. *For every $\varphi \in \mathcal{L}^{\mathsf{P}}$, if $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \vdash \varphi$.*

Proof. Let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. By means of the rule ($\mathbb{V}$i), from α_i we can infer $\alpha_1 \mathbb{V} \dots \mathbb{V} \alpha_n$, and thus, by the previous lemma, we can infer φ. □

Next, notice that Proposition 2.4.3 ensures that, if $\Phi, \psi \not\models \chi$, then there is a specific resolution $\alpha \in \mathcal{R}(\psi)$ such that $\Phi, \alpha \not\models \chi$. Intuitively, this means that, if an entailment from ψ does not hold, the failure can always be traced to some specific piece of information of type ψ.

For instance, consider our hospital protocol example. By itself, the patient's symptoms do not determine whether the treatment is prescribed. Thus, we have γ, $?s_1 \wedge ?s_2 \not\models ?t$. In this case, the failure of the entailment can be traced to the resolution $s_1 \wedge \neg s_2$, since the fact that the patient presents only symptom S_1 does not determine whether the treatment is prescribed. That is, we have $\gamma, s_1 \wedge \neg s_2 \not\models ?t$. The following lemma shows that this feature of entailment is shared by the provability relation.

3.3.6. Lemma. *If $\Phi, \psi \nvdash \chi$, then $\Phi, \alpha \nvdash \chi$ for some $\alpha \in \mathcal{R}(\psi)$.*

Proof. We will prove the contrapositive: if $\Phi, \alpha \vdash \chi$ for all $\alpha \in \mathcal{R}(\psi)$, then $\Phi, \psi \vdash \chi$. Let $\mathcal{R}(\psi) = \{\alpha_1, \dots, \alpha_n\}$. The rule ($\mathbb{V}$e) ensures that, if we have $\Phi, \alpha_i \vdash \chi$ for $1 \leq i \leq n$, we also have $\Phi, \alpha_1 \mathbb{V} \dots \mathbb{V} \alpha_n \vdash \chi$. Since the previous lemma gives $\psi \vdash \alpha_1 \mathbb{V} \dots \mathbb{V} \alpha_n$, we also get $\Phi, \psi \vdash \chi$. □

The next lemma extends this result from a single assumption to the whole set.

3.3.7. Lemma (Traceable deduction failure).
If $\Phi \nvdash \psi$, there is some resolution $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$.

Proof. Let us fix an enumeration of Φ, say $(\varphi_n)_{n \in \mathbb{N}}$. We are going to define a sequence $(\alpha_n)_{n \in \mathbb{N}}$ of classical formulas in $\mathcal{L}^{\mathsf{P}}$ such that, for all $n \in \mathbb{N}$:

- $\alpha_n \in \mathcal{R}(\varphi_n)$;
- $\{\alpha_i \mid i \leq n\} \cup \{\varphi_i \mid i > n\} \nvdash \psi$.

Let us apply inductively the previous lemma. Assume we have defined α_i for $i < n$ and let us proceed to define α_n. The induction hypothesis tells us that $\{\alpha_i \mid i < n\} \cup \{\varphi_i \mid i \geq n\} \nvdash \psi$, that is, $\{\alpha_i \mid i < n\} \cup \{\varphi_i \mid i > n\}, \varphi_n \nvdash \psi$. Now the previous lemma tells us that we can "specify" the formula φ_n to a resolution, i.e., we can find a formula $\alpha_n \in \mathcal{R}(\varphi_n)$ and $\{\alpha_i \mid i < n\} \cup \{\varphi_i \mid i > n\}, \alpha_n \nvdash \psi$. But this means that $\{\alpha_i \mid i \leq n\} \cup \{\varphi_i \mid i > n\} \nvdash \psi$, completing the inductive proof.

Now let $\Gamma := \{\alpha_n \mid n \in \mathbb{N}\}$. By construction, $\Gamma \in \mathcal{R}(\Phi)$. Moreover, we claim that $\Gamma \nvdash \psi$. To see this, suppose towards a contradiction $\Gamma \vdash \psi$: then for some

n it should be the case that $\alpha_1, \ldots, \alpha_n \vdash \psi$; but this is impossible, since by construction we have $\{\alpha_1, \ldots, \alpha_n\} \cup \{\varphi_i \mid i > n\} \nvdash \psi$. Thus, we have $\Gamma \nvdash \psi$, which shows that some resolution of the assumptions does not derive ψ. □

Using the existence of the Resolution Algorithm (Theorem 3.2.1) on the one hand, and the Traceable Failure Lemma on the other, we obtain an analogue of the Resolution Theorem (Theorem 2.5.12) for provability: a set of assumptions Φ derives a formula ψ iff any resolution of Φ derives some resolution of ψ.

3.3.8. Lemma (Resolution Lemma).
$\Phi \vdash \psi \iff$ *for all* $\Gamma \in \mathcal{R}(\Phi)$ *there is* $\alpha \in \mathcal{R}(\psi)$ *such that* $\Gamma \vdash \alpha$.

Proof. The left-to-right direction of the lemma follows immediately from Theorem 3.2.1. For, suppose there is a proof $P : \Phi \vdash \psi$ and take $\Gamma \in \mathcal{R}(\Phi)$: the theorem describes how to use P and Γ to construct a proof $Q : \Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

For the converse, suppose $\Phi \nvdash \psi$: the previous lemma tells us that there is a resolution $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$. Since from any resolution $\alpha \in \mathcal{R}(\psi)$ we can derive ψ (Corollary 3.3.5), we must also have $\Gamma \nvdash \alpha$ for every $\alpha \in \mathcal{R}(\psi)$. Thus, it is not the case that any resolution of Φ derives some resolution of ψ. □

The next lemma shows that the Split Property is shared by provability, at least for the case in which the assumptions are classical formulas.

3.3.9. Lemma (Provable Split).
If Γ *is a set of classical formulas and* $\Gamma \vdash \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$*, then* $\Gamma \vdash \varphi$ *or* $\Gamma \vdash \psi$.

Proof. Suppose $\Gamma \vdash \varphi \vee\!\!\!\vee \psi$. Since Γ is a set of classical formulas, Proposition 2.4.10 ensures that $\mathcal{R}(\Gamma) = \Gamma$. So, by Lemma 3.3.8 we have $\Gamma \vdash \beta$ for some $\beta \in \mathcal{R}(\varphi \vee\!\!\!\vee \psi)$. Since $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$ we have either $\beta \in \mathcal{R}(\varphi)$ or $\beta \in \mathcal{R}(\psi)$. In the former case, by Corollary 3.3.5 we have $\beta \vdash \varphi$, and thus also $\Gamma \vdash \varphi$. In the latter case, we have $\beta \vdash \psi$ and thus also $\Gamma \vdash \psi$. □

3.3.2 Canonical model

Let us now turn to the definition of our canonical model for InqB. As usual in classical and modal logic, we will construct our possible worlds out of complete theories. While we could take complete theories in the whole language $\mathcal{L}^{\mathsf{P}}$, the proof is more straightforward if we just consider complete theories in the *classical* language $\mathcal{L}^{\mathsf{P}}_c$. To be fully precise, we make the following definitions.

3.3.10. Definition. [Theories of classical formulas]
A *theory of classical formulas* is a set $\Gamma \subseteq \mathcal{L}^{\mathsf{P}}_c$ which is closed under deduction of classical formulas, that is, if $\alpha \in \mathcal{L}^{\mathsf{P}}_c$ and $\Gamma \vdash \alpha$, then $\alpha \in \Gamma$.

3.3.11. Definition. [Complete theories of classical formulas]
A *complete* theory of classical formulas is a theory of classical formulas Γ s.t.:

- $\bot \notin \Gamma$;
- for any $\alpha \in \mathcal{L}_c^{\mathsf{P}}$, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$.

The following lemma is familiar from completeness proofs in classical logic, and can be proved by means of the usual inductive procedure.

3.3.12. Lemma. *Any consistent set of classical formulas can be extended to a complete theory of classical formulas.*

If S is a set of theories of classical formulas, we will denote by $\bigcap S$ the intersection of all the theories $\Gamma \in S$, with the convention that the intersection of the empty set of theories is the set of *all* classical formulas: $\bigcap \emptyset = \mathcal{L}_c^{\mathsf{P}}$. A simple fact that will be useful in our proof is that $\bigcap S$ is itself a theory of classical formulas.

3.3.13. Lemma.
If S is a set of theories of classical formulas, $\bigcap S$ is a theory of classical formulas.

Proof. It is obvious that $\bigcap S$ is a set of classical formulas. Moreover, suppose $\bigcap S \vdash \alpha$ and α is a classical formula. Take any $\Theta \in S$: since $\bigcap S \subseteq \Theta$, we also have $\Theta \vdash \alpha$, and since Θ is a theory of classical formulas, we have $\alpha \in \Theta$. Finally, since this is the case for all $\Theta \in S$, we have $\alpha \in \bigcap S$. This shows that $\bigcap S$ is closed under deduction of classical formulas. □

Our canonical model will have complete theories of classical formulas as worlds, while the canonical valuation will, as usual, equate truth at a world with membership in it.

3.3.14. Definition. [Canonical model]
The canonical model for InqB is the model $M^c = \langle W^c, V^c \rangle$ defined as follows:

- W^c is the set of complete theories of classical formulas;
- $V^c : W^c \times \mathcal{P} \to \{0,1\}$ is defined by: $V^c(\Delta, p) = 1 \iff p \in \Delta$

3.3.3 Completeness

Usually, the next step in the completeness proof is to prove the *truth lemma*, a result connecting truth at a possible world in the canonical model with provability from that world. However, in inquisitive semantics the fundamental semantic notion is not truth at a possible world, but support at an information state. Thus, what we need is a *support lemma* that characterizes the notion of support at a state in M^c in terms of provability. What should this characterization be?

We may think of the information available in a state S as being captured by those statements that are true at all the worlds in S. Syntactically, truth at a world will correspond to membership in it. Thus, the information available in a state S is captured syntactically by the theory of classical formulas $\bigcap S$, which consists of those statements that belong to all the worlds in S.

For a formula φ, to be supported at S is to be settled by the information available in S. Syntactically, this would then correspond to φ being derivable from $\bigcap S$. Thus, we expect to have the following connection: $S \models \varphi \iff \bigcap S \vdash \varphi$. The following lemma states that this connection does indeed hold in M^c.[4]

3.3.15. Lemma (Support Lemma). *For any state $S \subseteq W^c$ and any $\varphi \in \mathcal{L}^{\mathsf{P}}$:*

$$S \models \varphi \iff \bigcap S \vdash \varphi$$

Proof. The proof is by induction on φ, simultaneously for all $S \subseteq W^c$.

- *Atoms.* By the support clause for atoms, we have $S \models p \iff V^c(\Gamma, p) = 1$ for all $\Gamma \in S$. By definition of the canonical valuation, this is the case if and only if $p \in \Gamma$ for all $\Gamma \in S$, i.e., if and only if $p \in \bigcap S$. Finally, by Lemma 3.3.13 we have $p \in \bigcap S \iff \bigcap S \vdash p$.

- *Falsum.* Suppose $S \models \bot$. This means that $S = \emptyset$. Recalling that we have defined $\bigcap \emptyset$ to be the set $\mathcal{L}_c^{\mathsf{P}}$ of all classical formulas, we have $\bigcap S \vdash \bot$. Conversely, suppose $S \not\models \bot$, that is, $S \neq \emptyset$. Then, take a $\Gamma \in S$: $\bigcap S \subseteq \Gamma$, and since $\Gamma \nvdash \bot$ by definition, also $\bigcap S \nvdash \bot$.

- *Conjunction.* The inference rules for conjunction imply that a conjunction is provable from a given set of assumptions iff both of its conjuncts are. Using this fact and the induction hypothesis, we obtain: $S \models \varphi \wedge \psi \iff S \models \varphi$ and $S \models \psi \iff \bigcap S \vdash \varphi$ and $\bigcap S \vdash \psi \iff \bigcap S \vdash \varphi \wedge \psi$.

- *Implication.* Suppose $\bigcap S \vdash \varphi \to \psi$. Consider any state $T \subseteq S$ with $T \models \varphi$. By induction hypothesis, this means that $\bigcap T \vdash \varphi$. Since $T \subseteq S$, we have $\bigcap S \subseteq \bigcap T$, and since we are assuming $\bigcap S \vdash \varphi \to \psi$, also $\bigcap T \vdash \varphi \to \psi$. Now, since from $\bigcap T$ we can derive both φ and $\varphi \to \psi$, by an application of ($\to$e) we can also derive ψ. Hence, by induction hypothesis we have $T \models \psi$. Since T was an arbitrary substate of S, we have shown that $S \models \varphi \to \psi$.

[4]If we had chosen to define the worlds as complete theories in the whole language $\mathcal{L}^{\mathsf{P}}$, rather than just in the classical language $\mathcal{L}_c^{\mathsf{P}}$, we would need a more convoluted support lemma. To see that support at a state S could not simply be equated with derivability from $\bigcap S$, notice that, since $\neg ?p$ is inconsistent, any complete theory Γ would have to contain $?p$. Thus, we would have $\bigcap S \vdash ?p$ for any S, while it is not the case that $?p$ is supported at any state in M^c. The reason for this complication is that, for a question, support at S does not amount to truth at all the worlds in S. So, we should not expect support to correspond to membership to all the worlds in S, i.e., to membership in $\bigcap S$.

For the converse, suppose $\bigcap S \nvdash \varphi \to \psi$. Given the rule ($\to$i), this implies $\bigcap S, \varphi \nvdash \psi$. But then, Lemma 3.3.6 ensures that there is a resolution $\alpha \in \mathcal{R}(\varphi)$ such that $\bigcap S, \alpha \nvdash \psi$.

Now let $T_\alpha = \{\Gamma \in S \,|\, \alpha \in \Gamma\}$. First, by definition we have $\alpha \in \bigcap T_\alpha$, whence $\bigcap T_\alpha \vdash \varphi$ by Corollary 3.3.5. By induction hypothesis we then have $T_\alpha \models \varphi$. Now, if we can show that $\bigcap T_\alpha \nvdash \psi$ we are done. For then, the induction hypothesis gives $T_\alpha \not\models \psi$: this would mean that T_α is a substate of S that supports φ but not ψ, showing that $S \not\models \varphi \to \psi$.

So, we are left to show that $\bigcap T_\alpha \nvdash \psi$. Towards a contradiction, suppose $\bigcap T_\alpha \vdash \psi$. Since $\bigcap T_\alpha$ is a set of classical formulas, it is a (in fact, the only) resolution of itself. Thus, Lemma 3.3.8 tells us that $\bigcap T_\alpha \vdash \beta$ for some resolution $\beta \in \mathcal{R}(\psi)$, which by Lemma 3.3.13 amounts to $\beta \in \bigcap T_\alpha$. So, for any $\Gamma \in T_\alpha$ we have $\beta \in \Gamma$ and thus also $\alpha \to \beta \in \Gamma$, since Γ is closed under deduction of classical formulas and $\beta \vdash \alpha \to \beta$. Now consider any $\Gamma \in S - T_\alpha$: this means that $\alpha \notin \Gamma$; then since Γ is complete we have $\neg\alpha \in \Gamma$, whence $\alpha \to \beta \in \Gamma$, because Γ is closed under deduction of classical formulas and $\neg\alpha \vdash \alpha \to \beta$. We have thus shown that $\alpha \to \beta \in \Gamma$ for any $\Gamma \in S$, whether $\Gamma \in T_\alpha$ or $\Gamma \in S - T_\alpha$. We can then conclude $\alpha \to \beta \in \bigcap S$, whence $\bigcap S, \alpha \vdash \beta$. And since β is a resolution of ψ we also have $\bigcap S, \alpha \vdash \psi$. But this is a contradiction since by assumption α is such that $\bigcap S, \alpha \nvdash \psi$.

- *Inquisitive disjunction.* Suppose $S \models \varphi \mathbin{\mathbb{V}} \psi$. By the support clause for $\mathbb{V}$, this means that either $S \models \varphi$, or $S \models \psi$. The induction hypothesis gives $\bigcap S \vdash \varphi$ in the former case, and $\bigcap S \vdash \psi$ in the latter. In either case, the rule ($\mathbb{V}$i) ensures that we also have $\bigcap S \vdash \varphi \mathbin{\mathbb{V}} \psi$.

  Conversely, suppose $\bigcap S \vdash \varphi \mathbin{\mathbb{V}} \psi$. Since $\bigcap S$ is a set of classical formulas, by Lemma 3.3.9 we have either $\bigcap S \vdash \varphi$ or $\bigcap S \vdash \psi$. The induction hypothesis gives $S \models \varphi$ in the former case, and $S \models \psi$ in the latter. In either case, we can conclude $S \models \varphi \mathbin{\mathbb{V}} \psi$. $\square$

Notice that, if we take our state S to be a singleton $\{\Gamma\}$, then $\bigcap S = \Gamma$, and we obtain the usual Truth Lemma as a particular case of the Support Lemma.

3.3.16. Corollary (Truth Lemma). *For any world $\Gamma \in W^c$ and formula φ:*

$$\Gamma \models \varphi \iff \Gamma \vdash \varphi$$

However, it is really the Support Lemma, and not just the Truth Lemma, that we need in order to establish completeness. This is because certain invalid entailments can only be falsified at non-singleton states. For instance, consider the polar question $?p$: although this formula is not logically valid, it is true at any possible world in any model—i.e., supported at any singleton state. Thus, in order to detect the invalidity of $?p$ in the canonical model, we really have to find a non-singleton state S in M^c at which $?p$ is not supported.

In general, given a set of formulas Φ and a formula ψ such that $\Phi \nvdash \psi$, the question is how to produce a state $S \subseteq W^c$ which refutes the entailment $\Phi \models \psi$. In proofs for classical logic, one starts with the observation that if $\Phi \nvdash \psi$, then $\Phi \cup \{\neg\psi\}$ is a consistent set of formulas. But in our logic, this is not generally true: for instance, the soundness of the logic ensures that $\nvdash ?p$, but it is easy to see that $\{\neg ?p\}$ is not a consistent set.

The key to produce the desired state is to make use of the Resolution Lemma: if $\Phi \nvdash \psi$, we can find a specific resolution Θ of the assumptions that derives none of the resolution $\alpha_1, \dots, \alpha_n$ of the conclusion. We can then build a number of worlds $\Gamma_1, \dots, \Gamma_n \in W^c$, each refuting one of the entailments $\Theta \models \alpha_1, \dots, \Theta \models \alpha_n$, and finally we can consider the state $S = \{\Gamma_1, \dots, \Gamma_n\}$. The following argument concludes the proof of the Completeness Theorem.

Proof of Theorem 3.3.2. Suppose $\Phi \nvdash \psi$. By the Resolution Lemma, there is a resolution Θ of Φ which does not derive any resolution of ψ. Now let $\mathcal{R}(\psi) = \{\alpha_1, \dots, \alpha_n\}$ and consider any α_i. Since $\Theta \nvdash \alpha_i$, we must have $\Theta \cup \{\neg\alpha_i\} \nvdash \bot$. For, if we had $\Theta \cup \{\neg\alpha_i\} \vdash \bot$, we would also have $\Theta \vdash \neg\neg\alpha_i$ and thus, since α_i is a classical formula, we would have $\Theta \vdash \alpha_i$, contrary to assumption. Hence, the set $\Theta \cup \{\neg\alpha_i\}$ is consistent, and thus extendible to a complete theory of classical formulas $\Gamma_i \in W^c$.

Now let S be the state $\{\Gamma_1, \dots, \Gamma_n\}$: we claim that $S \models \Phi$ but $S \not\models \psi$. To see that $S \models \Phi$, note that by construction we have $\Theta \in \bigcap S$, which implies $S \models \Theta$ by the Support Lemma. But since $\Theta \in \mathcal{R}(\Phi)$, Proposition 2.4.3 implies $S \models \Phi$.

To see that $S \not\models \psi$, suppose S supported ψ: then by Proposition 2.4.3 it should also support α_i for some i. By the Support Lemma, that would mean that $\bigcap S \vdash \alpha_i$, and so also $\alpha_i \in \Gamma_i$, since $\bigcap S \subseteq \Gamma_i$ and Γ_i is closed under deduction of classical formulas. But that is impossible, since Γ_i is consistent and contains $\neg\alpha_i$ by construction. Hence, we have $S \models \Phi$ but $S \not\models \psi$, which witnesses that $\Phi \not\models \psi$. □

3.4 On the role of questions in proofs

Let us now abstract away from the specific setting of InqB, and let us turn to examine what our investigations in this chapter bring out about the role of questions in logical proofs. Traditionally, logicians have tacitly assumed questions to have no such role to play; occasionally, the meaningfulness of letting questions participate in logical proofs has even been overtly denied, as in the following passage, drawn from the introduction to Belnap and Steel (1976):

> Absolutely the wrong thing is to think [the logic of questions] is a logic in the sense of a deductive system, since one would then be

> driven to the pointless task of inventing an inferential scheme in which questions, or interrogatives, could serve as premises and conclusions.

One thing I hope to have achieved with this chapter is to show that Belnap and Steel were wrong: not only is it possible to give a deductive system in which questions serve as premisses and conclusions, and a well-behaved one at that; but also, proofs in such a system are perfectly meaningful, and worthy of investigation.

For one thing, we saw in Chapter 1 that, if we adopt a support semantics, entailments involving questions capture interesting logical relations. This, in itself, would already be sufficient to grant interest in a syntactic calculus that tracks the entailment relation. However, in this section we have seen that the role of question in proofs goes beyond this: inferences involving questions are themselves interesting logical objects, which capture meaningful arguments.

In a nutshell, what we found is that questions make it possible to perform inferences with information which is *generic*. To appreciate this point, it might be useful to draw a connection with the generic individual constants used in natural deduction systems for standard first-order logic.[5] For instance, in order to infer ψ from $\exists x\varphi(x)$, one can make a new assumption $\varphi(c)$, where c is fresh in the proof and not occurring in ψ, and then try to derive ψ from this assumption. Here, the idea is that c stands for a *generic* object in the extension of $\varphi(x)$. If ψ can be inferred from $\varphi(c)$, then it must follow no matter which specific "object of type φ" the constant c denotes, and thus it must follow from the mere existence of such an object.

Questions allow us to do something similar, except that instead of a generic *individual*, a question may be viewed as denoting a generic *piece of information* of a given type. For instance, the question *what the patient's symptoms are*, may be viewed as denoting a generic specification of the patient's symptoms.

When assuming a question μ, what we are supposing is *some* information of type μ. Thus, e.g., by assuming the question *what the patient's symptoms are*, we are supposing to be given a complete specification of the patient's symptoms. We are not assuming anything specific about what these symptoms are—we are not supposing, say, that the patient has symptom S_1 but not S_2. Rather, we are just assuming some generic complete information on the subject.

Similarly, consider the move of drawing a conclusion. In concluding a question, what we are establishing is that, under the given assumptions, we are guaranteed to have *some* information of the corresponding type—though precisely *what* information this is may depend on how our question assumptions are instantiated by specific pieces of information.

As we saw, by manipulating such generic information in the form of questions, it is possible to construct proofs that witness the existence of certain dependence relations and which, moreover, actually describe how to compute the relevant

[5] Thanks to Justin Bledin for suggesting this analogy.

dependencies. Thus, far from being useless from a proof-theoretic perspective, questions turn out to be very interesting tools for logical inference.

Chapter 4

Questions in First-Order Logic

In Chapter 1, we have discussed at an abstract level how classical logic may be enriched with questions, and what this enrichment allows us to capture. In Chapter 2, we have discussed a particular system which extends classical propositional logic with questions, and in Chapter 3 we have looked at a proof system for this logic, focusing in particular on the role of questions in logical proofs. In this chapter, we are going to see how the approach can be extended further to the setting of first-order predicate logic. In other words, we are going to show how the support approach allows us to extend classical first-order logic with questions.

Such an extension is interesting not only because classical first-order logic itself is a much more expressive basis than propositional logic, but also because, through quantification, many interesting kinds of questions become expressible, besides the propositional, *whether... or* questions that we have analyzed in the previous chapter. For instance, in the language considered in this chapter, we will have questions that ask for an instance of a property, such as (1-a); questions that ask for the unique instance of a property, such as (1-b); and, finally, questions that ask for the *extension* of a property, such as (1-c).

(1) a. What is one European capital?
   b. What is the capital of France?
   c. What are all the European capitals?

The system that we will describe, denoted InqBQ, where Q refers to quantification, has appeared for the first time in Ciardelli (2009). It has then been motivated from an algebraic perspective in Roelofsen (2013), and it has been adopted in Ciardelli et al. (2012, 2015a) as the logical backbone of the linguistic framework of inquisitive semantics.

Just as we did for the propositional system InqB, we will take a new perspective on this system, regarding it as a conservative extension of classical logic with new operators. That is, we will first provide a semantics for classical first-order logic based on the relation of support, and then we will exploit the move to the support setting to enrich classical first-order logic with two question-forming logical operators, the inquisitive disjunction $\vee\!\!\!\vee$ that we saw in Chapter 2, and a quantifier counterpart of it, the inquisitive existential $\overline{\exists}$.

The system InqBQ is not as well-understood as its propositional counterpart. While Ciardelli (2009) and Roelofsen (2013) provided a characterization of the general structure of this logic and point out some notable features, no complete axiomatization is known. In this chapter, we will make progress in understanding the logical features of InqBQ. We will isolate two fragments of InqBQ that contain classical first-order logic as well as a natural stock of questions that are interesting in practice. The first of these fragments, that we will call the *mention-some* fragment, contains, among others, all propositional questions that we know from the previous chapters, as well as mention-some questions like (1-a) and unique-instance questions like (1-b). The other fragment, that we will call the *mention-all* fragment, contains polar questions as well as mention-all questions like (1-c)—coinciding in expressive power with a precursor of inquisitive logic, the Logic of Interrogation of Groenendijk (1999). For each of these fragments, we will provide a complete axiomatization in the form of a natural-deduction proof system, where each operator, including the ones involved in question formation, is handled by standard introduction and elimination rules.

The chapter is structured as follows. We begin in Section 4.1 by providing a support semantics for classical first-order logic. In Section 4.2 we exploit the support setting to extend first-order logic with two question-forming operators, and we illustrate the range of questions that these operators can be used to express. In Section 4.3 we examine a number of basic features of the resulting system. In Section 4.4 we turn to the associated entailment relation, exploring its general properties, and providing a few simple examples of the sort of logical dependencies that can be captured as entailments among first-order questions. In Section 4.5 we examine a stronger notion of entailment which arises from restricting to models in which the extension of the identity relation is settled. In Section 4.6 we identify a number of inference rules for InqBQ, providing a candidate axiomatization. Finally, in Section 4.7 and 4.8 we focus on two interesting fragments of InqBQ which jointly cover the most interesting cases of first-order questions, and we provide a completeness result for each of them.

4.1 Support for classical first-order logic

Let us start out by describing how classical first-order logic may be given a support semantics. As usual, our language is based on a signature $\mathcal{S}$, consisting a set $\mathfrak{F}_{\mathcal{S}}$ of function symbols, and a set $\mathfrak{R}_{\mathcal{S}}$ of relation symbols, where each of these symbols has a certain *arity* $n \geq 0$.[1] Moreover, we assume that among the function symbols we have a specified set $\mathfrak{F}^{R}_{\mathcal{S}}$ of *rigid* function symbols, whose interpretation is

[1]As usual, individual constants are regarded as function symbols of arity 0.

required to be fixed across different possible worlds. We will refer to the remaining function symbols, whose interpretation may vary across different possible worlds, as *non-rigid* function symbols.

We will make the simplifying assumption that the signature $\mathcal{S}$ is countable, so that the resulting language is also countable.[2] In addition to the symbols provided by the signature, we also assume a countably infinite stock of variables, $\mathsf{Var} = \{x_1, x_2, \dots\}$. The set $\mathsf{Ter}(\mathcal{S})$ of *terms* in the language is defined in the standard way. We will say that a term t is *rigid* in case it is built up from variables and rigid function symbols (including rigid individual constants).

The set of classical first-order formulas in the signature $\mathcal{S}$ is also defined as usual, where we take $\bot$, $\wedge$, and $\to$ as our primitive propositional connectives, and $\forall$ as our primitive quantifier.

4.1.1. Definition. [Classical formulas]
The set $\mathcal{L}_c^{\mathsf{Q}}(\mathcal{S})$ of classical first-order formulas is defined recursively as follows:

$$\varphi \ ::= \ R(t_1, \dots, t_n) \mid (t{=}t') \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi$$

where R is an n-ary relation symbol in $\mathcal{S}$, $t_1, \dots, t_n \in \mathsf{Ter}(\mathcal{S})$, and $x \in \mathsf{Var}$.

When there is no need to emphasize the signature $\mathcal{S}$, we will drop reference to it and simply refer to the set of classical first-order formulas as $\mathcal{L}_c^{\mathsf{Q}}$. As in the propositional case, we will take the remaining operators of classical logic to be defined from our primitive ones, as follows.

4.1.2. Definition. [Defined classical operators]

- $\neg\varphi := \varphi \to \bot$
- $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$
- $\exists x\varphi := \neg\forall x\neg\varphi$

It will be very useful to introduce some notational conventions. First, we will often write $\overline{t}$ for a sequence $\langle t_1, \dots, t_n\rangle$ of terms, and $\overline{x}$ for a sequence $\langle x_1, \dots, x_n\rangle$ of variables. Moreover, if Q is a quantifier and $\overline{x} = \langle x_1, \dots, x_n\rangle$ a sequence of variables, we will write $Q\overline{x}\varphi$ for $Qx_1 \dots Qx_n\varphi$.

Free and bound occurrences of a variable x in a formula are defined as usual. Given a formula φ, we write $FV(\varphi)$ for the set of variables which are free in φ.

[2] This assumption is dispensable, but it simplifies some of our proofs. More specifically, in the completeness proof for the mention-some fragment (Section 4.7), it would not be enough to introduce countably many new constants: we would need as many new constants as there are formulas in the language, and Lemma 4.7.19 would have to be established by ordinal induction rather than by induction on natural numbers.

Moreover, if $x \in \mathsf{Var}$ and $t \in \mathsf{Ter}(\mathcal{S})$, we write $\varphi[t/x]$ for the formula that results from replacing each free occurrence of x in φ by t. More generally, if $\overline{x} = \langle x_1, \ldots, x_n \rangle$ is a tuple of variables and $\overline{t} = \langle t_1, \ldots, t_n \rangle$ is a tuple of terms, we write $\varphi[\overline{t}/\overline{x}]$ for the formula that results from simultaneously replacing each occurrence of a variable x_i in φ by the term t_i. As usual, we say that a term t is *free for* a variable x in a formula φ in case no free occurrence of x in φ lies within the scope of a quantifier which binds a variable y occurring in t.

Let us now turn to the structures that are to serve as models for our language. From the perspective of our language, a complete state of affairs consists of those things that are usually specified by a first-order model: a domain of entities, together with suitable denotations for function symbols, i.e., functions over the domain, and for relation symbols, i.e., relations over the domain. So, our models will consist of a set of possible worlds, each described by a first-order model.

In order to simplify the issues arising from quantification across different possible worlds, we will assume a fixed domain of individuals. However, we will take each world w to come with a possibly different notion $\sim_w$ of identity between these individuals: this is because, even if we do not allow for uncertainty about whether or not certain individuals exist, we still want to allow for uncertainty about the relation of identity. For instance, we want to allow for information states where it is uncertain whether two individuals *Hesperus* and *Phosphorus* are the same. Thus, we may regard the domain D shared among all the worlds as a domain of "epistemic" individuals, which may be found out to be identical, or to be distinct, as more information is acquired, while the "actual" individuals existing at a possible world w are the equivalence classes $\{[d]^{\sim_w} \mid d \in D\}$. Obviously, if two individuals are identical at a world, then they must share exactly the same properties, and applying a function to them should give identical results: this means that we should require the relation $\sim_w$ to be a *congruence*. This condition ensures that we may identify a world w with a first-order model w^* having as its domain the set $D/{\sim_w}$ of equivalence classes modulo $\sim_w$.[3]

4.1.3. Definition. [First-order information models]
A first-order information model is a quadruple $M = \langle W, D, I, \sim \rangle$, where:

- W is a set, the elements of which we call *possible worlds*;
- D is a non-empty set, the elements of which we call *individuals*;

[3]There are, of course, interesting philosophical issues about these epistemic objects that we will put aside here. In this chapter, we will use such objects to model the way in which, in an information state, properties are attributed to entities that do not directly coincide with the entities existing in a particular world. An analogy can be drawn with the *discourse referents* used in dynamic semantics (Heim, 1982; Landman, 1986; Groenendijk et al., 1996); these are objects that sentences provide information about, and which in turn are supposed to correspond, but not necessarily in a one-to-one fashion, to entities in the world. More work is needed to assess the potential and the limitations of the simple kind of modeling used here.

- I is a map assigning to each $w \in W$ a function I_w defined on $\mathcal{S}$ such that:
  - $I_w(f) : D^n \to D$ for an n-ary function symbol $f \in \mathcal{S}$, with the constraint that if $f \in \mathfrak{F}^R_{\mathcal{S}}$, then for any $w, w' \in W$, $I_w(f) = I_{w'}(f)$.
  - $I_w(R) \subseteq D^n$ for an n-ary relation symbol $R \in \mathcal{S}$.
- $\sim$ is a map assigning to each $w \in W$ an equivalence relation $\sim_w \subseteq D \times D$, with the condition that $\sim_w$ be a *congruence*, that is:
  - for any n-ary function symbol f, if $d_1 \sim_w d'_1, \ldots, d_n \sim_w d'_n$, then

$$I_w(f)(d_1, \ldots, d_n) \sim_w I_w(f)(d'_1, \ldots, d'_n)$$

  - for any n-ary relation symbol R, if $d_1 \sim_w d'_1, \ldots, d_n \sim_w d'_n$, then

$$\langle d_1, \ldots, d_n \rangle \in I_w(R) \iff \langle d'_1, \ldots, d'_n \rangle \in I_w(R)$$

As in the previous chapters, an information state s in an information model M will be a set $s \subseteq W$ of possible worlds.

As customary, the specification of the semantics is given relative to an assignment function g, which fixes the interpretation of the variables in the language. Notice that, since the domain D is shared among the possible worlds in the model M, assignments can be defined as usual, as functions $g : \mathsf{Var} \to D$. If $d \in D$, we write $g[x \mapsto d]$ for the assignment which maps x to d, and otherwise coincides with g. Using assignments, all terms can be assigned a referent in a natural way.

4.1.4. Definition. [Referent of a term]
The referent of a term t in world w of an information model M under an assignment g is the individual defined inductively as follows:

- $[x]^w_g = g(x)$
- $[f(t_1, \ldots, t_n)]^w_g = I_w(f)([t_1]^w_g, \ldots, [t_n]^w_g)$

Finally we are now ready to define the relation of support between states and formulas, which specifies what information it takes to settle a first-order formula.

4.1.5. Definition. [Support]
The relation of *support* between states s of an information model M and formulas $\varphi \in \mathcal{L}^{\mathsf{Q}}_c$, relative to an assignment g into M, is defined recursively as follows:

- $M, s \models_g R(t_1, \ldots, t_n) \iff$ for all $w \in s$, $\langle [t_1]^w_g, \ldots, [t_n]^w_g \rangle \in I_w(R)$
- $M, s \models_g (t = t') \iff$ for all $w \in s$, $[t]^w_g \sim_w [t']^w_g$
- $M, s \models_g \bot \iff s = \emptyset$

- $M, s \models_g \varphi \wedge \psi \iff M, s \models_g \varphi$ and $M, s \models_g \psi$
- $M, s \models_g \varphi \to \psi \iff$ for all $t \subseteq s,\ M, t \models_g \varphi$ implies $M, t \models_g \psi$
- $M, s \models_g \forall x\varphi \iff$ for all $d \in D,\ M, s \models_{g[x \mapsto d]} \varphi$

Intuitively, the definition may be read as follows. An atomic formula $R\overline{t}$ is settled in an information state s if the information available in s implies that the tuple of individuals denoted by $\overline{t}$ belongs to the extension of R. An identity formula $(t = t')$ is settled in s if the information available in s implies that t and t' denote the same individual. The falsum constant is only settled in the inconsistent information state. A conjunction $\varphi \wedge \psi$ is settled in s in case both conjuncts are. An implication $\varphi \to \psi$ is settled in s in case φ entails ψ relative to s, that is, in case enhancing s so as to settle φ is bound to lead to a state where ψ is settled as well. Finally, a universal formula $\forall x\varphi(x)$ is settled in case $\varphi(d)$ is settled for every epistemic individual $d \in D$.

When no confusion arises, we will allow ourselves to drop reference to the model M, and simply write support relative to a state. As usual, we refer to the set of states supporting φ in a model M, now relative to an assignment g, as the *support-set* of φ, notation $[\varphi]^g_M$. The *alternatives* for φ in M relative to g are the $\subseteq$-maximal elements of $[\varphi]^g_M$, and the set of these alternatives is denoted $\textsc{Alt}^g_M(\varphi)$. Reference to the assignment g will be dropped when φ is a sentence, in which case all the semantic notions are easily seen to be independent of g.

Also, we will say that a state s is *compatible* with a formula φ relative to g, notation $s \between_g \varphi$, if s can be enhanced consistently to support φ relative to g, i.e., if there is a non-empty state $t \subseteq s$ such that $t \models_g \varphi$. The clauses for the defined connectives can be given an insightful formulation in terms of compatibility.

- $s \models_g \neg\varphi \iff$ it is not the case that $s \between_g \varphi$
- $s \models_g \varphi \vee \psi \iff$ for any consistent $t \subseteq s,\ t \between_g \varphi$ or $t \between_g \psi$
- $s \models_g \exists x\varphi \iff$ for any consistent $t \subseteq s,\ t \between_{g[x \mapsto d]} \varphi$ for some $d \in D$

The clauses for negation and disjunction are as in the propositional case. The clause for the existential quantifier says that $\exists x\varphi(x)$ is settled in s iff any consistent enhancement of s is bound to be compatible with $\varphi(d)$ for some $d \in D$, i.e., iff it is impossible to consistently enhance s so as to rule out $\varphi(d)$ for all $d \in D$.

As usual, truth at a world w in a model M is defined as support at the state $\{w\}$:

$$w \models_g \varphi \iff \{w\} \models_g \varphi$$

The truth-set of φ in M relative to g is the set: $|\varphi|^g_M = \{w \in W \mid w \models_g \varphi\}$. Simply by spelling out our support conditions for the special case of a singleton state $\{w\}$, we can see that our semantics determines the following truth-conditional clauses.

4.1.6. Proposition (Truth-conditions for classical formulas).
For any information model M, any world w in M, and any assignment g:

- $w \models_g R(t_1, \ldots, t_n) \iff \langle [t_1]^w_g, \ldots, [t_n]^w_g \rangle \in I_w(R)$
- $w \models_g (t = t') \iff [t]^w_g \sim_w [t']^w_g$
- $w \not\models_g \bot$
- $w \models_g \varphi \wedge \psi \iff w \models_g \varphi$ *and* $w \models_g \psi$
- $w \models_g \varphi \to \psi \iff w \not\models_g \varphi$ *or* $w \models_g \psi$
- $w \models_g \forall x \varphi \iff$ *for all* $d \in D$, $w \models_{g[x \mapsto d]} \varphi$

To make the connection with truth in classical logic fully precise, let us associate with each world $w \in W$ a first-order model w^* obtained by taking the quotient of $\langle D, I_w \rangle$ with respect to the congruence $\sim_w$.

4.1.7. Definition. [First-order model associated with a world]
The first-order model w^* associated with a world w is the pair $\langle D^*_w, I^*_w \rangle$, where:

- $D^*_w = \{[d]_{\sim_w} \mid d \in D\}$ is the set of equivalence classes modulo $\sim_w$
- $I^*_w(f)([d_1]_{\sim_w}, \ldots, [d_n]_{\sim_w}) = [I_w(f)(d_1, \ldots, d_n)]_{\sim_w}$
- $\langle [d_1]_{\sim_w}, \ldots, [d_n]_{\sim_w} \rangle \in I^*_w(R) \iff \langle d_1, \ldots, d_n \rangle \in I_w(R)$

Since $\sim_w$ is a congruence, this model is well-defined, in the sense that the definitions do not depend on the choice of representatives within an equivalence class. If t is a term, let us write $[t]^{w^*}_g$ for the referent of t in w^*, namely, $[[t]^w_g]_{\sim_w}$.

It is easy to verify that the above truth-conditions amount precisely to the standard truth-conditions in the first-order model w^*.

4.1.8. Proposition.
For any world w in an information model M, any assignment g, and any $\varphi \in \mathcal{L}^{\mathsf{Q}}_c$:

$$w \models_g \varphi \iff w^* \models_g \varphi \text{ holds in classical first-order logic}$$

Now we can prove by induction that all formulas in our first-order language are truth-conditional, that is, support at a state s simply amounts to truth at each world in s, for all formula $\varphi \in \mathcal{L}^{\mathsf{Q}}_c$.

4.1.9. Proposition (Classical formulas are truth-conditional).
For any $\varphi \in \mathcal{L}^{\mathsf{Q}}_c$, any state s in an information model M, and any assignment g:

$$s \models_g \varphi \iff w \models_g \varphi \text{ for all } w \in s$$

In view of the previous proposition, this shows that the relation of support is inter-definable with the relation of truth of classical first-order logic. Moreover, we saw in Section 1.1.2 that this connection ensures that the two semantics give rise to the same notion of entailment. This means that what we have provided is a support semantics for classical first-order logic. Following the same strategy that we pursued in the propositional case, in the next section we will take advantage of the move from truth to support in order to enrich classical logic with questions.

4.2 Adding questions to first-order logic

In the propositional case, questions are introduced into the system by means of a new connective, the inquisitive disjunction $\vee\!\!\vee$. In the first-order case, it is natural to also consider a quantifier counterpart of $\vee\!\!\vee$, denoted $\overline{\exists}$, which we will call *inquisitive existential quantifier*. The full language of our system is obtained by enriching the language of classical first-order logic with these two operators.

4.2.1. Definition.
The set $\mathcal{L}^{\mathsf{Q}}(\mathcal{S})$ of first-order formulas of InqBQ is defined recursively as follows, where R is an n-ary relation symbol in $\mathcal{S}$, $t_1, \dots, t_n \in \mathsf{Ter}(\mathcal{S})$, and $x \in \mathsf{Var}$.

$$\varphi \ ::= \ R(t_1, \dots, t_n) \mid (t = t') \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \forall x \varphi \mid \varphi \vee\!\!\vee \varphi \mid \overline{\exists} x \varphi$$

Recall that a state s supports an inquisitive disjunction $\varphi \vee\!\!\vee \psi$ in case it supports one of the disjuncts. Similarly, a state s will support an inquisitive existential $\overline{\exists} x \varphi(x)$ in case it supports $\varphi(d)$ for some specific individual $d \in D$. That is, the definition of support is extended to the whole language $\mathcal{L}^{\mathsf{Q}}$ by augmenting Definition 4.1.5 with the following two clauses.

4.2.2. Definition. [Support clauses for inquisitive connectives]

- $s \models_g \varphi \vee\!\!\vee \psi \iff s \models_g \varphi$ or $s \models_g \psi$
- $s \models_g \overline{\exists} x \varphi \iff$ for some $d \in D$, $s \models_{g[x \mapsto d]} \varphi$

As in the propositional case, we will refer to a formula as a *statement* if it is truth-conditional, and as a *question* if it is not. While Proposition 4.1.9 implies that all classical formulas are statements, the operators $\vee\!\!\vee$ and $\overline{\exists}$ bring questions into the picture.

We have discussed in Chapter 2 the kind of questions that can be expressed by means of $\vee\!\!\vee$. To illustrate the questions that can be expressed by means of $\overline{\exists}$, consider a predicate symbol P. The inquisitive existential sentence $\overline{\exists} x P(x)$ is settled in a state s iff s establishes of some individual that it has property P.

$$\begin{aligned} s \models_g \overline{\exists} x P(x) \quad &\iff \quad \text{for some } d \in D, s \models_{g[x \mapsto d]} P(x) \\ &\iff \quad \text{for some } d \in D, \text{ for all } w \in s, \ d \in I_w(P) \end{aligned}$$

Thus, what is needed to settle $\overline{\exists}xP(x)$ is to establish an instance of property P. If P is, say, the property of being a European capital, then $\overline{\exists}xP(x)$ corresponds to the *mention-some* question in (2).

(2) What is one European capital?

For a simple model M containing just two distinct individuals, and four possible worlds corresponding to the four possible interpretations for the predicate P, the formula $\overline{\exists}xP(x)$ has two alternatives, depicted in Figure 4.1(c). Assuming the two individuals are named by two rigid constants a and b, these two alternatives correspond to the classical formulas Pa and Pb, which provide just enough information to establish an instance of property P.

Similarly, if R is a binary relation symbol, then the formula $\overline{\exists}x\overline{\exists}yR(x,y)$ is supported in s just in case s establishes of a specific pair that it stands in the relation R. Thus, what is needed to settle $\overline{\exists}x\overline{\exists}yR(x,y)$ is an instance of a pair which belongs to the relation R.

For another example, consider an identity formula $(x = a)$, where a is a non-rigid constant symbol. The inquisitive existential sentence $\overline{\exists}x(x = a)$ is supported in a state s in case s establishes of some individual that it is identical to the referent of a. Since a can have only one referent, this is simply to say that s settles to what individual the constant refers.

$$
\begin{aligned}
s \models_g \overline{\exists}x(x = a) \quad &\iff \quad \text{for some } d \in D, s \models_{g[x\mapsto d]} (x = a) \\
&\iff \quad \text{for some } d \in D, \text{ for all } w \in s, \; I^*_w(a) = [d]_{\sim_w}
\end{aligned}
$$

Thus, the formula $\overline{\exists}x(x = a)$ is a question which asks for the identity of a.[4]

For another interesting example of the sort of questions expressible by means of the inquisitive existential quantifier, let us introduce the following abbreviation, analogous to the one commonly used for the classical existential quantifier:

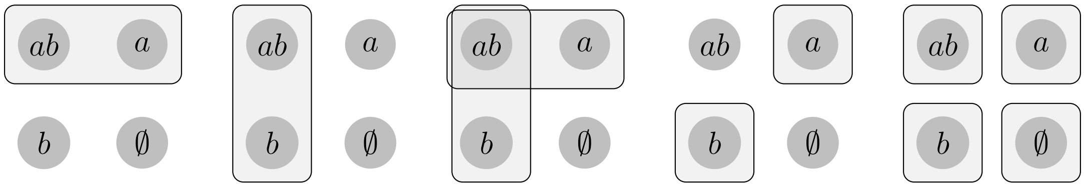
$$
\overline{\exists}!x\varphi \;:=\; \overline{\exists}x(\varphi \wedge \forall y(\varphi[y/x] \to (y = x)))
$$

Now consider again a predicate symbol P. The sentence $\overline{\exists}!xP(x)$ is supported in a state s in case s establishes of some individual $d \in D$ that d is *the* individual who has property P.[5] Omitting the details of the calculation, we have:

$$
\begin{aligned}
s \models_g \overline{\exists}!xP(x) \quad &\iff \quad \text{for some } d \in D, \text{ for all } w \in s : d \in I_w(P) \text{ and} \\
&\qquad\quad \text{for all } d' \in D, \; d' \in I_w(P) \text{ implies } d' \sim_w d \\
&\iff \quad \text{for some } d \in D, \text{ for all } w \in s : \; I^*_w(P) = \{[d]_{\sim_w}\}
\end{aligned}
$$

[4] Incidentally, *rigid* terms may be characterized precisely as those for which the question $\overline{\exists}x(x = t)$ is logically valid, i.e., settled in all states of every model.

[5] More precisely, if s establishes that d is the only *actual* individual having property P. That is, s establishes that if any other epistemic individual d' has property P, then d' and d are actually the same individual.



(a) Pa (b) Pb (c) $\overline{\exists}xP(x)$ (d) $\overline{\exists}!xPx$ (e) $\forall x?Px$

Figure 4.1: The alternatives for some first-order formulas in an information model consisting of two individuals and four possible worlds. The two individuals are distinct at each world, and denoted by two rigid constants a and b. The label ab in the pictures stands for a world w in which $I_w(P) = \{I_w(a), I_w(b)\}$, the label a stands for a world w in which $I_w(P) = \{I_w(a)\}$, and so on.

Thus, $\overline{\exists}x!P(x)$ is a question that asks for the identity of the only individual having property P. If P is the property of being a capital of France, for instance, $\overline{\exists}x!P(x)$ corresponds to a unique-instance question such as (3).

(3) What is the capital of France?

In the simple model illustrated in Figure 4.1, the formula $\overline{\exists}!xP(x)$ has two alternatives, depicted in Figure 4.1(d). These alternatives correspond to the classical formulas $\forall x(P(x) \leftrightarrow x = a)$ and $\forall x(P(x) \leftrightarrow x = b)$, which provide just enough information to establish of some individual that it is the unique P.

These examples illustrate the sort of questions that can be formed in InqBQ by means of the inquisitive existential quantifier, $\overline{\exists}$. However, It is not only through the use of $\overline{\exists}$ that our language can express new, interesting classes of questions, in addition to the propositional questions examined in the previous chapter. For, notice that the classical logical operators in our language may in turn apply to the questions that can be formed by means of $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$ and $\overline{\exists}$. In Chapter 2, we have discussed in some detail the effect of embedding questions under conjunction and implication. These operations apply in much the same way to the richer repertoire of questions available in the current setting. By means of conjunction we can, e.g., form conjunctive questions like $\overline{\exists}xP(x) \wedge \overline{\exists}xQ(x)$ which asks at once for an instance of property P and an instance of property Q, while means of implication we can form conditional questions like $\exists xP(x) \to \overline{\exists}xP(x)$, which asks for an instance of property P under the assumption that there is one; we may also form questions such as $\overline{\exists}xP(x) \to \overline{\exists}xQ(x)$, which may be seen as asking for a method for turning an instance of property P into an instance of property Q.

The novelty introduced by InqBQ is that it also becomes possible to *universally quantify* over questions. To see what sort of questions become expressible in this way, consider again a predicate symbol P. We know from Chapter 2 that $?P(x)$, which abbreviates $P(x) \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}} \neg P(x)$, expresses the polar question whether x has property P. Now consider the sentence $\forall x?P(x)$: this sentence is supported in a state s in case s settles the question $?P(x)$ for all values of x, that is, in case s establishes of every individual $d \in D$ whether or not d has property P. This amounts to saying that s must settle what the extension of the property P is.

$$
\begin{aligned}
s \models_g \forall x?P(x) &\iff \text{for all } d \in D: \ s \models_{g[x\mapsto d]} P(x) \text{ or } s \models_{g[x\mapsto d]} \neg P(x)\\
&\iff \text{for all } d \in D: \ (\text{for all } w \in s,\, d \in I_w(P)) \text{ or}\\
&\phantom{\iff \text{for all } d \in D: \ } (\text{for all } w \in s,\, d \notin I_w(P))\\
&\iff \text{for all } d \in D, \text{ for all } w, w' \in s: \ d \in I_w(P) \iff d \in I_{w'}(P)\\
&\iff \text{for all } w, w' \in s, \text{ for all } d \in D: \ d \in I_w(P) \iff d \in I_{w'}(P)\\
&\iff \text{for all } w, w' \in s: I_w(P) = I_{w'}(P)
\end{aligned}
$$

Thus, $\forall x?P(x)$ is a question that asks for a specification of the extension of the property P. If we once again take P to stand for the property of being a European capital, $\forall x?P(x)$ corresponds to the *mention-all* question in (4).

(4) What are all the European capitals?

In the toy model of Figure 4.1, this formula has four alternatives, depicted in figure 4.1(e). Given that the constant symbols a and b are interpreted rigidly, these alternatives correspond to the classical formulas $\forall xPx$, $\forall x(Px \leftrightarrow x = a)$, $\forall x(Px \leftrightarrow x = b)$ and $\forall x\neg Px$, each of which provides just enough information to completely specify the extension of the property P in our model.

In Section 4.8 we will see that the questions that can be obtained by universally quantifying over polar questions are precisely those which ask for the extension of a property or relation. As we will see, these coincide exactly with the questions expressible in the *Logic of Interrogation* of Groenendijk (1999).

Of course, we may also universally quantify over different kinds of questions. For instance, by universally quantifying over the mention-some question $\overline{\exists} yRxy$ we obtain the question $\forall x\overline{\exists} yRxy$, which is settled in case we provide, for each individual d, an instance of an individual d' to which d is related. Thus, for instance, if Rxy stands for "*y is a city located in country x*", then $\forall x\overline{\exists} yRxy$ asks to provide for each country an instance of a city located in that country. Similarly, if Rxy stands for "*y is a capital of country x*", the question $\forall x\overline{\exists}!yRxy$ asks to specify each country's unique capital.

This completes our illustration of the broad range of questions which are expressible in InqBQ by combining the inquisitive operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$ with the classical logical operators extended to the support setting.

4.3 General features of InqBQ

Let us now take a look at some basic features of the system InqBQ which we have defined. We will see that, while many features of the propositional system

InqB carry over, there are also some interesting complication that arise when we are dealing with infinite domains. Furthermore, the first-order setting brings into play some subtle issues having to do with reference and with identity.

4.3.1 Support and alternatives

Let us start by examining the features of the support relation. As we expect, support is persistent, that is, it is always preserved when we move from a state to an enhancement of it. In the limit case constituted by the inconsistent state, all formulas become supported.

4.3.1. Proposition.
For any information model M, assignment g and formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$, we have:

- *Persistence property: $s \models_g \varphi$ and $t \subseteq s$ implies $t \models_g \varphi$*
- *Empty state property: $\emptyset \models_g \varphi$*

This ensures that the support-set $[\varphi]^g_M$ of a formula is always an inquisitive proposition, i.e., a non-empty and downward closed set of states.

The main feature of the propositional system that does not carry over to the first-order setting is *normality*, that is, the fact that we have $[\varphi]^g_M = \text{Alt}^g_M(\varphi)^{\downarrow}$. We will show this by adapting a counterexample given in Ciardelli (2009).

4.3.2. Proposition.
For some signature $\mathcal{S}$, there is a model M, a state s, and a sentence $\varphi \in \mathcal{L}^{\mathsf{Q}}(\mathcal{S})$ such that $s \models \varphi$ but s is not included in any alternative for φ.

Proof. Let $\mathcal{S}$ consist of a binary relation symbol R and of a property symbol P. Let us denote by $\mathbb{N}$ the set of natural numbers and by $\leq$ the usual order on natural numbers. Consider a model $M = \langle W, D, I, id\rangle$, where:

- $W = \{w_n \mid n \in \mathbb{N}\}$
- $D = \mathbb{N}$
- $I_{w_n}(R) = \leq$
- $I_{w_n}(P) = \{0, \ldots, n\}$

and id maps any world to the identity relation. Now let $B_P(x) := \forall y(Py \to Ryx)$. $B_P(x)$ is a classical formula stating that x is an upper bound for P. Now consider the formula $\overline{\exists} x B_P(x)$ which asks for an instance of an upper bound to P. We will show that there is no alternative for this formula. Since we know that $\emptyset \models \overline{\exists} x B_P(x)$ by the empty state property, this suffices to show that normality fails.

We will show that no supporting state for s can be a *maximal* supporting state. To see this, suppose $s \models \overline{\exists} x B_P(x)$. By the support conditions for $\overline{\exists}$, there should be some $n \in \mathbb{N}$ such that $s \subseteq |B_P(x)|^{[x\mapsto n]}$, i.e., such that $I_w(P) \subseteq \{0, \ldots, n\}$ for all $w \in s$. Now, by definition of I, this implies that $s \subseteq \{w_0, \ldots, w_n\}$.

Now consider the state $s_{n+1} = \{w_0, \dots, w_{n+1}\}$. Clearly, since $s \subseteq \{w_0, \dots, w_n\}$, s_{n+1} is a proper superset of s. Moreover, since $s_{n+1} \subseteq |B_P(x)|^{[x\mapsto n+1]}$, we have $s_{n+1} \models \overline{\exists} x B_P(x)$. Thus, s is not an alternative for $\overline{\exists} x B_P(x)$. Since this holds for any state s, this shows that there are no alternatives for $\overline{\exists} x B_P(x)$ in M. $\Box$

This shows that, unlike in the propositional setting, in the first-order setting the meaning of a formula is not always captured by its set of alternatives.[6]

4.3.2 Truth-conditional formulas

We saw above that classical formulas $\varphi \in \mathcal{L}^{\mathsf{Q}}$ are always truth-conditional. Conversely, as in the propositional case, we can show that any truth-conditional formula in InqBQ is equivalent to a classical formula. In order to show this, let us first spell out the truth-conditional clauses that arise from our support definition. We have already seen in Proposition 4.1.6 that for the classical operators, these clauses are the standard ones; it is easy to check that this remains true when these operators are generalized to the full language $\mathcal{L}^{\mathsf{Q}}$. As for the inquisitive operators, each of them has the same truth-conditions as its classical counterpart.

4.3.3. Proposition (Truth conditions for inquisitive operators).

- $w \models_g \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi \iff w \models_g \varphi$ *or* $w \models_g \psi$
- $w \models_g \overline{\exists} x\varphi \iff$ *for some* $d \in D$, $w \models_{g[x\mapsto d]} \varphi$

This shows that replacing either inquisitive connective by its classical counterpart may affect the support conditions of the formula, but not its truth-conditions. This allows us to associate with each formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ a classical formula φ^{cl} having the same truth-conditions.

4.3.4. Definition. [Classical variant of a first-order formula]
If φ is a formula in our first-order language $\mathcal{L}^{\mathsf{Q}}$, its *classical variant* is the formula $\varphi^{cl} \in \mathcal{L}^{\mathsf{Q}}_c$ obtained by replacing occurrences of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ by $\vee$, and occurrences of $\overline{\exists}$ by $\exists$.

[6] This is not a problem as long as we are only concerned with the logical notion of entailment, which only requires us to specify in what circumstances a question is settled. However, for some purposes, such as getting at a logical notion of *relatedness* of a response to a question, it is important to be able to distinguish a set of states that settle the question without containing any irrelevant information. One way of doing this is described in Ciardelli (2009), where such a set is obtained via a direct inductive definition; another is described in Ciardelli et al. (2013b), where it is obtained by adopting a more fine-grained view of information states. In particular, it is worth noticing that the definition given in Ciardelli (2009) provides us with a *generator* $T^g_M(\varphi)$ of the proposition $[\varphi]^g_M$ in InqBQ (see Definition 1.2.5) i.e., with a specific way of viewing each formula in our semantics as describing a type of information. This line of work is closely related with the *intuitionistic truth-maker semantics* of Fine (2014), which aims at characterizing the notion of a state being an *exact verifier* of a formula.

4.3.5. PROPOSITION.
For any formula $\varphi \in \mathcal{L}^Q$*, model* M *and assignment* g*:* $w \models_g \varphi \iff w \models_g \varphi^{cl}$

If φ itself is truth-conditional, this implies that φ and φ^{cl} are equivalent. Thus, any truth-conditional formula is equivalent to a classical formula. Conversely, if a formula φ is equivalent to a classical formula, then since classical formulas are truth-conditional, φ must be truth-conditional as well.

4.3.6. PROPOSITION.
For any $\varphi \in \mathcal{L}^Q$*,* φ *is truth-conditional* $\iff \varphi \equiv \alpha$ *for some* $\alpha \in \mathcal{L}^Q_c$

This shows that, while our question-forming operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$ enrich the expressive power of the language, allowing us to form questions, they do not allow us to express any new *statements*. If we only look at truth-conditions, then InqBQ is no more expressive than standard first-order logic.

In Chapter 2, we saw that, like classical formulas, negations are always truth-conditional. Since negation works in exactly the same way in the first-order setting, this will still be true in InqBQ. Since truth-conditions are classical, $\neg\neg\varphi$ is a truth-conditional formula having the same truth-conditions as φ. Thus, $\neg\neg\varphi$ must be equivalent with the classical variant φ^{cl}.

4.3.7. PROPOSITION. *For any* $\varphi \in \mathcal{L}^Q$*,* $\neg\neg\varphi \equiv \varphi^{cl}$

This shows that negations, just like classical formulas, are representative of all truth-conditional formulas in InqBQ.

4.3.8. PROPOSITION.
For any $\varphi \in \mathcal{L}^Q$*,* φ *is truth-conditional* $\iff \varphi \equiv \neg\psi$ *for some* ψ

Now consider questions, which by definition are not truth-conditional. If μ is a question, the classical variant μ^{cl} is not equivalent to μ: rather, μ^{cl} is a statement which expresses the presupposition of μ (cf. Section 1.3). As we did in Chapter 2, with some abuse of terminology will also refer to μ^{cl} as *the presupposition* of μ. Let us illustrate this notion by means of three examples.

- The presupposition of the question $\overline{\exists}xPx$ is the statement $\exists xPx$. This captures the fact that the question $\overline{\exists}xPx$, asking for an instance of a P, is only soluble at worlds where the extension of P is non-empty.

- The presupposition of the question $\overline{\exists}!xPx$ is the statement $\exists!xPx$. This captures the fact that the question $\overline{\exists}!xPx$, asking for the identity of the unique P, is only soluble at worlds where the extension of P is a singleton.

- The presupposition of the question $\forall x?Px$ is the statement $\forall x(Px \vee \neg Px)$, which is a tautology. This captures the fact that the question $\forall x?Px$, which asks for the extension of P, is always soluble at any world.[7]

4.3.3 Resolutions?

A key property of the propositional logic InqB is that we can associate any formula φ with a set $\mathcal{R}(\varphi)$ of classical formulas such that to settle φ is to establish that α is true for some $\alpha \in \mathcal{R}(\varphi)$. Is something similar possible the first-order case? That is, can we define for each first-order formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ a set $\mathcal{R}(\varphi) \subseteq \mathcal{L}^{\mathsf{Q}}_c$ with the property that for any M and g, the following connection holds?

$$s \models_g \varphi \iff s \models_g \alpha \text{ for some } \alpha \in \mathcal{R}(\varphi)$$

The answer is *no*, and for two different reasons. The first reason is that we are not assured to have ways to rigidly designate any individual in the model. Thus, even if a state s settles of some individual d that it has property P—thus supporting the question $\overline{\exists} xPx$—we may not be able to trace this to the support of a classical formula, because we may lack a name for the individual d.

Clearly, this problem may be obviated by extending the language with rigid names for all entities in the domain. But even this does not allow us to associate with all formulas φ a set $\mathcal{R}(\varphi)$ with the required properties. To see this, consider a mention-all question $\forall x?Px$, which as we saw asks for the extension of property P. Even if we have names for all individuals in the domain, we do not in general have names for all possible extensions for a property, as a simple cardinality argument shows. Now, suppose we have no formula stating that the extension of P is a certain set $X \subseteq D$: if s is an information state consisting only of worlds in which the extension of P is X, then the question $\forall x?Px$ will be supported at s, but this will not be traceable to the support of any specific classical formula.

Nevertheless, we will see in Section 4.7 that, for an interesting fragment of the system InqBQ, it is still possible to give a helpful notion of resolutions. Even if only a weak version of the above property obtains, the logical properties of questions in this fragment are still determined by the properties of their set of resolutions, which will provide the key to an interesting completeness result.

4.3.4 Identity

We have set up our models in such a way as to be able to capture uncertainty about the extension of the identity predicate. The fundamental idea is that, if we regard a set of worlds s as an information state, there is no reason why

[7]At least, that is true in our setting, where a property symbol P is assigned an extension at each world; if we also allowed for properties which are undefined at some worlds, the question $\forall x?Px$ may also have a non-trivial presupposition.

the objects to which information is attributed should correspond precisely to the individuals existing in some particular world. Rather, one of the ways in which an information state may be partial is that it may include information about different objects, while leaving open the issue of whether they correspond to the same actual individual. For an example, consider a Greek astronomer who has just discovered the identity of Hesperus and Phosphorus. Before the discovery, this astronomer knew what *Hesperus* referred to, and also knew what *Phosphorus* referred to (he could, say, point to the referent of each name). Yet, prior to the discovery, he might have been wondering whether Hesperus and Phosphorus were in fact the same celestial body. Thus, our astronomer may have been in an information state s characterized by the following features:

$$s \models \overline{\exists} x(x = h) \qquad s \models \overline{\exists} x(x = p) \qquad s \not\models ?(p = h)$$

In our models, such a situation can be captured. It suffices to assume that, in the state s, the constants h and p denote the same epistemic individuals $\mathtt{h}$ and $\mathtt{g}$ at each world, but in some worlds $w \in s$ we have $\mathtt{h} \sim_w \mathtt{g}$, while at other worlds we have $\mathtt{h} \not\sim_w \mathtt{g}$. Intuitively, this captures the fact that our astronomer attributes certain properties to two epistemic objects, $\mathtt{h}$ and $\mathtt{g}$, which are the referents of the names Hesperus and Phosphorus in his state, and is uncertain between worlds in which these objects are two manifestation of the same individual, and worlds in which they correspond to two distinct actual individuals.[8]

The following observation about identity will turn out useful in the following: if it is settled in a state s that two terms t and t' denote the same actual individual, replacing t by t' in a formula does not affect support at s.

4.3.9. Proposition (Substitution of known identicals).
Let $\varphi \in \mathcal{L}^Q$ and let t, t' be two terms free for x in φ. For any M, s and g:

$$s \models_g t = t' \quad \Longrightarrow \quad (\ s \models_g \varphi[t/x] \iff s \models_g \varphi[t'/x]\ )$$

Proof. Straightforward, using the fact that $\sim_w$ is required to be a congruence. □

4.3.5 Dispensing with non-rigid function symbols

In our definition of InqBQ, we have allowed for non-rigid function symbols, whose interpretation in a model may vary from world to worlds. This makes it possible to model uncertainty about the value of a constant, and to model questions asking for this value, as we saw in Section 4.2. However, to provide a proof-system for

[8]In fact, the way in which the relevant individuals are represented plays a crucial role in the interpretation of questions, as demonstrated by Aloni (2001). Capturing this role in a satisfactory way is likely to require formal tools that go beyond those provided by our simple models. Refining our logic with such tools is an enterprise that must be left for future work.

our logic, it will be much more convenient to work with a language which only contains rigid function symbols (and in which, therefore, all terms are rigid).

The advantage of this assumption is that is allows us to handle quantifiers by means of the standard rules. For instance, from a formula $\varphi[t/x]$ we will be allowed to infer $\overline{\exists}x\varphi$, while from a universal $\forall x\varphi$ will be allowed to infer $\varphi[t/x]$.

These inference rules are not sound if t is a non-rigid term. To see that the above $\overline{\exists}$-introduction rule is not sound for non-rigid terms, notice that $Pa \not\models \overline{\exists}xPx$ if a is a non-rigid constant; intuitively, this is because, if we do not know who the constant a refers to, the information that a has property P does not provide us with a witness for property P: it merely establishes a dependency between the question $\overline{\exists}xPx$ and the question $\overline{\exists}x(x = a)$ about the identity of a.

To see that the $\forall$-elimination rule is not sound for non-rigid t, consider the formula $\forall x\overline{\exists}y(x = y)$. This formula is logically valid: for any $d \in D$ we always have a witness for the property of being identical to d, namely, d itself. However, if we instantiate the universal quantifier to a non-rigid constant a, we get $\overline{\exists}y(a = y)$: as we discussed, this formula is not a logical validity, but rather a question asking for the referent of a. Hence, $\forall x\overline{\exists}y(x = y) \not\models \overline{\exists}y(a = y)$.

Fortunately, the assumption that our language only contains rigid function symbols is not restrictive. For, non-rigid function symbols may be dispensed with in a way that is commonly used in classical logic: namely, a non-rigid function symbol f of arity n may be replaced by a relation symbol Z^f of arity $n+1$, appropriately constrained with a uniqueness condition $\forall x_1 \dots \forall x_n \exists! y Z^f(x_1, \dots, x_n, y)$.

As an illustration, consider the case of a non-rigid constant a. Instead of working with a, we can work with a unary predicate Z^a, standing for the property of being the referent of a.[9] With any formula $\varphi \in \mathcal{L}^{\mathsf{Q}}(\mathcal{S} \cup \{a\})$ we may then associate a corresponding formula $\varphi^* \in \mathcal{L}^{\mathsf{Q}}(\mathcal{S} \cup \{Z^a\})$, in which the predicate Z^a occurs instead of a. For instance, we may let:

$$(Pa)^* = \exists x(Z^a x \wedge Px)$$

It is then easy to show that entailments between formulas in the signature $\mathcal{S} \cup \{a\}$ can be re-cast as entailments in the signature $\mathcal{S} \cup \{Z^a\}$ which involve the additional assumption $\exists! x Z^a x$. That is, for any set of formulas $\Phi \cup \{\psi\} \in \mathcal{L}^{\mathsf{Q}}(\mathcal{S} \cup \{a\})$:

$$\Phi \models \psi \iff \Phi^*, \overline{\exists}! x Z^a x \models \psi^*$$

Non-rigid function symbols of positive arity can be eliminated in a similar way.[10]

[9] Less formally, we could characterize Z^a as the property of being the individual called a.

[10] Rigid function symbols f may be dispensed with as well, but the corresponding relation symbols must be constrained not only with a uniqueness axiom $\forall x_1 \dots \forall x_n \exists! y Z^f(x_1, \dots, x_n, y)$, which ensures that the interpretation of Z^f is indeed a function, but also with a rigidity axiom $\forall x_1 \dots \forall x_n \forall y ?Z^f(x_1, \dots, x_n, y)$, which ensures that the interpretation of Z^f is indeed rigid. However, while it will be convenient for our purposes to dispense with non-rigid function symbols, we will have no reason to dispense with rigid function symbols.

4.4 Entailment

Let us now turn to examine the features of the entailment relation to which our first order system gives rise.

4.4.1 Entailment to and from truth-conditional formulas

Many of the features of inquisitive entailment that we discussed in detail in the setting of propositional logic carry over straightforwardly to the first-order case. To start with, entailment towards truth-conditional formulas is truth-conditional.

4.4.1. Proposition (Entailment to a truth-conditional conclusion). *Let $\Phi \cup \{\alpha\} \subseteq \mathcal{L}^Q$, where α is truth-conditional. We have:*

$$\Phi \models \alpha \iff \textit{for any model } M, \textit{ world } w, \textit{ assignment } g: \ w \models_g \Phi \textit{ implies } w \models_g \psi$$

The proof is the same as in the propositional case. In particular, since classical formulas are truth-conditional, and since truth conditions are the same as those given by classical logic (Proposition 4.1.8), entailment among classical formulas coincides with entailment in classical first-order logic.

4.4.2. Proposition (Entailment of classical formulas is classical). *If $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^Q_c$, then $\Gamma \models \alpha \iff \Gamma$ entails α in classical first-order logic*

Also, as in the propositional case, Proposition 4.4.1 implies that, when the conclusion is truth-conditional, any assumption φ may just as well be replaced by its classical variant φ^{cl}.

4.4.3. Proposition. *If α is truth-conditional, for any Φ: $\Phi \models \alpha \iff \Phi^{cl} \models \alpha$*

In particular, this means that a question μ entails all and only the statements that follow from its presupposition. Thus, for instance, the question $\overline{\exists} xPx$ entails its presupposition $\exists xPx$, any statements that follows from it, and no other statements. For another example, consider the mention-all question $\forall x?Px$: we saw that the presupposition of this question is a tautology; thus, it follows that tautologies are the only statements entailed by $\forall x?Px$.

The characterization of entailments *from* truth-conditional assumptions given in Chapter 2 also carries over: a set Γ of truth-conditional formulas entails a formula φ in case, in any model M and with respect to any assignment g, φ is settled in the state $|\Gamma|^g_M$ encoding the information that the formulas in Γ are true.

4.4.4. Proposition (Entailment from truth-conditional assumptions). *Let $\Gamma \cup \{\varphi\} \subseteq \mathcal{L}^Q$, where all formulas in Γ are truth-conditional. We have:*

$$\Gamma \models \varphi \iff \textit{for all models } M \textit{ and assignments } g, \ |\Gamma|^g_M \models_g \varphi$$

In particular, as we saw in Chapter 1, a statement entails a question in case the information that the statement is true suffices to settle the question. For instance, if a is a rigid constant symbol, the statement Pa entails the question $\overline{\exists}xPx$, since it provides an instance of property P, but Pa does not entail $\forall x?Px$, since it does not fully specify the extension of property P.

One interesting question that we will leave open is whether the Split Property discussed in Chapter 2 for InqB still holds. That is, is it still true that, if a set Γ of truth-conditional formulas entails an inquisitive disjunction, it must entail one of the disjuncts? An analogous question arises for the inquisitive existential quantifier. Is it true that, whenever a set Γ of truth-conditional formulas entails an inquisitive existential $\overline{\exists}x\varphi$, this can be traced to the fact that Γ entails $\varphi[t/x]$ for some term t? The workings of the semantics suggest that they do.

4.4.5. Conjecture (Split properties).
Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}^{\mathsf{Q}}$, where all formulas in Γ are truth-conditional.

- $\vee\!\!\!\vee$ split property: *if $\Gamma \models \varphi \vee\!\!\!\vee \psi$, then $\Gamma \models \varphi$ or $\Gamma \models \psi$.*
- $\overline{\exists}$ split property: *if $\Gamma \models \overline{\exists}x\varphi$, then $\Gamma \models \varphi[t/x]$ for some term t.*

From the point of view of a state in a model, these properties are easily provable. That is, we have the following *local* split properties.

4.4.6. Proposition (Local split properties).
Let α be a classical formula, and let φ, ψ be arbitrary formulas.

- $\vee\!\!\!\vee$ split: $(\alpha \to \varphi \vee\!\!\!\vee \psi) \models (\alpha \to \varphi) \vee\!\!\!\vee (\alpha \to \psi)$
- $\overline{\exists}$ split: *if $x \notin FV(\alpha)$*, $(\alpha \to \overline{\exists}x\varphi) \models \overline{\exists}x(\alpha \to \varphi)$

Proof. The proof is completely analogous to that of Proposition 2.5.10. The key is the fact that, if α is truth-conditional, then we have $s \models_g \alpha \to \varphi \iff s \cap |\alpha|^g_M \models \varphi$ (cf. Proposition 2.2.9). This means that, when assessing the conditional at s, there is actually only *one* enhancement of s that we need to look at, namely, $s \cap |\alpha|_M$. It is this state that "decides" the relevant witness for the inquisitive disjunction, or for the inquisitive existential. □

4.4.2 Examples of first-order dependencies

Let us now examine two concrete examples of InqBQ-entailments in which questions play a crucial role. We will use two variations of the hospital protocol example that accompanied our exposition in the previous chapters.

4.4.7. EXAMPLE. Suppose there is a range of diseases a patient may have, and suppose what we have to do is, quite simply, administer all and only the treatments for the diseases that the patient has. We assume the specification of the treatments corresponding to each disease to be given. We can formalize this example in a language equipped with two predicate symbols, H and A, and a relation symbol T, to be interpreted as follows:

- Hx: x is a disease the patient has
- Txy: x is a treatment for disease y
- Ax: x is a treatment to be administered

Now, the background assumption is that we have access to the information about what counts as a treatment for any given disease. Formally speaking, this means that we have access to the extension of the relation T, i.e., that we are given information of type $\forall xy?Txy$.

The instruction to administer all and only the treatments for the diseases the patient has is captured by the following classical formula:

$$\gamma := Ax \leftrightarrow \exists y(Txy \land Hy)$$

Now, under these background assumptions, what diseases a patient has fully determines what treatments must be administered. This is captured by the following inquisitive entailment.

$$\gamma,\ \forall xy?Txy,\ \forall x?Hx\ \models\ \forall x?Ax$$

This entailment is indeed valid, as can be verified by a semantic argument. Moreover, all the formulas involved in this example are part of a fragment of InqBQ that we will call the *mention-all* fragment. In Section 4.8, we will look at this fragment in detail; in particular, we will provide a sound and complete proof system, which ensures that the validity of our entailment may also be given an explicit syntactic proof.

Notice an interesting fact about this example: we used a question assumption not just to capture the "truly indeterminate" information of type $\forall x?Hx$, which will vary from one patient to the other and which cannot be specified once and for all, but also to capture the fact that we have a complete specification of the set of disease-treatment pairs, without taking the trouble of stating what this specification is. For the purpose of establishing that a dependency exists, this does not matter: this shows that questions can also be used as convenient tools to omit irrelevant (and possibly lengthy) details about the available information. In this case, all that matters is that the information we have is of a certain type.

4.4.8. EXAMPLE. Now let us consider yet another variation of our example. Suppose we have now a fixed range of diseases $d_1, \ldots, d_n$, each of which may give rise to a certain set of symptoms, where any symptom is traceable to only one of the diseases. Now, each disease d_i may be countered by a corresponding treatment t_i. Quite simply, if a patient has a symptom for a certain disease, the corresponding treatment should be administered.

Now, in this scenario, as soon as we get the information that the patient has a certain symptom, together with the information about what disease it indicates, we are in the position to prescribe a corresponding treatment. Let us see how this fact can be captured as a case of entailment. We will use rigid constants $d_1, \dots, d_n$ to refer to the diseases, $t_1, \dots, t_n$ to refer to the treatments, two predicate symbols P and A, and a relation symbol S, where:

- Px: x is a symptom the patient presents
- Sxy: x is a symptom of disease y
- Ax: x is a treatment to be administered

The following formula encodes the assumption that, if a patient shows a symptom of a certain disease, the treatment for that disease should be administered.

$$\gamma := \forall x[\,(Sxd_1 \land Px \to At_1)\ \land \dots \land\ (Sxd_n \land Px \to At_n)\,]$$

Now, given γ, the information that a patient has a certain symptom, together with information about what disease this symptom indicates, determines a corresponding treatment. This fact is captured by the following entailment.

$$\gamma,\ \ \overline{\exists} x[Px \land (Sxd_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} Sxd_n)]\ \models\ \overline{\exists} x Ax$$

The formulas involved in this example are part of another fragment of InqBQ, the *mention-some* fragment. In Section 4.7, we will provide a complete axiomatization of this fragment, and we will illustrate this axiomatization by providing a formal proof of this entailment. We will also see that, as in the propositional case, such a proof may be seen as encoding a method for computing the dependency.

4.5 Decidable identity

We saw that our semantics allows for information states where the extension of the identity relation is uncertain. However, in many concrete scenarios, such as the ones of the examples we just considered, the identity of individuals is not at stake. To model this sort of scenarios, we can simply take the relation $\sim_w$ to be the identity relation $id_D = \{\langle d, d\rangle \mid d \in D\}$ at any world. If our information model M is of this form, we will say that M is an id-model.

4.5.1. Definition. [id-models]
An information model $M = \langle W, V, I, \sim\rangle$ is an id-model if for all $w \in W$, $\sim_w = \mathsf{id}_D$.

Now, on id-models, some of the characterizations given above can be simplified. For instance, consider again the question $\overline{\exists}x(x = a)$ asking for the identity of a non-rigid constant a. If our model is an id-model, we can characterize this question as being supported in a state just in case all worlds in the state assign the same referent to a.

$$\begin{aligned} s \models \overline{\exists}x(x = a) &\iff \text{there is some } d \in D \text{ s.t. for all } w \in s,\ I_w(a) = d \\ &\iff \text{for any } w, w' \in s,\ I_w(a) = I_{w'}(a) \end{aligned}$$

Similarly, consider the unique-instance question $\overline{\exists}!xPx$. If our model is an id-model, we can characterize $\overline{\exists}!xPx$ as being supported in a state s iff the extension of P is the same singleton set in all the worlds in s.

$$s \models \overline{\exists}!xPx \iff \text{there is some } d \in D \text{ s.t. for all } w \in s,\ I_w(P) = \{d\}$$

If we are not interested in cases where the extension of identity is at stake, we will want to work with a stronger notion of entailment, one that only takes id-models into account. We will refer to this stronger notion as id-entailment.

4.5.2. Definition. [id-entailment]
$\Phi \models_{\mathsf{id}} \psi \iff$ for all id-models M, all states s in M, and all assignments g, $M, s \models_g \Phi$ implies $M, s \models_g \psi$.

In an id-model, characterizing a predicate in terms of identity with a rigid term counts as settling the extension of the predicate. In terms of id-entailment, this can be expressed as follows.

4.5.3. Proposition.
Let α and β be classical formulas, where β is built up by means of $\bot, \wedge, \rightarrow$, and $\forall$ from identity formulas that only involve rigid terms. Then:

$$\forall x(\alpha \leftrightarrow \beta) \models_{\mathsf{id}} \forall x?\alpha$$

Proof. It is easy to verify that the following entailment is valid on all models, and thus also on all id-models.

$$\forall x(\alpha \leftrightarrow \beta), \forall x?\beta \models \forall x?\alpha$$

We can verify inductively that, for any β constructed as above, the extension of β is always constant throughout the worlds of an id-model. That is means that $\models_{\mathsf{id}} \forall x?\beta$. But clearly, $\forall x(\alpha \leftrightarrow \beta), \forall x?\beta \models_{\mathsf{id}} \forall x?\alpha$ and $\models_{\mathsf{id}} \forall x?\beta$ implies $\forall x(\alpha \leftrightarrow \beta) \models_{\mathsf{id}} \forall x?\alpha$. $\square$

As notable cases of this pattern, we have the following, where $t_1, \ldots, t_n$ are rigid:

- $\forall x[\alpha \leftrightarrow (x{=}t_1 \vee \cdots \vee x{=}t_n)] \models_{\mathsf{id}} \forall\overline{x}?\alpha$
- $\forall x[\alpha \leftrightarrow (x{\neq}t_1 \vee \cdots \vee x{\neq}t_n)] \models_{\mathsf{id}} \forall\overline{x}?\alpha$

The same is not true for general entailment: a formula like $\forall x(Px \leftrightarrow x = t)$, with t rigid, fully ties the extension of P to the extension of identity. But this does not necessarily settle the extension of P if the extension of identity itself is not settled. For instance, we may well have the information that Hesperus is the only planet where a certain chemical x is found, and still wonder whether Phosphorus is a planet where x is found. All our information achieves is to make this question dependent on the question of the identity of Hesperus and Phosphorus.

To get a feel for the role of decidability of identity, it is also illuminating to consider the following pair of examples.

4.5.4. Example. Let a_1, a_2, b_1, b_2, and c be rigid constants, and let γ be the following formula:

$$\gamma = \forall x[Px \to (x{=}a_1 \vee x{=}a_2)] \wedge (Pa_1 \to Qb_1) \wedge (Pa_2 \to Qb_2)$$

That is, γ says that any P coincides with one among a_1 and a_2, and moreover, if a_1 is a P then b_1 is a Q, and if a_2 is a P then b_2 is a Q. Given γ, it might at first seem that from a witness for P we can derive a witness for Q, and so, that we should have $\gamma, \overline{\exists} xPx \models \overline{\exists} xQx$. However, unless we know the extension of the identity relation, that is, unless we can decide whether two individuals are identical, this is not true. For, suppose we are told that c is a witness of P: we know that either $c{=}a_1$, in which case b_1 is a witness for Q, or $c{=}a_2$, in which case b_2 is a witness for P. But without any further information, we don't know which is the case. Thus, we are unable to provide a specific witness for Q. And indeed, by setting up this scenario and computing the semantic clauses, we can show that:

$$\gamma,\ \overline{\exists} xPx\ \not\models\ \overline{\exists} xQx$$

On the other hand, if we can decide whether two given individuals are identical, then we *can* turn any witness for P into a witness for Q: when given c as a witness for P, we just have to decide the identity question $?(c{=}a_1)$: if $c{=}a_1$, then b_1 is a witness for Q; if $c{\neq}a_1$, then γ implies $c{=}a_2$, and b_2 is a witness for Q. And, indeed, it is easy to verify that the following id-entailment is valid:

$$\gamma,\ \overline{\exists} xPx\ \models_{\mathsf{id}}\ \overline{\exists} xQx \qquad \lhd$$

It is interesting to contrast this example with the following minimal variant of it. The difference may at first seem immaterial, but it turns out to be a crucial one.

4.5.5. Example. In the same context as before, let γ' be the following formula:

$$\gamma' = \forall x[Px \to (x{=}a_1 \vee x{=}a_2)] \wedge (Pa_1 \to Qa_1) \wedge (Pa_2 \to Qa_2)$$

Everything is like in the previous example, except that now, if a_1 is a P then it is a_1 *itself* which is a Q, and similarly for a_2. It might seem that just what instance of a Q we are given in each case should not matter. But this is not so. For, γ' entails that $\forall x(Px \to Qx)$. Thus, when given, say, c as witness for P, we can return c itself as a witness for Q. In this case, we don't need to decide whether $c{=}a_1$ or $c{=}a_2$, so the decidability of identity does not play a role, and we have:

$$\gamma',\ \overline{\exists} xPx \models \overline{\exists} xQx \qquad \triangleleft$$

Now, what is the precise relation between plain entailment and id-entailment? To answer this question, let us start with an interesting observation: our language does contain a question that asks for the extension of the identity relation, namely,

$$\forall x \forall y ?(x = y)$$

Indeed, if we spell out the support conditions for this formula, we find that it is supported in a state s in case the extension of the identity relation is settled in s, that is, if for any two epistemic individuals it is settled whether or not they correspond to the same actual individual.

$$\begin{aligned} s \models \forall x \forall y ?(x = y) \quad &\iff \quad \text{for } d, d' \in D \text{ and } w, w' \in s : \ (d \sim_w d') \iff (d \sim_{w'} d') \\ &\iff \quad \text{for all } w, w' \in s : \ \sim_w = \sim_{w'} \end{aligned}$$

This means that we could also characterize a model where there is no uncertainty about the identity relation as one in which this formula is settled in any state. By persistency, this simply means that $\forall x \forall y ?(x = y)$ is supported by the set W of all worlds in the model. Let us say that such a model has *decidable identity.*

4.5.6. Definition. [Decidable identity]
A model $M = \langle W, D, I, \sim \rangle$ has decidable identity in case $M, W \models \forall x \forall y ?(x = y)$.

Clearly, id-models have decidable identity. On the other hand, a model M with decidable identity is not necessarily an id-model: for, M may have the same congruence $\sim_w$ at any world w, and still this congruence might not be the identity relation. However, if this is the case, we may as well simplify M and *make it* an id-model by taking its quotient modulo the relation $\sim_w$.

4.5.7. Definition. [Turning a model with decidable identity into an id-model]
Let $M = \langle W, D, I, \sim \rangle$ have decidable identity. Let us write $\sim$ for $\sim_w$ where w is an arbitrary world, and let us write $[d]$ for the equivalence class of d modulo $\sim$. The *id-contract* of M is the model $M^{\mathsf{id}} = \langle W, D/{\sim}, I^{\sim}, id \rangle$, where:

- $D/{\sim} = \{[d] \,|, d \in D\}$
- $I^{\sim}_w(f)([d_1], \dots, [d_n]) = [I_w(f)(d_1, \dots, d_n)]$

- $\langle [d_1], \ldots, [d_n] \rangle \in I^{\sim}_w(R) \iff \langle d_1, \ldots, d_n \rangle \in I_w(R)$
- $id_w = id_D$ for any $w \in W$

The fact that $\sim$ is a congruence at each world guarantees that this is a good definition, i.e., that the definition of $I^{\sim}_w(f)$ and $I^{\sim}_w(R)$ does not depend on the choice of representatives for each equivalence class. Moreover, M^{id} is an id-model by definition. The following proposition ensures that this transformation does not affect the satisfaction of formulas. The straightforward proof is omitted.

4.5.8. Proposition.
Let $M = \langle W, D, I, \sim \rangle$ have decidable identity and let s be a state and g a valuation. Let $g^{id} : \mathsf{Var} \to W/\!\sim$ be the valuation $x \mapsto [g(x)]$. For any formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$:

$$M, s \models_g \varphi \iff M^{id}, s \models_{g^{id}} \varphi$$

We can now prove that the relation between general entailment and id-entailment is extremely tight. Working with id-entailment is tantamount to adding an explicit assumption of decidability of identity to standard entailment.

4.5.9. Proposition (id-entailment and standard entailment).

$$\Phi \models_{id} \psi \iff \Phi, \forall x \forall y ?(x = y) \models \psi$$

Proof The right-to-left direction of the theorem follows from the fact that $\forall x \forall y ?(x = y)$ is valid on id-models. For the converse, we reason by contraposition. Suppose $\Phi, \forall x \forall y ?(x = y) \not\models \psi$. Then, there must be a model $M = \langle W, D, I, \sim \rangle$ and a state s such that $M, s \models \Phi$ and $M, s \models \forall x \forall y ?(x = y)$, but $M, s \not\models \psi$.

Now consider the model $M_{|s} = \langle s, D, I_{|s}, \sim_{|s} \rangle$ obtained by restricting M to the worlds in s in the obvious way. It is easy to see that for any formula χ, $M, s \models \chi \iff M_{|s}, s \models \chi$. Now, since $M_{|s}, s \models \forall x \forall y ?(x = y)$, the model $M_{|s}$ is an id-model. And since $M_{|s}, s \models \Phi$ but $M_{|s}, s \not\models \psi$, this shows that $\Phi \not\models_{\mathsf{id}} \psi$. □

4.6 Inferences in InqBQ

Let us now turn to the task of providing a proof system for our first-order logic. A sound, but possibly incomplete natural deduction system is described in Figure 4.2. Let us comment briefly on the ingredients of this system.

Connectives and quantifiers As in the propositional system InqB, we may let each of the connectives of our system, $\wedge, \to, \bot$, and $\mathbin{\rotatebox[origin=c]{-90}{$\geq$}}$, be handled by its standard introduction and elimination rules. Furthermore, our treatment of quantification allows us to adopt the standard rules for quantifiers—with one *caveat*: as we saw in Section 4.3.5, we must assume that our language contains only rigid function symbols; for, the rules ($\forall x\mathsf{e}$) and ($\overline{\exists} x\mathsf{i}$) are not sound when applied to a non-rigid term t. However, we saw that this is not a restrictive assumption, since we can replace non-rigid function symbols by suitably constrained relation symbols.

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\varphi]\\ \vdots \\ \psi\end{array}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Universal quantifier

$$\frac{\varphi[y/x]}{\forall x\varphi} \qquad \frac{\forall x\varphi}{\varphi[t/x]}$$

Falsum

$$\frac{\bot}{\varphi}$$

Identity

$$\frac{}{t = t} \qquad \frac{\varphi[t/x] \quad t = t'}{\varphi[t'/x]}$$

Classical $\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \qquad \frac{\psi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \qquad \frac{\varphi \mathbin{\vee\!\!\!\vee} \psi \quad \begin{array}{c}[\varphi]\\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi]\\ \vdots \\ \chi\end{array}}{\chi}$$

Inquisitive existential

$$\frac{\varphi[t/x]}{\overline{\exists} x\varphi} \qquad \frac{\overline{\exists} x\varphi \quad \begin{array}{c}[\varphi[y/x]]\\ \vdots \\ \psi\end{array}}{\psi}$$

$\mathbin{\vee\!\!\!\vee}$-split

$$\frac{\alpha \to \varphi \mathbin{\vee\!\!\!\vee} \psi}{(\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi)}$$

$\overline{\exists}$-split

$$\frac{\alpha \to \overline{\exists} x\varphi}{\overline{\exists} x(\alpha \to \varphi)}\; x \notin \mathrm{FV}(\alpha)$$

Constant domains

$$\frac{\forall x(\varphi \mathbin{\vee\!\!\!\vee} \psi)}{(\forall x\varphi) \mathbin{\vee\!\!\!\vee} \psi}\; x \notin \mathrm{FV}(\psi)$$

Classical negation

$$\frac{\neg\varphi}{\neg\varphi^{cl}}$$

Figure 4.2: A sound but possibly incomplete natural deduction system for InqBQ. In the rules, terms are assumed to be rigid. The variable α ranges over all classical formulas, while φ and ψ range over arbitrary formulas. The restrictions on variables are the familiar ones: in ($\forall$e) and ($\overline{\exists}$i), t must be free for x in φ. In ($\forall$i), x must not occur free in any undischarged assumption. In ($\exists$e), x must not occur free in ψ or in any undischarged assumption.

Identity predicate Standard introduction and elimination rules will also take care of the identity predicate. For, on the one hand, the identity predicate is obviously reflexive—which provides the introduction rule for identity. And, on the other hand, Proposition 4.3.9, ensuring the substitutability of known identicals, gives us the standard elimination rule. Notice that other features of identity, such as symmetry and transitivity, are easily provable by means of these rules.

Connection with intuitionistic logic So far, what we have described is simply a natural-deduction system for intuitionistic first-order logic, with $\mathbin{\vee\!\!\!\vee}$ and $\overline{\exists}$ in the role of intuitionistic disjunction and existential quantifier, respectively. The soundness of these rules implies that anything that is intuitionistically valid is also valid in InqBQ. This observation is worth stating as a proposition.

4.6.1. Proposition (InqBQ includes intuitionistic logic).
If Φ entails ψ in intuitionistic logic when $\mathbin{\vee\!\!\!\vee}$ and $\overline{\exists}$ are identified with intuitionistic disjunction and inquisitive existential respectively, then $\Phi \models \psi$.

On top of this intuitionistic skeleton, InqBQ validates a number of other principles.

Double negation elimination We saw that the classical fragment of InqBQ coincides with classical logic. To capture this, we endow our system with a rule of double negation elimination, whose application is restricted to classical formulas.

Split rules In the propositional case, our system contains the $\mathbin{\vee\!\!\!\vee}$-split rule, which distributes a classical antecedent over an inquisitive-disjunctive consequent. In the first-order case, we will still need this rule, and we also need an analogous $\overline{\exists}$-split rule for the inquisitive existential quantifier. The role of these rules is to capture the local split properties of Proposition 4.4.6. As we saw, these properties express important features of the system: the local $\mathbin{\vee\!\!\!\vee}$-split property captures the fact that, if a statement resolves an inquisitive disjunction $\varphi \mathbin{\vee\!\!\!\vee} \psi$ in a certain context, it must do so by entailing a specific disjunct. Similarly, the logical $\overline{\exists}$-split property captures the fact that, if a statement resolves an inquisitive existential $\overline{\exists} x\varphi(x)$ in a context, it must entail that $\varphi(x)$ holds for a specific individual $d \in D$.

Constant domains In making the domain D of individuals world-independent, we have incorporated into our logic the simplifying assumption that the set of epistemic individuals is fixed. As is known in the field of intermediate logics, assuming a fixed domain has repercussions on the logic, rendering valid any entailment of the form $\forall x(\varphi \mathbin{\vee\!\!\!\vee} \psi) \models (\forall x\varphi) \mathbin{\vee\!\!\!\vee} \psi$, where $x \notin \mathrm{FV}(\psi)$ (see Görnemann, 1971; Gabbay, 1981). To capture the restriction to constant domains, we will thus take this principle to be a rule of our system.

In Section 4.8 we will see that this rule allows us to prove a more specific kind of validity which will play an key role in that section, namely $\forall x?\varphi \models ?\forall x\varphi$.

When φ is a classical formula, this amounts to the fact that settling the extension of a property P implies settling whether the extension of P is the whole domain. This entailment is valid in our setting, where the domain is fixed, but it may fail in situations in which the domain itself is unknown: for instance, we may know that the extension of P is the set $\{d, d'\}$, and yet we may be unsure whether d and d' are the only individuals in the domain.

Classical negation Recall that Proposition 4.3.8 ensures that a negation $\neg\varphi$ is always truth-conditional. Thus, $\neg\varphi$ will be equivalent with its classical variant, $\neg\varphi^{cl}$. The classical negation rule allows us to capture this classicality of negations. To see this, first notice that by means of this rule, the equivalence $\neg\neg\varphi \equiv \varphi^{cl}$ established by Proposition 4.3.7 becomes provable. Writing $\vdash$ for derivability in our system, and $\dashv\vdash$ for inter-derivability, we have the following.[11]

4.6.2. Proposition. *For any $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $\neg\neg\varphi \dashv\vdash \varphi^{cl}$*

Proof. By the classical negation rule, $\neg\varphi \vdash \neg\varphi^{cl}$, which easily gives $\varphi^{cl} \vdash \neg\neg\varphi$. Conversely, suppose $\neg\neg\varphi$. By the classical negation rule we can infer $\neg\neg\varphi^{cl}$. Since φ^{cl} is a classical formula, we can use double negation elimination to infer φ^{cl}. Hence, $\neg\neg\varphi \vdash \varphi^{cl}$. □

It follows from this that our system proves that all negations are classical.

4.6.3. Corollary. *For all $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $\neg\varphi \dashv\vdash \neg\varphi^{cl}$*

Proof. By the previous proposition, from $\neg\varphi^{cl}$ we can infer $\neg\neg\neg\varphi$, whence by standard intuitionistic reasoning we can infer $\neg\varphi$. The converse is immediate by the classical negation rule. □

We can also use Proposition 4.6.2 to show that from a formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ we can always infer its classical variant, φ^{cl}.

4.6.4. Proposition. *For all $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $\varphi \vdash \varphi^{cl}$*

Proof. The proof is by induction on φ. The only interesting case is the inductive step for $\varphi = \psi \to \chi$. The formal proof of $\psi \to \chi \vdash \psi^{cl} \to \chi^{cl}$ goes as follows. Assume $\psi \to \chi$. Assume ψ^{cl}. By the previous lemma, we can infer $\neg\neg\psi$. Using

[11] If we regard InqBQ as an intermediate logic, identifying $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\exists$ with constructive disjunction and existential quantifier, this proposition is an analogue of Glivenko's theorem for intuitionistic propositional logic, which states that $\neg\neg\varphi$ is intuitionistically valid iff φ is classically valid. In first-order intuitionistic logic, Glivenko's theorem fails, essentially due to the fact that $\forall x\neg\neg\varphi \not\models \neg\neg\forall x\varphi$. In InqBQ, however, the entailment $\forall x\neg\neg\varphi \models \neg\neg\forall x\varphi$ is valid, and the classical negation rule allows us to provide a simple proof of it. Conversely, it is possible to show that, by assuming the rule $\forall x\neg\neg\varphi \vdash \neg\neg\forall x\varphi$ instead of our classical negation rule, the latter becomes generally provable.

the assumption $\psi \to \chi$, it is easy to infer $\neg\neg\chi$, which in turn by the previous lemma gives χ^{cl}. Finally we can discharge our assumption ψ^{cl} and conclude $\psi^{cl} \to \chi^{cl}$, as we wanted. $\Box$

This result can be used to show that for the case in which our conclusion is a classical formula, our proof system is complete.[12]

4.6.5. Theorem (Completeness for classical conclusions).
Let $\Phi \subseteq \mathcal{L}^{\mathsf{Q}}$ *and* $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$. *If* $\Phi \models \alpha$, *then* $\Phi \vdash \alpha$.

Proof. Suppose $\Phi \models \alpha$. As classical formulas are truth-conditional, it follows from Proposition 4.4.3 that $\Phi^{cl} \models \alpha$. Since entailment among classical formulas coincides with entailment in classical first-order logic, and since our proof system includes a complete proof system for first-order logic, it follows that $\Phi^{cl} \vdash \alpha$. This means that there must be $\varphi_1, \dots, \varphi_n \in \Phi$ such that $\varphi_1^{cl}, \dots, \varphi_n^{cl} \vdash \alpha$. But the previous proposition ensures that $\varphi_i \vdash \varphi_i^{cl}$ for each $i \leq n$. Hence, $\varphi_1, \dots, \varphi_n \vdash \alpha$, which implies $\Phi \vdash \alpha$. $\Box$

On the other hand, the question of whether our system is complete for the full language is a challenging one, which will not be settled in this thesis. However, in the next sections we will turn our attention to two especially interesting fragments of InqBQ for which our axiomatization can be shown to be complete.

4.7 The mention-some fragment

4.7.1 Definition and basic features

The first fragment that we will investigate consists of formulas built up from classical formulas by means of the operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$, as well as conjunction and conditionalization with a classical antecedent. Following the discussion in Section 4.3.5, we will assume that our signature only contains rigid function symbols.

4.7.1. Definition. [Mention-some fragment]
The mention-some fragment of InqBQ is the set $\mathcal{L}_{\exists}$ defined inductively as follows, where $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$:

$$\varphi \ ::= \ \alpha \,|\, \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \varphi \,|\, \overline{\exists} x \varphi \,|\, \varphi \wedge \varphi \,|\, \alpha \to \varphi$$

Thus, in addition to all formulas of classical first-order logic, the mention-some fragment contains a broad range of questions, including all questions that can be formed by means of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, such as $?\alpha$ and $\alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \beta$, all mention-some questions $\overline{\exists}\overline{x}\alpha(\overline{x})$, asking for an instance of a property or relation, and all unique-instance questions $\overline{\exists}!\overline{x}\alpha(\overline{x})$, asking for the unique instance of a certain property or relation. Thus, e.g., (5-a-d) are all captured in the fragment.

[12]This result is the counterpart in our setting of a result by Kontinen and Väänänen (2013), who axiomatized the classical consequences of first-order dependence logic.

(5) a. Does Alice like Bob?
b. Does Alice like Bob, or Charlie?
c. Who is one person Alice likes?
d. Who is the person Alice likes?

What the fragment does *not* include are formulas of the form $\forall x\mu$ or $\mu \to \varphi$, where $\mu \notin \mathcal{L}_c^{\mathsf{Q}}$, as well as any formula built up from such formulas. Thus, e.g., the formulas $\forall x?Px$ and $\overline{\exists} xPx \to \overline{\exists} xQx$ lie outside of the mention-some fragment.

In this section, we will examine the logic of this fragment. As we will see, this fragment shares many features with the propositional system InqB; this will allow us to establish a completeness result, and also to provide a computational interpretation of proofs analogous to the one investigated in the previous chapter. As a first step in this investigation, recall that in InqB we had an important normal form result: any formula is equivalent with an inquisitive disjunction $\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n$ of classical formulas. Now, one key feature of the mention-some fragment is that it allows for a similar normal form.

4.7.2. Proposition (Normal form for $\mathcal{L}_\exists$).
For any $\varphi \in \mathcal{L}_\exists$ there are classical formulas $\alpha_1, \ldots, \alpha_n$ and tuples of variables $\overline{x}_1, \ldots, \overline{x}_n$ such that:

$$\varphi \equiv \overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n$$

This result can be proved semantically by induction on φ. We will not spell out the proof here, but rather we will regard it as a corollary of a result that we will establish in a moment: given a sound proof system for $\mathcal{L}_\exists$, each formula is provably equivalent to a formula of the form $\overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n$.

This normal form result brings out the fact that all questions in the mention-some fragment express requests of a particular kind: they ask to provide some witness for at least one out of a number of properties or relations.

An interesting fact is that the mention-some fragment essentially contains propositional inquisitive logic, in the sense specified by the following proposition.

4.7.3. Proposition.
Any quantifier-free formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ is equivalent to some $\psi \in \mathcal{L}_\exists$.

Proof. It is easy to see that, just as in the propositional case, any quantifier-free formula φ is equivalent to an inquisitive disjunction $\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \alpha_n$ of classical formulas. By definition, such a disjunction is in $\mathcal{L}_\exists$. □

Moreover, as for propositional logic, we can associate each formula $\varphi \in \mathcal{L}_\exists$ with a set $\mathcal{R}(\varphi)$ of classical formulas that we will refer to as *resolutions*.

4.7.4. Definition. [Resolutions]
Let $\varphi \in \mathcal{L}_\exists$. The set $\mathcal{R}(\varphi)$ of resolutions of φ is defined as follows:

- $\mathcal{R}(\alpha) = \{\alpha\}$ where if $\alpha \in \mathcal{L}^{\mathbf{Q}}_{c}$
- $\mathcal{R}(\varphi \mathbin{\vee\!\!\!\vee} \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$
- $\mathcal{R}(\overline{\exists} x\varphi) = \bigcup\{\mathcal{R}(\varphi[t/x]) \,|\, t \text{ a term free for } x \text{ in } \varphi\}$
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \,|\, \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\alpha \to \varphi) = \{\alpha \to \beta \,|\, \beta \in \mathcal{R}(\varphi)\}$

Notice that the clauses for the propositional connectives are identical to the ones given in the propositional case. Only, the clause for implication can be simplified due to the restriction to classical antecedents. Moreover, we have added a clause for inquisitive existential: to give a resolution of $\overline{\exists} x\varphi$, we need to first instantiate x to some specific term t, and then give a resolution of $\varphi[t/x]$.

At this point, we will have to establish a few technical facts about resolutions that will be needed later on. The first observation says that the set $\mathcal{R}(\varphi)$ is closed under the operation of substituting a term for a variable which is not free in φ.

4.7.5. Lemma.
Suppose $\alpha \in \mathcal{R}(\varphi)$. If $x \notin FV(\varphi)$ and t is free for x in α, then $\alpha[t/x] \in \mathcal{R}(\varphi)$.

Proof. Straightforward, by induction on $\varphi \in \mathcal{L}_{\overline{\exists}}(\mathcal{S})$. $\Box$

The second fact that we will need is that, if we take a resolution of φ and substitute x by a term t, we obtain a resolution of $\varphi[t/x]$.

4.7.6. Lemma.
If $\alpha \in \mathcal{R}(\varphi)$ and t is free for x in φ, then $\alpha[t/x] \in \mathcal{R}(\varphi[t/x])$.

Proof. By induction on the complexity of φ. The only inductive case that requires some work is the one for $\overline{\exists}$. So, let $\varphi = \overline{\exists} y\psi$ and suppose t is a variable free for x in $\overline{\exists} y\psi$. This means, in particular, that t does not contain any occurrences of y, otherwise these occurrences would end up bound in $(\overline{\exists} y\psi)[t/x]$.

Now consider any $\alpha \in \mathcal{R}(\overline{\exists} y\psi)$. By definition of resolutions, this means that $\alpha \in \mathcal{R}(\psi[t'/y])$ for some term t' free for y in ψ. Since $\psi[t'/y]$ has lower complexity than $\overline{\exists} y\psi$, the induction hypothesis applies, yielding $\alpha[t/x] \in \mathcal{R}(\psi[t'/y][t/x])$. Now the crucial observation is that, since y does not occur in t, we have:

$$\psi[t'/y][t/x] = \psi[t/x][t''/y]$$

where $t'' = t'[t/x]$. Notice that t'' is free for y in $\psi[t/x]$: for, if some variable in t'' ended up bound in $\psi[t/x][t/x'']$, this would mean that some variable in t or some variable in t' would end up bound in $\psi[t'/y][t/x]$, which is impossible by the assumptions on the terms t and t'.

We have thus seen that, $\alpha[t/x] \in \mathcal{R}(\psi[t/x][t''/y])$ and t'' is free for y in $\psi[t/x]$. Now, notice that:

$$\mathcal{R}((\overline{\exists} y\psi)[t/x]) = \mathcal{R}(\overline{\exists} y(\psi[t/x])) = \{\psi[t/x][u/y] \mid u \text{ free for } y \text{ in } \psi[t/x]\}$$

Therefore, we can conclude $\alpha[t/x] \in \mathcal{R}((\overline{\exists} y\psi)[t/x])$, as we wanted. $\square$

Finally, we need to establish the following property: under the assumption that $t = t'$, our system proves of each resolution of $\varphi[t/x]$ that it is equivalent to some resolution of $\varphi[t'/x]$, and vice versa.

4.7.7. Lemma. *Let $\varphi \in \mathcal{L}_{\exists}$ and let $\bar{t} = \langle t_1, \ldots, t_n\rangle$ and $\bar{t}' = \langle t_1', \ldots, t_n'\rangle$ be two tuples of terms free for $\overline{x}$ in φ. For any $\alpha \in \mathcal{R}(\varphi[\bar{t}/\overline{x}])$ there is some $\beta \in \mathcal{R}(\varphi[\bar{t}'/\overline{x}])$ such that we have:*

$$(t_1 = t_1') \wedge \cdots \wedge (t_n = t_n') \vdash \alpha \leftrightarrow \beta$$

Proof. By induction on φ. The proof requires some caution with handling substitutions, but is otherwise completely straightforward, and thus omitted. $\square$

In the propositional case, a state supports a formula just in case it supports some resolution of it. The connection between formulas and resolutions is not as strong in the present setting. On the one hand, if a specific resolution is supported, then the formula itself is supported, as can easily be verified by induction.

4.7.8. Proposition. *For all $\varphi \in \mathcal{L}_{\exists}$ and any $\alpha \in \mathcal{R}(\varphi)$, $\alpha \models \varphi$.*

On the other hand, a formula may be supported without any resolution of it being supported. For instance, suppose a state s consists entirely of worlds w where the extension of P is $\{d\}$, but suppose no term in the language has d as its referent. Then, s will support $\overline{\exists} xP(x)$ but it will fail to support a resolution $P(t)$.

However, we do have the desired connection as soon as we allow ourselves access to the witnesses we need. To make this claim precise, let us introduce the following notation: if g and g' are two assignments, let us write $g \equiv_{\varphi} g'$ in case $g(x) = g'(x)$ for any variable $x \in \mathrm{FV}(\varphi)$. Then, we have the following fact.

4.7.9. Proposition.
Let $\varphi \in \mathcal{L}_{\exists}(\mathcal{S})$. For any model M, state s and assignment g:

$$s \models_g \varphi \iff \textit{there is some } \alpha \in \mathcal{R}(\varphi) \textit{ and some } g' \equiv_{\varphi} g \textit{ such that } s \models_{g'} \alpha$$

Proof. The right-to-left direction follows immediately from the previous proposition: if $s \models_{g'} \alpha$ for some resolution $\alpha \in \mathcal{R}(\varphi)$ and for some $g' \equiv_{\varphi} g$, then the previous proposition gives $s \models_{g'} \varphi$ and thus, since $g' \equiv_{\varphi} g$, $s \models_g \varphi$.

Now consider the left-to-right direction. Proposition 4.7.2 tells us that $\varphi \equiv \overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n$, where $\alpha_1, \ldots, \alpha_n$ are classical formulas. We will show in

Proposition 4.7.11 that the variables in the tuples $\overline{x}_1, \dots, \overline{x}_n$ may be taken not to occur free in φ, and that the formulas $\alpha_1, \dots, \alpha_n$ may be taken to be in $\mathcal{R}(\varphi)$.

Now suppose $s \models_g \varphi$. Then $s \models_g \overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n$, which by the semantic clauses for $\vee\!\!\!\vee$ and $\overline{\exists}$ means that $s \models_{g[\overline{x}_i \mapsto \overline{d}_i]} \alpha_i$ for some $1 \leq i \leq n$ and some tuple $\overline{d}_i$ of elements of the appropriate size. Now, α_i is a resolution of φ, and since the variables $\overline{x}_i$ are taken not to occur free in φ, we have $g[\overline{x}_i \mapsto \overline{d}_i] \equiv_\varphi g$. $\square$

We will also need a notion of a resolution of a set $\Phi \subseteq \mathcal{L}_{\overline{\exists}}$ of mention-some formulas. As in Chapter 2, this is simply a set of classical formulas obtained by replacing any formula $\varphi \in \Phi$ by a resolution of it (see 2.4.9 for the formal definition). The set of resolutions of Φ is denoted by $\mathcal{R}(\Phi)$. Notice that, if Γ is a set of classical formulas, then for any $\alpha \in \Gamma$ we have $\mathcal{R}(\alpha) = \{\alpha\}$. As a consequence, the only resolution of Γ is Γ itself: $\mathcal{R}(\Gamma) = \{\Gamma\}$.

4.7.2 Proof system and provable normal form

Let us now turn our attention to inferences in the mention-all fragment. Figure 4.3 describes a proof system for $\mathcal{L}_{\overline{\exists}}$, obtained simply by restricting the proof system of Figure 4.2 for full InqBQ to the fragment we are now considering. The result is a system that consists of the following ingredients:

- the standard inference rules for the operators $\bot, \wedge, \to, \forall$, $\vee\!\!\!\vee$ and $\overline{\exists}$;
- the standard inference rules for the identity predicate;
- the double negation elimination rule restricted to classical formulas, which captures the fact that classical formulas are truth-conditional;
- the $\vee\!\!\!\vee$-split and $\overline{\exists}$-split rules, which capture the local split properties.

Within this section, let us write $\vdash$ for provability in this system. We are now going to prove that our system allows us to turn each formula $\varphi \in \mathcal{L}_{\overline{\exists}}$ into the sort of normal form described by Proposition 4.7.2. Let us start from the easy observation that any formula can be inferred from a resolution of it. The straightforward inductive proof is omitted.

4.7.10. Lemma. *If $\varphi \in \mathcal{L}_{\overline{\exists}}(\mathcal{S})$ and $\alpha \in \mathcal{R}(\varphi)$, then $\alpha \vdash \varphi$.*

We will make use of this lemma in the following provable normal form result.

4.7.11. Proposition (Provable normal form).
For any $\varphi \in \mathcal{L}_{\overline{\exists}}(\mathcal{S})$ there exist classical formulas $\alpha_1, \dots, \alpha_n$ and tuples $\overline{x}_1, \dots, \overline{x}_n$ of variables not occurring free in φ such that:

1. $\varphi \dashv\vdash \overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n$

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad\qquad \frac{\varphi \wedge \psi}{\varphi} \quad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{array}{c}[\alpha] \\ \vdots \\ \varphi\end{array}}{\alpha \to \varphi} \qquad\qquad \frac{\alpha \quad \alpha \to \varphi}{\varphi}$$

Universal quantifier

$$\frac{\alpha[y/x]}{\forall x \alpha} \qquad\qquad \frac{\forall x \alpha}{\alpha[t/x]}$$

Falsum

$$\frac{\bot}{\varphi}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \quad \frac{\psi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \qquad\qquad \frac{\varphi \mathbin{\vee\!\!\!\vee} \psi \quad \begin{array}{c}[\varphi] \\ \vdots \\ \chi\end{array} \quad \begin{array}{c}[\psi] \\ \vdots \\ \chi\end{array}}{\chi}$$

Inquisitive existential

$$\frac{\varphi[t/x]}{\overline{\exists} x \varphi} \qquad \frac{\overline{\exists} x \varphi \qquad \begin{array}{c}[\varphi[y/x]] \\ \vdots \\ \psi\end{array}}{\psi}$$

Identity

$$\frac{}{t = t} \qquad\qquad \frac{\varphi[t/x] \quad t = t'}{\varphi[t'/x]}$$

Classical $\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

$\mathbin{\vee\!\!\!\vee}$-split

$$\frac{\alpha \to \psi \mathbin{\vee\!\!\!\vee} \chi}{(\alpha \to \varphi) \mathbin{\vee\!\!\!\vee} (\alpha \to \psi)}$$

$\overline{\exists}$-split

$$\frac{\alpha \to \overline{\exists} x \varphi}{\overline{\exists} x (\alpha \to \varphi)}$$

Figure 4.3: A sound and complete natural-deduction system for the mention-some fragment of InqBQ. In these rules, α ranges over all classical formulas, while φ and ψ range over arbitrary formulas in $\mathcal{L}_{\overline{\exists}}$. In the rules for quantifiers, the usual restrictions apply. In the $\overline{\exists}$-split rule, x must not occur free in α.

2. *$\alpha_1, \ldots, \alpha_n \in \mathcal{R}(\varphi)$*

3. *for any tuple $\overline{t}$ of closed terms, $\alpha_i[\overline{t}/\overline{x}_n] \in \mathcal{R}(\varphi)$*

Proof. By induction on the formula $\varphi \in \mathcal{L}_{\overline{\exists}}(\mathcal{S})$. The base case is obvious: if $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$, we have $\alpha \dashv\vdash \alpha$, and conditions 2 and 3 are satisfied trivially.

For the inductive step, suppose the claim holds for φ and ψ. That is, assume:

$$\varphi \dashv\vdash \overline{\exists}\overline{x}_1\alpha_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \overline{\exists}\overline{x}_n\alpha_n \qquad \psi \dashv\vdash \overline{\exists}\overline{y}_1\beta_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \overline{\exists}\overline{y}_m\beta_m$$

where the two normal forms satisfy the statement of the lemma. Without loss of generality, we may furthermore assume that the variables in $\overline{x}_i$ do not occur, either free or bound, in any formula $\overline{\exists}\overline{y}_j\beta_j$, and that, conversely, the variables in $\overline{y}_i$ do not occur in any formula $\overline{\exists}\overline{x}_j\alpha_j$. If this is not the case, we simply use the rules for $\overline{\exists}$ to rename the problematic variables to fresh ones. Clearly, this does not affect condition 1. Moreover, since the variables in $\overline{x}_i$ do not occur free in φ by induction hypothesis, Lemma 4.7.5 ensures that this does not affect conditions 2 and 3 either. Now let us consider the various inductive steps.

- The inductive step for $\mathbin{\vee\!\!\!\vee}$ is obvious. Given the induction hypothesis, using the rules for $\mathbin{\vee\!\!\!\vee}$ we can see that:

  $$\varphi \mathbin{\vee\!\!\!\vee} \psi \dashv\vdash \overline{\exists}\overline{x}_1\alpha_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \overline{\exists}\overline{x}_n\alpha_n \mathbin{\vee\!\!\!\vee} \overline{\exists}\overline{y}_1\beta_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \overline{\exists}\overline{y}_m\beta_m$$

  and each α_i and β_j satisfies the statement of the lemma by induction hypothesis and the fact that $\mathcal{R}(\varphi \mathbin{\vee\!\!\!\vee} \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

- Consider the inquisitive existential formula $\overline{\exists}y\varphi$. Without loss of generality, we may assume that the variables in each tuple $\overline{x}_i$ are distinct from y—if not, we can simply use the rules for $\overline{\exists}$ to rename the variables $\overline{x}_i$. Using the rules of our system, it is not hard to show that we have:

  $$\overline{\exists}y\varphi \dashv\vdash \overline{\exists}y\overline{x}_1\alpha_1 \mathbin{\vee\!\!\!\vee} \ldots \mathbin{\vee\!\!\!\vee} \overline{\exists}y\overline{x}_n\alpha_n$$

  The formulas $\alpha_1, \ldots, \alpha_n$ satisfy Condition 2, since $\alpha_i \in \mathcal{R}(\varphi) \subseteq \mathcal{R}(\overline{\exists}y\varphi)$. As for Condition 3, consider any sequence $t\overline{t}'$ of closed terms of the same size as the sequence of variables $y\overline{x}_i$. Since $t\overline{t}'$ are *closed* terms, it holds that $\alpha_i[t\overline{t}'/y\overline{x}_i] = (\alpha_i[\overline{t}'/\overline{x}_i])[t/y]$. By induction hypothesis we know that $\alpha_i[\overline{t}'/\overline{x}_i] \in \mathcal{R}(\varphi)$. By Lemma 4.7.6, then, $(\alpha_i[\overline{t}'/\overline{x}_i])[t/y] \in \mathcal{R}(\varphi[t/y])$. Since $\mathcal{R}(\varphi[t/y]) \subseteq \mathcal{R}(\overline{\exists}y\varphi)$ by definition of resolutions, this shows that $\alpha_i[t\overline{t}'/y\overline{x}_i] \in \mathcal{R}(\overline{\exists}y\varphi)$, as we wanted.

- Consider the conjunction $\varphi \wedge \psi$. Using the rules of our system, it is easy to show that:

  $$\varphi \wedge \psi \dashv\vdash \textstyle\bigvee\!\!\!\bigvee_{i \leq n, j \leq m} \overline{\exists}\overline{x}_i\overline{y}_j(\alpha_i \wedge \beta_j)$$

Notice that, since $\alpha_i \in \mathcal{R}(\varphi)$ and $\beta_j \in \mathcal{R}(\psi)$ by induction hypothesis, we have $\alpha_i \wedge \beta_j \in \mathcal{R}(\varphi \wedge \psi)$, so Condition 2 is satisfied. As for Condition 3, consider any sequence $\overline{t}$ of closed terms of the same size as $\overline{x}_i\overline{y}_j$. This means that $\overline{t}$ consists of a sequence $\overline{t}_1$ of the same size as $\overline{x}_i$ and of a sequence $\overline{t}_2$ of the same size as $\overline{y}_j$. Now since we made sure that the variables $\overline{x}_i$ do not occur in β_j and the variables $\overline{y}_j$ do not occur in α_i, we have:

$$(\alpha_i \wedge \beta_j)[\overline{t}/\overline{x}_i\overline{y}_j] = (\alpha_i \wedge \beta_j)[\overline{t}_1\overline{t}_2/\overline{x}_i\overline{y}_j] = \alpha_i[\overline{t}_1/\overline{x}_i] \wedge \beta_j[\overline{t}_2/\overline{y}_j]$$

By induction hypothesis we have $\alpha_i[\overline{t}_1/\overline{x}_i] \in \mathcal{R}(\varphi)$ and $\alpha_j[\overline{t}_2/\overline{y}_j] \in \mathcal{R}(\psi)$, which, by definition of the resolutions for a conjunction, finally implies $(\alpha_i \wedge \beta_j)[\overline{t}/\overline{x}_i\overline{y}_j] \in \mathcal{R}(\varphi \wedge \psi)$, as required.

- Finally, consider the implication $\gamma \to \varphi$, where $\gamma \in \mathcal{L}_c^{\mathsf{Q}}$. We will assume that no variable in any of the tuples $\overline{x}_i$ occurs free in γ; if not, just rename all such variables to fresh ones not occurring in γ or in φ. Using our proof system, and in particular by making use of the split rules, we can show:

$$\gamma \to \varphi \dashv\vdash \overline{\exists} x_1(\gamma \to \alpha_1) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \overline{\exists} x_n(\gamma \to \alpha_n)$$

Since $\alpha_i \in \mathcal{R}(\varphi)$ by induction hypothesis, we have $\gamma \to \alpha_i \in \mathcal{R}(\gamma \to \varphi)$, and so Condition 2 is satisfied. It remains to be shown that for any closed term $\overline{t}$ we have $(\gamma \to \alpha_i)[\overline{t}/\overline{x}_i] \in \mathcal{R}(\gamma \to \varphi)$. This is easy: since the variables in $\overline{x}_i$ do not occur free in γ, $(\gamma \to \alpha_i)[\overline{t}/\overline{x}_i] = \gamma \to \alpha_i[\overline{t}/\overline{x}_i]$; by induction hypothesis, $\alpha_i[\overline{t}/\overline{x}_i] \in \mathcal{R}(\varphi)$, whence $\gamma \to \alpha_i[\overline{t}/\overline{x}_i] \in \mathcal{R}(\gamma \to \varphi)$. □

4.7.3 Constructive content of proofs

The next thing that we want to show is that the Resolution Algorithm described by the proof of Theorem 3.2.1 can be extended to the mention-some fragment. Thus, whenever we are given a proof $P : \overline{\varphi} \vdash \psi$ of an entailment, we can define recursively on P a method F_P which, when given resolutions $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$, returns a proof $F_P(\overline{\alpha}) : \overline{\alpha} \vdash \beta$, where $\beta \in \mathcal{R}(\psi)$.

4.7.12. Theorem (Existence of a Resolution Algorithm).
Let $\overline{\varphi}, \psi \subseteq \mathcal{L}_{\exists}(\mathcal{S})$ and let $P : \overline{\varphi} \vdash \psi$. Recursively on P, we can define a procedure which turns any $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$ into a proof $Q : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\overline{\varphi})$.

In proving our theorem, we will make use of the following technical lemma, which tells us that we can turn a proof of $\overline{\varphi} \vdash \psi$ into a proof of $\overline{\varphi}[t/x] \vdash \psi[t/x]$ without altering its complexity.[13] Since this lemma does not depend on any specific features of our system, but only on general facts about the use of variables in natural deduction systems, we will omit the details of the proof.

[13]As a measure of the complexity of a proof, we take the size of its longest branch. However, it seems that other notions, as the total number of inference steps, would also do for our purposes.

4.7.13. Lemma. *If $P : \overline{\varphi} \vdash \psi$, x is a variable and t is a term free for x in $\overline{\varphi}$ and ψ, there exists a proof $Q : \overline{\varphi}[t/x] \vdash \psi[t/x]$ of the same complexity as P.*

Equipped with this lemma, we are now ready to describe how the Resolution Algorithm of Theorem 3.2.1 can be extended to the mention-some fragment.

Proof of Theorem 4.7.12. We proceed by induction on the complexity of the proof $P : \overline{\varphi} \vdash \psi$. The base case in which P consists of an undischarged assumption is trivial. For the inductive step, we need to consider a number of cases depending on the last rule applied in P. If the last rule applied in P is a rule for the connectives $\wedge$, $\rightarrow$, $\bot$, or $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$, or if it is an instance of $\neg\neg$-elimination or of $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$-split, then the algorithm proceeds as in the propositional case.[14] Now let us consider the case in which the last rule in P is one of the remaining rules of our deduction system.

- $\psi = \forall x\beta$ was obtained by ($\forall$i) from $\beta[y/x]$, where β is a classical formula and the variable y does not occur in $\overline{\varphi}$ or in β. Then, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \beta[y/x]$.

  Now consider any resolution $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$. The subtlety here is that, although y does not occur free in $\overline{\varphi}$, it may occur free in $\overline{\alpha}$. So, let us consider a different variable z which does not occur free in $\overline{\alpha}, \overline{\varphi}$, or β. Now, if we replace any free occurrence of y in P' by z we obtain a proof $P'' : \overline{\varphi} \vdash \beta[z/x]$. Notice that since y did not occur free in $\overline{\varphi}$, we did not change our assumptions. Moreover, the substitution does not alter the complexity of the proof, and so it is still possible to apply our induction hypothesis. Since $\beta[z/x]$ is a classical formula and thus the only resolution of itself, this gives us a proof $Q'' : \overline{\alpha} \vdash \beta[z/x]$.

  Finally, since we have made sure that z does not occur free in $\overline{\alpha}$ or β, we can extend Q'' with an application of $\forall$i and obtain $Q : \overline{\alpha} \vdash \forall x\beta$, which is what we wanted, since $\forall x\beta$ is a resolution of itself.

- $\psi = \beta[t/x]$ was obtained by ($\forall$e) from $\forall x\beta$. Then the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \forall x\beta$. Now consider any resolutions $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$. Since $\forall x\beta$ is the only resolution of itself, the induction hypothesis gives us a proof $Q' : \overline{\alpha} \vdash \forall x\beta$. Finally, by extending Q' with an application of ($\forall$e) we obtain a proof $Q : \overline{\alpha} \vdash \beta[t/x]$, which is what we wanted, since $\beta[t/x]$, being a classical formula, is a resolution of itself.

- $\psi = (t = t)$ was obtained by (=i). Then, $\mathcal{R}(\psi) = \{\psi\}$; given any $\overline{\alpha} \in \mathcal{R}(\psi)$ we can take Q to consist simply of an application of (=i) to conclude $(t = t)$.

- $\psi = \chi[t'/x]$ was obtained by (=e) from $\chi[t/x]$ and $(t = t')$. Then, the immediate subproofs of P are proofs $P' : \overline{\varphi} \vdash \chi[t/x]$ and $P'' : \overline{\varphi} \vdash (t = t')$.

[14] The description of the steps corresponding to the rules ($\rightarrow i$) and ($\rightarrow e$) can be simplified, since we are restricted to classical antecedents, which have themselves as their unique resolution.

Now, let $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$. The induction hypothesis on P' gives us a proof $Q' : \overline{\alpha} \vdash \beta$, for $\beta \in \mathcal{R}(\chi[t/x])$. Moreover, since $(t{=}t')$ is the only resolution of itself, the induction hypothesis on P'' gives us a proof $Q'' : \overline{\alpha} \vdash (t{=}t')$.

Now, Lemma 4.7.7 ensures that we can find a resolution $\gamma \in \mathcal{R}(\chi[t'/x])$ such that $(t = t') \vdash \beta \leftrightarrow \gamma$. Now, by plugging in our proof $Q'' : \overline{\alpha} \vdash (t = t')$ for the assumption $(t = t')$ of a proof $(t = t') \vdash \beta \leftrightarrow \gamma$ we obtain a proof $Q''' : \overline{\alpha} \vdash (\beta \leftrightarrow \gamma)$. Finally, by extending the proofs $Q' : \overline{\alpha} \vdash \beta$ and $Q''' : \overline{\alpha} \vdash \beta \leftrightarrow \gamma$ by means of $(\wedge\mathsf{e})$ and $(\rightarrow\mathsf{e})$, we obtain the desired proof $Q : \overline{\alpha} \vdash \gamma$, where $\gamma \in \mathcal{R}(\chi[t'/x]) = \mathcal{R}(\psi)$.

- $\psi = \overline{\exists} x\chi$ was obtained by $(\overline{\exists}\mathsf{i})$ from $\chi[t/x]$, where t is free for x in χ. Thus, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \chi[t/x]$. Now take any $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$. The induction hypothesis gives us a proof $Q : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\chi[t/x])$. By definition of resolutions, $\beta \in \mathcal{R}(\overline{\exists} x\chi)$, which shows that Q is a proof of the desired kind.

- ψ was obtained by $(\overline{\exists}\mathsf{i})$ from $\overline{\exists} x\chi$. Then the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash \overline{\exists} x\chi$ and a proof $P'' : \overline{\varphi}, \chi[y/x] \vdash \psi$, where y is free for x in χ and not occurring free in $\overline{\varphi}, \chi$ or ψ.

  Now consider resolutions $\overline{\alpha} \in \mathcal{R}(\overline{\varphi})$. We may assume without loss of generality that y does not occur in $\overline{\alpha}$. (If y does occur in $\overline{\alpha}$, we can just replace all free occurrences of y in P'' by occurrences of a fresh variable z, and work with the resulting proof $P''' : \overline{\varphi}, \chi[z/x] \vdash \psi$.)

  Now our induction hypothesis applied to P' gives us a proof $Q' : \overline{\alpha} \vdash \beta$ for some $\beta \in \mathcal{R}(\overline{\exists} x\chi)$. By definition of resolutions, $\beta \in \mathcal{R}(\chi[t/x])$ for some term t free for x in χ.

  By Lemma 4.7.13, we have a proof $P''' : \overline{\varphi}[t/y], (\chi[y/x])[t/y] \vdash \psi[t/y]$ of the same complexity as P''. Now since by assumption y does not occur free in $\overline{\varphi}, \chi$, or ψ, this means that $P''' : \overline{\varphi},\, \chi[t/x] \vdash \psi$. Moreover, since P''' has the same complexity as P'', the induction hypothesis is applicable to P'''. Now, since $\beta \in \mathcal{R}(\chi[t/x])$, we have $\overline{\alpha}, \beta \in \mathcal{R}(\overline{\varphi}, \chi[t/x])$. Thus, the induction hypothesis on P''' gives us a proof $Q'' : \overline{\alpha}, \beta \vdash \delta$ for some $\delta \in \mathcal{R}(\psi)$.

  Finally, replace any undischarged assumption of β in Q'' by an occurrence of the proof $Q' : \overline{\alpha} \vdash \beta$. The result is a proof $Q : \overline{\alpha} \vdash \delta$, where $\delta \in \mathcal{R}(\psi)$.

- $\psi = \overline{\exists} x(\alpha \rightarrow \chi)$ was obtained by $\overline{\exists}$−split from $\alpha \rightarrow \overline{\exists} x\chi$, where $x \notin \mathrm{FV}(x)$. Then, the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash \alpha \rightarrow \overline{\exists} x\chi$. Now consider any $\overline{\beta} \in \mathcal{R}(\overline{\varphi})$: the induction hypothesis gives us a proof $Q : \overline{\beta} \vdash \gamma$, where $\gamma \in \mathcal{R}(\alpha \rightarrow \overline{\exists} x\chi)$. But it is easy to check that, since $x \notin \mathrm{FV}(\alpha)$, we have $\mathcal{R}(\alpha \rightarrow \overline{\exists} x\chi) = \mathcal{R}(\overline{\exists} x(\alpha \rightarrow \chi))$. So, we have $\gamma \in \mathcal{R}(\overline{\exists} x(\alpha \rightarrow \chi))$, which shows that Q itself is a proof of the desired kind. $\square$

As in the propositional case, this result implies that any proof $P : \Phi \vdash \psi$ in our fragment encodes a resolution function, i.e., a function $f_P : \mathcal{R}(\Phi) \to \mathcal{R}(\psi)$ which maps each $\Gamma \in \mathcal{R}(\Phi)$ to a resolution $f_P(\Gamma) \in \mathcal{R}(\psi)$ with the property that $\Gamma \models f_P(\Gamma)$. So, like proofs in InqB, proofs in the present system may be seen as encoding methods for computing dependencies.

To illustrate this, let α be a classical formula. We saw that the entailment $\alpha, \overline{\exists}xPx \models \overline{\exists}xQx$ captures the fact that, on the basis of α, any witness t for property P determines a corresponding witness t' for property Q. Our result shows that a proof $P : \alpha, \overline{\exists}xPx \vdash \overline{\exists}xQx$ of this entailment encodes a method for computing this witness t'. More precisely, such a proof determines a function that turns any formula of the form $P(t)$ into a formula of the form $Q(t')$ with the property that $\alpha, Pt \models Qt'$.

Notice that since a set of classical formulas is the only resolution of itself, Theorem 4.7.12 yields the following corollary, which will play an important role in our completeness proof: if a set Γ of classical formulas derives a formula ψ, then Γ must derive some specific resolution of ψ.

4.7.14. Corollary (Provable resolution split).
Let $\Gamma \cup \{\psi\} \subseteq \mathcal{L}_{\overline{\exists}}(\mathcal{S})$, where Γ is a set of classical formulas. If $\Gamma \vdash \psi$, then $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

In turn, as immediate consequences of this corollary we get the following facts, which will be crucial in our completeness proof: a set of classical formulas derives an inquisitive disjunction only if it derives one of the disjuncts; and it derives an inquisitive existential $\overline{\exists}x\varphi(x)$ only if it derives $\varphi(t)$ for some specific term t.

4.7.15. Corollary (Provable Split).
Let $\Gamma \cup \{\varphi, \psi\} \subseteq \mathcal{L}_{\overline{\exists}}$, where Γ is a set of classical formulas.

- *If $\Gamma \vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$, then $\Gamma \vdash \varphi$ or $\Gamma \vdash \psi$.*
- *If $\Gamma \vdash \overline{\exists}x\varphi$, then $\Gamma \vdash \varphi[t/x]$ for some term t in $\mathcal{L}^{\mathsf{Q}}$.*

Proof. Suppose $\Gamma \vdash \varphi \mathbin{\vee\!\!\!\vee} \psi$. By the previous corollary, $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\varphi \mathbin{\vee\!\!\!\vee} \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$. If $\alpha \in \mathcal{R}(\varphi)$, then by Lemma 4.7.10 we have $\alpha \vdash \varphi$, and thus also $\Gamma \vdash \varphi$. Similarly, if $\alpha \in \mathcal{R}(\psi)$ we have $\Gamma \vdash \psi$. Now suppose $\Gamma \vdash \overline{\exists}x\varphi$. By the previous corollary we have $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\varphi[t/x])$. By Lemma 4.7.10 we have $\alpha \vdash \varphi[t/x]$, and thus also $\Gamma \vdash \varphi[t/x]$. $\square$

4.7.4 Completeness

Let us now turn to proving that our proof system completely axiomatized entailment between formulas in the mention-some fragment. At the end of this section, we will establish the following theorem.

4.7.16. THEOREM (SOUNDNESS AND COMPLETENESS).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\exists}(\mathcal{S})$, $\Phi \models \psi \iff \Phi \vdash \psi$

The proof follows roughly the same strategy as in the propositional case but, as we will see, it involves some interesting complications. To start with, recall that in the propositional case, one of our key results was that whenever $\Phi \nvdash \psi$, we can trace this to the fact that some particular resolution of Φ does not derive ψ. Intuitively, this captures the fact that, if information of type Φ is not guaranteed to yield information of type ψ, this is because there is some particular piece of information of type Φ—a resolution—which does not yield information of type ψ.

In the present setting, this result does not hold in general. To see how it may fail, consider the following set of formulas:

$$\Phi = \{\neg P(t) \mid t \in \mathsf{Ter}(\mathcal{S})\} \cup \{\overline{\exists} x P(x)\}$$

Clearly, this set is consistent, so by soundness, $\Phi \nvdash \bot$. However, a resolution of Φ is a set of the form $\Gamma = \{\neg P(t) \mid t \in \mathsf{Ter}(\mathcal{S})\} \cup \{P(t_0)\}$ for some term $t_0 \in \mathsf{Ter}(\mathcal{S})$. It is clear that for any such Γ we have $\Gamma \vdash \bot$.

The problem is that, in general, in order to instantiate an inquisitive existential $\overline{\exists} x\varphi$ to a suitable resolution, we may need a new name for the witness for x, a name about which the rest of our assumptions imply nothing specific. If we allow ourselves access to an infinite stock of fresh constants, then the problem disappears: now, any assumptions that fail to derive a conclusion can be instantiated to resolutions which do not derive the conclusion.

4.7.17. DEFINITION. [Extended language]
Let $\mathcal{S}^+$ be the signature obtained by expanding our original signature $\mathcal{S}$ with countably many new rigid constants, $c_i, i \in \mathbb{N}$. Let us denote by $\mathcal{R}^+(\psi)$ the set of resolutions of a formula ψ as computed in the extended language $\mathcal{L}^{\mathsf{Q}}(\mathcal{S}^+)$.

4.7.18. LEMMA.
Suppose $\Phi, \psi \nvdash \chi$. *If there are infinitely many constants in our signature that do not occur in* Φ *or* χ, *we can find an* $\alpha \in \mathcal{R}(\psi)$ *s.t.* $\Phi, \alpha \nvdash \chi$.

Proof. Suppose $\Phi, \psi \nvdash \chi$ and let $\overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n$ be the normal form of ψ as given by Proposition 4.7.11. Since $\psi \dashv\vdash \overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n$, we must have:

$$\Phi,\ \overline{\exists}\overline{x}_1\alpha_1 \vee\!\!\!\vee \ldots \vee\!\!\!\vee \overline{\exists}\overline{x}_n\alpha_n\ \nvdash\ \chi$$

Now, by the rule ($\vee\!\!\!\vee$e), this implies that $\Phi, \overline{\exists}\overline{x}_i\alpha_i \nvdash \chi$ for some $i \leq n$. Now let $\overline{c}$ be a sequence of constants which do not occur in Φ or χ, of the same size as $\overline{x}_i$. By Proposition 4.7.11, $\alpha_i[\overline{c}/\overline{x}_i] \in \mathcal{R}(\psi)$. Moreover, since the constants in $\overline{c}$ do not occur in Φ or χ, we must have $\Phi, \alpha_i[\overline{c}/\overline{x}_i] \nvdash \chi$. For, suppose towards a contradiction that there was a proof $P : \Phi, \alpha_i[\overline{c}/\overline{x}_i] \vdash \chi$. Then we could replace throughout the proof the constant $\overline{c}_i$ with a corresponding fresh variable z_i. This would give us a proof $P' : \Phi, \alpha_i[\overline{z}/\overline{x}_i] \vdash \chi$. Since each variable z_i does not occur free in Φ or χ, we could then apply ($\exists$e) to obtain a proof $P'' : \Phi, \overline{\exists}\overline{x}_i\alpha_i \vdash \chi$, which is impossible, since $\Phi, \overline{\exists}\overline{x}_i\alpha_i \nvdash \chi$. Thus, $\alpha_i[\overline{c}/\overline{x}_i]$ is the resolution we needed. $\square$

By using this lemma inductively, we can show that if Φ fails to derive ψ we can always trace this to some specific resolution of Φ failing to derive ψ.

4.7.19. Lemma (Traceable deduction failure).
Let $\Phi, \psi \subseteq \mathcal{L}_{\exists}(\mathcal{S})$. *If* $\Phi \nvdash \psi$, *there exists a resolution* $\Gamma \in \mathcal{R}^+(\Phi)$ *s.t.* $\Gamma \nvdash \psi$.

Proof. The proof is analogous to the proof of Lemma 3.3.7, using the previous lemma inductively to produce a resolution of the whole set of assumptions.[15] $\square$

We can now see that, if we look at resolutions in the extended language $\mathcal{L}^{\mathsf{Q}}(\mathcal{S}^+)$, then we have the same crucial connection between formulas and resolutions that we used in the previous chapter to establish completeness: Φ derives ψ if and only if any resolution of Φ derives some corresponding resolution of ψ.

4.7.20. Lemma (Resolution Lemma).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\exists}(\mathcal{S})$:

$$\Phi \vdash \psi \iff \textit{for all } \Gamma \in \mathcal{R}^+(\Phi) \textit{ there is some } \alpha \in \mathcal{R}^+(\psi) \textit{ such that } \Gamma \vdash \alpha$$

Proof. First consider the left-to-right direction. If $\Phi \vdash \psi$, then we have some proof $P : \Phi \vdash \psi$. Given any $\Gamma \in \mathcal{R}^+(\Phi)$, the existence of a resolution algorithm ensures that $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}^+(\psi)$.

For the converse, suppose $\Phi \nvdash \psi$. By the Traceable Deduction Failure Lemma, we have a resolution $\Gamma \in \mathcal{R}^+(\Phi)$ such that $\Gamma \nvdash \psi$. But by Lemma 4.7.10, we have $\alpha \vdash \psi$ for any $\alpha \in \mathcal{R}^+(\psi)$. It follows that Γ derives no resolution $\alpha \in \mathcal{R}^+(\psi)$. $\square$

We now have all the ingredients to prove the completeness of our system by constructing a suitable canonical model. As in the propositional case, the possible worlds in the canonical model will be complete theories of classical formulas. However, now we also need to impose another requirement on these theories, familiar from the completeness proof for classical logic: they need to be *Henkin theories*, that is, they need to contain suitable witnesses for all the existential formulas they contain. Since in our language it is $\forall$ that is taken as primitive, we will formulate this property as follows: if the theory contains a formula of the form $\neg\forall x\varphi$, it must also contain the formula $\neg\varphi[c/x]$ for some constant c.

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[15] Recall that we made the simplifying assumption that our language is countable. If we lifted this assumption, this proof would have to be replaced by an argument using ordinal induction, and the enriched signature $\mathcal{S}^+$ should contain as many fresh constants as there are formulas in our initial language.

4.7.21. Definition. [Complete Henkin theories]
A *complete Henkin theory* is a set Γ of classical first-order formulas with the following properties.

- Closure under classical deduction: if $\Gamma \vdash \alpha$ and α is classical, then $\alpha \in \Gamma$;
- Consistency: $\bot \notin \Gamma$;
- Completeness: for any classical α, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$;
- Henkin property: if $\neg\forall x\alpha \in \Gamma$, there is a constant c such that $\neg\alpha[c/x] \in \Gamma$.

It is a well-known fact that any consistent set of classical first-order formulas in a certain signature can be extended to a complete Henkin theory in a signature having as many fresh constant symbols as there are formulas in the original language. Since we are assuming our language to be countable, countably many fresh constants will suffice.

4.7.22. Definition. [Expanding the language, again]
Let $\mathcal{S}^{++}$ be the signature obtained by expanding $\mathcal{S}^{+}$ with new constants $(c'_i)_{i\in\mathbb{N}}$, and let $\mathcal{R}^{++}(\varphi)$ denote the set of resolutions of φ as computed in $\mathcal{S}^{++}$.

For our purposes, it will be useful to state the Henkin Lemma in a slightly stronger form, as concerning not just consistent sets Γ in the language $\mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{+})$, but any consistent sets Γ which such that infinitely many c'_i do not occur in Γ.

4.7.23. Lemma (Henkin Lemma).
Let $\Gamma \subseteq \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$ be a consistent set such that infinitely many of the constants c'_i do not occur in Γ. Then $\Gamma \subseteq \Delta$ for some complete Henkin theory $\Delta \subseteq \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$.

In fact, for our completeness proof we need a stronger fact about complete Henkin extensions: a formula is in any complete Henkin extension of a given Γ if and only if it follows from Γ. Even though this is just a fact about classical first-order logic, it may not be familiar to the reader in this form, so I include a proof sketch.

4.7.24. Definition. [Set of Henkin extensions]
If $\Gamma \subseteq \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$, let us denote by S^Γ the set of all complete Henkin theories in $\mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$ which include Γ.

Let us denote by $\bigcap S^\Gamma$ the set of formulas which belong to each $\Delta \in S^\Gamma$, with the convention that if $S^\Gamma = \emptyset$, then $\bigcap S^\Gamma = \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$.

4.7.25. Lemma (Henkin Extensions Lemma).
Let $\Gamma \subseteq \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$ be a set in which only finitely many of the constants c'_i occur. Then for any classical formula $\alpha \in \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$:

$$\alpha \in \bigcap S^\Gamma \iff \Gamma \vdash \alpha$$

Proof. If $\Gamma \vdash \alpha$, then obviously α will be an element of any theory which includes Γ, so $\alpha \in \bigcap S^{\Gamma}$. For the converse, suppose $\Gamma \nvdash \alpha$. Since α is a classical formula, this means that $\Gamma \cup \{\neg\alpha\}$ is consistent. Now, since there are countably many constants c_i' in our signature S^{++} which do not occur in $\Gamma \cup \{\neg\alpha\}$, we can use them to extend $\Gamma \cup \{\neg\alpha\}$ to a complete Henkin theory $\Delta \subseteq \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$ in the standard way. By construction, $\Gamma \cup \{\neg\alpha\} \subseteq \Delta$, which means that $\Delta \in S^{\Gamma}$ and $\alpha \notin \Delta$. This shows that $\alpha \notin \bigcap S^{\Gamma}$, as we wanted. $\square$

Notice that the standard Henkin Lemma can be seen as a particular case of the Henkin Extensions Lemma: if $\Gamma \nvdash \bot$, then $\bot \notin \bigcap S^{\Gamma}$, so S^{Γ} must be non-empty. We are now finally ready to define our canonical model for $\mathcal{L}_{\exists}(\mathcal{S})$.

4.7.26. Definition. [Canonical Model]
The canonical model for a signature $\mathcal{S}$ is the tuple $M^c = \langle W^c, D^c, I^c, \sim^c\rangle$, where:

- W^c is the set of complete Henkin theories in the language $\mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$;
- D^c is the set of closed terms in $\mathcal{L}^{\mathsf{Q}}(\mathcal{S}^{++})$, i.e., terms containing no variables;
- if f is an n-ary function symbol and $\Gamma \in W^c$:

$$I^c_\Gamma(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$$

- if R is an n-ary relation symbol and $\Gamma \in W^c$:

$$\langle t_1, \ldots, t_n\rangle \in I^c_\Gamma(R) \iff R(t_1, \ldots, t_n) \in \Gamma$$

- if $\Gamma \in W^c$, $t \sim^c_\Gamma t' \iff (t = t') \in \Gamma$

One peculiarity of the canonical model is that an assignment into the domain can be simulated by a syntactic replacement of the free variables by closed terms. To state this concisely, we will use the following notational convention.

Notation. Let $g : \mathsf{Var} \to D^c$. If $\mathrm{FV}(\varphi) = \{x_1, \ldots, x_n\}$, we let:

$$\varphi_g \;:=\; \varphi[g(x_1)/x_1, \ldots, g(x_n)/x_n]$$

It is easy to verify that, in the canonical model, support of a formula φ relative to an assignment g amounts to support of the sentence φ^g.

4.7.27. Lemma.
For all formulas $\varphi \in \mathcal{L}_{\exists}(\mathcal{S}^{++})$, all states $S \subseteq W^c$, and all assignments g:

$$S \models_g \varphi \iff S \models \varphi_g$$

This yields the following corollary, stating that in evaluating a formula φ, to alter the relevant assignment is tantamount to performing a syntactic substitution.

4.7.28. Lemma. *For any $\varphi \in \mathcal{L}_{\exists}(\mathcal{S}^{++})$, any $t \in D^c$ and any $S \subseteq W^c$, we have:*

$$S \models_{g[x \mapsto t]} \varphi \iff S \models_g \varphi[t/x]$$

We will also make use of the observation that the intersection of a set of worlds in the canonical model is closed under deduction of classical formulas.

4.7.29. Lemma. *Let $S \subseteq W^c$ and $\alpha \in \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$. If $\bigcap S \vdash \alpha$, then $\alpha \in \bigcap S$.*

Equipped with these observations, we can now turn to the Support Lemma, which connects support at a state S in the canonical model to provability from the "common core" of the theories in S.

4.7.30. Lemma (Support Lemma).
For all $\varphi \in \mathcal{L}_{\exists}(\mathcal{S}^{++})$, all states $S \subseteq W^c$, and all assignments g:

$$S \models_g \varphi \iff \bigcap S \vdash \varphi_g$$

Proof of Lemma 4.7.30. We proceed by induction on the complexity of $\varphi \in {\mathcal{L}_{\exists}}^{+}$, simultaneously for all states S and assignments g. The base cases, concerning atomic formulas of the form $R(t_1, \ldots, t_n)$ or $(t = t')$, follow immediately from the definition of the canonical model and the lemmata we have just established. The inductive steps for the connectives $\bot$, $\wedge$, and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ are exactly parallel to those we gave in the propositional case. Let us now examine the remaining cases.

- Suppose the claim holds for a classical formula $\alpha \in \mathcal{L}_c^{\mathsf{Q}}(\mathcal{S}^{++})$ and let us prove it also holds for $\forall x \alpha$.[16] First suppose $\bigcap S \vdash (\forall x \alpha)_g$. By the rule ($\forall$e), for any term $t \in D^c$ we have $\bigcap S \vdash (\alpha[t/x])_g$. By induction hypothesis, this means that $S \models_g \alpha[t/x]$, which amounts to $S \models_{g[x \mapsto t]} \alpha$ by Lemma 4.7.28. So, for any $t \in D^c$ we have $S \models_{g[x \mapsto t]} \alpha$, which shows that $S \models_g \forall x \alpha$.

  Conversely, suppose $\bigcap S \not\vdash (\forall x \alpha)_g$. Then there must be some $\Gamma \in S$ such that $(\forall x \alpha)_g \notin \Gamma$. Since Γ is a complete theory we must have $(\neg \forall x \alpha)_g \in \Gamma$; moreover, since Γ has the Henkin property we must have $(\neg \alpha[c/x])_g \in \Gamma$ for some constant c; finally, since Γ is consistent we must have $(\alpha[c/x])_g \notin \Gamma$. Since $\Gamma \in S$, we have $(\alpha[c/x])_g \notin \bigcap S$. By lemma 4.7.29, $\bigcap S \not\vdash (\alpha[c/x])_g$. By the induction hypothesis, this yields $S \not\models_g \alpha[c/x]$, which by Lemma 4.7.28 amounts to $S \not\models_{g[x \mapsto c]} \alpha$. This shows that $S \not\models_g \forall x \alpha$.

[16] Recall that in the language $\mathcal{L}_{\exists}(\mathcal{S}^{+})$, the universal quantifier is restricted to range over classical formulas, so this is the only case that we need to consider in our inductive proof.

- Suppose the claim holds for formulas $\alpha \in \mathcal{L}^{\mathsf{Q}}_c(\mathcal{S}^{++})$ and $\varphi \in \mathcal{L}_{\exists}(\mathcal{S}^{++})$, and consider the implication $\alpha \to \varphi$. First suppose $\bigcap S \vdash (\alpha \to \varphi)_g$. Now consider any state $T \subseteq S$ such that $T \models_g \alpha$. The induction hypothesis gives $\bigcap T \vdash \alpha_g$. Since $T \subseteq S$, we have $\bigcap S \subseteq \bigcap T$, and since $\bigcap S \vdash (\alpha \to \varphi)_g$, also $\bigcap T \vdash (\alpha \to \varphi)_g$. Now, from $\bigcap T$ we can derive both α_g and $(\alpha \to \varphi)_g$, which is identical to $\alpha_g \to \varphi_g$; by the rule $(\to\mathsf{e})$, we thus have $\bigcap T \vdash \varphi_g$. Finally, the induction hypothesis gives $T \models_g \varphi$. Since this holds for all $T \subseteq S$, we can conclude $S \models_g \alpha \to \varphi$.

  Conversely, suppose $S \models_g \alpha \to \varphi$. Let $T_{\alpha_g} = \{\Gamma \in S \mid \alpha_g \in \Gamma\}$. Now, since $\alpha_g \in \bigcap T_{\alpha_g}$, we have $\bigcap T_{\alpha_g} \vdash \alpha_g$, which by the induction hypothesis implies $T_{\alpha_g} \models_g \alpha$. Since $T_{\alpha_g} \subseteq S$ and $S \models_g \alpha \to \varphi$, we have $T_{\alpha_g} \models_g \varphi$.

  Now, the induction hypothesis on φ gives us $\bigcap T_{\alpha_g} \vdash \varphi_g$. Since $\bigcap T_{\alpha_g}$ is a set of classical formulas, Corollary 4.7.14 ensures that $\bigcap T_{\alpha_g} \vdash \beta$ for some resolution $\beta \in \mathcal{R}^{++}(\varphi_g)$, which by Lemma 4.7.29 amounts to $\beta \in \bigcap T_{\alpha_g}$. So, for any $\Gamma \in T_{\alpha_g}$ we have $\beta \in \Gamma$, which implies $\alpha_g \to \beta \in \Gamma$, since Γ is closed under deduction of classical formulas and $\beta \vdash \alpha_g \to \beta$.

  Now consider any $\Gamma \in S - T_{\alpha_g}$: this means that $\alpha_g \notin \Gamma$; then since Γ is complete we have $\neg\alpha_g \in \Gamma$, whence $\alpha_g \to \beta \in \Gamma$, because Γ is closed under deduction of classical formulas and $\neg\alpha_g \vdash \alpha_g \to \beta$.

  We have thus shown that $\alpha_g \to \beta \in \Gamma$ for any $\Gamma \in S$, whether $\Gamma \in T_\alpha$ or $\Gamma \in S - T_\alpha$. We can then conclude $\alpha_g \to \beta \in \bigcap S$, whence $\bigcap S, \alpha_g \vdash \beta$. Since $\beta \vdash \varphi_g$ by Lemma 4.7.10, we also have $\bigcap S, \alpha_g \vdash \varphi_g$, which by the rule $(\to\mathsf{i})$ implies $\bigcap S \vdash \alpha_g \to \varphi_g$. Since $\alpha_g \to \varphi_g$ coincides with $(\alpha \to \varphi)_g$, this is the conclusion we wanted.

- Finally, consider the formula $\overline{\exists}x\varphi$. First suppose $S \models_g \overline{\exists}x\varphi$. This means that there is some $t \in D^c$ such that $S \models_{g[x\mapsto t]} \varphi$, that is, by Lemma 4.7.28, such that $S \models_g \varphi[t/x]$. Now the induction hypothesis gives $\bigcap S \vdash (\varphi[t/x])_g$, whence by the rule $(\overline{\exists}\mathsf{i})$ we have $\bigcap S \vdash (\overline{\exists}x\varphi)_g$.

  Conversely, suppose $\bigcap S \vdash (\overline{\exists}x\varphi)_g$. Lemma 4.7.15 ensures that for some term t we have $\bigcap S \vdash (\varphi[t/x])_g$. Our induction hypothesis gives $S \models_g \varphi[t/x]$, which by Lemma 4.7.28 implies $S \models_{g[x\mapsto t]} \varphi$, showing that $S \models_g \overline{\exists}x\varphi$. $\square$

The Support Lemma is thus proved. Notice that, if φ is a sentence, then $\varphi_g = \varphi$, so the Support Lemma simply states that, for $S \subseteq W^c$:

$$S \models \varphi \iff \bigcap S \vdash \varphi$$

However, as in the propositional case, there is still some work to do in order to prove the completeness theorem. If $\Phi \not\vdash \psi$, we cannot directly try to extend the theory $\Phi \cup \{\neg\psi\}$, as in classical logic: for on the one hand, even if $\Phi \not\vdash \psi$, the set $\Phi \cup \{\neg\psi\}$ may be inconsistent. Moreover, what we need in order to disprove the entailment $\Phi \models \psi$ is a state in our canonical model, i.e., a set of theories of classical formulas, while $\Phi \cup \{\psi\}$ will in general contain questions. In order to construct the state we need, we need to rely on the connection between a formula and its resolutions, and in particular on the Traceable Deduction Failure Lemma.

Proof of Theorem 4.7.16. First let us establish our completeness theorem for *sentences*. Suppose $\Phi \cup \{\psi\} \subseteq \mathcal{L}_\exists(\mathcal{S})$ is a set of sentences and suppose $\Phi \not\vdash \psi$. By the Traceable Deduction Failure Lemma (4.7.19) there is a resolution $\Gamma \in \mathcal{R}^+(\Phi)$ such that $\Gamma \not\vdash \alpha$ for any $\alpha \in \mathcal{R}^+(\psi)$. Now let $S^\Gamma \subseteq W^c$ be the state consisting of all complete Henkin extensions of Γ. We claim that $S^\Gamma \models \Phi$ but $S^\Gamma \not\models \psi$.

To see that $S^\Gamma \models \Phi$, consider any $\varphi \in \Phi$. Since $\Gamma \in \mathcal{R}(\Phi)$, Γ contains a resolution α of φ. Now, by construction, α is contained in any extension of Γ, and thus, $\alpha \in \bigcap S^\Gamma$. Since $\alpha \vdash \varphi$ by Lemma 4.7.10, we have $\bigcap S^\Gamma \vdash \varphi$, which by the Support Lemma gives $S^\Gamma \models \varphi$. This establishes $S^\Gamma \models \Phi$.

To see that $S^\Gamma \not\models \psi$, suppose towards a contradiction that $S^\Gamma \models \psi$. By the Support Lemma, this would mean $\bigcap S^\Gamma \vdash \psi$. Now since $\bigcap S^\Gamma$ is a set of classical formulas, Corollary 4.7.14 ensures that $\bigcap S^\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}^{++}(\psi)$. By Lemma 4.7.29, we thus have $\alpha \in \bigcap S^\Gamma$; and by the Henkin Extension Lemma (4.7.25), it follows that $\Gamma \vdash \alpha$. But then, since $\alpha \vdash \psi$ by Lemma 4.7.10, we would have $\Gamma \vdash \psi$, while Γ was chosen such that $\Gamma \not\vdash \psi$. We can thus conclude $S^\Gamma \not\models \psi$.

Summing up, we have shown that at the state S^Γ in the canonical model, all the sentences in Φ are supported, while ψ is not. This shows that $\Phi \not\models \psi$.

Now consider the case in which the formulas in $\Phi \cup \{\psi\}$ may contain free variables. If $\varphi \in \mathcal{L}_\exists(\mathcal{S})$, denote by φ^* be the sentence obtained by replacing each variable x_i with a corresponding fresh constant c_i. It is easy to see that if $\Phi \models \psi$, then $\Phi^* \models \psi^*$. Since the latter is an entailment among sentences, our completeness result for sentences implies that $\Phi^* \vdash \psi^*$. But it is easy to see that a proof $P : \Phi^* \vdash \psi^*$ can be straightforwardly turned into a proof $Q : \Phi \vdash \psi$. $\Box$

This theorem shows that at least an interesting fragment of the logic InqBQ, which includes many of the first-order questions discussed in the previous sections, shares the good logical behavior displayed by propositional inquisitive logic: in particular, we have a complete proof system consisting of simple inference rules; moreover, proofs in this system have an interesting constructive interpretation, which allows us to regard them as encoding dependence functions.

4.7.5 Illustration, repercussions, and an open problem

Illustration For an example of our proof system at work, recall the dependency discussed in Example 4.4.8. The context is provided by the classical formula:

$$\gamma := \forall x[(Sxd_1 \land Px \to At_1) \land \cdots \land (Sxd_n \land Px \to At_n)]$$

which expresses the fact that, if a patient presents a symptom which indicates disease d_i, treatment t_i should be administered. In the presence of γ, establishing a symptom of the patient, together with what disease it indicates, implies establishing some corresponding treatment to be administered. This is witnessed by the validity of the following entailment:

$$\gamma,\ \overline{\exists}x[Px \land (Sxd_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} Sxd_n)] \models \overline{\exists}xAx$$

Both questions involved in this example, as well as the classical formula γ, belong to the mention-some fragment. Thus, this entailment can be formally proved in our system. A proof of our entailment is displayed below. Parts of the proof that just involve inferences in classical logic have been omitted and denoted $(C_1), \ldots, (C_n)$. Moreover, the formula $Sxd_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} Sxd_n$ has been abbreviated as $\mathbb{V}_i Sxd_i$, and most rule labels have been omitted.

$$\dfrac{\overline{\exists}x(Px \land \mathbb{V}_i Sxd_i) \qquad \dfrac{\dfrac{\dfrac{\gamma \quad [Syd_1] \quad \dfrac{[Py \land \mathbb{V}_i Syd_i]}{Py}}{At_1}\,(C_1)}{\overline{\exists}xAx} \quad \ldots \quad \dfrac{\dfrac{\gamma \quad [Syd_n] \quad \dfrac{[Py \land \mathbb{V}_i Syd_i]}{Py}}{At_n}\,(C_n)}{\overline{\exists}xAx} \quad \dfrac{[Py \land \mathbb{V}_i Syd_i]}{\mathbb{V}_i Syd_i}}{\overline{\exists}xAx}\,(\mathbb{V}\mathsf{e})}{\overline{\exists}xAx}\,(\overline{\exists}\mathsf{e})$$

The resolution algorithm ensures that this proof encodes a function which takes information of type $\overline{\exists}x[Px \land (Sxd_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} Sxd_n)]$ and returns information of type $\overline{\exists}xAx$ on the basis of γ. Intuitively, we can read the proof from the bottom up as encoding the following procedure.

We are given the assumption γ together with information of type $\overline{\exists}x[Px \land (Sxd_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} Sxd_n)]$, and we have to return information of type $\overline{\exists}xAx$. Let y be the object for which we have information of type $Py \land (Syd_1 \mathbin{\mathbb{V}} \ldots \mathbin{\mathbb{V}} Syd_n)$. This means that we have the information that Py, and we also have the information that Syd_i for some $i \leq n$. Consider each of these n cases. If we have the information that Py and that Syd_1, then we can infer from γ that At_1; this is information of type $\overline{\exists}xAx$, as required. And similarly for the other $n-1$ cases.

Thus, the dependence function that we could automatically extract from this proof is, as we expect, the function that maps a resolution $Pu \land Sud_i$ of the question assumption to the resolution At_i of the conclusion.

Repercussions Theorem 4.7.16 also allows us to infer some important properties of the mention-some fragment of InqBQ. First, since a proof always has finitely many assumptions, the compactness of the logic follows.

4.7.31. COROLLARY (COMPACTNESS).
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\exists}(\mathcal{S})$. If $\Phi \models \psi$, there are $\varphi_1, \ldots, \varphi_n \in \Phi$ s.t. $\varphi_1, \ldots, \varphi_n \models \psi$.

Second, the connections between formulas and resolutions that we established for provability can now be transferred to entailment. In particular, the Resolution

Lemma shows how entailment among formulas in $\mathcal{L}_\exists$ is rooted in entailment in classical first-order logic.

4.7.32. Corollary (Resolution Theorem).
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}_\exists(\mathcal{S})$.

$$\Phi \models \psi \iff \textit{for all } \Gamma \in \mathcal{R}^+(\Phi) \textit{ there is some } \alpha \in \mathcal{R}^+(\psi) \textit{ s.t. } \Gamma \models \alpha$$

As a particular case, we obtain the Resolution Property (cf. Proposition 2.5.8), which follows immediately from Lemma 4.7.14: if a set of classical formulas entails a formula $\varphi \in \mathcal{L}_\exists$, then it must entail a particular resolution of it. Notice that for the case in which $\Gamma = \emptyset$, this tells us that the validity of a formula in $\mathcal{L}_\exists$ is always traceable to the validity of a specific resolution.[17]

4.7.33. Corollary (Resolution Property).
Let $\Gamma \in \mathcal{L}_c^Q(\mathcal{S})$ *and* $\varphi \in \mathcal{L}_\exists(\mathcal{S})$. *If* $\Gamma \models \varphi$, *then* $\Gamma \models \alpha$ *for some* $\alpha \in \mathcal{R}(\psi)$.

In turn, the Resolution Property ensures us that, at least for formulas in $\mathcal{L}_\exists$, the split properties that we conjectured for the whole language do indeed hold.

4.7.34. Corollary (Split Properties). *Let* $\Gamma \subseteq \mathcal{L}_c^Q(\mathcal{S})$ *and* $\varphi, \psi \in \mathcal{L}_\exists(\mathcal{S})$:

- $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-split: *if* $\Gamma \models \varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi$, *then* $\Gamma \models \varphi$ *or* $\Gamma \models \psi$;
- $\overline{\exists}$-split: *if* $\Gamma \models \overline{\exists} x\varphi$, *then* $\Gamma \models \varphi[t/x]$ *for some term* t.

Proof. Analogous to the proof of Lemma 4.7.15, using the previous corollary. □

Notice that by taking $\Gamma = \emptyset$ in this corollary, we obtain both the disjunction property for $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and the existence property for $\overline{\exists}$.

An open problem: id-entailment Let us close this section with a problem. We would like to extend our completeness result to the relation of id-entailment, which incorporates the assumption that the identity relation is decidable. We may expect this to pose no problem, as Proposition 4.5.9 implies that id-entailment coincides with standard entailment with the extra assumption $\forall x \forall y ?(x = y)$. But this formula does not belong to $\mathcal{L}_\exists$, so it is not covered by our completeness result.

However, it seems that from the point of view of the mention-some fragment, the decidability of identity surfaces only via the validity of two classes of polar questions, which suggests that these questions may be assumed as axioms for id-entailment, rather than the stronger but inexpressible $\forall x \forall y ?(x = y)$.

[17] In this case, the relevant resolution can be taken from the language $\mathcal{L}_c^Q(\mathcal{S})$. This is because the left-to-right direction is based on the Resolution Algorithm (Theorem 4.7.12), which does not require an extension of the language.

First, if the extension of identity is settled, then given two terms t and t', it is settled whether or not they refer to the same individual (recall that we are assuming a signature in which all terms are rigid). Thus, a polar question of the form $?(t = t')$ is always settled in an id-model.

Second, if the extension of identity is settled, it is also settled how many individuals there are in the domain. Let κ_n be the usual first-order formula stating that the domain contains at least n individuals: $\kappa_n := \exists x_1 \ldots \exists x_n \bigwedge_{i \neq j}(x_i \neq x_j)$ Then, each polar question of the form $?\kappa_n$ is always settled in an id-model.

Furthermore, it seems that valid id-entailments between mention-some formulas can generally be proved by means of our proof system, if we can use questions of the form $?(t = t')$ and $?\kappa_n$ as axioms. This suggests the following conjecture.

4.7.35. CONJECTURE.
The relation $\models_{\mathsf{id}}$ of id-entailment in the mention-some fragment is completely axiomatized by supplementing the system of Figure 4.3 with the following axioms.

- *Decidable identity axioms:* $?(t = t')$ *for all* $t, t' \in \mathsf{Ter}(\mathcal{S})$
- *Decidable cardinality axioms:* $?\kappa_n$ *for all* $n \in \mathbb{N}$

As an illustration of proofs involving identity axioms $?(t = t')$, consider again the id-entailment of Example 4.5.4. The context was given by the classical formula γ:

$$\gamma = \forall x[Px \to (x = a_1 \lor x = a_2)] \land (Pa_1 \to Qb_1) \land (Pa_2 \to Qb_2)$$

We remarked that, if the extension of the identity relation is settled, γ allows us to turn any witness for P into a witness for Q. This is captured by the following id-entailment:

$$\gamma,\ \overline{\exists} xPx \models_{\mathsf{id}} \overline{\exists} xQx$$

If we allow ourselves to use any identity question $?(t = t')$ as an axiom, we can provide a proof of this id-entailment. This proof is displayed below, where subproofs involving only classical logic have been omitted and denoted (C_1) and (C_2).

$$\dfrac{\overline{\exists} xPx \qquad \dfrac{\dfrac{}{?(z = a_1)}\,(\mathsf{id}) \qquad \dfrac{\dfrac{\gamma \quad [Pz] \quad [z = a_1]}{Qb_1}\,(\mathsf{C}_1)}{\overline{\exists} xQx}\,(\overline{\exists}\mathsf{i}) \qquad \dfrac{\dfrac{\gamma \quad [Pz] \quad [z \neq a_1]}{Qb_2}\,(\mathsf{C}_2)}{\overline{\exists} xQx}\,(\overline{\exists}\mathsf{i})}{\overline{\exists} xQx}\,(\mathbb{V}\mathsf{e})}{\overline{\exists} xQx}\,(\overline{\exists}\mathsf{e})$$

This proof describes how we can obtain a witness for Q from a witness for P, given γ and given an oracle that decides identities. Intuitively, we can read the proof from the bottom up, as follows: let z be the given witness for P; use the identity oracle to decide whether $(z = a_1)$; if the information obtained from the oracle is that $z = a_1$, then b_1 is a witness of Q; if the information obtained from the oracle is $z \neq a_1$, then b_2 is a witness for Q.

In general, given a proof that makes use of identity axioms $?(t = t')$ or cardinality axioms $?\kappa_n$, it will not be possible to obtain a resolution of the conclusion from any given resolution of the assumptions. This is easily seen, since these axioms themselves are provable from the empty set of assumptions, but no resolution of them is provable. Rather, what we can always do, using the algorithm of Theorem 4.7.12, is to obtain a resolution of the conclusion when given resolutions of the assumptions *and* resolutions of each of the questions $?(t=t')$ or $?\kappa_n$ used as axioms in the proof.

4.8 The mention-all fragment

4.8.1 Definition and basic features

In the last section we have focused on the *mention-some* fragment of InqBQ, exploring its logical properties and providing a complete axiomatization of it. This fragment contains many interesting kinds of questions, including all sorts of propositional questions formed by means of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and the connectives, mention-some questions, and unique-instance questions. Among the kinds of questions discussed in Section 4.2, the one notable class of questions that the mention-some fragment does *not* include is the class of *mention-all* questions, which ask for the extension of a certain property or relation. In this chapter, we will look in detail at the fragment of InqB obtained by extending classical first-order logic with precisely these questions—plus polar questions.[18] We will refer to this fragment as the *mention-all* fragment, and denote it $\mathcal{L}_\forall(\mathcal{S})$.

4.8.1. Definition. [Mention-all fragment]
The set $\mathcal{L}_\forall$ of mention-all formulas is defined as follows, where $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$:

$$\varphi \ ::= \ \alpha \,|\, ?\alpha \,|\, \forall x\varphi$$

Thus, the formulas in $\mathcal{L}_\forall$ are either classical formulas, or formulas of the form $\forall\overline{x}?\alpha$, where $\overline{x}$ is a (possibly empty) tuple of variables, and α is a classical formula. We will denote by $\mathcal{L}_\forall^?$ the set of formulas of this second kind; that is, $\mathcal{L}_\forall^? = \mathcal{L}_\forall - \mathcal{L}_c^{\mathsf{Q}}$. Moreover, we will write $\mathcal{L}_\forall^\sigma$ for the set of sentences in $\mathcal{L}_\forall$, i.e., formulas without free variables. As in the previous section, we will assume that our language only contains rigid function symbols, so that all terms in the language are rigid.

Now, let M be an information model and g an assignment. If α is a classical formula and $\overline{x} = \langle x_1, \dots, x_n\rangle$ a tuple of variables, let $\alpha_g^{\overline{x}}$ be the intensional relation determined by α with respect to $\overline{x}$, i.e., the function which maps any world $w \in W$ to the set of tuples $\overline{d} \in D^n$ that satisfy α in w relative to g.

$$\alpha_g^{\overline{x}}(w) = \{\overline{d} \in D^n \,|\, w \models_{g[\overline{x}\mapsto\overline{d}]} \alpha\}$$

[18]In fact, polar questions are naturally seen as limit cases of a mention-all question asking for the extension of a sentence, that is, of a 0-place property.

Clearly, if α contains no variables besides those in $\overline{x}$ then the assignment g plays no role, so we can drop reference to it. The following proposition states that, given a classical formula α and a tuple $\overline{x}$ of variables, the question $\forall\overline{x}?\alpha$ asks for the extension of the relation $\alpha_g^{\overline{x}}$. That is, the question is settled in a state s if any two worlds $w, w' \in s$ agree on the extension they assign to $\alpha_g^{\overline{x}}$.

4.8.2. Proposition (Support-conditions for mention-all questions). *Let $\forall\overline{x}?\alpha \in \mathcal{L}_\forall^?$. For any information model M, state s and assignment g:*

$$s \models_g \forall\overline{x}?\alpha \iff \textit{for all } w, w' \in s : \alpha_g^{\overline{x}}(w) = \alpha_g^{\overline{x}}(w')$$

Proof. We have the following sequence of equivalences.

$$\begin{aligned}
s \models_g \forall x?\alpha &\iff \text{for all } \overline{d} \in D^n : \; s \models_{g[\overline{x}\mapsto\overline{d}]} \alpha \text{ or } s \models_{g[\overline{x}\mapsto\overline{d}]} \neg\alpha \\
&\iff \text{for all } \overline{d} \in D^n : \; (\text{for all } w \in s,\, \overline{d} \in \alpha_g^{\overline{x}}(w)) \text{ or} \\
&\qquad\qquad\qquad\qquad (\text{for all } w \in s,\, \overline{d} \notin \alpha_g^{\overline{x}}(w)) \\
&\iff \text{for all } \overline{d} \in D^n,\; \text{for all } w, w' \in s : \; \overline{d} \in \alpha_g^{\overline{x}}(w) \iff \overline{d} \in \alpha_g^{\overline{x}}(w') \\
&\iff \text{for all } w, w' \in s,\; \text{for all } \overline{d} \in D^n : \; \overline{d} \in \alpha_g^{\overline{x}}(w) \iff \overline{d} \in \alpha_g^{\overline{x}}(w') \\
&\iff \text{for all } w, w' \in s : \alpha_g^{\overline{x}}(w) = \alpha_g^{\overline{x}}(w')
\end{aligned}$$ □

This result shows that we can decide whether the formula $\forall\overline{x}?\alpha$ is settled in a state s by looking only at pairs of possible worlds in s. We can make this observation more precise by means of the notion of *pair distributivity* (cf. Groenendijk, 2011): we say that φ is pair-distributive if, in order to decide whether φ is supported at a state, it is sufficient to look at sub-states which are either singletons, or pairs.

4.8.3. Definition. [Pair distributivity]
A formula φ is *pair-distributive* if for any model M, state s, and assignment g:

$$\text{if for all } t \subseteq s \text{ with } |t| \leq 2 \text{ we have } t \models_g \varphi, \;\text{ then }\; s \models_g \varphi$$

It is easy to give examples of formulas that are not pair distributive, both in propositional inquisitive logic (e.g., $p \vee\!\!\!\vee\, q \vee\!\!\!\vee\, r$), and in first-order inquisitive logic (e.g., $\overline{\exists} xPx$). On the other hand, Proposition 4.8.2 implies that all formulas in the mention-all fragment *are* pair-distributive.

4.8.4. Proposition. *Any formula in $\mathcal{L}_\forall$ is pair-distributive.*

Proof. If $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$, then α is truth-conditional, and thus *a fortiori* pair distributive: if $s \not\models_g \alpha$, we know there must be a singleton $\{w\} \in s$ such that $\{w\} \not\models_g \alpha$.

Consider then a formula $\forall\overline{x}?\alpha \in \mathcal{L}_\forall^?$ and suppose $s \not\models_g \forall x?\alpha$. By the previous proposition, there must be $w, w' \in s$ such that $\alpha_g^{\overline{x}}(w) = \alpha_g^{\overline{x}}(w')$. But then, the previous proposition also tells us that $\{w, w'\} \not\models_g \forall x?\alpha$. □

A related fact about questions in $\mathcal{L}^?_\forall$ which follows from Proposition 4.8.2 is that mention-all questions are always normal and moreover, their alternatives form a partition of the logical space.[19]

4.8.5. PROPOSITION. *Any $\mu \in \mathcal{L}^?_\forall$ is normal. Moreover, for any information model $M = \langle W, D, I\rangle$ and assignment g, the set $\text{ALT}^g_M(\mu)$ forms a partition of W.*

Proof. If $\mu \in \mathcal{L}^?_\forall$, then $\mu = \forall\overline{x}?\alpha$ for some tuple $\overline{x}$ of variables and some classical formula α. Now, consider an information model $M = \langle W, D, I, \sim\rangle$ and an assignment g. Define a relation $\equiv^\mu_g$ on the domain W as follows:

$$w \equiv^\mu w' \iff \alpha^{\overline{x}}_g(w) = \alpha^{\overline{x}}_g(w')$$

Clearly, $\equiv^\mu$ is an equivalence relation over W. Now, for any world $w \in W$, consider its equivalence class $[w]^{\equiv^\mu} := \{w' \,|\, w' \equiv^\mu w\}$. The equivalence classes form a partition of W. Moreover, Proposition 4.8.2 tells us that $s \models \mu$ if and only if s is included in some equivalence class $[w]^{\equiv^\mu}$ for some $w \in W$. In particular, we have that (i) each equivalence class $[w]^{\equiv^\mu}$ supports μ and (ii) each equivalence class $[w]^{\equiv^\mu}$ is maximal among the states that support μ; otherwise, by Proposition 4.8.2, $[w]^{\equiv^\mu}$ would have to be included in some different equivalence class $[w']^{\equiv^\mu}$; but this is impossible, since distinct equivalence classes are disjoint. It follows that the alternatives for μ are precisely the equivalence classes for the relation $\equiv^\mu$:

$$\text{ALT}^g_M(\mu) = \{[w]^{\equiv^\mu} \,|\, w \in W\}$$

Thus, the alternatives for μ form a partition of the logical space. Finally, notice that, given this fact, Proposition 4.8.2 tells us that any supporting state for μ in any information model must be included in one of these alternatives. Since this is the case in any model M, the question μ is normal. □

Now, equivalence relations on the logical space—and the partitions they induce—are precisely the objects taken to capture question meanings in Groenendijk and Stokhof's (1984) theory of questions. A logical system based on this theory, called the *Logic of Interrogation* (LoI) has been developed by Groenendijk (1999), and axiomatized by ten Cate and Shan (2007). In this chapter, we will show that the mention-all fragment of InqBQ is expressively equivalent to LoI. We will exploit this to transfer the completeness result for LoI to the mention-all fragment. Interestingly, the fact that mention-all questions are decomposed into three more basic operations—namely, universal quantifier, inquisitive disjunction, and negation—brings out how the special rules given for the question operator in ten Cate and Shan's axiomatization of LoI actually arise from familiar logical rules for these more basic operations. Thus, breaking down the question operator into simpler pieces allows us to bring out an elegant underlying logical structure.

[19] Recall that a formula φ is said to be normal if, in any model and with respect to any assignment, a state supporting φ is included in some alternative for φ (cf. Definition 1.2.4).

4.8.2 Relations with the Logic of Interrogation

We will now give an overview of the syntax, semantics and proof system of LoI, following the presentation of ten Cate and Shan (2007), and relate this system to the mention-all fragment of InqBQ.

The syntax of LoI is simple: the language consists of first-order sentences, plus formulas of the form $Q\alpha$, where Q is a special question operator, and α is a first-order formula—possibly containing free variables. The formula $Q\alpha$ is classified as a sentence, since the operator Q is taken to bind all free variables in α.

4.8.6. Definition. [Language of the Logic of Interrogation]

- If $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$ is a sentence, then $\alpha \in \mathcal{L}_{\mathsf{LoI}}$
- If $\alpha \in \mathcal{L}_c^{\mathsf{Q}}$, then $Q\alpha \in \mathcal{L}_{\mathsf{LoI}}$

The semantics of LoI is formulated relative to models that are precisely the same as our id-models of Section 4.5, that is, information models where the identity predicate is interpreted by means of the actual identity relation at each world.

Now, in its original formulation, LoI is presented as a dynamic semantics (in the tradition of Groenendijk and Stokhof, 1991; Groenendijk et al., 1996, a.o.). This means that the semantics of a sentence is given in terms of its *update potential*, that is, by associating the sentence with a map from contexts to contexts, where a context is modeled in LoI as an equivalence relation over a subset of W.

However, in the case of LoI, the dynamic coating is not essential: as remarked also by ten Cate and Shan, the semantics may be given an equivalent formulation in which formulas are interpreted with respect to ordered pairs of worlds. A classical sentence α is satisfied at a pair $\langle w, w'\rangle$ in case it is true at both worlds, while a question $Q\alpha$ is satisfied in case the worlds w and w' agree on the extension they assign to the relation $\alpha^{\overline{x}}$—where $\overline{x}$ are all the variables free in α.

4.8.7. Definition. [Static semantics of the Logic of Interrogation]
If M is an id-model, the satisfaction relations between pairs of worlds in M and formulas $\varphi \in \mathcal{L}_{\mathsf{LoI}}$ is defined as follows.

- $\langle w, w'\rangle \models \alpha \iff w \models_g \alpha$ and $w' \models_g \alpha$
- $\langle w, w'\rangle \models Q\alpha \iff \alpha^{\overline{x}}(w) = \alpha^{\overline{x}}(w')$, where $\overline{x}$ enumerates $\mathrm{FV}(\alpha)$

As a consequence of these definitions, one can see that the set $[\varphi]_M^{\mathsf{LoI}}$ of pairs that satisfy φ in LoI is always an equivalence relation over a subset of the logical space, i.e., the same kind of object that a context is taken to be. The result of updating a context c with a sentence φ can then be obtained from the static meaning of the sentence by setting $c[\varphi] := c \cap [\varphi]_M^{\mathsf{LoI}}$. Entailment in LoI can be defined in the expected way in terms of the given static semantics.

4.8.8. Definition. [Entailment in the Logic of Interrogation]
Let $\overline{\varphi}, \psi \in \mathcal{L}_{\mathsf{LoI}}$. $\overline{\varphi} \models_{\mathsf{LoI}} \psi$ in case for all pairs $\langle w, w' \rangle$ of worlds in an id-model M, $\langle w, w' \rangle \models \overline{\varphi}$ implies $\langle w, w' \rangle \models \psi$.

In terms of expressive power, LoI and the sentential part of the mention-all fragment of InqBQ are essentially equivalent: any sentence in LoI corresponds to a sentence in $\mathcal{L}_\forall$ that has an isomorphic semantics, and vice versa.[20] To make this precise, we define the following translations between $\mathcal{L}_{\mathsf{LoI}}$ and $\mathcal{L}_\forall^\sigma$.

4.8.9. Definition. [Translations]
We define two translations $(\cdot)^\sharp : \mathcal{L}_\forall^\sigma \to \mathcal{L}_{\mathsf{LoI}}$ and $(\cdot)^\flat : \mathcal{L}_{\mathsf{LoI}} \to \mathcal{L}_\forall^\sigma$, as follows:[21]

- $\alpha^\sharp = \alpha$
- $(\forall \overline{x} ?\alpha)^\sharp = Q\alpha$

- $\alpha^\flat = \alpha$
- $(Q\alpha)^\flat = \forall \overline{x} ?\alpha$, where $\mathrm{FV}(\alpha) = \{\overline{x}\}$

In what sense exactly are these maps *translations*? Well, we saw above that support of a formula $\varphi \in \mathcal{L}_\forall$ at a state s amounts to support at all states of the form $\{w, v\}$, where w and v are possibly equal. The next lemma says that $\{w, v\}$ supports φ precisely in case the pair $\langle w, v \rangle$ satisfies the translation $\varphi^\sharp$ in LoI.

4.8.10. Lemma.
For any sentence $\varphi \in \mathcal{L}_\forall^\sigma$ and ordered pair $\langle w, w' \rangle$ of worlds in an id-model M:

$$\{w, w'\} \models \varphi \iff \langle w, w' \rangle \models \varphi^\sharp$$

Proof. Immediate by Proposition 4.8.2 and by the semantic clauses for LoI. □

Conversely, satisfaction of a sentence φ at a pair $\langle w, v \rangle$ corresponds to support of its translation $\varphi^\flat$ in the state $\{w, v\}$.

4.8.11. Lemma.
For any sentence $\varphi \in \mathcal{L}_{\mathsf{LoI}}$ and ordered pair $\langle w, w' \rangle$ of worlds in an id-model M:

$$\langle w, w' \rangle \models \varphi \iff \{w, w'\} \models \varphi^\flat$$

Proof. By definition of the maps, we have $\varphi = \varphi^{\flat\sharp}$. Using this and the previous lemma, we have: $\langle w, w' \rangle \models \varphi \iff \langle w, w' \rangle \models \varphi^{\flat\sharp} \iff \{w, w'\} \models \varphi^\flat$. □

This ensures that entailment for mention-all sentences in InqBQ corresponds to entailment in LoI. However, there is a subtlety: since the semantics of LoI is only defined for id-models, the relevant result concerns id-entailment in InqBQ.

[20] Here, we will restrict to id-models, because this is the setting for which ten Cate and Shan proved their completeness result. However, the correspondence itself would hold more generally if we only extended the semantics of LoI in the obvious way to arbitrary information models.

[21] To make $(Q\alpha)^\flat$ completely specified, we assume the variables in the language are indexed, and that $\overline{x}$ enumerates the free variables of α according to their index. In the following, I will not pay attention to the way variables are arranged in the sequence $\overline{x}$: it is clear that our quantifier rules allow us to infer $\forall \overline{x}' \alpha$ from $\forall \overline{x} \alpha$ where $\overline{x}'$ is a permutation of $\overline{x}$.

$$\frac{\alpha_1 \quad \dots \quad \alpha_n}{\beta}\ (\mathrm{CT}) \qquad \text{where } \alpha_1, \dots, \alpha_n \models \beta \text{ in classical logic}$$

$$\frac{Q\alpha}{Q\neg\alpha}\ (\neg) \qquad \frac{Q\alpha \quad Q\beta}{Q(\alpha \wedge \beta)}\ (\wedge) \qquad \frac{Q\alpha}{Q\forall x\alpha}\ (\forall)$$

$$\frac{}{Q(x = t)}\ (\mathrm{Eq}) \qquad \frac{\forall\overline{x}(\alpha \leftrightarrow \beta) \quad Q\alpha}{Q\beta}\ (\mathrm{Equiv})$$

Figure 4.4: A natural deduction presentation of ten Cate and Shan's proof system for Lol. In the rule (Equiv), the tuple $\overline{x}$ enumerates $\mathrm{FV}(\alpha) \cup \mathrm{FV}(\beta)$.

4.8.12. Lemma. *For all* $\overline{\varphi}, \psi \in \mathcal{L}^{c}_{\forall}$, $\quad \overline{\varphi} \models_{\mathsf{id}} \psi \iff \overline{\varphi}^{\sharp} \models_{\mathsf{Lol}} \psi^{\sharp}$

Proof. The left-to-right direction is straightforward from Lemma 4.8.10. For the converse, suppose $\overline{\varphi}^{\sharp} \models_{\mathsf{id}} \psi^{\sharp}$. Consider any state s in an id-model M and suppose $s \models \overline{\varphi}$. Now let $\{w, w'\}$ be an arbitrary substate of s of cardinality at most 2, where w and w' may be identical. By persistence, we have $\{w, w'\} \models \overline{\varphi}$, and thus by Lemma 4.8.10, $\langle w, w'\rangle \models \overline{\varphi}^{\sharp}$. Since $\overline{\varphi}^{\sharp} \models_{\mathsf{Lol}} \psi^{\sharp}$, this implies $\langle w, w'\rangle \models \psi^{\sharp}$, and thus again by Lemma 4.8.10, $\{w, w'\} \models \psi$. So, ψ is supported at any substate $\{w, w'\} \subseteq s$ of cardinality at most 2. Since ψ is pair-distributive by Proposition 4.8.4, this implies $s \models \psi$. Hence, we have $\overline{\varphi} \models_{\mathsf{id}} \psi$. □

We will use this correspondence between id-entailment in the mention-all fragment of InqBQ and entailment in Lol to transfer the axiomatization result of ten Cate and Shan (2007) for Lol to the mention-all fragment of InqBQ.

A natural deduction version of ten Cate and Shan's system is given in Figure 4.4. This presentation allows us to do without the explicit structural rules of the original presentation. Moreover, I have replaced ten Cate and Shan's rule (∃) by the rule (∀) which involves our primitive classical quantifier, ∀. It is easy to see that, given the rules (CT) and (Equiv), these two rules are equivalent.

$$\frac{Q\varphi}{Q\exists x\varphi}\ (\exists) \qquad \frac{Q\varphi}{Q\forall x\varphi}\ (\forall)$$

We will write $P : \overline{\varphi} \vdash_{\mathsf{Lol}} \psi$ to mean that P is a proof of ψ from the set of assumptions $\overline{\varphi}$ in this system, and $\overline{\varphi} \vdash_{\mathsf{Lol}} \psi$ to mean that such a proof exists.

Let us comment briefly on the rules of this system. The rule (CT) outsources derivation among classical formulas to a system for classical first-order logic. The rule (¬) captures the fact that the extension of the relation $\alpha^{\overline{x}}$ determines the extension of the relation $(\neg\alpha)^{\overline{x}}$. Analogously, the rule (∧) captures the fact that the extension of the relations $\alpha^{\overline{x}}$ and $\beta^{\overline{x}}$ determines the extension of $(\alpha\wedge\beta)^{\overline{x}}$, while the rule ($\forall$) captures the fact that the extension of $\alpha^{\overline{x}}$ determines the extension of $(\forall x\alpha)^{\overline{x}}$. The rule (Equiv) captures the fact that, if α and β coincide on each tuple, then the extension of $\alpha^{\overline{x}}$ determines the extension of $\beta^{\overline{x}}$. Finally, the rule (Eq) captures the restriction to models with decidable identity. Notice that the rules for the operator Q in this system are quite special. Unlike the other connectives, Q is not given introduction and elimination rules, but only "transmission rules", which lead from one or more Q sentences to another Q sentence.

4.8.3 Proof system

Now let us turn our attention to the mention-all fragment. The set of rules for InqBQ described in Figure 4.2 suggests a natural axiomatization of this fragment, obtained essentially by restricting the rules to formulas in $\mathcal{L}_\forall$. In addition, since we are trying to characterize id-entailment, we must add a rule that captures the decidability of identity. The resulting natural-deduction system is given in Figure 4.5. Within this section, we will use the notation $\vdash$ for provability in this system. Let us comment briefly on the various inference rules.

Standard rules. First, our system includes the standard rules for the classical logical constants $\wedge, \to, \bot, , \forall$, and $=$. Notice that the rules for $\forall$ also apply to questions, without any amendment.

Question mark rules. In the mention-all fragment we do not have arbitrary inquisitive disjunctions, but only inquisitive disjunctions of a particular form, namely, of the form $?\alpha := \alpha \vee\!\!\!\vee \neg\alpha$. The rules for ? are simply the rules for $\vee\!\!\!\vee$ specialized to this particular case.

Distributivity of $\forall$ over ?. Like the rules for ? are the shadow of the more general rule for $\vee\!\!\!\vee$ within the mention-all fragment, so this rule is the shadow of the more general Constant Domains rule of Figure 4.2 within the fragment.

To see this, first notice that, since $\neg\alpha \vdash \neg\forall x\alpha$, we have $\alpha \vee\!\!\!\vee \neg\alpha \vdash \alpha \vee\!\!\!\vee \neg\forall x\alpha$, that is, $?\alpha \vdash \alpha \vee\!\!\!\vee \neg\forall x\alpha$. As a consequence, it is easy to see that $\forall x?\alpha \vdash \forall x(\alpha \vee\!\!\!\vee \neg\forall x\alpha)$. Since x does not occur free in $\forall x\alpha$, we can apply the Constant Domains rule to $\forall x(\alpha \vee\!\!\!\vee \neg\forall x\alpha)$ to obtain $\forall x\alpha \vee\!\!\!\vee \neg\forall x\alpha$, i.e., $?\forall x\alpha$. Putting things together, we thus have that $\forall x?\alpha \vdash ?\forall x\alpha$. Since $\mathcal{L}_\forall$ does not contain arbitrary inquisitive disjunction, the Constant Domains rule cannot be formulated: the task of capturing the restriction to constant domains within the fragment is thus delegated to the more specific rule of distributivity of $\forall$ over ?.

Decidable identity. Finally, our system contains the decidable identity axiom $?(t = t')$ for any two terms t and t'. This is sufficient to capture the restriction to id-models. For, given that $?(x = y)$ can be derived from the empty set of assumptions, the rule ($\forall$i) allows us to infer $\forall x\forall y?(x\!=\!y)$, and we have seen above (Proposition 4.5.9) that this formula characterizes the decidability of identity.

Conjunction

$$\frac{\alpha \quad \beta}{\alpha \wedge \beta} \qquad \frac{\alpha \wedge \beta}{\alpha} \qquad \frac{\alpha \wedge \beta}{\beta}$$

Implication

$$\frac{\begin{array}{c}[\alpha]\\ \vdots \\ \beta\end{array}}{\alpha \to \beta} \qquad \frac{\alpha \quad \alpha \to \beta}{\beta}$$

Universal quantifier

$$\frac{\varphi[y/x]}{\forall x\varphi} \qquad \frac{\forall x\varphi}{\varphi[t/x]}$$

Falsum

$$\frac{\bot}{\varphi}$$

Identity

$$\frac{}{t = t} \qquad \frac{\varphi[t/x] \quad t = t'}{\varphi[t'/x]}$$

Classical $\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha}$$

Question mark

$$\frac{\alpha}{?\alpha} \qquad \frac{\neg\alpha}{?\alpha} \qquad \frac{?\alpha \quad \begin{array}{c}[\alpha]\\ \vdots \\ \varphi\end{array} \quad \begin{array}{c}[\neg\alpha]\\ \vdots \\ \varphi\end{array}}{\varphi}$$

Distributivity of $\forall$ over ?

$$\frac{\forall x?\alpha}{?\forall x\alpha}$$

Decidable identity

$$\frac{}{?(t = t')}$$

Figure 4.5: A natural deduction system the mention-all fragment of InqBQ. In these rules, α and β range over classical formulas, while φ and ψ range over arbitrary formulas in $\mathcal{L}_\forall$. The restrictions on variables are the standard ones.

Notice that, interestingly, the only two non-standard ingredients of our proof system can actually be traced to specific restrictions that we are making on top of a more general semantic architecture: decidable identity, and constant domains. Also, remark that decomposing the operator Q of LoI into three distinct operators, $\neg$, $\vee\!\!\!\vee$, and $\forall$, makes it possible to appreciate how the unusual logical properties of Q actually derive from the familiar logical properties of these operators.

4.8.4 Completeness

Let us now show that this proof system gives a sound and complete axiomatization of the mention-all fragment of InqBQ over id-models.

4.8.13. Theorem (Completeness for the mention-all fragment).
For any $\overline{\varphi}, \psi \in \mathcal{L}_\forall$:

$\overline{\varphi} \models_{\mathsf{id}} \psi \iff \psi$ *is provable from $\overline{\varphi}$ by means of the rules described in Figure 4.5*

The soundness of the system follows immediately from the soundness of the more general rules for InqBQ given in Figure 4.2, plus the fact that any formula of the form $?(t = t')$, where t and t' are rigid terms, is id-valid over models with decidable identity. As for completeness, we will first show that, if an entailment $\overline{\varphi} \models \psi$ is provable in ten Cate and Shan's system for LoI, then the entailment $\overline{\varphi}^\flat \models_{\mathsf{id}} \psi^\flat$ between the translations of the formulas is provable in our system.

4.8.14. Lemma. *For all $\overline{\varphi}, \psi \in \mathcal{L}_{LoI}$, if $\overline{\varphi} \vdash_{\mathsf{LoI}} \psi$, then $\overline{\varphi}^\flat \vdash \psi^\flat$.*

To establish this, we first notice that our system can simulate the rules $(\neg)$, $(\wedge)$, and (Equiv) of the LoI system.

4.8.15. Lemma.
For any classical formulas α, β and any tuples of variables $\overline{x}, \overline{y}$:

- $\forall\overline{x}?\alpha \vdash \forall\overline{x}?\neg\alpha$
- $\forall\overline{x}?\alpha, \forall\overline{y}?\beta \vdash \forall\overline{x}\overline{y}(\alpha \wedge \beta)$
- $\forall\overline{x}(\alpha \leftrightarrow \beta), \forall\overline{x}?\alpha \vdash \forall\overline{x}?\beta$

Proof. The proof is straightforward. By way of example, we show the third item. Suppose $\forall\overline{x}(\alpha \leftrightarrow \beta)$ and $\forall\overline{x}?\alpha$. By ($\forall$e) we obtain $\alpha \leftrightarrow \beta$ and $?\alpha$. Now, if we further assume α, by classical reasoning we can get β, and thus also $?\beta$ by ($\vee\!\!\!\vee$i). Similarly, if we assume $\neg\alpha$, by classical reasoning we can get $\neg\beta$, and thus also $?\beta$. Since we have $?\alpha$, we can use ($\vee\!\!\!\vee$e) and conclude $?\beta$. Finally, since the variables in $\overline{x}$ do not occur free in our assumptions, we get $\forall\overline{x}?\beta$. □

Equipped with this lemma, we are ready to show that any inference in ten Cate and Shan's proof system can be reproduced in our system.

Proof of Lemma 4.8.14. By induction on the proof $P : \overline{\varphi} \vdash_{\mathsf{LoI}} \psi$ in the LoI system. If the proof is identical to one of the assumptions, the conclusion is obvious. Now consider a proof P and suppose the claim holds for all proper subproofs of P. We distinguish a number of cases depending on the last rule applied in P.

- Suppose the last rule used in P is [CT]. Then ψ is a classical sentence β and the immediate subproofs of P are $P_1, \ldots, P_n$, where $P_i : \overline{\varphi} \vdash_{\mathsf{LoI}} \alpha_i$, and where $\alpha_1, \ldots, \alpha_n$ entail β in classical logic.

  Now the induction hypothesis ensures that $\overline{\varphi}^\flat \vdash \alpha_i$ for $1 \leq i \leq n$. Moreover, since $\alpha_1, \ldots, \alpha_n$ classically entail β, and since our proof system includes a sound and complete system for classical logic, we have $\alpha_1, \ldots, \alpha_n \vdash \beta$. Putting things together, we have $\overline{\varphi}^\flat \vdash \beta$, i.e., since $\beta^\flat = \beta$, we have $\overline{\varphi}^\flat \vdash \psi^\flat$.

- Suppose the last rule used in P is the rule $[\neg]$. Then $\psi = Q\neg\alpha$ and the immediate subproof of P is a proof $P' : \overline{\varphi} \vdash_{\mathsf{LoI}} Q\alpha$. The induction hypothesis then tells us that we have $\overline{\varphi}^\flat \vdash \forall\overline{x}?\alpha$, where $\overline{x}$ is the sequence of variables occurring free in α. But by Lemma 4.8.15 we have $\forall\overline{x}?\alpha \vdash \forall\overline{x}?\neg\alpha$. Thus, we also have $\overline{\varphi}^\flat \vdash \forall\overline{x}?\neg\alpha$, which amounts precisely to $\overline{\varphi}^\flat \vdash \psi^\flat$.

- Suppose the last rule used in P is the rule $[\wedge]$. This case is similar to the previous one, using the second item of Lemma 4.8.15.

- Suppose the last rule applied in P is the rule $[\forall]$. Then $\psi = Q\forall y\alpha$ and the immediate subproof of P is $P' : \overline{\varphi} \vdash_{\mathsf{LoI}} Q\alpha$.

  By the induction hypothesis, we know that $\overline{\varphi}^\flat \vdash \forall\overline{x}?\alpha$. We must now distinguish two cases according to whether or not y is free in α. Assume first that this is not the case. Then, by the rules for $\forall$ we have $\alpha \dashv\vdash \forall y\alpha$. As a consequence, we also get $\overline{\varphi}^\flat \vdash \forall\overline{x}?\forall y\alpha$. Now since y was not free in α, the free variables in α and $\forall\overline{x}?\forall y\alpha = (\forall y\alpha)^\flat = \psi^\flat$. In this case, we have thus obtained the desired conclusion, $\overline{\varphi}^\flat \vdash \psi^\flat$.

  Assume now that y *is* free in α. Then by definition of $(\cdot)^\flat$, y is among the variables in $\overline{x}$. Since our rules for $\forall$ allow us to rearrange a sequence of universal quantifiers in any way we like, we may assume that $\overline{x} = \overline{x}'y$, where the tuple $\overline{x}'$ enumerates the free variables in α, except for y. Now, these variables are precisely the variables which are free in $\forall y\alpha$. Now, the rule of distribution of $\forall$ over ? guarantees that $\forall y?\alpha \vdash ?\forall y\alpha$, whence it is easy to see that $\forall\overline{x}'y?\alpha \vdash \forall\overline{x}'?\forall y\alpha$. Putting things together, we thus have $\overline{\varphi}^\flat \vdash \forall\overline{x}'?\forall y\alpha$. Finally, since $\overline{x}'$ enumerates the free variables in $\forall y\alpha$, we have $\forall x'?\forall y\alpha = (\forall y\alpha)^\flat = \psi^\flat$. Thus, in this case too we have reached the desired conclusion, $\overline{\varphi}^\flat \vdash \psi^\flat$.

- Suppose the last rule applied in P is the rule $[Eq]$. Then $\psi = Q(x = y)$. Then, $\psi^\flat = \forall xy?(x{=}y)$. Now, the Decidable Identity rule allows us to infer $?(x = y)$ without any assumption, whence by two applications of $(\forall\mathsf{i})$ we have $\forall xy?(x = y)$. This shows that $\vdash \psi^\flat$, which also implies $\overline{\varphi}^\flat \vdash \psi^\flat$.

- Finally, suppose the last rule applied in P is $[Equiv]$. This means that $\psi = Q\beta$, and the immediate subproofs of P are a proof $P' : \overline{\varphi} \vdash_{\mathsf{LoI}} \forall\overline{x}(\alpha \leftrightarrow \beta)$ and a proof $P'' : \overline{\varphi} \vdash_{\mathsf{LoI}} Q\alpha$ where $\overline{x}$ lists $\mathrm{FV}(\alpha) = \mathrm{FV}(\beta)$.

  The induction hypothesis tells us that $\overline{\varphi}^\flat \vdash \forall\overline{x}(\alpha \leftrightarrow \beta)$ and that $\overline{\varphi}^\flat \vdash \forall\overline{y}?\alpha$, where $\overline{y}$ lists the variables occurring free in α. Since the variables in $\overline{y}$ are among the variables in $\overline{x}$, and since $\overline{\varphi}^\flat$ is a set of sentences, by some applications of $(\forall\mathsf{i})$ we have $\overline{\varphi}^\flat \vdash \forall\overline{x}?\alpha$. Now by the third item of Lemma 4.8.15 we have $\forall\overline{x}(\alpha \leftrightarrow \beta), \forall\overline{x}?\alpha \vdash \forall\overline{x}?\beta$. Putting things together, we thus have $\overline{\varphi}^\flat \vdash \forall\overline{x}?\beta$. Finally, let $\overline{z}$ list the variables free in β: since the variables in $\overline{z}$ are among the variables in $\overline{x}$, by some applications of $(\forall\mathsf{e})$ we have $\overline{\varphi}^\flat \vdash \forall\overline{z}?\beta$, that is, $\overline{\varphi}^\flat \vdash \psi^\flat$. □

We have now almost reached a completeness result for the mention-all fragment. For, the previous lemma ensures that our system proves all valid entailments among translations of LoI sentences. But the next lemma states that any mention-all sentence is provably equivalent to the translation of a LoI sentence.

4.8.16. LEMMA. *For all $\varphi \in \mathcal{L}^\sigma_\forall$, $(\varphi)^{\sharp\flat} \dashv\vdash \varphi$*

Proof. By definition of the translations, the only case in which φ and $(\varphi)^{\sharp\flat}$ are not identical is when $\varphi = \forall\overline{x}?\alpha$ and $\varphi^{\sharp\flat} = \forall\overline{y}?\alpha$ where the variables in $\overline{y}$ are a proper subset of the variables in $\overline{x}$. Obviously, $\forall\overline{x}?\alpha \vdash \forall\overline{y}?\alpha$. Conversely, notice that since $\overline{y}$ by definition enumerate all the free variables in α, all the variables in $\overline{x}$ which are not in $\overline{y}$ do not occur free in $\forall\overline{y}?\alpha$. Given this, from $\forall\overline{y}?\alpha$ we can simply introduce the missing quantifications to obtain $\forall\overline{x}?\alpha$. □

Now we finally have all the ingredients needed to prove the completeness result.

Proof of Theorem 4.8.13. First suppose our formulas $\overline{\varphi}, \psi$ are *sentences*. Suppose $\overline{\varphi} \models_{\mathsf{id}} \psi$: by Lemma 4.8.12, we have $\overline{\varphi}^\sharp \models_{\mathsf{LoI}} \psi^\sharp$. By the completeness of $\vdash_{\mathsf{LoI}}$, this ensures $\overline{\varphi}^\sharp \vdash_{\mathsf{LoI}} \psi^\sharp$. By Lemma 4.8.14, in turn, this yields $\overline{\varphi}^{\sharp\flat} \vdash \psi^{\sharp\flat}$. Finally, Lemma 4.8.16 ensures that $\overline{\varphi}^{\sharp\flat} \dashv\vdash \overline{\varphi}$ and $\psi^{\sharp\flat} \dashv\vdash \psi$. Putting the pieces together, we thus have $\overline{\varphi} \vdash \psi$. This proves completeness for sentences. We can then extend this result to formulas with free variables by means of the same argument used in the proof of Theorem 4.7.16. □

So, just as for the mention-some fragment, by restricting to a specific interesting fragment of the language, we can show that the proof system given in Figure 4.2 is sufficient to capture all valid entailments in the fragment.

Curiously, the situation in the mention-all fragment is opposite to the one we found for the mention-some fragment. In the latter case, we obtained a complete axiomatization of general entailment, and we conjectured that a completeness result for id-entailment can be obtained by extending the system with decidable identity axioms and decidable cardinality axioms. In the present section, we have obtained a completeness result for id-entailment. We conjecture that dropping the decidable identity axiom results in a complete system for general entailment.

4.8.17. CONJECTURE.
For all $\overline{\varphi}, \psi \in \mathcal{L}_\forall$, $\overline{\varphi} \models \psi \iff \psi$ is derivable from $\overline{\varphi}$ in the system of Figure 4.5 deprived of the decidable identity axioms.

Of course, for formulas which do not contain the identity predicate, entailment and id-entailment coincide, which gives the following corollary.

4.8.18. COROLLARY.
Let $\overline{\varphi}, \psi \in \mathcal{L}_\forall$ be formulas without identity. Then $\overline{\varphi} \models \psi \iff \overline{\varphi} \vdash \psi$.

4.8.5 Illustration and an open problem

Let us conclude this Section with an illustration of our proof system in action. Recall Example 4.4.7 of a logical dependency in the mention-all fragment. We had two predicate symbols, H and A, where Hx stands for "x is a disease the patient has" and Ax for "x is a treatment to be administered". Moreover, we had a relation symbol T, where Txy stands for "x is a treatment for disease y". The context was given by the formula:

$$\gamma := \forall x(Ax \leftrightarrow \exists y(Txy \wedge Hy))$$

which says that we must administer x if and only if x is a treatment for a disease the patient has. Now, we noticed that, given this information as background, a specification of the set of treatments for each disease ($\forall xy?Txy$), together with a specification of the diseases the patient has ($\forall x?Hx$), determines which treatments should be administered ($\forall x?Ax$). That is, the following entailment is valid:

$$\gamma,\ \forall xy?Txy,\ \forall x?Hx\ \models\ \forall x?Ax$$

By the previous corollary, we should be able to provide a proof of this entailment in our proof system. Let us present such a proof, breaking it up in several parts. First, we prove that the questions $\forall xy?Txy$ and $\forall x?Hx$ logically determine the question $\forall y?\neg(Txy \wedge Py)$. As usual, we omit portions of the proof that only involve inferences in classical logic, and denote them $(\mathsf{C}_1), (\mathsf{C}_2), (\mathsf{C}_3)$.

$$
\dfrac{
\dfrac{\forall xy?Txy}{?Txy}\,(\forall\mathsf{e})
\quad
\dfrac{
\dfrac{\forall x?Hx}{?Py}\,(\forall\mathsf{e})
\quad
\dfrac{\dfrac{[Txy]\quad[Hy]}{\neg\neg(Txy\wedge Hy)}\,(\mathsf{C}_1)}{?\neg(Txy\wedge Hy)}\,(\mathbb{V}\mathsf{i})
\quad
\dfrac{\dfrac{[\neg Hy]}{\neg(Txy\wedge Hy)}\,(\mathsf{C}_2)}{?\neg(Txy\wedge Hy)}\,(\mathbb{V}\mathsf{i})
}{?\neg(Txy\wedge Hy)}\,(\mathbb{V}\mathsf{e})
\quad
\dfrac{\dfrac{[\neg Txy]}{\neg(Txy\wedge Hy)}\,(\mathsf{C}_3)}{?\neg(Txy\wedge Hy)}\,(\mathbb{V}\mathsf{i})
}{
\dfrac{?\neg(Txy\wedge Hy)}{\forall y?\neg(Txy\wedge Hy)}\,(\forall\mathsf{i})
}\,(\mathbb{V}\mathsf{e})
$$

Let us refer to this proof as P_1. Second, we show how P_1 can be used to prove that the questions $\forall xy?Txy$ and $\forall x?Hx$ logically determine the question $?\exists y(Txy \wedge Hy)$ of whether a given x is a treatment of some disease the patient has.

$$
\dfrac{
\dfrac{\dfrac{\forall xy?Txy\quad \forall x?Hx}{\forall y?\neg(Txy\wedge Hy)}\,(\mathsf{P}_1)}{?\forall y\neg(Txy\wedge Hy)}\,(\forall?\mathsf{dist})
\quad
\dfrac{\dfrac{[\forall y\neg(Txy\wedge Hy)]}{\neg\exists y(Txy\wedge Hy)}\,(\mathsf{C}_4)}{?\exists y(Txy\wedge Hy)}\,(\mathbb{V}\mathsf{i})
\quad
\dfrac{\dfrac{[\neg\forall y\neg(Txy\wedge Hy)]}{\exists y(Txy\wedge Hy)}\,(\mathsf{C}_5)}{?\exists y(Txy\wedge Hy)}\,(\mathbb{V}\mathsf{i})
}{?\exists y(Txy\wedge Hy)}\,(\mathbb{V}\mathsf{e})
$$

Call this proof P_2. Finally, we show how P_2 figures in a proof of our dependency.

$$
\dfrac{
\dfrac{\forall xy?Txy\quad \forall x?Hx}{?\exists y(Txy\wedge Hy)}\,(\mathsf{P}_2)
\quad
\dfrac{\dfrac{\gamma\quad[\exists y(Txy\wedge Hy)]}{Ax}\,(\mathsf{C}_6)}{?Ax}\,(\mathbb{V}\mathsf{i})
\quad
\dfrac{\dfrac{\gamma\quad[\neg\exists y(Txy\wedge Hy)]}{\neg Ax}\,(\mathsf{C}_7)}{?Ax}\,(\mathbb{V}\mathsf{i})
}{
\dfrac{?Ax}{\forall x?Ax}\,(\forall\mathsf{i})
}\,(\mathbb{V}\mathsf{e})
$$

Notice that it is natural to regard our proof as encoding a method to obtain information of type $\forall x?Ax$ from information of type $\forall xy?Txy$ and $\forall x?Hx$, i.e., to compute the extension of A from the extensions of T and H. Focusing on the last bit of the proof, we can see that it essentially describes the following procedure. In order to determine the extension of A, do the following for any individual x: use the specification of T and H to decide $?\exists y(Txy \wedge Hy)$, i.e, to determine whether or not x is a treatment of some symptom the patient has, according to the procedure described by P_2. If the answer is positive, put x in the extension of A; if the answer is negative, leave it out.

Since we lack a suitable notion of resolutions for mention-all questions, however, the constructive content of such proofs cannot be spelled out in terms of the existence of a Resolution Algorithm, like the one presented for propositional logic and for the mention-some fragment. It is an interesting task for future work to investigate how to best formalize the intuition that in the mention-all fragment, too, proofs seem to have an algorithmic aspect to them.

Chapter 5

Relations with Dependence Logic

In the previous chapters we saw how expanding logic to cover questions allows us to extend the notion of entailment to capture not only the standard relation of consequence, but also the relation of *dependency*. This relation is also the focus of the recently developed, and currently very active, field of *dependence logic* (see, a.o., Väänänen, 2007, 2008; Abramsky and Väänänen, 2009; Kontinen and Väänänen, 2009; Galliani, 2012; Grädel and Väänänen, 2013; Yang, 2014).

In this chapter, based on Ciardelli (2015a), we discuss the connections between these two approaches to dependency. We will see that, formally, the systems developed in the two traditions are deeply connected, which opens the way for a transfer of insights and results. At the same time, the perspective taken on dependency is quite different in the two cases, and it is particularly on this point that our discussion will focus. We will consider, in turn, the system PD of propositional dependence logic (Yang, 2014; Yang and Väänänen, 2014), and the system FOD of first-order dependence logic (Väänänen, 2007). In each case, we will see that the fundamental element of the system, the so-called *dependence atom*, captures a particular pattern of dependency in our sense. In the propositional case, the dependence atom $=\!\!(p_1, \dots, p_n, q)$ expresses a dependency between the atomic polar questions $?p_1, \dots, ?p_n$ and $?q$; in the first-order case, the atom $=\!\!(x_1, \dots, x_n, y)$ expresses a dependency between questions $\lambda x_1, \dots, \lambda x_n$ and λy, each asking for the value of the corresponding variable. By recognizing this, it also becomes clear that the dependence atoms capture only a special instance of a more general phenomenon. In both settings, we will discuss a range of dependence patterns which go beyond these "standard" ones, and which can be captured in a natural way within a language equipped with questions. In the propositional case, e.g., we can capture not only dependencies such as (1-a), but also dependencies such as (1-b). In the first-order case, we can capture not only the standard dependency (2-a), but also dependencies such as (2-b-d).

(1) a. Whether it is Monday determines whether Alice is in the office.
    b. Which day it is determines whether Alice or Bob is in the office.

(2) a. The value of x determines the value of y.
b. The value of x determines the parity of y.
c. The value of x and whether $x < y$ determines the value of y.
d. The value of x determines a prime factor of y.

Additionally, we will see that, in dependence logic, there is a mismatch between the propositional case, in which dependencies concern sentences, as in (1-a), and the first-order case, in which dependencies concern first-order variables, as in (2-a). This has the puzzling consequence that things that are expressible in propositional dependence logic are no longer expressible in first-order dependence logic. We will propose a more general semantic framework for a first-order logic of dependency in which (1-a) and (2-a) (as well as (1-b) and (2-b-d)) can be captured as instances of one and the same relation.

Finally, we will abstract away from the details of a specific formal system, and discuss the main conceptual difference between the present approach to dependence and the one adopted in systems of dependence logic, namely, the fact that we regard dependency as a relation that holds between questions, rather than between variables. We will argue that, while the two views are equivalent in some respects, the question-view also has some important advantages.

The chapter is structured as follows. We start in Section 5.1 by providing a short historical introduction to dependence logic. In Section 5.2 we discuss the system PD of propositional dependence logic and its connections with InqB. In Section 5.3 we discuss the system FOD of first-order dependence logic, and we consider a related system InqBQT, which is the analogue of InqBQ in a semantic setting in which the uncertainty does not concern the state of affairs but rather the value of variables. In Section 5.4, we discuss a mismatch between the dependencies of propositional dependence logic, and those of the first-order system, and we look at how it can be resolved by adopting a more general semantic framework. Finally, in Section 5.5 we compare two ways of construing the notion of dependency, namely, as a relation between questions and as a relation between variables, and we spell out several arguments in favor of the former view.

5.1 Historical notes

The line of work leading to dependence logic originates with Henkin's observation that certain patterns of quantification over individuals are not expressible in first-order logic. For instance, it is impossible to write a first-order formula expressing that for every x and x', there exist a y determined only by x and a y' determined only by x', such that a certain formula $\varphi(x, x', y, y')$ holds. To provide the tools to express such patterns, Henkin (1961) introduced so-called *branching quantifiers*, and Hintikka and Sandu (1997) later developed this work in the framework of Independence Friendly (IF) logic, which allows for existentially quantified variables to be explicitly marked as independent of other variables. Väänänen (2007) proposed a new approach to the issue: he noticed that dependency and quantification may be separated out. In the resulting Dependence Logic, quantifiers have the standard form, but the language is enriched with a new kind of atomic formula $=(x_1, \ldots, x_n, y)$, expressing the fact that the value of y is determined by the values of $x_1, \ldots, x_n$. Thus, in Dependence Logic, the pattern of quantification mentioned above may be expressed as follows:

$$\forall x \forall x' \exists y \exists y' (\, =(x, y) \wedge =(x', y') \wedge \varphi(x, x', y, y'))$$

Clearly, in the standard semantic context consisting of a model and an assignment, every variable simply gets a specific value, and there is no way to make sense of the idea that some variables are determined by others. Instead, building on work by Hodges (1997a,b) on semantics for IF logic, Dependence Logic is interpreted with respect to a model and a *set* of assignments, called a *team*. Relative to a team X, we can say that the value of y is determined by the values of $x_1, \ldots, x_n$ in case the value that an assignment $g \in X$ assigns to y is fully determined by the values it assigns to $x_1, \ldots, x_n$. Dependence atoms are thus interpreted by means of the following clause:

$$M \models_X =(x_1, \ldots, x_n, y) \iff \forall g, g' \in X\text{: } [g(x_i) = g'(x_i) \text{ for all } i] \text{ implies } g(y) = g'(y)$$

Due to the similarity between assignments for individual variables in predicate logic and valuations for propositional variables in propositional logic, dependence atoms have later been considered also in the setting of propositional and modal logic (Väänänen, 2008; Yang, 2014). In this setting, a dependence atom has the form $=(p_1, \ldots, p_n, q)$, and it is interpreted, relative to a set s of possible worlds, as expressing that the truth-value that a world $w \in s$ assigns to q is determined by the truth-values it assigns to $p_1, \ldots, p_n$. It is from this simpler logical setting that we will start our discussion, in the next section.

5.2 Propositional logic

5.2.1 The system PD

Propositional dependence logic, PD, appears implicitly as a basis for the modal dependence logic of Väänänen (2008), but it has been first investigated systematically by Yang (2014). Like InqB, this logic may be interpreted within a propositional information model, that is, a pair $M = \langle W, V \rangle$ consisting of a set W of worlds and a valuation function $V : W \times \mathcal{P} \to \{0, 1\}$, where $\mathcal{P}$ is the given set of propositional variables.[1] On the other hand, the language $\mathcal{L}^{\mathsf{PD}}$ of this logic is

[1] As for propositional inquisitive logic, the standard presentation of PD actually assumes a unique, "canonical" information model, having the valuations themselves as possible worlds. As we discussed in Chapter 2, this does not make a difference: for, any situation that arises in an arbitrary propositional information model may be reproduced within this particular model.

quite different from the language $\mathcal{L}^{\mathsf{P}}$ of InqB. First, there are three sorts of logical atoms: positive atoms $p \in \mathcal{P}$, negative atoms $\overline{p}$ for $p \in \mathcal{P}$, and dependence atoms $=\!\!(p_1, \ldots, p_n, q)$ for $p_1, \ldots, p_n, q \in \mathcal{P}$. Starting with these atoms, complex formulas may be obtained by means of two connectives: conjunction, $\wedge$, and *tensor disjunction*, $\otimes$.[2]

$$\varphi \ ::= \ p \mid \overline{p} \mid \ =\!\!(p_1, \ldots, p_n, q) \mid \varphi \wedge \varphi \mid \varphi \otimes \varphi$$

We will write $\mathcal{L}_c^{\mathsf{PD}}$ for the set of formulas in $\mathcal{L}^{\mathsf{PD}}$ in which no dependence atom occurs, and we will refer to such formulas as the *classical* formulas of $\mathcal{L}^{\mathsf{PD}}$.

As for InqB, the semantics is given by a relation between states and formulas. In the interest of consistency with our exposition, we will refer to this relation as *support*, even though this term is not used in the dependence logic literature. The system PD is obtained by means of the following support conditions.

5.2.1. Definition. [Semantics of PD]

- $M, s \models p \iff V(w, p) = 1$ for all $w \in s$
- $M, s \models \overline{p} \iff V(w, p) = 0$ for all $w \in s$
- $M, s \models\ =\!\!(p_1, \ldots, p_n, q) \iff$ for all $w, w' \in s$: if $V(w, p_i) = V(w', p_i)$ for each i then $V(w, q) = V(w', q)$
- $M, s \models \varphi \wedge \psi \iff M, s \models \varphi$ and $M, s \models \psi$
- $M, s \models \varphi \otimes \psi \iff s = t \cup t'$ for some t, t' s.t. $M, t \models \varphi$ and $M, t' \models \psi$

We may read the clauses as follows. A positive atom p is settled in a state s if all the worlds in s agree that p is true. A negative atom $\overline{p}$ is settled in s if all the worlds in s agree that p is false. A dependence atom $=\!\!(p_1, \ldots, p_n, q)$ is settled in s in case whenever two worlds in s agree on the value of $p_1, \ldots, p_n$, they must also agree on the value of q. This means precisely that, within s, the truth-value of q is a function of the truth-values of $p_1, \ldots, p_n$. To make this precise, let us say that f is a dependence function from $p_1, \ldots, p_n$ to q in case, within s, the truth-value of q is obtained by applying f to the truth-values of $p_1, \ldots, p_n$.

5.2.2. Definition. [Dependence functions for propositional variables]
$f : \{0, 1\}^n \to \{0, 1\}$ is a *dependence function* from $p_1, \ldots, p_n$ to q in a state s, notation $f : p_1, \ldots, p_n \rightsquigarrow_s q$, if for all $w \in s$, $V(w, q) = f(V(w, p_1), \ldots, V(w, p_n))$.

[2] In the dependence logic literature, tensor disjunction is also called *split-junction*, or simply *disjunction*, and denoted $\vee$. We will use the notation $\otimes$, which, besides avoiding conflict with our defined disjunction, $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$, brings out the fact that, from an algebraic point of view, $\otimes$ is not a join operation, but a *quantale multiplication*, as first noted by Abramsky and Väänänen (2009). This notation and the term *tensor* have already been used systematically in previous work on propositional dependence logic (see Yang, 2014; Yang and Väänänen, 2014).

Then, the following proposition says that $=\!\!(p_1, \dots, p_n, q)$ is supported in a state s iff there exists some dependence function from $p_1, \dots, p_n$ to q in s.

5.2.3. Proposition.
$s \models =\!\!(p_1, \dots, p_n, q) \iff$ *for some* $f : \{0,1\}^n \to \{0,1\},\ f : p_1, \dots, p_n \rightsquigarrow_s q$

As for the connectives, a conjunction $\varphi \wedge \psi$ is settled in s in case both conjuncts are settled in s, just as in InqB, while a tensor disjunction $\varphi \otimes \psi$ is settled in s if we can subdivide s into two states t and t', each supporting one of the disjuncts.[3]

The system PD shares both of the fundamental semantic features of InqB: support is persistent, and moreover, the inconsistent state supports all formulas. This means that the support-set $[\varphi]_M$ of a formula $\varphi \in \mathcal{L}^{\mathsf{PD}}$ is always a non-empty and downward closed set of states, i.e., an *inquisitive proposition* in the sense of Definition 1.2.2.

5.2.4. Proposition. *For any model M and formula $\varphi \in \mathcal{L}^{PD}$, we have:*

- *Persistency property: $M, s \models \varphi$ and $t \subseteq s$ implies $M, t \models \varphi$*
- *Empty state property: $M, \emptyset \models \varphi$*

Since the semantics of PD is formulated in the same framework as that of InqB, many of the notions that we introduced in the inquisitive setting also make sense for PD. One example is the notion of truth at a world, defined as support at the corresponding singleton state:

$$M, w \models \varphi \overset{def}{\iff} M, \{w\} \models \varphi$$

Also, as in InqB, we will say that a formula is *truth-conditional* if support for it always amounts to truth at each world.[4]

By spelling out the support clauses with respect to singletons, we find that PD gives rise to the following truth-conditions.

5.2.5. Proposition (Truth-conditions in PD).

- $M, w \models p \iff V(w, p) = 1$
- $M, w \models \overline{p} \iff V(w, p) = 0$
- $M, w \models =\!\!(p_1, \dots, p_n, q)$ *always*
- $M, w \models \varphi \wedge \psi \iff M, w \models \varphi$ *and* $M, w \models \psi$
- $M, w \models \varphi \otimes \psi \iff M, w \models \varphi$ *or* $M, w \models \psi$

[3] The clause does not require the two states t and t' to be disjoint. However, it is easy to see that, within the present language, requiring t and t' to be disjoint would not make a difference.

[4] In the dependence logic literature, truth-conditional formulas are called *flat* formulas.

As in InqB, so also in PD it is easy to isolate a syntactic fragment which essentially coincides with classical propositional logic. For, the following proposition states that any classical formula in $\mathcal{L}^{\mathsf{PD}}$ is truth-conditional.

5.2.6. Proposition. *Any formula $\alpha \in \mathcal{L}_c^{\mathsf{PD}}$ is truth-conditional.*

By reading $\otimes$ as disjunction and a negative atom $\overline{p}$ as $\neg p$, any formula $\alpha \in \mathcal{L}_c^{\mathsf{PD}}$ may be regarded as a formula of classical propositional logic. According to Proposition 5.2.5, α receives the standard truth-conditions, and its support-conditions just amount to truth at each world. Thus, the semantics of a classical formula is essentially the same as in classical propositional logic. Moreover, it is not hard to see that the formulas in $\mathcal{L}_c^{\mathsf{PD}}$ are representative of *all* formulas of classical propositional logic, in the sense that one can define a translation $(\cdot)^*$ from classical propositional logic to $\mathcal{L}_c^{\mathsf{PD}}$ such that φ and φ^* have the same truth-conditions.

Thus, the situation in PD is very similar to that in InqB. We have regarded InqB as arising from first providing a support-based implementation of classical propositional logic, based on the primitive connectives $\bot, \wedge$, and $\to$, and then extending this system with a new connective $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, which takes us beyond the truth-conditional realm. Similarly, we may regard PD as arising from first providing a support-based implementation of classical propositional logic, based on positive and negative atoms and the connectives $\wedge$ and $\otimes$, and then extending this system with a new kind of atoms. As in the case of inquisitive disjunction, adding such atoms takes us beyond the truth-conditional realm: clearly, a dependence atom $=\!(p_1, \dots, p_n, q)$ cannot be truth-conditional, since its truth-conditions are always satisfied, but its support conditions are non-trivial.

As in InqB, it is possible to associate with each formula φ a classical formula φ^{cl} which has the same truth-conditions as φ. In the literature on dependence logic, φ^{cl} is called the *flattening* of φ.

5.2.7. Definition. [Flattening]
For any $\varphi \in \mathcal{L}^{\mathsf{PD}}$, the *flattening* of φ is the classical formula $\varphi^{cl} \in \mathcal{L}_c^{\mathsf{PD}}$ obtained by replacing each occurrence of a dependence atom in φ by the formula $p \otimes \overline{p}$.

5.2.8. Proposition (Flattening preserves truth-conditions).
For any model M and formula $\varphi \in \mathcal{L}^{\mathsf{PD}}$: $|\varphi^{cl}|_M = |\varphi|_M$.

Since InqB and PD are both interpreted relative to information states in a propositional information model, it also makes good sense to say that a formula φ of InqB is equivalent to a formula ψ of PD: this simply means that for any model M and state s, $M, s \models \varphi \iff M, s \models \psi$.

Given this fact, it is natural to ask whether one of these logics can be translated into the other. In fact, Yang (2014) proved that these logics are inter-translatable.

5.2.9. Theorem (InqB and PD are equi-expressive).
There exist maps $(\cdot)^i : \mathcal{L}^{PD} \to \mathcal{L}^{P}$ *and* $(\cdot)^d : \mathcal{L}^{P} \to \mathcal{L}^{PD}$ *such that:*

- *for all* $\varphi \in \mathcal{L}^{PD}$, $\varphi \equiv \varphi^i$
- *for all* $\varphi \in \mathcal{L}^{P}$, $\varphi \equiv \varphi^d$

However, this should not lead one to think that InqB and PD are essentially two formulations of the same logic. The maps $(\cdot)^i$ and $(\cdot)^d$ are not simple, recursively defined translations. To go from one system to the other, one must first compute the meaning of a formula, and then construct a formula in the other system that expresses that meaning. Thus, even simple formulas may be translated to very complex ones; moreover, no syntactic structure is preserved by the translation.

To see why this is so, recall what we have discussed in Section 2.5.5 for InqB: due to the fact that the logic is not closed under uniform substitution, there is a difference between the definability of a connective, and its *uniform definability*. For instance, it is shown in Ciardelli (2009) that, if we take $\neg$ as a primitive connective in InqB, then $\{\neg, \vee\!\!\!\vee\}$ is an expressively complete set of connectives. However, the connectives $\wedge$ and $\to$ are not uniformly definable from $\neg$ and $\vee\!\!\!\vee$, which means that the language generated by $\{\neg, \vee\!\!\!\vee\}$ does not allow us to express the operations which are performed by $\wedge$ and $\to$, which are fundamental algebraic operations on the space of inquisitive propositions (Roelofsen, 2013).

The situation is similar in PD: in this system, too, atoms are truth-conditional. Thus, they do not stand for arbitrary formulas, but only for arbitrary statements. Due to the restrictions existing on the syntax of PD, atoms cannot be replaced by arbitrary formulas to yield a well-formed formula of $\mathcal{L}^{\mathsf{PD}}$, which makes it somehow tricky to make sense of the notion of uniform definability. However, Yang (2014) shows how to formulate such a notion, and proves that neither implication $\to$ nor inquisitive disjunction $\vee\!\!\!\vee$ is uniformly definable in InqB. Thus, the operations corresponding to these connectives are not expressible in PD.

Conversely, it seems highly implausible that $\otimes$ is uniformly definable in InqB, though this is only a conjecture at this point. If this is correct, then the operation corresponding to tensor disjunction is not expressible in InqB. In Section 5.2.4, we will consider briefly the effect of adding $\otimes$ as a primitive connective to InqB.

Thus, InqB and PD should really be regarded as different logics, in which different operations on inquisitive propositions can be expressed. In the next section we will see that the operations corresponding to $\vee\!\!\!\vee$ and $\to$, which are expressible in InqB, but not in PD, play a crucial role in a general analysis of dependencies in propositional logic.

5.2.2 Dependencies in propositional logic

In PD, the task of capturing dependencies is entrusted to dependence atoms, $=\!(p_1, \dots, p_n, q)$. But what is the relation that these atoms capture? In the previous chapters, we have analyzed dependency as a relation holding between several questions, namely, the relation of contextual entailment. The following proposition shows that dependence atoms do indeed capture a relation of dependency in this sense, whose protagonists are atomic polar questions, $?p_1, \dots, ?p_n, ?q$.

5.2.10. Proposition (Dependence atoms capture dependencies).
For any model M and state s: $M, s \models =\!\!(p_1, \dots, p_n, q) \iff ?p_1, \dots, ?p_n \models_s ?q$

Proof. It is easy to see that an atomic polar question $?p$ is supported at a state s iff all the worlds in s agree on the truth-value of p.

$$s \models ?p \iff [V(w,p) = V(w',p) \text{ for all } w, w' \in s]$$

Now suppose $M, s \models =\!\!(p_1, \dots, p_n, q)$ and consider any state $t \subseteq s$ which supports $?p_1, \dots, ?p_n$. This means that each p_i has the same truth-value in all the worlds in t. Since $t \subseteq s$ and $s \models =\!\!(p_1, \dots, p_n, q)$, it follows that q must also have the same truth-value in all the worlds in t. Thus, we must have $M, t \models ?q$. This shows that $?p_1, \dots, ?p_n \models_s ?q$.

Conversely, suppose $?p_1, \dots, ?p_n \models_s ?q$. Let $w, w' \in s$ and suppose $V(w, p_i) = V(w', p_i)$ for each i. This means that, for each i, we have $\{w, w'\} \models ?p_i$. Since $?p_1, \dots, ?p_n \models_s ?q$ and $\{w, w'\} \subseteq s$, it follows that $\{w, w'\} \models ?q$. But this means precisely that $V(w, q) = V(w', q)$, which shows that $s \models =\!\!(p_1, \dots, p_n, q)$. $\square$

Notice that, due to the support-conditions for $\wedge$, we have $?p_1, \dots, ?p_n \models_s ?q \iff ?p_1 \wedge \dots \wedge ?p_n \models_s ?q$. Moreover, we saw in Chapter 1 that contextual entailments can always be internalized in the language by means of the support-based implication operation: for any formulas φ, ψ, we have $\varphi \models_s \psi \iff M, s \models \varphi \to \psi$. Thus, the previous proposition has the following corollary.

5.2.11. Corollary. $=\!\!(p_1, \dots, p_n, q) \equiv ?p_1 \wedge \dots \wedge ?p_n \to ?q$

This shows that dependence atoms, the fundamental ingredient of PD, may be defined in a uniform way in InqB. While this fact was already observed by Yang (2014), the conceptual picture developed in the previous chapters of this thesis casts new light on this connection in several ways.

First, we can now see that this decomposition reflects a more fundamental connection between dependencies and questions: a dependency is a case of contextual entailment having questions as its protagonists; since contextual entailment can be internalized by means of implication, this relation is then expressible in the language in the form of an implication between questions.

Second, the above definition decomposes the dependence atom into operators that are simpler and more fundamental: the implication $\to$, the conjunction $\wedge$, the inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$, and the falsum constant $\bot$ (the latter two being hidden within the shorthand '?'). The mathematical "simplicity" of these operators is witnessed by the fact that each of them corresponds to a familiar algebraic operation on the space of inquisitive propositions, which forms a Heyting algebra (Abramsky and Väänänen, 2009; Roelofsen, 2013). This has a concrete payoff when it comes to proof theory, since as we saw, these operations can all be handled by means of natural, and in fact standard, inference rules. Thus, the above decomposition of dependence atoms is also useful in order to handle dependencies in inference in a natural way.

Third, our perspective brings out that dependence atoms capture only a special case of a more general phenomenon. What dependence atoms capture are dependencies among *atomic polar questions*, i.e., questions of the form $?p$ for $p \in \mathcal{P}$. However, dependencies may concern all sorts of questions. Expressing dependencies by means of implication is completely general in this respect: for any questions $\mu_1, \ldots, \mu_n, \nu$ expressible in InqB, the fact that $\mu_1, \ldots, \mu_n$ determine ν can be captured in InqB as a case of entailment in context, and expressed within the language by means of the implication $\mu_1 \wedge \cdots \wedge \mu_n \to \nu$.

To make this point concrete, let us consider a few instances of dependencies that involve questions other than atomic polar questions.

5.2.12. Example. [Non-atomic polar questions] Let the atoms a, b, c stand, respectively, for the statement that Alice, Bob, and Charles will go to the party. Suppose that whether Charles will go to the party is determined by whether at least one of Alice and Bob will go there. The dependency holds in a context s in case $?(a \vee b) \models_s ?c$, and it may thus be expressed by the following implication:

$$?(a \vee b) \;\to\; ?c$$

This example does not yet witness a serious gap in generality between capturing dependencies by means of dependence atoms, and capturing them by means of implications among questions. This is because, even though the dependence atom of PD is restricted to atomic sentences, there is no serious reason for this limitation. One may naturally consider so-called *extended* dependence atoms of the form $=(\varphi_1, \ldots, \varphi_n, \psi)$, where $\varphi_1, \ldots, \varphi_n, \psi \in \mathcal{L}^{\mathsf{PD}}$.[5] Making use of the generalized notion of truth defined above, the clause for dependence atoms can be generalized as follows:

$$s \models =(\varphi_1, \ldots, \varphi_n, \psi) \iff \forall w, w' \in s : \text{ if } (w \models \varphi_i \iff w' \models \varphi_i) \text{ for all } i \text{ then } w \models \psi \iff w' \models \psi$$

That is, $=(\varphi_1, \ldots, \varphi_n, \psi)$ is supported in s if, relative to s, the truth-value of ψ is completely determined by the truth-values of $\varphi_1, \ldots, \varphi_n$. Making use of these

[5] For the case in which $\varphi_1, \ldots, \varphi_n, \psi$ are classical formulas, such atoms have indeed been investigated, especially in the context of modal dependence logic, where they enhance the expressive power of the language (Ebbing et al., 2013; Hella et al., 2014). The reason for the restriction to classical formulas is discussed in detail in the next sub-section.

extended dependence atoms, the dependency of our previous example can be captured by means of the formula $=(a \vee b\,, c)$.

Dependencies among non-atomic polar questions can generally be captured by means of extended dependence atoms in a similar way. At the heart of this is the fact that a polar question $?\alpha$ is a question about the truth-value of a statement α. As a consequence, in this particular case the relation $?\alpha \models_s ?\beta$ of dependency can also be construed as a relation $\alpha \rightsquigarrow_s \beta$ between two statements, which holds in case the truth-value of the first determines the truth-value of the second.

However, not all questions ask for the truth-value of some statement. For instance, this is not the case for an alternative question *whether p or q* and for a conditional question *whether q if p* (in visual terms, these questions do not induce a bi-partition of the logical space, as illustrated by figures 2.2(a) and 2.2(d)). As a consequence, relations of dependence that involve such questions cannot be re-conceptualized as relations among some underlying statements. However, such questions can very well participate in interesting dependency relations, as the following examples illustrate. It is such examples that show that the question-based approach to dependency is strictly more general than the one based on dependence atoms.

5.2.13. EXAMPLE. [Alternative questions] Let a and b stand, respectively, for the statement that Alice is in the office, and for the statement that Bob is. Moreover, let mo, tu, we, th, fr stand, respectively, for the statement that it is Monday, Tuesday, etc. Now, suppose that whether Alice or Bob is in the office (read as an alternative question) depends on what day of the week it is. The question what day of the week it is can be captured by the following inquisitive disjunction:[6]

$$day \;:=\; mo \mathbin{\vee\!\!\!\vee} tu \mathbin{\vee\!\!\!\vee} we \mathbin{\vee\!\!\!\vee} th \mathbin{\vee\!\!\!\vee} fr$$

The dependency holds in a context s in case $day \models_s a \mathbin{\vee\!\!\!\vee} b$, and it may thus be expressed by means of the following implication:

$$day \;\to\; a \mathbin{\vee\!\!\!\vee} b$$

The various alternatives for the conditional correspond to ways in which the dependency may hold, such as the following one: if it is Monday, Tuesday, or Friday, then Alice is in the office; if it is Wednesday or Thursday, Bob is.

$$(mo \vee tu \vee fr \;\to\; a) \wedge (we \vee th \;\to\; b)$$

5.2.14. EXAMPLE. [Conditional questions] Let a stand for the statement that Bob asks Alice to dance, d for the statement that Alice will dance, and g for the statement that Alice is in a good mood. Now, suppose that whether Alice will

[6]If we wish, it is unproblematic to modify the questions involved in this example so that they presuppose that exactly one of the disjuncts is true.

dance if John asks her is determined by whether or not she is in a good mood. This dependency holds in a context s in case we have $?g \models_s a \to ?d$, and thus it may be expressed by means of the following implication:

$$?g \to (a \to ?d)$$

The various alternatives for the conditional correspond to the various ways in which the dependency may hold, such as the following one: if Mary is in a good mood, she will dance if John asks her, but if she is not is a good mood, she won't.

$$(g \to (a \to d)) \wedge (\neg g \to (a \to \neg d))$$

Incidentally, notice that $?g \to (a \to ?d) \equiv a \to (?g \to ?d)$: this shows that a dependency of a conditional question $a \to ?d$ on a polar question $?g$ can be equivalently regarded as a dependency among polar questions which holds conditionally on the statement a. That is, our situation can be equivalently described as one where, conditionally on John asking Mary to dance, whether she will dance depends on whether she is in a good mood. Notice that $?g \to ?d$ is nothing but the dependence atom $=(g, d)$: so, this is an example in which the sort of dependency expressed by a dependence atom holds, not *simpliciter*, but rather conditionally on a statement, in the sense of Section 1.4.2.

Summing up, then, regarding dependency as question entailment gives rise to a principled and general view of dependencies, which encompasses the relation captured by the dependence atoms of PD as a particular case, but which also encompasses many other natural dependence relations which are not covered by the dependence atoms. The operations expressed by implication and inquisitive disjunction, which are available in InqB, but not in PD, play a crucial role in this respect: the latter allows us to form questions in a natural way, while the former allows us to express dependencies between such questions within the language.[7]

5.2.3 Higher-order dependencies and embedding problems

If we try to regard $=(\cdot, \cdot)$ as a logical operator, we will immediately notice one rather odd feature: its application is restricted to a particular kind of formulas, namely, propositional atoms. Since this restriction is syntactic in nature, it is natural to consider how it may be lifted. In the previous section, we have already

[7]Of course, it follows from Proposition 5.2.9 that, given a formula φ that expresses the dependency in InqB, we can find some equivalent formula φ^d in $\mathcal{L}^{\mathsf{PD}}$. However, such a formula does not provide a convenient way of representing the dependency: first, φ^d is typically very cumbersome; second, there is no systematic way of obtaining φ^d from formulas expressing the questions involved; rather, φ^d has to be cooked up a posteriori by first computing all the ways for the dependency to obtain; finally, there is no way of recognizing from its syntactic form that φ^d expresses a dependency, and to read off what dependency this is.

mentioned that there is a natural way to make the semantic clause for $=\!(\cdot,\cdot)$ applicable to arbitrary formulas. Since truth at a world is defined for all formulas in the language, we can give the following definition:

$$s \models {=}(\varphi_1,\dots,\varphi_n,\psi) \iff \forall w,w' \in s: \text{ if } (w \models \varphi_i \iff w' \models \varphi_i) \text{ for all } i \text{ then } (w \models \psi \iff w' \models \psi)$$

As we mentioned in footnote 5, such a generalization of the dependence operator has indeed been considered in the literature on modal dependence logic (see Ebbing et al., 2013; Hella et al., 2014, among others). Even in this work, however, the application of the operator $=\!(\cdot,\cdot)$ is taken to be restricted to classical formulas, i.e., to formulas which do not in turn contain occurrences of the dependence operator. This is because, the moment the above clause is applied to formulas which themselves contain occurrences of the dependence operator, the results are no longer satisfactory. For, recall that a formula ${=}(\varphi_1,\dots,\varphi_n,\psi)$ is always true at any given world. As a consequence, the above clause predicts that any higher-order dependence formula such as the following is a tautology.

$${=}(\ {=}(\varphi_1,\dots,\varphi_n,\psi)\ ,\ {=}(\chi_1,\dots,\chi_m,\xi)\ )$$

This shows that, in propositional (and modal) dependence logic, it is hard to make sense of higher-order dependencies—dependencies among dependence formulas. This issue is sometimes called the *embedding problem* for dependence atoms.

Now let us take a look at this problem from the perspective of our system InqB. We have seen that, in InqB, a dependency is expressed by a conditional $\mu \to \nu$ involving two questions as antecedent and consequent. As Figure 2.2(e) exemplifies, typically such a conditional is itself a question: for, $\mu \to \nu$ can be settled in several ways, corresponding to the different ways in which the dependency may obtain, i.e., to the different possible dependence functions from μ to ν.[8] E.g., we saw in Chapter 2 that to settle $?p \to ?q$ is to establish one of the following statements, i.e., to establish of some function $f : \mathcal{R}(?p) \to \mathcal{R}(?q)$ that it is a dependence function.

1. $(p \to q) \wedge (\neg p \to q) \ \equiv\ q$
2. $(p \to q) \wedge (\neg p \to \neg q) \ \equiv\ q \leftrightarrow p$
3. $(p \to \neg q) \wedge (\neg p \to q) \ \equiv\ q \leftrightarrow \neg p$
4. $(p \to \neg q) \wedge (\neg p \to \neg q) \ \equiv\ \neg q$

[8] One may be tempted to read $\mu \to \nu$ as "μ determines ν" and thus to think that $\mu \to \nu$ should qualify as a statement. That is not quite right, as witnessed by the fact that $?p \to ?q$ is always true, and that its negation is a contradiction. The issue of how to provide a proper logical representation of dependence statements is discussed in detail in the next chapter.

Being a question, a conditional $\mu \to \nu$ can itself stand in the relation of dependency with other questions. E.g., consider a contextual entailment of the form:

$$\mu \to \nu \ \models_s \ \mu' \to \nu'$$

This captures the fact that, in the context s, settling $\mu \to \nu$ implies settling $\mu' \to \nu'$. Now, to settle $\mu \to \nu$ is to establish a specific way for μ to determine ν, i.e., a dependence function from μ to ν. Similarly, to settle $\mu' \to \nu'$ is to establish a way for μ' to determine ν', i.e., a dependence function from μ' to ν'. Thus, the above dependency amounts to the following: relative to s, any specific way for μ to determine ν yields a corresponding way for μ' to determine ν'.

As a concrete example, suppose in our state s it is settled that $r \leftrightarrow q \vee p$. Then, as soon as we settle a way for $?p$ to determine $?q$, this yields a way for $?p$ to determine $?r$: if we learn that $q \leftrightarrow p$, then we know that $r \leftrightarrow p$; if we learn that $q \leftrightarrow \neg p$, then we know that r; if we learn that q, then we know that r; and, finally, if we learn that $\neg q$, then we know that $r \leftrightarrow p$. Thus, any resolution of $?p \to ?q$ yields a corresponding resolution of $?p \to ?r$, which shows that the following higher-order dependency holds in the given context:

$$?p \to ?q \ \models_s \ ?p \to ?r$$

A similar, but more concrete example of a higher-order dependency will be discussed in Section 5.3.4 within the setting of first-order logic. For the time being, let us just point out that, if dependencies are regarded as relations between questions and captured by means of implication, the embedding problem simply does not arise. In InqB, all connectives apply unrestrictedly to all sorts of formulas. Moreover, we can make good sense of the relation of dependency among two dependence implications $\mu \to \nu$ and $\mu' \to \nu'$: this simply means that any way for μ to determine ν yields a corresponding way for μ' to determine ν'. Since nothing prevents us from nesting implications, such a higher-order dependency can then be expressed straightforwardly as $(\mu \to \nu) \to (\mu' \to \nu')$.

To conclude, it is worth pausing to consider why the embedding problem arises for dependence atoms. Our perspective suggests the following diagnosis. In our view, dependency is a relation between questions. Question meaning is captured by support conditions. Hence, to check whether a dependency holds, we must in general look at the support conditions for the questions involved. However, if our dependency concerns *polar* questions $?\alpha_1, \dots, ?\alpha_n, ?\beta$, then dependency boils down to a relation among the statements $\alpha_1, \dots, \alpha_n, \beta$, which involves only their *truth*-conditions. It is this truth-conditional relation which is detected by the dependence atom. But this is only a special case: as we have seen, not all questions are polar questions. In particular, a dependence implication $\mu \to \nu$ is not a polar question, as Figure 2.2(e) shows. In order to account for the relation of dependency between such questions, a support-sensitive treatment of dependency seems indispensable.

5.2.4 Importing tensor in InqB

Besides dependence atoms, the other ingredient of the system PD which is novel from the perspective of InqB is the tensor disjunction $\otimes$. In this section, we briefly consider the prospects of adding this operator to InqB.

Let us denote by $\mathcal{L}^{\mathsf{P}}_{\otimes}$ the language obtained by adding $\otimes$ to $\mathcal{L}^{\mathsf{P}}$, and let us denote $\mathsf{InqB}^{\otimes}$ the system obtained by interpreting all the operators as in InqB, and interpreting $\otimes$ according to the clause in Definition 5.2.1. A first fact about tensor is that it preserves truth-conditionality.

5.2.15. Proposition ($\otimes$ preserves truth-conditionality).
If $\alpha, \beta \in \mathcal{L}^{\mathsf{P}}_{\otimes}$ are truth-conditional, then $\alpha \otimes \beta$ is truth-conditional.

Moreover, as in the context of PD, $\otimes$ has the same truth-conditions as $\vee$ and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

5.2.16. Proposition (Truth-conditions for $\otimes$).
$M, w \models \varphi \otimes \psi \iff M, w \models \varphi$ *or* $M, w \models \psi$

Thus, when applied to truth-conditional formulas α, β, tensor disjunction returns a truth-conditional formula $\alpha \otimes \beta$ which is equivalent to the classical disjunction $\alpha \vee \beta$. This means that, in setting up a support-based implementation for classical logic, we could have used $\otimes$ as our classical disjunction, rather than defining disjunction in terms of the other connectives. This option is indeed pursued in Ciardelli (2015a). While $\otimes$ and $\vee$ coincide on statements, however, they yield different results when it comes to questions. The effect of $\vee$ on questions is very dull: since $\varphi \vee \psi$ abbreviates the negation $\neg(\neg\varphi \wedge \neg\psi)$, and since negations are always statements, $\varphi \vee \psi$ is always a statement, equivalent to the classical formula, $\varphi^{cl} \vee \psi^{cl}$. On the other hand, $\otimes$ does allow us to combine two questions to yield a new question. To see what effect $\otimes$ has on questions, consider first an example. We have:

$$?p \otimes ?q \equiv (p \vee q) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\neg p \vee q) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (p \vee \neg q) \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} (\neg p \vee \neg q)$$

That is, $?p \otimes ?q$ is a question that is settled as soon as we provide a disjunction of a resolution of $?p$ with a resolution of $?q$. In fact, this illustrates a general fact about the result of combining two formulas by means of $\otimes$. To spell this out in general, let us extend the notion of *resolutions* of a formula to $\mathsf{InqB}^{\otimes}$.

5.2.17. Definition.
The set $\mathcal{R}(\varphi)$ of resolutions of a formula $\varphi \in \mathcal{L}^{\mathsf{P}}_{\otimes}$ is the set of classical formulas obtained by augmenting Definition 2.4.1 with the following clause:

- $\mathcal{R}(\varphi \otimes \psi) = \{\alpha \vee \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$

As in InqB, to settle a formula φ is to establish some resolution of it, which gives us an inquisitive normal form.

$$
\dfrac{\varphi}{\varphi \otimes \psi}\,(\otimes\mathsf{i}_1)
\qquad
\dfrac{\psi}{\varphi \otimes \psi}\,(\otimes\mathsf{i}_2)
\qquad\qquad
\dfrac{\begin{array}{ccc}[\varphi] & [\psi] & \\ \vdots & \vdots & \\ \alpha & \alpha & \varphi \otimes \psi\end{array}}{\alpha}\,(\otimes\mathsf{e})
$$

$$
\dfrac{\begin{array}{ccc}[\varphi] & [\psi] & \\ \vdots & \vdots & \\ \varphi' & \psi' & \varphi \otimes \psi\end{array}}{\varphi' \otimes \psi'}\,(\otimes\mathsf{r})
\qquad\qquad
\dfrac{\varphi \otimes \psi}{\psi \otimes \varphi}\,(\otimes\mathsf{c})
$$

$$
\dfrac{\varphi \otimes (\psi \otimes \chi)}{(\varphi \otimes \psi) \otimes \chi}\,(\otimes\mathsf{a})
\qquad\qquad
\dfrac{\varphi \otimes (\psi \mathbin{\mathbb{V}} \chi)}{(\varphi \otimes \psi) \mathbin{\mathbb{V}} (\varphi \otimes \chi)}\,(\otimes\mathsf{d})
$$

Figure 5.1: Inference rules for $\otimes$, where α is restricted to classical formulas.

5.2.18. Proposition.
For any $\varphi \in \mathcal{L}^P_\otimes$, if $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$, then $\varphi = \alpha_1 \mathbin{\mathbb{V}} \dots \mathbin{\mathbb{V}} \alpha_n$

In particular, if μ and ν are two questions, then the previous proposition gives:

$$\mu \otimes \nu \equiv \mathbb{V}\{\alpha \vee \beta \mid \alpha \in \mathcal{R}(\mu), \beta \in \mathcal{R}(\nu)\}$$

That is, $\mu \otimes \nu$ is settled in a state if a disjunction of a resolution of μ with a resolution of ν is established in the state. It is unclear to me whether there are contexts in which such "pointwise disjunctive" questions play a role. To the extent that there are, such questions can be obtained naturally by means of $\otimes$.

Extending our proof system for InqB to handle $\otimes$ is fairly straightforward, even though the set of rules needed for $\otimes$ is not as parsimonious as those for the other connectives.[9] The relevant set of rules, described in Figure 5.1 is essentially due to Yang (2014), though it has been made more minimal by omitting some redundant rules.

Let us comment on these rules briefly. First, a tensor disjunction can be introduced in the standard way, inferring it from a specific disjunct. However, as for our defined disjunction $\vee$, the standard elimination rule has to be restricted to

[9] At least, that is so within a natural-deduction style system. Plausibly, $\otimes$ can be handled nicely in a sequent calculus with two types of contexts, additive and multiplicative, such as the one proposed by O'Hearn and Pym (1999) for their logic of bunched implications.

conclusions which are classical formulas.[10] Once again, this is needed to prevent unsound derivations such as $p \otimes \neg p \vdash ?p$: the formula $p \otimes \neg p$ is a tautology, and thus, it does not logically resolve the question $?p$.

Since the elimination rule restricted in this way only allows us to infer classical consequences of an inquisitive disjunction, we need some additional rules to characterize those non-classical formulas that follow from it. This is the role of the remaining rules in Figure 5.1. First, the rule ($\otimes$r), called *tensor replacement*, allows us to replace each disjunct of a tensor disjunction by a consequence of it. The rules ($\otimes$c) and ($\otimes$a), called *tensor commutativity* and *associativity* respectively, stipulate that $\otimes$ is commutative and associative. Finally the rule ($\otimes$d) of *tensor distribution* allows us to distribute a tensor disjunction over an inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. The following theorem states that adding these rules to our proof system for InqB yields a sound and complete proof system for $\mathsf{InqB}^{\otimes}$. For the proof of this result, the reader is referred to Ciardelli (2015a).

5.2.19. Theorem (Completeness for $\mathrm{InqB}^{\otimes}$).
The proof system obtained by expanding our natural deduction system for InqB with the rules of Figure 5.1 is sound and complete for $\mathsf{InqB}^{\otimes}$.

5.3 First-order logic

5.3.1 The system FOD

As we saw in Section 5.1, dependence logic was first developed for the language of first-order logic, and later on the same tools were applied to the setting of modal and, ultimately, propositional logic. However, there is a fundamental difference between propositional and modal dependence logic on the one hand, and first-order dependence logic on the other: in the former case, dependencies concern the truth-values of (atomic) formulas; in the latter case, they concern the value of first-order variables.

The language $\mathcal{L}^{\mathsf{FOD}}$ of first-order dependence logic is obtained by introducing, besides the usual atomic formulas $R(t_1, \ldots, t_n)$ and $t = t'$ of first-order logic, a new kind of atomic formulas, called *dependence atoms*, and having the syntactic form $=(x_1, \ldots, x_n, y)$, where $x_1, \ldots, x_n, y \in \mathsf{Var}$. Instead of having a negation operator, $\mathcal{L}^{\mathsf{FOD}}$ includes negative atoms, that we will denote $\overline{R(t_1, \ldots, t_n)}$ and $t \neq t'$. Complex formulas may be formed by means of the connectives $\wedge$ and $\otimes$, and be means of the quantifiers $\forall^d$ and $\exists^d$.[11] Thus, the language $\mathcal{L}^{\mathsf{FOD}}$ is

[10]Here, classical formulas are defined as in Chapter 2, as formulas built up from atoms and $\bot$ by means of $\wedge$ and $\rightarrow$. However, nothing would change if we also allowed $\otimes$ in this list. As we saw, any formula $\varphi \in \mathcal{L}^{\mathsf{P}}_{\otimes}$ is truth-conditional as long as it contains no occurrence of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

[11]These quantifiers are simply denoted $\forall$ and $\exists$ in dependence logic. We use superscripts to distinguish them from the quantifiers of inquisitive logic, which have different semantic clauses.

given by the following definition, where $\mathbf{t} = t_1, \ldots, t_n$ is a tuple of terms, and $\mathbf{x} = x_1, \ldots, x_n$ is a tuple of variables.

$$\varphi \ := \ R\mathbf{t} \mid \overline{R\mathbf{t}} \mid t = t' \mid t \neq t' \mid {=}(\mathbf{x}, y) \mid \varphi \wedge \varphi \mid \varphi \otimes \varphi \mid \forall^d x\varphi \mid \exists^d x\varphi$$

To make sense of dependencies between the values of variables, it is not necessary to consider multiple first-order models at once; instead, what we need to consider is multiple assignments. Following the standard terminology of dependence logic, we will refer to a set of assignments as a *team*.[12]

5.3.1. Definition. [Teams]
A *team* over a domain D is a set of assignments into D, i.e., a set of functions $g : \mathsf{Var} \to D$.

From our perspective, we may think of this semantic setting as one in which one has total information about the model, but only partial information about the value of each variable. In order to spell out the semantics of FOD, we need to introduce some operations on teams.

5.3.2. Definition. [Operations on teams]
Let X be a team over a domain D, $x \in \mathsf{Var}$, $d \in D$, and $f : X \to D$. Then:

- $X[x \mapsto d] = \{g[x \mapsto d] \mid g \in X\}$
- $X[x \mapsto f] = \{g[x \mapsto f(g)] \mid g \in X\}$
- $X[x \mapsto D] = \{g[x \mapsto d] \mid g \in X, d \in D\}$

In words, $X[x \mapsto d]$ is the team that results from setting the value of x throughout the team to d; $X[x \mapsto f]$ is the team obtained by modifying the value of x throughout the team according to f, that is, replacing g by $g[x \mapsto f(g)]$; finally $X[x \mapsto D]$ is the team obtained by replacing each $g \in X$ by a number of assignments $g[x \to d]$, one for each $d \in D$, i.e., by erasing any information available in X about the value of x.

The system FOD is obtained by evaluating the sentences of $\mathcal{L}^{\mathsf{FOD}}$ with respect to a first-order model $M = \langle D, I\rangle$ and to a team X over D by means of the following clauses, where the denotation $[t]^M_g$ of a term is defined as usual.

5.3.3. Definition. [Semantics of FOD]

- $M \models_X R(t_1, \ldots, t_n) \iff$ for all $g \in X$, $\langle [t_1]^M_g, \ldots, [t_n]^M_g\rangle \in I(R)$

[12]In dependence logic, a team is a set of *partial* assignments, i.e., partial functions from Var to the domain of the given model. While this is indeed very convenient in practice, here we will stick with total assignments, simply to avoid having to make stipulations about cases in which the value of a term is undefined. This difference is not essential to the points discussed below.

- $M \models_X (t = t') \iff$ for all $g \in X$, $[t]^M_g = [t']^M_g$
- $M \models_X \overline{R(t_1, \ldots, t_n)} \iff$ for all $g \in X$, $\langle [t_1]^M_g, \ldots, [t_n]^M_g \rangle \notin I(R)$
- $M \models_X (t \neq t') \iff$ for all $g \in X$, $[t]^M_g \neq [t']^M_g$
- $M \models_X =\!\!(x_1, \ldots, x_n, y) \iff$ for all $g, g' \in X$: if $g(x_i) = g'(x_i)$ for each i then $g(y) = g'(y)$
- $M \models_X \varphi \wedge \psi \iff M \models_X \varphi$ and $M \models_X \psi$
- $M \models_X \varphi \otimes \psi \iff X = Y \cup Y'$ for some Y, Y' s.t. $M \models_Y \varphi$ and $M \models_{Y'} \psi$
- $M \models_X \forall^d x\varphi \iff M \models_{X[x \mapsto D]} \varphi$
- $M \models_X \exists^d x\varphi \iff M \models_{X[x \mapsto f]} \varphi$ for some $f : X \to D$

We may read these clauses as follows. A positive atom $R\bar{t}$ or $t = t'$ is settled with respect to a team X in case it is true under any $g \in X$; similarly, a negative atom $\overline{R\mathbf{t}}$ or $t \neq t'$ is settled with respect to X in case the corresponding positive atom is false under any $g \in X$. A dependence atom $=\!\!(x_1, \ldots, x_n, y)$ is settled with respect to X in case the value that an assignment $g \in X$ assigns to y is determined by the values it assigns to $x_1, \ldots, x_n$, in the sense that if two assignments in X agree on the values of $x_1, \ldots, x_n$, they must also agree on the value of y. This amounts to saying that, within X, the value of y is a function of the values of $x_1, \ldots, x_n$. To make this precise, let us introduce the notion of a dependence function among first-order variables.

5.3.4. Definition. [Dependence functions for first-order variables]
A function $f : D^n \to D$ is a *dependence function* from $x_1, \ldots, x_n$ to y in a team X, notation $f : x_1, \ldots, x_n \rightsquigarrow_X y$, in case for all $g \in X$, $g(y) = f(g(x_1), \ldots, g(x_n))$.

The next proposition says that $=(x_1, \ldots, x_n, y)$ is supported with respect to a team X iff in X there exists some dependence function from $x_1, \ldots, x_n$ to y.

5.3.5. Proposition.
$M \models_X =\!\!(x_1, \ldots, x_n, y) \iff$ *for some* $f : D^n \to D,\ f : x_1, \ldots, x_n \rightsquigarrow_X y$

The clauses for the connectives $\wedge, \otimes$ are essentially as in the propositional case, except that the set structure needed for the interpretation of $\otimes$ now concerns the team X, rather than a state s. A universal formula $\forall^d x\varphi$ is settled with respect to X in case φ is settled in the team $X[x \mapsto D]$ that results from erasing any information that X contains about the value of x. An existential formula $\exists^d x\varphi$ is settled with respect to X if φ is settled in some state that may be obtained by suitably re-setting the value of x throughout X.

Although the relevant information ordering now concerns a team X, rather than a set s of possible worlds, the relation of support still has the familiar features: support is preserved as information grows (persistence property) and in the limit case of inconsistent information, every formula is supported (empty team property).

5.3.6. Proposition. *For any FO-model M and formula $\varphi \in \mathcal{L}^{\mathsf{FOD}}$, we have:*

- Persistence property: $M \models_X \varphi$ *and* $Y \subseteq X$ *implies* $M \models_Y \varphi$
- Empty team property: $M \models_\emptyset \varphi$

In analogy to what we did in the case of states, from this definition of support with respect to a team we may obtain a definition of *truth* with respect to an assignment g simply by looking at support with respect to the corresponding singleton team:

$$M \models_g \varphi \overset{def}{\iff} M \models_{\{g\}} \varphi$$

If we spell out the support conditions with respect to singletons, we find the following clauses for truth-conditions in FOD.

5.3.7. Definition. [Truth-conditions for FOD]

- $M \models_g R(t_1, \ldots, t_n) \iff \langle [t_1]^M_g, \ldots, [t_n]^M_g \rangle \in I(R)$
- $M \models_g (t = t') \iff [t]^M_g = [t']^M_g$
- $M \models_g \overline{R(t_1, \ldots, t_n)} \iff \langle [t_1]^M_g, \ldots, [t_n]^M_g \rangle \notin I(R)$
- $M \models_g (t \neq t') \iff [t]^M_g \neq [t']^M_g$
- $M \models_g {=}(x_1, \ldots, x_n, y)$ always
- $M \models_g \varphi \wedge \psi \iff M \models_g \varphi$ and $M \models_g \psi$
- $M \models_g \varphi \otimes \psi \iff M \models_g \varphi$ or $M \models_g \psi$
- $M \models_g \forall^d x\varphi \iff M \models_{\{g[x \mapsto d] \,|\, d \in D\}} \varphi$
- $M \models_g \exists^d x\varphi \iff M \models_{g[x \mapsto d]} \varphi$ for some $d \in D$

Here, it is important to notice that the truth-conditions for a universal formula $\forall^d\varphi$ depend on the support-conditions of φ, and not only on its truth-conditions. This means that, unlike in the other systems we encountered so far (but similarly to systems that we will encounter in the next chapters), truth does not admit a direct recursive characterization; rather, computing the truth-conditions of some formula φ in general requires computing the support-conditions of some subformula ψ with respect to non-singleton teams.

We will say that a formula is *truth-conditional* in case support at a team X simply amounts to truth under each assignment $g \in X$.[13]

5.3.8. Definition. [Truth-conditionality]
We call a formula $\varphi \in \mathcal{L}^{\mathsf{FOD}}$ *truth-conditional* if for any model M and team X:

$$M \models_X \varphi \iff M \models_g \varphi \text{ for all } g \in X$$

As in propositional dependence logic, it is easy to isolate a fragment of FOD which is fully truth-conditional, and which may be identified with classical first-order logic. Let us say that a formula $\varphi \in \mathcal{L}^{\mathsf{FOD}}$ is *classical* in case it contains no occurrence of a dependence atom, and let $\mathcal{L}_c^{\mathsf{FOD}}$ be the set of such formulas. Now, any classical formula φ may be identified with a formula $\varphi^{\#}$ of classical first-order logic: it suffices to replace each negative atom $\overline{\alpha}$ in φ by $\neg\alpha$, each tensor $\otimes$ by $\vee$, and each quantifier $\forall^d, \exists^d$ by $\forall, \exists$. The following proposition tells us that classical formulas are assigned the same truth-conditions as in classical first-order logic.

5.3.9. Proposition.
For all M, g, and all $\varphi \in \mathcal{L}_c^{\mathsf{FOD}}$, $M \models_g \varphi \iff M \models_g \varphi^{\#}$ in first-order logic

Moreover, classical formulas are always truth-conditional.

5.3.10. Proposition. *Any $\varphi \in \mathcal{L}_c^{\mathsf{FOD}}$ is truth-conditional.*

Thus, classical formulas receive essentially the same meaning in FOD as they do in classical first-order logic. Conversely, it is easy to see that each formula φ of first-order logic may be associated with a classical formula $\varphi^* \in \mathcal{L}_c^{\mathsf{FOD}}$ which has the same truth-conditions. This means that the fragment of FOD consisting only of classical formulas may be identified with classical first-order logic.

As we expect, the support-conditions for a formula φ relative to a team X are only sensitive to the values that the assignments in X assign to the variables which actually occur free in φ. To make this precise, let us define a notion of equivalence of teams with respect to a set V of variables as follows.

5.3.11. Definition.
Let V be a set of variables. We say that two assignments g, g' are equivalent on V, notation $g \equiv_V g'$, in case $g(x) = g'(x)$ for all $x \in V$. We say that two teams X, X' are equivalent on V, notation $X \equiv_V X'$, in case:

- for any $g \in X$ there is $g' \in X'$ such that $g \equiv_V g'$

[13] Again, the term used in the dependence logic literature is *flat*, but I find the term *truth-conditional* more convenient in view of the conceptual perspective adopted here.

- for any $g' \in X'$ there is $g \in X$ such that $g \equiv_V g'$

Then, if two teams X and X' are equivalent on the set of variables which occur free in φ, the difference between X and X' is immaterial for the support of φ.

5.3.12. Proposition.
Let $\varphi \in \mathcal{L}^{\mathsf{FOD}}$. For any M and any $X \equiv_{FV(\varphi)} X'$, $M \models_X \varphi \iff M \models_{X'} \varphi$

In particular, consider the case in which φ is a sentence, i.e., $\mathrm{FV}(\varphi) = \emptyset$. Then we have $g \equiv_{\mathrm{FV}(\varphi)} g'$ for any two assignments g, g', which implies $X \equiv_{\mathrm{FV}(\varphi)} X'$ for any non-empty teams X and X'. Thus, if $\varphi \in \mathcal{L}^{\mathsf{FOD}}$ is a sentence, we can simply write $M \models \varphi$ for $M \models_X \varphi$, where X is any non-empty team. Making use of this fact, it is easy to show that sentences in FOD are generally truth-conditional.

5.3.13. Corollary. *All sentences in $\mathcal{L}^{\mathsf{FOD}}$ are truth-conditional.*

This should not lead one to think that sentences in FOD are essentially the same as sentences in first-order logic. This is not the case. The reason is the fact, observed above, that the truth-conditions of a sentence may depend crucially on the support conditions of a sub-formula of it with respect to non-singleton teams. In particular, universal quantifiers $\forall^d$ "create a range", i.e., they lead us to evaluate their immediate sub-formula in a non-singleton team, where dependence atoms can then be used to express facts that are not first-order expressible.

Väänänen (2007) shows how to use this feature to construct sentences which express properties of the model that are not expressible in first-order logic. For instance, given any signature we can express that the domain of the model is infinite, while given a signature including a relation symbol R, we can express the fact that R is disconnected, as well as the fact that R is an ill-founded ordering.

In fact, it was proved by Väänänen (2007) that, as far as sentences are concerned, FOD has the same expressive power as Σ^1_1, the existential fragment of second-order logic, consisting of second-order formulas of the form:

$$\exists X_1 \dots \exists X_n \varphi$$

where $X_1, \dots, X_n$ are second-order variables, and φ contains no second-order quantifiers. To state Väänänen's result, let us introduce the following terminology: if φ is a sentence in $\mathcal{L}^{\mathsf{FOD}}$ and ψ a sentence in Σ^1_1 (over the same signature), we will say that φ and ψ are equivalent, and write $\varphi \equiv \psi$, in case for any model M we have $M \models \varphi \iff M \models \psi$. Then, we have the following theorem.

5.3.14. Theorem (Väänänen, 2007).
There exist computable maps $(\cdot)^{so} : \mathcal{L}^{\mathsf{FOD}} \to \Sigma^1_1$ and $(\cdot)^d : \Sigma^1_1 \to \mathcal{L}^{\mathsf{FOD}}$ such that:

- *for any sentence $\varphi \in \mathcal{L}^{\mathsf{FOD}}$, $\varphi \equiv \varphi^{so}$*

- *for any sentence* $\varphi \in \Sigma^1_1$, $\varphi \equiv \varphi^d$

This theorem implies that first-order dependence logic is not axiomatizable. For, if it were, then the set of its valid sentences would be recursively enumerable. But by the previous theorem, this would imply that the set of valid Σ^1_1 sentences is recursively enumerable as well, contrary to fact. In fact, Väänänen (2007) shows that the set of (Gödel numbers of) theorems of FOD is not only not recursively enumerable, but not even arithmetical, i.e., the property of being a code of a valid FOD-sentence is not expressible in the language of Peano arithmetic.

5.3.2 Inquisitive logic in team semantics

In the previous section, we were able to compare propositional inquisitive logic InqB and propositional dependence logic PD very precisely: since these systems are interpreted in the same semantic setting, formulas in one system may be equivalent to formulas in the other, operations used in one system can be imported in the other, and so on. We cannot directly compare FOD to InqBQ in the same way, since there is a very fundamental difference in the semantic setting of the two systems: in InqBQ, formulas are interpreted with respect to a set of possible worlds—each having the structure of a first-order model—and by means of a single assignment. In FOD, on the other hand, formulas are interpreted with respect to a single model and by means of a set of assignments.

However, we may try to factor out this fundamental difference. Obviously, we cannot reformulate FOD in a framework using a single assignment, otherwise it would be impossible to provide a suitable interpretation for the dependence atoms $=\!\!(x_1, \ldots, x_n, y)$. However, nothing prevents us from considering a version of InqBQ based on a single model and multiple assignments. We will refer to this system as InqBQT, where Q indicates the presence of quantifiers, while T indicates that formulas are interpreted with respect to teams. In fact, this system has been investigated previously by Yang (2014) under the name of WID (for *weak intuitionistic dependence logic*), and most of the technical results mentioned here can be traced to her work. At the same time, the view on dependency developed in the previous chapters will allow us to take a novel perspective on this system, and to make some points about its features and its scope as a logic for dependencies.

The language of InqBQT will be the same as the language $\mathcal{L}^{\mathsf{Q}}$ of InqBQ: a standard first-order language having as primitives the operators $\bot, \wedge, \rightarrow$, and $\forall$, and enriched with the inquisitive operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$. As usual, we will take $\neg, \vee$, and $\exists$ to be defined operators. As in FOD, formulas will be evaluated with respect to a first-order model and a team. The clauses are identical to those in InqBQ, except that it is now the team that plays the role of the information state.

5.3.15. Definition. [Semantics of InqBQT]

- $M \models_X R(t_1, \ldots, t_n) \iff$ for all $g \in X$, $\langle [t_1]^M_g, \ldots, [t_n]^M_g \rangle \in I(R)$

- $M \models_X (t = t') \iff$ for all $g \in X$, $[t]^M_g = [t']^M_g$
- $M \models_X \bot \iff X = \emptyset$
- $M \models_X \varphi \wedge \psi \iff M \models_X \varphi$ and $M \models_X \psi$
- $M \models_X \varphi \to \psi \iff$ for all $Y \subseteq X : M \models_Y \varphi$ implies $M \models_Y \psi$
- $M \models_X \varphi \vee\!\!\!\vee \psi \iff M \models_X \varphi$ or $M \models_X \psi$
- $M \models_X \forall x \varphi \iff M \models_{X[x \mapsto d]} \varphi$ for all $d \in D$
- $M \models_X \overline{\exists} x \varphi \iff M \models_{X[x \mapsto d]} \varphi$ for some $d \in D$

The clauses for atoms are the same as in FOD: an atomic sentence is settled with respect to a team X if it is true under any assignment $g \in X$. The clauses for the connectives are the familiar inquisitive clauses, except that now, the relevant information ordering concerns the team X. The clauses for the quantifiers are also very similar to those we used in InqBQ, except that instead of setting the value of x to d in just one assignment, we have to do this for all assignments $g \in X$. Now, setting the value of x to d throughout X amounts to stipulating that x denotes d; this allows us to look at what is settled in X about the individual d, rather than about the variable x. Thus, the clauses for the quantifiers may be read as follows: $\forall x \varphi(x)$ (respectively, $\overline{\exists} x \varphi(x)$) is settled with respect to X in case X settles $\varphi(x)$ of every (respectively, some) individual $d \in D$.

Let us take a quick look at the fundamental features of the system. As usual, support is persistent, and any formula is supported relative to the empty team.

5.3.16. PROPOSITION. *For any first-order model M and formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ we have:*

- Persistence property: $M \models_X \varphi$ *and* $Y \subseteq X$ *implies* $M \models_Y \varphi$
- Empty team property: $M \models_\emptyset \varphi$

As in FOD, we define the notion of truth by setting $M \models_g \varphi \iff M \models_{\{g\}} \varphi$. Then, the above support definition implies that all the standard operators have the familiar truth-conditions, while the inquisitive operators $\vee\!\!\!\vee$ and $\overline{\exists}$ have the same truth-conditions as the corresponding classical operators $\vee$ and $\exists$.

5.3.17. PROPOSITION (TRUTH-CONDITIONS FOR INQBQT).

- $M \models_g R(t_1, \dots, t_n) \iff \langle [t_1]^M_g, \dots, [t_n]^M_g \rangle \in I(R)$
- $M \models_g (t = t') \iff [t]^M_g = [t']^M_g$
- $M \not\models_g \bot$

- $M \models_g \varphi \wedge \psi \iff M \models_g \varphi$ *and* $M \models_g \psi$
- $M \models_g \varphi \to \psi \iff M \not\models_g \varphi$ *or* $M \models_g \psi$
- $M \models_g \varphi \mathbin{\vee\!\!\!\vee} \psi \iff M \models_g \varphi$ *or* $M \models_g \psi$
- $M \models_g \forall x\varphi \iff M \models_{g[x\mapsto d]} \varphi$ *for all* $d \in D$
- $M \models_g \overline{\exists} x\varphi \iff M \models_{g[x\mapsto d]} \varphi$ *for some* $d \in D$

Notice that now, unlike in FOD, the truth-conditions for a formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$ depend only on the truth-conditions for the sub-formulas of φ: computing the truth-conditions for φ never requires moving to non-singleton teams. Also, recall from the previous chapter that φ^{cl} is the classical formula obtained from φ by replacing each occurrence of $\mathbin{\vee\!\!\!\vee}$ and $\overline{\exists}$ with $\vee$ and $\exists$, respectively. Then, it follows from the previous proposition that φ and φ^{cl} always have the same truth-conditions, and thus, that any formula has the same truth-conditions as some classical formula.

5.3.18. Corollary.
For any model M, assignment g, and formula $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $M \models_g \varphi \iff M \models_g \varphi^{cl}$

As we did for FOD, we will say that φ is *truth-conditional* if in any model M, φ is supported by a team X as soon as it is true with respect to any $g \in X$. As in InqBQ and in FOD, classical formulas are always truth-conditional.

5.3.19. Proposition. *Any $\alpha \in \mathcal{L}^{\mathsf{Q}}_c$ is truth-conditional in InqBQT.*

Since the truth-conditions for classical formulas are the standard ones, this means that, as far as classical formulas are concerned, the support semantics of InqBQT is essentially equivalent to the standard truth-conditional semantics.

Also as in FOD, moreover, the support-conditions for a formula φ depend only on the values that the assignments in the team X give for those variables which actually occur free in φ.

5.3.20. Proposition.
Let $\varphi \in \mathcal{L}^{\mathsf{Q}}$. For all M and for $X \equiv_{FV(\varphi)} X'$, $M \models_X \varphi \iff M \models_{X'} \varphi$.

In particular, having no free variables, a sentence is not sensitive to the team of evaluation at all, as long as this team is consistent. If φ is a sentence, we can thus write $M \models \varphi$ as a shorthand for $M \models_X \varphi$, where X is any non-empty team. As a corollary we get that, just as in FOD, all sentences are truth-conditional.

5.3.21. Corollary. *Any sentence $\varphi \in \mathcal{L}^{\mathsf{Q}}$ is truth-conditional in InqBQT.*

Thus, in InqBQT, no sentence is a question, even if it contains occurrences of $\mathbin{\vee\!\!\!\vee}$ and $\overline{\exists}$. Notice that, since a sentence φ and its classical variant φ^{cl} have the same truth-conditions, and since both are truth-conditional, we always have $\varphi \equiv \varphi^{cl}$.

5.3.22. Proposition. *For any sentence $\varphi \in \mathcal{L}^{\mathsf{Q}}$, $\varphi \equiv \varphi^{cl}$ in InqBQT.*

Thus, any sentence of InqBQT is equivalent to a classical first-order sentence. Moreover, recall that classical sentences receive the standard truth-conditions. Thus, in this respect InqBQT is very different from FOD: at the level of sentences, InqBQT is essentially equivalent to standard first-order logic—a result which was first proved by Yang (2014).

On the other hand, things become interesting as soon as we consider formulas with free variables, which are no longer truth-conditional: although questions concerning features of the model cannot be captured, questions concerning features of the assignment can. We will now examine three interesting classes of questions expressible in the system. These classes by no means exhaust the range of questions expressible in InqBQT, but they offer an insightful illustration, and they will suffice to provide some interesting examples of first-order dependencies which go beyond the ones expressed by the dependence atoms of FOD.

For our illustration we will make use of Figure 5.2, which depicts four teams over a domain of natural numbers. As customary in the dependence logic literature, we represent a team X graphically as a table, where the rows correspond to the assignments $g \in X$, while the columns correspond to the variables. In our examples, the questions we will consider have x as their only free variable. Proposition 5.3.20 ensures that the value of the assignments in X on the variable x is all that matters to decide on the support of these questions, which is why our tables consist of only one column, the one corresponding to the variable x.

5.3.23. Example. [Whether t lies within P]
Consider the formula $?Pt$. We have:

$$\begin{array}{lll} M \models_X ?Pt & \iff & M \models_X Pt \text{ or } M \models_X \neg Pt \\ & \iff & [\, M \models_g Pt \text{ for all } g \in X\,] \text{ or } [\, M \not\models_g Pt \text{ for all } g \in X\,] \\ & \iff & [\, [t]^M_g \in I(P) \text{ for all } g \in X\,] \text{ or } [\, [t]^M_g \notin I(P) \text{ for all } g \in X\,] \end{array}$$

Thus, $?Pt$ is settled relative to a team X if it is settled in X that the value of t lies within the extension of P, or it is settled in X that the value of t lies outside of the extension of P. It is easy to see that, as soon as t contains some free variable, $?Pt$ is a question, which we may read as the question of whether or not the value of t is located within P.

For instance, suppose our domain is the set $\mathbb{N}$ of natural numbers, and E is a predicate symbol interpreted as the set of even numbers. Then, $?Ex$ expresses the question whether the value of x is even or odd, and we have $M \models_X ?Ex$ in case the parity of x is settled in X. Thus, for instance, the question $?Ex$ is settled in the teams X_a and X_c of Figure 5.2, where it is settled that x is even, and also in X_b, where it is settled that x is odd, but not in X_d, since both even and odd values of x are compatible with X_d.

x
16
16
16
16

(a)

x
7
13
19
25

(b)

x
12
24
30
36

(c)

x
5
10
15
35

(d)

Figure 5.2: Four teams X_a, X_b, X_c, X_d, each consisting of four assignments into the domain $\mathbb{N}$ of natural numbers. For simplicity, we only display the value that each assignment in the team gives for the variable x.

Question	X_a	X_b	X_c	X_d
$?Ex$	✓	✓	✓	
λx	✓			
$\lambda \mathrm{mod}_3(x)$	✓	✓	✓	
$\overline{\exists} y \mathrm{Pf}(y, x)$	✓		✓	✓

Figure 5.3: The support conditions for the questions in our examples with respect to the four assignments of Figure 5.2.

5.3.24. Example. [Value of t]
Let t be a term, and let y be any variable which does not occur in t. Consider the formula $\overline{\exists} y(t = y)$. We have:

$$\begin{aligned}
M \models_X \overline{\exists} y(t = y) \quad &\iff \quad \text{for some } d \in D : M \models_{X[y \mapsto d]} (t = y) \\
&\iff \quad \text{for some } d \in D : \text{for all } g \in X[y \mapsto d],\ [t]^M_g = g(y) \\
&\iff \quad \text{for some } d \in D : \text{for all } g \in X,\ [t]^M_{g[y \mapsto d]} = g[y \mapsto d](y) \\
&\iff \quad \text{for some } d \in D : \text{for all } g \in X,\ [t]^M_g = d \\
&\iff \quad \text{for all } g, g' \in X,\ [t]^M_g = [t]^M_{g'}
\end{aligned}$$

Thus, $\overline{\exists} y(t = y)$ is settled in X in case all $g \in X$ agree about the value of t, that is, in case the value of t is settled in X. If t contains free variables, then $\overline{\exists} y(t = y)$ is a question, which we refer to as the *identity question* about t, and which we will denote by λt:

$$\lambda t := \overline{\exists} y(t = y) \qquad [y \notin FV(t)]$$

In particular, the question λx is settled in a team X if all the assignments $g \in X$ agree on the value of x, that is, in case the value of x is settled in X. Thus, among the teams depicted in Figure 5.2, λx is settled in X_a, where it is settled that the value of x is 16, but not in X_b, X_c, and X_d, since each of these teams is compatible with several values for x.

For a slightly more complex example, suppose again that our domain is $\mathbb{N}$, and suppose mod_3 is a unary function symbol interpreted in the model as the remainder of the division by 3. Then the question $\lambda\mathrm{mod}_3(x)$ is a question which is settled in a team X in case all the assignments $g \in X$ agree on the value of $\mathrm{mod}_3(x)$, that is, in case the equivalence class of x modulo 3 is settled in X.

Among the teams in Figure 5.2, the question $\lambda\mathrm{mod}_3(x)$ is settled in teams X_a and X_b, where it is settled that $\mathrm{mod}_3(x) = 1$, and also in team X_c, where it is settled that $\mathrm{mod}_3(x) = 0$. The question is not settled in team X_d, since according to the information available in X_d, the value of x may belong to any of the equivalence classes modulo 3.

5.3.25. Example. [Instance of a property]
Let R be an n-ary relation symbol, $\overline{x}$ a tuple of $n-1$ variables, and y a variable distinct from each x_i. Consider the formula $\overline{\exists} yR(\overline{x}, y)$. We have:

$$\begin{aligned} M \models_X \overline{\exists} yR(\overline{x}, y) \quad &\iff \quad \text{for some } d \in D : M \models_{X[y\mapsto d]} R(\overline{x}, y) \\ &\iff \quad \text{for some } d \in D : \text{for all } g \in X[y \mapsto d], \langle g(\overline{x}), g(y)\rangle \in R \\ &\iff \quad \text{for some } d \in D : \text{for all } g \in X, \langle g(\overline{x}), d\rangle \in R \end{aligned}$$

Thus, $\overline{\exists} yR(\overline{x}, y)$ is settled in the team X in case there is an individual d of which it is settled that it stands in the relation R to the tuple denoted by $\overline{x}$.

As an example, suppose again that our domain is the set $\mathbb{N}$ of natural numbers, and suppose our language contains a relation symbol Pf which is interpreted as the relation *being a prime factor of*. Then, the question $\overline{\exists} y\mathrm{Pf}(y, x)$ is settled relative to a team X in case there is a number $n \in \mathbb{N}$ of which it is settled that it is a prime factor of x. Thus, the question $\overline{\exists} y\mathrm{Pf}(y, x)$ is settled in team X_a above, where it is settles that 2 is a prime factor of x, as well as in team X_d, where it is settled that 5 is, and in team X_c, where it is settled of both 2 and 3 that they are prime factors of x. The question is not settled in team X_b, since there is no number which is settled in X_b to be a prime factor of x.

Recall that, in our approach, the relation of dependency amounts to question entailment in context, where the context encodes a relevant body of information. Now, in the systems InqB and InqBQ, this information concerns the state of affairs, and it is modeled by a set of worlds. In InqBQT, by contrast, the information at stake concerns the value of variables, and it is modeled by a team. Thus, in InqBQT it is natural to consider a notion of entailment relativized to the context provided by a given team: when we assess an entailment relative to a team X, it is only assignments $g \in X$, and teams made up of such assignments, that are taken into account.

5.3.26. Definition. [Entailment relative to a team]
We say that Φ entails ψ relative to a FO-model $M = \langle D, I\rangle$ and to a team X, notation $\Phi \models_{M,X} \psi$, in case for all $Y \subseteq X$, $M \models_Y \Phi$ implies $M \models_Y \psi$. If the model M is clear from the context, we also drop reference to it and write $\Phi \models_X \psi$.

Notice that the connection between contextual entailment and implication is preserved. An implication $\varphi \to \psi$ is supported in M with respect to a team X iff φ entails ψ relative to M and X.

$$M \models_X \varphi \to \psi \iff \varphi \models_{M,X} \psi$$

Thus, just as we did for InqB and InqBQ, we can capture dependencies between InqBQT-questions as cases of entailment in context, and we can express them by means of implication. In the next section we turn to examine such dependencies.

5.3.3 Dependencies involving first-order variables

In FOD, dependencies are expressed by means of a dedicated kind of atomic formulas, the dependence atoms $=\!\!(x_1, \dots, x_n, y)$. The following proposition shows that what these atoms capture is a dependency relation in our sense, i.e., a case of contextual entailment whose participants are identity questions about the variables $x_1, \dots, x_n, y$.

5.3.27. Proposition (Dependence atoms capture dependencies).
For any first-order model M, team X, and variables $x_1, \dots, x_n, y$:

$$M \models_X =\!\!(x_1, \dots, x_n, y) \iff \lambda x_1, \dots, \lambda x_n \models_X \lambda y$$

Proof. Suppose $M \models_X =\!\!(x_1, \dots, x_n, y)$, and consider a team $Y \subseteq X$ that supports $\lambda x_1, \dots, \lambda x_n$. As we saw in Example 5.3.24, this means that any $g, g' \in Y$ agree on the value of x_i, for each i. Since $g, g' \in X$ and since $M \models_X =\!\!(x_1, \dots, x_n, y)$, it follows that g and g' must also agree on the value of y. Since this is true for all $g, g' \in Y$, it follows that $M \models_Y \lambda y$. Since this holds for all $Y \subseteq X$, we have $\lambda x_1, \dots, \lambda x_n \models_X \lambda y$.

Conversely, suppose $\lambda x_1, \dots, \lambda x_n \models_X \lambda y$. Consider any $g, g' \in X$ such that $g(x_i) = g'(x_i)$ for all $i \leq n$. This means that $M \models_{\{g,g'\}} \lambda x_i$ for each i. Since $\{g, g'\} \subseteq X$ and $\lambda x_1, \dots, \lambda x_n \models_X \lambda y$, it follows $M \models_{\{g,g'\}} \lambda y$, which means that $g(y) = g'(y)$. Hence, we have $M \models_X =\!\!(x_1, \dots, x_n, y)$ □

Thus, dependence atoms capture a relation of dependency among identity questions. By making use of conjunction, this relation can be turned into a relation of dependency among two InqBQT-questions, namely, $\lambda x_1 \wedge \cdots \wedge \lambda x_n \models_X \lambda y$. Finally, this contextual entailment amounts to the fact that the corresponding implication $\lambda x_1 \wedge \cdots \wedge \lambda x_n \to \lambda y$ is supported relative to X. This shows how a dependence atom of FOD can be expressed in InqBQT.

5.3.28. Corollary. $=\!\!(x_1, \dots, x_n, y) \equiv \lambda x_1 \wedge \cdots \wedge \lambda x_n \to \lambda y$

The possibility of defining dependence atoms in this way by means of the operators $\overline{\exists}, \wedge$, and $\rightarrow$ was noted in Yang (2014), based on a similar decomposition given by Abramsky and Väänänen (2009):

$$=\!(x_1, \ldots, x_n, y) \;\equiv\; =\!(x_1) \wedge \cdots \wedge =\!(x_n) \;\rightarrow\; =\!(y)$$

However, our analysis of the relation of dependency in terms of question entailment casts new light on this fact. First, we can now see that this equivalence stems from a deeper fact, namely, that the configuration expressed by dependence atoms amounts to a relation of entailment between identity questions.

Moreover, we can now see that, as in the propositional case, dependence atoms are only a special case of a general pattern. To illustrate this, let us take a look at some examples of the range of dependencies that can be captured by combining implication and the questions which are expressible in InqBQT. To make our examples more concrete, let us assume a specific language and a specific model for it. The domain of our model is the set $\mathbb{N}$ of natural numbers. Our language contains the following non-logical vocabulary:

- a predicate symbol E, interpreted as the set of even numbers;
- a binary relation symbol $<$, interpreted as the relation *strictly less than*;
- a binary relation symbol Pf, interpreted as the relation *prime factor of.*

To illustrate our examples, we will make use of a few teams, depicted in Figure 5.4. Since our examples will involve just two free variables x and y, only the columns corresponding to these variables are displayed in the figures.

5.3.29. Example. [Value of x determines parity of y]
In the team X_a of Figure 5.4(a), the value x does not completely determine the value of y. Yet, it does determine something about y, namely, the *parity* of y: if the value of x is 0, then y is even; if the value of x is 1 or 2, then y is odd. So, in this context an interesting dependency relation between x and y holds, but one which is much weaker than the relation expressed by the dependence atom $=\!(x, y)$. This dependency amounts to the following:

$$\lambda x \models_{X_a} ?Ey$$

This weak dependency amounts to the fact that, relative to X, complete information about x yields some partial information about y. By means of implication, this sort of dependency may be expressed as $\lambda x \rightarrow ?Ey$.

5.3.30. Example. [Parity of x determines value of y]
In the team X_b of Figure 5.4(b), the value of y is completely determined by the value of x. But in fact, something much stronger holds: to determine the value of

x	y
0	0
0	4
1	1
1	3
2	5
2	7

(a)

x	y
0	1
1	2
2	1
3	2
4	1
5	2

(b)

x	y
1	0
1	2
2	1
2	3
3	2
3	4

(c)

x	y
1	2
2	2
3	3
4	3
5	4
6	4

(d)

x	y
1	6
1	12
1	14
2	6
2	12
2	15

(e)

Figure 5.4: Five teams, each consisting of six assignments into the domain $\mathbb{N}$. For simplicity, only the value of the assignments on the variables x and y is displayed.

Dependency	X_a	X_b	X_c	X_d	X_e
$\lambda x \rightarrow \lambda y$		✓		✓	
$\lambda x \rightarrow ?Ey$	✓	✓	✓	✓	
$?Ex \rightarrow \lambda y$		✓			
$?Ex \rightarrow ?Ey$		✓	✓		
$\lambda x \wedge ?(x<y) \rightarrow \lambda y$		✓	✓	✓	
$\lambda x \rightarrow \overline{\exists} z \mathrm{Pf}(z, y)$				✓	✓

Figure 5.5: A table that shows in which of the teams of Figure 5.4 each of the dependencies we consider holds.

y, we don't even need to know the value of x; knowing the *parity* of x is sufficient. If x is even, then the value of y is 1; if x is odd, then the value of y is 2. Thus, in this context we have a dependency relation which is much stronger than the one expressed by the dependence atom $=\!\!(x, y)$. This dependency amounts to the following relation of contextual entailment:

$$?Ex \models_{X_b} \lambda y$$

In this case, some partial information about x yields complete information about y. By means of implication, we can express this strong dependency as $?Ex \rightarrow \lambda y$.

5.3.31. Example. [Parity of x determines parity of y]
We may also consider dependencies that are neither weaker not stronger than the standard ones. In the team X_c of Figure 5.4(c), the parity of x determines the parity of y: if x is even, then y is odd, and if x is odd, y is even. Thus, the following relation holds:

$$?Ex \models_{X_c} ?Ey$$

This dependence relation is neither stronger nor weaker than the standard dependence $\lambda x \models_X \lambda y$: for, on the one hand, we have $?Ex \models_{X_c} ?Ey$ but $\lambda x \not\models_{X_c} \lambda y$; on the other hand, we have $\lambda x \models_{X_d} \lambda y$ but $?Ex \not\models_{X_d} ?Ey$.

In this case, the dependency relation amounts to the fact that some partial information about x yields some other partial information about y. By means of implication, this relation may be expressed as $?Ex \to ?Ey$.

5.3.32. Example. [Value of x and whether $x<y$ determine value of y]
In the team X_c, the value of x does not completely determine the value of y. However, knowing the value of x and knowing, in addition, whether x is smaller than y *is* sufficient to identify y. This means that the following relation holds:

$$\lambda x,\ ?(x<y) \models_{X_c} \lambda y$$

In this case, complete information about y follows from complete information about x paired with some information about the relation between x and y. Of course, this dependency can be expressed by the corresponding implication:

$$\lambda x \wedge ?(x<y) \ \to\ \lambda y$$

5.3.33. Example. [Values of x determines some prime factor of y]
In the team X_e of Figure 5.4(e), the value of x does not completely determine the value of y. However, also in this case, the value of x does determine something interesting about y. For, as soon as we know the value of x, we know some prime factor of y: if the value of x is 1, then 2 is a prime factor of y, while if the value of x is 2, then 3 is a prime factor of y. This means that, in the context of X_e, the identity question λx determines the question $\overline{\exists} z\mathrm{Pf}(z,y)$ asking for an instance of a prime factor of y. That is, the following dependency holds in X_e:

$$\lambda x \models_{X_e} \overline{\exists} z\mathrm{Pf}(z,y)$$

As usual, the dependency may be expressed by means the formula $\lambda x \to \overline{\exists} z\mathrm{Pf}(z,y)$.

Summing up, then, our perspective on dependency allows us to recognize that the dependence pattern expressed by the dependence atoms in FOD is just a particular instance of a much broader spectrum of interesting informational connections which may be detected within a certain body of information. By means of the inquisitive operators $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ and $\overline{\exists}$, many interesting questions concerning the value of variables and the mutual relations between such values may be expressed. Dependencies among these questions are captured by the relation of contextual entailment, and expressed within the language by means of corresponding implications.

5.3.4 Higher-order dependencies in InqBQT

Within the setting of InqBQT, we can also give a natural example of the higher-order dependencies that we discussed in Section 5.2.3. For this example, consider an empty signature, and let our domain be the set $\mathbb{R}$ of real numbers.

Consider three variables p, v, and t, which stand respectively for the pressure, volume, and temperature of a given gas. Suppose it is settled in the team X that the gas obeys the ideal gas law, $pv = kt$, where $k \in \mathbb{R}$ is a constant depending on the amount of gas. Concretely, what this means is that for any assignment $g \in X$ we have $g(p)g(v) = kg(t)$.

We saw that the formula $\lambda t \to \lambda p$ is supported in a team Y if, relative to Y, there is a dependence function which yields the value of p from the value of t:

$$M \models_Y \lambda t \to \lambda p \iff \text{there is } f : \mathbb{R} \to \mathbb{R} \text{ such that } f : t \rightsquigarrow_Y p$$

Thus, we may regard $\lambda t \to \lambda p$ as a question asking for such a dependence function, i.e., for a way to compute p from t. Of course, the situation is completely analogous for the formula $\lambda t \to \lambda v$.

Now, for any sub-team $Y \subseteq X$, if we have a dependence function $f : t \rightsquigarrow_Y p$, we also have a dependence function $h : t \rightsquigarrow_Y v$. For, consider a sub-team $Y \subseteq X$ and suppose $f : t \rightsquigarrow_Y p$. Let $g \in Y$. On the one hand, since $Y \subseteq X$, we have $g(p)g(v) = kg(t)$; on the other hand, since $f : t \rightsquigarrow_Y p$, we have $g(p) = f(g(t))$. But then, it follows that:[14]

$$g(v) = \frac{kg(t)}{g(p)} = \frac{kg(t)}{f(g(t))}$$

If h is the function defined by $x \mapsto \frac{kx}{f(x)}$, we then have $g(v) = h(g(t))$. Since this is true for each $g \in Y$, we have $h : t \rightsquigarrow_Y v$.

This shows that any sub-team $Y \subseteq X$ which supports $\lambda t \to \lambda p$ also supports $\lambda t \to \lambda v$, which means that the following higher-order dependency holds in X.

$$\lambda t \to \lambda p \models_X \lambda t \to \lambda v$$

This captures the fact that any given way of computing the pressure of an ideal gas from its temperature yields a way of computing the volume of the gas from its temperature. This dependency may then be expressed by means of the formula:

$$(\lambda t \to \lambda p) \to (\lambda t \to \lambda v)$$

This example is meant to show that higher-order dependencies may be meaningful features of an informational scenario. In FOD, it is hard to see how such dependencies may be captured, since dependence atoms can only be applied to individual variables, not to other dependence atoms. In InqBQT, on the other hand, higher-order dependencies are handled straightforwardly: for, dependence formulas are themselves questions, which may very well stand in the relation of dependency with other questions; moreover, since implications may be nested, such dependencies can be represented in a straightforward way within the language.

[14]We assume that it is settled in X that the pressure p always has a positive value, so we need not worry about division by zero.

5.3.5 On the choice of the logical repertoire

It is important to emphasize that the main points made in this section are not strictly tied to the specific system InqBQ that we have used to illustrate them. For, while our considerations highlight the role of certain logical operators, they certainly do not determine the complete repertoire of operations that our logic of dependency should include. That is a matter that depends largely on one's ultimate goals.

If one's goal is to provide the tools to express patterns of quantification that are not expressible in classical first-order logic, the dependence quantifiers $\forall^d$ and $\exists^d$ are indispensable. For, we saw that sentences in InqBQT are not more expressive than sentences in first-order logic. By adding the dependence quantifiers to this system, on the other hand, Henkin-type patterns of quantification would become expressible in an elegant way. E.g., a sentence presenting the Henkin-type quantification pattern discussed in Section 5.1 may be written as follows:

$$\forall^d x \forall^d x' \exists^d y \exists^d y' ((\lambda x \to \lambda y) \wedge (\lambda x' \to \lambda y') \wedge \varphi(x, x', y, y'))$$

On the other hand, one may of course have other goals in mind. In the previous chapters of this thesis, the enterprise has been to develop logics capable of expressing a range of interesting questions, and thus to capture a range of interesting dependencies as cases of entailment; we have aimed at providing proof systems whereby dependencies may be formally proved to hold, and we have analyzed the constructive content of such proofs. The dependence quantifiers are not indispensable for this enterprise. In fact, even having formulas which *express* dependencies is not strictly necessary for the task of reasoning about dependencies: it is the availability of questions which is the key factor, since it is by manipulating questions that dependencies may be formally proved to hold.[15]

From this perspective, the fact that sentences in InqBQT are not more expressive than sentences of first-order logic is interesting, as it leaves open the possibility of obtaining a complete axiomatization of the logic. Even if this turns out to be impossible, the results obtained in the previous section for the related system InqBQ provide some reason to be optimistic about the prospect of axiomatizing at least some interesting fragments of this logic. Thus, from this point of view, there is actually some reason to investigate a system like InqBQT, where powerful operators such as the dependence quantifiers are lacking. However, it should be clear that this is only one of the many specific systems in which the general approach advocated here can be incarnated.

[15] Of course, that is not to say that having formulas expressing dependencies is not important. Thanks to such formulas, we can not only prove that certain dependencies hold; we can also, e.g., prove that whenever certain dependencies $D_1, \ldots, D_n$ hold, some other dependency D_{n+1} holds as well.

5.4 A unified framework for propositional and individual dependencies

A striking difference exists between propositional (and modal) dependence logic on the one hand, and first-order dependence logic on the other: in the propositional case, the relation of dependency concern sentences, while in the first-order case, it concerns individual variables. This is slightly puzzling, as it is not clear that it is one and the same logical relation that is at stake in both contexts. Besides, this also has the strange consequence that some things that are expressible in propositional dependence logic are no longer expressible in first-order dependence logic. For instance, suppose that whether Alice will go to the party determines whether Bob will go. If we capture the two sentences involved as propositional atoms p and q, the dependency at hand can be expressed in PD by means of the formula $=(p, q)$. On the other hand, as soon as the same sentences are captured in first-order logic as Pa and Pb, the dependency can no longer be expressed. Superficially, this is due to the fact that, in FOD, the dependence atom only embeds individual variables, not formulas. At a deeper level, the reason is that in FOD, formulas are evaluated with respect to just one, fixed model. Whether Alice will go to the party, and whether Bob will go, are two questions which are simply settled once and for all in this model, and there is no way to make sense of one of them depending on the other.

Our perspective on dependency brings the two settings closer together. In both propositional and first-order logic, dependency is a relation between questions, that is, between a particular class of sentences. The dependence atoms of PD express relations between polar questions of the form $?p$, while the dependence atoms of FOD express relations between identity questions of the form λx. Moroever, any dependence that can be captured in the propositional logic InqB can also be captured in the first-order inquisitive logic InqBQ of the previous chapter.

However, there still remains a fundamental difference between the first-order system InqBQ and the system InqBQT considered in the previous section: in the former system, dependencies involve questions about the relevant state of affairs, such as *who will go to the party*, or *whether Alice will go*; in the latter system, they involve questions about the value of variables, such as *what the value of x is*, or *whether x is an even number*. Clearly, this difference is rooted in the difference in semantic setup. In InqBQ, we consider multiple states of affairs, corresponding to multiple possible worlds, but only a single assignment. As a consequence, we can interpret questions about the features of the state of affairs, but not questions about the value of variables. On the other hand, in InqBQT, we consider a single state of affairs—a single first-order model—but multiple assignments; as a consequence, we can interpret questions about the value of variables, but not questions about the state of affairs.

This section explores a natural way to reconcile these two approaches within a unique, general system InqBQ^+ which can handle both questions concerning the state of affairs and questions concerning the value of variables, as well as questions about their connection. To achieve this, we need to evaluate formulas with respect to objects that capture partial information about both the state of affairs and the value of variables. This can be achieved by taking our points of evaluation to be sets of world-assignment pairs. We will refer to such an object as an *information states with referents*, abbreviated as *r-state*.[16,17]

5.4.1. Definition. [Indices and r-states]
Let $M = \langle W, D, I, = \rangle$ be a first-order information id-model.[18]

- An *index* is a pair $i = \langle w_i, g_i \rangle$, where $w_i \in W$ and $g_i : \mathsf{Var} \to D$.
- An information state with referents, or *r-state* for short, is a set s of indices.

In the system InqBQ^+, sentences will be interpreted relative to r-states. Notice that an r-state determines both an ordinary information state and a team.

5.4.2. Definition. Let s be an information state with referents. Then:

- the state associated with s is $\pi_1[s] = \{w \mid \langle w, g \rangle \in s \text{ for some } g\}$
- the team associated with s is $\pi_2[s] = \{g \mid \langle w, g \rangle \in s \text{ for some } w\}$

However, an r-state is not uniquely determined by the state $\pi_1[s]$ and the team $\pi_2[s]$. This is because, in general, s also encodes information about the relation between the state of affairs and the value of variables, information that

[16]A similar semantic setup has been independently proposed by Väänänen (2014) with a rather different motivation in mind. Väänanen's goal is to develop a logic capable of expressing interesting properties of a set-theoretic *multiverse*, i.e., a structure containing a multitude of distinct models of set theory. In his system, formulas are evaluated with respect to a multiset of first-order models and to a function mapping each of these models to an assignment into the corresponding domain. While seemingly more complex, this setup is essentially equivalent to our setup based on sets of model-assignment pairs, provided that we allow different worlds to have different domains. For simplicity, in this section we stick to the case of a constant domain.

[17]Information states with referents are a fundamental notion in *dynamic semantics* (see, e.g., Heim, 1982; Dekker, 1993; Groenendijk et al., 1996; Aloni, 2000), where they are simply called *information states*. In this line of work, the standard way to think about such an object s is as follows: s encodes not only information about features of the world, but also about the possible values of certain *discourse referents*, which stand for individuals that the discourse is about, but whose identity is not necessarily known. For instance, if we hear that "a girl was running", and if we use variable x to store the new discourse referent that this sentence introduces, the resulting state s will only contain pairs $\langle w, g \rangle$ such that the individual $g(x)$ is a girl who was running in world w.

[18]For simplicity, we spell out the proposal for the case of id-models, in which the extension of the relation of identity among the relevant individuals is settled. It is obvious how to generalize this system to allow for models in which identity may be uncertain, as in Chapter 4.

is not reflected by the projections $\pi_1[s]$ and $\pi_2[s]$. For instance, the r-states $s_1 = \{\langle w, g\rangle, \langle w', g'\rangle, \langle w, g'\rangle, \langle w', g\rangle\}$ and $s_2 = \{\langle w, g\rangle, \langle w', g'\rangle\}$ have the same projections, but encode different information: s_1 contains only the information that the actual world is one of w, w', and the actual assignment one of g, g', while s_2 also contains, in addition, the information that w is the actual world iff g is the actual assignment.

The language of our system InqBQ^+ is the same first-order language $\mathcal{L}^{\mathsf{Q}}$ that we have for the systems InqBQ and InqBQT. The value of a term t at an index i is simply $[t]^i := [t]^{w_i}_{g_i}$. Moreover, given an r-state s and an individual $d \in D$, we write $s[x \mapsto d]$ for the r-state obtained by modifying the valuation at each index in s from g to $g[x \mapsto d]$:

$$s[x \mapsto d] = \{\langle w, g[x \mapsto d]\rangle \mid \langle w, g\rangle \in s\}$$

Then, the fundamental semantic relation of support between r-states s in a model M, and formulas $\varphi \in \mathcal{L}^{\mathsf{Q}}$ is defined as follows.

5.4.3. Definition. [Support in InqBQ^+]

- $M, s \models R(t_1, \dots, t_n) \iff \langle [t_1]^i, \dots, [t_n]^i\rangle \in I_{w_i}(R)$ for all $i \in s$
- $M, s \models (t = t') \iff [t]^i = [t']^i$ for all $i \in s$
- $M, s \models \bot \iff s = \emptyset$
- $M, s \models \varphi \wedge \psi \iff M, s \models \varphi$ and $M, s \models \psi$
- $M, s \models \varphi \to \psi \iff$ for all $t \subseteq s, M, t \models \varphi$ implies $M, t \models \psi$
- $M, s \models \varphi \mathbin{\vee\!\!\!\vee} \psi \iff M, s \models \varphi$ or $M, s \models \psi$
- $M, s \models \forall x \varphi \iff M, s[x \mapsto d] \models \varphi$ for all $d \in D$
- $M, s \models \exists\!\!\!\exists x \varphi \iff M, s[x \mapsto d] \models \varphi$ for some $d \in D$

This system is similar to InqBQ in many respects. First of all, the semantics is persistent, and every formula is supported by the empty r-state. We can define truth with respect to an index i as support with respect to $\{i\}$, and we can call a formula truth-conditional if support for it always amounts to truth at each world. Then, we still have that all classical formulas are truth-conditional.

5.4.4. Proposition (Classical formulas are truth-conditional).
For any $\alpha \in \mathcal{L}^{\mathsf{Q}}_c$, model M and r-state s: $M, s \models \alpha \iff M, i \models \alpha$ for all $i \in s$

Moreover, the truth-conditions for a classical formula at an index $i = \langle w, g\rangle$ are the ones given by classical first-order logic with respect to the first-order model w^* associated with w and to the assignment g.[19]

[19] w^* denotes the first-order model associated with the world w, as given by Definition 4.1.7

5.4.5. Proposition (Truth-conditions for classical formulas).
For any classical formula α, any model M and index $\langle w, g\rangle$:

$$M, \langle w, g\rangle \models \alpha \iff w^* \models_g \alpha \text{ in classical logic}$$

This shows that, as far as classical formulas are concerned, InqBQ^+ is yet another informational semantics for classical first-order logic: when we look at classical formulas, the entailment relation that arises from InqBQ^+ is just the classical one.

Moreover, it is worth remarking that the systems InqBQ and InqBQT both arise from restricting the semantics of InqBQ^+ to particular kinds of r-states. The system InqBQ is obtained by restricting the semantics to r-states in which all indices i have the same assignment component g_i.

5.4.6. Proposition.
Suppose $\pi_2[s] = \{g\}$. Then $M, s \models \varphi \iff M, \pi_1[s] \models_g \varphi$ in InqBQ

Similarly, the system InqBQT is obtained by restricting the semantics to r-states in which all indices i have the same world component w_i.

5.4.7. Proposition.
Suppose $\pi_1[s] = \{w\}$. Then $M, s \models \varphi \iff w^* \models_{\pi_2[s]} \varphi$ in InqBQT

As we expect, the interpretation of sentences is insensitive to the assignment component of an r-state, which means that, as far as sentences are concerned, InqBQ^+ boils down to InqBQ.

5.4.8. Proposition.
If φ is a sentence and s is an r-state, $M, s \models \varphi \iff M, \pi_1[s] \models \varphi$ in InqBQ

Notice all the questions that we discussed in the previous chapter were sentences. As usual in first-order logic, we were only interested in variables insofar as these would ultimately get bound. The previous proposition implies that all those questions receive essentially the same interpretation in InqBQ^+. In particular, this includes the following classes of questions, where $FV(\alpha(\overline{x})) \subseteq \{\overline{x}\}$:

- questions formed by means of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ from classical sentences, e.g., $?\beta$ or $\beta \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \gamma$;
- mention-some questions $\overline{\exists}\overline{x}\alpha(\overline{x})$, asking for instances of a certain relation;
- mention-all question $\forall\overline{x}\,?\alpha(\overline{x})$, asking for the extension of a certain relation.

Thus, all the interesting dependence patterns which we have seen to be captured in InqBQ can also be captured in the same way in InqBQ^+. In addition, however, InqBQ^+ can also interpret questions concerning the value of variables. E.g., it is easy to verify that the following holds: an r-state s supports λx in case the value of x is settled in s, i.e., in case all indices $i \in s$ assign the same value to x.

$$M, s \models \lambda x \iff \text{for all } g, g' \in \pi_2[s] : \; g(x) = g'(x)$$

Thus, while support for questions that are sentences depends exclusively on the state $\pi_1[s]$ determined by s, support for identity questions about variables depends exclusively on the team $\pi_2[s]$ determined by s. However, this is only a special case: in general, the interpretation of a formula with free variables really depends on the r-state as a whole. To see this, consider the simple example of a polar question $?Px$.

5.4.9. Example.
It is immediate to verify that $?Px$ is settled in an r-state s if it is settled either that the value of x lies in P, or that the value of x lies outside of P.

$$M, s \models ?Px \iff [\, g_i(x) \in I_{w_i}(P) \text{ for all } i \in s] \text{ or } [\, g_i(x) \notin I_{w_i}(P) \text{ for all } i \in s]$$

Thus, whether $?Px$ is supported depends both on what s settles about the value of x, and on what it settles about the extension of P. Thinking of s as an epistemic state, we may know exactly what the value of x is, yet fail to support $?Px$ because we don't know whether the extension of P includes this value; conversely, we may know exactly what the extension of P is, yet fail to support $?Px$ because we don't know what individual x denotes. Conversely, notice that our state s may well support $?Px$, without it being settled of any particular individual whether it has property P. All that is needed is for a certain relation to hold, across s, between the value of x and the extension of P.

Within the system InqBQ^+, we obtain a uniform analysis of the different sorts of dependencies that we encountered so far. Let us start out with the kinds of dependencies expressed by the dependence atoms $=\!(x_1, \dots, x_n, y)$ of FOD. Since identity questions about variables can be expressed in InqBQ^+ just as in InqBQT, these dependencies are still captured by the implication $\lambda x_1 \wedge \dots \wedge \lambda x_n \to \lambda y$:

$$\begin{aligned} M, s \models \lambda x_1 \wedge \dots \wedge \lambda x_n \to \lambda y \iff & \text{ for all } g, g' \in \pi_2[s] : \\ & g(x_k) = g'(x_k) \text{ for each } k \text{ implies } g(y) = g'(y) \end{aligned}$$

At the same time, however, we no longer have the puzzling situation which we observed in FOD and InqBQT, where dependencies that are expressible in propositional dependence/inquisitive logic are no longer expressible. For instance, consider the sort of dependency expressed by a propositional dependence atom in PD: this relation amounts to the fact that the truth-value of $\alpha_1, \dots, \alpha_n$ determines the truth-value of β, where $\alpha_1, \dots, \alpha_n, \beta$ are classical formulas. We can express this relation straightforwardly as an implication among polar questions, $?\alpha_1 \wedge \cdots \wedge ?\alpha_n \to ?\beta$

$$\begin{aligned} M, s \models ?\alpha_1 \wedge \cdots \wedge ?\alpha_n \to ?\beta \quad \iff \quad & \text{for all } i, i' \in s, \\ & \text{if: } M, i \models \alpha_k \iff M, i' \models \alpha_k \text{ for each } k \\ & \text{then : } M, i \models \beta \iff M, i' \models \beta \end{aligned}$$

Thus, the propositional-level dependencies of PD and the individual-level dependencies of FOD can be captured in a uniform framework, as two instances of one and the same logical relation, namely, dependency, construed as contextual question entailment. All that is different between the two cases is the specific questions that are involved in this relation: questions about the truth-value of sentences in the former case, questions about the value of first-order variables in the latter.

Moreover, notice that these two kinds of dependencies can be mixed. As an example, we may want to express the fact that, in a given r-state, the truth value of a formula $\alpha(x_1, \ldots, x_n)$ is completely determined by the value of the variables $x_1, \ldots, x_n$. This is expressed by the formula $\lambda x_1 \wedge \cdots \wedge \lambda x_n \to ?\alpha$.

$$\begin{aligned} M, s \models \lambda x_1 \wedge \cdots \wedge \lambda x_n \to ?\alpha \quad \iff \quad & \text{for all } i, i' \in s, \\ & \text{if: } g_i(x_k) = g_{i'}(x_k) \text{ for each } k \\ & \text{then : } M, i \models \alpha \iff M, i' \models \alpha \end{aligned}$$

Summing up, then, we have seen that, simply by transposing the system InqBQ to a semantic framework that represents partial information about both the state of affairs and the value of variables, we obtain a logic capable of representing on the one hand sentence-level dependencies, such as those expressed by the dependence atoms $=\!\!(p_1, \ldots, p_n, q)$ of PD and the many others we discussed in Section 5.2.2 and in Chapter 4, and on the other hand individual-level dependencies, such as those expressed by the dependence atoms $=\!\!(x_1, \ldots, x_n, y)$ of FOD, and the many others we discussed in Section 5.3.3. Clearly, this calls for a more systematic investigation of the logic of the system InqBQ^+, but this is a task that will have to be left for future work.

5.5 Relation between variables or relation between questions?

Abstracting away from the technical aspects of the systems we have discussed, in this final section we turn to some more conceptual considerations. It is worth pointing out that what we have been concerned with, in comparing our approach to dependency to the one adopted in systems of dependence logic, is not primarily a difference in technical implementation, but rather a difference in the way the relation of dependency is construed: we have regarded dependency as a relation between questions, whereas in dependence logic, dependency is essentially regarded as a relation between variables. In the former view, a dependency holds in a context s when any information of the types $\mu_1, \dots, \mu_n$ yields, relative to s, some corresponding information of type ν. In the latter view, a dependency holds in s in case the value of the tuple $x_1, \dots, x_n$ determines the value of the variable y. In this section I will argue at a general level that the former view, while coinciding with the latter in many respects, has some important advantages to it.

Uniformity First, variables have different *types*, i.e., they stand for different sorts of semantic objects. This underlies the mismatch we remarked between PD and FOD: in the former system, dependencies concern *propositional* variables and their truth-values, while in the latter system, they concern *first-order* variables, and their individual-level values. This means that, to account for dependencies between different types of variables, different logical operations (or atoms), are needed. On the other hand, questions all have the same type: a question always denotes an inquisitive proposition, regardless of what information it takes to settle it. This makes it possible to capture dependencies of different "types" in a uniform framework, and also to straightforwardly mix these different types, as we saw in the previous section. Thus, the question view has the advantage of offering a more uniform account of the phenomenon of dependency.

Generality Given a variable x of any sort, we can associate it with a corresponding question Qx, the question of what the value of x is. To say that the value of x determines the value of y in a context c is to say that, relative to c, the question Qx determines the question Qy. Thus, any instance of dependency construed as a relation between variables can be re-conceptualized straightforwardly as an instance of a dependence relation between questions.

What about the converse direction? If μ is a question which has a unique true answer at each point (where by *point* I mean a world or an assignment, depending on the setting), we can associate μ with a variable x_μ, whose value at a point is the unique true answer to μ at that point. To say that μ determines ν in a context c is to say that, relative to c, the value of the variable x_μ determines the value of the variable x_ν. Thus, for unique-answer questions, the question view and the variable view are inter-translatable. However, as we discussed in Chapter 1, not all questions are unique-answer questions. Thus, for instance, consider the following dependency relation, which we considered above:

(3) The value of x determines a prime factor of y

Here, the determined question is a mention-some question: what is one prime factor of y? Clearly, this question does not in general have a unique complete answer at a given assignment g: if $g(y) = 6$, for instance, we have two complete answers, corresponding to the two prime factors 2 and 3. For such questions, we cannot let the variable x_μ denote the true answer to μ at any given point: for, in general, there is no unique true answer. Since the value of a variable at a point is bound to be unique, it seems that dependencies involving such questions cannot be construed as relations involving variables.[20] Thus, one advantage of regarding dependency as a relation between questions is the extra generality that it affords, allowing us to cover also dependencies involving non-unique answer questions, such as the one in (3), among many others.

Composability Unlike variables, which are atomic objects, questions have structure to them: they are formed by means of logical operators, and they can in turn be combined by logical operators to form more complex questions. The fact that questions are formed by specific operators allows us to manipulate them in inferences according to the rules for these operations. The fact that questions can be combined by means of operators is useful for a number of reasons: it allows us, e.g., to pack several questions into one by means of conjunction, so that we may always view a dependency as a relation involving *two* questions, a determined question and a determining question, rather than as involving multiple determining questions; it allows us to condition a question to a statement by means of implication, which allows us, among other things, to capture conditional dependencies, as we saw in Section 1.4.2; it also allows us to capture the dependency between two question μ and ν by means of another question $\mu \to \nu$, which may in turn stand in the relation of dependency to other questions; as we saw, this allows us to capture higher-order dependencies. Thus, the fact that questions are syntactically composed, and composable, is a valuable asset of the question view.

Taking dependency to the core of logic Finally, and perhaps most importantly, by construing dependency as involving questions we can capture this relation as a facet of entailment—of the very same notion of entailment with which we are acquainted in classical logic, only applied to a different class of sentences. This does not only give rise to a neat conceptual picture, but allows us to treat dependency by means of familiar logical tools. The possibility of expressing dependencies by means of implication is a prime example of this. Another important example is the fact that logical dependencies, being instances of entailment among questions, can be formally proved within a deduction system that, in addition to classical formulas, allows us to make inferences with questions.

[20] One may think of letting x_μ denote at a world w the set of all true answers to μ at w. A moment's reflection reveals that this will not work: saying that μ determines ν does not amount to saying that the set of answers to μ determines the set of answers to ν, but merely to saying that any answer to μ determines some corresponding answer to ν.

Chapter 6

Kripke modalities

In this chapter we investigate how InqB can be extended with modal operators to be interpreted on standard Kripke models. Or, to put it the other way around, we look at the result of enriching standard modal logic with questions. Crucially, in the framework that we will describe, modalities and questions will not just exist side by side; rather, we will enable modalities to embed questions, besides statements. More specifically, we will see that, by replacing the standard truth-conditional clause for a universal modality $\Box$ by a natural support clause, we obtain a semantics that handles in a uniform way a modal statement like (1), which is given the usual truth-conditions, and a modal statement like (2).

(1) John knows that Mary is home.

(2) John knows whether Mary is home.

In standard modal logic, (2) is usually translated in an ad-hoc way as a disjunction $\Box p \vee \Box \neg p$. In the system we will describe, there is no need for a paraphrase: we can simply treat (2) as obtained by applying the modality $\Box$ to the question $?p$. This predicts the desired truth-conditions in a compositional way. The equivalence $\Box ?p \equiv \Box p \vee \Box \neg p$ is obtained as a matter of fact, rather than taken as a definition.

In terms of technical results, we will not only give an axiomatization of the resulting general modal logic, InqBK, but we will also give a general recipe for turning a complete axiomatization for a canonical modal logic, such as KT, K4, S5, etc., into a complete axiomatization of the corresponding inquisitive modal logic, obtained by restricting the semantics to particular classes of Kripke frames.

We will consider an application of this modal logic to the modeling of dependence statements such as (3) and (4), and of questions such as (5).

(3) Which day it is determines whether Alice or Bob is in the office.

(4) Which day it is does not determine whether Alice or Bob is in the office.

(5) Does which day it is determine whether Alice or Bob is in the office?

We will argue that a dependence statement like (3) should be formalized as a modal statement $\mu \Rightarrow \nu \equiv \Box(\mu \to \nu)$ which is true at a world w in case μ determines ν in the context of the associated information state $\sigma(w)$. We will see that the modal implication $\Rightarrow$ and the universal modality $\Box$ are inter-definable, and we will investigate the logical properties of $\Rightarrow$, leading to a completeness result for the logic obtained by taking $\Rightarrow$ as our fundamental modal operator.

Finally, we will see that there exists a translation of the propositional system InqB into standard modal logic, which can be compared with the Gödel translation of intuitionistic propositional logic into the modal logic S4; we will argue that, however, analyzing questions and dependencies within a support semantics a number of crucial advantages over a formalization in modal logic.

The chapter is structured as follows: we start in Section 6.1 by looking at how standard modal logic can be reconstructed based on a support semantics. In Section 6.2, questions enter the stage; we will focus in particular on the novelty allowed for by this system, the fact that Kripke modalities can now operate on questions. In Section 6.3 we provide some useful normal form results. These results are put to use in Section 6.4, where we show how to transform a complete axiomatization of an arbitrary canonical modal logic into a complete axiomatization for the corresponding inquisitive system. In Section 6.5 we apply our theory of inquisitive Kripke modal logic to the analysis of dependence statements, and investigate the logic of the modal implication $\Rightarrow$. In Section 6.6, we present the modal translation of propositional inquisitive logic. Finally, in Section 6.7 we relate our work to a number of recent developments in the field of modal logic, and in Section 6.8 we outline an extension of our system which is left as a topic for future work.

The ideas presented in this chapter are partly based on Ciardelli (2014a) and Ciardelli and Roelofsen (2015b).

6.1 Support semantics for Kripke modalities

Let us start out by considering how standard modal logic can be reconstructed from a support-conditional foundation. Let us take our language $\mathcal{L}_c^{\mathsf{K}}$ of classical modal logic to be obtained from a set $\mathcal{P}$ of atomic sentences and the falsum constant by means of the connectives $\wedge$ and $\to$ and the modal operator $\Box$. The classical connectives $\neg$ and $\vee$ are defined as in propositional logic; moreover, we take $\Diamond$ to be defined by $\Diamond\varphi := \neg\Box\neg\varphi$. Let us recall the standard definitions of Kripke frames and models.

6.1.1. Definition. [Kripke frames]
A *Kripke frame* is a pair $F = \langle W, \sigma \rangle$, where:

- W is a set whose elements we call *possible worlds*

- $\sigma : W \to \wp(W)$ is an *accessibility map* which associates with each world w a set $\sigma(w)$ of worlds that we call its *successors*.

6.1.2. Definition. [Kripke models]
A *Kripke model* is a triple $M = \langle W, \sigma, V\rangle$, where $\langle W, \sigma\rangle$ is a Kripke frame and $V : W \times \mathcal{P} \to \{0, 1\}$ is a *valuation function* which specifies at each world which atomic sentences are true and which are false. We say that the Kripke model $M = \langle W, \sigma, V\rangle$ is *based on* the frame $F = \langle W, \sigma\rangle$.

In the most common presentation of Kripke models, the successors of a given point are taken to be determined by a relation $R \subseteq W \times W$, rather than by a state map $\sigma : W \to \wp(W)$. It is easy to realize that this difference is completely inessential. On the one hand, a relation R determines a state map σ_R defined by:

$$\sigma_R(w) = \{v \mid wRv\}$$

Conversely, a state map σ determines a corresponding relation R_σ, defined as:

$$wR_\sigma v \iff v \in \sigma(w)$$

Clearly, these transformations are inverse to each other, so that a binary relation and an accessibility map may be seen as equivalent representations of the same object. While binary relations have the advantage of being easier to visualize, the presentation in terms of state maps has an advantage which will be important for our purposes: it brings out how a state map is one of the ingredients that contribute to describe the state of affairs at each possible world. So, in a Kripke model, a state of affairs at a world w is determined not only—as in propositional logic—by an assignment of truth-values of the atomic facts, but also by a certain information state $\sigma(w)$ attached to the world—which may represent, for instance, the information available to an agent at that world. In the next chapter, this perspective on Kripke models will allow for a natural transition to *inquisitive modal models*, in which a world is equipped not with an information state $\sigma(w)$, but with an *inquisitive* state $\Sigma(w)$, describing both information and issues.

Let us now come back to the task of providing a support semantics for our classical modal language $\mathcal{L}_c^{\mathsf{K}}$. The clauses for atoms and propositional connectives are already familiar from Chapter 2. Now, how should we define the support clause for a modal formula $\Box\alpha$ in such a way that support for all classical formulas amounts to truth at every world? Well, suppose the desired connection holds for α. In order for the connection to also hold for $\Box\alpha$, we must have:

$$\begin{aligned} M, s \models \Box\alpha \quad &\iff \quad \text{for all } w \in s,\ M, w \models \Box\alpha \\ &\iff \quad \text{for all } w \in s,\ \text{for all } v \in \sigma(w),\ M, v \models \alpha \\ &\iff \quad \text{for all } w \in s,\ M, \sigma(w) \models \alpha \end{aligned}$$

This suggests the following support semantics for classical modal logic.

6.1.3. Definition. [Support]
Let M be a Kripke model. The relation of support between states s in M and formulas $\varphi \in \mathcal{L}_c^{\mathsf{K}}$ is given by augmenting the clauses for the connectives given in Definition 2.2.2 with the following clause:

- $M, s \models \Box\varphi \iff M, \sigma(w) \models \alpha$ for all $w \in s$

As usual, we define truth at a world w as support with respect to $\{w\}$, and we say that a formula is truth-conditional if support for it simply amounts to truth at each world. Notice that the support clause for $\Box\varphi$ makes this formula obviously truth-conditional, with the following truth-conditions.

6.1.4. Proposition (Truth-conditions for $\Box$).
$M, w \models \Box\varphi \iff M, \sigma(w) \models \varphi$

I.e., $\Box\varphi$ is true at w in case φ is settled in the information state associated with w. The clause becomes particularly perspicuous if we interpret $\sigma(w)$ as encoding the epistemic state of an agent, and we read $\Box\varphi$ as "the agent knows φ". Then, the clause says that an agent knows φ just in case her knowledge state settles φ.

It is easy to see that, given this clause for $\Box$, the derived operator $\Diamond$ is truth-conditional as well, and has the following truth-conditions: $\Diamond\varphi$ is true at w iff the state $\sigma(w)$ is compatible with φ.[1]

6.1.5. Proposition (Truth-conditions for $\Diamond$).
$M, w \models \Diamond\varphi \iff \sigma(w) \between_M \varphi$

Given this support definition, it is straightforward to check that all formulas in our classical modal language $\mathcal{L}_c^{\mathsf{K}}$ are indeed truth-conditional. That is, support for a formula $\alpha \in \mathcal{L}_c^{\mathsf{K}}$ always amounts to truth at each world.

6.1.6. Proposition (Formulas in $\mathcal{L}_c^{\mathsf{K}}$ are truth-conditional).
For any $\alpha \in \mathcal{L}_c^{\mathsf{K}}$, any Kripke model M, and any state s:

$$M, s \models \alpha \iff M, w \models \alpha \textit{ for all } w \in s$$

The truth-conditionality of each $\alpha \in \mathcal{L}_c^{\mathsf{K}}$ also implies that the truth-conditions that Proposition 6.1.4 assigns to $\Box$ coincide with the standard ones, as long as the argument of $\Box$ is a classical modal formula.[2]

6.1.7. Proposition. *For any $\alpha \in \mathcal{L}_c^{\mathsf{K}}$, any Kripke model M and world w:*

$$M, w \models \Box\alpha \iff M, v \models \alpha \textit{ for all } v \in \sigma(w)$$

[1] Recall that a state s is said to be *compatible* with a formula φ, notation $s \between \varphi$, in case there exists a consistent enhancement $t \subseteq s$ that supports φ.

[2] As a consequence, the truth-conditions of the defined operator $\Diamond$ are also the familiar ones.

Proof. Immediate from Propositions 6.1.4 and 6.1.6. $\Box$

Since we know that the connectives $\bot, \wedge$, and $\rightarrow$ also have standard truth-conditions, this shows that the support semantics we have just given is equivalent to the familiar truth-conditional semantics: for all classical modal formulas, our semantics determines the standard truth-conditions, and conversely, these truth-conditions fully determine the support relation as given by Definition 6.1.3.

As we saw, the interesting feature of the support setting is that it allows us to introduce questions. The use of giving classical logical operators a direct, recursive support clause is that the effect of these operators can be generalized beyond the truth-conditional realm, so that they can operate uniformly on statements and questions. We are now going to look at this for the case of the modality $\Box$.

6.2 Kripke modalities over questions

The full language of inquisitive Kripke modal logic, $\mathcal{L}^{\mathsf{K}}$, is obtained by extending classical modal logic with inquisitive disjunction.

6.2.1. Definition. [Language of InqBK]

$$\varphi \ ::= \ p \,|\, \bot \,|\, \varphi \wedge \varphi \,|\, \varphi \rightarrow \varphi \,|\, \varphi \vee\!\!\!\vee \varphi \,|\, \Box\varphi$$

So, in addition to classical modal formulas, our full language $\mathcal{L}^{\mathsf{K}}$ contains questions formed by means of inquisitive disjunction, such as $p \vee\!\!\!\vee q$, $?p$ and $?\Box p$. The main novelty of this system with respect to propositional inquisitive logic is that such questions can themselves be embedded under modalities. Now, what is the outcome of applying $\Box$ to a question? First, it is immediately visible from the support clause for $\Box$ that, even if μ is a question, embedding μ under $\Box$ results in a truth-conditional formula.

6.2.2. Proposition. *For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, $\Box\varphi$ is truth-conditional.*

Thus, in order to understand the meaning of the formula $\Box\mu$ when μ is a question, it suffices to look at its truth-conditions. Now, the clause for the operator $\Box$ tells us that $\Box\mu$ is true at w iff the information associated with w settles the question μ. Of course, different interpretations of this clause are possible depending on what we take the accessibility map σ to encode. For the sake of concreteness, in the following discussion we will take an epistemic perspective on σ, regarding the information state $\sigma(w)$ as representing the information state of a given agent at the world w.

6.2.3. Example. [Knowing whether p]
Consider the modal formula $\Box?p$. We have:

$$\begin{aligned} M, w \models \Box ?p &\iff M, \sigma(w) \models ?p \\ &\iff M, \sigma(w) \models p \text{ or } M, \sigma(w) \models \neg p \\ &\iff \sigma(w) \subseteq |p|_M \text{ or } \sigma(w) \subseteq |\neg p|_M \end{aligned}$$

That is, $\Box ?p$ is true at w iff the state $\sigma(w)$ settles whether p is true or false. Under an epistemic interpretation, $\Box ?p$ captures the fact that the agent *knows whether* p.

6.2.4. Example. [Knowing whether p or q]
As a second example, consider the formula $\Box(p \mathbin{\vee\!\!\!\vee} q)$. We have:

$$\begin{aligned} M, w \models \Box(p \mathbin{\vee\!\!\!\vee} q) &\iff M, \sigma(w) \models p \mathbin{\vee\!\!\!\vee} q \\ &\iff \sigma(w) \subseteq |p|_M \text{ or } \sigma(w) \subseteq |q|_M \end{aligned}$$

Thus, $\Box(p \mathbin{\vee\!\!\!\vee} q)$ is true at w iff the state $\sigma(w)$ settles a specific one among p and q. Under an epistemic interpretation, we may take this to capture the fact that the agent *knows whether p or q*.[3]

6.2.5. Example. [Knowing whether q conditionally on p]
Consider the formula $\Box(p \to ?q)$. We have:

$$\begin{aligned} M, w \models \Box(p \to ?q) &\iff M, \sigma(w) \models p \to ?q \\ &\iff M, \sigma(w) \cap |p|_M \models ?q \\ &\iff \sigma(w) \cap |p|_M \subseteq |q|_M \text{ or } \sigma(w) \cap |p|_M \subseteq |\neg q|_M \end{aligned}$$

That is, $\Box(p \to ?q)$ is true at a world w in case extending $\sigma(w)$ with the information that p results in a state that settles whether q is true or false. Under an epistemic interpretation, $\Box(p \to ?q)$ captures the fact that the agent *knows whether q conditionally on p*.

Notice that this particular example may be generalized. If α is any truth-conditional formula, we have the following truth-conditions for $\Box(\alpha \to \varphi)$:

$$M, w \models \Box(\alpha \to \varphi) \iff M, \sigma(w) \cap |\alpha|_M \models \varphi$$

Thus, under an epistemic interpretation, $\Box(\alpha \to \varphi)$ expresses the fact that the agent knows φ conditionally on α. If φ is itself a truth-conditional formula β, this boils down to the agent knowing the material conditional $\alpha \to \beta$. However, if φ is *not* truth-conditional, but a question, the material conditional interpretation is not adequate: to get the right meaning, it is crucial to be able to interpret the conditional antecedent as restricting the information state on which the consequent is going to be interpreted. This brings out the importance of having

[3] If we take the view that 'knowing whether p or q' requires knowing one of p and q *to the exclusion of the other*, we can capture this by means of the formula $\Box(p \underline{\mathbin{\vee\!\!\!\vee}} q)$, that is, $\Box((p \wedge \neg q) \mathbin{\vee\!\!\!\vee} (q \wedge \neg p))$.

a conditional operator which operates at the support-level, and not just at the truth-conditional level.[4] The latter example in particular brings out the potential of having a conditional $\to$ which can operate both at the truth-conditional level, as a material conditional, and at the level of non-singleton states, where the antecedent has the effect of restricting the evaluation state.

6.2.6. Example. [Knowing whether q conditionally on whether p]
In the previous example, we saw how we can capture knowledge of a question conditional on a statement. However, in InqBK we can also capture knowledge of a question *conditionally on another question*. To see this, consider the formula $\Box(?p \to ?q)$. It is not hard to see that we have:

$$M, w \models \Box(?p \to ?q) \iff M, \sigma(w) \cap |p|_M \models ?q \text{ and } M, \sigma(w) \cap |\neg p|_M \models ?q$$

Thus, $\Box(?p \to ?q)$ is true in w in case extending the state $\sigma(w)$ with either answer to $?p$ results in a state which settles $?q$. Thus, under an epistemic interpretation, $\Box(?p \to ?q)$ captures the fact that the agent *knows whether q conditionally on an answer to whether p*. Notice that this amounts to the fact that a dependency holds in the state of the agent, with the question $?p$ determining the question $?q$.

$$\begin{aligned} M, w \models \Box(?p \to ?q) &\iff ?p \models_{\sigma(w)} ?q \\ &\iff f : ?p \rightsquigarrow_{\sigma(w)} ?q \text{ for some } f \end{aligned}$$

Of course, all of this is not specific to the questions $?p$ and $?q$, but it generalizes straightforwardly to arbitrary questions μ and ν.

We have thus illustrated what sort of facts one can express by letting the universal modality $\Box$ apply to questions. In standard modal logic, it is not possible to directly apply a modality to a question—since the language contains no formulas expressing questions. The fact that an agent *knows whether* p can, nevertheless, be expressed by means of the paraphrase $\Box p \vee \Box\neg p$. In our system, in which questions *can* be embedded within modalities, the equivalence $\Box?p \equiv \Box p \vee \Box\neg p$ is obtained as a logical fact, rather than stipulated by definition. In fact, this equivalence is an instance of a more general fact about the interaction of $\Box$ and $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$: namely, $\Box$ can be always be distributed over an inquisitive disjunction turning it into a classical one.

6.2.7. Proposition (Distributivity of $\Box$ over $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$).
For any $\varphi, \psi \in \mathcal{L}^{\mathsf{K}}$, $\Box(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi) \equiv \Box\varphi \vee \Box\psi$

[4] Conditional knowledge of the answer to a question is the focus of recent research by Wang and Fan (2013, 2014), driven by applications in computer science and artificial intelligence. The relation with their work is spelled out in some detail in Section 6.7.4.

Proof. Since both $\Box(\varphi \vee\!\!\!\vee \psi)$ and $\Box\varphi \vee \Box\psi$ are truth-conditional, we just have to show that they have the same truth-conditions. We have:

$$\begin{aligned} M, w \models \Box(\varphi \vee\!\!\!\vee \psi) &\iff M, \sigma(w) \models \varphi \vee\!\!\!\vee \psi \\ &\iff M, \sigma(w) \models \varphi \text{ or } M, \sigma(w) \models \psi \\ &\iff M, w \models \Box\varphi \text{ or } M, w \models \Box\psi \\ &\iff M, w \models \Box\varphi \vee \Box\psi \end{aligned}$$

□

Using this fact, we can show that a Kripke modality applied to a question can always be paraphrased away as a disjunction of modalities applied to truth-conditional formulas—the question's resolutions. However, some work is needed to make this precise. To this we turn in the next section.

6.3 Resolutions and normal forms

We will start out by isolating a syntactic fragment of InqBK whose formulas are guaranteed to be truth-conditional. We will refer to these formulas as *declaratives.*

6.3.1. Definition. [Declarative fragment of $\mathcal{L}^{\mathsf{K}}$]
The set $\mathcal{L}^{\mathsf{K}}_!$ of declarative formulas in $\mathcal{L}^{\mathsf{K}}$ is given by the following definition, where φ is any formula from $\mathcal{L}^{\mathsf{K}}$.

$$\alpha \ ::= \ p \,|\, \bot \,|\, \Box\varphi \,|\, \alpha \wedge \alpha \,|\, \alpha \to \alpha$$

That is, declaratives are formulas built up from atoms, $\bot$, and modal formulas by means of conjunction and implication. Another way to characterize this class is as follows: α is a declarative if all occurrences of $\vee\!\!\!\vee$ in α are within the scope of a $\Box$ operator. Notice that any classical formula is a declarative, but not the other way around: for instance, $\Box ?p$ is a declarative, but not a classical formula, since it contains an occurrence of $\vee\!\!\!\vee$.

Now, we know that atoms, $\bot$, and modal formulas are truth-conditional, and we know that $\wedge$ and $\to$ preserve truth-conditionality (Proposition 2.3.9, which carries over to InqBK). It follows that all declaratives are truth-conditional.

6.3.2. Proposition. *Any formula $\alpha \in \mathcal{L}^{\mathsf{K}}_!$ is truth-conditional.*

We are now ready to extend the notion of resolutions we had in InqB to all formulas of $\mathcal{L}^{\mathsf{K}}$.

6.3.3. Definition. [Resolutions]
The set $\mathcal{R}(\varphi)$ of resolutions for a formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ is obtained by augmenting Definition 2.4.1 with the following clause:

- $\mathcal{R}(\Box\varphi) = \{\Box\varphi\}$

That is, we let a modal formula $\Box\varphi$ be the only resolution of itself. Notice that according to this definition, resolutions are not necessarily classical formulas. However, by definition $\mathcal{R}(\varphi)$ is always a set of declaratives. Moreover, it is easy to check by induction that a declarative α is always the only resolution of itself.

6.3.4. Proposition. *If $\alpha \in \mathcal{L}^{\mathsf{K}}_{!}$, then $\mathcal{R}(\alpha) = \{\alpha\}$.*

It is easy to verify that a formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ is always equivalent to the inquisitive disjunction of its resolutions.

6.3.5. Proposition (Normal form for InqBK).
For all formulas $\varphi \in \mathcal{L}^{\mathsf{K}}$, $\varphi \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$

As in the propositional case, the notion of resolutions can be extended to sets of formulas by letting a resolution of Φ be obtained by replacing each formula $\varphi \in \Phi$ by a resolution of it.

6.3.6. Definition. [Resolutions for sets]
If $\Phi \subseteq \mathcal{L}^{\mathsf{K}}$, a *resolution function* for Φ is a function $f : \Phi \to \mathcal{L}^{\mathsf{K}}_{!}$ which maps each $\varphi \in \Phi$ to a formula $f(\varphi) \in \mathcal{R}(\varphi)$. The set of resolutions of Φ is defined as follows:

$$\mathcal{R}(\Phi) = \{f[\Phi] \mid f \text{ is a resolution function for } \Phi\}$$

Again, each resolution $\Gamma \in \mathcal{R}(\Phi)$ is a set of declaratives, and moreover, a set Γ of declaratives is always the only resolution of itself, i.e., $\mathcal{R}(\Gamma) = \{\Gamma\}$.

Now that we have a specific notion of resolutions, we can make our previous claim precise: a modality $\Box$ applying over a question can always be paraphrased away by distributing it over the question's resolutions.

6.3.7. Proposition ($\Box$ distributes over resolutions).
If $\varphi \in \mathcal{L}^{\mathsf{K}}$ and $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$, then $\Box\varphi \equiv \Box\alpha_1 \vee \dots \vee \Box\alpha_n$.

Proof. Since $\varphi \equiv \alpha_1 \mathbin{\mathbb{V}} \dots \mathbin{\mathbb{V}} \alpha_n$, we have $\Box\varphi \equiv \Box(\alpha_1 \mathbin{\mathbb{V}} \dots \mathbin{\mathbb{V}} \alpha_n)$. By Proposition 6.2.7, we obtain $\Box\varphi \equiv \Box\alpha_1 \vee \dots \vee \Box\alpha_n$. $\Box$

This result suggests that letting Kripke modalities apply to questions does not allow InqBK to express any new truth-conditional meanings. That is, it is natural to expect any truth-conditional formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ to be equivalent to some classical modal formula $\varphi^{cl} \in \mathcal{L}^{\mathsf{K}}_{c}$. However, unlike in propositional and first-order inquisitive logic, we cannot simply take φ^{cl} to be obtained by replacing any occurrence of inquisitive disjunction with classical disjunction: for, within the scope of a modality, replacing $\mathbin{\mathbb{V}}$ by $\vee$ may well affect the truth-conditions of the resulting formula. For instance, the formula $\Box ?p$ does not have the same truth-conditions as $\Box(p \vee \neg p)$, which is a tautology.[5] Instead, a more subtle inductive definition of φ^{cl} is necessary.

To provide such a definition, we need a suitable ordering on modal formulas. As standard in modal logic, we define the *modal depth* of a formula φ, notation $d(\varphi)$, as the maximum number of nested occurrences of $\Box$ in φ. Then, we define the following ordering on our language.

6.3.8. Definition. [Modal simplicity ordering]
Given two formulas $\varphi, \psi \in \mathcal{L}^{\mathsf{K}}$, we say that φ is modally simpler than ψ, notation $\varphi \prec \psi$, in case either of the following holds:

- $d(\varphi) < d(\psi)$
- φ is a proper sub-formula of ψ

It is easy to see that $\prec$ is a well-founded strict partial order on $\mathcal{L}^{\mathsf{K}}$, and thus a suitable basis for inductive definitions and proofs. The following definition is by induction on the modal simplicity of the formula φ.[6]

6.3.9. Definition. [Classical variant of a formula in $\mathcal{L}^{\mathsf{K}}$]

- $p^{cl} = p$
- $\bot^{cl} = \bot$
- $(\varphi \circ \psi)^{cl} = \varphi^{cl} \circ \psi^{cl}$ for $\circ \in \{\wedge, \to\}$
- $(\varphi \mathbin{\vee\!\!\!\vee} \psi)^{cl} = \varphi^{cl} \vee \psi^{cl}$
- $(\Box\varphi)^{cl} = \Box\alpha_1^{cl} \vee \dots \vee \Box\alpha_n^{cl}$ where $\{\alpha_1, \dots, \alpha_n\} = \mathcal{R}(\varphi)$

By definition, φ^{cl} is always a classical modal formula. Also, notice that if φ is a propositional formula, then indeed, φ^{cl} is simply the formula obtained from φ by substituting all occurrences of $\mathbin{\vee\!\!\!\vee}$ by $\vee$, as in Chapter 2. So, this definition is a conservative extension of the notion of classical variant from Chapter 2. Moreover, a straightforward inductive proof shows that φ and φ^{cl} always have the same truth-conditions.

[5]This illustrates an interesting fact: in the system InqBK, the truth-conditions of a formula $\Box\varphi$ are not determined only by the truth-conditions of φ, but crucially depend on the *support* conditions of φ. Thus, unlike in InqB and InqBQ, we cannot in general compute the truth-conditions of a formula by means of recursive clauses that only appeal to the truth-conditions of its sub-formulas. In a way, the operator $\Box$ resembles the dependence logic universal quantifier $\forall^d$ that we encountered in Section 5.3: both are capable of "creating a range", i.e., of shifting the evaluation from a singleton state to a possibly non-singleton one.

[6]To see that this is a good inductive definition, we have to make sure that the resolutions $\alpha \in \mathcal{R}(\varphi)$ are all modally simpler than the formula $\Box\varphi$. This is indeed the case: for, the definition of resolutions never increases the modal depth of a formula. Thus, $\alpha \in \mathcal{R}(\varphi)$ implies $d(\alpha) \leq d(\varphi) < d(\Box\varphi)$ and so $\alpha \prec \Box\varphi$.

6.3.10. Proposition.
For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, any Kripke model M and world w: $M, w \models \varphi \iff M, w \models \varphi^{cl}$

As a corollary, we immediately get that the truth-conditional formulas in $\mathcal{L}^{\mathsf{K}}$ are all and only the formulas which are equivalent to a classical modal formula.

6.3.11. Corollary.
For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, φ is truth-conditional $\iff$ $\varphi \equiv \alpha$ for some $\alpha \in \mathcal{L}^{\mathsf{K}}_c$.

Thus, InqBK is no more expressive than standard modal logic as far as statements are concerned. Notice that, since declaratives are truth-conditional, any declarative formula α is equivalent to a classical formula, α^{cl}. This allows us to associate with any formula $\varphi \in \mathcal{L}^{\mathsf{K}}$ a set of *classical* resolutions, obtained by replacing each resolution of φ with its classical variant.

6.3.12. Definition. [Classical resolutions]
Let $\varphi \in \mathcal{L}^{\mathsf{K}}$. If $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$, the set of classical resolutions of φ is:

$$\mathcal{R}^{cl}(\varphi) = \{\alpha_1^{cl}, \dots, \alpha_n^{cl}\}$$

This definition is extended to sets in the standard way: $\Gamma \in \mathcal{R}^{cl}(\Phi)$ in case Γ is obtained by replacing each formula in Φ by a classical resolution of it.

Since any resolution is equivalent to its classical variant, we have a second normal form result, which represents any formula in InqBK as an inquisitive disjunction of classical modal formulas.

6.3.13. Proposition (Normal form with classical resolutions).
For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, $\varphi \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}^{cl}(\varphi)$.

This is the normal form result that we are going to use in our completeness proof.[7]

Summing up, then, allowing Kripke modalities to embed questions does not expand the truth-conditional expressive power of our logic. This will be different for the *inquisitive* modalities to be introduced in the next chapter. However, already in the present context, the support implementation of the modality makes one important contribution: it shows that there is a natural formulation of modal logic in which Kripke modalities operate uniformly on statements and questions, allowing us to handle sentences like (6) and (7) in exactly the same way.

(6) Alice knows that Bob is home.

[7] The reader may perhaps wonder why we introduced the class of declaratives, given that classical formulas are already representative of all truth-conditional meanings in the system. This notion is introduced primarily in preparation for the next chapter, where we will have declaratives which do not correspond to any classical formula. In that setting, it will be declaratives—and not classical formulas—that play the role of truth-conditional backbone of the logic. Since the logic presented in the next chapter is an extension of InqBK, I have set up the notion of resolutions in such a way that it does not have to be revised, but can simply be extended.

(7) Alice knows whether Bob is home.

Whether φ is a statement or a question, $\Box\varphi$ is a statement which is true iff φ is settled in the relevant information state. If φ is truth-conditional, this means that φ is true everywhere in the state, and we recover the clause of standard modal logic. On the other hand, if φ is a question, support does not boil down to truth at each world, and the support formulation of the clause is essential.[8]

6.4 Axiomatizing inquisitive Kripke logics

In this section, we investigate the logic of the system InqBK, providing a sound and complete axiomatization of it. In fact, we will prove a more general result: given any canonical normal modal logic L, we will consider the inquisitive logic InqBL obtained by restricting our semantics to Kripke frames for L. We will show that there is a general recipe for turning an axiomatization of L into an axiomatization of the corresponding inquisitive system InqBL.

6.4.1 Some standard notions of modal logic

Let us start out by recalling some standard notions and facts of ordinary modal logic. For further details and proofs, the reader is referred to Blackburn et al. (2002). First, we recall the notion of a *normal modal logic.*

6.4.1. Definition. [Normal modal logics]
A set $\mathsf{L} \subseteq \mathcal{L}_c^{\mathsf{K}}$ of classical modal formulas is a *normal modal logic* in case:

- L contains any instance of a propositional tautology
- L contains any formula of the form $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$
- if $\alpha \in \mathsf{L}$ and $\alpha \to \beta \in \mathsf{L}$, then $\beta \in \mathsf{L}$
- if $\alpha \in \mathsf{L}$, then $\Box\alpha \in \mathsf{L}$

[8] Another proposal to generalize the $\Box$ operator of epistemic logic to questions is put forward by Aloni et al. (2013). In this approach, formulas are evaluated not with respect to information states, but rather with respect to pairs of worlds, following a proposal of Lewis (1982). This is essentially the same semantic setup adopted in the Logic of Interrogation of Groenendijk (1999), and in early systems of inquisitive semantics (Groenendijk, 2009; Mascarenhas, 2009). As we discussed in Section 1.6.3, this approach is essentially equivalent to ours whenever it is applicable, but it is more restricted in scope, since not all questions may be seen as expressing binary relations. For instance, a sentence expressing knowledge of a mention-some question, such as *Alice knows who has a bike that she can borrow* is out of reach for an approach based on pairs of worlds, while it can be modeled in a simple way taking the approach described here.

If L is a normal modal logic, we say that a Kripke frame F is an L-frame in case L is valid over F.[9] If $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_c^{\mathsf{K}}$, we say that Γ *L-entails* α, notation $\Gamma \models_{\mathsf{L}} \alpha$, if for any Kripke model M based on an L-frame, and for any w in M, $M, w \models \Gamma$ implies $M, w \models \alpha$. On the other hand, we say that Γ *L-derives* α, notation $\Gamma \vdash_{\mathsf{L}} \alpha$, in case there are $\beta_1, \ldots, \beta_n \in \Gamma$ such that $\beta_1 \wedge \cdots \wedge \beta_n \to \alpha \in \mathsf{L}$.

It is easy to see that if Γ L-derives α, then Γ also L-entails α. If the converse is also true, we say that L is *Kripke complete*.

6.4.2. Definition. [Kripke completeness]
We say that a normal modal logic L is *Kripke complete* if for any $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}_c^{\mathsf{K}}$, $\Gamma \models_{\mathsf{L}} \alpha$ implies $\Gamma \vdash_{\mathsf{L}} \alpha$.

Kripke completeness is usually proven by constructing a canonical model for L. The canonical model construction is modular with respect to L, and uses the notion of a *complete L-theory*. An *L-theory* is a set $\Gamma \subseteq \mathcal{L}_c^{\mathsf{K}}$ which is closed under L-derivation, i.e., if $\Gamma \vdash_{\mathsf{L}} \alpha$, then $\alpha \in \Gamma$. A *complete* L-theory is an L-theory Γ such that (i) $\bot \notin \Gamma$ and (ii) for all $\alpha \in \mathcal{L}_c^{\mathsf{K}}$, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$. Complete L-theories are used as possible worlds in the canonical model for L.

6.4.3. Definition. [Canonical model for a normal modal logic]
The canonical Kripke model for a normal modal logic L is the triple $M_{\mathsf{L}}^c = \langle W_{\mathsf{L}}^c, V_{\mathsf{L}}^c, \sigma_{\mathsf{L}}^c \rangle$ defined as follows:

- W_{L}^c is the set of complete L-theories;
- $V_{\mathsf{L}}^c(\Gamma, p) = 1 \iff p \in \Gamma$
- $\sigma_{\mathsf{L}}^c(\Gamma) = \{\Gamma' \mid \alpha \in \Gamma' \text{ whenever } \Box\alpha \in \Gamma\}$

For any normal modal logic L, truth at a world Γ in the canonical model corresponds to membership in Γ.

6.4.4. Lemma (Truth Lemma).
For any normal modal logic L, any $\Gamma \in W_{\mathsf{L}}^c$, and any formula $\alpha \in \mathcal{L}_c^{\mathsf{K}}$:

$$M_{\mathsf{L}}^c, \Gamma \models \alpha \iff \alpha \in \Gamma$$

The truth-lemma implies that L is true at every world in the canonical model. However, it is not always true that L is also valid on the underlying frame. If this is the case, we say that L is *canonical*.

6.4.5. Definition. [Canonicity]
A normal modal logic L is said to be *canonical* if $F_{\mathsf{L}}^c = \langle W_{\mathsf{L}}^c, \sigma_{\mathsf{L}}^c \rangle$ is an L-frame.

[9] L is valid over a Kripke frame F in case any formula of L is true at any world in any Kripke model M over F.

If a normal modal logic is canonical, then it is Kripke complete. For, suppose $\Gamma \nvdash_{\mathsf{L}} \alpha$. Then, $\Gamma \cup \{\neg\alpha\}$ can be extended in the standard way to a complete L-theory Δ. By the Truth Lemma, $M^c_{\mathsf{L}}, \Delta \models \Gamma$ but $M^c_{\mathsf{L}}, \Delta \not\models \alpha$. Since L is canonical, M^c_{L} is a model based on an L-frame, which shows that $\Gamma \not\models_{\mathsf{L}} \alpha$.

6.4.6. Proposition. *If a normal modal logic is canonical, it is Kripke complete.*

As an example, here are four well-known schemata from standard modal logic, with their traditional names.

- D: $\neg\Box\bot$
- T: $\Box\alpha \to \alpha$
- $\mathsf{4}$: $\Box\alpha \to \Box\Box\alpha$
- $\mathsf{5}$: $\neg\Box\alpha \to \Box\neg\Box\alpha$

Several notable modal logics L can be obtained by adding one or more of these schemes to the normal modal logic K. That is, more precisely, they can be characterized as the smallest normal modal logic which contains all instances of one or more of these schemata. These logics are normally named by postfixing to K the names of the corresponding axioms: thus, for instance, $\mathsf{K4}$ is the minimal normal modal logic containing all formulas of the form $\Box\alpha \to \Box\Box\alpha$. However, some logics have historical names which deviate from this rule: in particular, it is standard to use the name $\mathsf{S4}$ instead of $\mathsf{KT4}$, and $\mathsf{S5}$ instead of $\mathsf{KT5}$.

With each of the above modal axioms we can associate a condition on the map σ, given in Figure 6.1. For any normal modal logic obtained by adding one or more of these axioms to K, the corresponding class of frames can then be given a simple characterization: a frame F is an L-frame just in case it satisfies the corresponding conditions. Thus, for instance, a frame F is an KT-frame in case we have $w \in \sigma(w)$ at any world w. Now, it is known that each of these logics is canonical, and thus also Kripke complete. We are going to see how to turn this completeness result into a completeness result for the inquisitive Kripke logic of the corresponding class of frames.

6.4.2 Inquisitive Kripke modal logics

We saw that each normal modal logic L determines a class of Kripke frames, namely, the class of frames on which L is valid. The inquisitive system InqBL corresponding to L is obtained by restricting inquisitive Kripke modal logic to models built over an L-frame. In particular, notice that InqBK is indeed the inquisitive system corresponding to the minimal normal modal logic K.

Scheme	Condition on σ
D	$\forall w : \sigma(w) \neq \emptyset$
T	$\forall w : w \in \sigma(w)$
4	$\forall w, v : v \in \sigma(w)$ implies $\sigma(v) \subseteq \sigma(w)$
5	$\forall w, v : v \in \sigma(w)$ implies $\sigma(v) \supseteq \sigma(w)$

Figure 6.1: Some prominent normal modal logics and the corresponding classes of Kripke frames, described in terms of conditions on the state map σ.

6.4.7. DEFINITION. [Entailment in InqBL]
Let L be a normal modal logic. For $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{K}}$, we say that Φ entails ψ in InqBL—notation $\Phi \models_{\mathsf{InqBL}} \psi$—if for any Kripke model M based on an L-frame and for any state s in M, $M, s \models \Phi$ implies $M, s \models \psi$.[10]

Now, how can we axiomatize the logic InqBL? To start with, notice that, since classical modal formulas are truth-conditional in our semantics, and since the truth-conditions they are assigned are the same as in Kripke semantics, InqBL coincides with the original logic L on classical formulas.

6.4.8. PROPOSITION (CONSERVATIVITY).
Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^{\mathsf{K}}_c$. Then $\Gamma \models_{\mathsf{L}} \alpha \iff \Gamma \models_{\mathsf{InqBL}} \alpha$

In particular, any formula valid in L is also valid in InqBL, that is, if we identify InqBL with its set of validities, we have $\mathsf{L} \subseteq \mathsf{InqBL}$.

Notice that this does *not* mean that, if a certain scheme is valid in L, then the same scheme will be generally valid in InqBL. This may or may not be the case. For a positive example, consider the logic $\mathsf{K4}$, axiomatized by $\Box\alpha \to \Box\Box\alpha$. We claim that the scheme $\Box\varphi \to \Box\Box\varphi$ is generally valid in $\mathsf{InqBK4}$, for all $\varphi \in \mathcal{L}^{\mathsf{K}}$. To see this, take a Kripke model M based on a $\mathsf{K4}$ frame, i.e., on a frame such that $v \in \sigma(w)$ implies $\sigma(v) \subseteq \sigma(w)$. Suppose $M, w \models \Box\varphi$. This means that $M, \sigma(w) \models \varphi$. Now take any $v \in \sigma(w)$: since the frame is $\mathsf{K4}$, we have $\sigma(v) \subseteq \sigma(w)$. By persistency, this implies $M, \sigma(v) \models \varphi$, and thus $M, v \models \Box\varphi$. So, the formula $\Box\varphi$ is true at any world $v \in \sigma(w)$. Since this formula is truth-conditional, this means that $M, \sigma(w) \models \Box\varphi$, and thus, $M, w \models \Box\Box\varphi$. This shows that the implication $\Box\varphi \to \Box\Box\varphi$ is true at any world in M. Since this formula is a declarative, it is truth-conditional, so this means that it is also supported at any state in M. This shows that $\Box\varphi \to \Box\Box\varphi$ is a validity of $\mathsf{InqBK4}$.

For a negative example, consider the modal logic KT, axiomatized by $\Box\alpha \to \alpha$. While each formula $\Box\alpha \to \alpha$ for $\alpha \in \mathcal{L}^{\mathsf{K}}_c$ must be valid in InqBKT by conservativity, we claim that the scheme $\Box\varphi \to \varphi$ is not valid in full generality. To see this, consider a KT Kripke model M with two worlds, $w_p, w_{\overline{p}}$, where

[10] Recall that, if Φ is a set of formulas, we write $M, s \models \Phi$ for "$M, s \models \varphi$ for all $\varphi \in \Phi$".

$\sigma(w_p) = \{w_p\}$, $\sigma(w_{\overline{p}}) = \{w_{\overline{p}}\}$, $V(w_p, p) = 1$ and $V(w_{\overline{p}}, p) = 0$. Now consider the state $W = \{w_p, w_{\overline{p}}\}$: it is easy to see that we have $M, W \models \Box ?p$ but $M, W \not\models ?p$. This shows that the implication $\Box ?p \to ?p$ is not generally valid in InqBKT.

This example shows that the characteristic schemes of a logic L do not in general transfer to the whole language in InqBL. However, the next Proposition states that the familiar distributive axiom of K does hold for all formulas in InqBK, statements and questions alike. The easy proof is omitted.

6.4.9. PROPOSITION (DISTRIBUTIVITY).
For any $\varphi, \psi \in \mathcal{L}^{\mathsf{K}}$, $\models_{\mathsf{InqBK}} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

A fortiori, distributivity will also be valid in any inquisitive Kripke logic InqBL. Moreover, recall from Proposition 6.2.7 that Kripke modalities satisfy another kind of distributivity in InqBK: namely, they distribute over inquisitive disjunctions, turning them into classical disjunctions. This means that the scheme $\Box(\varphi \vee\!\!\!\vee \psi) \to \Box\varphi \vee \Box\psi$ is valid in InqBK.

Finally, we have that, if ψ is a logical consequence of $\varphi_1, \ldots, \varphi_n$, then $\Box\psi$ is a logical consequence of $\Box\varphi_1, \ldots, \Box\varphi_n$.

6.4.10. PROPOSITION (MONOTONICITY OF $\Box$).
If $\Phi \models_{\mathsf{InqBL}} \psi$, *then* $\Box\Phi \models_{\mathsf{InqBL}} \Box\psi$, *where* $\Box\Phi = \{\Box\varphi \mid \varphi \in \Phi\}$.

Proof. Suppose $\Phi \models_{\mathsf{InqBL}} \psi$. Since all formulas in $\Box\Phi$ as well as $\Box\psi$ are truth-conditional, we just have to check that $\Box\Phi$ truth-conditionally entails $\Box\psi$ in models based on L-frames. So, let M be such a model, and w a world in it. Suppose $M, w \models \Box\Phi$. Given the truth-conditions for $\Box$, what this means is that $M, \sigma(w) \models \Phi$. Since $\Phi \models_{\mathsf{InqBL}} \psi$, we have $M, \sigma(w) \models \psi$, whence $M, w \models \Box\psi$. □

As it turns out, if our starting logic L is canonical, the properties of InqBL listed so far—conservativity over L, the two distribution laws, and monotonicity—suffice to give a complete axiomatization of InqBL.

6.4.3 Proof system

A proof system for the inquisitive Kripke logic InqBL is described in Figure 6.2 and Figure 6.3.[11] Figure 6.2 simply recalls our inference rules for the propositional connectives, with one important *caveat*: in the rules of $\vee\!\!\!\vee$-split and $\neg\neg$-elimination, we take α to range over declaratives—not just classical formulas. This is because we need to encode the truth-conditionality of modal formulas.

[11] For the sake of consistency with the previous chapters, we use a natural deduction system. In modal logics, Hilbert-style proof systems are more commonly used. It is not hard to turn our system into an equivalent Hilbert-style one, having *modus ponens* and $\Box$-monotonicity (or $\Box$-necessitation) as inference rules, and turning the rest of our rules into corresponding axioms.

Conjunction

$$\frac{\varphi \quad \psi}{\varphi \wedge \psi} \qquad \frac{\varphi \wedge \psi}{\varphi} \qquad \frac{\varphi \wedge \psi}{\psi}$$

Implication

$$\frac{\begin{matrix}[\varphi]\\ \vdots\\ \psi\end{matrix}}{\varphi \to \psi} \qquad \frac{\varphi \quad \varphi \to \psi}{\psi}$$

Inquisitive disjunction

$$\frac{\varphi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \qquad \frac{\psi}{\varphi \mathbin{\vee\!\!\!\vee} \psi} \qquad \frac{\varphi \mathbin{\vee\!\!\!\vee} \psi \quad \begin{matrix}[\varphi]\\ \vdots\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \vdots\\ \chi\end{matrix}}{\chi}$$

Falsum

$$\frac{\bot}{\varphi}$$

$\mathbin{\vee\!\!\!\vee}$ −split

$$\frac{\alpha \to \psi \mathbin{\vee\!\!\!\vee} \chi}{(\alpha \to \psi) \mathbin{\vee\!\!\!\vee} (\alpha \to \chi)} \; (\alpha \in \mathcal{L}^{\mathsf{K}}_{!})$$

$\neg\neg$ elimination

$$\frac{\neg\neg\alpha}{\alpha} \; (\alpha \in \mathcal{L}^{\mathsf{K}}_{!})$$

Figure 6.2: Rules for the propositional connectives, repeated from Chapter 3. In extending these rules to InqBL, it is important to specify the right restriction for the formula α occurring in the split rule and in the $\neg\neg$-elimination rule: we take α to stand for any declarative formula.

Notice that, since $\mathbin{\vee\!\!\!\vee}$-split and double negation elimination are valid for any truth-conditional formula, and since declaratives are truth-conditional, these inference rules are sound. Moreover, notice that we are not modifying the proof system we had for propositional logic: for, when we restrict to propositional formulas, the class of declaratives coincides with the class of classical formulas.

Figure 6.3 contains the rules for the Kripke modality, which mirror the properties of $\Box$ we discussed so far. First, we have two rules corresponding to the two distribution properties of $\Box$, the standard one over implication, and the one over inquisitive disjunction. Second, we have a rule capturing the monotonicity of $\Box$. This rule comes with an important *caveat*: it is crucial here that $\varphi_1, \ldots, \varphi_n$ be *all* the undischarged assumptions in the sub-proof leading to ψ. Without this condition, the monotonicity rule would not be sound, allowing us to derive invalid formulas such as $p \to \Box p$.[12] Finally, we have a rule that allows us to import a

[12] To obtain completeness, a particular instance of the monotonicity rule would be sufficient, namely, the one in which ψ was derived from the empty set of assumptions. This particular case corresponds to the standard *necessitation rule* $\varphi/\Box\varphi$ of Hilbert-style systems of modal logic.

$\Box\!\rightarrow$ distributivity

$$\frac{\Box(\varphi \to \psi)}{\Box\varphi \to \Box\psi}$$

$\Box\!\mathbin{\vee\!\!\!\vee}$ distributivity

$$\frac{\Box(\varphi \mathbin{\vee\!\!\!\vee} \psi)}{\Box\varphi \vee \Box\psi}$$

L-axioms

$$\overline{\alpha}\ (\alpha \in Ax(\mathsf{L}))$$

$\Box$ monotonicity

$$\begin{array}{c} [\varphi_1] \quad \dots \quad [\varphi_n] \\ \vdots \\ \psi \end{array} \qquad \frac{\qquad \Box\varphi_1 \quad \dots \quad \Box\varphi_n}{\Box\psi}$$

Figure 6.3: Rules for $\Box$ in the logic InqBL. In the monotonicity rule, $\varphi_1, \dots, \varphi_n$ must be the *only* undischarged assumptions in the sub-proof leading to ψ. In the L-axioms rule, $Ax(\mathsf{L})$ stands for an arbitrary set of axioms for L.

complete set of axioms for our base modal logic L. In this rule, the set of axioms $Ax(\mathsf{L})$ is any set of classical formulas which axiomatizes L, i.e., such that L can be characterized as the least normal modal logic containing the set $Ax(\mathsf{L})$. Thus, e.g., for the logic InqBKT it is sufficient to allow as axioms classical modal formulas of the form $\Box\alpha \to \alpha$, while for $\mathsf{InqBK4}$ it is sufficient to allow classical formulas of the form $\Box\alpha \to \Box\Box\alpha$.[13]

As usual, we write $P : \Phi \vdash_{\mathsf{InqBL}} \psi$ to mean that P is a proof in this system whose conclusion is ψ and whose assumptions are among Φ; we write $\Phi \vdash_{\mathsf{InqBL}} \psi$ to mean that some such proof exists; and we write $\varphi \dashv\vdash_{\mathsf{InqBL}} \psi$ to mean that $\varphi \vdash_{\mathsf{InqBL}} \psi$ and $\psi \vdash_{\mathsf{InqBL}} \varphi$. The next proposition ensures that our system is sound for entailment in InqBL.

6.4.11. Proposition (Soundness).
For any normal modal logic L and for any formulas $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{K}}$:

$$\Phi \vdash_{\mathsf{InqBL}} \psi \implies \Phi \models_{\mathsf{InqBL}} \psi$$

However, working with monotonicity is convenient for our purposes, and just as natural.

[13]Notice that taking $Ax(\mathsf{L}) = \mathsf{L}$ is a possible choice here, that is, we may choose to just import all L-validities. However, it is convenient to allow for just a specific set of L-validities to be taken as axioms in our system for InqBL. This is because there are modal logics that have a decidable set of axioms, but whose set of validities is not decidable. If we had to allow all L-validities as axioms, this would make it undecidable whether a given formula is an axiom of our system or not. By taking on board only formulas that are *axioms* of the original logic, this problem is avoided. Thanks to Valentin Goranko for pointing this out.

Proof. As usual, it suffices to check the soundness of each inference rule. Focusing only on the modal ingredients of our proof system, the soundness of the two distribution rules is guaranteed by Proposition 6.4.9 and Proposition 6.2.7. The soundness of the monotonicity rule follows from Proposition 6.4.10. Finally, the validity of all L-valid formulas is guaranteed by the conservativity of InqBL over L (Proposition 6.4.8). $\Box$

The remainder of this section is devoted to show that, for any canonical modal logic L, the system given for InqBL is also complete.

6.4.4 Completeness

To prove completeness for inquisitive Kripke logics, we need to retrace some of the steps that led to our completeness proof for propositional logic. As a first step, let us show that the Resolution Algorithm we had in the propositional case can be extended to the proof system $\vdash_{\mathsf{InqBL}}$.

6.4.12. Lemma.
Suppose $\Phi \vdash_{\mathsf{InqBL}} \psi$. For all $\Gamma \in \mathcal{R}(\Phi)$ there exists $\alpha \in \mathcal{R}(\psi)$ s.t. $\Gamma \vdash_{\mathsf{InqBL}} \alpha$.

Proof. As in the propositional case, the proof goes by induction on the proof $P : \Phi \vdash_{\mathsf{InqBL}} \psi$. The basis case is trivial, and for the inductive case, we need to distinguish a number of cases depending on what is the last rule applied in P. Now, if the last rule applied in P is one of the rules in Figure 6.2, we refer to the argument given in the propositional case (see the proof of Theorem 3.2.1). We can then focus on the rules concerning the modality $\Box$.

- Suppose ψ was obtained by either of the distributivity rules. Now, notice that each of these rules allows us to infer a declarative from another. To keep this in mind, let us write α instead of ψ. Then, the immediate subproof of P is a proof $P' : \Phi \vdash \beta$, where β is a declarative from which α can be inferred by one of the distributivity rules.

  Now, take any $\Gamma \in \mathcal{R}(\Phi)$. By the induction hypothesis, there is some $Q' : \Gamma \vdash \beta'$ for some $\beta' \in \mathcal{R}(\beta)$. Since β is a declarative, we know from Proposition 6.3.4 that $\mathcal{R}(\beta) = \{\beta\}$, so in fact we must have $Q' : \Gamma \vdash \beta$. Now, we know that α can be inferred from β by the distributivity rule; so, extending Q' with an application of this rule we get a proof $Q : \Gamma \vdash \alpha$. Since α is a declarative, we have $\mathcal{R}(\alpha) = \{\alpha\}$, so Q is a proof of the kind we need.

- Suppose $\psi = \Box\chi$ was obtained by the monotonicity rule. This means that the immediate subproofs of P are a proof $P' : \varphi_1, \ldots, \varphi_n \vdash \chi$ and, for $1 \leq i \leq n$, a proof $P_i : \Phi \vdash \Box\varphi_i$.

Now let $\Gamma \in \mathcal{R}(\Phi)$. Since $\mathcal{R}(\Box\varphi_i) = \{\Box\varphi_i\}$ by Proposition 6.3.4, the induction hypothesis applied to the proof P_i tells us that we have a proof $Q_i : \Gamma \vdash \Box\varphi_i$. Now we can apply $\Box$-monotonicity to the proof P' and the proofs $Q_1, \dots, Q_n$, obtaining a proof $Q : \Gamma \vdash \Box\chi$. Since $\mathcal{R}(\Box\chi) = \{\Box\chi\}$, this is what we need.

- Finally, suppose ψ is an axiom α of the logic L. Then it must be a classical formula, and thus also a declarative. This ensures that $\mathcal{R}(\alpha) = \{\alpha\}$. Since α is an axiom of our system, for any $\Gamma \in \mathcal{R}(\Phi)$ we trivially have $\Gamma \vdash \alpha$. $\Box$

Notice that this lemma has the following corollary, which plays a crucial role in proving the Support Lemma for inquisitive disjunction: if a set of declaratives derives a formula, then it derives some specific resolution of it.

6.4.13. Corollary (Provable resolution split).
Let $\Gamma \subseteq \mathcal{L}^{\mathsf{K}}_!$. If $\Gamma \vdash_{\mathsf{InqBL}} \varphi$, then $\Gamma \vdash_{\mathsf{InqBL}} \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$.

Proof. It follows easily from Proposition 6.3.4 that, if $\Gamma \subseteq \mathcal{L}^{\mathsf{K}}_!$, then $\mathcal{R}(\Gamma) = \{\Gamma\}$. Using this fact, the claim follows immediately from the previous lemma. $\Box$

Next, we remark that our system can prove the equivalence between a formula and its normal form, as given by Proposition 6.3.5

6.4.14. Lemma. *For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, $\varphi \dashv\vdash_{\mathsf{InqBL}} \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.*

Proof. The proof is by induction on φ. If φ is an atom, the constant $\bot$, or a modal formula, the claim is obvious by definition of resolutions. The inductive cases for the connectives go as in the propositional case (see Lemma 3.3.4). $\Box$

From this, it follows easily that our system allows us to distribute $\Box$ over the resolutions of its argument.

6.4.15. Lemma (Provable $\Box$ distribution over resolutions).
For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, if $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$, then $\Box\varphi \dashv\vdash_{\mathsf{InqBL}} \Box\alpha_1 \vee \dots \vee \Box\alpha_n$

Proof. We know from the previous lemma that $\varphi \dashv\vdash_{\mathsf{InqBL}} \alpha_1 \mathbin{\mathpalette\dovee\relax} \dots \mathbin{\mathpalette\dovee\relax} \alpha_n$. By monotonicity, we get $\Box\varphi \dashv\vdash_{\mathsf{InqBL}} \Box(\alpha_1 \mathbin{\mathpalette\dovee\relax} \dots \mathbin{\mathpalette\dovee\relax} \alpha_n)$. At this point, the rule of $\Box\mathbin{\mathpalette\dovee\relax}$ distributivity gives $\Box(\alpha_1 \mathbin{\mathpalette\dovee\relax} \dots \mathbin{\mathpalette\dovee\relax} \alpha_n) \vdash_{\mathsf{InqBL}} \Box\alpha_1 \vee \dots \vee \Box\alpha_n$. It is an easy exercise to show that the converse holds as well, that is, $\Box\alpha_1 \vee \dots \vee \Box\alpha_n \vdash_{\mathsf{InqBL}} \Box(\alpha_1 \mathbin{\mathpalette\dovee\relax} \dots \mathbin{\mathpalette\dovee\relax} \alpha_n)$. $\Box$

The next step is to show that our system is capable of turning any declarative into its classical variant.

6.4.16. Lemma. *For any $\alpha \in \mathcal{L}^{\mathsf{K}}_!$, $\alpha \dashv\vdash_{\mathsf{InqBL}} \alpha^{cl}$*

Proof. The proof proceeds by induction on the modal simplicity ordering. So, assume the claim is true for all declaratives $\beta \prec \alpha$, and let us proceed to show that it must hold for α as well. To do this, we distinguish a number of cases, depending on the form of α. If α is an atom or $\bot$, the claim is obvious, since $\alpha^{cl} = \alpha$. The cases in which α is a conjunction or an implication are also straightforward.

Finally, let us consider the case in which α is a modal formula $\Box\varphi$. Let $\mathcal{R}(\varphi) = \{\beta_1, \dots, \beta_m\}$. By the previous Lemma we know that $\Box\varphi \dashv\vdash_{\mathsf{InqBL}} \Box\beta_1 \vee\cdots\vee \Box\beta_m$. Now, by definition, resolutions are declaratives. Moreover, since by definition a resolution never has a higher modal depth than the original formula, we have $d(\beta_i) \leq d(\varphi) < d(\Box\varphi) = d(\alpha)$. This shows that for each $\beta_i \in \mathcal{R}(\varphi)$ we have $\beta_i \prec \alpha$. Thus, the induction hypothesis applies, yielding $\beta_i \dashv\vdash \beta_i^{cl}$.

Since we had $\Box\varphi \dashv\vdash_{\mathsf{InqBL}} \Box\beta_1 \vee \cdots \vee \Box\beta_m$, it follows easily that $\Box\varphi \dashv\vdash_{\mathsf{InqBL}} \Box\beta_1^{cl} \vee \cdots \vee \Box\beta_m^{cl}$, which by definition amounts precisely to $\Box\varphi \dashv\vdash_{\mathsf{InqBL}} (\Box\varphi)^{cl}$. This completes the inductive proof. $\Box$

This allows us to show that our system also proves a formula to be equivalent to the inquisitive disjunction of its *classical* resolutions, and that $\Box$ distributes over the *classical* resolutions of a formula.

6.4.17. Lemma (Provable classical normal form).
For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, $\varphi \dashv\vdash_{\mathsf{InqBL}} \mathbb{V}\mathcal{R}^{cl}(\varphi)$.

Proof. Immediate from Lemma 6.4.14 and Lemma 6.4.16. $\Box$

6.4.18. Lemma (Provable $\Box$ distribution over classical resolutions).
For any $\varphi \in \mathcal{L}^{\mathsf{K}}$, if $\mathcal{R}^{cl}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$, then $\Box\varphi \dashv\vdash_{\mathsf{InqBL}} \Box\alpha_1 \vee \cdots \vee \Box\alpha_n$

Proof. Analogous to that of Lemma 6.4.15, using the previous lemma. $\Box$

From now on, it will be convenient to always work with classical resolutions. This is because we are going to prove our completeness result for InqBL using the canonical model M^c_{L} for the standard modal logic L, whose worlds are made out of complete sets of classical formulas. Thus, it will be important to be able to relate the proof-theoretic features of an arbitrary formula φ of our language to the proof theoretic features of a set of classical formulas.[14]

By using the provability of classical normal form, we can then prove the traceable deduction failure lemma for our logic, showing that, if Φ fails to derive ψ, this can be traced to the fact that some specific resolution Γ of Φ fails to derive ψ.

[14]In fact, the same result could be achieved by defining a canonical model whose worlds are complete theories of declaratives; this is the strategy that will be pursued in the next section. Once we follow that strategy, there is no reason to be particularly concerned with classical formulas. However, in this section we stick to the standard canonical model construction from standard modal logic.

6.4.19. Lemma (Traceable deduction failure).
If $\Phi \not\vdash_{\mathsf{InqBL}} \psi$, there is a $\Gamma \in \mathcal{R}^{cl}(\Phi)$ such that $\Gamma \not\vdash_{\mathsf{InqBL}} \psi$.

Proof. Completely analogous to the proof of Lemma 3.3.7, the analogue of the present lemma in the propositional case. □

The next step towards completeness is to prove that the canonical model for $\mathcal{L}$ does not only satisfy the standard truth-lemma for classical formulas (Lemma 6.4.4, which transfers to InqBK by conservativity), but also the more general support-lemma for the whole language $\mathcal{L}^{\mathsf{K}}$.

6.4.20. Lemma (Support Lemma for M^c_{L}).
For any state S in M^c_{L}, the canonical model for L, and for any $\varphi \in \mathcal{L}^{\mathsf{K}}$:

$$M^c_{\mathsf{L}}, S \models \varphi \iff \bigcap S \vdash_{\mathsf{InqBL}} \varphi$$

Proof. As usual, the proof is by induction on φ. The basic cases and the induction steps for the connectives are just as in the propositional case (see Lemma 3.3.15), so we only spell out the induction step for $\Box$. To reduce clutter, throughout the proof we omit subscripts to the provability relation.

Suppose first $\bigcap S \vdash \Box\varphi$. Consider a $\Gamma \in S$. Since $\bigcap S \subseteq \Gamma$, we have $\Gamma \vdash \Box\varphi$. Now let $\mathcal{R}^{cl}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. Lemma 6.4.18 gives us $\Box\varphi \dashv\vdash \Box\alpha_1 \vee \dots \vee \Box\alpha_n$, so we also have $\Gamma \vdash \Box\alpha_1 \vee \dots \vee \Box\alpha_n$. Now, since Γ is closed under deduction of classical modal formulas, this implies $\Box\alpha_1 \vee \dots \vee \Box\alpha_n \in \Gamma$. And since complete theories have the disjunction property, we must have $\Box\alpha_i \in \Gamma$ for some i. By definition of the canonical state map σ^c_{L}, this means that for any $\Delta \in \sigma^c_{\mathsf{L}}(\Gamma)$ we must have $\alpha_i \in \Delta$. This shows that $\alpha_i \in \bigcap \sigma^c_{\mathsf{L}}(\Gamma)$. But $\alpha_i \vdash \varphi$ by Lemma 6.4.17. So, $\bigcap \sigma^c_{\mathsf{L}}(\Gamma) \vdash \varphi$, which by induction hypothesis gives $M^c_{\mathsf{L}}, \sigma^c_{\mathsf{L}}(\Gamma) \models \varphi$. Since this is true for any $\Gamma \in S$, by the support clause for $\Box$ we have $M^c_{\mathsf{L}}, S \models \Box\varphi$.

For the converse, suppose $\bigcap S \not\vdash \Box\varphi$. Again, let $\mathcal{R}^{cl}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. By Lemma 6.4.18, this implies $\bigcap S \not\vdash \Box\alpha_1 \vee \dots \vee \Box\alpha_n$, which means that $\Box\alpha_1 \vee \dots \vee \Box\alpha_n \notin \Gamma$ for some $\Gamma \in S$. Now, by the standard truth lemma for L (Lemma 6.4.4), which applies since $\Box\alpha_1 \vee \dots \vee \Box\alpha_n \in \mathcal{L}^{\mathsf{K}}_c$, we get $M^c_{\mathsf{L}}, \Gamma \not\models \Box\alpha_1 \vee \dots \vee \Box\alpha_n$. Since $\Box\varphi \equiv \Box\alpha_1 \vee \dots \vee \Box\alpha_n$ by Proposition 6.2.7 and Proposition 6.3.13, this implies $M^c_{\mathsf{L}}, \Gamma \not\models \Box\varphi$. Since truth amounts to support with respect to singleton states, we have $M^c_{\mathsf{L}}, \{\Gamma\} \not\models \Box\varphi$. Since $\{\Gamma\} \subseteq S$, the persistency of the semantics implies $M^c_{\mathsf{L}}, S \not\models \Box\varphi$. □

Equipped with the lemmata established so far, we are now ready to prove our completeness result. The proof follows the same argument that we have already used in the propositional case.

6.4.21. Theorem (Completenss).
Let L be a canonical normal modal logic. For any $\Phi \subseteq \mathcal{L}^{K}$ and $\psi \in \mathcal{L}^{K}$:

$$\Phi \models_{\mathsf{InqBL}} \psi \;\; \textit{implies} \;\; \Phi \vdash_{\mathsf{InqBL}} \psi$$

Proof. Suppose $\Phi \not\models_{\mathsf{InqBL}} \psi$. By Lemma 6.4.19, we have a $\Gamma \in \mathcal{R}^{cl}(\Phi)$ such that $\Gamma \not\models_{\mathsf{InqBL}} \psi$. Now, let $\mathcal{R}^{cl}(\psi) = \{\alpha_1, \dots, \alpha_n\}$. For any $i \leq n$, we must have $\Gamma \nvdash \alpha_i$: otherwise, given that by Lemma 6.4.17 we have $\alpha_i \vdash \psi$, Γ would derive ψ. Since we have double negation elimination for α_i, this means that $\Gamma \not\models \neg\neg\alpha_i$, whence also $\Gamma, \neg\alpha_i \nvdash \bot$. Thus, the set $\Gamma \cup \{\neg\alpha_i\}$ is a consistent set of classical modal formulas, and can be extended in the usual way to a complete theory of classical formulas $\Delta_i \in W^c_{\mathsf{L}}$.

Now consider the state $S = \{\Delta_1, \dots, \Delta_n\}$. Since $\Gamma \subseteq \Delta_i$ for $1 \leq i \leq n$, we have $\Gamma \subseteq \bigcap S$. Now take any $\varphi \in \Phi$. Since $\Gamma \in \mathcal{R}^{cl}(\Phi)$, Γ contains a classical resolution $\beta \in \mathcal{R}^{cl}(\varphi)$. So, $\beta \in \bigcap S$, and since $\beta \vdash \varphi$, also $\bigcap S \vdash \varphi$. By the Support Lemma, this means that $M^c_{\mathsf{L}}, S \models \varphi$. Thus, we have $M^c_{\mathsf{L}}, S \models \Phi$.

On the other hand, suppose for a contradiction that $M^c_{\mathsf{L}}, S \models \psi$. Then, by Proposition 6.3.13 we should have $M^c_{\mathsf{L}}, S \models \alpha_i$ for some $\alpha_i \in \mathcal{R}^{cl}(\psi)$. By the Support Lemma, this would mean that $\bigcap S \vdash \alpha_i$. Since $\bigcap S \subseteq \Delta_i$, we would have $\Delta_i \vdash \alpha_i$. This is impossible, since by construction Δ_i contains $\neg\alpha_i$ and is consistent.

Thus, we have proved that $M^c_{\mathsf{L}}, S \models \Phi$ but $M^c_{\mathsf{L}}, S \not\models \psi$. Since L is canonical, M^c_{L} is an L-model. Thus, the existence of our state S shows that $\Phi \not\models_{\mathsf{InqBL}} \psi$. $\Box$

Notice in particular that, for the general system InqBK of inquisitive Kripke modal logic, no axioms from the corresponding modal logic K are needed. Thus, for InqBK we have the following completeness result, showing that the only non-standard feature of $\Box$ needed to axiomatize InqBK is the distributivity of $\Box$ over $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

6.4.22. Corollary (Completeness for InqBK).
InqBK is completely axiomatized by extending the system for connectives of Figure 6.2 with the rules of $\Box\!\to$ distributivity, $\Box\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ distributivity, and $\Box$ monotonicity.

Moreover, for each normal modal logic InqBL obtained by imposing one or more of the frame conditions in Figure 6.1, a complete axiomatization is obtained by adding the standard axioms for the logic L: thus, e.g., InqBKT is axiomatized by adding axioms $\Box\alpha \to \alpha$ for all $\alpha \in \mathcal{L}^{\mathsf{K}}_c$; a complete axiomatization of InqBK4 is obtained by adding $\Box\alpha \to \Box\Box\alpha$; etc. In this way, we get simple complete systems for the inquisitive version of many notable normal modal logics.

6.5 A modal account of dependence statements

In the previous chapters, we saw that implications $\mu \to \nu$ among questions capture dependencies: a state s supports $\mu \to \nu$ just in case the question μ determines the question ν relative to s. However, in this section we will see that a formula $\mu \to \nu$ should not be taken directly to be a formalization of a dependence statement, i.e., of a sentence of the form "μ determines ν". Instead, we investigate a modal account of dependence statements, based on a binary modal operator $\Rightarrow$.

6.5.1 Motivations

Let the atom s_1 stand for "the patient has symptom S_1" and the atom t stand for "the treatment is prescribed". We will consider four reasons why the formula $?s_1 \to ?t$ is not a suitable formal rendering of the statement in (8).[15]

(8) Whether the patient has symptom S_1 determines whether the treatment is prescribed.

Reason 1. Statement (8) can very well be false: as a matter of fact, given the protocol considered in Chapter 1, this statement *is* false. By contrast, the formula $?s_1 \to ?t$ is true at all possible worlds. Indeed, $?s_1 \to ?t$ has the same truth-conditions as its classical variant, $(s_1 \vee \neg s_1) \to (t \vee \neg t)$, which is a tautology.

Reason 2. If we take $?s_1 \to ?t$ to translate (8), we would want (9) to be translated as $\neg(?s_1 \to ?t)$. However, the statement (9) is clearly consistent; by contrast, the negation $\neg(?s_1 \to ?t)$ is a logical contradiction: $\neg(?s_1 \to ?t) \equiv \bot$

(9) Whether the patient has symptom S_1 does not determine whether the treatment is prescribed.

Reason 3. If we take $?s_1 \to ?t$ to translate (8), we would want the polar question in (10) to be translated as $?(?s_1 \to ?t)$. However, it is easy to see that $?(?s_1 \to ?t) \equiv ?s_1 \to ?t$. Thus, we would wrongly predict the question (10) to mean the same as the statement (8).

(10) Does whether the patient has symptom S_1 determine whether the treatment is prescribed?

Reason 4. An information state s may well settle (8)—that is, it may be known in the state that $?s_1$ determines $?t$—while s does not settle exactly in what way $?s_1$ determines $?t$—whether, say, the treatment is prescribed iff symptom S_1 is present, or the treatment is prescribed iff symptom S_1 is absent. Once again, we find a discrepancy here between (8) and the formula $?s_1 \to ?t$, which is only settled if some *specific way* for $?t$ to depend on $?s_1$ is settled:

$$\begin{aligned} M, s \models ?s_1 \to ?t \quad \iff \quad & M, s \models (s_1 \to t) \wedge (\neg s_1 \to t) \text{ or} \\ & M, s \models (s_1 \to t) \wedge (\neg s_1 \to \neg t) \text{ or} \\ & M, s \models (s_1 \to \neg t) \wedge (\neg s_1 \to t) \text{ or} \\ & M, s \models (s_1 \to \neg t) \wedge (\neg s_1 \to \neg t) \end{aligned}$$

The conceptual view developed in this thesis suggests the following diagnosis of the problem we are faced with: a sentence like (8) is a statement. Statements

[15] In the dependence logic literature, the dependence atom $=\!\!(s_1, t)$ appears to be often implicitly assumed to be a formal rendition of statements like (8). Since $=\!\!(s_1, t)$ is equivalent with $?s_1 \to ?t$, the arguments given in this section apply equally well to the dependence atom.

are truth-conditional: thus, we need to associate (8) with a truth-conditional formula, not with a question like $?s_1 \to ?t$. If a formula α is to capture (8), its semantics must be completely determined at the level of single worlds.

At the same time, we saw that propositional dependencies only manifest themselves when we consider a *set* of worlds, i.e., in the context of an information state.

These two facts can be reconciled by taking each world w to be associated with a relevant information state $\sigma(w)$, i.e., by interpreting such statements in the context of a Kripke model.[16] A sentence like (8), can then be judged true or false at a world w depending on whether a certain dependency holds at the associated state $\sigma(w)$. In a slogan, *dependence statements are modal statements.*[17]

6.5.2 Modal implication

In the setting of inquisitive Kripke modal logic, there is a natural way to implement this idea. Let $\mathcal{L}^{\Rightarrow}$ be the language obtained by expanding the propositional language $\mathcal{L}^{\mathsf{P}}$ by means of a binary operator $\Rightarrow$, which we will refer to as *modal implication.* A formula $\varphi \Rightarrow \psi$ will be read as "φ determines ψ".

$$\varphi \ ::= \ p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\vee\!\!\!\vee} \varphi \mid \varphi \Rightarrow \varphi$$

As for InqBK, formulas of $\mathsf{InqB}^{\Rightarrow}$ will be interpreted with respect to a state s drawn from a Kripke model M. Atoms and connectives are interpreted as usual. For the interpretation of $\Rightarrow$, recall that we have the following definition for when a formula φ entails ψ relative to a state s:[18]

$$\varphi \models_s \psi \iff \text{for any } t \subseteq s : M, t \models \varphi \text{ implies } M, t \models \psi$$

The clause for $\Rightarrow$ states that a formula $\varphi \Rightarrow \psi$ is supported at s in case s implies that the actual world w is one in which φ entails ψ relative to $\sigma(w)$.

6.5.1. Definition. [Support clause for $\Rightarrow$]
$M, s \models \varphi \Rightarrow \psi \iff \varphi \models_{\sigma(w)} \psi$ for all $w \in s$

This makes any formula $\varphi \Rightarrow \psi$ truth-conditional, with the following truth-conditions.

6.5.2. Proposition (Truth-conditions for $\Rightarrow$).
$M, w \models \varphi \Rightarrow \psi \iff \varphi \models_{\sigma(w)} \psi$

[16] What the information $\sigma(w)$ represents will depend on the flavor of the dependency at hand: e.g., it could be the information state of an agent, the set of possible futures for a world, the set of configurations compatible with some physical law.

[17] Goranko and Kuusisto have independently come to the same conclusion. Their formal implementation of this idea, currently under development, is somewhat different from the one give here, in that their system does not include questions. The relations between the two approaches are discussed in Section 6.7.2.

[18] The most appropriate notation would be $\varphi \models_{M,s} \psi$, since of course it matters which model M the state s is drawn from. However, to reduce clutter we will leave the model M implicit.

First, let us consider the particular case of a formula $\alpha \Rightarrow \beta$, where both α and β are truth-conditional. In this case, $\alpha \models_{\sigma(w)} \beta$ boils down to the fact that any α-world in $\sigma(w)$ is also a β-world—equivalently, that the material conditional $\alpha \to \beta$ is true throughout the state $\sigma(w)$. Thus, we have the following fact.

6.5.3. Proposition. *If α and β are truth-conditional,*

$$\begin{aligned} M, w \models \alpha \Rightarrow \beta \quad &\iff \quad \sigma(w) \cap |\alpha|_M \subseteq |\beta|_M \\ &\iff \quad M, v \models \alpha \to \beta \textit{ for all } v \in \sigma(w) \end{aligned}$$

Thus, in this case $\alpha \Rightarrow \beta$ is simply a strict conditional. Intuitively, if $\sigma(w)$ represents the state of an agent at w, we may read $\alpha \Rightarrow \beta$ as expressing that for the agent, the truth of α implies the truth of β.

It is also interesting to consider the case of a formula $\alpha \Rightarrow \mu$ in which α is truth-conditional and μ is a question. In this case, $\alpha \models_{\sigma(w)} \mu$ boils down to the fact that μ is settled in the state $\sigma(w) \cap |\alpha|_M$, i.e., to the fact that relative to $\sigma(w)$, the information that α is true suffices to settle the question μ.

6.5.4. Proposition. *If α is truth-conditional,*

$$M, w \models \alpha \Rightarrow \varphi \iff M, \sigma(w) \cap |\alpha|_M \models \varphi$$

Now let us turn to the case of a formula $\mu \Rightarrow \nu$ where μ and ν are two questions. In this case, we know from the discussion in Chapter 1 that $\mu \models_{\sigma(w)} \nu$ means that μ determines ν relative to $\sigma(w)$. Thus $\mu \Rightarrow \nu$ is a statement with the desired property of being true at a world if and only if μ determines ν in the state $\sigma(w)$ associated with the world. Thus, the formula $\mu \Rightarrow \nu$ is a suitable representation for a dependence statement "μ determines ν".

Let us now verify that the four problems pointed out above no longer arise if dependence statements are translated by means of $\Rightarrow$.

1. Unlike $?s_1 \to ?t$, the formula $?s_1 \Rightarrow ?t$ may well be false at a world w: all that it takes is an associated state $\sigma(w)$ in which the question $?s_1$ fails to determine the question $?t$. In particular, suppose we model our hospital protocol example by means of a Kripke model, where the state $\sigma(w)$ associated with a world consists of the worlds which are compatible with what the hospital's protocol at w prescribes. Now, we know by our description of the protocol that the actual world w is one where $?s_1 \not\models_{\sigma(w)} ?t$. This ensures that the modal implication $?s_1 \Rightarrow ?t$ is indeed false at w, as we expect.

2. Consider (9), the negation of (8). This statement can be translated simply as the negation of (8), $\neg(?s_1 \Rightarrow ?t)$. Unlike $\neg(?s_1 \to ?t)$, this formula is perfectly consistent: as we expect, it is true at a world w in case the question $?s_1$ fails to determine the question $?t$ relative to $\sigma(w)$.

3. Similarly, the polar question (10), can be translated simply by applying a question mark to the translation of (8). It is easy to see that the formula $?(?s_1 \Rightarrow ?t)$ does not boil down to $?s_1 \Rightarrow ?t$; rather, as we expect, it is a polar question which is settled in a state s just in case s settles one among (8) and (9).

4. By formalizing (8) as $?s_1 \Rightarrow ?t$, we correctly predict that in a state it can be settled *that* $?s_1$ determines $?t$, while it is not settled *how* $?s_1$ determines $?t$: for, it is quite possible that at any world $w \in s$ we have $?s_1 \models_{\sigma(w)} ?t$, that is, for any $w \in s$ there is a dependence function f_w from $?s_2$ to $?t$ in $\sigma(w)$, yet there is no function $f : \mathcal{R}(?s_1) \to \mathcal{R}(?t)$ which is settled in s to be a dependence function, i.e., no function which is a dependence function relative to $\sigma(w)$ for each world $w \in s$.

Summing up, then, the dependence operator $\Rightarrow$ generalizes the strict conditional operator from statements to questions. For any formulas φ and ψ, $\varphi \Rightarrow \psi$ is a statement, which is true at a world w iff φ entails ψ relative to the state $\sigma(w)$ attached to the world. In particular, $\mu \Rightarrow \nu$ is a statement which is true at w precisely in case ν is determined by μ relative to $\sigma(w)$. In this way, we obtain a simple formalization of dependence statements which avoids the range of problems discussed above for a more naïve formalization in terms of plain implication or dependence atoms.

6.5.3 Axiomatizing modal implication

Now that we have seen the relevance of the operator $\Rightarrow$, let us turn to the investigation of its logical features. A rather obvious but important fact is that $\Rightarrow$ is uniformly definable in terms of the Kripke modality $\Box$ and implication. For, if φ and ψ are propositional formulas, we have:

$$\begin{aligned} M, w \models \varphi \Rightarrow \psi \quad &\iff \quad \varphi \models_{\sigma(w)} \psi \\ &\iff \quad M, \sigma(w) \models \varphi \to \psi \\ &\iff \quad M, w \models \Box(\varphi \to \psi) \end{aligned}$$

Since both $\varphi \Rightarrow \psi$ and $\Box(\varphi \to \psi)$ are truth-conditional, this implies that they are equivalent. This fact immediately yields a translation of $\mathsf{InqB}^{\Rightarrow}$ into InqBK, where $\circ$ stands for any of the connectives $\wedge, \to, \vee\!\!\!\vee$.

6.5.5. Definition. [Translation of $\mathsf{InqB}^{\Rightarrow}$ into InqBK]

- $p^{\Box} = p$
- $\bot^{\Box} = \bot$
- $(\varphi \circ \psi)^{\Box} = \varphi^{\Box} \circ \psi^{\Box}$
- $(\varphi \Rightarrow \psi)^{\Box} = \Box(\varphi^{\Box} \to \psi^{\Box})$

It is immediate to verify that we indeed have $\varphi \equiv \varphi^{\Box}$ for all $\varphi \in \mathcal{L}^{\Rightarrow}$. Thus, there is no need to regard $\Rightarrow$ as a new operator. We can capture dependence statements in inquisitive modal logic, as modal formulas of the form $\Box(\varphi \to \psi)$. This allows us to take advantage of the logical results established above for InqBK.

At the same time, it is also interesting to consider what logic we obtain if we take the binary operator $\Rightarrow$, rather than the unary operator $\Box$, as our primary modal operator. A first observation in this respect is that $\Box$ is in turn uniformly definable in terms $\Rightarrow$. If φ and ψ are propositional formulas, we have:

$$\begin{aligned} M, w \models \Box\varphi \quad &\iff \quad M, \sigma(w) \models \varphi \iff M, \sigma(w) \models \top \to \varphi \\ &\iff \quad \top \models^{M}_{\sigma(w)} \varphi \iff M, w \models \top \Rightarrow \varphi \end{aligned}$$

Again, by the truth-conditionality of both formulas, this implies that $\Box\varphi \equiv \top \Rightarrow \varphi$. This suggests the following translation from InqBK to $\mathsf{InqB}^{\Rightarrow}$, where again $\circ$ stands for any connective.

6.5.6. Definition. [Translation from InqBK to $\mathsf{InqB}^{\Rightarrow}$]

- $p^{\Rightarrow} = p$
- $\bot^{\Rightarrow} = \bot$
- $(\varphi \circ \psi)^{\Rightarrow} = \varphi^{\Rightarrow} \circ \psi^{\Rightarrow}$
- $(\Box\varphi)^{\Rightarrow} = (\top \Rightarrow \varphi^{\Rightarrow})$

Again, it is easy to verify that we have $\varphi \equiv \varphi^{\Rightarrow}$ for all $\varphi \in \mathsf{InqBK}$. This shows that the systems are equivalent, and that either of $\Rightarrow$ and $\Box$ may be taken as the primitive modal operator of our inquisitive Kripke modal logic.

Now, in the language $\mathcal{L}^{\Rightarrow}$, like in $\mathcal{L}^{\mathsf{K}}$, we can isolate a class of *declaratives*, formulas whose syntactic form ensures that they are truth-conditional.

6.5.7. Definition. [Declarative fragment of $\mathcal{L}^{\Rightarrow}$]
The set $\mathcal{L}^{\Rightarrow}_!$ of declarative formulas in $\mathcal{L}^{\Rightarrow}$ is given by the following rewrite rules, where φ, ψ are formulas in $\mathcal{L}^{\Rightarrow}$:

$$\alpha \ ::= \ p \mid \bot \mid \varphi \Rightarrow \psi \mid \alpha \land \alpha \mid \alpha \to \alpha$$

That is, declaratives are built up from atoms, $\bot$, and arbitrary modal implications, by means of the connectives $\land$ and $\to$. Alternatively, they may be defined as formulas whose sole occurrences of $\vee\!\!\!\vee$ are within the scope of a modal implication.

It is easy to check that, indeed, declaratives are always truth-conditional. Moreover, it will be useful to remark that both translations $(\cdot)^{\Box}$ and $(\cdot)^{\Rightarrow}$ map declaratives to declaratives: that is, $\alpha \in \mathcal{L}^{\mathsf{K}}_!$ implies $\alpha^{\Rightarrow} \in \mathcal{L}^{\Rightarrow}_!$, and conversely, $\alpha \in \mathcal{L}^{\Rightarrow}_!$ implies $\alpha^{\Box} \in \mathcal{L}^{\mathsf{K}}_!$.

As usual, we can associate any formula $\varphi \in \mathcal{L}^{\Rightarrow}$ with a finite set $\mathcal{R}(\varphi)$ of declaratives such that $\varphi \equiv \bigvee\!\!\!\bigvee \mathcal{R}(\varphi)$. To do this, it suffices to extend Definition 2.4.1 with a clause $\mathcal{R}(\varphi \Rightarrow \psi) = \{\varphi \Rightarrow \psi\}$. Thus, as for InqB and InqBK, we have a normal form result which allows us to represent any formula as an inquisitive disjunction of statements.

Now, let us turn to examine the logical properties of our modal implication. First, notice that this operation is *transitive* in the following sense.

6.5.8. Proposition (Transitivity). $\varphi \Rightarrow \psi,\ \psi \Rightarrow \chi \ \models\ \varphi \Rightarrow \chi$

Proof. Since all formulas involved are truth-conditional, we just need to show that the truth of the premisses implies the truth of the conclusion. By spelling out the relevant truth-conditions, the claim follows immediately. □

Second, we have a split property for $\Rightarrow$. However, this property is slightly different from the one we have for plain implication: a declarative antecedent does not simply distribute over an inquisitive disjunction in the consequent; rather, in distributing, it turns the inquisitive disjunction into a classical disjunction.

6.5.9. Proposition ($\Rightarrow\!\mathbb{V}$ split).
If α is truth-conditional, $\alpha \Rightarrow (\varphi \mathbin{\mathbb{V}} \psi) \ \models\ (\alpha \Rightarrow \varphi) \vee (\alpha \Rightarrow \psi)$

Proof. Since both formulas are truth-conditional, we just have to verify entailment at the level of truth-conditions. We have:

$$\begin{aligned} M, w \models \alpha \Rightarrow \varphi \mathbin{\mathbb{V}} \psi \quad &\iff \quad \alpha \models_{\sigma(w)} \varphi \mathbin{\mathbb{V}} \psi \quad \iff \quad M, \sigma(w) \models \alpha \to \varphi \mathbin{\mathbb{V}} \psi \\ &\iff \quad M, \sigma(w) \models \alpha \to \varphi \text{ or } M, \sigma(w) \models \alpha \to \psi \\ &\iff \quad M, w \models \alpha \Rightarrow \varphi \text{ or } M, w \models \alpha \Rightarrow \psi \\ &\iff \quad M, w \models (\alpha \Rightarrow \varphi) \vee (\alpha \Rightarrow \psi) \end{aligned}$$

□

Third, we have that a modal implication $\varphi \Rightarrow \psi$ is logically valid just in case φ logically entails ψ.

6.5.10. Proposition. $\models \varphi \Rightarrow \psi \iff \varphi \models \psi$

Proof. If $\varphi \models \psi$, then then φ will entail ψ in any context. Thus, $\varphi \Rightarrow \psi$ must be true at any world, and since it is truth-conditional, it must be a logical validity.

For the converse, suppose $\varphi \not\models \psi$. This means that there is a Kripke model M and a state s such that $M, s \models \varphi$ but $M, s \not\models \psi$. Now consider the model $M' = \langle W', \sigma', V' \rangle$ which is just like M, except that we add a world w_0 and we let $\sigma'(w_0) = s$ (as for the valuation, we may set the value of atoms at w_0 at will). Since we have not changed the valuation or the set of successors of any world in M, we still have that $M', s \models \varphi$ and $M', s \not\models \psi$. Since $s = \sigma'(w_0)$, this implies $M', w_0 \not\models \varphi \Rightarrow \psi$, which shows that $\varphi \Rightarrow \psi$ is not logically valid. □

Recall that the same property holds for plain implication. Thus, we have that a formula $\varphi \Rightarrow \psi$ is logically valid if and only if the formula $\varphi \to \psi$ is. However, Transitivity

$$\frac{\varphi \Rightarrow \psi \quad \psi \Rightarrow \chi}{\varphi \Rightarrow \chi}$$

Split

$$\frac{\alpha \Rightarrow (\varphi \mathbin{\mathbb{V}} \psi)}{(\alpha \Rightarrow \varphi) \vee (\alpha \Rightarrow \psi)}$$

Logical Dependency

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \psi\end{array}}{\varphi \Rightarrow \psi}$$

Internalization

$$\frac{(\varphi \wedge \psi) \Rightarrow \chi}{\varphi \Rightarrow (\psi \to \chi)}$$

Figure 6.4: A complete set of rules for the operator $\Rightarrow$. In the Split rule, α is restricted to declarative formulas. In the Logical Dependency rule, φ must be the *only* undischarged assumption in the proof leading to ψ. Finally, notice that Internalization makes $(\varphi \wedge \psi) \Rightarrow \chi$ and $\varphi \Rightarrow (\psi \to \chi)$ inter-derivable.

as we saw, the two implications interact differently with the other connectives. For instance, since $\varphi \Rightarrow \psi$ is a statement, we have $\neg\neg(\varphi \Rightarrow \psi) \equiv (\varphi \Rightarrow \psi)$, while $\neg\neg(\varphi \to \psi) \not\equiv (\varphi \to \psi)$.

Finally, the following proposition shows that there is an interaction between the modal conditionals. A conjunction $\varphi \wedge \psi$ determines a formula χ if and only if φ determines the conditional $\psi \to \chi$. This corresponds to the fact that $\to$ internalizes entailment in context, so we will refer to this property as *internalization.*

6.5.11. Proposition (Internalization). $(\varphi \wedge \psi) \Rightarrow \chi \;\equiv\; \varphi \Rightarrow (\psi \to \chi)$

Proof. Since both formulas are truth-conditional, we just have to verify that they have the same truth-conditions. Now, we have $M, w \models (\varphi \wedge \psi) \Rightarrow \chi \iff M, \sigma(w) \models (\varphi \wedge \psi) \to \chi$ and $M, w \models \varphi \Rightarrow (\psi \to \chi) \iff M, \sigma(w) \models \varphi \to (\psi \to \chi)$. Thus, the claim follows from the propositional equivalence $(\varphi \wedge \psi) \to \chi \equiv \varphi \to (\psi \to \chi)$. □

We are now going to see that these four features of modal implication suffice to completely characterize its logic. We will take our proof system for $\mathsf{InqB}^{\Rightarrow}$ to be given by adding to the familiar rules for the propositional connectives of Figure 6.3 (where the role of declaratives is played by formulas $\alpha \in \mathcal{L}_!^{\Rightarrow}$) four inference rules for $\Rightarrow$, corresponding to the four logical features of the operator $\Rightarrow$ that we just discussed. Notice that, in the Logical Dependency rule, φ is required to be the *only* undischarged assumption in the sub-proof leading to ψ.

Clearly, the soundness of this proof system is guaranteed by propositions 6.5.8-6.5.11. We are now going to show that this system is also complete for $\mathsf{InqB}^{\Rightarrow}$.

Let us write $\vdash_{\mathsf{InqB}^{\Rightarrow}}$ for provability in our system, omitting the subscript when no confusion arises. First, notice that our system allows us to replace provably equivalent formulas as arguments of $\Rightarrow$.

6.5.12. Lemma (Replacement of equivalents in $\Rightarrow$).
If $\varphi \dashv\vdash \varphi'$ and $\psi \dashv\vdash \psi'$, then $\varphi \Rightarrow \psi \dashv\vdash \varphi' \Rightarrow \psi'$.

Proof Suppose $\varphi \dashv\vdash \varphi'$ and $\psi \dashv\vdash \psi'$. By the Logical Dependency rule we have $\vdash \varphi' \Rightarrow \varphi$ and $\vdash \psi \Rightarrow \psi'$. Now, if we assume $\varphi \Rightarrow \psi$, by two applications of the Transitivity rule we obtain $\varphi' \Rightarrow \psi'$. This shows $\varphi \Rightarrow \psi \vdash \varphi \Rightarrow \chi$. The converse direction is proved similarly, using the fact that, by the Logical Dependency rule, we have $\vdash \varphi \Rightarrow \varphi'$ and $\vdash \psi' \Rightarrow \psi$. ☐

Second, for all $\varphi \in \mathcal{L}^{\Rightarrow}$, our system proves φ to be equivalent to $(\varphi^{\Box})^{\Rightarrow}$, which we will abbreviate as $\varphi^{\Box\Rightarrow}$.

6.5.13. Lemma. *For any $\varphi \in \mathcal{L}^{\Rightarrow}$, $\varphi \dashv\vdash \varphi^{\Box\Rightarrow}$.*

Proof. The proof is by induction on φ. The only non-trivial case is the inductive step for $\varphi = (\psi \Rightarrow \chi)$. By definition, we have $\varphi^{\Box\Rightarrow} = (\top \Rightarrow (\psi^{\Box\Rightarrow} \to \chi^{\Box\Rightarrow}))$. The induction hypothesis gives $\psi \dashv\vdash \psi^{\Box\Rightarrow}$ and $\chi \dashv\vdash \chi^{\Box\Rightarrow}$, whence $\psi \to \chi \dashv\vdash \psi^{\Box\Rightarrow} \to \chi^{\Box\Rightarrow}$. By Replacement of Equivalents, we get $\varphi^{\Box\Rightarrow} = (\top \Rightarrow (\psi^{\Box\Rightarrow} \to \chi^{\Box\Rightarrow})) \dashv\vdash (\top \Rightarrow (\psi \to \chi))$. Now by Internalization, we have $\top \Rightarrow (\psi \to \chi) \dashv\vdash (\top \wedge \psi) \Rightarrow \chi$. Obviously, we have $\top \wedge \psi \dashv\vdash \psi$, whence by replacement of equivalents we get $(\top \wedge \psi) \Rightarrow \chi \dashv\vdash \psi \Rightarrow \chi$. Putting all these pieces together, we have:

$$\varphi^{\Box\Rightarrow} \dashv\vdash \top \Rightarrow (\psi \to \chi) \dashv\vdash (\top \wedge \psi) \Rightarrow \chi \dashv\vdash \psi \Rightarrow \chi = \varphi$$

☐

The key step of our completeness proof for $\mathsf{InqB}^{\Rightarrow}$ consists in showing that any proof in our complete system for InqBK can be simulated by our system for $\mathsf{InqB}^{\Rightarrow}$.

6.5.14. Lemma.
For $\varphi_1, \dots, \varphi_n, \psi \in \mathcal{L}^{\mathsf{K}}$, $\varphi_1, \dots, \varphi_n \vdash_{\mathsf{InqBK}} \psi$ implies $\varphi_1^{\Rightarrow}, \dots, \varphi_n^{\Rightarrow} \vdash_{\mathsf{InqB}^{\Rightarrow}} \psi^{\Rightarrow}$.

Proof. We just have to make sure that each rule of our system for InqBK can be simulated by our system for $\mathsf{InqB}^{\Rightarrow}$. This is straightforward for the rules for the connectives, which are the same in the two systems.[19] Now consider the three inference rules for the modality $\Box$, namely: (i) $\Box\!\to$ distributivity, (ii) $\Box\mathbb{V}$ distributivity and (iii) $\Box$ monotonicity.

[19]For the rules of $\mathbb{V}$-split and $\neg\neg$-elimination, we have to rely on the above observation that the map $(\cdot)^{\Rightarrow}$ maps declaratives in $\mathcal{L}^{\mathsf{K}}$ to declaratives in $\mathcal{L}^{\Rightarrow}$. For instance, suppose α is a declarative in $\mathcal{L}^{\mathsf{K}}$ and consider the rule $\neg\neg\alpha \vdash_{\mathsf{InqBK}} \alpha$. Then, *since $\alpha^{\Rightarrow}$ is a declarative in $\mathsf{InqB}^{\Rightarrow}$*, we can apply the $\neg\neg$-elimination rule, and obtain $(\neg\neg\alpha)^{\Rightarrow} = \neg\neg\alpha^{\Rightarrow} \vdash_{\mathsf{InqB}^{\Rightarrow}} \alpha^{\Rightarrow}$.

For $\Box\!\rightarrow$ distributivity, we need to prove that from the formula $(\Box(\varphi \to \psi))^{\Rightarrow} = (\top \Rightarrow (\varphi^{\Rightarrow} \to \psi^{\Rightarrow}))$ we can infer $(\Box\varphi \to \Box\psi)^{\Rightarrow} = (\top \Rightarrow \varphi^{\Rightarrow}) \to (\top \Rightarrow \psi^{\Rightarrow})$. To see this, assume $\top \Rightarrow (\varphi^{\Rightarrow} \to \psi^{\Rightarrow})$. By Internalization and replacement of equivalents we get $\varphi^{\Rightarrow} \Rightarrow \psi^{\Rightarrow}$. If we assume $\top \Rightarrow \varphi^{\Rightarrow}$, by Transitivity we obtain $\top \Rightarrow \psi^{\Rightarrow}$. Discharging the assumption, we get $(\top \Rightarrow \varphi^{\Rightarrow}) \to (\top \Rightarrow \psi^{\Rightarrow})$, as we wanted.

For $\Box\!\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ distributivity, we need to show that from the formula $(\Box(\varphi \mathbin{\vee\!\!\!\vee} \psi))^{\Rightarrow} = (\top \Rightarrow (\varphi^{\Rightarrow} \mathbin{\vee\!\!\!\vee} \psi^{\Rightarrow}))$ we can infer $(\Box\varphi \vee \Box\psi)^{\Rightarrow} = (\top \Rightarrow \varphi^{\Rightarrow}) \vee (\top \Rightarrow \psi^{\Rightarrow})$. This is obvious, since this inference is an instance of the Split rule for $\Rightarrow$.

Finally, we have to show that we can simulate the monotonicity rule, i.e., that if we have $\varphi_1^{\Rightarrow}, \dots, \varphi_n^{\Rightarrow} \vdash \psi^{\Rightarrow}$, we also have $(\Box\varphi_1)^{\Rightarrow}, \dots, (\Box\varphi_n)^{\Rightarrow} \vdash (\Box\psi)^{\Rightarrow}$, which by definition amounts to:

$$(\top \Rightarrow \varphi_1^{\Rightarrow}), \dots, (\top \Rightarrow \varphi_n^{\Rightarrow}) \vdash (\top \Rightarrow \psi^{\Rightarrow})$$

Let us omit the $\Rightarrow$ superscripts, which play no role in the argument. So, suppose $\varphi_1, \dots, \varphi_n \vdash \psi$. Propositional reasoning gives $\top \vdash \varphi_1 \to (\cdots \to (\varphi_n \to \psi))$. Using the rule of Logical Dependency, we have $\vdash \top \Rightarrow (\varphi_1 \to (\cdots \to (\varphi_n \to \psi)))$. But we have shown above that from a formula $\top \Rightarrow (\chi \to \xi)$ we can always infer the formula $(\top \Rightarrow \chi) \to (\top \Rightarrow \xi)$. Using this distribution fact multiple times, we finally obtain $\vdash (\top \Rightarrow \varphi_1) \to (\cdots \to ((\top \Rightarrow \varphi_n) \to (\top \Rightarrow \psi)))$, whence it is clear that $(\top \Rightarrow \varphi_1), \dots, (\top \Rightarrow \varphi_n) \vdash (\top \Rightarrow \psi)$. $\Box$

These lemmata allow us to transfer to $\mathsf{InqB}^{\Rightarrow}$ our completeness result for InqBK.

6.5.15. Theorem (Completeness for $\mathrm{InqB}^{\Rightarrow}$).
For $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\Rightarrow}$, $\Phi \models \psi \iff \Phi \vdash_{\mathsf{InqB}^{\Rightarrow}} \psi$.

Proof. The soundness direction of the theorem follows from our discussion of the inference rules for $\Rightarrow$, together with the soundness of the rules for the connectives.

As for the completeness direction, suppose $\Phi \models \psi$. Since the translation $(\cdot)^{\Box}$ preserves support conditions, we have $\Phi^{\Box} \models \psi^{\Box}$. By the completeness for InqBK, we have $\Phi^{\Box} \vdash_{\mathsf{InqBK}} \psi^{\Box}$, whence the previous lemma gives $\Phi^{\Box\Rightarrow} \vdash_{\mathsf{InqB}^{\Rightarrow}} \psi^{\Box\Rightarrow}$. Finally, Lemma 6.5.13 gives $\Phi \vdash_{\mathsf{InqB}^{\Rightarrow}} \psi$. $\Box$

Thus, the set of rules in Figure 6.4 provides a simple, complete axiomatization of our modal implication operator. Thus, taking this operation as a primitive leads to a suitable alternative formulation of inquisitive Kripke modal logic, one that directly has dependence statements at its core. A further investigation of this modal system and of its extensions to particular classes of Kripke frames is an interesting task for future work.

6.6 Modal translation of inquisitive logic

In this section, we show that there exists a translation of InqBK, and thus also of InqB, into the standard modal logic K. This translation may be compared with the so-called Gödel translation of intuitionistic logic into the modal logic $\mathsf{S4}$. To define this translation, let us start with the following observation: in InqBK, φ entails ψ if and only if $\Box\varphi$ entails $\Box\psi$.

6.6.1. Proposition. *For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{K}}$, $\Phi \models_{\mathsf{InqBK}} \psi \iff \Box\Phi \models_{\mathsf{InqBK}} \Box\psi$*

Proof. The left-to-right direction is the monotonicity of $\Box$. As for the left-to-right direction, suppose $\Phi \not\models \psi$. This means that there exists a Kripke model M and a state s such that $M, s \models \Phi$ but $M, s \not\models \psi$. Now, consider a new Kripke model M' which is just like M, except that it contains a new world w with $\sigma(w) = s$; the valuation at w does not matter. Since we have not altered the valuation or the accessibility for any of the worlds in M, we still have $M', s \models \Phi$ but $M', s \not\models \psi$. Since $s = \sigma(w)$, this implies $M, w \models \Box\Phi$ but $M, w \not\models \Box\psi$, which shows that $\Box\Phi \not\models \Box\psi$. ☐

This means that any entailment $\Phi \models \psi$ in InqBK has a corresponding "boxed" counterpart $\Box\Phi \models \Box\psi$, and vice versa. On the other hand, any modal formula $\Box\varphi$ is truth-conditional, and so, by Proposition 6.3.11, equivalent to a classical modal formula, $(\Box\varphi)^{cl}$. This suggests the possibility of defining a translation $(\cdot)^{tr}$ from InqBK to the modal logic K, as follows.

6.6.2. Definition. [Modal translation of an inquisitive formula]
The translation of a formula $\varphi \in \mathsf{InqBK}$ into the modal logic K is the formula

$$\varphi^{tr} = (\Box\varphi)^{cl}$$

The previous proposition, coupled with the fact that InqBK is conservative over K, implies that this is indeed a translation, in the sense that it preserves entailments.

6.6.3. Theorem (The modal translation is entailment-preserving).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{K}}$, $\Phi \models_{\mathsf{InqBK}} \psi \iff \Phi^{tr} \models_{\mathsf{K}} \psi^{tr}$

Let us focus in particular on the translation of propositional InqB into K. In this case, the translation takes a rather simple form.[20]

6.6.4. Proposition (Modal translation of a propositional formula).
For any $\varphi \in \mathcal{L}^{\mathsf{P}}$, if $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$, then $\varphi^{tr} = \Box\alpha_1 \vee \dots \vee \Box\alpha_n$

[20]It is worth remarking that, for the translation of propositional formulas, it does not matter whether we take K or another modal logic such as $\mathsf{T}, \mathsf{D}, \mathsf{K4}, \mathsf{S4}, \mathsf{S5}$, and many others. Since the translation of a propositional formula is a Boolean combination of boxed propositional formulas, it is easy to see that all that matters is that, given set s of worlds equipped with a propositional valuation, we can always see it as $s = \sigma(w)$ for some w in some Kripke model for our logic; in particular, any logic whose class Kripke frames includes all frames with a total accessibility relation will do. Thanks to Johannes Marti for discussion of this point.

This shows how propositional dependencies, which we have construed as entailments involving questions, may be captured in modal logic. For instance, consider the following entailment, capturing the fact that, given the information that $p \leftrightarrow q$, the question whether p determines the question whether q.

$$p \leftrightarrow q,\ ?p \models ?q$$

This can be translated into the modal logic K as follows:

$$\Box(p \leftrightarrow q),\ \Box p \vee \Box\neg p \models \Box q \vee \Box\neg q$$

However, just like the existence of Gödel's translation does not mean that intuitionistic reasoning is best formalized within $\mathsf{S4}$, so the existence of this translation does not mean that dependencies are best formalized within K. In the following, I discuss some respects in which an inquisitive-style formalization of questions and dependencies is more suitable than one based on plain modal logic.

Insightfulness

In Chapter 1, we saw that once we bring questions into play, dependency can be understood as a facet of entailment. Besides being an important insight in itself, this allows us to use standard ideas and techniques of logic in the analysis of the dependency relation. For instance, we saw that, since entailment can be generally internalized in the language by means of implication, dependencies can be expressed by means of implications between questions. This gives us a well-behaved logical representation of dependencies, and suggests natural rules for reasoning with these formulas. As an example of the explanatory power of the approach, this perspective shows that the well-known Armstrong's axioms for dependency used in database theory are nothing but the familiar rules for implication in disguise (a point first made by Abramsky and Väänänen, 2009).

Economy

Sheer familiarity with modal logic may lead one to think that, since dependencies can be formalized in modal logic, we don't need *more* than that. But in fact, the inquisitive approach is more parsimonious than the modal one. In order to interpret a question such as $?p$, we just need a set of worlds equipped with a propositional valuation. In order to interpret a modal formula $\Box p \vee \Box\neg p$, we also need an accessibility map σ. This map plays no role whatsoever in the analysis of the dependency relation: its only role is to take us from a world w where the formulas are evaluated, to a state $\sigma(w)$ where the action actually takes place. This is only necessary because we insist on evaluating formulas at worlds. The inquisitive approach is more parsimonious: by evaluating formulas directly at states, we do not need to assume an accessibility map: in this way, we can do with simpler models, and the analysis of the dependency becomes more direct and more fundamental.

Uncovering structure

In intuitionistic logic there is a wealth of interesting structure that becomes rather invisible from the $\mathsf{S4}$ translation. Similarly, we have seen that the inquisitive approach leads to the discovery of interesting structural features at the support level: in particular, truth-conditional operations such as classical conjunction, implication, universal quantification, and Kripke modalities all generalize to this setting in a natural way, so that they can also manipulate questions. These generalizations do not only give nice results in practice: they are also natural from an algebraic and from a proof-theoretic perspective. To give just one example, let us focus on the conditional operator of InqB. Consider the following sentences:

(11) a. Alice will come to the party.
b. Bob will come to the party.
c. Will Bob come to the party?
d. If Alice comes to the party, Bob will come to the party.
e. If Alice comes to the party, will Bob come to the party?

In inquisitive logic, these sentences can be formalized in a simple, compositional way. If we translate (11-a) as p and (11-b) as q, then we can translate (11-c) as $?q$, (11-d) as $p \to q$, and (11-e) as $p \to ?q$. Notice that one and the same operation is at play in (11-d) and (11-e): this operation has a uniform semantic clause, a natural algebraic characterization (see Roelofsen, 2013), and can be manipulated in inferences by means of the standard introduction and elimination rules.

In the modal approach, the translation of (11-a) is $\Box p$, and the translation of (11-b) is $\Box q$. The translation of (11-c) is $\Box p \vee \Box\neg p$. The translation of (11-d) is not $\Box p \to \Box q$, but rather $\Box(p \to q)$; similarly, the translation of (11-e) is not $\Box p \to (\Box q \vee \Box\neg q)$, but rather $\Box(p \to q) \vee \Box(p \to \neg q)$.[21]

Although modal logic does have an implication connective, this cannot be used to interpret the conditional construction here. More importantly, it is not clear that there is any structural similarity between (11-d) and (11-e), nor that there is an obvious relation between these two sentences and the simpler sentences (11-a-c). Clearly, an important piece of structure—the existence of a neat conditional operation that applies uniformly to statements and questions—is being missed by modeling things in this way.

Inferences and computational interpretation of proofs

We saw in Chapter 3 that inquisitive logic allows us to manipulate questions in inference by means of simple and familiar logical rules. By means of these rules, we can provide formal proofs of the validity of certain dependencies. We also saw

[21]Notice that this example highlights the fact that the modal translation $(\cdot)^{tr}$, unlike the Gödel translation of intuitionistic logic, is not fully compositional: e.g., the translation of $p \to q$, which is $\Box(p \to q)$ is not computed from the translations p and q, which are $\Box p$ and $\Box q$.

that such proofs have a computational interpretation: a proof of a dependency can always be seen as describing an algorithm to compute that dependency.

It is not clear that the modal approach has an equally attractive framework to offer. First, we would have to reason with a more complex language—including modalities in addition to just connectives (or, in addition to whatever other constants the language includes). Second, it is unclear whether a computational interpretation of proofs is still available in the modal approach.

Independent motivation for questions

The point of the modal approach to dependencies would presumably be to leave questions out of the scope of logic, paying the price by using modalities instead. However, besides dependencies, there are other reasons to be interested in logics equipped with questions. The first has to do with embedded questions. We saw in this section that by having questions in the language, and by providing an appropriate clause for modalities, we can model in a uniform way expressions like *knowing that* p, *knowing whether* p, and *knowing whether* p *or* q. Furthermore, we will see in the next chapter that, by using questions embedded under a new kind of modal operator, we can naturally talk about the issues that agents entertain, expressing facts such as *agent* a *wonders whether* p. Finally, in Chapter 8 we will see that having questions in the language, a simple generalization of the treatment of *public announcements* in dynamic epistemic logic suffices to model the conversational effect of *asking a question* as leading to the raising of an issue. So, questions are not just useful to capture dependencies, but also (at least) to describe agent's attitudes, and for a logical dynamics of information exchange. Since these aspects all work well together, there does not seem to be much of a point in trying to avoid using the logical potential that questions make available.

6.7 Related work

In this section we connect the systems InqBK and $\mathsf{InqB}^{\Rightarrow}$ investigated in this chapter to a number of other recent developments in modal logic. In Section 6.7.1 we consider the relations with the framework of *modal dependence logic*, which is similar to our systems in that formulas are interpreted relative to a state in a Kripke model; we will establish a very precise connection with this framework, which allows us to import an interesting result about frame definability. In Section 6.7.2 we discuss the work of Goranko and Kuusisto, who are currently developing a modal approach of dependence statements closely related to that of Section 6.5. In Section 6.7.3 we discuss the relation with the work of Holliday (2014), who investigates a semantics for modal logic based on an abstract space of partial states, rather than on Kripke models. Finally, in Section 6.7.4 we discuss connections with the non-standard epistemic logics of *knowing whether* and *knowing what* investigated by Wang and Fan (2014) and Fan et al. (2015).

6.7.1 Modal dependence logic

A set of systems that are closely related to the logic InqBK developed in this chapter is the family of *modal dependence logics*. This line of research originated with Väänänen (2008), who defined a modal dependence logic, MDL, as a logic for reasoning about propositional dependencies which hold possibly or necessarily. The language of this logic is obtained by augmenting the language of propositional dependence logic, discussed in Section 5.2, with two modalities, $\Diamond$ and $\Box$. Just as in the system InqBK, formulas are interpreted with respect to a state s drawn from a Kripke model $M = \langle W, \sigma, V\rangle$. However, the interpretation of the modalities is different from the one given in InqBK. First, the accessibility map $\sigma : W \to \wp(W)$ is taken to give rise to a map $\sigma[\cdot] : \wp(W) \to \wp(W)$ on information states and to a relation $R^\sigma \subseteq \wp(W) \times \wp(W)$ on information states, defined as follows:

- $\sigma[s] = \bigcup_{w\in s} \sigma(w)$
- $sR^\sigma t \iff$ for all $w \in s : \sigma(w) \cap t \neq \emptyset$

The semantics of MDL is obtained by augmenting Definition 5.2.1 with the following clauses for modalities:

- $M, s \models \Box\varphi \iff M, \sigma[s] \models \varphi$
- $M, s \models \Diamond\varphi \iff M, t \models \varphi$ for some state t such that $sR^\sigma t$

Like InqBK, also MDL may be seen as obtained from enriching standard modal logic with a non-standard logical constant, namely, the dependence atom $=(p_1, \dots, p_n, q)$. For, it is easy to verify that any *classical* formula in MDL, i.e., any formula φ which does not include dependence atoms, is truth-conditional, and it is assigned the same truth-conditions as in standard modal logic.

An *extended* modal dependence logic (EMDL) was introduced by Ebbing et al. (2013), who considered not only standard dependence atoms $=(p_1, \dots, p_n, q)$, but also *generalized* dependence atoms $=(\alpha_1, \dots, \alpha_n, \beta)$, where $\alpha_1, \dots, \alpha_n, \beta$ are classical modal formulas. These atoms are interpreted as discussed in Section 5.2.2: they are supported in a team s if throughout s, the truth-value of β is completely determined by the truth-values of $\alpha_1, \dots, \alpha_n$. Ebbing et al. (2013) show that generalizing the dependence atoms in this way results in a strictly more expressive system: for instance, they show that the formula $=(\Diamond p)$, equivalent to the question $?\Diamond p$ of InqBK, is not expressible in MDL.

Ebbing et al. (2013) also consider a system, ML[$\mathbin{\rotatebox[origin=c]{180}{$\wedge$}}$], obtained by expanding the classical fragment of MDL with inquisitive disjunction (which they refer to, curiously, as *classical* disjunction). Thus, the language of ML[$\mathbin{\rotatebox[origin=c]{180}{$\wedge$}}$] is as follows:

$$\varphi ::= p \mid \overline{p} \mid \varphi \wedge \varphi \mid \varphi \otimes \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \varphi \mathbin{\rotatebox[origin=c]{180}{$\wedge$}} \varphi$$

In this system, the operators $\otimes, \Diamond$, and $\Box$ are interpreted as in MDL, while $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is interpreted as in InqBK. Just like InqBK, this system can be regarded as extending classical modal logic with a non-standard connective, $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. However, there are significant differences between ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$] and InqBK: besides containing different propositional connectives—which cannot emulate each other in a uniform way when it comes to questions—modalities are treated differently in the two systems; in particular, while modal formulas are always statements in InqBK, the same is not true in ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$], or in the other systems of modal dependence logic.

In spite of these differences, the systems ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$] and InqBK have the same expressive power. This follows from Proposition 6.3.13, stating that any formula in InqBK is equivalent to an inquisitive disjunction of classical modal formulas, and from analogous result for ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$], due to Lohmann and Vollmer (2013).

6.7.1. Proposition (Normal form for ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$]).
For any formula φ in ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$], $\varphi \equiv \alpha_1 \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \alpha_n$, where $\alpha_1, \dots, \alpha_n$ are classical (i.e., $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-free) formulas in ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$].

We can use these normal forms to establish the equi-expressivity result.

6.7.2. Proposition. *The systems ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$] and InqBK are equi-expressive.*

Proof. I only show explicitly how to translate a formula of ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$] into InqBK. The converse direction is established by means of a similar argument.

For any formula φ in ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$], let us denote by φ^{nf} an inquisitive disjunction $\alpha_1 \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \alpha_n$ of classical formulas which is equivalent to φ; the existence of such a formula is ensured by Proposition 6.7.1. If α is a classical formula, we may assume $\alpha^{\mathrm{nf}} = \alpha$. Now, let us first define our translation $\alpha^{\#}$ inductively for the case in which α is a classical formula in ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$].

- $(p)^{\#} = p$
- $(\overline{p})^{\#} = \neg p$
- $(\alpha \wedge \beta)^{\#} = \alpha^{\#} \wedge \beta^{\#}$
- $(\alpha \otimes \beta)^{\#} = \alpha^{\#} \vee \beta^{\#}$
- $(\Diamond\alpha)^{\#} = \Diamond\alpha^{\#}$
- $(\Box\alpha)^{\#} = \Box\alpha^{\#}$

It is easy to verify inductively that, for any classical formula α in ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$], $\alpha \equiv \alpha^{\#}$. Next, consider an arbitrary formula φ in ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$]. We let:

- $\varphi^{\#} := \alpha_1^{\#} \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \alpha_n^{\#}$, where $\varphi^{\mathrm{nf}} = \alpha_1 \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \alpha_n$

Clearly, $\varphi^{\#} \in \mathcal{L}^{\mathsf{K}}$. Since $\varphi \equiv \varphi^{\mathrm{nf}}$ by definition, and for each i we have $\alpha_i^{\#} \equiv \alpha_i$, we have: $\varphi \equiv \alpha_1 \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \alpha_n \equiv \alpha_1^{\#} \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \dots \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \alpha_n^{\#} \equiv \varphi^{\#}$. $\Box$

Moreover, it was shown by Hella et al. (2014) that the systems EMDL and ML[$\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$], are equi-expressive. This implies that InqBK is also equi-expressive with EMDL, and strictly more expressive than MDL.

6.7.3. Corollary. *The systems EMDL and InqBK are equi-expressive.*

However, as for InqB and PD, the fact that the modal systems InqBK, ML[$\mathbb{V}$], and EMDL are equi-expressive should not lead one to regard them as being essentially the same system. The reason is the one we discussed in Section 5.2: since these logics are not closed under uniform substitution, the fact that they can express the same meanings does not mean that they can express the same *operations* on these meanings. For instance, it seems likely that the operator $\Box$ of InqBK cannot be uniformly defined in ML[$\mathbb{V}$] and EMDL. Thus, these systems do not provide a good starting point, e.g., to extend epistemic logic to capture *knowing whether*. Conversely, since the modal operators of EMDL and ML[$\mathbb{V}$] are likely not to be uniformly definable in InqBK, our logic will not be suitable for applications that make crucial use of such operators.

An immediate benefit of the previous equi-expressivity results is that we can transfer to InqBK some interesting results obtained for modal dependence logics. For instance, importing results recently established by Sano and Virtema (2015) for ML[$\mathbb{V}$], we obtain a structural characterization of the classes of Kripke frames which can be defined by means of formulas of InqBK, in the style of the well-known Goldblatt-Thomason theorem for standard modal logic. To state this theorem, let us define the notion of a class of frames being defined by a formula of InqBK.

6.7.4. Definition. [Frame validity, definable frame classes]
Let $F = \langle W, \sigma \rangle$ be a Kripke frame and $\varphi \in \mathcal{L}^{\mathsf{K}}$. We say that φ is valid on F in case for any model M based on F we have $M, W \models \varphi$.[22] We say that a class $\mathbb{F}$ of Kripke frames is defined by $\varphi \in \mathcal{L}^{\mathsf{K}}$ if $\mathbb{F}$ is the set of frames on which φ is valid. We say that $\mathbb{F}$ is definable in InqBK if $\mathbb{F}$ is defined by some $\varphi \in \mathcal{L}^{\mathsf{K}}$.

By combining the results of Sano and Virtema (2015) with Proposition 6.7.2, we obtain the following theorem for InqBK.[23]

6.7.5. Theorem (Characterization of frame definability in InqBK). *A class $\mathbb{F}$ of frames is definable in InqBK iff the following conditions are satisfied:*

- $\mathbb{F}$ *is closed under generated subframes:*
  if $F \in \mathbb{F}$ and F' is a generated sub-frame of F, then $F' \in \mathbb{F}$;
- $\mathbb{F}$ *is closed under bounded morphic images:*
  if $F \in \mathbb{F}$ and $F' = f[F]$ where f is a bounded morphism, then $F' \in \mathbb{F}$;
- $\mathbb{F}$ *reflects ultrafilter extensions:*
  if $ue(F) \in \mathbb{F}$, then $F \in \mathbb{F}$, where $ue(F)$ is the ultrafilter extension of F;

[22] Notice that, by downward closure, this amounts to $M, s \models \varphi$ for any $s \subseteq W$.
[23] For definitions of the (standard) notions involved in the following theorem, we refer the reader to Sano and Virtema (2015), or to a modal logic textbook, e.g., Blackburn et al. (2002).

- $\mathbb{F}$ *reflects finitely generated subframes:* *if all finitely generated sub-frames of F are in $\mathbb{F}$, then $F \in \mathbb{F}$.*

Notice that InqBK can define more frame classes than standard modal logic can. E.g., it is easy to verify that $?p$ defines the class of frames $F = \langle W, \sigma \rangle$ in which W is a singleton, while $p \vee\!\!\!\vee \Diamond\neg p$ defines the class of frames $F = \langle W, \sigma \rangle$ in which $\sigma(w) = W$ for all $w \in W$ (i.e., in which the accessibility relation is total). Both these frame classes are not closed under taking disjoint unions, and thus, by the Goldblatt-Thomason theorem, they are not definable in standard modal logic.

6.7.2 Goranko and Kuusisto's modal approach to propositional dependency

In forthcoming work, Goranko and Kuusisto (henceforth G&K) develop an approach to propositional dependency which is similar in many respects to the account of dependence statements we gave in Section 6.5. Indeed, the motivations for their work are very similar to the ones that led us to investigate the modal implication $\Rightarrow$: for instance, G&K also aim at a system in which we can make good sense of negated dependence statements, such as (9) above.

G&K's system is based on a language built up from atoms, propositional connectives $\neg, \vee$, and a dependence operator D of flexible arity. Formulas are evaluated relative to a propositional information model $M = \langle W, V \rangle$ and a world $w \in W$.[24] The truth-conditions for atoms and propositional connectives are the obvious ones, while the operator D is interpreted by means of the following clause:

$$M, w \models D(\alpha_1, \dots, \alpha_n, \beta) \iff \begin{array}{l} \text{for all } v, v' \in W: \\ \text{if } [M, v \models \alpha_i \iff M, v' \models \alpha_i] \text{ for each } i \\ \text{then } M, v \models \beta \iff M, v' \models \beta \end{array}$$

Now, with any propositional information model $M = \langle W, V \rangle$, we can associate a Kripke model $M^\tau = \langle W, \tau, V \rangle$, where τ is the total accessibility map, $\tau(w) = W$ for all $w \in W$. Then, it is easy to see that we have the following equivalence.

$$M, w \models D(\alpha_1, \dots, \alpha_n, \beta) \iff M^\tau, w \models (?\alpha_1 \wedge \dots \wedge ?\alpha_n) \Rightarrow ?\beta$$

This connection allows us to bring out precisely what the differences are between the approach of G&K and the one of Section 6.5. First of all, G&K interpret dependency by means of the universal accessibility map, while we allow it to be interpreted by means of an arbitrary accessibility map, so that only a set of successors of a given world are relevant to the truth of a dependence statement at that world. This difference may well be important for some applications; e.g.,

[24] As in propositional inquisitive and dependence logic, we may if we wish identify a possible world with the corresponding propositional valuation; this is immaterial to the resulting logic.

having non-total accessibility relations may be necessary to capture epistemic dependencies in a multi-agent setting. However, I do not regard this as an essential difference, since each of the two approaches could easily be made to work with either a restricted or an unrestricted interpretation of the dependency operator.

The central difference between the two systems is that in G&K's system, dependency is construed as a relation between statements, while in our system, it is construed as a relation between questions. The advantage of G&K's approach is that the semantics of the system can be kept completely truth-conditional. No detour at the level of information states is necessary to compute the semantics of dependence statements. In $\mathsf{InqB}^{\Rightarrow}$, such a detour is necessary; for, although a dependence statement $\varphi \Rightarrow \psi$ is fully truth-conditional, its truth-conditions depend on the semantics of φ and ψ: if φ and ψ are questions, their semantics is only fully captured at the support level.

However, $\mathsf{InqB}^{\Rightarrow}$ also has some important advantages, which are similar to the ones we discussed in Section 5.2 while comparing InqB to propositional dependence logic. First, the account of dependence statements provided by $\mathsf{InqB}^{\Rightarrow}$ is more general, allowing us to capture dependence statements such as (12) and (13). The dependencies which are at play in these examples are not dependencies between the truth-values of some statements, and thus, they are out of the scope of G&K's theory. In $\mathsf{InqB}^{\Rightarrow}$, these sentences can be formalized, respectively, as $(mo \mathbin{\vee\!\!\!\!\vee} tu \mathbin{\vee\!\!\!\!\vee} we \mathbin{\vee\!\!\!\!\vee} th \mathbin{\vee\!\!\!\!\vee} fr) \Rightarrow (a \mathbin{\vee\!\!\!\!\vee} b)$ and $?g \Rightarrow (a \rightarrow ?d)$.

(12) What weekday it is determines whether Alice or Bob is in the office.

(13) Whether Alice is in a good mood determines whether she will dance if Bob asks her.

Second, $\mathsf{InqB}^{\Rightarrow}$ allows for a better proof theory. If we compare G&K's analysis of a dependence statement with our own, we can see that G&K's operator D bears the burden of performing many operations at once: inquisitive disjunction, negation, conjunction, and $\Rightarrow$, which in turn compounds a box modality and an implication, as the equivalence $(\varphi \Rightarrow \psi) \equiv \Box(\varphi \rightarrow \psi)$ shows. As a consequence, this operator is not very well-behaved from a proof-theoretic point of view. In particular, G&K's axiomatization of it contains the following axiom:

$$D(\alpha_1, \ldots, \alpha_n, \beta) \longleftrightarrow \bigvee_{\chi \in \mathrm{Bool}(\alpha_1, \ldots, \alpha_n)} (\chi \leftrightarrow \beta) \wedge D(\chi \leftrightarrow \beta)$$

where $\mathrm{Bool}(\alpha_1, \ldots, \alpha_n)$ is the set of Boolean combinations of $\alpha_1, \ldots, \alpha_n$. Of course, such an axiom is hardly the sort of characterization one would want for the proof-theoretic behavior of a fundamental logical operator. By contrast, we saw in Section 6.5 that in $\mathsf{InqB}^{\Rightarrow}$, dependence statements are analyzed in terms of much more basic building blocks, each of which, including the dependence modal operator $\Rightarrow$, obeys very simple logical rules.

In spite of these differences in implementation, however, it is remarkable that our approach and the one of G&K converge on a very fundamental conceptual point: *dependence statements are modal statements.* The fact that, in modal logic, each world is associated with a corresponding information state—a set of worlds—allows us to declare a dependence statement true or false at a world w depending on whether the claimed dependency holds in the associated set of worlds $\sigma(w)$. On the one hand, this approach allows us to capture dependencies, which only arise when a plurality of alternative states of affairs is brought into play. On the other hand, it allows us to assign dependence statements with meaningful truth-conditions with respect to worlds, and to manipulate such statements in a standard way by means of the classical connectives, delivering the expected reading for negated dependence statements, disjoined dependence statements and, in $\mathsf{InqB}^{\Rightarrow}$, polar questions about dependence statements.

6.7.3 Functional possibility semantics for modal logic

An interesting connection exists between the modal logic InqBK developed in this chapter and the *functional possibility semantics for modal logic* developed by Holliday (2014) on the basis of a similar proposal made by Humberstone (1981).

Like us, Holliday investigates a semantics for modal logic in which formulas are evaluated not with respect to possible worlds, but rather with respect to partial states, which he calls *possibilities*, and which may be regarded as information states.[25] Thus, part of Holliday's proposal is similar to our development in Section 6.1 of a support semantics for standard modal logic.

However, Holliday's goal is quite different from the one pursued here. For us, re-implementing standard modal logic in terms of support relative to states was a first step in the development of a system of modal logic enriched with questions. By contrast, Holliday's focus is on the standard language of modal logic, and his aim is to construct a theory of modal logic based on partial-state structures, where states are not construed as sets of worlds, but taken as primitive entities drawn from a partially ordered set $\langle \mathcal{S}, \geq \rangle$, where the ordering $\geq$ is the counterpart of our enhancement relation $\subseteq$. In Holliday's system, a modal operator $\Box$ is interpreted by means of a function $f : \mathcal{S} \to \mathcal{S}$ on the space of information states (whence the term *functional* possibility semantics), via the following clause:

$$M, s \models \Box\varphi \iff M, f(s) \models \varphi$$

Thus, Holliday starts with what we dubbed the *abstract* view of information states in Section 1.6.2, and then investigates what conditions the space $\langle \mathcal{S}, \geq \rangle$ and the function f must satisfy in order for the semantics to give rise to classical modal

[25]Incidentally, notice that modeling states as sets of worlds, too, does not commit one to regard states as representing bodies of information. While the development of our theory was driven by an interpretation of states as bodies of information, one may abstract away from it.

logic, and what the theory of modal logic looks like when such spaces are taken as our fundamental semantic structures.[26]

Notice how each of the two lines of work is relevant to the other. From the inquisitive perspective, it seems interesting to investigate what features of the space of information states are actually needed in order to have a semantics for InqBK, or other inquisitive logics. Allowing for a broader class of models would make it easier to provide counter-models for invalid formulas, and it may also allow for simpler and more finitary canonical model constructions, like those given by Holliday (2014) for standard modal logic.

From the point of view of possibility semantics, on the other hand, it is interesting to ask whether, besides providing an alternative foundation for standard modal logic, the framework allows us to capture things that are out of reach for truth-conditional semantics. We have seen in this thesis that questions are a prime example of sentences which have a natural interpretation with respect to partial states, but not to total worlds. Thus, it would be a natural move to exploit the possibility semantics framework to bring questions into play. Ideally, the two enterprises should then converge on the same result, namely, a possibility semantics for InqBK.

This is an interesting direction to be pursued in future work. We can, however, already point out a way in which considering questions may lead to reconsider the interpretation of modal operators in possibility semantics. To illustrate this point, let us adopt an epistemic interpretation of the modality $\Box$. In InqBK, $\Box$ is interpreted by means of a map σ assigning to each world w an information state $\sigma(w)$—the epistemic state of the agent at w. A formula $\Box\varphi$ is true at a world w if the agent's state $\sigma(w)$ settles φ. As any statement, $\Box\varphi$ is then settled in a state s in case it is true everywhere in the state. Thus, we have:

$$M, s \models \Box\varphi \iff M, \sigma(w) \models \varphi \text{ for all } w \in s$$

The natural way to lift $\sigma : W \to \wp(W)$ to a function $f^\sigma : \wp(W) \to \wp(W)$ of the kind needed for the interpretation of $\Box$ in possibility semantics is to define:

$$f^\sigma(s) = \sigma[s] = \bigcup_{w \in s} \sigma(w)$$

Notice that, when f^σ is applied to a singleton state $\{w\}$, we have $f^\sigma(\{w\}) = \sigma(w)$. Thus, $f^\sigma(\{w\})$ describes the information state of the agent at w. However, when f^σ is applied to a non-singleton state s, two different sources of partiality are conflated in the state $f^\sigma(s)$: on the one hand, the state of an agent at a world is typically a state of partial information; on the other hand, the state s only partly

[26] A closely related proposal was made by Punčochář (2014), whose system coincides with functional possibility semantics when the space $\langle \mathcal{S}, \geq, f \rangle$ forms a Boolean algebra with operators. Thus, the peculiarity of Punčochář's approach is that it uses the standard *algebraic* structures for modal logic to give a *relational* interpretation of the language.

determines what this state is. In general, by looking only at the state $f^\sigma(s)$, we will not be able to tell apart these two sources of partiality.

As long as we are only interested in interpreting the modality $\Box$ over statements, as in the system of Holliday (2014), this is not a problem. Since a statement α is truth-conditional, we have:

$$\begin{aligned} M, s \models \Box\alpha &\iff M, \sigma(w) \models \alpha \text{ for all } w \in s \\ &\iff M, v \models \alpha \text{ for all } w \in s \text{ and all } v \in \sigma(w) \\ &\iff M, v \models \alpha \text{ for all } v \in f^\sigma(s) \\ &\iff M, f^\sigma(s) \models \alpha \end{aligned}$$

However, the moment we are also interested in capturing the effect of $\Box$ as applied to a question, telling apart the two sources of partiality that are conflated in $f(s)$ becomes crucial, at least if we want $\Box\mu$ to express that the agent knows μ. To see this, consider two Kripke models $M_1 = \langle W, \sigma_1, V\rangle$ and $M_2 = \langle W, \sigma_2, V\rangle$, where:

- $W = \{w_p, w_{\overline{p}}\}$
- $V(w_p, p) = 1,\ \ V(w_{\overline{p}}, p) = 0$
- $\sigma_1(w_p) = \{w_p\},\ \ \sigma_1(w_{\overline{p}}) = \{w_{\overline{p}}\}$
- $\sigma_2(w_p) = \sigma_2(w_{\overline{p}}) = W$

M_1 is a model in which at each world, the agent knows whether p is the case, while M_2 is a model in which at each world, the agent is uncertain about whether p. It follows that in the pair M_1, W it is settled that the agent knows whether p, while in M_2, W, the opposite is settled.

The clause for $\Box$ in InqBK, which looks at the state of the agent at each world, correctly predicts this, yielding $M_1, W \models \Box?p$ and $M_2, W \not\models \Box?p$. However, this difference between M_1 and M_2 is impossible to capture if our clause for $\Box$ only looks at one global state, $\sigma[W]$. For, notice that $\sigma_1[W] = \sigma_2[W] = W$: the state associated with W is the same in the two models, obliterating the difference between a situation in which it is known that the agent does not know whether p, and a situation in which it is known the agent knows whether p, but in which it is not known whether p is actually the case.[27]

Now, can we adapt the clause for $\Box$ in possibility semantics so that it delivers the expected support-conditions also when modalities apply to questions? In InqBK, the clause for $\Box$ checks the state $\sigma(w)$ at each world $w \in s$. However, in possibility semantics states are not sets of worlds; moreover, an information state may not have any complete extension which we may identify with a world. In

[27]Notice that the problem does not just arise at the level of support-conditions, but also at the level of truth-conditions. In a setting in which we have two agents a and b, we can easily turn the above example into one in which we cannot assign the right truth-conditions to the formula $\Box_a\Box_b?p$, since the clause for $\Box$ would conflate the uncertainty of the two agents a and b, and make it impossible to distinguish a world in which a knows that b doesn't know whether p from a world in which a knows that b knows whether p, but a herself doesn't know whether p.

spite of this, there exists a way of rephrasing our clause which carries over to the abstract setting. For, the following holds for any Kripke model and any state:

$$M, s \models \Box\varphi \iff \begin{array}{l}\text{for all consistent states } t \subseteq s \\ \text{there is a consistent state } u \subseteq t \text{ s.t. } M, \sigma[u] \models \varphi\end{array}$$

Notice that this support clause for $\Box$, while being equivalent to our original one on Kripke models, does not mention worlds, but only the enhancement ordering, the notion of a consistent state, and the map $\sigma[\cdot]$ from states to states. As all of these notions have a counterpart in any possibility model, we can use this as our clause for $\Box$ in the abstract setting.[28]

$$M, s \models \Box\varphi \iff \begin{array}{l}\text{for all consistent states } t \geq s \\ \text{there is a consistent state } u \geq t \text{ s.t. } M, f(u) \models \varphi\end{array}$$

If we interpret $\Box$ by means of this clause, we obtain as a fact that, for any classical modal formula α, we have $M, s \models \Box\alpha \iff M, f(s) \models \alpha$. Thus, the support conditions for all classical modal formulas are the same as in Holliday's system. At the same time, as we saw, this clause coincides with the one of InqBK in the context of a Kripke model, and it allows us to assign the intended semantics to formulas in which an epistemic modality is applied to a question.

6.7.4 Non-standard epistemic logics

A recent trend in the field of modal logic is the investigation of so-called *non-standard* epistemic logics. These are logics which, besides the standard *knowing that* modality of epistemic logic, contain other knowledge operators. We will focus on the two operators of this kind which have been investigated in more detail: the *knowing whether* operator of Fan et al. (2015) and the *conditionally knowing what* operator of Wang and Fan (2013, 2014).[29,30] The former operator, denoted

[28]It is interesting to note that the $\forall\exists$ pattern, which in this clause allows us to factor out the uncertainty which is present in s about the agent's state, also plays a role in Holliday's clauses for the (defined) operators $\vee$ and $\Diamond$.

[29]In fact, both these operators are not of recent introduction. The first, under the name of *non-contingency* modality, was first introduced by Montgomery and Routley (1966), and is the subject of many papers (among others, Cresswell, 1988; Humberstone, 1995, 2002; Zolin, 1999; Pizzi, 2007); more recently, it has been considered from an epistemic perspective by Demri (1997), van der Hoek and Lomuscio (2004)—who actually consider its negation—and, finally, Fan et al. (2015), who also provide a survey of this line of work. The *knowing what* operator, in its non-conditional version, has been proposed by Plaza (1989); its conditional version and its axiomatization are due to Wang and Fan (2013, 2014). While the discussion here concerns these operators in general, for the sake of concreteness we will take as our starting point the specific logics of Fan et al. (2015) and Wang and Fan (2013, 2014).

[30]For simplicity, I will consider the single agent case, but the present discussion would be completely analogous in the multi-agent case.

Δ, is interpreted on a Kripke model as follows:

$$M, w \models \Delta\alpha \iff \text{for all } v, v' \in \sigma(w) : M, v \models \alpha \iff M, v' \models \alpha$$

If $\sigma(w)$ is regarded as the epistemic state of an agent, the clause says $\Delta\alpha$ is true at w iff the agent knows whether α, i.e., knows what the truth-value of α is.

The *conditionally knowing what* operator, Kv, combines with a formula α and a constant symbol c to form a formula $Kv(\alpha, c)$, read as: "conditionally on the information that α, a knows what the value of c is." This operator is interpreted on (S5) Kripke models equipped with a domain D of individuals and with an interpretation map I which assigns to any world w and constant c an object $I_w(c) \in D$. The truth-conditions for $Kv(\alpha, c)$ are as follows:

$$M, w \models Kv(\alpha, c) \iff \text{for all } v, v' \in \sigma(w) \cap |\alpha|_M : I_v(c) = I_{v'}(c)$$

Under an epistemic interpretation of the map σ, the clause says that $Kv(\alpha, c)$ is true if combining the agent's knowledge with the information that α is true leads to a state which settles what the value of c is; in short, $Kv(\alpha, c)$ is true in case the agent knows what value c has if α is true.

The intended applications for this logic are in the domain of formal verification: for instance, in a security setting, one may need to verify that a certain agent knows (or doesn't know) what a certain password is.

Now, the view that underlies the development of these logics, reflected in the name *non-standard* epistemic logics, is that *knowing whether*, *knowing what*, etc., are forms of knowledge which differ from the *knowledge that* of standard epistemic logic. However, our work in this chapter suggests a different diagnosis of the situation: it is the same, simple notion of knowledge, captured by the modality $\Box$, which may be taken to be at play in all these cases. This accounts for the fact that all of these cases involve the verb *to know*, which suggests that they all share a common core.[31] The difference between the various cases arises from the specific sort of complement to which this operator is applied. The standard epistemic logic account of *knowing that* results from the combination of $\Box$ with a statement α, as we saw in Proposition 6.1.7. So-called non-standard modal operators arise from the combination of $\Box$ with different classes of questions. We will demonstrate this for the particular case of the two logics considered above.

First consider the *knowing whether* operator, Δ. Given a standard modal formula α, we obtain a formula equivalent to $\Delta\alpha$ if we combine the standard modality $\Box$ (*a knows*) with the polar question $?\alpha$ (*whether* α). Indeed, we know from Proposition 6.2.2 that $\Box?\alpha$ is a truth-conditional formula, and we have seen above that this formula has the same truth-conditions as $\Delta\alpha$:

$$M, w \models \Delta\alpha \iff M, w \models \Box?\alpha$$

[31] Incidentally, this uniformity is not specific to English, but is a solid fact that holds cross-linguistically, which suggests that *know* is not merely ambiguous between different meanings.

Besides allowing us to isolate the core contribution of knowledge across the various kinds of epistemic statements, this decomposition allows us to analyze the logic of *knowing whether* as resulting from the interaction of a relatively standard modal operator, $\Box$, with a relatively standard disjunction operator, $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$.

Furthermore, this strategy has the merit of capturing in a uniform way not only the occurrence of *knowing whether* in (14-a), but also those in (14-b,c). These sentences cannot be modeled by the operator Δ but as we saw, they can be captured in InqBK by means of the formulas $\Box(p \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} q)$ and $\Box(r \to ?p)$.

(14) a. Alice knows whether Bob will move to Paris.
b. Alice knows whether Bob will move to Paris or to Lyon.
c. Alice knows whether Bob will move to Paris if he is offered the job.

Now consider the operator Kv. We saw that a formula $Kv(\alpha, c)$ is true at a world w in case the agent knows the answer to the following conditional question:

(15) If α is true, what is the value of c?

Since such a question concerns the value of an individual constant c, it cannot be modeled in the system InqBK, which contains no reference to individuals. However, the treatment of Kripke modalities developed in this chapter can be transferred straightforwardly to the first-order setting developed in Chapter 4. Let us sketch what the resulting logic InqBQK looks like. The language is the one of InqBQ, extended with the operator $\Box$.[32] Our models will be tuples $M = \langle W, D, I, \sim, \sigma\rangle$, where $\langle W, D, I, \sim\rangle$ is a first-order information model in the sense of Definition 4.1.3, and $\sigma : W \to \wp(W)$ is an accessibility map. The modality $\Box$ is interpreted as in InqBK, while the other support clauses are the same as in InqBQ.

A serious investigation of this system lies beyond the scope of this thesis. However, let us illustrate how *knowing the value of* c *conditionally on* α may be captured in this system. First, recall that, if c is a non-rigid individual constant, the formula $\overline{\exists}x(c = x)$ captures the question what the value of c is: the sentence is settled in s if s determines the value of c, i.e., if s implies for some particular individual $d \in D$ that c is identical to d.

$$M, s \models \overline{\exists}x(c = x) \iff M, s \models_{[x\mapsto d]} c = x \text{ for some } d \in D$$

Next, recall that our conditional operator allows us to form conditional questions. In particular, if α is truth-conditional, then the formula $\alpha \to \overline{\exists}(c = x)$ is settled in a state s if extending s with the information that α is true results in a state which determines the value of c.

$$M, s \models \alpha \to \overline{\exists}x(c = x) \iff M, s \cap |\alpha|_M \models \overline{\exists}x(c = x)$$

[32]Or, with operators $\Box_a$, one for each agent a. We stick to a single modality for simplicity.

Thus, the conditional $\alpha \to \overline{\exists}(c = x)$ captures the question (15).

Finally, by embedding this question under the modality $\Box$, we obtain a truth-conditional formula with the following truth-conditions.

$$M, w \models \Box(\alpha \to \overline{\exists}x(c = x)) \iff M, \sigma(w) \cap |\alpha|_M \models \overline{\exists}x(c = x)$$

That is, $\Box(\alpha \to \overline{\exists}x(c = x))$ is true at a world w in case the agent's information, $\sigma(w)$, combined with the information that α is true settles the question what the value of c is; in short, in case the agent knows the value of c conditionally on α.

In the particular case in which M is an id-model, we obtain a clause which is literally the same as the one given by Wang and Fan for $Kv(\alpha, c)$.[33]

$$M, w \models \Box(\alpha \to \overline{\exists}x(c = x)) \iff \text{for all } v, v' \in \sigma(w) \cap |\alpha|_M : I_v(c) = I_{v'}(c)$$

This shows that we can express *knowing the value of* c *conditionally on* α by dividing the labor between the modality $\Box$, capturing knowledge, and the formula $\alpha \to \overline{\exists}x(c = x)$, capturing the conditional question in (15).

Notice that, like in the case of statements, conditional knowledge of a question amounts to knowledge of a conditional question. However, this only becomes visible once our implication connective is lifted to operate at the support level, so that it can be applied not only to statements, but also to questions.

As in the case of *knowing whether*, one of the advantages of this decomposition is that the logical properties of the *conditionally knowing what* configuration can be seen as arising from those of a set of fundamental operators, each of which has rather standard logical features: the Kripke modality $\Box$, the implication $\to$, the inquisitive existential $\overline{\exists}$, and the identity predicate $=$.

The other advantage is that this account generalizes straightforwardly to other cases of *conditionally knowing what*. First, we can generalize the kind of condition to which the relevant knowledge is relativized: we can capture not only the fact that the agent knows the value of c given that a certain statement is true, but also that she knows the value of c *given an answer to another question*. E.g., we may capture that the agent knows the value of c conditionally on the value of another constant c'. For this, it suffices to replace the statement α in the above formula by the question $\overline{\exists}x(c' = x)$. The resulting statement,

$$\Box(\overline{\exists}x(c' = x) \to \overline{\exists}x(c = x))$$

is true at a world w iff, whenever the information state $\sigma(w)$ is extended with information settling the value of c', the resulting state ends up settling the value

[33]Notice, however, that one may have reasons for considering models which are not id-models. E.g., we may want to capture a situation in which an agent knows the value of 'Hesperus' (e.g., she can actually point at the planet) and she similarly knows the value of 'Phosphorus', yet she does not know that Hesperus is Phosphorus. Such a situation can never arise in an id-model, but it can be modeled naturally in a model in which the extension of identity is uncertain.

of c. This may be phrased equivalently by saying that, in the state $\sigma(w)$ of the agent, the value of c is functionally determined by the value of c', in the following sense:[34]

$$M, w \models \Box(\overline{\exists}x(c'=x) \to \overline{\exists}x(c=x)) \iff \text{there is } f : D \to D \text{ s.t. for all } v \in \sigma(w)\ I_v(c) = f(I_v(c'))$$

Of course, if an agent knows the value of c conditionally on c', then if the agent knows the value of c', she also knows the value of c. Our formulation shows that this inference is nothing but an instance of the standard distributivity of $\Box$.

$$\Box(\overline{\exists}x(c'=x) \to \overline{\exists}x(c=x)) \models \Box\overline{\exists}x(c'=x) \to \Box\overline{\exists}x(c=x)$$

This exemplifies the point made above, that we can trace the logical properties of the *knowing what* configuration to familiar properties of more primitive operators, in this case, modality and implication.

Besides extending the range of possible conditions, in InqBQK one may also consider a broader range of *knowing what* patterns. E.g., we can capture not only a sentence such as (16-a), but also sentences such as (16-b,c), which may be translated, respectively, as $\Box\overline{\exists}xPx$ and $\Box\forall x?Px$.

(16) a. Alice knows what the value of c is.
b. Alice knows what an instance of property P is.
c. Alice knows what the extension of property P is.

It is important to note, however, that Wang and Fan (2014) have a good reason for looking at a language without quantifiers, and with limited expressive power: they want to ensure that the resulting logic is *decidable*—a property which is crucial for some of the intended applications. Since the system InqBQK includes classical first-order logic as a fragment, it certainly does not share this feature. This, then, raises an important research question: can we identify a decidable fragment of InqBQK which is still expressive enough for the intended purposes? Alternatively, can we devise another system based on the inquisitive architecture which can express the desired *wh*-questions, while still giving rise to a decidable logic? If so, such a system would allow us to combine the benefits of decidability with the various advantages that we described for a decompositional approach.

6.8 Further work: generalizing the modality $\Diamond$

Throughout this chapter, we have focused on the universal modality, $\Box$. In modal logic, an equally important role is played by the existential modality, $\Diamond$. We have

[34]This formulation is relative to id-models. If our model is not an id-model, the condition $I_v(c) = f(I_v(c'))$ should be weakened to $I_v(c) \sim_v f(I_v(c'))$.

taken the latter to be defined in terms of the former, by letting $\Diamond\varphi := \neg\Box\neg\varphi$. This yields an existential modality that behaves in the standard way on classical formulas. However, this operator is not very interesting when it applies to questions. For, the formula $\Diamond\varphi$ is always truth-conditional, and its truth-conditions depend only on the truth-conditions of φ. Indeed, it is easy to see that the persistency of the semantics gives the following result:

$$M, w \models \Diamond\varphi \iff \sigma(w) \cap |\varphi|_M \neq \emptyset$$

Thus, in the scope of $\Diamond$, a formula can always be replaced by its classical variant, which shares the same truth-conditions: $\Diamond\varphi \equiv \Diamond\varphi^{cl}$. If μ is a question, all $\Diamond\mu$ expresses is that the relevant information state is compatible with the question's presupposition. For instance, $\Diamond ?p$ would simply be equivalent to $\Diamond(p \vee \neg p) \equiv \Diamond\top$.

However, there is a more interesting way to generalize the existential modality to the inquisitive setting. To get the idea, consider the standard truth-conditional clause for $\Diamond\alpha$, where α is a classical modal formula.

$$M, w \models \Diamond\alpha \iff \sigma(w) \cap |\alpha|_M \neq \emptyset$$

Now, for a classical formula α, we know that the truth-set $|\alpha|_M$ is the unique alternative for α in M.[35] Thus, the clause requires the information state $\sigma(w)$ to be consistent with the unique alternative for α. Now, questions in InqBK do not have a unique alternative, but rather, they have multiple alternatives. We could generalize the clause for $\Diamond$ to questions by requiring $\sigma(w)$ to be consistent with each of these alternatives. That is, we may introduce a modality $\Diamond^{\forall}$, to be interpreted by means of the following clause.

6.8.1. Definition. [Support conditions for $\Diamond^{\forall}$]
$M, s \models \Diamond^{\forall}\varphi \iff$ for all $w \in s : \sigma(w) \cap a \neq \emptyset$ for all $a \in \textsc{Alt}_M(\varphi)$

Clearly, this clause makes an existential modal formula $\Diamond^{\forall}\varphi$ truth-conditional, with the following truth-conditions.

6.8.2. Definition. [Truth-conditions for $\Diamond^{\forall}$]
$M, w \models \Diamond^{\forall}\varphi \iff \sigma(w) \cap a \neq \emptyset$ for all $a \in \textsc{Alt}_M(\varphi)$

If α is a classical formula, we have $\textsc{Alt}_M(\alpha) = \{|\alpha|_M\}$, and we obtain the usual truth-conditions for the formula $\Diamond^{\forall}\alpha$. On the other hand, if μ is a question, then in order for $\Diamond^{\forall}\mu$ to be true at a world w, the information state $\sigma(w)$ must be consistent with *each* alternative for μ. For instance, consider the formula $\Diamond^{\forall}?p$. Suppose the model M contains both p-worlds and $\neg p$-worlds, so that $\textsc{Alt}_M(?p) = \{|p|_M, |\neg p|_M\}$. Then, we have:

$$M, w \models \Diamond^{\forall}?p \iff \sigma(w) \cap |p|_M \neq \emptyset \text{ and } \sigma(w) \cap |\neg p|_M \neq \emptyset$$

[35]Recall that the alternatives for a formula φ in a model M are defined as the maximal states in M which support φ. The set of alternatives for φ in M is denoted $\textsc{Alt}_M(\varphi)$.

In particular, $\Diamond^{\forall}?p$ is not equivalent with $\Diamond^{\forall}(p \vee \neg p)$: the latter is true in case $\sigma(w)$ is non-empty, while the former is true in case $\sigma(w)$ is compatible both with p and with $\neg p$. Under an epistemic interpretation, $\Diamond^{\forall}?p$ captures the fact that the agent is ignorant as to whether p.[36]

Besides having this ignorance-related reading, the modality $\Diamond^{\forall}$ is of interest from a linguistic perspective. For, it is easy to see that, as long as p and q are logically independent in the model M, we have the following equivalence:

$$M, w \models \Diamond^{\forall}(p \mathbin{\mathbb{V}} q) \iff M, w \models \Diamond^{\forall}p \wedge \Diamond^{\forall}q$$

In Section (1), we discussed the possibility of taking natural language *or* to contribute inquisitive disjunction in all contexts, and to view classical disjunction $\vee$ as resulting from the combination of $\mathbb{V}$ with a declarative operator '!'. If we adopt this perspective, and if we interpret a modal *may* as by means of the operator $\Diamond^{\forall}$, then we predict that (17-a) entails (17-b) and (17-c).

(17) a. You may take an apple or a pear.
b. You may take an apple.
c. You may take a pear.

Inferences like the ones from (17-a) to (17-b,c), known as *free-choice inferences* for disjunction, are one of the main outstanding puzzles in the theory of modals (see Kamp, 1973; Zimmermann, 2000; Aloni, 2007, among many others). The issue is not just that standard accounts of modals fail to predict these inferences. Rather, it is that, if disjunction is treated classically and modalities operate compositionally, these inferences *cannot* be predicted, essentially because the two sets of worlds $|p|_M$ and $|q|_M$ that the modal claim is about are not recoverable from the meaning of the classical disjunction $p \vee q$ which the modal takes as input.

By interpreting disjunction as $\mathbb{V}$, this problem is resolved: from the meaning of an inquisitive disjunction $p \mathbin{\mathbb{V}} q$, the two sets $|p|_M$ and $|q|_M$ are recoverable as two separate alternatives, and the modal operator $\Diamond^{\forall}$ is thus able to predicate consistency with each of them. In fact, this is essentially the account of free choice inferences proposed, using different formal tools, by Aloni (2007). The advantage of the present formalization is that this account may be obtained in the context of a more structured logical theory of connectives and modalities. We leave it as a task for future work to investigate the logic of the modality $\Diamond^{\forall}$, and the relationship between this modality and natural language modals.

[36]Indeed, formulas of the form $\Diamond^{\forall}?\alpha$ have the same truth-conditions as the modal formulas $I\alpha$ of the logic of ignorance of van der Hoek and Lomuscio (2004).

Chapter 7

Inquisitive modalities

In the previous chapter, we have seen how to extend propositional inquisitive logic to incorporate Kripke modalities, or, to put it the other way around, how to extend standard modal logic with questions, making Kripke modalities capable of embedding in a uniform way both statements like p and questions like $?p$. However, the inquisitive perspective also suggest a different and more novel way to interpret modal operators. In Kripke models, each point w is associated with an information state, the set $\sigma(w)$ of its successors. In this chapter, we will consider a new kind of models in which each point w is associated with an *inquisitive state* $\Sigma(w)$, encoding both information and issues. On these models we may then interpret an inquisitive modality $\boxplus$ which allows us to express properties of the inquisitive state $\Sigma(w)$. The resulting system presents an important novelty with respect to the other inquisitive systems considered so far: by embedding questions under the modality $\boxplus$, we can express new truth-conditional meanings which would not be expressible if we did not have questions in the language; thus, the presence of questions now also results in a richer truth-conditional fragment.

This new, inquisitive modal logic has a clear relevance in the epistemic setting: we will discuss a system of *inquisitive epistemic logic* (IEL) in which agents are not just represented as having certain information, but also as entertaining certain issues. Our logic provides the tools to express facts about the issues that agents entertain, and to reason about them. Moreover, it allows us to generalize the standard common knowledge construction to a *public state* construction, which describes both the group's common knowledge and the group's *common issues*.

The chapter is structured as follows: in Section 7.1, we introduce the framework of inquisitive modal logic and discuss its fundamental features; in Section 7.2 we discuss a natural epistemic interpretation of this framework; in Section 7.3, we investigate and axiomatize the general modal logic arising from the framework: we will find out that, in spite of the departure from Kripke models, the modality $\boxplus$ is proof-theoretically standard, being completely characterized by distributivity and necessitation; in Section 7.4 we consider four natural constraints on inquisitive modal models suggested by the epistemic interpretation, and we axiomatize the various logics which arise from imposing one or more of them, again finding striking parallels with standard modal logic; finally, in Section 7.5 we discuss the relevance of this enriched view of modalities in various fields.

This chapter is partly based on two papers on *inquisitive epistemic logic*, namely, Ciardelli (2014a) and Ciardelli and Roelofsen (2015b): however, the ideas presented in these papers are distilled here to obtain a general theory of inquisitive modal logic, and further extended in several ways.

7.1 Inquisitive modal logic

Let us start out with an observation about standard modal logic. Our presentation in terms of state maps brings out the fact that a Kripke model may be seen as a structure in which any world is associated with an information state $\sigma(w)$. Classically, the proposition expressed by a formula α is itself an information state, namely, the set $|\alpha|_M$ of worlds at which the formula is true. Thus, a Kripke modality expresses a relation between two information states, or *classical propositions*: the state $\sigma(w)$ associated with the evaluation world, and the proposition expressed by its argument. In particular, $\Box$ expresses the fact that the former is a subset of the latter—i.e., that the former *entails* the latter–while $\Diamond$ expresses the fact that the two have a non-empty intersection—i.e., that they are *consistent*.

$$\begin{array}{lcl} M, w \models \Box\alpha & \iff & \sigma(w) \subseteq |\alpha|_M \\ M, w \models \Diamond\alpha & \iff & \sigma(w) \cap |\alpha|_M \neq \emptyset \end{array}$$

In inquisitive semantics, the proposition expressed by a formula φ in a model M is not a simple set of worlds; rather, it is a set of information states, the set $[\varphi]_M$ of states at which the formula is supported. More specifically, due to the persistency of support and to the empty state property, it is an *inquisitive proposition*, in the sense of Definition 1.2.2, which we repeat below.

7.1.1. Definition. [Inquisitive proposition]
An *inquisitive proposition* over a universe W of possible worlds is a non-empty set P of subsets of W, with the property of being *downward closed*, that is: if $s \in P$ and $t \subseteq s$, then $t \in P$. The set of inquisitive propositions over W is denoted Π_W.

With an inquisitive proposition P we can always associate an information state, namely, the union $\bigcup P$ of all the elements of P. We refer to this state as the *informative content* of P, and denote it by $\mathsf{info}(P)$.

7.1.2. Definition. [Informative content]
The *informative content* of an inquisitive proposition P is the state $\mathsf{info}(P) := \bigcup P$.

However, there is more to an inquisitive proposition than its informative content. Namely, we can think of an inquisitive proposition as also describing an *issue* over this informative content. The way in which P describes this issue is by laying out its resolution conditions: the issue is resolved in a state s if and only if $s \in P$.

Having recalled the notion of an inquisitive proposition, let us now turn back to modal logic. In the previous chapter, we have described how to extend Kripke modal logic to the inquisitive setting. In the resulting system InqBK, modal operators no longer express a relation between two semantic objects of the same kind, as in classical modal logic; rather, they express a relation between two semantic objects of different kinds, namely, the information state $\sigma(w)$ associated with the evaluation world, and the inquisitive proposition $[\varphi]_M$ expressed by the argument. In particular, $\Box\varphi$ expresses the fact that $\sigma(w)$ is an element of $[\varphi]_M$, $\Diamond\varphi$ expresses the consistency of $\sigma(w)$ with some element of $[\varphi]_M$, while $\Diamond^{\forall}\varphi$ expresses consistency of $\sigma(w)$ with all the maximal elements of $[\varphi]_M$.

$$\begin{aligned}
M, w \models \Box\varphi &\iff \sigma(w) \in [\varphi]_M \\
M, w \models \Diamond\varphi &\iff \sigma(w) \cap a \neq \emptyset \text{ for some } a \in [\varphi]_M \\
M, w \models \Diamond^{\forall}\varphi &\iff \sigma(w) \cap a \neq \emptyset \text{ for all } a \in \textsc{Alt}[\varphi]_M
\end{aligned}$$

Thus, in InqBK we have an asymmetry: the semantic object $\sigma(w)$ attached to a world is still a classical proposition, encoding only a body of information, while the semantic object $[\varphi]_M$ expressed by a formula is an inquisitive proposition, encoding both information and issues.

This suggests that the most natural framework for inquisitive modal logic is given not by a Kripke frame, but by a structure in which each world w is associated with an *inquisitive proposition* $\Sigma(w)$, describing both information and issues. We will call this kind of structure an *inquisitive modal frame*.

7.1.3. Definition. [Inquisitive modal frame]
An *inquisitive modal frame* is a pair $F = \langle W, \Sigma\rangle$, where:

- W is a set, whose elements we refer to as *possible worlds*;
- $\Sigma : W \to \Pi_W$ is an *inquisitive state map* which assigns to each world $w \in W$ an inquisitive proposition $\Sigma(w)$.

The corresponding notion of a model, on which formulas are interpreted, is obtained by equipping an inquisitive modal frame with a valuation function, which specifies the truth-value of each atomic sentence at each world.

7.1.4. Definition. [Inquisitive modal model]
An *inquisitive modal model* is a triple $M = \langle W, \Sigma, V\rangle$, where:

- $\langle W, \Sigma\rangle$ is an inquisitive modal frame;

	K	InqBK	InqBM
states	classical	classical	inquisitive
meanings	classical	inquisitive	inquisitive

Figure 7.1: A summary of the setup differences between standard modal logic, InqBK, and InqBM. In standard modal logic, the states used for the interpretation of modalities and the meaning of formulas are both classical propositions. In InqBK, states are still classical propositions, while meanings are inquisitive proposition. In InqBM, both states and meanings are inquisitive propositions.

- $V : W \times \mathcal{P} \to \{0, 1\}$ is a *valuation function*.

We say that the model $M = \langle W, \Sigma, V\rangle$ is a model *over* the frame $F = \langle W, \Sigma\rangle$.

Notice that any inquisitive modal frame determines a corresponding Kripke frame. For, if we simply look at the informative content $\mathsf{info}(\Sigma(w))$ of the proposition $\Sigma(w)$ associated with each world, what we obtain is an accessibility map.

7.1.5. Definition. [Kripke frame determined by an inquisitive modal frame]
Let $F = \langle W, \Sigma\rangle$ be an inquisitive modal frame. Let $\sigma : W \to \wp(W)$ be the information state map defined as follows:

$$\sigma(w) := \mathsf{info}(\Sigma(w)) = \bigcup \Sigma(w)$$

We refer to the Kripke frame $F^K = \langle W, \sigma\rangle$ as the *Kripke frame determined by* F. Similarly, if $M = \langle W, \Sigma, V\rangle$ is an inquisitive modal model, we refer to the Kripke model $M^K = \langle W, \sigma, V\rangle$ as the *Kripke model determined by* M.

Now that we have enriched our notion of state maps, we can equip our logic with modalities that express relations between two inquisitive propositions. Since inquisitive propositions have more structure to them than classical propositions, several such relations are conceivable—each of which gives rise to a corresponding inquisitive modality. We will focus here on a specific system, denoted InqBM, which is equipped with two modal operators that seem especially natural. One of them is just the Kripke modality $\Box$ associated with the Kripke model underlying the given inquisitive modal model; this operator and its logical features are familiar from the previous chapter. The other modal operator that we will consider will be referred to as the *inquisitive modality*, and denoted $\boxplus$.

7.1.6. Definition. [Language of Inquisitive Modal Logic]
The language $\mathcal{L}^{\mathsf{M}}$ of inquisitive modal logic is given by the following inductive definition, where $p \in \mathcal{P}$:

$$\varphi \ ::= \ p \mid \bot \mid \varphi \land \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee\, \varphi \mid \Box\varphi \mid \boxplus\varphi$$

Let us now specify the semantics for our language. As expected, the only novelty concerns the interpretation of the modalities.

7.1.7. Definition. [Support for InqBM]
Let $M = \langle W, \Sigma, V \rangle$ be an inquisitive modal model. The relation of *support* between information states $s \subseteq W$ and formulas $\varphi \in \mathcal{L}^{\mathsf{M}}$ is defined recursively by extending InqB with the following clauses for modalities:

- $M, s \models \Box\varphi \iff \text{for all } w \in s:\ M, \sigma(w) \models \varphi$
- $M, s \models \boxplus\varphi \iff \text{for all } w \in s,\ \text{for all } t \in \Sigma(w):\ M, t \models \varphi$

It is immediate to verify that support in InqBM is still persistent, and that the empty state supports any formula. This ensures that the support-set $[\varphi]_M$ of a formula in a model M is indeed an inquisitive proposition. As for the modalities, our clauses ensure that modal formulas are always truth-conditional, with the following truth-conditions.

7.1.8. Proposition (Truth-conditions for modalities in InqBM).

- $M, w \models \Box\varphi \iff M, \sigma(w) \models \varphi$
- $M, w \models \boxplus\varphi \iff \textit{for all } t \in \Sigma(w):\ M, t \models \varphi$

In words, $\Box\varphi$ is true at w when the informative content of the proposition $\Sigma(w)$ settles φ, while $\boxplus\varphi$ is true if settling the proposition $\Sigma(w)$ implies settling φ.

These truth-conditions can be restated in such a way that it becomes clear that both $\Box$ and $\boxplus$ express relations between the inquisitive state $\Sigma(w)$ attached to the evaluation world and the proposition $[\varphi]_M$ expressed by the argument.

7.1.9. Proposition (Truth-conditions for modalities, rephrased).

- $M, w \models \Box\varphi \iff \mathit{info}(\Sigma(w)) \in [\varphi]_M$
- $M, w \models \boxplus\varphi \iff \Sigma(w) \subseteq [\varphi]_M$

Notice that $\boxplus\varphi$ expresses the fact that $\Sigma(w) \subseteq [\varphi]_M$, just like in standard modal logic $\Box\varphi$ expresses the fact that $\sigma(w) \subseteq |\varphi|_M$. Thus, in InqBM it is $\boxplus$, and not $\Box$, that corresponds to the most fundamental semantic relation, that of entailment.

Let us now look in some detail at the properties of $\boxplus$ and at its relations with $\Box$. First, $\boxplus\varphi$ is always entailed by $\Box\varphi$.

7.1.10. Proposition. *For any $\varphi \in \mathcal{L}^{\mathsf{M}}$, $\Box\varphi \models \boxplus\varphi$*

Proof. Since modal formulas are truth-conditional, we just have to show that $\boxplus\varphi$ is true whenever $\Box\varphi$ is. So, suppose $M, w \models \Box\varphi$. This means that $M, \sigma(w) \models \varphi$. But now for any $t \in \Sigma(w)$ we have $t \subseteq \bigcup\Sigma(w) = \sigma(w)$, whence the persistency of support gives $M, t \models \varphi$. Thus, we also have $M, w \models \boxplus\varphi$. □

Furthermore, when applied to a truth-conditional argument, our two modalities boil down to the same thing.

7.1.11. PROPOSITION. *If $\alpha \in \mathcal{L}^{\mathsf{M}}$ is truth-conditional, then $\boxplus\alpha \equiv \Box\alpha$.*

Proof. Since we already know that $\Box\alpha \models \boxplus\alpha$, we just have to show that, if α is truth-conditional, the converse entailment holds as well. And since modal formulas are truth-conditional, we just have to show that $\Box\alpha$ is true whenever $\boxplus\alpha$ is. So, suppose $M, w \models \boxplus\alpha$. This means that $M, t \models \alpha$ for any $t \in \Sigma(w)$. Now consider any world $w \in \sigma(w)$. Since $\sigma(w) = \bigcup\Sigma(w)$, we must have $w \in t$ for some $t \in \Sigma(w)$. Since $t \in \Sigma(w)$ we must have $M, t \models \alpha$, and thus, by the persistency of support, also $M, \{w\} \models \alpha$, i.e., $M, w \models \alpha$. So, we have shown that $M, w \models \alpha$ for all $w \in \sigma(w)$. Since α is truth-conditional, this means that $M, \sigma(w) \models \alpha$, whence it follows that $M, w \models \boxplus\alpha$. □

Given what we know from the previous chapter about $\Box$, this shows that, when applied to a truth-conditional formula, both modalities of InqBM coincide with the standard universal modality.

7.1.12. COROLLARY (MODALITIES ON TRUTH-CONDITIONAL FORMULAS). *Let $\alpha \in \mathcal{L}^{\mathsf{M}}$ be truth-conditional, M an inquisitive modal model, and w a world:*

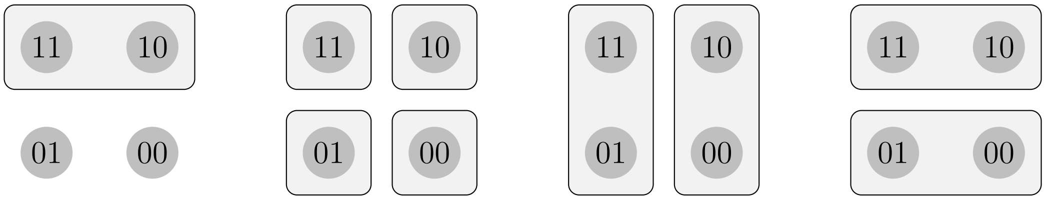
$$M, w \models \boxplus\alpha \iff M, w \models \Box\alpha \iff M, v \models \alpha \text{ for all } v \in \sigma(w)$$

Thus, both $\Box$ and $\boxplus$ extend the standard universal modality to questions. On the other hand, when applied to questions, $\Box$ and $\boxplus$ express very different things: to appreciate this, let us look at an example.

7.1.13. EXAMPLE. Let $M = \langle W, \Sigma, V\rangle$ be the inquisitive modal model defined as follows. W consists of four worlds, $w_{00}, w_{10}, w_{01}, w_{11}$, where the subscripts reflect the valuation of two atomic sentences p and q: at w_{00} both atoms are false, at w_{10} p is true and q is false, and so on. Since the interpretation of purely propositional formulas does not depend on Σ, we can describe $\Sigma(w)$ in terms of the proposition expressed by some non-modal formulas. We let:

$$\Sigma(w_{10}) = [p]_M \qquad \Sigma(w_{01}) = [?p \wedge ?q]_M \qquad \Sigma(w_{11}) = [?q]_M$$

The value of $\Sigma(w_{00})$ will not matter for our example. Roughly, we can describe the situation as follows: at w_{10} we have the information that p, and no open issues; at w_{01} we have no information, and the issue of which among p and q are true; finally, at w_{11} we have no information, and the issue of whether q. The



(a) $\Sigma(w_{10})$ (b) $\Sigma(w_{01})$ (c) $\Sigma(w_{11})$ (d) $[?p]$

Figure 7.2: The inquisitive states associated with three worlds, and the proposition expressed by $?p$. As usual, only the maximal elements of these propositions are displayed. The question $?p$ is closed at w_{10}, open at w_{01}, and absent at w_{11}.

inquisitive states associated with the three worlds are depicted in figures 7.2(a-c), with the usual convention that only the *alternatives* for each state are displayed. Now consider the truth value of the formulas $\Box ?p$ and $\boxplus ?p$ at each world.

- At w_{10}, the available information, $\sigma(w_{10}) = |p|_M$, settles the question $?p$. So, we have $M, w_{10} \models \Box ?p$. Since $\Box ?p \models \boxplus ?p$, we also have $M, w_{10} \models \boxplus ?p$.
- At w_{01}, the available information, $\sigma(w_{01}) = |\top|_M$, does not settle $?p$. So, we have $M, w_{01} \not\models \Box ?p$. However, the open issues at w_{01} *determine* $?p$, in the sense that reaching a state in $\Sigma(w_{01})$ implies settling $?p$, as can be seen by comparing figures 7.2(b) and 7.2(d). Thus, we do have $M, w_{01} \models \boxplus ?p$.[1]
- At w_{11}, the available information, $\sigma(w_{11}) = |\top|_M$ does not settle $?p$. So, we have $M, w_{11} \not\models \Box ?p$. Moreover, in this case the open issues at w_{11} do not determine $?p$ either: in order to reach a state in $\Sigma(w_{11})$ it is not necessary to settle $?p$. For instance, we have $|q|_M \in \Sigma(w_{11})$, but $M, |q|_M \not\models ?p$. This shows that we have $M, w_{11} \not\models \boxplus ?p$.

These three worlds exemplify the three possible situations that we can find at a world w with respect to a question μ. First, it may be the case that the information available at w already settles μ; in this case, captured by the formula $\Box\mu$, we may say that the question μ is *closed* at w. Second, it may be the case that the information available at w does not settle μ, but the open issues at w determine μ; in this case, captured by the formula $\neg\Box\mu \wedge \boxplus\mu$, we may say that the question μ is *open* at w. As this configuration plays an important role in the interpretations of inquisitive modal logic considered later on, it will be useful to introduce a defined modality $\boxtimes$ which allows us to express it concisely:

$$\boxtimes\varphi := \neg\Box\varphi \wedge \boxplus\varphi$$

[1]The notation for the two modalities, $\Box$ and $\boxplus$, is intended to evoke the fact that for its truth-conditions, $\boxplus$ tests support at each element of the inquisitive state $\Sigma(w)$—in the case of Figure 7.2(b), at each of the four blocks—while $\Box$ tests support at the union of these states.

Finally, it may be the case that the information available at w does not settle μ, and that the open issues at w do not determine μ either; in this case, captured by the formula $\neg\boxplus\mu$, we may say that the question μ is *absent* at w.[2]

There is an important difference between the modalities $\Box$ and $\boxplus$, which is visible from their support conditions, and which this example brings out. A formula like $\Box?p$ essentially describes a property of an information state. In this particular case, this is the property of settling the question whether p, i.e., of being included in either of $|p|_M$ and $|\neg p|_M$. The formula $\Box?p$ is true at a world w if the associated information state $\sigma(w)$ has this property. By contrast, the formula $\boxplus?p$ describes a property of inquisitive propositions—namely, the property of *determining* whether p, that is, of being a subset of $[?p]_M$. The formula $\boxplus?p$ is true at a world w if the associated inquisitive proposition $\Sigma(w)$ has this property.

The results in the previous chapter show that questions are not essential in order to describe information states. As we saw, any occurrence of a question under $\Box$ can be paraphrased away: for instance, we have $\Box?p \equiv \Box p \vee \Box\neg p$. The outcome of this is that any truth-conditional formula in InqBK is equivalent to a classical modal formula, as Proposition 6.3.10 ensures.

We are now going to see that, by contrast, questions are indispensable for the task of expressing properties of inquisitive propositions. As an example, consider again the formula $\boxplus?p$: while this formula is truth-conditional, we will see that it is not equivalent to any formula α which does not contain inquisitive disjunction. Let us first observe that $\boxplus$ does not distribute over $\mathbin{\vee\!\!\!\vee}$ in the way that $\Box$ does. For instance, $\boxplus?p \not\equiv \boxplus p \vee \boxplus\neg p$. We know this, because Proposition 7.1.11 implies that $\boxplus p \vee \boxplus\neg p \equiv \Box p \vee \Box\neg p \equiv \Box?p$, and we have just seen that $\boxplus?p \not\equiv \Box?p$.

Of course, this does not yet rule out the possibility that questions under $\boxplus$ may be paraphrased away in a different way. In order to prove this in general, let us first give a precise definition of classical formulas in $\mathcal{L}^{\mathsf{M}}$.

7.1.14. Definition. [Classical formulas in $\mathcal{L}^{\mathsf{M}}$]
We say that a formula $\varphi \in \mathcal{L}^{\mathsf{M}}$ is *classical* if it contains no occurrences of $\mathbin{\vee\!\!\!\vee}$. The set of all classical formulas in $\mathcal{L}^{\mathsf{M}}$ is denoted by $\mathcal{L}^{\mathsf{M}}_c$.

That is, classical formulas are just formulas built out of the classical connectives and the modalities. The following fact can be easily verified by induction.

7.1.15. Proposition. *Any classical formula is truth-conditional.*

Moreover, it follows from Corollary 7.1.12 that any classical formula has the standard truth-conditions, where both modalities are interpreted as the universal Kripke modality on the Kripke model M^K determined by the given inquisitive

[2]Notice that these three possibilities are exhaustive: since $\Box\varphi \models \boxplus\varphi$, we cannot have a fourth configuration in which $\Box\varphi$ is true but $\boxplus\varphi$ is false.

modal model. This leads to an interesting observation: the truth-value of a classical formula at a world w of an inquisitive modal model M depends exclusively on the underlying Kripke model M^K. Let us make this precise.

7.1.16. Definition. [Issue-insensitivity]
We say that a formula $\varphi \in \mathcal{L}^{\mathsf{M}}$ is *issue-insensitive* if for any inquisitive modal models M_1, M_2 with $M_1^K = M_2^K$, and for any state s, we have:

$$M_1, s \models \varphi \iff M_2, s \models \varphi$$

Since classical formulas are truth-conditional, and since their truth-conditions in a model depend only on the underlying Kripke model M^K, we have the following.

7.1.17. Proposition. *Any classical formula in* InqBM *is issue-insensitive.*

Thus, no classical formula expresses a genuine property of the inquisitive proposition $\Sigma(w)$ attached to a world: for the truth of a classical formula, all that matters is the information state $\sigma(w)$ that $\Sigma(w)$ determines.

On the other hand, we have seen that $\boxplus ?p$ *does* express a genuine property of inquisitive propositions: the truth of $\boxplus ?p$ at a world w does not depend only on $\sigma(w)$, as witnessed by the difference between the worlds w_{01} and w_{11} in Example 7.1.13. This shows that, in our system, questions are essential in order to express certain properties of the inquisitive proposition attached to a world.

7.1.18. Proposition (Questions under $\boxplus$ are essential).
There is no $\alpha \in \mathcal{L}_c^{\mathsf{M}}$ *such that* $\boxplus ?p \equiv \alpha$.

Proof. By the previous proposition, it suffices to prove that $\boxplus ?p$ is issue-sensitive. This can be established easily by constructing two inquisitive modal models which determine the same Kripke model, and on which $\boxplus ?p$ differs in truth-value. $\square$

Notice that the formula $\boxplus ?p$ is truth-conditional. Thus, the previous proposition shows that not all truth-conditional formulas are equivalent to a classical formula. In this respect, InqBM is very different from all the other systems discussed so far: in those systems, questions could be seen as being added "on top" of an independent truth-conditional core; in InqBM, by contrast, questions contribute to the truth-conditional expressivity of the system, making it possible to build up modal statements that talk about issues, statements whose truth-conditions depend crucially on the support conditions of the question that they embed.[3]

[3] This is analogous to the situation we find in natural language, where the truth-conditional meaning of some statements, such as "Mary wonders whether John will go" or "John has not decided whether he will go", depends on the meaning of the question embedded in them, "whether John will go". This shows that question semantics is not something that can be studied *besides* the more familiar truth-conditional semantics for statements. Rather, the two are interwoven, and neither can be fully understood in isolation from the other. This point was emphasized, among many others, by Belnap (1990) and Groenendijk and Stokhof (1997).

Potentially, the fact that classical formulas are not representative of all truth-conditional meanings could be a problem. After all, our completeness proofs in the previous chapters relied on the availability of a syntactically distinguished set of formulas that can be taken as descriptions of states of affairs, and out of which we build the possible worlds in our canonical model. Fortunately, in InqBM it is still possible to isolate such a class: only, this class will now contain not only classical formulas, but also any modal formula, including formulas such as $\boxplus ?p$, which have no classical counterpart.

7.1.19. Definition. [Declarative fragment of $\mathcal{L}^{\mathsf{M}}_{!}$]
The set $\mathcal{L}^{\mathsf{M}}_{!}$ of *declarative formulas* in $\mathcal{L}^{\mathsf{M}}$ is given by the following inductive definition, where $p \in \mathcal{P}$ and $\varphi \in \mathcal{L}^{\mathsf{M}}$:

$$\alpha \ ::= \ p \mid \bot \mid \Box\varphi \mid \boxplus\varphi \mid \alpha \wedge \alpha \mid \alpha \to \alpha$$

That is, declaratives are formulas constructed out of atoms, $\bot$, and modal formulas by means of conjunction and implication. Equivalently, they may be characterized as formulas in which all the occurrences of $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ are within the scope of a modality. Notice that if we restrict ourselves to the language of propositional logic, declaratives are just classical propositional formulas, while if we restrict ourselves to the language $\mathcal{L}^{\mathsf{K}}$ of inquisitive Kripke modal logic, the set of declaratives coincides with the set $\mathcal{L}^{\mathsf{K}}_{!}$ defined in the previous chapter.

Since atoms, $\bot$, and modal formulas are truth-conditional, and since $\wedge$ and $\to$ preserve truth-conditionality, any declarative formula is truth-conditional.

7.1.20. Proposition. *Any declarative formula is truth-conditional.*

Next, we want to show that declarative formulas, unlike classical formulas, are representative of all truth-conditional meanings expressible in InqBM. To show this, we associate any formula $\varphi \in \mathcal{L}^{\mathsf{M}}$ with a declarative $\varphi^! \in \mathcal{L}^{\mathsf{M}}_{!}$ having the same truth-conditions as φ.

7.1.21. Definition. [Declarative variant]
The declarative variant $\varphi^!$ of a formula $\varphi \in \mathcal{L}^{\mathsf{M}}$ is defined inductively as follows:

- $\alpha^! = \alpha$ if α is an atom, $\bot$, or a modal formula
- $(\varphi \wedge \psi)^! = \varphi^! \wedge \psi^!$
- $(\varphi \to \psi)^! = \varphi^! \to \psi^!$
- $(\varphi \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \psi)^! = \varphi^! \vee \psi^!$

It is immediate from the definition that $\varphi^!$ is indeed a declarative for any $\varphi \in \mathcal{L}^{\mathsf{M}}$. Moreover, we can verify inductively that φ and $\varphi^!$ have the same truth-conditions.

7.1.22. Proposition.
For any $\varphi \in \mathcal{L}^{\mathsf{M}}$, any inquisitive modal model M, and any world w:

$$M, w \models \varphi \iff M, w \models \varphi^{!}$$

In particular, if φ itself is truth-conditional, we have $\varphi \equiv \varphi^{!}$. This shows that declaratives are indeed representative of all truth-conditional formulas in $\mathcal{L}^{\mathsf{M}}$.

7.1.23. Corollary.
A formula $\varphi \in \mathcal{L}^{\mathsf{M}}$ is truth-conditional $\iff$ $\varphi \equiv \alpha$ for some $\alpha \in \mathcal{L}^{\mathsf{M}}_{!}$

In the setting of InqBM, declaratives play the same role as classical formulas play in the systems discussed in the previous chapters. In particular, just like in InqB and InqBK a formula is always equivalent with an inquisitive disjunction of classical formulas, in InqBM a formula is always equivalent with an inquisitive disjunction of declaratives.

To show this, let us associate with each formula a set of declaratives that we will call its resolutions. Atoms and modal formulas will have themselves as their unique resolution, while the inductive clauses for connectives are the usual ones.

7.1.24. Definition. [Resolutions]

- $\mathcal{R}(\alpha) = \{\alpha\}$ if α is an atom, $\bot$, or a modal formula;
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$

By definition, $\mathcal{R}(\varphi)$ is always a set of declaratives; notice that resolutions are not necessarily classical: for instance, we have $\mathcal{R}(\boxplus ?p) = \boxplus ?p$. Indeed, it is immediate to verify inductively that a declarative α always has itself as unique resolution.

7.1.25. Proposition. *For all $\alpha \in \mathcal{L}^{\mathsf{M}}_{!}$, $\mathcal{R}(\alpha) = \{\alpha\}$.*

A straightforward inductive proof suffices to establish that a formula φ is related to its resolutions in the usual way: φ is supported precisely when some resolution of it is supported.

7.1.26. Proposition.
For any $\varphi \in \mathcal{L}^{\mathsf{M}}$, any M and s: $M, s \models \varphi \iff M, s \models \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$

This gives as a corollary the following normal form result, which shows that any formula in $\mathcal{L}^{\mathsf{M}}$ is equivalent to an inquisitive disjunction of declaratives.

7.1.27. Corollary. *For any $\varphi \in \mathcal{L}^{\mathsf{M}}$, $\varphi \equiv \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.*

The definition of resolution is then extended in the usual way from single formulas to sets of formulas.

7.1.28. Definition. [Resolutions for sets]
A *resolution function* for a set $\Phi \subseteq \mathcal{L}^{\mathsf{M}}$ is a map $f : \Phi \to \mathcal{L}^{\mathsf{M}}_!$ such that for any $\varphi \in \Phi$, $f(\varphi) \in \mathcal{R}(\varphi)$. The set of resolutions for Φ is then defined as follows:

$$\mathcal{R}(\Phi) = \{f[\Phi] \mid f \text{ is a resolution function for } \Phi\}$$

By construction, each resolution $\Gamma \in \mathcal{R}(\Phi)$ is a set of declaratives. Moreover, as for single declaratives, a set of declaratives always has itself as unique resolution.

7.1.29. Proposition. *For all $\Gamma \subseteq \mathcal{L}^{\mathsf{M}}_!$, $\mathcal{R}(\Gamma) = \{\Gamma\}$.*

Proof. Since each $\alpha \in \Gamma$ has itself as unique resolution, there is only one resolution function for Γ, namely, the identity function on Γ, id_Γ. As a consequence, $\mathcal{R}(\Gamma) = \{id_\Gamma[\Gamma]\} = \{\Gamma\}$. □

Recall that we speak of a set of formulas Φ as being supported in a state s in case all formulas $\varphi \in \Phi$ are supported in s. It is then easy to see that Proposition 7.1.26 implies the following generalized version of it.

7.1.30. Proposition.
For any $\Phi \subseteq \mathcal{L}^{\mathsf{M}}$, any M and s: $M, s \models \Phi \iff M, s \models \Gamma$ for some $\Gamma \in \mathcal{R}(\Phi)$

This shows that InqBM still retains some of the crucial features of the inquisitive systems discussed so far: in particular, with each formula φ we can associate a set $\mathcal{R}(\varphi)$ of resolutions—truth-conditional formulas that jointly capture the meaning of φ. On the other hand, there is also an important novelty that InqBM introduces: now, the truth-conditional fragment of our logic is not independent of, and more fundamental than, the layer of questions. Instead, the two are tightly interwoven: inquisitive disjunction allows us to form questions out of truth-conditional formulas, and conversely, modal operators allow us to form new truth-conditional formulas out of questions, which express meanings that would not be expressible *without* questions. As a consequence, the logic of the declarative fragment of the language cannot be separated out from the logic of questions; thus, characterizing the logic of questions becomes crucial, even if our goal were merely to capture entailments among declaratives in InqBM.

7.2 Inquisitive epistemic logic

In our description of InqBM, we have so far been rather neutral about the intuitive reading of our modalities. This is because the significance of a modality crucially depends on what we take the corresponding state map Σ to represent. This is just like in standard modal logic, where a modal formula $\Box\alpha$ can be read in many different ways depending on what we take the accessibility map σ to represent. Each of these interpretations suggests specific constraints on the associated class of Kripke models, giving rise to a specific modal logic. In this section, we look at a concrete interpretation of inquisitive modal logic, which arises from regarding an inquisitive state map as describing an agent's information and issues. We will focus on two particular logics arising from this interpretation, which are generalizations to the inquisitive setting of standard versions of doxastic logic—the logic of belief—and epistemic logic—the logic of knowledge. In the literature, it is rather common to use *epistemic logic* also as a broad term, encompassing various systems of epistemic as well as doxastic modal logic. Analogously, I will use *inquisitive epistemic logic* not only as a name for a specific system of inquisitive modal logic, but also as a general term for the sort of interpretation explored in this section.[4]

7.2.1 Standard epistemic logic

Let us start by recalling a few facts about standard doxastic and epistemic logic (for a short introduction, the reader is referred to Egré, 2011; Holliday, 2015). In *doxastic logic*, a multi-modal Kripke model $M = \langle W, \{\sigma_a \mid a \in \mathcal{A}\}, V\rangle$ is interpreted as follows: each label $a \in \mathcal{A}$ is interpreted as the name of an agent, and the corresponding map σ_a is interpreted as giving, for each possible world w, the information state of agent a at w: that is, the set $\sigma_a(w)$ is read as consisting of those worlds that are consistent with the agent's information at w. Accordingly, a modal formula $\Box_a\alpha$ is read as "a believes that α".

Given this interpretation, specific assumptions about what an agent's information should be like give rise to corresponding constraints on the maps σ_a, which in turn determine a logic which is stronger than the basic normal modal logic K. For instance, one may assume that the information of an agent is always consistent, and that agents have complete information of what their information state is. Formally, the first of these assumptions amounts to the fact that the state $\sigma_a(w)$ of an a agent is always non-empty, while the second assumption amounts to the

[4] As mentioned in the introduction to this chapter, *inquisitive epistemic logic* has been defined and investigated in Ciardelli and Roelofsen (2015b) and Ciardelli (2014a). The system presented in these papers differs slightly from the one presented here, as it is based on a so-called *dichotomous* implementation of inquisitive semantics (the system InqD_π of Ciardelli et al., 2015b), in which formulas are categorized into two syntactic classes, *declaratives* and *interrogatives*. The difference between the two presentations is not crucial.

following: if a world v might be the actual world according to a's information at w, then v must agree with w in the state it assigns to a. For, suppose a's state at v were different than it is at w: then, at world w, a would be able to tell that v is not the actual world, on the ground that v misrepresents her information state.

7.2.1. Definition. [Doxastic models]
A multi-modal Kripke model $M = \langle W, \{\sigma_a \,|\, a \in \mathcal{A}\}, V\rangle$ is a (KD45) doxastic model if the following conditions are satisfied for all $a \in \mathcal{A}$:

- *Consistency:* for all $w \in W$, $\sigma_a(w) \neq \emptyset$
- *Introspection:* for all $w, v \in W$, $v \in \sigma_a(w)$ implies $\sigma_a(v) = \sigma_a(w)$

The logic of such doxastic models is known as (multi-modal) KD45, and it is axiomatized by adding the following axioms to the minimal normal modal logic K:

- Consistency (D): $\neg\Box_a\bot$
- Positive introspection (4): $\Box_a\alpha \to \Box_a\Box_a\alpha$
- Negative introspection (5): $\neg\Box_a\alpha \to \Box_a\neg\Box_a\alpha$

The *epistemic* interpretation of modal logic is just like the doxastic one, except that the information that the state $\sigma_a(w)$ represents is taken to be *knowledge*, and a modal formula $\Box_a\alpha$ is read accordingly as "*a knows* that α". Now, while information in general can be false, and misrepresent the world, information that counts as knowledge should be truthful. This means that the information state $\sigma_a(w)$ of an agent at a world w must always be a true description of the world itself, that is, we must have $w \in \sigma_a(w)$. This condition is known as *factivity.* Incidentally, notice that in the presence of factivity, the consistency condition becomes superfluous: since $w \in \sigma_a(w)$, we are guaranteed to have $\sigma_a(w) \neq \emptyset$. If, in addition to factivity, we also assume that agents know what their knowledge state is, we arrive at the following standard notion of epistemic models.[5]

7.2.2. Definition. [Epistemic models]
A multi-modal Kripke model $M = \langle W, \{\sigma_a \,|\, a \in \mathcal{A}\}, V\rangle$ is an (S5) epistemic model if the following conditions are satisfied for all $a \in \mathcal{A}$:

- *Factivity:* for all $w \in W$, $w \in \sigma_a(w)$
- *Introspection:* for all $w, v \in W$, $v \in \sigma_a(w)$ implies $\sigma_a(v) = \sigma_a(w)$

The logic of such epistemic models is known as (multi-modal) S5, and it is axiomatized by adding to the following axioms to the minimal normal modal logic K:[6]

[5]The latter assumption, while commonly made in applications, is widely regarded as being too strong for a general logic of knowledge, and replaced by weaker conditions. We could also take one of these weaker epistemic logics as our starting point. Since this choice is rather orthogonal to the novelties introduced by the inquisitive approach, we will just build on the standard system S5. However, the completeness result obtained below concerns not only the inquisitive version of S5, but also the inquisitive version of weaker logics, such as S4.

[6]In fact, it is easy to see that axiom 4 is not needed, since it is provable from T and 5.

- Factivity (T): $\Box_a\alpha \to \alpha$
- Positive introspection ($\mathsf{4}$): $\Box_a\alpha \to \Box_a\Box_a\alpha$
- Negative introspection ($\mathsf{5}$): $\Box_a\alpha \to \Box_a\neg\Box_a\alpha$

7.2.2 Inquisitive epistemic models

Doxastic and epistemic logics have important applications in the formal modeling of information-related processes, such as information exchange and scientific inquiry. However, there is a crucial feature of such processes that is not represented in their modeling by means of standard doxastic and epistemic logics: namely, both information exchange and scientific inquiry are processes of information-seeking; the agents involved in them are typically concerned with some issues, which they hope to settle through the process. Thus, in order to analyze such processes, it is important to keep track not only of the information that an agent already has, but also of the information that an agent wants to obtain, that is, of the issues that the agent aims to settle. Inquisitive modal models allow us to represent this aspect in a simple and elegant way: just like the map σ of a standard doxastic model can be seen as describing the agent's information at each world, in the inquisitive setting we can read the state map Σ as describing the agent's state at each world as consisting of both information and issues. More explicitly, if an agent's state at w is described by the inquisitive proposition $\Sigma(w)$, we will take this to mean that:

- the agent's information state is $\sigma(w) = \mathsf{info}(\Sigma(w))$;
- the agent's goal is to get her information state to be one of the $s \in \Sigma(w)$.

We will refer to this as the *doxastic* interpretation of inquisitive modal logic. The *epistemic* interpretation of inquisitive modal logic is completely analogous, except that the relevant information is taken to be knowledge. Thus, in the epistemic case we will read the state $\Sigma_a(w)$ of an agent a at a world w as follows:

- the agent's knowledge state is $\sigma(w) = \mathsf{info}(\Sigma(w))$;
- the agent's goal is to get her knowledge state to be one of the $s \in \Sigma(w)$.

In this way, we can capture in an integrated way not only the information that an agent has, but also the issues that she is interested in.

As in standard epistemic logic, we want to model scenarios involving several agents. We can do this by generalizing the notion of inquisitive modal model. Rather than being equipped with a unique state map Σ, an inquisitive *multi*-modal model is equipped with a number of state maps Σ_a, one for each item $a \in \mathcal{A}$ in a given set of labels, which we think of as names for the relevant agents.

7.2.3. Definition. [Inquisitive multi-modal models]
An *inquisitive multi-modal model* for a given set $\mathcal{A}$ of labels is a triple $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$, where W and V are as usual, and for each label $a \in \mathcal{A}$, $\Sigma_a : W \to \Pi_W$ is a map assigning to each possible world an inquisitive proposition. As usual, we then let σ_a be the accessibility map defined by $\sigma_a(w) = \bigcup \Sigma_a(w)$.

Now, given a set of labels $\mathcal{A}$, the corresponding modal language $\mathcal{L}^{\mathsf{M}}(\mathcal{A})$ is defined just like the language $\mathcal{L}^{\mathsf{M}}$, except for the fact that instead of having just two modalities $\Box$ and $\boxplus$, we have for each label $a \in \mathcal{A}$ two corresponding modalities $\Box_a$ and $\boxplus_a$, to be interpreted by means of the corresponding state map:

- $M, s \models \Box_a\varphi \iff$ for all $w \in s: \ M, \sigma_a(w) \models \varphi$
- $M, s \models \boxplus_a\varphi \iff$ for all $w \in s$, for all $t \in \Sigma_a(w): \ M, t \models \varphi$

All the definitions given in the previous sections can be adapted in an obvious way to the multi-modal case, and all the results proved carry over straightforwardly.

Let us now consider how the standard constraints on doxastic and epistemic models should be adapted to the inquisitive setting. Let us start from the doxastic case, where models were required to satisfy consistency and introspection. Consistency is essentially a condition on an agent's information, so it does not seem to require any modification. As for introspection, it seems natural to assume that agents are informed about their overall state, that is, not just about what information they have, but also about what issues they are interested in. To capture this, we will require that if an agent a considers world v to be possible at w, this must be because a's inquisitive state is the same in v as in w; otherwise, a would be able to rule out v as a possibility, on the ground that her inquisitive state is misrepresented by v. This leads us to the following definition.

7.2.4. Definition. [Inquisitive doxastic models]
An inquisitive multi-modal model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ is called an *inquisitive doxastic model* if the following conditions are satisfied for all $a \in \mathcal{A}$:

- *Consistency:* for all $w \in W$, $\sigma_a(w) \neq \emptyset$
- *Introspection:* for all $w, v \in W$, $v \in \sigma_a(w)$ implies $\Sigma_a(v) = \Sigma_a(w)$

It is easy to see that if M is an inquisitive doxastic model, the underlying Kripke model M^K is a standard doxastic model. Now consider the epistemic case. Like the consistency condition for doxastic models, the factivity condition for epistemic models seems to be inherently about an agent's information, so it does not require any adjustment. This leads to the following definition.

7.2.5. Definition. [Inquisitive epistemic models]
An inquisitive multi-modal model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ is called an *epistemic* model if the following conditions are satisfied for all $a \in \mathcal{A}$:

- *Factivity:* for all $w \in W,\ w \in \sigma_a(w)$
- *Introspection:* for all $w, v \in W,\ v \in \sigma_a(w)$ implies $\Sigma_a(v) = \Sigma_a(w)$

It is easy to verify that, if M is an inquisitive epistemic model, then the underlying Kripke model M^K is a standard epistemic model.

In the following, we will refer to the logic of inquisitive doxastic models as *inquisitive doxastic logic*, IDL, and to the logic of inquisitive epistemic models as *inquisitive epistemic logic*, IEL. A sound and complete axiomatization of these two logics will be established in Section 7.4.

7.2.3 Inquisitive epistemic modalities

Let us now consider what the modalities $\Box_a$ and $\boxplus_a$ allow us to express in the context of inquisitive doxastic and epistemic logic. First, consider a truth-conditional formula α: we know from Corollary 7.1.12 that, in this case, the formulas $\Box_a\alpha$ and $\boxplus_a\alpha$ boil down to the same thing, namely, to the standard $\Box_a$ modality of doxastic logic. So, both $\Box_a\alpha$ and $\boxplus_a\alpha$ simply express the fact that according to a's information, α is true. As usual, we can then read $\Box_a\alpha$ as "a believes that α" in the doxastic setting, and as "a knows that α" in the epistemic setting, where information is taken to be knowledge.

In order to see what our modalities allow us to express when they are applied to a question μ, it is useful to distinguish the three types of situation in which an agent a may find herself with respect to a question μ. These types of situations correspond to the three cases we discussed in Example 7.1.13, and are illustrated by the three states depicted in Figure 7.1.[7]

1. $M, \sigma_a(w) \models \mu$. This means that a's information at w settles the question μ. This case corresponds precisely to the truth of the formula $\Box_a\mu$, which we may thus read as "question μ is *closed* for agent a".

   In the epistemic setting, we may also read $\Box_a\mu$ simply as "agent a knows μ". For instance, the formula $\Box_a ?p$ may be read as "agent a knows whether p".[8]

2. $M, \sigma_a(w) \not\models \mu$ but for every $t \in \Sigma_a(w)$, $M, t \models \mu$. This means that the agent's information at w does not settle μ, but μ follows from the agent's issues at w, that is, settling the agent's issues at w implies settling μ.

[7] The model of Example 7.1.13 is neither a doxastic nor an epistemic model, since it does not satisfy the introspection condition. But this is irrelevant to the present discussion: we may just as well take the three pictures to represent states in three different models.

[8] An analogous direct rendering in English is not available in the doxastic case, where we would expect to read $\Box_a$ as *believes*. This difference between *know* and *believe* is not just an accident of English, but a fact attested systematically across natural languages and not fully understood. It has been observed (Hintikka, 1975; Boër, 1978; Egré, 2008) that there is a systematic connection between the factivity of a verb and its ability to license both declarative and interrogative complements. For a possible explanation of this, see Sæbø (2007).

Intuitively, what this situation captures is that μ is not settled for a, and settling μ is part of a's epistemic goals at w.

This situation corresponds precisely to the truth of the formula $\neg\Box_a\mu \wedge \boxplus_a\mu$. Remember that above we have introduced a modality $\boxtimes_a$ which allows us to express this situation more concisely:

$$\boxtimes_a\varphi := \neg\Box_a\varphi \wedge \boxplus_a\varphi$$

In the context of inquisitive doxastic and epistemic logic, we can thus read $\boxtimes_a\mu$ as "question μ is *open* for agent a" or also as "agent a *wonders about* question μ". In the context of inquisitive doxastic and epistemic logic, we will refer to $\boxtimes_a$ as the *wondering modality*.

Notice that if α is truth conditional we have $\Box_a\alpha \equiv \boxplus_a\alpha$, which implies that $\boxtimes_a\alpha = \neg\Box_a\alpha \wedge \boxplus_a\alpha \equiv \neg\Box_a\alpha \wedge \Box_a\alpha \equiv \bot$. So, the wondering modality only results in a consistent formula provided its argument is a question.

3. For some $t \in \Sigma_a(w)$, $M, t \not\models \mu$. This means that the agent's information at w does not settle w, and moreover, it is possible to settle the agent's issues at w without also settling μ. This means that, although the agent's information does not settle μ, settling μ is not a goal for the agent at w.

   This case corresponds to the truth of the formula $\neg\boxplus_a\mu$, which we will read, using a rather technical term, as "question μ is *absent* for agent a".

Now, what about the significance of a simple modal formula $\boxplus_a\mu$? Well, our choice to read $\neg\boxplus_a\mu$ as "question μ is absent for a" suggests to read $\boxplus_a\mu$ as "question μ is *present* for a". This is fine, but it is still technical terminology: what does it mean for a question to be *present* for an agent? Using the fact that $\Box_a\varphi \models \boxplus_a\varphi$ and the definition of $\boxtimes_a$, we can give a clearer characterization.

$$\begin{aligned}
\boxplus_a\varphi &\equiv \boxplus_a\varphi \wedge (\Box_a\varphi \vee \neg\Box_a\varphi) \\
&\equiv (\boxplus_a\varphi \wedge \Box_a\varphi) \vee (\boxplus_a\varphi \wedge \neg\Box_a\varphi) \\
&\equiv \Box_a\varphi \vee \boxtimes_a\varphi
\end{aligned}$$

Thus, for μ to be *present* for a means that either a has information resolving μ, or it is a goal for a to obtain such information. Intuitively, it means that a is not simply disregarding the question μ: if μ is not already settled, then she is taking an interest in it. To see the relevance of this notion, suppose the question $?p$ is asked in a cooperative conversation: then, an agent a should not simply disregard the question; this means that, after the question is posed, $\boxplus_a?p$ should hold. It is not necessarily the case that a should come to wonder about $?p$, though: for, if she has an answer to $?p$, there is no reason for her to give it up. Thus, $\boxtimes_a\mu$ should not necessarily hold after the question is posed.

A crucial feature of standard doxastic and epistemic logic is that it enables us to represent not only information about primitive, non-epistemic facts, but also *higher-order* information, that is, information about the information that other agents have. For instance, in epistemic logic a sentence like $\Box_a \neg \Box_b\, p$ expresses the fact that a knows that b doesn't know that p, and a sentence like $\Box_a(p \to \Box_b\, p)$ expresses the fact that a knows that if p is true, then b knows it.

Similarly, in inquisitive doxastic and epistemic logic, we can represent not only information and issues that agents have about some primitive facts, but also information and issues that agents have about one another's information and issues. Let us see in some more detail what this means. For concreteness, I will spell this out for inquisitive epistemic logic, but the situation is analogous for the doxastic case. We can distinguish four aspects of an agent's higher-order state:

- Information about other agents' information: this is as in standard epistemic logic, except for the fact that now, an agent's information can also be described directly in terms of the questions that it settles. E.g., the formula $\Box_a ?\Box_b ?p$ expresses the fact that a knows whether b knows whether p.

- Information about other agents' issues: for instance, the formula $\Box_a \boxtimes_b ?p$ expresses the fact that a knows that b wonders whether p, and the formula $\Box_a(\Box_b(p \veebar q) \to \boxtimes_b(p \mathbin{\vee\!\!\!\vee} q))$ expresses the fact that a knows that, if b knows that exactly one of p and q holds, then he wonders which one holds.[9]

- Issues about other agents' information: e.g., the formula $\boxtimes_a ?\Box_b p$ expresses that a wonders whether b knows that p, while the formula $\boxtimes_a(\Box_b\, p \mathbin{\vee\!\!\!\vee} \Box_c\, p)$ expresses that a wonders whether it is agent b or agent c who knows p.

- Issues about other agents' issues: e.g., the formula $\boxtimes_a ?\boxtimes_b ?p$ expresses that agent a wonders whether agent b wonders whether p.

Let us now illustrate the workings of inquisitive epistemic logic with the help of two concrete example models.

7.2.6. Example. [a wonders whether b knows whether p]
Consider the following inquisitive epistemic model M for two agents a and b. The universe W consists of four worlds: w_1 and w_1', where the atom p is true, and w_0 and w_0', where p is false. The state maps Σ_a and Σ_b of the agents are described graphically in Figure 7.3. These figures should be read as follows: given any world w, the epistemic state of the agent at w is represented by the dashed area to which w belongs. Thus, the epistemic state of a at each world is $\sigma_a(w) = W$, while for agent b we have $\sigma_b(w_1') = \{w_1'\}$, $\sigma_b(w_0') = \{w_0'\}$, $\sigma_b(w_1) = \sigma_b(w_0) = \{w_1, w_0\}$.

The inquisitive state of the agent at a world w consists of the solid blocks which are included in the agent's epistemic state at w, plus all subsets thereof. Formally, the state maps depicted in the figures are defined as follows:[10]

[9]Recall that $\veebar$ abbreviates exclusive disjunction: $p \veebar q := (p \wedge \neg q) \vee (q \wedge \neg p)$.

[10]Notice that this graphical representation of the state map Σ_a of an agent is made possible by

(a) agent a (b) agent b

Figure 7.3: The inquisitive state maps of the two agents in Example 7.2.6.

- $\Sigma_a(w_1) = \{\{w_1, w_0\}, \{w'_1, w'_0\}\}^{\downarrow}$
- $\Sigma_a(w_0) = \{\{w_1, w_0\}, \{w'_1, w'_0\}\}^{\downarrow}$
- $\Sigma_a(w'_1) = \{\{w_1, w_0\}, \{w'_1, w'_0\}\}^{\downarrow}$
- $\Sigma_a(w'_0) = \{\{w_1, w_0\}, \{w'_1, w'_0\}\}^{\downarrow}$
- $\Sigma_b(w_1) = \{\{w_1, w_0\}\}^{\downarrow}$
- $\Sigma_b(w_0) = \{\{w_1, w_0\}\}^{\downarrow}$
- $\Sigma_b(w'_1) = \{\{w'_1\}\}^{\downarrow}$
- $\Sigma_b(w'_0) = \{\{w'_0\}\}^{\downarrow}$

Now, notice that at w'_0 and w'_1, the information available to b settles $?p$, while this is not the case at w_1 and w_0. Thus, $\Box_b ?p$ is true at w'_1 and w'_2, while $\neg\Box_b ?p$ is true at w_1 and w_0. Since both $\Box_b ?p$ and $\neg\Box_b ?p$ are truth-conditional, we have:

- $M, \{w'_1, w'_0\} \models \Box_b ?p$
- $M, \{w_1, w_0\} \models \neg\Box_b ?p$

Thus, both states $\{w_1, w_0\}$ and $\{w'_1, w'_0\}$ settle, in different ways, the question whether b knows whether p. This means that both support the formula $?\Box_b ?p$.

Now consider the state of agent a at any world w. First, notice that the information of a at w does not settle the question $?\Box_b ?p$. For, $\sigma_a(w) = W$ contains worlds where $\Box_b ?p$ is true and worlds where it is false. Thus, we have:

$$M, w \models \neg\Box_a ?\Box_b ?p$$

However, notice that resolving a's issues implies settling $?\Box_b ?p$. To see this, consider any state $s \in \Sigma_a(w)$. This state must be included in either $\{w_1, w_0\}$ or $\{w'_1, w'_0\}$. By persistency, it follows that $s \models ?\Box_b ?p$. This means that we have:

$$M, w \models \boxplus_a ?\Box_b ?p$$

the fact that inquisitive *epistemic* models satisfy factivity and introspection. These assumptions ensure that (i) the sets $\{\sigma_a(w) \mid w \in W\}$ form a partition of W; (ii) for any $w \in W$, the information state $\sigma_a(w)$ is the unique cell of this partition which includes w; and (iii) the inquisitive state $\Sigma_a(w)$ is the same for all worlds w within one partition cell. If we had been working with a more general class of inquisitive modal models, it might not have been possible to give a single picture which simultaneously represents the state of the agent at each world.

(a) agent a (b) agent b

Figure 7.4: The inquisitive state maps of the two agents in Example 7.2.7.

Since $\boxtimes_a\varphi$ is defined as $\neg\Box_a\varphi \wedge \boxplus_a\varphi$, it follows that for any world $w \in W$:

$$M, w \models \boxtimes_a ?\Box_b ?p$$

This shows that, in any world in M, a is wondering whether b knows whether p. If, say, the actual world is w'_1, then our model describes the following situation: p is true and b knows this; a doesn't know whether b knows whether p, and she wonders whether this is the case.

7.2.7. Example. [a wonders whether b wonders whether p]
Now consider a model M which is exactly like the one of the previous example, except that the state map of agent b is different, as illustrated in Figure 7.4. Now, the inquisitive state of b at w'_1 and w'_0 is the following one:

- $\Sigma_b(w'_1) = \Sigma_b(w'_0) = \{\{w'_1\}, \{w'_0\}\}^{\downarrow}$

That is, at w'_1 and w'_0, b considers both worlds w'_0 and w'_1 possible, and he wonders which one is the actual world. It is easy to verify that at the worlds w'_1 and w'_0, the formula $\boxtimes_b ?p$ is true, while at the worlds w_1 and w_0 it is false. Thus, we have:

- $M, \{w_1, w_0\} \models \neg\boxtimes_b ?p$
- $M, \{w'_1, w'_0\} \models \boxtimes_b ?p$

Thus, both $\{w_1, w_0\}$ and $\{w'_1, w'_0\}$ settle, in different ways, the question $?\boxtimes_b ?p$.

Now consider agent a. At any world, a's information state does not determine whether b wonders whether p. For, $\sigma_a(w) = W$ contains the worlds w'_1, w'_0, where $\boxtimes_b ?p$ is true, as well as worlds w_1, w_0, where $\boxtimes_b ?p$ is false. Hence, we have:

$$M, w \models \neg\Box_a ?\boxtimes_b ?p$$

On the other hand, all the states where the agent's issues are settled ($\{w_1, w_0\}$, $\{w'_1, w'_0\}$, and their subsets) are states that settle the question $?\boxtimes_b ?p$. This gives:

$$M, w \models \boxplus_a ?\boxtimes_b ?p$$

Combining these two things we obtain that for any $w \in W$, we have:

$$M, w \models \boxtimes_a ?\boxtimes_b ?p$$

This shows that in any world in M, a is wondering whether b wonders whether p. If the actual world is, say, w_1', the situation may be described as follows: p is true; agent b does not know whether p is true and wonders about this; agent a does not know that b wonders whether p, and wonders whether this is the case.

7.2.4 Common knowledge, common issues

In standard doxastic and epistemic logic, it is possible to consider not only some agents' individual information, but also various notions of group information which play an important role in the analysis of multi-agent epistemic scenarios.

One notion that is of particular importance is that of *common information*, i.e., the information that is publicly shared among the members of a group. We will focus here on the properly epistemic setting, in which the group's common information is referred to as *common knowledge*.[11]

A sentence α is common knowledge among a group of agents in case every agent a knows that α, and every agent a knows that every agent b knows that α, and every agents a knows that every agent b knows that every agent c knows that α, and so on ad infinitum. Thus, a modality $\Box_*$ for common knowledge should obey the following condition for any model M and world w:

$$M, w \models \Box_* \alpha \iff M, w \models \Box_{a_1} \Box_{a_2} \dots \Box_{a_n} \alpha \text{ for any } a_1, \dots, a_n \in \mathcal{A},\ n \geq 1$$

This result can be achieved in a very elegant way. Given a standard epistemic model $M = \langle W, \{\sigma_a \mid a \in \mathcal{A}\}, V\rangle$, we define a common knowledge map σ_*, determined by the individual maps $\{\sigma_a \mid a \in \mathcal{A}\}$, which describes what the group's common knowledge is at any given world.

7.2.8. Definition. [Common knowledge map]
Given an epistemic model $M = \langle W, \{\sigma_a \mid a \in \mathcal{A}\}, V\rangle$, the group's common knowledge map $\sigma_* : W \to \wp(W)$ is defined as follows:

$$\begin{aligned}\sigma_*(w) = \{\, v \mid\ & \text{there exist } u_0, \dots, u_n \in W \text{ and } a_0, \dots, a_n \in \mathcal{A} \\ & \text{such that } u_0 = w,\ u_{i+1} \in \sigma_{a_i}(u_i) \text{ for all } i < n, \text{ and } v \in \sigma_{a_n}(u_n)\}\end{aligned}$$

It is easy to see that, like the individual epistemic maps σ_a, also the common knowledge map σ_* satisfies factivity and introspection.

Now, just like an agent's individual knowledge modality $\Box_a$ is interpreted by means of the agent's epistemic state map σ_a, so the common knowledge modality

[11] In principle, the discussion in this section applies to the doxastic case as well. However, cases of conflicting beliefs introduce some subtle issues have to be left for future work.

is interpreted by means of the common knowledge map σ_*: $\Box_*\alpha$ is true at a world w in case α follows from the group's common knowledge at w.

$$M, w \models \Box_*\alpha \iff M, v \models \alpha \text{ for all } v \in \sigma_*(w)$$

It is easy to verify that, indeed, the modality $\Box_*$ so defined conforms to the above condition. What this shows is that any epistemic model M for a group $\mathcal{A}$ of agents also carries with it a derived map σ_*, describing the group's common knowledge at each world, and that this map can be used to interpret a common knowledge modality.[12]

In inquisitive epistemic logic, the private state of an agent consists not only of a certain body of information, but also of a certain body of issues. Analogously, when we think of what is publicly shared among the group of agents, it is natural to consider not only the group's common information, but also the group's common issues, that is, those issues that are open for the group as a whole. To get a handle on this notion, it is useful to start from considering what a public version of the inquisitive modality $\boxplus$ should satisfy. Intuitively, we should have $\boxplus_*\varphi$ in case $\boxplus_a\varphi$ holds for all agents a, and all agents b know that this is the case, and all agents c know that all agents b know, and so on. That is, the modality $\boxplus_*$ should satisfy the following condition:

$$M, w \models \boxplus_*\varphi \iff M, w \models \Box_{a_1} \dots \Box_{a_{n-1}} \boxplus_{a_n} \varphi \text{ for any } a_1, \dots, a_n \in \mathcal{A},\ n \geq 1$$

Now, recall that $\Box_a$ and $\boxplus_a$ are equivalent on truth-conditional formulas (Proposition 7.1.11); since modal formulas are truth-conditional, it follows that the above condition on $\boxplus_*$ can be made completely analogous to the condition we had for the common knowledge modality $\Box_*$.

$$M, w \models \boxplus_*\varphi \iff M, w \models \boxplus_{a_1} \dots \boxplus_{a_n} \varphi \text{ for any } a_1, \dots, a_n \in \mathcal{A},\ n \geq 1$$

Now, in the standard case, it was possible to achieve the desired result by defining a common knowledge map σ_*, and by associating the common knowledge modality with this map. Is it possible to have an analogous construction in our case? That is, is there a way to (i) define a *public state map* Σ_* which describes at each world the group's public state, consisting of common knowledge and common issues, and then (ii) interpret the modality $\boxplus_*$ by means of this map, so that the above condition is satisfied? The answer is *yes*. The definition of the public state map Σ_* is in fact quite similar to that of the common knowledge map σ_*.

[12]For simplicity, here we focus on the common knowledge modality for the whole group. In fact, this is only a particular case: for any set of agents $\mathcal{G} \subseteq \mathcal{A}$, we can define a map $\sigma_*^{\mathcal{G}}$ which describes at any world what information is common knowledge among the agents in $\mathcal{G}$. With this map we can then associate a corresponding $\mathcal{G}$-common knowledge modality $\Box_*^{\mathcal{G}}$. The discussion in the present section can be adapted straightforwardly if we wanted to generalize this group-relative common knowledge construction to the inquisitive setting.

7.2.9. Definition. [Public state map]
Given an inquisitive epistemic model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$, the group's public state map $\Sigma_* : W \to \Pi_W$ is defined as follows:

$$\begin{aligned}\Sigma_*(w) = \{\, s \mid\ & \text{there exist } v_0, \dots, v_n \in \mathcal{W} \text{ and } a_0, \dots, a_n \in \mathcal{A} \\ & \text{such that } v_0 = w,\ v_{i+1} \in \sigma_{a_i}(v_i) \text{ for all } i < n, \text{ and } s \in \Sigma_{a_n}(v_n)\}\end{aligned}$$

This definition can be spelled out in a simpler way in terms of the common knowledge map σ_*: the states $s \in \Sigma_*(w)$ can be characterized as being precisely those that belong to the state $\Sigma_a(v)$ of some agent in some world which is compatible with the group's public information.

7.2.10. Proposition. $\Sigma_*(w) = \{s \mid s \in \Sigma_a(v)$ *for some* $a \in \mathcal{A}$ *and* $v \in \sigma_*(w)\}$

Proof. Suppose $s \in \Sigma_*(w)$. This means that there is a sequence of worlds $v_0, \dots, v_n$ such that $v_0 = w$, $v_{i+1} \in \sigma_{a_i}(v_i)$ for some agent a_i for each $i < n$, and $s \in \Sigma_{a_n}(v_n)$ for some agent a_n. Now, if $n > 0$, we have $v_n \in \sigma_*(w)$ and the claim is proved. So, suppose $n = 0$: this means that $s \in \Sigma_{a_0}(w)$. Now, in the epistemic case, $w \in \sigma_*(w)$, so the claim is proved. In the doxastic case, we reason as follows: by definition of inquisitive doxastic model we have $\sigma_{a_0}(w) \neq \emptyset$. Let $v \in \sigma_{a_0}(w)$: by introspection we have $\Sigma_{a_0}(v) = \Sigma_{a_0}(w)$, and so $s \in \Sigma_{a_0}(v)$. Moreover, since $\sigma_{a_0}(w) \subseteq \sigma_*(w)$ by construction, we have $v \in \sigma_*(w)$, and we are done. The proof of the converse inclusion is straightforward. □

The following proposition states that, as in standard epistemic logic, the public state map inherits the features of factivity and introspection which characterize the individual state maps. The proof is straightforward.[13]

7.2.11. Proposition.
For any inquisitive epistemic model M, Σ_* satisfies factivity and introspection.

Now, how should one read the definition of the public state $\Sigma_*(w)$? First, the following proposition, which follows easily from Proposition 7.2.10, ensures that the informative content $\mathsf{info}(\Sigma_*(w))$ of the public state at a world coincides precisely with the information state $\sigma_*(w)$ given by the standard common knowledge construction on the underlying Kripke model. In short, the informative content of the group's public state is just the group's common knowledge.

7.2.12. Proposition.
For any inquisitive epistemic model M and world w in M: $\sigma_*(w) = \mathsf{info}(\Sigma_*(w))$

[13] The analogue of this result fails in the doxastic case: even if each individual map Σ_a satisfies consistency and introspection, the public state map Σ_* might fail to satisfy the introspection condition. This is analogous to the situation that we find in standard doxastic logic.

As for the issues encoded by $\Sigma_a(w)$, Proposition 7.2.10 tells us that an information state s settles the group's common issues at w as soon as there is some world compatible with the group's common knowledge where s settles the issues of some agent. In other words, what is *required* in order to settle the group's common issues is only what is commonly known to be required to settle each agent's individual issues. Thus, just like the group's common knowledge is typically much weaker than the knowledge of each individual agent, so the group's common issues are typically much weaker than the issues of each individual agent.

Now that we have a public state map at our disposal, describing the group's common information and common issues, we are in a position to enrich the basic language of inquisitive epistemic logic with some corresponding public modalities $\Box_*$ and $\boxplus_*$. These modalities will be interpreted just like their private counterparts, but now relative to the group's public state.

7.2.13. DEFINITION. [Support conditions for the public modalities]

- $M, s \models \Box_* \varphi \iff$ for all $w \in s$, $M, \sigma_*(w) \models \varphi$
- $M, s \models \boxplus_* \varphi \iff$ for all $w \in s$ and all $t \in \Sigma_*(w)$, $M, t \models \varphi$

Clearly, these clauses make the formulas $\Box_* \varphi$ and $\boxplus_* \varphi$ truth-conditional, with the following truth-conditions.

7.2.14. PROPOSITION (TRUTH-CONDITIONS FOR PUBLIC MODALITIES).

- $M, w \models \Box_* \varphi \iff M, \sigma_*(w) \models \varphi$
- $M, w \models \boxplus_* \varphi \iff M, t \models \varphi$ *for all* $t \in \Sigma_*(w)$

Now we can prove that, indeed, the modalities defined in this way satisfy the expected conditions. Incidentally, notice that this is not completely obvious for $\Box_*$ either; for, although this modality coincides with the standard common knowledge modality when applied to formulas in the language of epistemic logic, it can now be applied to a broader range of formulas, including questions.

7.2.15. PROPOSITION.
Let M be an inquisitive epistemic model, w a world in M, and φ a formula in the language of inquisitive epistemic logic enriched with public modalities. We have:

- $M, w \models \Box_* \varphi \iff M, w \models \Box_{a_1} \dots \Box_{a_n} \varphi$ *for any* $a_1, \dots, a_n \in \mathcal{A}$, $n \geq 1$
- $M, w \models \boxplus_* \varphi \iff M, w \models \boxplus_{a_1} \dots \boxplus_{a_n} \varphi$ *for any* $a_1, \dots, a_n \in \mathcal{A}$, $n \geq 1$

Proof. In both cases, it suffices to spell out the truth-conditions for the relevant formulas and to unpack the definition of Σ_* to see that the left and the right hand side of the equivalence amount to the same thing. $\Box$

Let us now take a look at what these modalities allow us to express. Consider first the case in which a public modality is applied to a truth-conditional formula α. In this case, we know from Corollary 7.1.12 that $\Box_*\alpha$ and $\boxplus_*\alpha$ boil down to the same thing, namely, to the standard common knowledge modality.

$$M, w \models \Box_*\alpha \iff M, w \models \boxplus_*\alpha \iff M, v \models \alpha \text{ for all } v \in \sigma_*(w)$$

So, when applied to a truth-conditional formula α, both $\Box_*\alpha$ and $\boxplus_*\alpha$ simply express the fact that α is common knowledge among the agents. When we consider a question μ, on the other hand, we can consider once again the three possible situations that we discussed when talking about the individual modalities.

1. $M, \sigma_*(w) \models \mu$. This means that the group's common knowledge settles the question μ. This situation is described by the formula $\Box_*\mu$, which we may thus read as "question μ is publicly closed". Thus, for instance, $\Box_*?p$ expresses the fact that the question whether p is publicly closed, that is, it is common knowledge whether p is true or false.

2. $M, \sigma_*(w) \not\models \mu$ but for every $t \in \Sigma_*(w)$, $M, t \models \mu$. This means that the group's common knowledge does not presently settle μ, but settling the group's common issues implies settling μ. In other words, settling μ is part of the group's common goals. This situation can be expressed by means of a common wondering modality $\boxtimes_*$, defined just like its private counterpart: $\boxtimes_*\varphi := \neg\Box_*\varphi \wedge \boxplus_*\varphi$. If μ is a question, we can thus read the formula $\boxtimes_*\mu$ as "question μ is publicly open".

3. For some $t \in \Sigma_*(w)$, $M, t \not\models \mu$. This means that the group's common knowledge does not currently settle μ, and moreover, settling the group's common issues does not imply settling μ. This situation is captured by the formula $\neg\boxplus\mu$, which we will read as "question μ is absent for the group".

As for its private counterpart, the intuitive significance of the public modality $\boxplus_*$ applied to a question μ needs some explanation. The formula $\boxplus_*\mu$ is equivalent to $\Box_*\mu \vee \boxtimes_*\mu$, and thus it expresses that the question μ is *publicly present*, in the sense that either the group's common knowledge settles μ already, or it is a common goal for the group to settle μ.

An interesting feature of the common wondering modality is worth remarking: $\boxtimes_*\mu$ does not—as one may at first expect—entail $\boxtimes_a\mu$ for an individual agent a. If μ is publicly open, this does *not* mean that μ is open for each agent and that this is common knowledge. Rather, it means that it is a public goal for the group to settle μ, and that μ is not already settled by the group's common knowledge. But this does not necessarily mean that μ is not settled by an agent's *private* knowledge, which is required for the truth of $\boxtimes_a\mu$. A moment's reflection reveals that this prediction is a welcome one: for, we want $\boxtimes_*\mu$ to capture the fact that μ is publicly open for the group as such; now, imagine that μ has been publicly raised but not publicly resolved: intuitively, μ should then be publicly open. It does not matter if some agent has private information settling the question: it is only once such private information is publicly shared, and made common knowledge, that the question μ becomes publicly closed, and $\boxtimes_* \mu$ stops being true. In fact, $\boxtimes_* \mu$ may even be the case while *each* individual agent can resolve μ. Just think of a situation in which a question has been raised and each participant, while knowing the answer, does not share it: from the perspective of the group, the question will remain open, while it will not be open for any individual agent.

Summing up, we saw that in inquisitive epistemic logic, the standard common knowledge construction generalizes smoothly to the construction of a *public state map* which describes at each world the state of the group, consisting not only of the group's common knowledge, but also of the group's common issues. Besides the common knowledge modality, in this framework it is possible to interpret a *common wondering* modality, which allows us to describe the issues that are open for the group as a whole—an important but subtle notion, which does not plainly reduce to the common knowledge that the question is open for each agent.

7.3 Axiomatizing inquisitive modal logic

In the previous section we have looked at a specific interpretation of inquisitive modal logic, showing how standard epistemic logic can be enriched with the logical tools needed to represent and reason about the issues that agents are interested in. Let us now come back to the general setting of inquisitive modal logic and investigate in detail the logic of the system InqBM.

7.3.1 Conservativity results

The first thing we are going to see is that inquisitive modal logic InqBM is a conservative extension of the system InqBK of inquisitive *Kripke* modal logic discussed in the previous chapter. To see this, the first thing to notice is that for all $\boxplus$-free formulas, support at a state s of an inquisitive modal model M coincides with support at s in the underlying Kripke model according to InqBK.

7.3.1. Proposition.
Let $\varphi \in \mathcal{L}^K$. For any inquisitive modal model M and any state s in M:

$$M, s \models \varphi \iff M^K, s \models \varphi$$

Moreover, notice that any Kripke model $M = \langle W, \sigma, V \rangle$ can be regarded as a special case of an inquisitive modal model in which the inquisitive proposition $\Sigma(w)$ associated with each world is the powerset of the state $\sigma(w)$.

7.3.2. Definition. [Powerset extension of a Kripke model]
The *powerset extension* of a Kripke model $M = \langle W, \sigma, V\rangle$ is the inquisitive modal model $M^{\wp} = \langle W, \Sigma^{\wp}, V\rangle$ defined by putting $\Sigma^{\wp}(w) := \wp(\sigma(w))$.

If we start from a Kripke model M, it is immediate to check that $(M^{\wp})^K = M$. So, it follows from Proposition 7.3.1 that support of a formula in $\mathcal{L}^{\mathsf{K}}$ is not affected by the move from a Kripke model to its powerset extension.

7.3.3. Corollary.
Let $\varphi \in \mathcal{L}^{\mathsf{K}}$. For any Kripke model M and state s: $M, s \models \varphi \iff M^{\wp}, s \models \varphi$

Putting together Proposition 7.3.1 and Corollary 7.3.3, we obtain the expected conservativity result: if we restrict ourselves to formulas in $\mathcal{L}^K$, it makes no difference whether we look at entailment over Kripke models or inquisitive modal models.

7.3.4. Proposition (Conservativity over InqBK).
If $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{K}}$, $\Phi \models_{\mathsf{InqBM}} \psi \iff \Phi \models_{\mathsf{InqBK}} \psi$

This ensures that we can safely omit subscripts from the entailment relation: we will just use $\models$ for entailment in InqBM; when the relevant formulas are in $\mathcal{L}^{\mathsf{K}}$, this simply coincides with entailment in InqBK; and of course, if the relevant formulas are purely propositional, then entailment coincides with entailment in InqB.

Besides the conservativity of InqBM over less expressive inquisitive systems, there is another kind of conservativity which is worth stating, namely, conservativity over standard modal logic. Indeed, we know from Proposition 7.1.15 that classical (i.e., $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-free) formulas in InqBM are truth-conditional. We know from Chapter 2 that the truth-conditional behavior of the connectives is standard, and we know from Corollary 7.1.12 that the truth-conditional behavior of both modalities is standard as well, provided the argument to which they apply is truth-conditional. This means that all classical formulas in InqBM are truth-conditional, with truth-conditions in accordance with standard modal logic, when both modalities $\Box$ and $\boxplus$ are interpreted as the universal modality. Hence, the classical fragment of InqBM coincides with the minimal normal modal logic K.

7.3.5. Proposition (Conservativity over K).
Let $\Gamma \cup \{\alpha\} \subseteq \mathcal{L}^{\mathsf{M}}_c$. Let $\alpha^{\Box}$ be the formula obtained by replacing all occurrences of $\boxplus$ with $\Box$, and let $\Gamma^{\Box} = \{\beta^{\Box} \mid \beta \in \Gamma\}$. Then:

$$\Gamma \models \alpha \iff \Gamma^{\Box} \models_{\mathsf{K}} \alpha^{\Box}$$

7.3.2 Properties of the modality $\Box$

Let us now zoom in on the logical features of our modal operators. Let us start from the Kripke modality $\Box$. In the previous chapter we saw that the following logical properties completely characterize this operator.[14]

- $\Box\!\to$ distributivity: $\Box(\varphi \to \psi) \models \Box\varphi \to \Box\psi$
- $\Box\,\mathbb{V}$ distributivity: $\Box(\varphi \,\mathbb{V}\, \psi) \models \Box\varphi \vee \Box\psi$
- $\Box$ monotonicity: if $\Phi \models \psi$, then $\Box\Phi \models \Box\psi$, where $\Box\Phi = \{\Box\varphi \mid \varphi \in \Phi\}$

It is easy to check that these properties of $\Box$ are not affected by the move to the richer language $\mathcal{L}^{\mathsf{M}}$. Moreover, perhaps unexpectedly, in the context of InqBM the modality $\Box$ is actually definable in terms of $\boxplus$.

7.3.6. Proposition ($\Box$ is definable from $\boxplus$).
Let $\varphi \in L^M$ and let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. Then:

$$\Box\varphi \equiv \boxplus\alpha_1 \vee \cdots \vee \boxplus\alpha_n$$

Proof. We know from Corollary 7.1.27 that $\varphi \equiv \alpha_1 \,\mathbb{V}\, \dots \,\mathbb{V}\, \alpha_n$, whence $\Box\varphi \equiv \Box(\alpha_1 \,\mathbb{V}\, \dots \,\mathbb{V}\, \alpha_n)$. By the distributivity of $\Box$ over $\mathbb{V}$ (Proposition 6.2.7) we have $\Box(\alpha_1 \,\mathbb{V}\, \dots \,\mathbb{V}\, \alpha_n) \equiv \Box\alpha_1 \vee \cdots \vee \Box\alpha_n$. Since resolutions are declaratives and declaratives are truth-conditional, Proposition 7.1.11 gives us $\Box\alpha_i \equiv \boxplus\alpha_i$ for each i, whence $\Box\alpha_1 \vee \cdots \vee \Box\alpha_n \equiv \boxplus\alpha_1 \vee \cdots \vee \boxplus\alpha_n$. ☐

Notice that the definition of $\Box$ in terms of $\boxplus$ is not a uniform one, but depends on the specific formula to which $\Box$ is applied. We are thus in the same situation that we discussed in Chapter 2 with regard to the connectives $\wedge$ and $\to$: on the one hand, we do not have a way of defining the operation performed by $\Box$ on inquisitive meanings by means of the operations performed by $\boxplus$ and by the connectives; on the other hand, each particular output of this operation can be obtained without resorting to this operation.

The definability of $\Box$ implies that expanding propositional inquisitive logic with the modality $\boxplus$ alone is sufficient to obtain a system which is as expressive as InqBM. To make this observation more precise, let us write $\mathcal{L}^{\mathsf{M}^-}$ for the language which is obtained by expanding InqB only with the modality $\boxplus$, and let InqBM^- be the corresponding fragment of InqBM. We can then define a map $(\cdot)^- : \mathcal{L}^{\mathsf{M}} \to \mathcal{L}^{\mathsf{M}^-}$ which turns any formula in $\mathcal{L}^{\mathsf{M}}$ into an equivalent $\Box$-free formula.

7.3.7. Definition. [$\Box$-free variant of a modal formula]
For $\varphi \in \mathcal{L}^{\mathsf{M}}$, the formula φ^- is defined inductively as follows.

[14] Strictly speaking, it is these properties together with truth-conditionality: $\neg\neg\Box\alpha \models \Box\alpha$. This is because, although $\neg\neg\Box\alpha \vdash \Box\alpha$ is not stated as a separate rule, it is indirectly stipulated in the proof system by making $\Box\alpha$ a declarative, and thus fit for the $\neg\neg$-elimination rule.

- $p^- = p$
- $\bot^- = \bot$
- $(\varphi \circ \psi)^- = \varphi^- \circ \psi^-$ for $\circ \in \{\wedge, \to, \mathbin{\vee\!\!\!\vee}\}$
- $(\boxplus\varphi)^- = \boxplus\varphi^-$
- $(\Box\varphi)^- = \boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n$ where $\{\alpha_1, \dots, \alpha_n\} = \mathcal{R}(\varphi^-)$

Notice that, since the resolutions of a formula in $\mathcal{L}^{\mathsf{M}^-}$ are themselves in $\mathcal{L}^{\mathsf{M}^-}$, the formula φ^- is always guaranteed to be in $\mathcal{L}^{\mathsf{M}^-}$. Moreover, it is easy to check inductively that φ^- is always equivalent to the original formula φ, which shows that the fragment InqBM^- is as expressive as the full system InqBM.

7.3.8. Proposition. *For all $\varphi \in \mathcal{L}^{\mathsf{M}}$, $\varphi \equiv \varphi^-$*

For the moment, we will keep $\Box$ as a primitive and investigate the logic of the full system InqBM. However, later on in this section we will see that the expressively equivalent system InqBM^- admits a surprisingly minimal axiomatization, essentially constituting an inquisitive analogue of the minimal normal modal logic K.

7.3.3 Properties of the modality $\boxplus$

Let us now turn our attention to the properties of the inquisitive modality $\boxplus$. We have seen that $\boxplus$ differs from $\Box$ in one crucial respect: unlike $\Box$, $\boxplus$ does *not* distribute over inquisitive disjunction, for instance, $\boxplus?p \not\equiv \boxplus p \vee \boxplus\neg p$. What is more, we saw that there is in general no way of paraphrasing away an occurrence of inquisitive disjunction within the scope of $\boxplus$. The inquisitive modality $\boxplus$ allows us to express genuine properties of issues, but to express such properties, we need to have questions at our disposal. Thus, the effect of $\boxplus$ on questions is not reducible to the effect of $\boxplus$ on truth-conditional formulas.

Other than distributivity over $\mathbin{\vee\!\!\!\vee}$, however, $\boxplus$ shares all the properties of $\Box$. First, we have already seen that, just like modal formulas headed by $\Box$, modal formulas headed by $\boxplus$ are always truth-conditional, which means that the double negation law as well as the $\mathbin{\vee\!\!\!\vee}$-split property holds for these formulas. Second, like $\Box$, the inquisitive modality $\boxplus$ distributes over implication.

7.3.9. Proposition (Distributivity of $\boxplus$).
For all $\varphi, \psi \in \mathcal{L}^{\mathsf{M}}$: $\boxplus(\varphi \to \psi) \models \boxplus\varphi \to \boxplus\psi$

Proof. Since both formulas involved in this entailment are truth-conditional, we only have to show that if $\boxplus(\varphi \to \psi)$ is true at a world, so is $\boxplus\varphi \to \boxplus\psi$. So, let $M = \langle W, \Sigma, V\rangle$ be an inquisitive modal model and let w be a world such that

$M, w \models \boxplus(\varphi \to \psi)$. To conclude that $M, w \models \boxplus\varphi \to \boxplus\psi$, we just have to show that, if $M, w \models \boxplus\varphi$, then $M, w \models \boxplus\psi$.

So, let us suppose $M, w \models \boxplus\varphi$. Now consider any $t \in \Sigma(w)$: since $M, w \models \boxplus(\varphi \to \psi)$ we must have $M, t \models \varphi \to \psi$, and since $M, w \models \boxplus\varphi$ we must have $M, t \models \varphi$: by the support clause for implication, this ensures that $M, t \models \psi$. Finally, as this is true for all $t \in \Sigma(w)$, we have $M, w \models \boxplus\psi$, as we wanted. $\square$

Furthermore, like $\Box$, the inquisitive modality $\boxplus$ is monotonic. The proof of this fact is straightforward, and omitted.

7.3.10. Proposition (Monotonicity of $\boxplus$).
For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{M}}$: if $\Phi \models \psi$, then $\boxplus\Phi \models \boxplus\psi$, where $\boxplus\Phi = \{\boxplus\varphi \mid \varphi \in \Phi\}$

It turns out that these simple—in fact utterly standard—logical properties suffice to completely characterize the modality $\boxplus$. Thus, while $\boxplus$ operates in a crucially richer semantic space than Kripke modalities do, enabling us to express things that are not expressible by means of standard Kripke modalities, the fundamental proof-theoretic features of $\boxplus$ are identical to those of a standard Kripke modality: essentially, they just amount to distributivity and monotonicity.

7.3.4 Proof system

For the sake of consistency with the previous chapters, we will work with a system of natural deduction, but it would be easy to replace this by an equivalent Hilbert style system if desired. The rules for the propositional connectives will be the familiar ones (see Figure 6.2), with the important stipulation that the formula α occurring in the rules $\vee\!\!\!\vee$-split and $\neg\neg$-elimination ranges over the set of declaratives $\alpha \in \mathcal{L}^{\mathsf{M}}_{!}$.

The inference rules for the modal operators are described in Figure 7.5. The rules for the Kripke modality $\Box$ will be the ones discussed in the previous chapter for the logic InqBK. As for the modality $\boxplus$, all that is needed are the rules of distributivity over implication, and monotonicity. As in the case of the modality $\Box$, the monotonicity rule comes with an important restriction: $\varphi_1, \dots, \varphi_n$ should be the only undischarged assumptions in the sub-proof leading to ψ; if we did not impose this condition, the rule would not be sound, allowing us, e.g., to derive $\boxplus p$ from p.[15] Finally, our system contains a rule witnessing the fact that the two modalities $\boxplus$ and $\Box$ coincide on declaratives.

Let us write $\vdash_{\mathsf{InqBM}}$ for derivability in the system consisting of the rules for propositional connectives plus the rules of Figure 7.5, omitting the subscript when

[15] Also, it is worth remarking that we could just as well restrict the monotonicity rule to the case where we have zero premises, corresponding to the rule of *necessitation* standardly used in modal logic. For, it is easy to see that the more general monotonicity rule is derivable from necessitation and distributivity. However, working with monotonicity seems equally natural, and it is slightly handier in practice.

$\boxplus \rightarrow$ distributivity

$$\frac{\boxplus(\varphi \rightarrow \psi)}{\boxplus\varphi \rightarrow \boxplus\psi}$$

$\boxplus\Box$ coincidence on declaratives

$$\frac{\boxplus\alpha}{\Box\alpha}\ (\alpha \in \mathcal{L}^{\mathsf{M}}_{!})$$

$\Box \rightarrow$ distributivity

$$\frac{\Box(\varphi \rightarrow \psi)}{\Box\varphi \rightarrow \Box\psi}$$

$\Box \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ distributivity

$$\frac{\Box(\varphi \vee\!\!\vee \psi)}{\Box\varphi \vee \Box\psi}$$

$\Box$ monotonicity

$$\frac{\begin{array}{c}[\varphi_1] \quad \dots \quad [\varphi_n] \\ \vdots \\ \psi\end{array} \qquad \Box\varphi_1 \quad \dots \quad \Box\varphi_n}{\Box\psi}$$

$\boxplus$ monotonicity

$$\frac{\begin{array}{c}[\varphi_1] \quad \dots \quad [\varphi_n] \\ \vdots \\ \psi\end{array} \qquad \boxplus\varphi_1 \quad \dots \quad \boxplus\varphi_n}{\boxplus\psi}$$

Figure 7.5: Rules for the two modalities in InqBM. In the monotonicity rules, $\varphi_1, \dots, \varphi_n$ must be the *only* undischarged assumptions in the sub-proof leading to ψ. Notice that the rule of $\Box\boxplus$ coincidence on declaratives makes $\Box\alpha$ and $\boxplus\alpha$ inter-derivable when α is a declarative.

no confusion arises. As usual, $P : \Phi \vdash \psi$ means that P is a proof whose undischarged assumptions are included in Φ and whose conclusion is ψ, and $\varphi \dashv\vdash \psi$ means that φ and ψ are inter-derivable. Our discussion so far shows the soundness of all the rules for modalities in our proof system. Given the soundness of the rules for the connectives, this implies the soundness of the whole proof system.

7.3.11. Proposition (Soundness).
For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{M}}$, $\Phi \vdash \psi$ implies $\Phi \models \psi$.

We now turn to proving that this system is also complete for InqBM.

7.3.5 Completeness

In order to establish the completeness of our system, we follow the same general strategy as in the previous chapters: first, we will establish a tight connection between derivability among formulas in InqBM and derivability among their resolutions: we will see that, as for the systems considered in the previous chapters, φ derives ψ in case each resolution of φ derives a corresponding resolution of ψ. Second, we will construct a canonical inquisitive modal model, whose worlds are complete theories made out of truth-conditional formulas. Third, we will prove the Support Lemma, connecting support at a state S of this canonical model to derivability from the "syntactic common core" $\bigcap S$ of the theories in S. Finally, we will use the Support Lemma to show that, whenever $\Phi \nvdash \psi$, the non-entailment $\Phi \not\models \psi$ is witnessed by some information state S in this canonical model.

Let us start out by showing that for InqBM we have a resolution algorithm, analogous to that discussed in detail in Chapter 3. That is, let us show that if $P : \Phi \vdash \psi$ and $\Gamma \in \mathcal{R}(\Phi)$, P can be used to construct a corresponding proof $Q : \Gamma \vdash \alpha$ whose conclusion is a resolution of ψ.

7.3.12. Lemma.
Suppose $\Phi \vdash \psi$. Then for all $\Gamma \in \mathcal{R}(\Phi)$ there exists $\alpha \in \mathcal{R}(\psi)$ such that $\Gamma \vdash \alpha$.

Proof. As usual, the proof is by induction on the complexity of a proof $P : \Phi \vdash \psi$. The basis case is obvious: if ψ is one of the formulas in Φ, then the claim follows immediately by definition of resolutions of a set. For the inductive step, we must consider each inference rule in our system. For rules involving the connectives, the argument goes as in the proof of the corresponding result for propositional logic (Theorem 3.2.1). For the rules concerning the modalities, too, the steps are very similar to those spelled out in the previous chapter for Lemma 6.4.12. We retrace these arguments here for the sake of exhaustiveness.

- Suppose ψ was obtained by a rule (r) which is either one of the three distributivity rules, or the rule of $\Box\boxplus$ coincidence on declaratives. Now, notice that each of these rules allows us to infer a declarative from another.

To keep this in mind, let us write α instead of ψ. Moreover, let us write β for the declarative from which α was obtained, and $P' : \Phi \vdash \beta$ for the immediate sub-proof of P.

Now, take any $\Gamma \in \mathcal{R}(\Phi)$. By the induction hypothesis, there is some $Q' : \Gamma \vdash \beta'$ for some $\beta' \in \mathcal{R}(\beta)$. However, since β is a declarative, we know that $\mathcal{R}(\beta) = \{\beta\}$, so in fact we must have $Q' : \Gamma \vdash \beta$. Now, we know that α can be inferred from β by means of the rule (r) of our system; so, extending Q' with an application of (r) we get a proof $Q : \Gamma \vdash \alpha$. Since α is a declarative, we have $\mathcal{R}(\alpha) = \{\alpha\}$, so Q is a proof of the kind we need.

- Suppose $\psi = \blacksquare\chi$ was obtained by one of the monotonicity rules, where $\blacksquare \in \{\Box, \boxplus\}$. This means that the immediate subproofs of P are a proof $P' : \varphi_1, \dots, \varphi_n \vdash \psi$ and, for $1 \leq i \leq n$, a proof $P_i : \Phi \vdash \blacksquare\varphi_i$.

  Now let $\Gamma \in \mathcal{R}(\Phi)$. Since $\mathcal{R}(\blacksquare\varphi_i) = \{\blacksquare\varphi_i\}$, the induction hypothesis applied to the proof P_i tells us that we have a proof $Q_i : \Gamma \vdash \blacksquare\varphi_i$. Now we can apply $\blacksquare$-monotonicity to the proof P' and the proofs $Q_1, \dots, Q_n$, obtaining a proof $Q : \Gamma \vdash \blacksquare\psi$. Since $\mathcal{R}(\blacksquare\psi)\{\blacksquare\psi\}$, Q is a proof with the desired features. $\Box$

Notice that, since a set of declaratives is always the unique resolution of itself, the previous lemma has the following corollary: if a set of declaratives derives a formula, it must also derive some specific resolution of it.

7.3.13. Lemma (Provable resolution split).
Let $\Gamma \subseteq \mathcal{L}^{\mathsf{M}}_{!}$ be a set of declaratives. If $\Gamma \vdash \psi$, then $\Gamma \vdash \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

Next, we show that the normal form result of Corollary 7.1.27, which represents a formula as an inquisitive disjunction of declaratives, is provable in our system.

7.3.14. Lemma (Provability of the normal form).
For all $\varphi \in \mathcal{L}^{\mathsf{M}}$, $\varphi \dashv\vdash \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$.

Proof. The proof is by induction on φ. The basic cases and the inductive steps for the modalities are trivial, since if α is an atom, the falsum constant, or a modal formula we have $\mathcal{R}(\alpha) = \{\alpha\}$. For the inductive steps for the connectives, see the proof of the analogous lemma in Chapter 3, Lemma 3.3.4. $\Box$

From this, it is easy to show that our system allows us to define $\Box$ in terms of $\boxplus$, according to Proposition 7.3.6.

7.3.15. Lemma (Provable definability of $\Box$).
Let $\varphi \in \mathcal{L}^{\mathsf{M}}$ and $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. We have:

$$\Box\varphi \dashv\vdash \Box\alpha_1 \vee \dots \vee \Box\alpha_n \dashv\vdash \boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n$$

Proof. First, notice that the monotonicity rule ensures that, whenever $\varphi \dashv\vdash \psi$, we also have $\Box\varphi \dashv\vdash \Box\psi$. Since the previous lemma tells us that $\varphi \dashv\vdash \alpha_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} \alpha_n$, it follows that $\Box\varphi \dashv\vdash \Box(\alpha_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} \alpha_n)$.

Now, by $\Box\mathbin{\vee\!\!\!\vee}$-distributivity we have $\Box(\alpha_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} \alpha_n) \vdash \Box\alpha_1 \vee \dots \vee \Box\alpha_n$. Conversely, notice that for all $i \leq n$, $\Box\alpha_i \vdash \Box(\alpha_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} \alpha_n)$ by $\Box$-monotonicity. It is easy to prove that the standard rule of $\vee$-elimination towards a declarative conclusion can be simulated in our system, by means of the rules for $\wedge$ and $\neg$. Using this, we get $\Box\alpha_1 \vee \dots \vee \Box\alpha_n \vdash \Box(\alpha_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} \alpha_n)$. So far, we have thus reached the following conclusion:

$$\Box\varphi \dashv\vdash \Box\alpha_1 \vee \dots \vee \Box\alpha_n$$

Now, each α_i is a declarative; so, the rule of $\boxplus\Box$ coincidence on declaratives gives $\Box\alpha_i \dashv\vdash \boxplus\alpha_i$. From this, it is straightforward to see that $\Box\alpha_1 \vee \dots \vee \Box\alpha_n \dashv\vdash \boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n$, which completes the proof of the claim. $\Box$

Notice that, as a corollary, a Kripke modal formula $\Box\varphi$ always derives the corresponding inquisitive modal formula $\boxplus\varphi$.

7.3.16. Corollary. *For any $\varphi \in \mathcal{L}^{\mathsf{M}}$, $\Box\varphi \vdash \boxplus\varphi$*

Proof. By the previous lemma we know that $\Box\varphi \vdash \boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n$, where $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. Now, $\alpha_i \vdash \varphi$ by Lemma 7.3.14, whence by monotonicity we obtain $\boxplus\alpha_i \vdash \boxplus\varphi$. Finally, since we can simulate the standard elimination rule for $\vee$ towards a declarative conclusion, it follows that $\boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n \vdash \boxplus\varphi$. We thus conclude $\Box\varphi \vdash \boxplus\varphi$. $\Box$

The fact that a formula is provably equivalent to its normal form also allows us to prove the traceable deduction failure lemma for InqBM, stating that, if Φ fails to derive ψ, this can be traced to the fact that some particular resolution Γ of Φ fails to derive ψ. We omit the proof, which is completely analogous to the one spelled out in detail for the propositional case (see Lemma 3.3.7).

7.3.17. Lemma (Traceable deduction failure for InqBM).
If $\Phi \nvdash \psi$, there is a $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \nvdash \psi$.

Together, Lemma 7.3.12 and Lemma 7.3.17 imply the Resolution Lemma for InqBM: a set of formulas Φ derives a formula ψ *if and only if* any resolution of Φ derives some corresponding resolution of ψ.

7.3.18. Lemma (Resolution Lemma for InqBM).
$\Phi \vdash \psi \iff$ *for all $\Gamma \in \mathcal{R}(\Phi)$ there exists some $\alpha \in \mathcal{R}(\psi)$ such that $\Gamma \vdash \alpha$.*

So far, we have thus made sure that the tight proof-theoretic connection between formulas and resolutions that was crucial to our completeness proofs is retained in the setting of inquisitive modal logic. We now want to construct a canonical model M^c for InqBM. To do this, the first question we face is what the possible worlds in this model should be. In the previous sections, we always took possible worlds to be complete theories of classical formulas. However, in the present setting, such theories are not suitable candidates for the role of possible worlds: this is because a complete theory of classical formulas does not specify a complete state of affairs from the perspective of InqBM: e.g., a complete theory of classical formulas need not determine whether $\boxplus ?p$ should be true or false, since $\boxplus ?p$ is not equivalent to any classical formula. Thus, a theory of classical formulas does not contain enough information to determine exactly what formulas the corresponding world in the canonical model should make true. If our interpretation is an epistemic one, we may say that, while a complete theory of classical formulas contains a complete description of the valuation, as well as a description of the agent's information, it lacks a description of the agent's issues, since the formulas describing such issues are not part of the classical fragment of the language.

Fortunately, we have seen that we can still identify a syntactic fragment which is representative of all the truth-conditional meanings expressible in InqBM. This fragment is given by the class $\mathcal{L}_!^{\mathsf{M}}$ of declaratives. Thus, we will take the possible worlds in our canonical model for InqBM to be complete theories of declaratives. Such complete theories specify a state of affairs in all the respects relevant to the interpretation of $\mathcal{L}^{\mathsf{M}}$.

7.3.19. Definition. [Theories of declaratives]
A set of declaratives $\Gamma \subseteq \mathcal{L}_!^{\mathsf{M}}$ is called a *theory of declaratives* if for any $\alpha \in \mathcal{L}_!^{\mathsf{M}}$, if $\Gamma \vdash \alpha$ then $\alpha \in \Gamma$. A theory of declaratives Γ is said to be *consistent* if $\bot \not\in \Gamma$.

7.3.20. Definition. [Complete theories of declaratives]
A consistent theory of declaratives Γ is said to be *complete* if for any $\alpha \in \mathcal{L}_!^{\mathsf{M}}$, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$.

It is easy to see that, just like complete theories of classical formulas, complete theories of declaratives have the disjunction property.

7.3.21. Lemma.
If Γ is a complete theory of declaratives and $\alpha \vee \beta \in \Gamma$, then $\alpha \in \Gamma$ or $\beta \in \Gamma$.

Moreover, any consistent theory of declaratives can be extended to a complete theory of declaratives by means of the standard Lindenbaum procedure.

7.3.22. Lemma.
If Γ is a consistent theory of declaratives, then $\Gamma \subseteq \Delta$ for some complete theory of declaratives Δ.

Now that we have the universe W^c of our canonical model, it is also clear how the canonical valuation function V^c should be defined: an atom p should be true at a world Γ just in case it belongs to Γ. We are then left with the task of deciding how the canonical state map Σ^c should be defined. Now, Σ^c should map each world $\Gamma \in W^c$ to an inquisitive proposition $\Sigma^c(\Gamma)$ over W^c; this proposition will be a set of information states in W^c, i.e., a set of subsets $S \subseteq W^c$. But how can we see from the formulas in Γ which states S should belong to $\Sigma^c(\Gamma)$?

To answer this, let us look at the role that the state $\Sigma^c(\Gamma)$ will have to play in our completeness proof. As in the previous chapters, our strategy will be to connect provability to support in the canonical model via a Support Lemma stating that support at a state S amounts to derivability from the intersection $\bigcap S$ of the theories in S. Now consider a formula of the form $\boxplus\varphi$. If the Support Lemma is to hold, we must have the following, where the first equivalence is due to the fact that $\boxplus\varphi$ is a declarative, and Γ is closed under deduction of declaratives:

$$\begin{aligned}
\boxplus\varphi \in \Gamma \quad &\iff \quad \Gamma \vdash \boxplus\varphi \\
&\iff \quad \bigcap\{\Gamma\} \vdash \boxplus\varphi \\
&\iff \quad M^c, \Gamma \models \boxplus\varphi \\
&\iff \quad \text{for all } T \in \Sigma^c(\Gamma),\ M^c, T \models \varphi \\
&\iff \quad \text{for all } T \in \Sigma^c(\Gamma),\ \bigcap T \vdash \varphi
\end{aligned}$$

Now, the left-to-right direction of this equivalence tells us that a state T should only be allowed in $\Sigma^c(\Gamma)$ provided that $\bigcap T$ proves φ whenever $\boxplus\varphi \in \Gamma$. This imposes one condition on states T to be member of $\Sigma^c(\Gamma)$.

On the other hand, if we read the equivalence from right to left, it tells us that, whenever $\boxplus\varphi \notin \Gamma$, we need to have some corresponding witness state $T_\varphi \in \Sigma^c(\Gamma)$ such that $\bigcap T_\varphi \nvdash \varphi$. This condition pushes us to include in $\Sigma^c(\Gamma)$ as many states as we can: after all, the more states $\Sigma^c(\Gamma)$ contains, the easier it will be to find the relevant witness state T_φ whenever $\boxplus\varphi \notin \Gamma$. This suggests to impose no other conditions for membership to $\Sigma^c(\Gamma)$ than what is needed for the left-to-right direction to hold, and define:

$$S \in \Sigma^c(\Gamma) \iff \bigcap S \vdash \varphi \text{ whenever } \boxplus\varphi \in \Gamma$$

This leads to the following definition of canonical model for InqBM.

7.3.23. Definition. [Canonical Model for InqBM]
The canonical model for InqBM is the inquisitive modal model $M^c = \langle W^c, \Sigma^c, V^c \rangle$ defined as follows:

- W^c is the set of complete theories of declaratives in $\mathcal{L}^{\mathsf{M}}$.

- $\Sigma^c : W^c \to \Pi_{W^c}$ is the inquisitive state map defined as follows:

$$\Sigma^c(\Gamma) = \{S \subseteq W^c \mid \bigcap S \vdash \varphi \text{ whenever } \boxplus\varphi \in \Gamma\}$$

  where if $S = \emptyset$, we let $\bigcap S = \mathcal{L}_!^{\mathsf{M}}$.

- $V^c : W^c \times \mathcal{P} \to \{0,1\}$ is the map defined by: $V^c(\Gamma, p) = 1 \iff p \in \Gamma$

Notice that in order for M^c to qualify as an inquisitive modal model, the value of Σ^c applied to any world must be an inquisitive proposition, that is, it must be a non-empty and downward closed set of information states. The next lemma ensures that this is indeed the case.

7.3.24. Lemma (M^c is an inquisitive modal model).
For any $\Gamma \in W^c$, $\Sigma^c(\Gamma)$ is an inquisitive proposition.

Proof. Consider any $\Gamma \in W^c$. First, we show $\Sigma^c(\Gamma) \neq \emptyset$. To see this, notice that $\bigcap \emptyset = \mathcal{L}_!^{\mathsf{M}}$ contains $\bot$. As a consequence, $\bigcap \emptyset$ derives everything; in particular, it derives φ whenever $\boxplus\varphi \in \Gamma$, which ensures $\emptyset \in \Sigma^c(\Gamma)$. So, $\Sigma^c(\Gamma)$ is non-empty.

Now let us show that $\Sigma^c(\Gamma)$ is downward closed. Suppose $S \in \Sigma^c(\Gamma)$ and $T \subseteq S$. Since $T \subseteq S$ we have $\bigcap S \subseteq \bigcap T$, and since $\bigcap S$ derives φ whenever $\boxplus\varphi \in \Gamma$, so will the stronger theory $\bigcap T$, which means that $T \in \Sigma^c(\Gamma)$. □

Second, we show that the inquisitive proposition $\Sigma^c(\Gamma)$ is large enough to ensure that whenever $\boxplus\varphi \notin \Gamma$, there is some witness state $T \in \Sigma^c(\Gamma)$ such that $\bigcap T \nvdash \varphi$—a condition that we saw was needed for the Support Lemma to go through.

7.3.25. Lemma.
Let $\Gamma \in W^c$. If $\boxplus\varphi \notin \Gamma$ there exists a state $T \in \Sigma^c(\Gamma)$ such that $\bigcap T \nvdash \varphi$.

Proof. Suppose $\boxplus\varphi \notin \Gamma$. Put $\Gamma^{\boxplus} = \{\psi \mid \boxplus\psi \in \Gamma\}$. We claim that $\Gamma^{\boxplus} \nvdash \varphi$. Towards a contradiction, suppose $\Gamma^{\boxplus} \vdash \varphi$. Let $\psi_1, \dots, \psi_n \in \Gamma^{\boxplus}$ be assumptions such that $\psi_1, \dots, \psi_n \vdash \varphi$. By monotonicity, we would then have $\boxplus\psi_1, \dots, \boxplus\psi_n \vdash \boxplus\varphi$. Now, $\psi_1, \dots, \psi_n \in \Gamma^{\boxplus}$ means that $\boxplus\psi_1, \dots, \boxplus\psi_n$ are in Γ. Since Γ is closed under deduction of declaratives, we should then have $\boxplus\varphi \in \Gamma$, contrary to assumption.

We have thus proved $\Gamma^{\boxplus} \nvdash \varphi$. By the Resolution Lemma (Lemma 7.3.18), we know that there exists some $\Theta \in \mathcal{R}(\Gamma^{\boxplus})$ which derives no resolution of φ.

Now take any $\alpha \in \mathcal{R}(\varphi)$. Since $\Theta \nvdash \alpha$, the set $\Theta \cup \{\neg\alpha\}$ must be consistent: for, if we had $\Theta, \neg\alpha \vdash \bot$, by the rules for negation it would follow $\Theta \vdash \neg\neg\alpha$, and since α is a declarative, by double negation elimination we would have $\Theta \vdash \alpha$, contrary to assumption. So, $\Theta \cup \{\neg\alpha\}$ is consistent, and Lemma 7.3.22 ensures that it can be extended to some complete theory of declaratives Δ_α.

Now consider the state $T = \{\Delta_\alpha \mid \alpha \in \mathcal{R}(\varphi)\}$. We claim that $T \in \Sigma^c(\Gamma)$ and that $\bigcap T \nvdash \varphi$. To see that $T \in \Sigma^c(\Gamma)$, notice that since $\Theta \subseteq \Delta_\alpha$ for each α, we have $\Theta \subseteq \bigcap T$. Now suppose $\boxplus\psi \in \Gamma$: then $\psi \in \Gamma^{\boxplus}$, and since Θ is a resolution of $\Gamma^{\boxplus}$, it contains some resolution β of ψ. But then, since $\beta \in \Theta \subseteq \bigcap T$ and since $\beta \vdash \psi$ by Lemma 7.3.14, we must also have $\bigcap T \vdash \psi$. We have thus proved that $\bigcap T \vdash \psi$ whenever $\boxplus\psi \in \Gamma$, which means that $T \in \Sigma^c(\Gamma)$.

It remains to be shown that $\bigcap T \not\vdash \varphi$. Towards a contradiction, suppose $\bigcap T \vdash \varphi$. Since $\bigcap T$ is a set of declaratives, Lemma 7.3.13 implies $\bigcap T \vdash \alpha$ for some $\alpha \in \mathcal{R}(\varphi)$. Since $\Delta_\alpha \in T$, we have $\bigcap T \subseteq \Delta_\alpha$, so we would also have $\Delta_\alpha \vdash \alpha$. But this is impossible, since by construction Δ_α contains $\neg\alpha$ and is a consistent theory. Hence, $\bigcap T \not\vdash \varphi$, and our lemma is proved. $\Box$

Recall that the information state $\sigma^c(\Gamma)$ associated with a world Γ is defined as the union of the inquisitive state $\Sigma^c(\Gamma)$. The following lemma gives a direct characterization of which theories $\Delta \in W^c$ belong to $\sigma^c(\Gamma)$.

7.3.26. LEMMA. $\sigma^c(\Gamma) = \{\Delta \mid \alpha \in \Delta$ *whenever* $\boxplus\alpha \in \Gamma$ *and* $\alpha \in \mathcal{L}_!^{\mathsf{M}}\}$

Proof. First, assume $\Delta \in \sigma^c(\Gamma)$. Since $\sigma^c(\Gamma) = \bigcup \Sigma^c(\Gamma)$, this means that $\Delta \in S$ for some state S such that $\bigcap S \vdash \varphi$ whenever $\boxplus\varphi \in \Gamma$. In particular, then, if $\boxplus\alpha \in \Gamma$ we have $\bigcap S \vdash \alpha$, whence also $\Delta \vdash \alpha$. Since Δ a theory of declaratives, it follows that $\alpha \in \Delta$.

Conversely, suppose $\alpha \in \Delta$ whenever $\boxplus\alpha \in \Gamma$ and $\alpha \in \mathcal{L}_!^{\mathsf{M}}$. We claim that the singleton state $\{\Delta\}$ belongs to $\Sigma^c(\Gamma)$, which implies $\Delta \in \bigcup \Sigma^c(\Gamma) = \sigma^c(\Gamma)$. Since $\bigcap\{\Delta\} = \Delta$, to show that $\{\Delta\} \in \Sigma^c(\Gamma)$ we must show for all $\varphi \in \mathcal{L}^{\mathsf{M}}$, $\boxplus\varphi \in \Gamma$ implies $\Delta \vdash \varphi$. So, suppose $\boxplus\varphi \in \Gamma$. Let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. We know from Lemma 7.3.14 that $\varphi \vdash \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$. Using the rules for inquisitive disjunction, it is easy to show that $\alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n \vdash \alpha_1 \vee \dots \vee \alpha_n$. By $\boxplus$-monotonicity we then have $\boxplus\varphi \vdash \boxplus(\alpha_1 \vee \dots \vee \alpha_n)$. But notice that $\alpha_1 \vee \dots \vee \alpha_n$ is a declarative, so by our assumption on Δ we have $\alpha_1 \vee \dots \vee \alpha_n \in \Delta$. Now, since Δ is a complete theory, by Lemma 7.3.21 we must have $\alpha_i \in \Delta$ for some i. But since $\alpha_i \in \mathcal{R}(\varphi)$, we have $\alpha_i \vdash \varphi$ by Lemma 7.3.14, whence $\Delta \vdash \varphi$.

Summing up, then, we have shown that $\varphi \in \Delta$ whenever $\boxplus\varphi \in \Gamma$. This means that $\{\Delta\} \in \Sigma^c(\Gamma)$, which in turn establishes $\Delta \in \sigma^c(\Gamma)$. $\Box$

The following lemma is the analogue of Lemma 7.3.25 for the modality $\Box$: it states that if $\Box\varphi \notin \Gamma$, this is witnessed by the fact that $\bigcap \sigma^c(\Gamma) \not\vdash \varphi$.

7.3.27. LEMMA.
Let $\Gamma \in W^c$. *If* $\Box\varphi \notin \Gamma$, *then* $\bigcap \sigma^c(\Gamma) \not\vdash \varphi$.

Proof. Suppose $\Box\varphi \notin \Gamma$ and let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. By Lemma 7.3.15, we know that $\Box\varphi \dashv\vdash \boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n$, so we have $\boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n \notin \Gamma$. Since Γ is closed under declarative deduction, it must contain none of $\boxplus\alpha_1, \dots, \boxplus\alpha_n$.

By Lemma 7.3.25, since $\boxplus\alpha_i \notin \Gamma$ there is a state $T_i \in \Sigma^c(\Gamma)$ such that $\bigcap T_i \not\vdash \alpha$. Now since $T_i \in \Sigma^c(\Gamma)$ we have $T_i \subseteq \bigcup \Sigma^c(\Gamma) = \sigma^c(\Gamma)$, whence $\bigcap \sigma^c(\Gamma) \subseteq \bigcap T_i$.

Since $\bigcap T_i \nvdash \alpha_i$, also $\bigcap \sigma^c(\Gamma) \nvdash \alpha_i$. But as $\bigcap \sigma^c(\Gamma)$ is a set of declaratives and does not derive any resolution of φ, Lemma 7.3.13 implies $\bigcap \sigma^c(\Gamma) \nvdash \varphi$. $\square$

We now have all the ingredients needed to prove the Support Lemma, connecting support in the canonical model to provability in our proof system.

7.3.28. Lemma (Support Lemma).
For all formulas $\varphi \in \mathcal{L}^{\mathsf{M}}$ and all states $S \subseteq W^c$:

$$M^c, S \models \varphi \iff \bigcap S \vdash \varphi$$

Proof. As usual, the proof is by induction on φ. The basic cases for atomic formulas and $\bot$ and the inductive steps for the connectives proceed exactly as in the propositional case, so we only spell out the inductive steps for the modalities.

- Suppose the claim holds for φ and consider the formula $\boxplus\varphi$. Suppose $\bigcap S \vdash \boxplus\varphi$ and consider any $\Gamma \in S$. Since $\Gamma \in S$, we have $\bigcap S \subseteq \Gamma$, and so also $\Gamma \vdash \boxplus\varphi$. Since $\boxplus\varphi$ is a declarative, this implies $\boxplus\varphi \in \Gamma$ and thus, by definition of Σ^c, it implies $\bigcap T \vdash \varphi$ for all $T \in \Sigma^c(\Gamma)$. But by induction hypothesis, this is equivalent to $M^c, T \models \varphi$ for all $T \in \Sigma^c(\Gamma)$. Since this is true for any $\Gamma \in S$, it follows that we have $M^c, S \models \boxplus\varphi$.

  Conversely, suppose $\bigcap S \nvdash \boxplus\varphi$. Then, $\boxplus\varphi \notin \bigcap S$ which means that $\boxplus\varphi \notin \Gamma$ for some $\Gamma \in S$. Then, Lemma 7.3.25 ensures that there is some $T \in \Sigma^c(\Gamma)$ such that $\bigcap T \nvdash \varphi$. The induction hypothesis tells us that this amounts to saying that $M^c, T \not\models \varphi$. This shows that it is not the case that for all $\Gamma \in S$ and for all $T \in \Sigma^c(\Gamma)$, $M^c, T \models \varphi$, which shows that $M^c, S \not\models \boxplus\varphi$.

- Now suppose the claim holds for φ and consider the formula $\Box\varphi$. First suppose $\bigcap S \vdash \Box\varphi$. Consider any $\Gamma \in S$: since $\bigcap S \subseteq \Gamma$, we have $\Gamma \vdash \Box\varphi$. Let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$: by Lemma 7.3.15 we have $\Box\varphi \dashv\vdash \boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n$. It follows that $\Gamma \vdash \boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n$ and, since Γ is closed under deduction of declaratives, $\boxplus\alpha_1 \vee \dots \vee \boxplus\alpha_n \in \Gamma$. Moreover, by Lemma 7.3.21 we know that Γ has the disjunction property, and so we must have $\boxplus\alpha_i \in \Gamma$ for some α_i. Now, Lemma 7.3.26 ensures that if $\Delta \in \sigma^c(\Gamma)$, we must have $\Delta \vdash \alpha_i$; in turn, since α_i is a declarative, this means that $\alpha_i \in \Delta$. Since this holds for each $\Delta \in \sigma^c(\Gamma)$, we thus have $\alpha_i \in \bigcap \sigma^c(\Gamma)$. Now, since $\alpha_i \in \mathcal{R}(\varphi)$, we have $\alpha_i \vdash \varphi$ by Lemma 7.3.14, so also $\bigcap \sigma^c(\Gamma) \vdash \varphi$. By induction hypothesis, this means that $M^c, \sigma^c(\Gamma) \models \varphi$. Since this is the case for any $\Gamma \in S$, we can conclude that $M^c, S \models \Box\varphi$.

  Conversely, suppose $\bigcap S \nvdash \Box\varphi$. Then $\Box\varphi \notin \bigcap S$, which implies $\Box\varphi \notin \Gamma$ for some $\Gamma \in S$. Now Lemma 7.3.27 gives $\bigcap \sigma^c(\Gamma) \nvdash \varphi$, which by induction hypothesis means that $M^c, \sigma^c(\Gamma) \not\models \varphi$. Thus, it is not the case that for all $\Gamma \in S$, $M^c, \sigma^c(\Gamma) \models \varphi$, which shows that $M^c, S \not\models \Box\varphi$. $\square$

As usual, by restricting the Support Lemma to singletons we obtain the Truth Lemma, stating that truth at a world in M^c amounts to derivability from it.

7.3.29. Corollary (Truth Lemma).
For all formulas $\varphi \in \mathcal{L}^{\mathsf{M}}$ and all worlds $\Gamma \in W^c$:

$$M^c, \Gamma \models \varphi \iff \Gamma \vdash \varphi$$

Finally, we can use the Support Lemma to establish the completeness of our proof system. We omit the argument, which is completely analogous to the one given for the completeness theorem in the propositional case (Theorem 3.3.2).

7.3.30. Theorem (Completeness Theorem for InqBM).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{M}}$,

$$\Phi \models \psi \iff \Phi \vdash \psi$$

This shows that, even though in this section we have abandoned the standard, Kripkean framework for modal logic in favor of a richer semantic framework, which allows us to interpret a more expressive language, the move to this richer framework does not compromise the simplicity and elegance of modal logic.

This point can be reinforced by remarking that the proof system we have used so far is somewhat more complicated than it needs to be. For, many of the rules of this system are concerned with the Kripke modality $\Box$, and with its connection to the inquisitive modality $\boxplus$. However, we have seen above that this modality is actually definable in the system. If we want to get at the core of inquisitive modal logic, we may thus turn to the system InqBM^- in which only the modality $\boxplus$ is a primitive operator, while the modality $\Box$ is used as an abbreviation, according to Proposition 7.3.6. We may then ask what features of $\boxplus$ are needed for completeness. The answer is quite striking: all we need are the standard rules of distributivity and monotonicity.[16] Adding these rules, repeated in Figure 7.6, to the rules for propositional connectives, we obtain a proof system $\vdash^-$ which is complete for entailment in InqBM^-. One way to establish this result is to retrace the steps leading to our completeness theorem, noticing that the only use for rules involving $\Box$ is in proving the inductive step for $\Box$ in the Support Lemma—which is no longer necessary if $\Box$ is defined away. Another simple way to prove the completeness of $\vdash^-$ is to show that, if $\Box$ is taken to be defined by means of the map $(\cdot)^-$, all the rules for $\Box$ in Figure 7.5 may be simulated by means of the rules for $\boxplus$ and the rules for the connectives.

7.3.31. Theorem (Completeness, simplified).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{M}^-}$, $\Phi \models \psi \iff \Phi \vdash^- \psi$

[16] Or, if we prefer, distributivity and necessitation, since monotonicity is then derivable.

$\boxplus\!\to$ distributivity

$$\frac{\boxplus(\varphi \to \psi)}{\boxplus\varphi \to \boxplus\psi}$$

$\boxplus$ monotonicity

$$\frac{\begin{matrix}[\varphi_1] & \dots & [\varphi_n] \\ & \vdots & \\ & \psi & \end{matrix} \qquad \boxplus\varphi_1 \quad \dots \quad \boxplus\varphi_n}{\boxplus\psi}$$

Figure 7.6: If we take $\Box$ to be a defined operator, adding the two rules above to the rules for connectives gives a complete system for inquisitive modal logic. As usual, in the $\boxplus$ monotonicity rule we require $\varphi_1, \dots, \varphi_n$ to be the only undischarged assumptions in the sub-proof leading to ψ.

We have thus reached the conclusion that, in essence, the expressively rich system of inquisitive modal logic can be seen as arising from expanding propositional inquisitive logic by means of a modal operator $\boxplus$ which is characterized completely by distributivity and monotonicity—or, by distributivity and necessitation. This brings out the fact that, from a proof-theoretic point of view, the fundamental inquisitive modality $\boxplus$ behaves just like the universal modality of the minimal modal logic K, which it extends conservatively beyond the truth-conditional realm.

7.4 Beyond the basic modal system

The axiomatization result established in the previous section gives us a complete characterization of the general inquisitive modal logic InqBM, the logic of the class of all inquisitive modal models. However, just as in the case of standard modal logic, particular interpretations of inquisitive modal logic may suggest certain restrictions on the class of suitable modal models, leading to inquisitive modal logics which are stronger than the plain system InqBM.

In Section 7.2.2, we have considered some conditions that one may want to impose on the state maps given a particular interpretation of the framework. The following definition recalls these conditions. Notice that, since they are motivated by a doxastic/epistemic interpretation of inquisitive modal logic, it will be convenient to examine them in the setting of inquisitive *multi*-modal logic, though this will make no substantial difference. Thus, throughout this section, we will assume an arbitrary set $\mathcal{A}$ of labels. Moreover, it will be useful to analyze the condition that we called *introspection* as the conjunction of two distinct conditions, called *positive* and *negative* introspection, as is customary in standard modal logic.

7.4.1. Definition.
Let $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ be an inquisitive multi-modal model. We say that:

- M satisfies *consistency* if for all $w \in W$, $\sigma_a(w) \neq \emptyset$;
- M satisfies *factivity* if for all $w \in W$, $w \in \sigma_a(w)$;
- M satisfies *positive introspection* if $v \in \sigma_a(w)$ implies $\Sigma_a(v) \subseteq \Sigma_a(w)$;
- M satisfies *negative introspection* if $v \in \sigma_a(w)$ implies $\Sigma_a(v) \supseteq \Sigma_a(w)$;

In this section, we consider the question of what inquisitive modal logic arises from restricting the set of admissible models by imposing one or more of these semantic properties. In particular, this will lead us to provide axiomatizations for the two specific flavors of inquisitive modal logic discussed in Section 7.2: *inquisitive doxastic logic* IDL, the logic of models that satisfy consistency as well as positive and negative introspection, and *inquisitive epistemic logic* IEL, the logic of models that satisfy factivity as well as positive and negative introspection. Of course, this is only a first step in a more general investigation of the spectrum of inquisitive modal logics, and of the correspondence between inquisitive modal formulas and features of inquisitive modal frames—a worthwhile enterprise that has to be left for future work.

To start with, it will be useful to have a notation for the logic L that results from imposing zero or more of the above conditions on top of InqBM. For this, let us associate a symbol with each of the above conditions: following the notation for the analogous conditions in standard modal logic, we will write D for consistency, T for factivity, $\mathsf{4}$ for positive introspection, and $\mathsf{5}$ for negative introspection. We will then refer to a logic by adding the corresponding symbol after InqBM: thus, for instance, $\mathsf{InqBMD4}$ is the logic of inquisitive modal models satisfying consistency and positive introspection. However, we will write IDL for inquisitive doxastic logic instead of $\mathsf{InqBMD45}$, and IEL for inquisitive epistemic logic instead of $\mathsf{InqBMT45}$. Entailment with respect to a logic L will be denoted $\models_{\mathsf{L}}$.

Let us start out by showing how each of the above conditions gives rise to some familiar logical validities. Consider first an inquisitive multi-modal model M satisfying consistency. Consider an arbitrary world w in M: by the consistency condition, $\sigma_a(w) \neq \emptyset$, which means that $M, \sigma_a(w) \not\models \bot$. By definition, this gives $M, w \models \neg\Box_a\bot$. Thus, the formula $\neg\Box_a\bot$ is true at any world in M; since it is truth-conditional, it must also be supported at any state in M. Since M was an arbitrary model satisfying consistency, this shows that $\neg\Box_a\bot$ is valid in the logic InqBMD. Moreover, since $\bot$ is truth-conditional, we have $\neg\boxplus_a\bot \equiv \neg\Box_a\bot$, so $\neg\boxplus_a\bot$ must also be a validity of InqBMD. We have thus proved the following.

7.4.2. Proposition.
$\models_{\mathsf{InqBMD}} \neg\blacksquare\bot$ *for* $\blacksquare \in \{\Box_a, \boxplus_a \mid a \in \mathcal{A}\}$

Next, consider an inquisitive multi-modal model M that satisfies factivity, and let α be a declarative formula. Suppose $M, w \models \Box_a\alpha$: since α is truth-conditional, by Corollary 7.1.12 this just means that $M, v \models \alpha$ for all $v \in \sigma_a(w)$. Now, since $w \in \sigma_a(w)$, this implies $M, w \models \alpha$. This means that the formula $\Box_a\alpha \to \alpha$ is true everywhere in M. Since this formula is a declarative, and thus truth-conditional, this ensures that $\Box_a\alpha \to \alpha$ is also supported at any state in M. And since M was an arbitrary model satisfying factivity, this shows that $\Box_a\alpha \to \alpha$ is a validity of the logic InqBT. Moreover, since $\boxplus_a$ and $\Box_a$ coincide on declaratives, the same is true of the formula $\boxplus_a\alpha \to \alpha$. We have thus reached the following conclusion.

7.4.3. Proposition.
$\models_{\mathsf{InqBMT}} \blacksquare\alpha \to \alpha$ *for* $\alpha \in \mathcal{L}^M_!(\mathcal{A})$ *and* $\blacksquare \in \{\Box_a, \boxplus_a \mid a \in \mathcal{A}\}$

In this proposition, the restriction to declaratives is essential: in general, neither the scheme $\Box_a\varphi \to \varphi$ nor the scheme $\boxplus_a\varphi \to \varphi$ are valid on the class of all factive models, as the following example shows.

7.4.4. Example. Let $M = \langle W, \Sigma, V\rangle$ be the factive model defined as follows:

- $W = \{w_0, w_1\}$
- $\Sigma(w_0) = \{\{w_0\}, \emptyset\}, \quad \Sigma(w_1) = \{\{w_1\}, \emptyset\}$
- $V(w_0, p) = 0, \quad V(w_1, p) = 1$

At any world $w \in W$, the state $\sigma(w)$ is a singleton, and so we have $M, \sigma(w) \models ?p$. By definition, this means that $M, W \models \Box ?p$, and since $\Box ?p$ entails $\boxplus ?p$, also that $M, W \models \boxplus ?p$. However, we have $M, W \not\models ?p$, since W contains both a p-world, and a $\neg p$-world. Thus, we have $M, W \not\models \Box ?p \to ?p$ and $M, W \not\models \boxplus ?p \to ?p$. ◁

Under an epistemic interpretation of the modalities, it is easy to see intuitively why $\Box\varphi \to \varphi$ is valid for a statement like p, but fails for a question like $?p$: for, by the factivity of knowledge, one cannot settle that the agent knows that p without also settling that p is the case; however, one may very well settle that the agent knows whether p without settling whether p is the case; this is precisely what happens at the state W in the above model.

Let us now suppose our inquisitive modal model M satisfies the *positive introspection* condition. Suppose $M, w \models \boxplus_a\varphi$. Consider a world $v \in \sigma_a(w)$: since M satisfies positive introspection, we have $\Sigma_a(v) \subseteq \Sigma_a(w)$. Since $M, w \models \boxplus_a\varphi$, it follows easily that $M, v \models \boxplus_a\varphi$. This shows that the formula $\boxplus_a\varphi$ is supported at all $v \in \sigma_a(w)$. Since $\boxplus_a\varphi$ is truth-conditional, it follows that $M, \sigma_a(w) \models \boxplus_a\varphi$, whence we have $M, w \models \boxplus_a\boxplus_a\varphi$. Thus, the implication $\boxplus_a\varphi \to \boxplus_a\boxplus_a\varphi$ is true at any world in M. Since this formula is truth-conditional, this means that it is also supported at any state in M. Since M was an arbitrary model satisfying positive introspection, this shows that $\boxplus_a\varphi \to \boxplus_a\boxplus_a\varphi$ is a validity of the logic $\mathsf{InqBM4}$. By means of a similar argument we can establish that $\Box_a\varphi \to \Box_a\Box_a\varphi$ is also a validity of this logic. We have thus reached the following conclusion.

7.4.5. PROPOSITION.
$\models_{\mathsf{InqBM4}} \blacksquare\varphi \to \blacksquare\blacksquare\varphi$ *for* $\varphi \in \mathcal{L}^{\mathsf{M}}(\mathcal{A})$ *and* $\blacksquare \in \{\Box_a, \boxplus_a \mid a \in \mathcal{A}\}$

Finally, suppose our inquisitive modal model M satisfies *negative introspection*. Suppose $M, w \models \neg \boxplus_a \varphi$. Now, consider a world $v \in \sigma_a(w)$: since M satisfies negative introspection, we have $\Sigma_a(v) \supseteq \Sigma_a(w)$. If we had $M, v \models \boxplus_a\varphi$, it would thus follow $M, w \models \boxplus_a\varphi$, contrary to assumption. Thus, we must have $M, v \models \neg \boxplus_a \varphi$. This shows that the formula $\neg \boxplus_a \varphi$ is true at all $v \in \sigma_a(w)$. As $\neg \boxplus_a \varphi$ is a declarative, it follows from Corollary 7.1.12 that $M, w \models \boxplus_a \neg \boxplus_a \varphi$. This shows that the implication $\neg\boxplus_a\varphi \to \boxplus_a\neg\boxplus_a\varphi$ is true at any world in M, and thus, since it is truth-conditional, it is supported everywhere in M. Since M is an arbitrary model satisfying negative introspection, $\neg\boxplus_a\varphi \to \boxplus_a\neg\boxplus_a\varphi$ is valid in the logic InqBM5. Moreover, a similar argument establishes that $\neg\Box_a\varphi \to \Box_a\neg\Box_a\varphi$ is also a validity of this logic. So, we have the following conclusion.

7.4.6. PROPOSITION.
$\models_{\mathsf{InqBM5}} \neg\blacksquare\varphi \to \blacksquare\neg\blacksquare\varphi$ *for* $\varphi \in \mathcal{L}^{\mathsf{M}}(\mathcal{A})$ *and* $\blacksquare \in \{\Box_a, \boxplus_a \mid a \in \mathcal{A}\}$

Notice that, while the scheme $\boxplus_a\alpha \to \alpha$ rendered valid by factivity is restricted to declaratives, the schemes $\boxplus_a\varphi \to \boxplus_a\boxplus_a\varphi$ and $\neg\boxplus_a\varphi \to \boxplus_a\neg\boxplus_a\varphi$ rendered valid by the introspection conditions concern all formulas $\varphi \in \mathcal{L}^{\mathsf{M}}(\mathcal{A})$. This witnesses the fact that, while factivity is only concerned with information, the introspection conditions concern not only information, but also issues.

Moreover, it is worth remarking here that, if we have *both* positive and negative introspection, then from the introspection schemes for both modalities $\Box_a$ and $\boxplus_a$ we can derive the validity of the following schemes for the wondering modality, which witness the fact that an agent knows just which questions are open for her.

- $\boxtimes_a\mu \to \Box_a\boxtimes_a\mu$
- $\neg\boxtimes_a\mu \to \Box_a\neg\boxtimes_a\mu$

So far, we have looked at some logical repercussions of the four semantic conditions that play a role in the definition of doxastic and epistemic models. We are now going to see that these logical validities are precisely what we need to characterize the logic of the corresponding class of models.

In Figure 7.7, the four validities that we have identified are turned into inference rules for the inquisitive modality $\boxplus$. We will refer to each of these rules using the name of the corresponding condition, preceded by the symbol $\boxplus$. Notice that we do not consider the corresponding inference rules for the Kripke modality $\Box$. This is because they are not needed: it will follow from the results in this section that, in each case, the relevant property of $\Box$ is derivable from the corresponding inference rule for $\boxplus$. This is not too surprising if we keep in mind that $\Box$ is

$\boxplus\mathsf{D}$	$\boxplus\mathsf{T}$	$\boxplus\mathsf{4}$	$\boxplus\mathsf{5}$
$\dfrac{}{\neg\boxplus_a\bot}$	$\dfrac{\boxplus_a\alpha}{\alpha}$	$\dfrac{\boxplus_a\varphi}{\boxplus_a\boxplus_a\varphi}$	$\dfrac{\neg\boxplus_a\varphi}{\boxplus_a\neg\boxplus_a\varphi}$

Figure 7.7: Some inference rules for $\boxplus$ that can be added to InqBM to obtain stronger inquisitive modal logics. In the rule $\boxplus\mathsf{T}$, α ranges over declaratives.

actually definable from $\boxplus$ in InqBM, which implies that the logical properties of $\boxplus$ on a certain class of models completely determine the logical properties of $\Box$.

Now, let L be one of the logics obtained by imposing a subset of the above conditions on inquisitive modal models. Let us write $\vdash_{\mathsf{L}}$ for derivability in the proof system obtained by extending the proof system for InqBM with the rules corresponding to these conditions. Thus, e.g., $\vdash_{\mathsf{InqBMD4}}$ stands for derivability in the proof system obtained by expanding the proof system for InqBM with the rules $\boxplus\mathsf{D}$ and $\boxplus\mathsf{4}$.[17]

We have just seen that each inference rule in Figure 7.7 is sound with respect to the associated inquisitive frame condition. It follows that the proof system $\vdash_{\mathsf{L}}$ is always sound for the corresponding entailment relation $\models_{\mathsf{L}}$.

7.4.7. Proposition (Soundness). *If $\Phi \vdash_{\mathsf{L}} \psi$ then $\Phi \models_{\mathsf{L}} \psi$.*

The rest of this section will be devoted to showing that $\vdash_{\mathsf{L}}$ is also complete for $\models_{\mathsf{L}}$. For this, we will follow the same steps that lead us to our completeness theorem for InqBM. The first thing to be verified is that L still admits a resolution algorithm.

7.4.8. Lemma.
Suppose $\Phi \vdash_{\mathsf{L}} \psi$. For all $\Gamma \in \mathcal{R}(\Phi)$ there exists $\alpha \in \mathcal{R}(\psi)$ such that $\Gamma \vdash_{\mathsf{L}} \alpha$.

Proof. We need to extend the proof of Lemma 7.3.12 with a few other cases corresponding to the new inference rules that our system L may contain. Now, notice that the rules $\boxplus\mathsf{T}$, $\boxplus\mathsf{4}$, and $\boxplus\mathsf{5}$ allow us to infer a declarative from another declarative. Thus, the argument spelled out in the first bullet item of the proof of Lemma 7.3.12 applies to these rules as well. We are left with the case in which L includes the rule $\boxplus\mathsf{D}$, and the conclusion $\psi = \neg\boxplus_a\bot$ was obtained by this rule. But in this case, ψ is a resolution of itself, and it is also a validity, so it is certainly provable from any resolution $\Gamma \in \mathcal{R}(\Phi)$. $\Box$

As a corollary, we have the Resolution Split for L, which plays a crucial role at several points in the completeness proof.

[17] To be fully precise, $\vdash_{\mathsf{L}}$ is obtained by adding the relevant axioms to the *multi-modal* version of the proof system given for InqBM. This means that our system must contain the rules given in Figure 7.5 for each pair of modalities $\Box_a$ and $\boxplus_a$, for $a \in \mathcal{A}$.

7.4.9. Lemma (Resolution Split).
Let $\Gamma \subseteq \mathcal{L}_!^M$ be a set of declaratives. If $\Gamma \vdash_L \psi$, then $\Gamma \vdash_L \alpha$ for some $\alpha \in \mathcal{R}(\psi)$.

Obviously, since we have not removed any of the rules for InqBM, our system still allows us to bring a formula into normal form, that is, we have $\varphi \dashv\vdash \bigvee\!\!\!\!\bigvee \mathcal{R}(\varphi)$. As we know, this is all it takes to prove the traceable deduction failure for L.

7.4.10. Lemma (Traceable deduction failure for L).
If $\Phi \not\vdash_L \psi$, there is a $\Gamma \in \mathcal{R}(\Phi)$ such that $\Gamma \not\vdash_L \psi$.

In turn, this together with Lemma 7.4.8 implies the Resolution Lemma for L.

7.4.11. Lemma (Resolution Lemma for L).
$\Phi \vdash_L \psi \iff$ *for all $\Gamma \in \mathcal{R}(\Phi)$ there exists some $\alpha \in \mathcal{R}(\psi)$ such that $\Gamma \vdash_L \alpha$.*

We can then define our canonical model M^c_{L} just as we did for InqBM. However, our worlds will now be complete L-theories of declaratives; that is, a possible world $\Gamma \in W^c_{\mathsf{L}}$ will have to be closed under deduction of declaratives in the system $\vdash_{\mathsf{L}}$.

7.4.12. Definition. [Canonical Model for L]
The canonical model for L is the inquisitive multi-modal model defined as follows: $M^c_{\mathsf{L}} = \langle W^c_{\mathsf{L}}, \{\Sigma^c_a \mid a \in \mathcal{A}\}, V^c\rangle$, where:

- W^c_{L} is the set of complete L-theories of declaratives.
- $\Sigma^c_a : W^c_{\mathsf{L}} \to \Pi_{W^c_{\mathsf{L}}}$ is the inquisitive state map given by:

$$\Sigma^c_a(\Gamma) = \{S \subseteq W^c_{\mathsf{L}} \mid \bigcap S \vdash \varphi \text{ whenever } \boxplus_a \varphi \in \Gamma\}$$

  where if $S = \emptyset$, we let $\bigcap S = \mathcal{L}_!^{\mathsf{M}}(\mathcal{A})$.
- $V^c : W^c_{\mathsf{L}} \times \mathcal{P} \to \{0,1\}$ is the map defined by: $V^c(\Gamma, p) = 1 \iff p \in \Gamma$

As above, we can easily prove that M^c_{L} is indeed an inquisitive multi-modal model, in the sense that for any $\Gamma \in W^c_{\mathsf{L}}$ and for any $a \in \mathcal{A}$, the set $\Sigma^c_a(\Gamma)$ is an inquisitive proposition over W^c_{L}. The canonical information state map σ^c_a can then be given a characterization identical to that given in Lemma 7.3.26.

7.4.13. Lemma. $\sigma^c_a(\Gamma) = \{\Delta \mid \alpha \in \Delta$ *whenever* $\boxplus_a \alpha \in \Gamma$ *and α is declarative*$\}$

The remaining part of the argument presents no novelties: proceeding as above, we can show the Support Lemma and, as a particular case, also the Truth Lemma.

7.4.14. Lemma (Support Lemma).
Let $\varphi \in \mathcal{L}^M(\mathcal{A})$ and let S be a state in M^c_L. We have: $M^c_L, S \models \varphi \iff \bigcap S \vdash_L \varphi$

7.4.15. Lemma (Truth Lemma).
Let $\varphi \in \mathcal{L}^{\mathsf{M}}(\mathcal{A})$ and let Γ be a world in M^c_{L}. We have: $M^c_{\mathsf{L}}, \Gamma \models \varphi \iff \Gamma \vdash_{\mathsf{L}} \varphi$

In turn, the connection between L-provability and support in M^c_{L} afforded by the Support Lemma makes it possible, whenever $\Phi \not\vdash_{\mathsf{L}} \psi$, to produce a state in M^c_{L} which supports all formulas in Φ but not ψ. The argument is the same one we used for completeness in propositional logic, cf. the proof of Theorem 3.3.2.

7.4.16. Lemma.
Let $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{M}}(\mathcal{A})$. If $\Phi \not\vdash_{\mathsf{L}} \psi$, there exists a state S in the canonical model M^c_{L} such that $M^c_{\mathsf{L}}, S \models \Phi$ but $M^c_{\mathsf{L}}, S \not\models \psi$.

However, even if there is a state in M^c_{L} which supports Φ but not ψ, this is not yet enough to conclude $\Phi \not\models_{\mathsf{L}} \psi$: to reach this conclusion, we must show that M^c_{L} is a model of the appropriate kind, i.e., one that satisfies the conditions associated with the logic L. In the standard jargon of modal logic, we want to show that each of the inference rules $\boxplus\mathsf{D}, \boxplus\mathsf{T}, \boxplus\mathsf{4}$ and $\boxplus\mathsf{5}$ is *canonical* for the associated property.

7.4.17. Lemma (Canonicity).

- *If L contains $\boxplus\mathsf{D}$, then M^c_{L} satisfies consistency;*
- *If L contains $\boxplus\mathsf{T}$, then M^c_{L} satisfies factivity;*
- *If L contains $\boxplus\mathsf{4}$, then M^c_{L} satisfies positive introspection;*
- *If L contains $\boxplus\mathsf{5}$, then M^c_{L} satisfies negative introspection;*

Proof.

- Suppose L contains $\boxplus\mathsf{D}$ and consider a world $\Gamma \in W^c_{\mathsf{L}}$. Since Γ is an L-theory we must have $\neg\boxplus_a\bot \in \Gamma$. By the Truth Lemma, this implies $M^c_{\mathsf{L}}, \Gamma \models \neg\boxplus_a\bot$. Now since $\bot$ is truth-conditional, $\neg\boxplus_a\bot \equiv \neg\Box_a\bot$, so we also have $M^c_{\mathsf{L}}, \Gamma \models \neg\Box_a\bot$. By definition, this amounts to $M^c_{\mathsf{L}}, \sigma^c_a(\Gamma) \not\models \bot$, which implies $\sigma^c_a(\Gamma) \neq \emptyset$. Since Γ was arbitrary, this shows that M^c_{L} satisfies consistency.

- Suppose L contains $\boxplus\mathsf{T}$ and consider a world $\Gamma \in W^c_{\mathsf{L}}$. Since Γ in an L-theory and L contains $\boxplus\mathsf{T}$, whenever we have $\boxplus_a\alpha \in \Gamma$ for a declarative α, we must also have $\alpha \in \Gamma$. But by the characterization of σ^c_a in Lemma 7.4.13, this means that $\Gamma \in \sigma^c_a(\Gamma)$. Since Γ was arbitrary, this shows that M^c_{L} satisfies factivity.

- Suppose L contains $\boxplus 4$ and suppose $\Delta \in \sigma_a^c(\Gamma)$. By Lemma 7.4.13, this means that Δ contains a declarative α whenever $\boxplus_a\alpha \in \Gamma$. In particular, since modal formulas are declaratives, this implies that $\boxplus_a\varphi \in \Delta$ whenever $\boxplus_a \boxplus_a \varphi \in \Gamma$. But now, since Γ is an L-theory and since L contains the inference rule $\boxplus 4$, we also have $\boxplus_a\boxplus_a\varphi \in \Gamma$ whenever $\boxplus_a\varphi \in \Gamma$. Thus, we have $\boxplus_a\varphi \in \Delta$ whenever $\boxplus_a\varphi \in \Gamma$. By definition of Σ_a^c, this implies that $\Sigma_a^c(\Delta) \subseteq \Sigma_a^c(\Gamma)$, which shows that M_{L}^c satisfies positive introspection.

- Suppose L contains $\boxplus 5$ and suppose $\Delta \in \sigma_a^c(\Gamma)$. Again, by Lemma 7.4.13, this means that Δ contains a declarative α whenever $\boxplus_a\alpha \in \Gamma$. In particular, since $\neg \boxplus_a \varphi$ is a declarative, we have $\neg \boxplus_a\varphi \in \Delta$ whenever $\boxplus_a\neg \boxplus_a\varphi \in \Gamma$. Now, suppose $\boxplus_a\varphi \notin \Gamma$: since Γ is complete, we must have $\neg\boxplus_a\varphi \in \Gamma$. Since Γ is an L-theory and L contains the rule $\boxplus 5$, this means that $\boxplus_a\neg \boxplus_a\varphi \in \Gamma$, which we have seen to imply $\neg \boxplus_a\varphi \in \Delta$. Finally, since Δ is consistent this implies $\boxplus_a\varphi \notin \Delta$. We have thus shown that if $\boxplus_a\varphi \notin \Gamma$, then $\boxplus_a\varphi \notin \Delta$. This means that, contrapositively, $\boxplus_a\varphi \in \Delta$ implies $\boxplus_a\varphi \in \Gamma$, for any formula φ. By definition of the canonical state map Σ_a^c, this implies that $\Sigma_a^c(\Gamma) \subseteq \Sigma_a^c(\Delta)$, which shows that M_{L}^c satisfies negative introspection. $\Box$

Finally, from lemmata 7.4.16 and 7.4.17 we obtain our completeness result for L.

7.4.18. Theorem (Completeness for L).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{M}(\mathcal{A})$,

$$\Phi \models_{L} \psi \iff \Phi \vdash_{L} \psi$$

This result holds for all inquisitive multi-modal logics L obtained by adding zero or more of the conditions in Definition 7.4.1 to InqBM.[18] Two such logics that are especially interesting for us are the inquisitive doxastic logic IDL and the inquisitive epistemic logic IEL that we introduced in Section 7.2. Let us spell out explicitly what the completeness result looks like for these two logics.

7.4.19. Corollary (Completeness for inquisitive doxastic logic).
IDL is completely axiomatized by enriching a system for InqBM with the rules $\boxplus D, \boxplus 4$ and $\boxplus 5$.

7.4.20. Corollary (Completeness for inquisitive epistemic logic).
IEL is completely axiomatized by enriching a system for InqBM with the rules $\boxplus T, \boxplus 4$ and $\boxplus 5$.[19]

[18]Notice that the case in which zero of these conditions are added is just the completeness result for the multi-modal version of InqBM.

[19]In fact, for IEL, adding $\boxplus$T and $\boxplus$5 is sufficient: $\boxplus$4 is provable from these rules. The proof of this fact is the essentially the same which can be used to show that 4 follows from T and 5 in standard modal logic (see, e.g., Hughes and Cresswell, 1996).

Thus, it is not only the basic inquisitive modal logic InqBM that is a proof theoretically smooth generalization of its standard counterpart, the logic K. Stronger inquisitive modal logics are also axiomatized by simple inference rules, which are the obvious counterparts of rules that are familiar from standard modal logic. This does not only provide us with well-behaved proof systems to reason about information and issues in a multi-agent scenario, but it also indicates that inquisitive modal logic gives rise to a mathematically elegant theory, which is worth exploring further.

7.5 Discussion

In this chapter, we saw how inquisitive semantics suggests a simple but radical enrichment of the notion of a modal operator, and we investigated a particular system of modal logic, InqBM, equipped with two natural modalities, $\Box$ and $\boxplus$. This section concludes with a discussion of the relevance of these ideas in various domains, from semantics and pragmatics to epistemology and computer science.

Question-directed attitudes

Traditionally, the framework of modal logic has been used to analyze propositional attitudes like *knowledge* and *belief*, which are construed as relations between an agent and a classical proposition, and to investigate the logic of these notions. However, there is an important class of attitudes whose object is not a classical proposition, but rather the sort of object which is denoted by a question. These attitudes, which Friedman (2013) calls *question-directed* attitudes, include *wonder*, *be curious about* and *be agnostic about*. While epistemologists have traditionally focused on propositional attitudes, Hookway (2008) and Friedman (2013) emphasize the crucial role of question-directed attitudes in the process of inquiry: indeed, this process is essentially *driven* by such attitudes, while standard propositional attitudes such as belief may be seen as the inquiry's end products.

As we saw, within our modal framework it is possible to analyze in a uniform way both standard propositional attitudes and question-directed attitudes. We illustrated this by providing a specific analysis of *wondering*; of course, different accounts are possible as well. The crucial feature of the framework is the fact that a modal operator can be seen as encoding a relation between two *inquisitive* propositions. This makes is possible to capture an attitude like *wondering* as a relation in which an agent may stand to an inquisitive proposition by virtue of the issues she entertains.[20]

[20]This is relevant not only in philosophy, but also in linguistics, allowing us to give a modal analysis of the semantics of attitude verbs like *wonder*, *be curious* and *be agnostic*.

Dynamics of information exchange

Within the framework of *dynamic epistemic logic* (see van Ditmarsch et al., 2007, for an overview), much attention has been devoted to modeling how an epistemic situation evolves when agents share information by making certain statements. However, the process of information exchange involves more than just making statements: in particular, it crucially involves raising issues for discussion, to which subsequent statements will then be expected to relate. Therefore, in order to model and reason about this process, our static logic should capture not only the information available to the agents, but also the issues that these agents entertain, both privately and publicly. As we saw, inquisitive epistemic logic provides such a logical framework. In the next chapter we will show how this framework can be taken as the basis for a simple dynamic logic which captures in a uniform way the conversational effect of making statements and asking questions.

Pragmatics

In a conversation, an utterance inevitably conveys more than its literal meaning. For, a hearer will draw certain conclusions from the very fact that the speaker chose to utter that particular sentence. Technically, the utterance is said to trigger certain *implicatures*. A typical case are *exhaustivity implicatures*, which arise by reasoning about stronger *relevant* things that the speaker could have said, but didn't. Now, whether or not a certain sentence would have been a relevant thing to say depends on the question that the utterance was intended to address. This means that in the context of different questions, the same statement gives rise to different exhaustivity implicatures, as illustrated by the following examples.[21]

(1) A Who is coming for dinner?
B Carol is. $\rightsquigarrow$ Dan is not coming

(2) A Which girls are coming for dinner?
B Carol is. $\not\rightsquigarrow$ Dan is not coming

This shows that in order to model the process of pragmatic enrichment of literal meaning, we need to keep track not only of information, but also of issues. Pragmaticists are well aware of this, and the standard models for pragmatics indeed assume a context to come with a stack of "questions under discussion" (e.g., Roberts, 1996). By bringing issues into the picture, inquisitive epistemic logic could provide a starting point for an account of pragmatic strengthening of literal meaning in a dynamic-logical setting. On the one hand, this would give rise to a precise logic of pragmatic reasoning, capable of making inferences about the indirect effects of utterances. On the other hand, it would also refine the standard

[21]For an account of exhaustivity implicatures in a setting related to inquisitive semantics, see Westera (2013).

models used in formal pragmatics, by representing not only public issues—the "questions under discussion"—but also private issues and higher-order information about such issues, which also play a role in actual pragmatic reasoning.

Formal epistemology

Besides communication, over the past decade dynamic epistemic logics have been fruitfully applied to the field of formal epistemology (see, e.g., van Ditmarsch, 2005; Baltag and Smets, 2006; van Benthem, 2007) providing a modeling of belief revision in inquiry which is more explicit that the traditional AGM framework (Alchourrón et al., 1985). However, Olsson and Westlund (2006) have emphasized that an account of inquiry should not focus solely on changes in an agent's beliefs, but should also take into account changes in the agent's *research agenda*, i.e., in the issues that the agent intends to investigate. By modeling both information and issues in an integrated way, inquisitive modal logic provides a natural starting point for a logic of inquiry of the kind envisaged by Olsson and Westlund. A refinement of inquisitive epistemic logic to deal with fallible beliefs was considered by Ciardelli and Roelofsen (2014). As an example of the applications of such a logic, consider the following quote from Olsson and Westlund (2006):

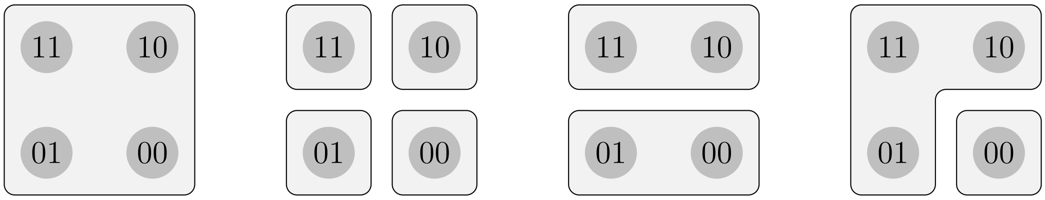
> [An] adequate model should keep track not only of questions in need of answers but also of beliefs that answer questions. The latter have a special status. It is natural to think of them as having a higher degree of informational value than other beliefs. [...] The special status of question-answering beliefs should arguably be reflected in a formal model.

Within an inquisitive logic for inquiry, this idea of a *pertinent belief* can be formalized: an agent's belief that α is *pertinent* if there is some question μ such that: (i) resolving μ is part of the agent's goals (ii) μ is currently settled by the agent's beliefs, but (iii) μ would no longer be settled if the agent were to give up the belief that α. Similarly, other interesting epistemic notions and procedures involving the inquirer's research agenda can be investigated in such a framework.

InqBM as a logic of agency

In this chapter we have considered only one intuitive interpretation of inquisitive modal logic, namely, the doxastic/epistemic one. However, as for standard modal logic, various interpretations are possible for InqBM, depending on what we take the state map Σ to encode. To illustrate this, I will sketch here an alternative intuitive interpretation of InqBM.

Consider an agent who is involved in a certain process, and who has certain actions at her disposal. The worlds $v \in \sigma(w)$ are the worlds that may possibly result from the process as taking place at w. The alternatives $a \in \textsc{Alt}(\Sigma(w))$



(a) (b) (c) (d)

Figure 7.8: Four possible states for an agent involved in a process: in the first figure, the agent has no control over the outcome; in the second, the agent has complete control; in the third and fourth figure, the agent has partial control.

correspond to the various courses of actions available to the agents, and the worlds $w \in a$ are the different outcomes that may result from a certain course of actions.

For instance, Figure 7.8(a) describes a state in which both p and q may turn out to be true or false, and the agent has no control over the outcome. At the opposite end of the spectrum, Figure 7.8(b) describes a state in which both p and q may turn out true or false, but the agent's choices determine exactly which configuration will obtain. Figures 7.8(c) and 7.8(d) both represent situations in which the agent has partial control over the outcome of the process. In Figure 7.8(c), the agent's choices determine the value of p, while the value of q is out of the agent's control. In Figure 7.8(d), the agent's choices completely determine the truth-value of $p \vee q$, but not the truth-value of either atom.

Let us consider what the modalities of InqBM express in this interpretation. If α is a statement, then both $\Box\alpha$ and $\boxplus\alpha$ express that α is true in all $v \in \sigma(w)$; this means that α is bound to be the case, regardless of what the agent does.

Now consider a question, say, $?p$. The formula $\Box?p$ expresses that whether p will be the case is pre-determined, and not subject to either the agent's choices or other external factors. In our examples, $\Box?p$ is always false, since in each scenario both p-outcomes and $\neg p$-outcomes are possible.

On the other hand, the formula $\boxplus?p$ expresses that whether p will be the case is completely determined by the agent's course of actions. Thus, $\boxplus?p$ is true in the scenarios of Figure 7.8(b) and Figure 7.8(c), in which the truth-value of p in the outcome world depends uniquely on the agent's choices, but it is false in the scenario of Figure 7.8(a), in which the agent has no control over the outcome, and in the scenario of Figure 7.8(c), since there is one course of actions for the agent which leaves the truth-value of p undetermined.

Finally, in this interpretation, a formula $\boxtimes?p = \neg\Box?p \wedge \boxplus?p$ expresses the fact that whether p will hold is determined by the agent's choices, and the dependency is non-trivial, i.e., the agent actually has some power to affect the truth-value of p.

Thus, InqBM allows us to model in a very simple way the difference between (i) matters that are predetermined in a certain way independently of the agent's choices; (ii) matters that are fully determined by the agent's choices; and (iii) matters that are partly beyond the agent's control. Capturing this sort of differences is the primary goal of several existing logics of agency, such as *alternating-time temporal logic* (Alur et al., 2002; Goranko and van Drimmelen, 2006) and *stit logic* (Belnap et al., 2001). Moreover, this interpretation is of course also available in the multi-agent setting, where we might be interested in reasoning about game-like situations, as in the *game logic* of Parikh (1985), or about what agents can achieve by forming coalitions, as in the *coalition logic* of Pauly (2001). Investigating exactly the potential of InqBM as a logic of agency and its precise relations with existing logics is an interesting goal for future work.

Issue-sensitivity of modals

On the standard view, the semantics of modal expressions like *must*, *should*, or *may* in natural language involves a set of accessible worlds—the *modal base*—and a *ranking* of these worlds—the *ordering source* (Kratzer, 1981). E.g., a sentence like *Alice should share her fortune* is taken to be true if all the highest-ranked worlds in the modal base are worlds where Alice shares her fortune.

Recently, however, it has been argued (Lassiter, 2011; Cariani et al., 2013; Charlow, 2013, 2014) that this view has trouble accounting for the truth conditions of deontic modals in the context of decision problems involving uncertainty, such as the *miners paradox* of Kolodny and MacFarlane (2010). In a nutshell, the problem is that, according to the standard view, whether one ought to bring about φ is determined only by the highest-ranked φ-worlds, that is, by the best (in some underspecified sense) outcomes compatible with φ. However, examples like the miners paradox highlight that, in judging what one should do, we actually take into account *all* possible outcomes of an action, not just a subset of them.

Cariani (2015) argues that to solve this problem, what we should really be ranking are not *worlds*, i.e., the possible outcomes of the actions available to the agent in the decision problem at hand, but rather *classical propositions*, which can be taken to correspond to the actions themselves. Thus, in Cariani's theory, the context for the evaluation of *should* needs to supply not just a set of worlds, but also a relevant issue, together with a ranking of its alternative resolutions.

From our point of view, this idea is particularly natural. For, inquisitive modal models may be seen as equipping each world not only with a set of worlds, but also with an issue, which in the deontic case we may regard as encoding the decision problem at hand. Thus, on the inquisitive view of modal operators, we actually *expect* that a modal like *should* may be sensitive to a contextual decision problem and, more generally, that a modal may be sensitive to a contextual issue.

We may thus regard Cariani's proposal as pointing towards an inquisitive incarnation of the standard treatment of modals. Now, the "modal base" would be a set of classical propositions, namely, the alternatives for the relevant issue. As in the standard account, a modal statement like *Alice should share her fortune* may then be evaluated relative to a modal base and to a ranking of its elements.

Chapter 8

Dynamics

In the previous chapter we saw how *epistemic logic*, broadly construed as the logic of information in the multi-agent setting, can be generalized in a natural way to an *inquisitive* epistemic logic (IEL) which allows us to reason not only about the information that agents have, but also about the issues that they entertain.

Epistemic logic forms the basis for a number of *dynamic* epistemic logics, which model how an epistemic situation changes when certain actions are performed. The simplest and most popular of these logics is the Public Announcement Logic (PAL) introduced by Plaza (1989), and further developed by Gerbrandy and Groeneveld (1997), Baltag et al. (1998), and van Ditmarsch (2000). This logic allows us to reason not only about a fixed epistemic situation, but also about the way in which such a situation changes when a sentence is *publicly announced*, where a public announcement of φ may be regarded as an assertion of φ directed at the whole group of agents. Thus, PAL may be regarded as an elementary logical account of the process of communication.

In this chapter, we show that IEL provides the basis for an *inquisitive dynamic epistemic logic* (IDEL) which models how an inquisitive-epistemic situation changes as a result of publicly uttering a sentence φ, which may be a statement or a question. In other words, in IDEL agents can not only make public assertions—the standard public announcements—but also ask public questions. This provides the basic means for a more faithful account of communication as a process of raising and resolving issues, in which agents request information by uttering questions, and provide information by uttering statements.[1]

Technically, the way in which the *public utterance* action is modeled is very similar to the modeling of public announcements in PAL: when φ is uttered, worlds where φ is false are dropped from the model, and the state of each agent is restricted by intersecting it with the proposition expressed by φ. Following the

[1]The view of communication as a process in which agents interact by raising and resolving issues has played an important role in the line of work leading to inquisitive semantics (see, e.g., Hulstijn, 1997, 2000; Groenendijk, 1999; Groenendijk and Roelofsen, 2009).

tradition of dynamic epistemic logics, we equip our logical language with dynamic modalities $[\varphi]$ which allow us to conditionalize a formula ψ to the public utterance of φ. We will show that, as in PAL and many other dynamic epistemic logics, this enrichment—while very convenient in practice—does not take us beyond the expressive power of the static language of IEL. We will exploit this fact to axiomatize IDEL by means of *reduction axioms* that inductively turn any formula $[\varphi]\psi$ into a formula of the static language of IEL, for which a sound and complete axiomatization was given in the previous chapter.

The chapter is structured as follows. Section 1 briefly sums up the workings of standard Public Announcement Logic, and recalls its most standard axiomatizations. Section 2 introduces the action of *public utterance*, which generalizes public announcements to the inquisitive setting. Section 3 investigates the resulting dynamic logic, IDEL, showing that it has the same expressive power as the underlying static system IEL, and thereby establishing a sound and complete axiomatization. Section 4 discusses the public utterance of non-epistemic formulas, showing in particular how the group's common knowledge and common issues change as a result. Section 5 proposes a pragmatic principle of division of labor between statements and questions, which constrains the situations in which it is appropriate to utter a sentence, and explores the consequences of this principle for the modeling of dialogue. Section 6 compares IDEL in some detail to another recent dynamic approach to questions, the *dynamic epistemic logic with questions* of van Benthem and Minică (2012). Finally, Section 7 summarizes the findings of the chapter, and lists some directions for future work.

This chapter is partly based on Ciardelli and Roelofsen (2015b). Throughout this chapter, I will not use the abstract notation for modalities employed in the previous chapters; rather, following Ciardelli (2014a) and Ciardelli and Roelofsen (2015b), I will use a notation which is more suggestive of the intended epistemic reading: we will write K_a, E_a, and W_a instead of $\Box_a, \boxplus_a$ and $\boxtimes_a$ respectively. Moreover, we will read $K_a\varphi$ as "a knows φ", $E_a\varphi$ as "a entertains φ", and $W_a\varphi$ "a wonders φ". I should remark that, while K_a and W_a are intended to be rather natural formalizations of the verbs *know* and *wonder*, the modality E_a does not seem to have a close natural language counterpart; thus, the term *entertain* for E is to be understood here as a purely technical term. I will also write K_a instead of $\Box_a$ in the context of standard epistemic logic.

8.1 Public Announcement Logic

The starting point of Public Announcement Logic is a system of epistemic logic, such as the system EL described in the previous chapter (i.e., multi-modal S5). The models on which PAL is interpreted are just models for this logic, that is, multi-modal Kripke models the epistemic maps of which satisfy the conditions of Factivity and Introspection. The essential extra ingredient of PAL is a special kind of model transformation, which is used to model the effect of the public announcement of a formula.

The fundamental idea is that a public announcement of φ has the effect of making it common knowledge that φ is true at the moment of utterance. That is, when φ is publicly announced in a model M, all agents learn that the actual world lies in the proposition $|\varphi|_M$ expressed by φ at the moment of utterance, and they learn that everyone else now knows this, and that everyone knows that everyone knows, and so on ad infinitum. Technically, this is achieved by letting a public announcement of φ have the effect of eliminating from the model all worlds where φ is false, and restricting the epistemic maps of the agents by intersecting them with the proposition $|\varphi|_M$.

8.1.1. Definition. [Updating an epistemic model with a classical proposition] Let $M = \langle W, \{\sigma_a \mid a \in \mathcal{A}\}, V\rangle$ be an epistemic model, and let $s \subseteq W$. The update of M with s is the epistemic model $M^s = \langle W^s, \{\sigma_a^s \mid a \in \mathcal{A}\}, V^s\rangle$, where:

- $W^s = W \cap s$
- $\sigma_a^s(w) = \sigma_a(w) \cap s$
- $V^s = V \restriction s$

It is easy to see that, given an epistemic model M, the model M^s which results from this transformation is also an epistemic model, that is, it still satisfies the factivity and introspection conditions. To reduce clutter, we will write $M^\varphi = \langle W^\varphi, \{\sigma_a^\varphi \mid a \in \mathcal{A}\}, V^\varphi\rangle$ instead of $M^{|\varphi|_M} = \langle W^{|\varphi|_M}, \{\sigma_a^{|\varphi|_M} \mid a \in \mathcal{A}\}, V^{|\varphi|_M}\rangle$ for the model resulting from updating M with the proposition $[\varphi]_M$ expressed by φ.

The language of PAL is obtained by enriching the language of standard epistemic logic with dynamic modalities $[\varphi]$, which enable the language to express facts about what will be the case after φ is publicly announced. That is, the language $\mathcal{L}^{\mathsf{PAL}}$ is given by the following inductive definition:

$$\varphi \ ::= \ p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid K_a\varphi \mid [\varphi]\varphi$$

The semantics of PAL is then obtained by adding to the clauses for standard epistemic logic the following clause for dynamic modalities.

8.1.2. Definition. [Truth-conditions for dynamic modalities]
$M, w \models [\varphi]\psi \iff M, w \models \varphi$ implies $M^\varphi, w \models \psi$

In words, a dynamic modal formula $[\varphi]\psi$ is true at w in case, provided w survives the update with φ, ψ is true at w in the resulting model. So, $[\varphi]\psi$ may be regarded as a sort of conditional, the antecedent of which we read not as "if φ is true", but as "if φ is announced". If ψ is a non-epistemic formula (i.e., does not contain any occurrence of a K_a modality) then $[\varphi]\psi$ can indeed be seen to boil down to the standard conditional $\varphi \to \psi$. However, if ψ is an epistemic formula, then in general $\varphi \to \psi$ and $[\varphi]\psi$ have a different meaning: for instance, $[p]K_a p$ is a logical validity, but $p \to K_a p$ is not; the mere fact that p is true implies nothing about a's knowledge; however, if the truth of p is publicly announced, then it is part of the way in which public announcements work that a will come to know that p.

Incidentally, notice that $[\varphi]K_a\varphi$ is *not* a general validity of PAL. This may seem paradoxical, given that in the updated model M^φ we always have $\sigma_a^\varphi(w) \subseteq |\varphi|_M$ by construction. But there is a subtlety: although updating a model with φ will not change truth values of proposition letters at a world, it does change the agents' epistemic states. As a consequence, the truth values of epistemic formulas at a given world in the updated model may differ from the truth values that these sentences received at the same world in the original model, which means that we might have $|\varphi|_{M^\varphi} \neq |\varphi|_M$. That is why, even though by definition we have $\sigma_a^\varphi(w) \subseteq |\varphi|_M$, we might have $\sigma_a^\varphi(w) \not\subseteq |\varphi|_{M^\varphi}$.

In words, although after the announcement the agent knows that φ was true in the original model, she might not know that φ is true in the updated model; and in fact, this may be for a good reason: in general, φ might no longer be true. To illustrate this, consider the sentence $\varphi := p \wedge \neg K_a p$. Let M be an epistemic model and w a world in M where φ is true. Then it is easy to see that after the update, $K_a p$ will be true at w, which however implies that $M^\varphi, w \not\models \varphi$. Thus, in this case the announcement of a true sentence leads to a state where the sentence is false—and *a fortiori* not known to the agents.

While thinking in terms of updates is convenient and cognitively natural, it is not essential: after all, what will be the case after an update with φ is performed is completely determined by the current features of an epistemic model. Thus, even a formula containing dynamic modalities actually expresses a property of the current epistemic situation. Can this property also be expressed by means of a standard epistemic formula? The answer is yes. To see this, let us first point out a few facts about how dynamic modalities behave when applied to an atom, a negation, a conjunction, and a K_a modality.

8.1.3. Proposition.
The following logical equivalences are valid in PAL for any φ, ψ, and χ.

- $[\varphi]p \equiv \varphi \to p$
- $[\varphi]\bot \equiv \neg\varphi$
- $[\varphi](\psi \to \chi) \equiv [\varphi]\psi \to [\varphi]\chi$
- $[\varphi](\psi \wedge \chi) \equiv [\varphi]\psi \wedge [\psi]\chi$
- $[\varphi]K_a\psi \equiv \varphi \to K_a[\varphi]\psi$

Now, these equivalences make it possible to translate any formula of PAL into a standard epistemic formula.

8.1.4. PROPOSITION.
Any $\varphi \in \mathcal{L}^{PAL}$ is logically equivalent to some $\varphi^* \in \mathcal{L}^{EL}$.

Proof. We proceed by induction on φ. All the steps are immediate except for the case in which $\varphi = [\psi]\chi$. Now, by induction hypothesis we have two formulas $\psi^*, \chi^* \in \mathcal{L}^{\mathsf{EL}}$ which are equivalent to ψ and χ respectively. It follows that $\varphi \equiv [\psi^*]\chi^*$. At this point we use the equivalences in the previous proposition to show by induction on χ^* that $[\psi^*]\chi^* \equiv \varphi^*$ for some $\varphi \in \mathcal{L}^{\mathsf{EL}}$. $\square$

This fact can be used to provide a complete axiomatization of PAL: all we need is to extend a complete system for EL with inference rules that enable us to turn any formula $\varphi \in \mathcal{L}^{\mathsf{PAL}}$ into the corresponding $\varphi^* \in \mathcal{L}^{\mathsf{EL}}$. This is achieved by turning each of the equivalences of Proposition 8.1.3 into an inference rule. But this is not enough, as Wang and Cao (2013) proved: we also need some other rule to ensure that once we have established $\varphi \leftrightarrow \varphi^*$ and $\psi \leftrightarrow \psi^*$ we can obtain $[\varphi]\psi \leftrightarrow [\varphi^*]\psi^*$. Here, there are several natural choices. The most straightforward choice, adopted by Plaza (1989), is to use a rule of replacement of equivalents; we will denote provability in this system by $\vdash_{\mathsf{PAL}^{\mathsf{RE}}}$.

A different option, adopted by van Ditmarsch et al. (2007) is to use a rule that allows us to compose two dynamic modalities into one, making use of the following feature of PAL.

8.1.5. PROPOSITION (COMPOSITION OF ANNOUNCEMENTS IN PAL).
$[\varphi][\psi]\chi \equiv [\varphi \wedge [\varphi]\psi]\chi$

We can make this equivalence into an inter-derivability rule, denoted !Comp; provability in the system equipped with this rule is denoted $\vdash_{\mathsf{PAL}^{!\mathsf{Comp}}}$

A third option is to have a monotonicity rule for the dynamic modality. This rule is just like the monotonicity rule we used in the previous chapters for static modalities: it states that a dynamic modality $[\varphi]$ is monotonic with respect to the entailment order, in the following sense.

8.1.6. PROPOSITION (MONOTONICITY OF DYNAMIC MODALITIES).
If $\psi_1, \ldots, \psi_n \models \chi$, then $[\varphi]\psi_1, \ldots, [\varphi]\psi_n \models [\varphi]\chi$.

We will denote provability in the system equipped with the monotonicty rule for dynamic modalities as $\vdash_{\mathsf{PAL}^{!\mathsf{Mon}}}$.

The relevant inference rules are collected in Figure 8.1. The following theorems state that all three systems $\vdash_{\mathsf{PAL}^{\mathsf{RE}}}, \vdash_{\mathsf{PAL}^{!\mathsf{Comp}}}$, and $\vdash_{\mathsf{PAL}^{!\mathsf{Mon}}}$ are complete for PAL.

8.1.7. THEOREM (PLAZA (1989)).
For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{PAL}$, $\Phi \models \psi \iff \Phi \vdash_{PAL^{RE}} \psi$

8.1.8. THEOREM (VAN DITMARSCH ET AL. (2007)).
For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{PAL}$, $\Phi \models \psi \iff \Phi \vdash_{PAL^{!Comp}} \psi$

!Atom

$$\dfrac{[\varphi]p}{\varphi \to p}$$

!⊥

$$\dfrac{[\varphi]\bot}{\neg\varphi}$$

!∧

$$\dfrac{[\varphi](\psi \wedge \chi)}{[\varphi]\psi \wedge [\varphi]\chi}$$

!→

$$\dfrac{[\varphi](\psi \to \chi)}{[\varphi]\psi \to [\varphi]\chi}$$

!K

$$\dfrac{[\varphi]K_a\psi}{\varphi \to K_a[\varphi]\psi}$$

RE

$$\dfrac{\varphi \leftrightarrow \psi}{\chi[\varphi/p] \leftrightarrow \chi[\psi/p]}$$

!Comp

$$\dfrac{[\varphi][\psi]\chi}{[\varphi \wedge [\varphi]\psi]\chi}$$

!Mon

$$\dfrac{\begin{array}{c}[\psi_1] \quad \dots \quad [\psi_n] \\ \vdots \\ \chi\end{array} \qquad [\varphi]\psi_1 \quad \dots \quad [\varphi]\psi_n}{[\varphi]\chi}$$

Figure 8.1: Inference rules for dynamic modalities. As usual, the monotonicity rule comes with the constraint that $\psi_1, \dots, \psi_n$ be the only undischarged assumptions in the sub-proof leading to χ. Notice that, except for RE and !Mon, all the other rules are inter-derivability rules. A complete system for PAL is obtained by enriching a system for epistemic logic with the rules on the first row, plus either one of RE, !Comp, or !Mon.

8.1.9. Theorem (Wang and Cao (2013)).
For all $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{PAL}$, $\Phi \models \psi \iff \Phi \vdash_{PAL^{!Mon}} \psi$

These three proof systems axiomatize PAL by providing the means for a reduction to EL. Similarly, in this chapter we will axiomatize inquisitive dynamic epistemic logic by providing the means for a reduction to IEL. Besides these, there is also a radically different axiomatization of PAL, due to Wang and Cao (2013), which does not use reduction axioms (or rules) at all. To prove the completeness of their system, Wang and Cao show that a modality $[\varphi]$ of PAL may be regarded as a Kripke modality operating in a suitable Kripke model, thus establishing a more standard semantics for PAL. We will leave it as an interesting task for future work to investigate whether a similar approach is possible in the case of IDEL, that is, whether the dynamic modalities $[\varphi]$ that we will introduce in the next section may be regarded as inquisitive modalities operating in a suitable inquisitive modal model.

8.2 Uttering statements and questions

Let us now turn to the inquisitive case. Our starting point will be the system of inquisitive epistemic logic, IEL, presented in the previous chapter. This means that our models describe a range of scenarios where each agent has both a certain body of knowledge, and a certain stock of issues she is interested in. Moreover, we have seen that, implicitly, such models also include a representation of the group's common knowledge and common issues.

As in PAL, we will focus on the action of publicly uttering a sentence to the whole group of agents.[2] The main novelty is that now, the sentence which is uttered may be either a statement, or a question. As we will see, this means that in IDEL, the utterance of a sentence may have the effect of providing new information—as in PAL—but may also have the effect of raising a new issue.

Let us start out by specifying how an inquisitive epistemic model changes as a result of the public utterance of a sentence φ. This works very much like in PAL: when φ is uttered, all the worlds in which φ is false are dropped from the model, and the state of each agent at a world is restricted by intersecting it with the proposition $[\varphi]_M$ expressed by φ in the model.

8.2.1. Definition. [Updating a model with an inquisitive proposition]
Let $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$ be an inquisitive epistemic model, and let P be an inquisitive proposition over W.[3] The update of M with P is the model $M^P = \langle W^P, \{\Sigma_a^P \mid a \in \mathcal{A}\}, V^P\rangle$ defined as follows:

- $W^P = W \cap \mathsf{info}(P)$
- $V^P = V\restriction_{\mathsf{info}(P)}$
- $\Sigma_a^P(w) = \Sigma_a(w) \cap P$

To reduce clutter, given a formula φ we will write $M^\varphi = \langle W^\varphi, \{\Sigma_a^\varphi \mid a \in \mathcal{A}\}, V^\varphi\rangle$ instead of $M^{[\varphi]_M} = \langle W^{[\varphi]_M}, \{\Sigma_a^{[\varphi]_M} \mid a \in \mathcal{A}\}, V^{[\varphi]_M}\rangle$.

The following proposition says that the information state $\sigma_a^P(w)$ of an agent at a world in the updated model is obtained just like in PAL, by restricting the original epistemic state $\sigma_a(w)$ to the set of worlds where P is true.

8.2.2. Proposition.
For any model M, world w, agent a and inquisitive proposition P we have:

$$\sigma_a^P(w) = \sigma_a(w) \cap \mathit{info}(P)$$

[2] We speak of *utterance* rather than *announcement* because the latter term suggests an informational interpretation. For instance, in our setting, the utterance of a question such as $?p$ has the effect of raising the issue whether p (for each agent, and ultimately also as a public issue). This should not be confused with an action of *announcing whether* p, i.e., announcing the true answer to $?p$, which is a more standard action of providing information.

[3] Recall that an inquisitive proposition is a non-empty and downward closed set of information states. If P is an inquisitive proposition, its *informative content* is $\mathsf{info}(P) := \bigcup P$. See Definition 7.1.1 and Definition 7.1.2.

Proof. First of all, it is immediate to check that the intersection of two inquisitive propositions, i.e., of two non-empty and downward closed sets of information states, is itself an inquisitive proposition. This ensures that $\Sigma_a^P(w) = \Sigma_a(w) \cap P$ is indeed an inquisitive proposition.

Second, notice that if Q is an inquisitive proposition, then due to downward closure we have $w \in \mathsf{info}(Q) \iff \{w\} \in Q$. Since $\Sigma_a(w), \Sigma_a^P(w)$ and P are all inquisitive propositions, we have:

$$\begin{aligned} v \in \sigma_a^P(w) &\iff v \in \mathsf{info}(\Sigma_a^P(w)) \\ &\iff \{v\} \in \Sigma_a^P(w) \\ &\iff \{v\} \in \Sigma_a(w) \cap P \\ &\iff \{v\} \in \Sigma_a(w) \text{ and } \{v\} \in P \\ &\iff v \in \mathsf{info}(\Sigma_a(w)) \text{ and } v \in \mathsf{info}(P) \iff v \in \sigma_a(w) \cap \mathsf{info}(P) \end{aligned}$$

$\Box$

Notice in particular that, since $\mathsf{info}([\varphi]_M) = |\varphi|_M$, we have $\sigma_a^\varphi(w) = \sigma_a(w) \cap |\varphi|_M$.

We can use the proposition we just established to show that the update operation turns inquisitive epistemic models into inquisitive epistemic models.

8.2.3. Proposition.
For any inquisitive epistemic model M and for any inquisitive proposition P, M^P is an inquisitive epistemic model.

Proof. We have already remarked that, since the intersection of two inquisitive propositions is itself an inquisitive proposition, the value $\Sigma_a^P(w)$ of the new state maps on a world is indeed an inquisitive proposition, as it should be.

To see that the updated maps Σ_a^P satisfy the factivity condition, consider a world $w \in W^P = W \cap \mathsf{info}(P)$. Since the original map satisfies factivity by assumption, we have $w \in \sigma_a(w)$, and since $w \in \mathsf{info}(P)$ we can conclude $w \in \sigma_a(w) \cap \mathsf{info}(P)$, that is, according to the previous proposition, $w \in \sigma_a^P(w)$.

Finally, we have to check that the updated maps Σ_a^P satisfy the introspection condition. Suppose $v \in \sigma_a^P(w) = \sigma_a(w) \cap \mathsf{info}(P)$. Since the original map satisfies introspection by assumption, $v \in \sigma_a(w)$ implies $\Sigma_a(v) = \Sigma_a(w)$, whence $\Sigma_a^P(v) = \Sigma_a(v) \cap P = \Sigma_a(w) \cap P = \Sigma_a^P(w)$. $\Box$

Let us now turn to the logical language. As in the case of PAL, we will extend the static language of inquisitive epistemic logic with dynamic modalities which allow us to conditionalize a formula to the utterance of another formula. Thus, the language $\mathcal{L}_{\mathsf{IDEL}}$ is given by the following definition in Backus-Naur form:

$$\varphi \;:=\; p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee\, \varphi \mid K_a\varphi \mid E_a\varphi \mid [\varphi]\varphi$$

When talking of a formula $[\varphi]\psi$, we will refer to φ as the *label* of the dynamic modality $[\varphi]$, and to ψ as the *argument* of the modality. Semantically, assessing a sentence of the form $[\varphi]\psi$ at a pair $\langle M, s\rangle$ amounts to assessing ψ at the pair $\langle M^\varphi, s \cap |\varphi|_M\rangle$ consisting of the model resulting from the utterance of φ and the information state resulting from enhancing s with the information that φ is true (or, equivalently, from restricting s to W^φ). Thus, the semantics of IDEL is given by extending the support-clauses for IEL with the following clause.

8.2.4. Definition. [Support conditions for dynamic modalities]
$M, s \models [\varphi]\psi \iff M^\varphi, s \cap |\varphi|_M \models \psi$

By specializing these support conditions for dynamic modalities to singletons, we recover the truth-conditions which are familiar from PAL: $[\varphi]\psi$ is true at a world w in case, if φ is true at w, an utterance of φ leads to ψ being true at w in the updated model.

8.2.5. Proposition (Truth conditions for dynamic modalities).
$M, w \models [\varphi]\psi \iff M, w \models \varphi$ *implies* $M^\varphi, w \models \psi$

In order to familiarize ourselves with the effects of public utterances, let us consider first the special case in which the formula being uttered is a statement α—i.e., a truth-conditional formula. Consider an inquisitive epistemic model M. If P is an inquisitive proposition in M and s is an information state, the *restriction of P to s*, denoted $P \restriction s$, is the inquisitive proposition defined as follows:

$$P \restriction s = \{t \in P \mid t \subseteq s\}$$

Now, the fact that α is a statement implies that $[\alpha]_M = \{t \in W \mid t \subseteq |\alpha|_M\}$. Thus, if we compute the state map for agent a updated with α, we have:

$$\Sigma_a^\alpha(w) = \Sigma_a(w) \cap [\alpha]_M = \{s \in \Sigma_a(w) \mid s \subseteq |\alpha|_M\} = \Sigma_a(w) \restriction_{|\alpha|_M}$$

That is, the inquisitive state of a at w after the utterance of α is nothing but the restriction of the original state to the set of worlds in M where α is true. Since we will make use of this fact several times later on, it will be convenient to state it as a proposition.

8.2.6. Proposition.
If $\alpha \in \mathcal{L}^{\mathsf{IDEL}}$ is truth-conditional, then $\Sigma_a^\alpha(w) = \Sigma_a(w) \restriction_{|\alpha|_M}$

This shows that there is nothing more to the utterance of a statement than there used to be in standard PAL: as a consequence of the utterance of α, worlds where α was false are removed from the model, and all the agents' states are restricted to the remaining worlds. In other words, just as in PAL, the only consequence of the public utterance of α is to establish common knowledge of the fact that α was true at the moment of utterance.

Now consider the case in which the uttered sentence is a question, say $?p$. In this case the set of worlds where $?p$ is true is the whole space W, so no world is removed from the model in updating with $?p$. Moreover, by Proposition 8.2.2 we have $\sigma_a^{?p}(w) = \sigma_a(w) \cap \mathsf{info}([?p]_M) = \sigma_a(w) \cap |?p|_M = \sigma_a(w)$, which means that no agent gains any information from the utterance of the question. However, this does not mean that the update is trivial. For any agent a and world w, the update changes the inquisitive state of the agent, from $\Sigma_a(w)$ to $\Sigma_a^{?p}(w) = \Sigma_a(w) \cap [?p]_M$; this means that a's issues after the utterance of $?p$ are more demanding than they were before: they are only settled by an information state s in case: (i) s settles a's previous issues and (ii) s settles the question $?p$. In other words, if the agent did not previously entertain the question whether p, as a result of the utterance of $?p$ she will come to entertain it. In fact, as we will see in Section 8.4, unless it is already common knowledge whether p, an utterance of the question $?p$ will make $?p$ an open question for the group. Thus, a public utterance of $?p$ does not provide any new information, but in general it does raise a new issue.

Now, not any question is true at all worlds, as $?p$ is. For instance, consider the question $p \mathbin{\vee\!\!\!\vee} q$: the truth-conditions for this question coincide with those of the declarative disjunction $p \vee q$, which we referred to in Chapter 2 as the *presupposition* of $p \mathbin{\vee\!\!\!\vee} q$. Now consider the update of a model M with $p \mathbin{\vee\!\!\!\vee} q$. If M contains any worlds where both p and q are false, then these worlds will be eliminated from the model as a result of the update. Moreover, all agents' information states will be restricted accordingly, that is, we will have $\sigma_a^{p \mathbin{\vee\!\!\!\vee} q}(w) = \sigma_a(w) \cap |p \mathbin{\vee\!\!\!\vee} q|_M = \sigma_a(w) \cap |p \vee q|_M$. That is, after a public utterance of the question $p \mathbin{\vee\!\!\!\vee} q$, the agents learn that the presupposition $p \vee q$ of the question is true. Moreover, in general the new inquisitive state $\Sigma_a^{p \mathbin{\vee\!\!\!\vee} q}(w)$ of an agent after the update is not simply the restriction $\Sigma_a(w) \upharpoonright_{|p \vee q|_M}$ of the old state to the new information: rather, after the update, each agent comes to entertain at each world the issue expressed by $p \mathbin{\vee\!\!\!\vee} q$.[4]

These examples illustrate that, while the typical effect of uttering a statement is to provide new information, the typical effect of uttering a question is to raise a new issue. Notice that, although it may be convenient to talk of *asserting* for the act of uttering a statement, and of *asking* for the act of uttering a question, asserting and asking are in this view not two different kinds of action, but rather one and the same kind of action—utterance—performed with sentences which have a different semantic content. It is the content of the sentence being uttered which determines whether the utterance will bring about new information, raise new issues, or both. Thus, our uniform semantic account of statements and questions allows us to keep the dynamic component of the system elementary,

[4]The fact that, by asking a question, one can establish the truth of the question's presupposition may not seem completely satisfactory. A more natural account would regard the presupposition of a question as something that has to be established *before* the question can be appropriately asked (see, e.g., Hintikka (1999), Hamami (2015)). A refinement of the present account of the public utterance action in this sense will be investigated in section 8.5.

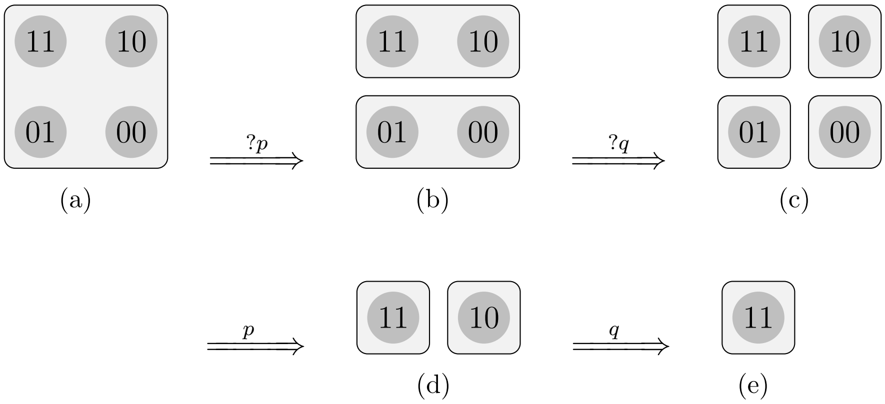


Figure 8.2: The effects of a series of simple utterances on a state.

while still capturing both the effect of asserting a statement, and the effect of asking a question. We will return to this important point when comparing our proposal with that of van Benthem and Minică (2012) in section 8.6.

Let us illustrate the effects of public utterances graphically for some very simple cases. Consider a model consisting of four worlds, $W = \{11, 10, 01, 00\}$, where the two digits correspond to the truth values of two atoms p and q: 11 makes both p and q true, 10 makes p true and q false, etc. Suppose that there is just one agent, a, and suppose that initially, in any $w \in W$, a's inquisitive state $\Sigma_a(w)$ is the tautological proposition $\wp(W)$, consisting of all subsets of W. This means that initially a has no (non-trivial) information and no (non-trivial) issues. Throughout this example, a's inquisitive state will always be the same for any $w \in W$, so we will simply denote it by Σ_a. Thus, a's initial state is depicted in Figure 8.2(a), where as usual we represent Σ_a by drawing its maximal elements.

Now suppose that a polar question $?p$ is publicly uttered. To capture the effect of this, Σ_a needs to be intersected with the proposition expressed by $?p$, which consist of all information states that support either p or $\neg p$. The resulting inquisitive state is depicted in Figure 8.2(b).

Next suppose that another polar question, $?q$, is publicly uttered. To compute the effect of this, Σ_a needs to be further intersected with the proposition expressed by $?q$, which consists of all information states that support either q or $\neg q$. The resulting state is depicted in Figure 8.2(c). Notice that a's *information* state, i.e., $\sigma_a = \bigcup \Sigma_a$, has not changed: no worlds have been eliminated, which means that no information has been gained. However, a's inquisitive state has been enhanced in a non-trivial way, capturing the fact that two issues have been raised: the issue whether p, and the issue whether q. The resulting inquisitive state consists precisely of those information states that resolve both these issues.

Now suppose that p is publicly uttered. The effect of this action is depicted in Figure 8.2(d). Now a's information state σ_a *does* change: worlds where p is false are eliminated. This resolves the issue initially raised by the question whether p. However, the resulting inquisitive state reflects the fact that the issues raised by the question whether q is still open.

Finally, suppose that q is publicly uttered. This leads to the inquisitive state in Figure 8.2(e). Again, σ_a changes through this utterance: all worlds where q is false are eliminated. This resolves the issue whether q, and leads to a situation in which a no longer entertains any (non-trivial) issues.

Now that we have some grasp of the effect of public utterances in IDEL, let us turn back to our logical language, and let us consider the significance of a dynamic formula of the form $[\varphi]\psi$. Let us start out by considering the special case in which ψ is truth-conditional.

8.2.7. PROPOSITION.
If ψ is truth-conditional, then $[\varphi]\psi$ is truth-conditional for any $\varphi \in \mathcal{L}_{\mathsf{IDEL}}$.

Proof. We need to show that $[\varphi]\psi$ is supported at a state as soon as it is true at every world in the state. So, suppose $[\varphi]\psi$ is true at any world in a state s of an inquisitive epistemic model M. This means that for all $w \in s$ we have $M, w \not\models \varphi$ or $M^\varphi, w \models \psi$. Now, this is just to say that for any $w \in s \cap |\varphi|_M$ we have $M^\varphi, w \models \psi$. Since ψ is truth-conditional, this is sufficient to have $M^\varphi, s \cap |\varphi|_M \models \psi$, which by definition amounts to $M, s \models [\varphi]\psi$. □

Thus, if ψ is a statement, then the semantics of the formula $[\varphi]\psi$ is fully captured by the truth-conditions for the formula, which are the same as in PAL. Thus, in this case $[\varphi]\psi$ is supported by a state if it is established that ψ will be true after an utterance of φ, if such an utterance is at all possible.

In particular, notice that $\mathcal{L}_{\mathsf{PAL}} \subseteq \mathcal{L}_{\mathsf{IDEL}}$ and that, if $[\varphi]\psi$ is actually a formula from the language of PAL, then the truth-conditions for this formula are exactly the same as in PAL. Proceeding by induction on the structure of a formula φ, we can then show that all formulas of PAL receive exactly the same treatment in IDEL: that is, they are truth-conditional, and their truth-conditions are the same as in PAL. This is made precise by the following proposition, where $M^K = \langle W, \{\sigma_a \mid a \in \mathcal{A}\}, V\rangle$ denotes the Kripke model determined by the inquisitive modal model $M = \langle W, \{\Sigma_a \mid a \in \mathcal{A}\}, V\rangle$.

8.2.8. PROPOSITION (IDEL IS CONSERVATIVE OVER PAL).
Let $\varphi \in \mathcal{L}^{\mathsf{PAL}}$. Then φ is truth-conditional in IDEL and for any inquisitive modal model M and world w:

$$M, w \models \varphi \iff M^K, w \models \varphi \text{ in } \mathsf{PAL}$$

Proof. By induction on the structure of formulas in $\mathcal{L}^{\mathsf{PAL}}$, making use of Proposition 8.2.5 and Proposition 8.2.7. □

We have seen that a dynamic modality applied to a statement has the standard significance. However, in IDEL, dynamic modalities can also be applied to questions. If μ is a question, then $[\varphi]\mu$ should be thought of as a "dynamically conditionalized" question, which asks to resolve μ under the assumption not just that φ is true, but that φ were actually uttered. As an example, consider the formula $[p]?K_aq$. We have the following support conditions, where for simplicity we drop the subscripts referring to M:

$$\begin{aligned}
M, s \models [p]?K_aq &\iff M^p, s \cap |p| \models ?K_aq \\
&\iff M^p, s \cap |p| \models K_aq \text{ or } M^p, s \cap |p| \models \neg K_aq \\
&\iff \text{for all } w \in s \cap |p|,\ M^p, w \models K_aq \text{ or} \\
&\qquad \text{for all } w \in s \cap |p|,\ M^p, w \models \neg K_aq \\
&\iff \text{for all } w \in s \cap |p|,\ \sigma_a^p(w) \subseteq |q|_{M^p} \text{ or} \\
&\qquad \text{for all } w \in s \cap |p|,\ \sigma_a^p(w) \not\subseteq |q|_{M^p} \\
&\iff \text{for all } w \in s \cap |p|,\ \sigma_a(w) \cap |p| \subseteq |q| \text{ or} \\
&\qquad \text{for all } w \in s \cap |p|,\ \sigma_a(w) \cap |p| \not\subseteq |q|
\end{aligned}$$

Thus, $[p]?K_aq$ is settled in case either (i) it is settled that, if p is true and if a were to learn that p, she would know that q, or (ii) it is settled that, if p is true and if a were to learn that p she would still not know that q. Thus, the formula $[p]?K_aq$ encodes the conditional question: "if p were truthfully and publicly uttered, would a know that q?".[5]

Compare $[p]?K_aq$ with the conditional $p \to ?K_aq$, which encodes the question "if p is true, does a know that q?". Knowing that $K_a(p \to q)$ is sufficient to resolve the former question, but not the latter: even assuming that p is true, there is no reason why this should be known to a, so we cannot tell whether a knows that q; on the other hand, we do know that, if p is true and if it were publicly uttered, then a would know that p, and as a consequence, since she would still know that $p \to q$, she would also know that q.

8.3 Axiomatizing IDEL

In this section, we will see that, like in PAL, the presence of dynamic modalities does not make IDEL more expressive than its static fragment, IEL. Any formula $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ can be recursively transformed into a corresponding formula $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$, and this also paves the way for a relatively simple completeness result, similar to those we reviewed above for PAL.

[5] Another way to put it is the following: $[p]?K_aq$ is equivalent to the inquisitive disjunction $[p]K_aq \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} [p]\neg K_aq$, which can be settled either by establishing that $[p]K_aq$, or by establishing that $[p]\neg K_aq$.

To see this, we first need to extend the notion of declaratives and resolutions to $\mathcal{L}^{\mathsf{IDEL}}$. Let us start out with declaratives: a declarative in IDEL will be a formula in which $\vee\!\!\!\vee$ only occurs in the following environments: (i) within the scope of a static modality K_a or E_a, or (ii) within the label of a dynamic modality.

8.3.1. Definition. [Declaratives]
The declarative fragment of $\mathcal{L}^{\mathsf{IDEL}}$, denoted $\mathcal{L}^{\mathsf{IDEL}}_!$, is the set of formulas defined inductively as follows:

- $p \in \mathcal{L}^{\mathsf{IDEL}}_!$
- $\bot \in \mathcal{L}^{\mathsf{IDEL}}_!$
- for any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, $K_a\varphi$ and $E_a\varphi$ are in $\mathcal{L}^{\mathsf{IDEL}}_!$
- if $\alpha, \beta \in \mathcal{L}^{\mathsf{IDEL}}_!$, then $\alpha \wedge \beta$ and $\alpha \to \beta$ are in $\mathcal{L}^{\mathsf{IDEL}}_!$
- if $\alpha \in \mathcal{L}^{\mathsf{IDEL}}_!$, then for any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, $[\varphi]\alpha \in \mathcal{L}^{\mathsf{IDEL}}_!$

Using Proposition 8.2.7, together with the fact that $\wedge$ and $\to$ preserve truth-conditionality (Proposition 2.3.9) and with the fact that formulas of the shape $K_a\varphi$ or $E_a\varphi$ are always truth-conditional, we can show inductively that any declarative formula is truth-conditional.

8.3.2. Proposition. *Any $\alpha \in \mathcal{L}^{\mathsf{IDEL}}_!$ is truth-conditional.*

As in IEL, declaratives are representative of all truth-conditional formulas in IDEL. As usual, we show this by associating to each formula φ a declarative $\varphi^!$ having the same truth-conditions.

8.3.3. Definition. [Declarative variant of a formula of $\mathcal{L}^{\mathsf{IDEL}}$]
The declarative variant $\varphi^!$ of a formula $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ is defined by augmenting definition 7.1.21 with the following clause:

- $([\varphi]\psi)^! = [\varphi]\psi^!$

It is immediate by the definition of declarative variant that $\varphi^!$ is a declarative for any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$. Moreover, the following proposition, which can be established by a straightforward induction, ensures that φ and $\varphi^!$ have the same truth-conditions.

8.3.4. Proposition.
For any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, any model M and world w: $M, w \models \varphi \iff M, w \models \varphi^!$

This shows that, like in IEL, any truth-conditional formula is equivalent to a declarative—namely, to its declarative variant. On the other hand, if μ is a question, then $\mu^!$ is a statement which is established in a state s just in case the information available in s ensures that μ can be truthfully resolved. In line with the discussion in Chapter 2, we will refer to $\mu^!$ as the *presupposition* of μ, and we will use the more suggestive notation π_μ instead of $\mu^!$.

8.3.5. Definition. [Presupposition]
If $\mu \in \mathcal{L}^{\mathsf{IDEL}}$ is a question, then the presupposition of μ is $\pi_\mu := \mu^!$

Moreover, as we did for IEL, we can associate with any formula $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ a set $\mathcal{R}(\varphi) \subseteq \mathcal{L}_!^{\mathsf{IDEL}}$ of declaratives, in such a way that settling φ amounts to establishing one (or more) of the declaratives $\alpha \in \mathcal{R}(\varphi)$.

8.3.6. Definition. [Resolutions]
The resolutions of a formula $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ are defined by augmenting Definition 7.1.24 with the following clause:

- $\mathcal{R}([\varphi]\psi) = \{[\varphi]\alpha \mid \alpha \in \mathcal{R}(\psi)\}$

Now, we want to show that, as usual, any formula in $\mathcal{L}^{\mathsf{IDEL}}$ can be represented as the inquisitive disjunction of its resolutions.

8.3.7. Proposition (Normal form for IDEL).
Let $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. Then $\varphi \equiv \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n$.

Proof. The proof is by induction on φ. We only spell out the inductive step for the dynamic modality. Let $\mathcal{R}(\psi) = \{\alpha_1, \dots, \alpha_n\}$, and consider the formula $[\varphi]\psi$. Using the induction hypothesis on ψ, we have:

$$\begin{aligned} M, s \models [\varphi]\psi \quad &\iff \quad M^\varphi, s \cap |\varphi|_M \models \psi \\ &\iff \quad M^\varphi, s \cap |\varphi|_M \models \alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee \alpha_n \\ &\iff \quad M^\varphi, s \cap |\varphi|_M \models \alpha_i \text{ for some } 1 \le i \le n \\ &\iff \quad M, s \models [\varphi]\alpha_i \text{ for some } 1 \le i \le n \\ &\iff \quad M, s \models [\varphi]\alpha_1 \vee\!\!\!\vee \dots \vee\!\!\!\vee [\varphi]\alpha_n \end{aligned}$$

which is what we need, since $\{[\varphi]\alpha_1, \dots, [\varphi]\alpha_n\} = \mathcal{R}([\varphi]\psi)$. □

One thing that will be useful later on is a characterization of the updated state of an agent in terms of resolutions. The following proposition states that the inquisitive state $\Sigma_a^\varphi(w)$ of an agent is obtained as the union, for α a resolution of φ, of the restriction of the state $\Sigma_a(w)$ to the α-worlds.

8.3.8. Proposition.
Let $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. Suppose M is an inquisitive epistemic model and w a world in it. Then:

$$\Sigma_a^\varphi(w) \;=\; \Sigma_a(w)\!\upharpoonright_{|\alpha_1|_M} \cup \dots \cup\; \Sigma_a(w)\!\upharpoonright_{|\alpha_n|_M}$$

Proof. The previous proposition ensures that $\varphi \equiv \alpha_1 \vee \ldots \vee \alpha_n$, which implies $[\varphi]_M = [\alpha_1 \vee \ldots \vee \alpha_n]_M = [\alpha_1]_M \cup \cdots \cup [\alpha_n]_M$. Using this fact, we have:

$$\begin{array}{rcl}\Sigma_a^\varphi(w) & = & \Sigma_a(w) \cap [\varphi]_M \quad = \quad \Sigma_a(w) \cap ([\alpha_1]_M \cup \cdots \cup [\alpha_n]_M) \\ & = & \bigcup\limits_{1 \le i \le n} (\Sigma_a(w) \cap [\alpha_i]_M) \quad = \quad \Sigma_a^{\alpha_1}(w) \cup \cdots \cup \Sigma_a^{\alpha_n}(w)\end{array}$$

Since any resolution is a declarative, and any declarative is truth-conditional, Proposition 8.2.6 ensures that $\Sigma_a^{\alpha_i}(w) = \Sigma_a(w) \upharpoonright_{|\alpha_i|_M}$. Thus, we can conclude $\Sigma_a^\varphi(w) = \bigcup_{1 \le i \le n} \Sigma_a(w) \upharpoonright_{|\alpha_i|_M}$. □

Let us now turn to showing how any occurrence of a dynamic modality can be paraphrased away, inductively on the structure of the argument. This is very easy if the argument is an atomic sentence, or $\bot$.

8.3.9. PROPOSITION. $[\varphi]p \equiv \varphi \to p$ *and* $[\varphi]\bot \equiv \neg\varphi$

Proof. Since p is truth-conditional, so are both $[\varphi]p$ (by Proposition 8.2.7) and $\varphi \to p$ (by Proposition 2.3.9). So, to establish the equivalence of these formulas we just have to show that they have the same truth-conditions. Since the update operation does not change the truth-value of atoms, we have:

$$\begin{array}{rcl}M, w \models [\varphi]p & \iff & M, w \models \varphi \text{ implies } M^\varphi, w \models p \\ & \iff & M, w \models \varphi \text{ implies } M, w \models p \iff M, w \models \varphi \to p\end{array}$$

The equivalence $[\varphi]\bot \equiv \neg\varphi$ is established by an analogous argument. □

As in PAL, dynamic modalities distribute smoothly over the connectives, which now also include inquisitive disjunction.

8.3.10. PROPOSITION. $[\varphi](\psi \wedge \chi) \equiv [\varphi]\psi \wedge [\varphi]\chi$

Proof. Immediate by inspecting the support conditions. □

8.3.11. PROPOSITION. $[\varphi](\psi \to \chi) \equiv [\varphi]\psi \to [\varphi]\chi$

Proof. We have the following, where the crucial passage from the second to the third line is justified by the simple set-theoretic facts that the subsets of $s \cap |\varphi|$ are all and only the sets of the form $t \cap |\varphi|$ for some $t \subseteq s$.

$$\begin{array}{rcl}M, s \models [\varphi](\psi \to \chi) & \iff & M^\varphi, s \cap |\varphi| \models \psi \to \chi \\ & \iff & \text{for any } t \subseteq s \cap |\varphi|, \ \text{ if } M^\varphi, t \models \psi \text{ then } M^\varphi, t \models \chi \\ & \iff & \text{for any } t \subseteq s, \ \text{ if } M^\varphi, t \cap |\varphi| \models \psi \text{ then } M^\varphi, t \cap |\varphi| \models \chi \\ & \iff & \text{for any } t \subseteq s, \ \text{ if } M, t \models [\varphi]\psi \text{ then } M, t \models [\varphi]\chi \\ & \iff & M, s \models [\varphi]\psi \to [\varphi]\chi\end{array}$$

□

8.3.12. PROPOSITION. $[\varphi](\psi \mathbin{\vee\!\!\!\vee} \chi) \equiv [\varphi]\psi \mathbin{\vee\!\!\!\vee} [\varphi]\chi$

Proof. Immediate by inspecting the support conditions. $\square$

A dynamic modality $[\varphi]$ over a K modality behaves just like in PAL: it can be brought within the scope of K, provided we condition the resulting formula on φ.

8.3.13. PROPOSITION. $[\varphi]K_a\psi \equiv \varphi \to K_a[\varphi]\psi$

Proof. Notice that, since $K_a\psi$ is truth-conditional, so are both $\varphi \to K_a[\varphi]\psi$ and $[\varphi]K_a\psi$, by propositions 8.2.7 and 2.3.9. Thus, in order to establish the equivalence we just have to show that they have identical truth-conditions. Making use of the fact that $\sigma_a^\varphi(w) = \sigma_a(w) \cap |\varphi|$ (Proposition 8.2.2), we have:

$$\begin{aligned}
M, w \models [\varphi]K_a\psi \quad &\iff \quad M, w \models \varphi \text{ implies } M^\varphi, w \models K_a\psi \\
&\iff \quad M, w \models \varphi \text{ implies } M^\varphi, \sigma_a^\varphi(w) \models \psi \\
&\iff \quad M, w \models \varphi \text{ implies } M^\varphi, \sigma_a(w) \cap |\varphi| \models \psi \\
&\iff \quad M, w \models \varphi \text{ implies } M, \sigma_a(w) \models [\varphi]\psi \\
&\iff \quad M, w \models \varphi \text{ implies } M, w \models K_a[\varphi]\psi \\
&\iff \quad M, w \models \varphi \to K_a[\varphi]\psi
\end{aligned}$$

$\square$

The reduction for the E modality is only slightly more complicated. However, the proof of the equivalence is rather subtle, and it relies on the use of resolutions.

8.3.14. PROPOSITION. $[\varphi]E_a\psi \equiv \varphi \to E_a(\varphi \to [\varphi]\psi)$

Proof. Let $\mathcal{R}(\varphi) = \{\alpha_1, \dots, \alpha_n\}$. First, notice that, by Proposition 8.3.8, the information states $s \in \Sigma_a^\varphi(w)$ are all and only the states of the form $s = t \cap |\alpha_i|_M$ for some $t \in \Sigma_a(w)$ and some α_i. Moreover, notice that since $\alpha_i \in \mathcal{R}(\varphi)$, we have $|\alpha_i|_M \subseteq |\varphi|_M$, whence $t \cap |\alpha_i|_M = t \cap |\alpha_i|_M \cap |\varphi|_M$. Finally, recall that the following equivalence is valid in inquisitive logic, for all formulas $\chi_1, \dots, \chi_n, \xi$:

$$(\chi_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} \chi_n) \to \xi \equiv (\chi_1 \to \xi) \wedge \dots \wedge (\chi_n \to \xi)$$

Now suppose that $M, w \models \varphi$, so that w is also a world in the updated model M^φ. Making use of all the facts we have just recalled, as well as of Proposition 8.3.7

we have:

$$
\begin{aligned}
M^{\varphi}, w \models E_a\psi \quad &\iff \quad \text{for all } s \in \Sigma_a^{\varphi}(w),\ M^{\varphi}, s \models \psi \\
&\iff \quad \text{for all } t \in \Sigma_a(w),\ \text{for all } \alpha_i,\ M^{\varphi}, t \cap |\alpha_i|_M \models \psi \\
&\iff \quad \text{for all } t \in \Sigma_a(w),\ \text{for all } \alpha_i,\ M^{\varphi}, t \cap |\alpha_i|_M \cap |\varphi|_M \models \psi \\
&\iff \quad \text{for all } t \in \Sigma_a(w),\ \text{for all } \alpha_i,\ M, t \cap |\alpha_i|_M \models [\varphi]\psi \\
&\iff \quad \text{for all } t \in \Sigma_a(w),\ \text{for all } \alpha_i,\ M, t \models \alpha_i \to [\varphi]\psi \\
&\iff \quad \text{for all } t \in \Sigma_a(w),\ M, t \models (\alpha_1 \to [\varphi]\psi) \wedge \dots \wedge (\alpha_n \to [\varphi]\psi) \\
&\iff \quad \text{for all } t \in \Sigma_a(w),\ M, t \models (\alpha_1 \mathbin{\vee\!\!\!\vee} \dots \mathbin{\vee\!\!\!\vee} \alpha_n) \to [\varphi]\psi \\
&\iff \quad \text{for all } t \in \Sigma_a(w),\ M, t \models \varphi \to [\varphi]\psi \\
&\iff \quad M, w \models E_a(\varphi \to [\varphi]\psi)
\end{aligned}
$$

Finally, using this equivalence we get, for any model M and world w:

$$
\begin{aligned}
M, w \models [\varphi]E_a\psi \quad &\iff \quad M, w \models \varphi \text{ implies } M^{\varphi}, w \models E_a\psi \\
&\iff \quad M, w \models \varphi \text{ implies } M, w \models E_a(\varphi \to [\varphi]\psi) \\
&\iff \quad M, w \models \varphi \to E_a(\varphi \to [\varphi]\psi)
\end{aligned}
$$

Since formulas headed by an E_a modality are truth-conditional, it follows from Proposition 8.2.7 and Proposition 2.3.9 that the formulas $\varphi \to E_a(\varphi \to [\varphi]\psi)$ and $[\varphi]E_a\psi$ are truth-conditional as well. Since we saw that these formulas have the same truth-conditions, it follows that they are equivalent. □

One may wonder whether the above equivalence for E_a may be simplified. After all, in PAL, it holds generally that $\varphi \to [\varphi]\psi \equiv [\varphi]\psi$, because $[\varphi]\psi$ is trivially true in all worlds where φ is false. If this was also true in IDEL, we could make the equivalence for the E_a modality simpler, and completely analogous to the equivalence we stated for K_a. This is indeed the case if φ is truth-conditional.

8.3.15. Proposition. *If $\alpha \in \mathcal{L}^{\mathsf{IDEL}}$ is truth-conditional, $\alpha \to [\alpha]\psi \equiv [\alpha]\psi$*

Proof. Recall that, if α is truth-conditional, then a conditional $\alpha \to \varphi$ is supported in a state s just in case φ is supported in the state $s \cap |\alpha|_M$ that results from enhancing s with the information that α is true (Proposition 2.2.9). Thus:

$$
\begin{aligned}
M, s \models \alpha \to [\alpha]\psi \quad &\iff \quad M, s \cap |\alpha|_M \models [\alpha]\psi \\
&\iff \quad M^{\alpha}, s \cap |\alpha|_M \cap |\alpha|_M \models \psi \\
&\iff \quad M^{\alpha}, s \cap |\alpha|_M \models \psi \quad \iff \quad M, s \models [\alpha]\psi
\end{aligned}
$$

□

However, this does not generalize to the case in which φ is a question. This is because there is an important difference between the role of questions as conditional antecedents, and as labels of dynamic modalities: in the former case, the consequent of the implication is evaluated with respect to those states that result from *settling* the question; in the latter case, the argument of the modality is evaluated in the model-state pair that results from *publicly asking* the question.

For a concrete illustration, consider the question $?p$. Since $|?p|_M$ always amounts to the set of all possible worlds, we have:

$$M, s \models [?p]\psi \iff M^{?p}, s \cap |?p|_M \models \psi \iff M^{?p}, s \models \psi$$

Thus, $[?p]\psi$ is supported at s in case ψ is supported at s after the question $?p$ has been asked. On the other hand, We have:

$$M, s \models ?p \to [?p]\psi \iff \text{for all } t \subseteq s: \ M, t \models ?p \text{ implies } M^{?p}, t \models \psi$$

Thus, $?p \to [?p]\psi$ is supported at s if *for any enhancement t of s that settles $?p$*, ψ is supported at t after $?p$ is asked. This means that, in particular, $?p \to [?p]?p \equiv \top$. On the other hand, using the above clause it is easy to see that $[?p]?p \equiv ?p$. Thus, $?p \to [?p]?p \not\equiv [?p]?p$. This establishes the following proposition.

8.3.16. Proposition. *In general, $\varphi \to [\varphi]\psi \not\equiv [\varphi]\psi$*

Let us now come back to showing that any IDEL formula is equivalent to an IEL-one. With propositions 8.3.9–8.3.14 in place, we are now ready to prove this.

8.3.17. Theorem. *For any $\varphi \in \mathcal{L}^{IDEL}$, $\varphi \equiv \varphi^*$ for some $\varphi^* \in \mathcal{L}^{IEL}$.*

Proof. By induction on the structure of φ. The only inductive step which is not immediate is the one for a formula $\varphi = [\psi]\chi$. By induction hypothesis, we have two formulas $\psi^*, \chi^* \in \mathcal{L}^{\mathsf{IEL}}$ which are equivalent to ψ and χ respectively. Thus, $\varphi \equiv [\psi^*]\chi^*$. Now we just have to show that $[\psi^*]\chi^* \equiv \varphi^*$ for some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$. For this, we proceed by induction on the complexity of χ^*. Notice that since $\chi^* \in \mathcal{L}^{\mathsf{IEL}}$, we only have to consider the basic cases (atoms and $\bot$) and the inductive steps for the connectives and the modalities K_a and E_a. Now, each of these cases corresponds precisely to one of the equivalences that we just established. We only discuss the basic cases and one inductive step, since the others are analogous.

- **Basic case.** If χ^* is an atom or $\bot$, then Proposition 8.3.9 ensures that we can take $\varphi^* := \psi^* \to \chi^*$.

- **Inductive step for E_a.** Suppose $\chi^* = E_a\xi$. By Proposition 8.3.14 we have $\varphi \equiv [\psi^*]\chi^* = [\psi^*]E_a\xi \equiv \psi^* \to E_a(\psi^* \to [\psi^*]\xi)$. Now, ξ is strictly less complex than χ^*, so by induction hypothesis we have a formula $\xi^* \in \mathcal{L}^{\mathsf{IEL}}$ such that $[\psi^*]\xi \equiv \xi^*$. It follows that $\varphi \equiv \psi^* \to E_a(\psi^* \to \xi^*)$. Since both ψ^* and ξ^* are in $\mathcal{L}^{\mathsf{EL}}$, so is the formula $\psi^* \to E_a(\psi^* \to \xi^*)$, which we can thus take to be the desired φ^*.

$\square$

!Atom	!$\bot$	!$\wedge$	!$\to$
$\dfrac{[\varphi]p}{\varphi \to p}$	$\dfrac{[\varphi]\bot}{\neg\varphi}$	$\dfrac{[\varphi](\psi \wedge \chi)}{[\varphi]\psi \wedge [\varphi]\chi}$	$\dfrac{[\varphi](\psi \to \chi)}{[\varphi]\psi \to [\varphi]\chi}$
!$\mathbin{\vee\!\!\!\vee}$	!K	!E	RE
$\dfrac{[\varphi](\psi \mathbin{\vee\!\!\!\vee} \chi)}{[\varphi]\psi \mathbin{\vee\!\!\!\vee} [\varphi]\chi}$	$\dfrac{[\varphi]K_a\psi}{\varphi \to K_a[\varphi]\psi}$	$\dfrac{[\varphi]E_a\psi}{\varphi \to E_a(\varphi \to [\varphi]\psi)}$	$\dfrac{\varphi \leftrightarrow \psi}{\chi[\varphi/p] \leftrightarrow \chi[\psi/p]}$

Figure 8.3: Inference rules for dynamic modalities in IDEL. Notice that all rules but the last one are bi-directional, that is, they make the formulas above and below the double line interderivable.

We can use this reduction of IDEL to IEL to provide a complete axiomatization for IDEL. All we need to do is to enrich the complete system for IEL described in the previous chapter (see Corollary 7.4.20) by adding inference rules which allow us to prove the equivalence between any formula $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and the corresponding formula $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$. The easiest way to achieve this, analogous to the most standard axiomatization of PAL, is to make all of the above equivalences into inference rules, and to equip our system with a rule of replacement of equivalents. Thus, our full system will consist of all the inference rules for IEL, plus the rules displayed in Figure 8.3. We will denote the relation of derivability in this system by $\vdash_{\mathsf{IDEL}^{\mathsf{RE}}}$, and the relation of inter-derivability by $\dashv\vdash_{\mathsf{IDEL}^{\mathsf{RE}}}$.

First of all, notice that the results we have proved so far show that the given proof system is sound for IDEL.

8.3.18. Proposition (Soundness).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IDEL}}$, $\Phi \vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \psi$ implies $\Phi \models \psi$.

Moreover, the next Proposition states that the rules of our system are indeed sufficient to prove any formula $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ to be equivalent to some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$.

8.3.19. Proposition.
For any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, $\varphi \dashv\vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \varphi^*$ for some $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$.

Proof. We retrace exactly the steps of the proof of Theorem 8.3.17, but now replacing the relation of logical equivalence with the relation $\dashv\vdash_{\mathsf{IDEL}^{\mathsf{RE}}}$ of inter-derivability in the system. $\square$

The completeness of our system follows immediately from this and from the completeness of our system with respect to formulas in IEL.

8.3.20. Theorem (Completeness).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IDEL}}$*,* $\Phi \models \psi$ *implies* $\Phi \vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \psi$*.*

Proof. For any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, let φ^* be a formula in $\mathcal{L}^{\mathsf{IEL}}$ such that $\varphi \dashv\vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \varphi^*$. The previous proposition guarantees that such a formula exists. Moreover, if $\Phi \subseteq \mathcal{L}^{\mathsf{IDEL}}$, let $\Phi^* = \{\varphi^* \mid \varphi \in \Phi\}$. Now, suppose $\Phi \models \psi$. This implies that $\Phi^* \models \psi^*$. Now this is an entailment which is valid in IEL. Since our system contains a complete system for IEL, we have $\Phi^* \vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \psi^*$. Now, let $\varphi_1^*, \ldots, \varphi_n^* \in \Phi^*$ be such that $\varphi_1^*, \ldots, \varphi_n^* \vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \psi^*$. Since we have $\varphi_i \dashv\vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \varphi_i^*$ for $1 \leq i \leq n$, as well as $\psi \dashv\vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \psi^*$, it follows that $\varphi_1, \ldots, \varphi_n \vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \psi$, which implies $\Phi \vdash_{\mathsf{IDEL}^{\mathsf{RE}}} \psi$. $\square$

This establishes a simple axiomatization for IDEL. As in the case of PAL, there are some interesting alternatives that we can use instead of the rule of replacement of equivalents to guarantee that our system for IDEL can perform the reduction. One option is to notice that, just like in PAL, so also in IDEL two subsequent utterances can be merged into a unique, complex one.

8.3.21. Proposition (Update composition).
For any formulas $\varphi, \psi \in \mathcal{L}^{\mathsf{IDEL}}$ *and for any model* M*:* $(M^\varphi)^\psi = M^{\varphi \wedge [\varphi]\psi}$

Proof. In order to show that $(M^\varphi)^\psi = M^{\varphi \wedge [\varphi]\psi}$, let us start out by showing that the sets of worlds $(W^\varphi)^\psi$ and $W^{\varphi \wedge [\varphi]\psi}$ are the same. We have

$$W^{\varphi \wedge [\varphi]\psi} = W \cap |\varphi \wedge [\varphi]\psi|_M = W \cap |\varphi|_M \cap |[\varphi]\psi|_M$$

But now, if we take a world $w \in W$ such that $M, w \models \varphi$, then $M, w \models [\varphi]\psi \iff M^\varphi, w \models \psi$. This means that $|\varphi|_M \cap |[\varphi]\psi|_M = |\varphi|_M \cap |\psi|_{M^\varphi}$. So, we have:

$$W \cap |\varphi|_M \cap |[\varphi]\psi|_M = W \cap |\varphi|_M \cap |\psi|_{M^\varphi} = W^\varphi \cap |\psi|_{M^\varphi} = (W^\varphi)^\psi$$

This shows that the models $(M^\varphi)^\psi$ and $M^{\varphi \wedge [\varphi]\psi}$ have the same universe of possible worlds. Since the truth-value of a propositional atom at a world is not affected by updates, the two also have the same valuation function. Now let us consider the state map for an agent a at a world w. We have:

$$\Sigma_a^{\varphi \wedge [\varphi]\psi}(w) = \Sigma_a(w) \cap [\varphi \wedge [\varphi]\psi]_M = \Sigma_a(w) \cap [\varphi]_M \cap [[\varphi]\psi]_M$$

Now, consider a state $s \in [\varphi]_M$. We have $s \subseteq \bigcup[\varphi]_M = |\varphi|_M$, so $s \cap |\varphi|_M = s$. This means that for such an s we have $M, s \models [\varphi]\psi \iff M^\varphi, s \cap |\varphi|_M \models \psi \iff M^\varphi, s \models \psi$. This shows that $[\varphi]_M \cap [[\varphi]\psi]_M = [\varphi]_M \cap [\psi]_{M^\varphi}$. Using this fact, we obtain:

$$\Sigma_a(w) \cap [\varphi]_M \cap [[\varphi]\psi]_M = \Sigma_a(w) \cap [\varphi]_M \cap [\psi]_{M^\varphi} = \Sigma_a^\varphi(w) \cap [\psi]_{M^\varphi} = (\Sigma_a^\varphi)^\psi(w)$$

Since this holds for any world w, this shows that the models $(M^\varphi)^\psi$ and $M^{\varphi \wedge [\varphi]\psi}$ also have the same state map for each agent, which completes the proof of the fact that they are identical. $\square$

The fact that two subsequent updates have the same effect as a single update with a complex formula gives rise to the following validity, which shows that two dynamic modalities can be combined into one.

8.3.22. Proposition (Composition of dynamic modalities).
$[\varphi][\psi]\chi \equiv [\varphi \wedge [\varphi]\psi]\chi$

Proof. Consider any model M and any state s. In the proof of the previous Proposition, we have established that $(M^{\varphi})^{\psi} = M^{\varphi\wedge[\varphi]\psi}$. Moreover, we have also established that $|\varphi \wedge [\varphi]\psi|_M = |\varphi|_M \cap |\psi|_{M^{\varphi}}$. Making use of these facts, we have:

$$\begin{aligned} M, s \models [\varphi][\psi]\chi &\iff M^{\varphi}, s \cap |\varphi|_M \models [\psi]\chi \\ &\iff (M^{\varphi})^{\psi}, s \cap |\varphi|_M \cap |\psi|_{M^{\varphi}} \models \chi \\ &\iff M^{\varphi\wedge[\varphi]\psi}, s \cap |\varphi \wedge [\varphi]\psi|_M \models \chi \\ &\iff M, s \models [\varphi \wedge [\varphi]\psi]\chi \end{aligned}$$

□

We can make this equivalence into a bi-directional inference rule, which we will denote !Comp, as in the case of PAL.

$$\frac{[\varphi][\psi]\chi}{\overline{[\varphi \wedge [\varphi]\psi]\chi}}$$

Now, consider the proof system $\vdash_{\mathsf{IDEL}^{!\mathrm{Comp}}}$ which is just like $\vdash_{\mathsf{IDEL}^{\mathsf{RE}}}$, except that the rule RE of replacement of equivalents is substituted by the rule !Comp. The previous proposition ensures that this proof system is sound. The next Theorem says that it is also complete.

8.3.23. Theorem (Completeness via !Comp).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathit{IDEL}}$, $\Phi \models \psi$ *implies* $\Phi \vdash_{\mathit{IDEL}^{!\mathit{Comp}}} \psi$.

Proof. The proof is completely analogous to the one given in van Ditmarsch et al. (2007) for PAL. I only give the proof strategy, and refer to Chapter 7.4 of van Ditmarsch et al. (2007) for the technical details. First, we define a non-standard measure $c : \mathcal{L}^{\mathsf{IDEL}} \to \mathbb{N}$ of complexity, which makes the following true:

- if ψ is a proper sub-formula of φ, then $c(\varphi) > c(\psi)$
- $c([\varphi]p) > c(\varphi \to p)$ and $c([\varphi]\bot) > c(\neg\varphi)$
- $c([\varphi](\psi \circ \chi)) > c([\varphi]\psi \circ [\varphi]\chi)$ for $\circ \in \{\wedge, \to, \mathbin{\vee\!\!\!\vee}\}$
- $c([\varphi]K_a\psi) > c(\varphi \to K_a[\varphi]\psi)$
- $c([\varphi]E_a\psi) > c(\varphi \to E_a(\varphi \to [\varphi]\psi))$

- $c([\varphi][\psi]\chi) > c([\varphi \wedge [\varphi]\psi]\chi)$

Proceeding by induction on this notion of complexity, we can define a translation $(\cdot)^* : \mathcal{L}^{\mathsf{IDEL}} \to \mathcal{L}^{\mathsf{IEL}}$ as follows.

- $p^* = p$
- $\bot^* = \bot$
- $(\varphi \circ \psi)^* = \varphi^* \circ \psi^*$ for $\circ \in \{\wedge, \to, \vee\!\!\!\vee\}$
- $(\blacksquare\varphi)^* = \blacksquare\varphi^*$ for $\blacksquare \in \{K_a, E_a\}$
- $([\varphi]p)^* = (\varphi \to p)^*$
- $([\varphi]\bot)^* = (\neg\varphi)^*$
- $([\varphi](\psi \circ \chi))^* = ([\varphi]\psi \circ [\varphi]\chi)^*$ for $\circ \in \{\wedge, \to, \vee\!\!\!\vee\}$
- $([\varphi]K_a\psi)^* = (\varphi \to K_a[\varphi]\psi)^*$
- $([\varphi]E_a\psi)^* = (\varphi \to E_a(\varphi \to [\varphi]\psi))^*$
- $([\varphi][\psi]\chi)^* = ([\varphi \wedge [\varphi]\psi]\chi)^*$

It is then straightforward to prove, by induction on the complexity of a formula, that for any $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ we have $\varphi \dashv\vdash_{\mathsf{IDEL}^{!\mathsf{Comp}}} \varphi^*$. Given this, the completeness results follows from the fact that $\dashv\vdash_{\mathsf{IDEL}^{!\mathsf{Comp}}}$ includes a complete system for IEL.$\square$

Furthermore, as for PAL, there is yet another alternative that we can use, instead of the rules RE and !Comp. Namely, we can add a rule that captures the monotonicity of dynamic modalities. The next proposition ensures that dynamic modalities are indeed monotonic. The straightforward proof is omitted.

8.3.24. Proposition (Dynamic modalities are monotonic).
Let $\varphi, \psi_1, \ldots, \psi_n, \chi \in \mathcal{L}^{\mathsf{IDEL}}$. If $\psi_1, \ldots, \psi_n \models \chi$, then $[\varphi]\psi_1, \ldots, [\varphi]\psi_n \models [\varphi]\chi$.

We can capture this property by means of a rule that allows us to infer $[\varphi]\chi$ from $[\varphi]\psi_1, \ldots, [\varphi]\psi_n$ provided we can deduce χ from $\psi_1, \ldots, \psi_n$. That is, we can use the following rule, where $\psi_1, \ldots, \psi_n$ are required to be the only undischarged assumptions in the proof leading to χ.

$$\dfrac{\begin{array}{c}[\psi_1] \quad \ldots \quad [\psi_n] \\ \vdots \\ \chi\end{array} \qquad [\varphi]\psi_1 \quad \ldots \quad [\varphi]\psi_n}{[\varphi]\chi}$$

Let us denote by $\vdash_{\mathsf{IDEL}^{!\mathsf{Mon}}}$ the proof system which is just like $\vdash_{\mathsf{IDEL}^{\mathsf{RE}}}$, except that the rule RE of replacement of equivalents is substituted by the rule !Mon. The previous proposition ensures that this system is sound. The next Theorem says that it is also complete.

8.3.25. Theorem (Completeness via !Mon).
For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}^{\mathsf{IDEL}}$, $\Phi \models \psi$ implies $\Phi \vdash_{\mathsf{IDEL}^{!\mathsf{Mon}}} \psi$.

Proof. The proof is very similar to that of Theorem 8.3.23. We modify the translation given in that proof slightly by setting:

$$([\varphi][\psi]\chi)^* = ([\varphi]([\psi]\chi)^*)^*$$

Using again the notion of complexity from p. 187 of van Ditmarsch et al. (2007), we can prove by induction on the complexity of φ that φ^* is well-defined, and that $\varphi^* \in \mathcal{L}^{\mathsf{IEL}}$. Since our system includes a complete system for IEL, we just have to prove that $\varphi \dashv\vdash_{\mathsf{IDEL}^{!\mathsf{Mon}}} \varphi^*$. Again, we proceed by induction on the complexity of φ. The only part which is not immediate is the inductive step for a formula of the form $[\varphi][\psi]\chi$. Since $[\psi]\chi$ is less complex than $[\varphi][\psi]\chi$, the induction hypothesis gives $[\psi]\chi \dashv\vdash_{\mathsf{IDEL}^{!\mathsf{Mon}}} ([\psi]\chi)^*$. By two applications of monotonicity for the dynamic modality we get $[\varphi][\psi]\chi \dashv\vdash_{\mathsf{IDEL}^{!\mathsf{Mon}}} [\varphi]([\psi]\chi)^*$. Now, since $[\varphi]([\psi]\chi)^*$ is less complex than $[\varphi][\psi]\chi$ (notice that we need to prove this in order to show that $([\varphi][\psi]\chi)^*$ is well-defined), the induction hypothesis gives $[\varphi]([\psi]\chi)^* \dashv\vdash_{\mathsf{IDEL}^{!\mathsf{Mon}}} ([\varphi]([\psi]\chi)^*)^*$. Putting things together, we have $[\varphi][\psi]\chi \dashv\vdash_{\mathsf{IDEL}^{!\mathsf{Mon}}} ([\varphi]([\psi]\chi)^*)^*$ which is what we need, since by definition $([\varphi]([\psi]\chi)^*)^* = ([\varphi][\psi]\chi)^*$. □

We have thus established three different axiomatizations for IDEL, which correspond precisely to the three axiomatizations for PAL described in Section 8.1.

8.4 Non-epistemic utterances

By uttering a statement α, an agent typically intends to make α common knowledge among the group. Similarly, by uttering a question μ, an agent typically intends to make μ an open issue for the group. However, it is not always the case that after an utterance of a statement α we have $K_*\alpha$, and after an utterance of a question μ we have $W_*\mu$. In other words, the formulas $[\alpha]K_*\alpha$ and $[\mu]W_*\mu$ are not generally valid in IDEL. In part, this is due to the presence of self-falsifying sentences such as the one we have mentioned in the context of PAL. Since uttering a statement essentially works as in PAL, it is easy to see that in IDEL, too, uttering the statement $\alpha = p \wedge \neg K_a p$ leads to a model in which α is false at all worlds, and therefore is not common knowledge.

Besides self-falsifying statements, in IDEL we also have self-resolving questions. As an example, consider the question $\mu = (p \wedge K_a\, p) \mathbin{\vee\!\!\!\vee} (p \wedge \neg K_a\, p)$, which asks whether p is true and a knows it, or whether p is true but a doesn't know it. Given that both "a knows that p" and "a doesn't know that p" normally presuppose p in natural language, this is actually a natural rendering of a question such as (1).

(1) Does Alice know that we are preparing a surprise party?

The presupposition of this question is $\pi_\mu = (p \wedge K_a p) \vee (p \wedge \neg K_a\, p) \equiv p$—in the case of our example, that we are preparing a surprise party; the question μ is settled if, in addition to this, we establish whether or not $K_a\, p$ holds.

Now, imagine that μ is uttered. As a result of this, the universe of our model M is restricted to those worlds where the presupposition of μ is true, i.e., to the p-worlds. But then, in the updated model M^μ, it must hold in any world that a knows that p—after all, all the worlds in the model are p-worlds. So, at any state in the resulting model, the question μ will be settled. In particular, μ will be settled by the group's common knowledge, which means that at any world w we will have $K_*\mu$. In turn, since $W_*\mu = \neg K_*\mu \wedge E_*\mu$, this implies $\neg W_*\mu$. In words, the result of uttering the question μ is not that μ becomes an open issue for the group, but rather that μ becomes publicly settled. This is quite intuitive if we think of the question (1) being publicly uttered to a group which includes Alice. After (1) has been asked, one may answer: "Well, she certainly does *now*!".

Self-falsifying sentences, also known as *Moore sentences*, are the subject of much interesting research in the dynamic epistemic logic community (see, e.g., van Ditmarsch and Kooi, 2006; Holliday and Icard, 2010). Similarly, there is probably much to understand about the phenomenon of self-resolving questions in IDEL. However, in this section I want to set aside the subtleties involved in the utterance of inquisitive-epistemic formulas, and focus on the utterance of more mundane formulas that do not contain any occurrence of the K and E modalities. We will refer to these formulas as *non-epistemic formulas*. More formally, the set of non-epistemic formulas is given by the following definition:

$$\varphi \ ::= \ p \,|\, \bot \,|\, \varphi \wedge \varphi \,|\, \varphi \to \varphi \,|\, \varphi \mathbin{\vee\!\!\!\vee} \varphi \,|\, [\varphi]\varphi$$

The reason for focusing on these formulas is that we want to get a more in-depth understanding of how an inquisitive-epistemic situation changes as a result of the utterance of an "ordinary" sentence, such as (2) or (3).

(2) The engine is broken.

(3) Is the engine broken?

Throughout this section and the following one, it will be useful to work with the extended language $\mathcal{L}_*^{\mathsf{IDEL}}$ which includes the common knowledge modality K_* and the common entertain modality E_*—introduced and discussed in Section 7.2.4—as well as the wondering modality W_*, defined as $W_*\varphi = \neg K_*\varphi \wedge E_*\varphi$.

The first thing that we want to show is that the support-conditions of a non-epistemic formula are not affected by the utterance of a sentence. To prove this, it is useful to first remark that non-epistemic formulas are essentially just formulas of propositional inquisitive logic: for, propositional formulas are by definition non-epistemic and conversely, the following proposition ensures that a non-epistemic formula is always equivalent to a propositional formula.

8.4.1. Proposition.
If $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ is non-epistemic, then $\varphi \equiv \varphi^*$ for some formula φ^* in the language of propositional inquisitive logic.

Proof. It is easy to verify by induction that, if φ is non-epistemic, then the translation φ^* of φ in IEL as defined in the proof of Theorem 8.3.23 is indeed a propositional formula. $\Box$

Using this fact, it is easy to establish the desired invariance result.

8.4.2. Proposition (Update-invariance).
Let $\varphi, \psi \in \mathcal{L}^{\mathsf{IDEL}}$ with ψ non-epistemic. For any model M and state $s \subseteq W^\varphi$:

$$M^\varphi, s \models \psi \iff M, s \models \psi$$

Proof. By the previous proposition, we can assume without loss of generality that ψ is a propositional formula, i.e., that ψ is built up from atoms and $\bot$ by means of the connectives $\wedge, \to$, and $\vee\!\!\!\vee$. Given this, a straightforward induction on the structure of ψ suffices to establish the claim. $\Box$

This proposition tells us that for any model M, any formula φ and any non-epistemic formula ψ, we have $[\psi]_{M^\varphi} = [\psi]_M \upharpoonright_{|\varphi|_M}$. In particular, when $\varphi = \psi$, $[\varphi]_M \upharpoonright_{|\varphi|_M}$ boils down to $[\varphi]_M$, because any $s \in [\varphi]_M$ is a subset of $|\varphi|_M = \bigcup[\varphi]_M$. So, we have the following fact.

8.4.3. Fact. If $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ is non-epistemic, then for any M: $[\varphi]_{M^\varphi} = [\varphi]_M$

Now consider the model M^φ resulting from the utterance of a non-epistemic formula φ. At any world $w \in W^\varphi$ we have by construction that $\Sigma^\varphi_a(w) \subseteq [\varphi]_M$ and thus by the previous proposition $\Sigma^\varphi_a(w) \subseteq [\varphi]_{M^\varphi}$. And since this is true for any agent a and for any world $w \in \mathcal{W}^\varphi$, it follows by definition of Σ^φ_* that we must also have $\Sigma^\varphi_*(w) \subseteq [\varphi]_{M^\varphi}$ at each world w. But this means precisely that $E_*\varphi$ will be true at any world in the updated model, and thus is also supported at any state in the updated model.

8.4.4. Proposition. *If φ is non-epistemic, $[\varphi]E_*\varphi$ is logically valid.*

Proof. We have $M, s \models [\varphi]E_*\varphi \iff M^\varphi, s \cap |\varphi|_M \models E_*\varphi$. But we have just argued that $E^*\varphi$ is supported in any state in M^φ. $\Box$

So, the utterance of a non-epistemic formula φ is guaranteed to lead to φ becoming commonly entertained. Now, if φ is a statement α, then we know that $E_*\alpha \equiv K_*\alpha$. Thus, uttering a non-epistemic statement α leads to α being common knowledge, just like in standard PAL.

8.4.5. Proposition.
If α is a non-epistemic statement, then $[\alpha]K_*\alpha$ is logically valid.

On the other hand, if φ is a question μ, then since $E_*\mu \equiv K_*\mu \vee W_*\mu$, what Proposition 8.4.4 shows is that uttering a non-epistemic question μ leads to a situation in which μ is either publicly settled, or a public open issue.

Now, the former will always be the case if μ was *already* publicly settled before the utterance took place. However, μ could also become publicly settled by the very information that it establishes. As an example, consider a model M and a world w such that $M, w \models q \wedge \neg K_*q \wedge K_*\neg p$, and let μ be the non-epistemic question $p \underline{\mathbin{\vee\!\!\!\vee}} q := (p \wedge \neg q) \mathbin{\vee\!\!\!\vee} (q \wedge \neg p)$. It is easy to see that $M, w \models \neg K_*\mu \wedge [\mu]K_*\mu$, that is, the question μ becomes publicly settled as a result of being uttered: for, the utterance of μ establishes common knowledge that exactly one of p and q is true, which, together with the previous common knowledge that $\neg p$ (which is not affected by the update) creates common knowledge of $\neg p \wedge q$, which is sufficient to resolve μ.

On the other hand, if the common knowledge of a group is sufficient to establish the presupposition π_μ of a certain non-epistemic question μ, but is not sufficient to resolve μ, then indeed uttering μ has the effect of making μ an open issue. In order to see this, we will first prove the following fact, which holds for any sentence φ: if φ is already common knowledge at a world, then a public utterance of φ does not enhance any information state at that world.

8.4.6. Proposition.
Let M be an inquisitive epistemic model, w a world in M and $\varphi \in \mathcal{L}_*^{\mathsf{IDEL}}$. If $\sigma_*(w) \subseteq |\varphi|_M$, then $\sigma_a^\varphi(w) = \sigma_a(w)$ for any $a \in \mathcal{A}$, and $\sigma_*^\varphi(w) = \sigma_*(w)$.

Proof. First, if $\sigma_*(w) \subseteq |\varphi|_M$, then since $w \in \sigma_*(w)$ we have $w \in |\varphi|_M$, which means that w is also a world in the updated model M^φ. Consider first the case of a private state $\sigma_a^\varphi(w)$. Since $\sigma_a(w) \subseteq \sigma_*(w) \subseteq |\varphi|_M$, Fact 8.2.2 yields

$$\sigma_a^\varphi(w) = \sigma_a(w) \cap |\varphi|_M = \sigma_a(w)$$

Now consider common knowledge. Obviously, $\sigma_*^\varphi(w) \subseteq \sigma_*(w)$. Conversely, suppose $v \in \sigma_*(w)$: then there is a sequence $u_0, \dots, u_{n+1}$ of worlds and a sequence $a_1, \dots, a_n$ of agents with $u_0 = w$, $u_{n+1} = v$ and $u_{i+1} \in \sigma_{a_i}(u_i)$ for $0 \leq i \leq n$. But it is easy to prove by induction on the index i that at each u_i we must have $\sigma_*(u_i) \subseteq |\varphi|_M$, whence by what we just established for private maps, $\sigma_{a_i}(u_i) = \sigma_{a_i}^\varphi(u_i)$. So, we have a sequence $u_0, \dots, u_{n+1}$ of worlds and a sequence $a_1, \dots, a_n$ of agents with $u_0 = w$, $u_{n+1} = v$ and $u_{i+1} \in \sigma_{a_i}^\varphi(u_i)$ for $0 \leq i \leq n$, which means that $v \in \sigma_*^\varphi(w)$. $\square$

With this fact in place we are ready to prove the fact mentioned before: if the presupposition of a non-epistemic question μ is common knowledge, but μ is not publicly settled, then a public utterance of μ results in μ becoming an open issue.

8.4.7. Proposition.
If μ is a non-epistemic question, then the following is logically valid:

$$(K_*\pi_\mu \wedge \neg K_*\mu) \to [\mu]W_*\mu$$

Proof. Since $W_*\mu$ is truth-conditional, propositions 8.2.7 and 2.3.9 imply that so is the above implication as a whole. So, in order to prove that it is valid it is sufficient to show that it is true at all worlds in all models. So, suppose $M, w \models K_*\pi_\mu \wedge \neg K_*\mu$. Since $M, w \models K_*\pi_\mu$, we must have that $\sigma_*(w) \in [\pi_\mu]_M$, which by Proposition 8.3.4 amounts to $\sigma_*(w) \subseteq |\pi_\mu|_M = |\mu|_M$. But, by factivity, $w \in \sigma_*(w)$, and thus $w \in |\mu|_M$, which means that w is a world in M^μ.

Now, since μ is non-epistemic, by Fact 8.4.4 we know that $M^\mu, w \models E_*\mu$. Moreover, since $\sigma_*(w) \subseteq |\mu|_M$, the previous Proposition yields $\sigma^\mu_*(w) = \sigma_*(w)$. On the other hand, the assumption $M, w \models \neg K_*\mu$ means that $\sigma_*(w) \notin [\mu]_M$, whence it follows $\sigma^\mu_*(w) \notin [\mu]_M$. Finally, since $[\mu]_M = [\mu]_{M^\mu}$ by Fact 8.4.3, we obtain $\sigma^\mu_*(w) \notin [\mu]_{M^\mu}$, which means that $M^\mu, w \models \neg K_*\mu$. Putting things together, we have $M^\mu, w \models \neg K_*\mu \wedge E_*\mu$, that is, $M^\mu, w \models W_*\mu$, which implies $M, w \models [\mu]W_*\mu$, as we set out to show. □

Thus, a non-epistemic question μ is guaranteed to become an open issue if uttered in a context in which it is not yet publicly settled, and in which its presupposition is common knowledge. As we will make more precise in the next section, this is indeed the natural situation in which it is appropriate to ask μ.

8.5 Division of labor

We saw that, in IDEL, an agent may provide new information by uttering a statement, and raise a new issue by uttering a question. This is as it should be. However, we also pointed out a feature of IDEL which is not as intuitive: by uttering a question, an agent may also provide some information—namely, the same information that would be conveyed by uttering the question's presupposition.

This does not seem to be faithful to how dialogue actually works. The linguistic consensus is that, when a speaker utters a sentence which has a certain presupposition, she is not explicitly conveying the information that this presupposition is true; rather, she is taking this piece of information for granted, that is, she is treating it as being *common ground* (see Stalnaker, 1974).[6]

Admittedly, sometimes the presupposition of a sentence uttered was not previously known to the other participants, and if so, they may indeed come to learn that it is true—or at least, that the speaker takes it to be true. However, this

[6]The common ground of a conversation is the body of information that is publicy shared among the participants. In the idealized world of (inquisitive) epistemic logic, it may be identified with the group's common knowledge—though in a more realistic model it would have to be treated as common belief, or even as common *pretense* of belief (see Stalnaker, 1998, 2002).

happens indirectly, via a pragmatic process known as *presupposition accommodation*, whereby the conversational participants quietly adjust their information states in order to make good sense of the speaker's utterance (see Beaver, 1999; von Fintel, 2008, among many others).

Accommodation is a famously complicated process, and formulating a proper formal account of it is beyond the scope of this chapter. However, in this section we will show that it is possible to make our account of public utterances more realistic by enforcing a strict division of labor between statements and questions. That is, we will explore the consequences of a pragmatic stipulation about when it is appropriate to assert a statement or to ask a question. The stipulation will amount to the following: uttering a statement is appropriate in a given context iff it provides new information and does not raise any new issues; uttering a question is appropriate iff it raises new issues and does not provide any new information.

To formulate this principle more precisely, we need to give a formal definition of what it is for a sentence to be *informative*—to provide new information—and to be *inquisitive*—to raise new issues. We will say that a sentence φ is *informative* in a world w in case uttering φ enhances the group's common knowledge at w.

8.5.1. Definition. [Informativeness]
Let $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and let M be a model and w a world at which φ is true.
We say that φ is *informative* in w in case $\sigma_*^\varphi(w) \subset \sigma_*(w)$.

Second, we will say that a sentence φ is *inquisitive* in a world w in case the common issues after an utterance of φ are not just the restriction of the old common issues to the new common knowledge state resulting from the utterance. In other words, φ is inquisitive at a world in case uttering φ would not just enhance the public information state (if it does at all) but also makes it harder to reach a state where the public issues are settled.

8.5.2. Definition. [Inquisitiveness]
Let $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$ and let M be a model and w a world at which φ is true.
We say that φ is *inquisitive* in w in case $\Sigma_*^\varphi(w) \subset \Sigma_*(w)\upharpoonright_{\sigma_*^\varphi(w)}$.

To understand these notions better, we will prove two facts that characterize when an utterance of φ brings about a change in the group's common knowledge, or in the group's public state. The first of these results states that an utterance of φ establishes new common knowledge at w iff it is not *already* common knowledge that φ is true.

8.5.3. Proposition.
Let $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, and let M be a model and w a world at which φ is true.

$$\varphi \text{ is informative in } w \iff \sigma_*(w) \not\subseteq |\varphi|_M$$

Proof. If $\sigma_*(w) \not\subseteq |\varphi|_M$ then since $\sigma_*^\varphi(w) \subseteq W^\varphi = |\varphi|_M$ we cannot have $\sigma_*^\varphi(w) = \sigma_*(w)$, and since obviously $\sigma_*^\varphi(w) \subseteq \sigma_*(w)$, we must have $\sigma_*^\varphi(w) \subset \sigma_*(w)$. The converse direction was proved as part of Proposition 8.4.6. □

The second fact says that an utterance of φ brings about a change in the public state at w just in case φ is not already commonly entertained at w.

8.5.4. Proposition.
Let $\varphi \in \mathcal{L}^{\mathsf{IDEL}}$, and let M be a model and w a world at which φ is true.

$$\Sigma_*^\varphi(w) \subset \Sigma_*(w) \iff \Sigma_*(w) \not\subseteq [\varphi]_M$$

Proof. If $\Sigma_*(w) \not\subseteq [\varphi]_M$, then since $\Sigma_*^\varphi(w) \subseteq [\varphi]_M$ by construction of the updated model, $\Sigma_*^\varphi(w)$ cannot be equal to $\Sigma_*(w)$, and so we must have $\Sigma_*^\varphi(w) \subset \Sigma_*(w)$. Conversely, suppose $\Sigma_*(w) \subseteq [\varphi]_M$. Then $\sigma_*(w) = \bigcup \Sigma_*(w) \subseteq \bigcup[\varphi]_M = |\varphi|_M$, so by the previous lemma $\sigma_*^\varphi(w) = \sigma_*(w)$. But if $s \in \Sigma_*(w)$, Proposition 7.2.10 ensures that there is a $v \in \sigma_*(w)$ s.t. $s \in \Sigma_a(v)$ for some agent a. Now, since $\sigma_*^\varphi(w) = \sigma_*(w)$, we also have $v \in \sigma_*^\varphi(w)$. Moreover, since $v \in \sigma_*(w)$, it follows from the definition of the public state map that $\Sigma_a(v) \subseteq \Sigma_*(w)$, so our assumption implies $\Sigma_a(v) \subseteq [\varphi]_M$, whence $\Sigma_a^\varphi(v) = \Sigma_a(v) \cap [\varphi]_M = \Sigma_a(v)$. Putting these pieces together, we have $v \in \sigma_*^\varphi(w)$ and $s \in \Sigma_a^\varphi(v)$, which means that $s \in \Sigma_*^\varphi(w)$. □

The notions of informativeness and inquisitiveness allow us to give a precise formulation of our pragmatic principle of division of labor.[7]

8.5.5. Definition. [Principle of division of communicative labor]

- The utterance of a statement α is appropriate in a world w in case α is true, informative, and not inquisitive in w.
- The utterance of a question μ is appropriate in a world w in case μ is true, inquisitive, and not informative in w.[8]

Let us now examine what these conditions amount to. First, we will show that a statement is never inquisitive.

8.5.6. Proposition. *A statement is never inquisitive in a world.*

[7] Notice that the principle as formulated here incorporates a *non-redundancy* requirement: an utterance is only appropriate at a world in case it enhances the public state. We could choose to separate out non-redundancy from division of labor proper, which would then amount simply to the following: the utterance of a statement α is appropriate in a world only if α is non-inquisitive, while an utterance of a question μ is appropriate in a world only if μ is non-informative.

[8] Recall that for a question μ, to be true at a world means that the question admits of a true resolution at w.

Proof. Let α be a statement. We must show that, for any model M and world $w \in |\alpha|_M$, $\Sigma_*^\alpha(w)$ coincides with $\Sigma_*(w) \upharpoonright_{\sigma_*^\alpha(w)}$. Since we always have that $\Sigma_*^\alpha(w) \subseteq \Sigma_*(w) \upharpoonright_{\sigma_*^\alpha(w)}$, we must show the opposite inclusion. For this, take a state $s \in \Sigma_*(w) \upharpoonright_{\sigma_*^\alpha(w)}$, that is, $s \in \Sigma_*(w)$ and $s \subseteq \sigma_*^\alpha(w)$. We want to prove that $s \in \Sigma_*^\alpha(w)$.

If $s = \emptyset$, then $s \in \Sigma_*^\alpha(w)$ is trivially true, so we may assume without loss of generality that s is non-empty. Now, by Proposition 7.2.10, $s \in \Sigma_*(w)$ means that there is a world $v \in \sigma_*(w)$ and an agent a such that $s \in \Sigma_a(v)$. Now, let u be any world in s: since $s \in \Sigma_a(v)$, we have $u \in \sigma_a(v)$, whence by introspection we get $\Sigma_a(u) = \Sigma_a(v)$. But then $s \in \Sigma_a(u)$, and since we also have that $s \subseteq \sigma_*^\alpha(w) \subseteq |\alpha|_M$, we have $s \in \Sigma_a(u) \upharpoonright_{|\alpha|_M}$, which by Proposition 8.3.8 amounts to $s \in \Sigma_a^\alpha(u)$. Finally, since $u \in s$ and $s \subseteq \sigma_*^\alpha(w)$, we have that $u \in \sigma_*^\alpha(w)$. From $u \in \sigma_*^\alpha(w)$ and $s \in \Sigma_a^\alpha(u)$ it follows that $s \in \Sigma_*^\alpha(w)$, as required. □

All that our principle requires of a statement α, then, is that it be true and informative in the world where it is uttered. By Proposition 8.5.3, the latter condition amounts to $\sigma_*(w) \not\subseteq |\alpha|_M$. Now, since statements are defined as truth-conditional formulas, we have $[\alpha]_M = \wp(|\alpha|_M)$. So, the condition that α should be informative in w boils down to $\sigma_*(w) \notin [\alpha]_M$, that is, to $M, w \models \neg K_*\alpha$. In conclusion, uttering a statement is appropriate just in case α is true, but is not already common knowledge.

8.5.7. Proposition (Appropriateness for statements).
Uttering a statement α is appropriate in a world w iff $M, w \models \alpha \wedge \neg K_*\alpha$.

Now let us consider what it takes for the utterance of a question μ to be appropriate in a world w. First, μ should be true at w, which by Proposition 8.3.4 means that its presupposition π_μ should be true at w.

Second, μ should not be informative in w. This means that we should have $\sigma_*^\mu(w) = \sigma_*(w)$. By Proposition 8.5.3, this is equivalent to $\sigma_*(w) \subseteq |\mu|_M = |\pi_\mu|_M$. Now, since π_μ is truth-conditional, we have $\sigma_*(w) \subseteq |\pi_\mu|_M \iff \sigma_*(w) \in [\pi_\mu]_M \iff M, w \models K_*\pi_\mu$. Thus, the second appropriateness condition for uttering μ is that the presupposition π_μ be common knowledge. Notice that this second condition subsumes the first: if π_μ is common knowledge at w, it must also be true at w.

Finally, the principle requires that μ be inquisitive in w. This means that we should have $\Sigma_*^\mu(w) \subset \Sigma_*(w) \upharpoonright_{\sigma_*^\mu(w)}$. But, assuming that the first condition is met, that is, μ is not informative in w, we have $\sigma_*^\mu(w) = \sigma_*(w)$, and thus $\Sigma_*(w) \upharpoonright_{\sigma_*^\mu(w)} = \Sigma_*(w) \upharpoonright_{\sigma_*(w)} = \Sigma_*(w)$. The requirement that μ be inquisitive in w can then be simplified as $\Sigma_*^\mu(w) \subset \Sigma_*(w)$. By Fact 8.5.4, this holds if and only if $\Sigma_*(w) \not\subseteq [\mu]_M$, that is, if and only if $M, w \models \neg E_*\mu$.

Thus, we have found that uttering a question μ is appropriate in a world w just in case (i) the presupposition π_μ of μ is common knowledge in w and (ii) μ is not already publicly entertained in w.

8.5.8. Proposition (Appropriateness for questions).
Uttering a question μ is appropriate in a world w iff $M, w \models K_* \pi_\mu \wedge \neg E_* \mu$.

Notice that since $E_*\mu \equiv K_*\mu \vee W_*\mu$, the second conjunct of the appropriateness condition can also be rewritten as $\neg K_*\mu \wedge \neg W_*\mu$: in order for an utterance of μ to be appropriate, μ should be neither already settled for the group, nor already an open issue.

Now let us consider how our logical language may be enriched to talk about the effects of public utterances in a way that takes appropriateness conditions into account. Presently, our logical language contains expressions of the form $[\varphi]\psi$, involving a dynamic modal operator that allows us to talk about the effects of a public utterance in a given situation. However, this operator does not take into account whether the utterance under consideration is appropriate in the given situation. To overcome this limitation, we may further extend our logical language with a second, 'pragmatically sensitive' dynamic operator, $[.]^p$. We will refer succinctly to $[\varphi]^p$ as a *pragmatic modality*.

Semantically, $[\varphi]^p\psi$ differs from $[\varphi]\psi$ in that the former does not only conditionalize ψ to the fact that φ is uttered, but also to the assumption that the utterance of φ is appropriate in the first place. To formulate this support condition precisely, let us define the *appropriateness set* of a sentence in a model.

8.5.9. Definition. [Appropriateness set]
The appropriateness set of φ in a model M, denoted $\lfloor\varphi\rfloor_M$, is the set of worlds in M where the utterance of φ is appropriate.

In order to check whether a state s in a model M supports $[\varphi]^p\psi$, we first enhance s by supposing that φ can indeed be appropriately uttered, obtaining $s \cap \lfloor\varphi\rfloor_M$, and then check whether the utterance of φ would lead to a state that supports ψ. Thus, the support conditions for $[\varphi]^p\psi$ can be concisely formulated as follows:

$$M, s \models [\varphi]^p\psi \iff M, s \cap \lfloor\varphi\rfloor_M \models [\varphi]\psi$$

Spelling out the support conditions for $[\varphi]\psi$, we get to the clause $M, s \models [\varphi]^p\psi \iff M^\varphi, s \cap \lfloor\varphi\rfloor_M \cap |\varphi|_M \models \psi$. But notice that, according to the principle of division of labor, the utterance of φ is only appropriate in worlds where φ is true; this means that $\lfloor\varphi\rfloor_M \subseteq |\varphi|_M$, which implies that $\lfloor\varphi\rfloor_M \cap |\varphi|_M = \lfloor\varphi\rfloor_M$. We thus get to the following clause, which we take as our official semantic clause for $[\varphi]^p\psi$.

8.5.10. Definition. [Support for pragmatic modalities]

- $M, s \models [\varphi]^p\psi \iff M^\varphi, s \cap \lfloor\varphi\rfloor_M \models \psi$

Thus, the clause for pragmatic modalities is the same as for standard dynamic modalities, except that the state of evaluation is made more informed by assuming not only that φ was true at the time of utterance, but also that φ could be appropriately uttered.

By specializing this support clause to a singleton information state, we obtain the following truth-conditions for $[\varphi]^p\psi$.

8.5.11. Proposition (Truth for pragmatic modalities).

- $M, w \models [\varphi]^p\psi \iff w \in \lfloor\varphi\rfloor_M$ *implies* $M^\varphi, w \models \psi$

That is, $[\varphi]^p\psi$ is true in case, if φ can be appropriately uttered at w, then uttering φ results in a situation in which ψ is true.

Notice that since $\lfloor\varphi\rfloor_M \subseteq |\varphi|_M$, persistency implies that the support conditions for $[\varphi]^p\psi$ are less demanding than those for $[\varphi]\psi$. As a consequence, the following entailment is valid:

$$[\varphi]\psi \models [\varphi]^p\psi$$

To illustrate the difference in logical behavior between $[.]^p$ and $[.]$, recall that Proposition 8.4.7 guarantees that the following is valid for every non-epistemic question μ:

$$(K_*\pi_\mu \wedge \neg K_*\mu) \to [\mu]W_*\mu$$

In words: if the presupposition of μ is common knowledge but μ is not yet publicly settled, then uttering μ results in μ being a common open issue. Now, since uttering μ is only appropriate if the presupposition of μ is indeed common knowledge and μ is not yet publicly settled, any appropriate utterance of a non-epistemic question μ results in μ being a common open issue. That is, for every non-epistemic question μ, the following formula is valid:

$$[\mu]^pW_*\mu$$

A question that naturally arises at this point is whether it is possible to give reduction axioms for these pragmatic modalities. Since the semantics of the operator $[\varphi]^p$ relies on the appropriateness conditions for φ, which in turn crucially involve the public state map Σ_*, it is natural to conjecture that such a reduction is only possible in the language $\mathcal{L}_*^{\mathsf{IEL}}$ enriched with the public modalities K_* and E_*. Since in the last chapter I have not provided an axiomatization for inquisitive epistemic logic with public modalities, I will also leave the question about the pragmatic modalities for future work.

In the next section we turn to the comparison between the logic IDEL and a recent alternative proposal to integrate questions in dynamic epistemic logic.

8.6 Related work

Though questions only played a very marginal role in early work on dynamic epistemic logic (with Baltag, 2001, as a notable exception), they did receive some attention in more recent work (van Eijck and Unger, 2010; Peliš and Majer, 2010, 2011; Ågotnes et al., 2011; van Benthem and Minică, 2012; Liu and Wang, 2013). Among these, the proposal that is most centered on the role of questions, and most closely related to our work in this chapter, is the *dynamic epistemic logic with questions* (DELQ) of van Benthem and Minică (2012). This section provides an overview of this logic, and compares it to IDEL.

8.6.1 Dynamic epistemic logic with questions

Epistemic logic with issues

Let us start out by describing the static component of the system DELQ, called *epistemic logic with issues.* The semantic structures that van Benthem and Minică consider are standard epistemic models enriched with a set of issues, one for each agent. Following Groenendijk and Stokhof (1984), issues are modeled as equivalence relations $\approx$ on the set of worlds. As we know, such relations may be equivalently regarded as partitions $\pi_\approx$ of the logical space, the cells of which correspond to the possible answers to the issue. For any world w, $\pi_\approx(w)$ is used to denote the unique cell of the partition containing w. Intuitively, $\pi_\approx(w)$ is the information state that results from minimally resolving the issue $\approx$ in w.

To stay closer to the presentation of van Benthem and Minică, we shift here to the standard presentation of epistemic models which uses epistemic accessibility relations $\sim_a$ instead of epistemic maps σ_a.

8.6.1. Definition. [Epistemic issue models]
An epistemic issue model M is a quadruple $\langle W, V, \sim_\mathcal{A}, \approx_\mathcal{A}\rangle$, where:

- W is a set whose elements are called possible worlds;
- $V : W \to \wp(\mathcal{P})$ is a valuation function;
- $\sim_\mathcal{A} = \{\sim_a \mid a \in \mathcal{A}\}$ is a set of equivalence relations on W, called *epistemic relations*;
- $\approx_\mathcal{A} = \{\approx_a \mid a \in \mathcal{A}\}$ is a set of equivalence relations on W, called *issue relations.*

The language that van Benthem and Minică use to describe their epistemic issue models is the standard language of epistemic logic enriched with a universal modality U, as well as a question modality Q_a and a resolution modality R_a for every agent a. The semantics of this language is given as usual in terms of truth-conditions at a world. The connectives and the K_a modalities are interpreted as usual, while the other modalities are interpreted as follows:

1. $M, w \models U\varphi$ iff $M, v \models \varphi$ for all $v \in W$
2. $M, w \models Q_a\varphi$ iff $M, v \models \varphi$ for all $v \in W$ such that $w \approx_a v$
3. $M, w \models R_a\varphi$ iff $M, v \models \varphi$ for all $v \in W$ such that $w \sim_a v$ and $w \approx_a v$

The universal modality, a standard tool in modal logic, quantifies over all worlds in the model. The question modality Q_a quantifies over all worlds $\approx_a$-equivalent to the evaluation world w, that is, all worlds in the state $\pi_{\approx_a}(w)$. We said above that this state represents the information state that would result from correctly resolving the issue $\approx_a$ at w. Thus, the question modality Q_a allows us to express what would be established if the issue entertained by a were resolved.

The resolution modality R_a, on the other hand, quantifies over all the worlds which are both $\sim_a$-equivalent and $\approx_a$-equivalent to w. These are the worlds that make up the information state resulting from pooling together the private information available to a at w and the information that would result from resolving a's issues at w. Thus, the resolution modality R_a allows us to express what agent a would know if her current issue were resolved.

Combining the modalities U and Q_a we can express facts about the issues that agent a entertains. For instance, consider the formula $U(Q_a\varphi \vee Q_a\neg\varphi)$. This formula says that any world w is such that, if a's private issues were resolved at w, either φ or $\neg\varphi$ would be established. Thus, it says that resolving a's private issues necessarily involves establishing an answer to the question whether φ is the case. Van Benthem and Minică suggest we take this to be a description of what it means for a to entertain the issue whether φ.

Dynamics

We have seen how van Benthem and Minică's models, just like ours, include a description of private issues and information. Here, too, both components may be affected by agents performing certain actions. Van Benthem and Minică consider a number of actions. We will focus our attention on two of these, the most fundamental ones: the action of *publicly announcing* that φ is the case, denoted $\varphi!$, and the action of *publicly asking* whether φ is the case, denoted $\varphi?$. So, whereas in IDEL we have only one action—public utterance—which can be performed with two kinds of sentences—statements and questions—in DELQ there is only one kind of sentence—statements—but there are two types of actions—announcing and asking. Let us see how these two actions work.

A public announcement of φ transforms a model M into the model $M^{\varphi!}$ which differs from M only in the agents' epistemic relations. The new epistemic relation $\sim_a^\varphi$ for agent a is $\sim_a \cap \equiv^\varphi$, where $\equiv^\varphi$ is the relation holding between two worlds just in case φ has the same truth value in both worlds. Thus, a public announcement of φ has the effect of making it common knowledge whether φ holds.[9] Publicly asking whether φ has a similar effect, but on the issue component of the model. That is, it tranforms a model M into the model $M^{\varphi?}$ which differs from M only in the agents' issue relations. The new issue relation $\approx_a^\varphi$ for agent a is $\approx_a \cap \equiv^\varphi$, where $\equiv^\varphi$ is as before. Thus, like in IDEL, a public question whether φ has the effect of making all agents entertain the question whether φ.

As customary, the system provides dynamic modalities $[\varphi!]$ and $[\varphi?]$ corresponding to these actions. The clauses for these modalities are as follows:

1. $M, w \models [\varphi!]\psi \iff M^{\varphi!}, w \models \psi$
2. $M, w \models [\varphi?]\psi \iff M^{\varphi?}, w \models \psi$

This concludes our minimal exposition of the system DELQ. We now turn to a comparison of the two approaches.

8.6.2 Comparison

As we saw, DELQ is very much in the same spirit as the system IDEL presented in this chapter: in their static component, both systems describe agents as having both information and issues; in their dynamic component, they both consider not only the standard action of publicly announcing a statement, but also the action of publicly asking a question, which is modeled as having the effect of raising an issue for all agents. However, alongside these fundamental similarities, the two systems also present several crucial differences.

Local versus global issues

The definition of our inquisitive epistemic models is driven by a simple but powerful idea: possible worlds represent complete states of affairs; when we consider an information exchange scenario, a state of affairs encompasses not just the external facts which constitute the basic topic of the exchange, but also any feature of the exchange itself which is relevant for the purpose at hand. The formal model should reflect this idea, equipping each world with a representation of all the relevant features. In epistemic logic, the relevant feature of an information exchange scenario is the knowledge that the agents have. Accordingly, in an epistemic model, a world comes equipped with a description of the agents' information states. In *inquisitive* epistemic logic, what matters is not only the information

[9]Notice that on this approach, a public announcement never removes any world from the model. This has the puzzling consequence that in a $\neg p$-world, announcing that p has the effect of making $\neg p$ common knowledge. This treatment of public announcements of declarative sentences is clearly different from the one we gave. However, since both systems are in principle compatible with either account of public announcements of declaratives, we do not take this difference to reflect an essential discrepancy between the two approaches.

that the agents have, but also the issues that they entertain. Accordingly, in our models, a world also comes equipped with a representation of the agents' issues.

In DELQ, on the other hand, a model comes with just one issue for each agent, which is not relativized to any particular world. Thus, while the information available to the agents may differ from world to world, the issues that the agents entertain are fixed and independent of the world under consideration.

Conceptually, it is difficult to see how this asymmetry could be motivated. Certainly, a particular distribution of issues among the agents partly determines what a world is like, no less than a particular distribution of information does. Moreover, it is natural to assume that agents may entertain different issues at different worlds.

These conceptual concerns also have important practical consequences. In particular, the asymmetric treatment of information and issues puts significant limitations on the descriptive power of DELQ. In DELQ, just like in IDEL, agents may have incomplete knowledge about other agents' knowledge, and if they do, they may indeed wonder what the other agents know. However, one would also like to be able to describe situations where agents have incomplete knowledge and wonder about the *issues* that the other agents entertain. In IDEL, such situations can be described straightforwardly. Indeed, the language of IDEL contains sentences such as $K_a W_b \mu$, expressing the fact that *a knows that b wonders about* μ, and $W_a ?W_b \mu$, expressing the fact that *a wonders whether b wonders about* μ.

In DELQ, such situations cannot be modeled appropriately. To see this, recall that the formula $U(Q_a\varphi \vee Q_a\neg\varphi)$ is used in DELQ to describe situations in which agent a entertains the issue whether φ is the case or not. Thus, the formula $K_b U(Q_a\varphi \vee Q_a\neg\varphi)$ is used to describe situations in which agent b *knows* that agent a entertains the issue whether φ holds or not. Now suppose that M is a model and w a world such that $M, w \models U(Q_a\varphi \vee Q_a\neg\varphi)$. That is, a entertains the issue whether φ in w. Then, since the universal modality U ranges over all worlds in M, we must also have for any world $v \neq w$ in M that $M, v \models U(Q_a\varphi \vee Q_a\neg\varphi)$. But then we must certainly have that $M, w \models K_b U(Q_a\varphi \vee Q_a\neg\varphi)$. That is, if an agent entertains a certain issue, then all the other agents automatically know this. Conversely, if a does *not* entertain the issue, that is, $M, w \models \neg U(Q_a\varphi \vee Q_a\neg\varphi)$, then $U(Q_a\varphi \vee Q_a\neg\varphi)$ must be false at all worlds, and thus we must also have $M, w \models K_b \neg U(Q_a\varphi \vee Q_a\neg\varphi)$. Thus, it is impossible to model situations where the agents have partial information about other agents' issues, let alone situations where the agents wonder about other agents' issues.

This limitation is not the only price that DELQ pays for its non-local treatment of issues. The other limitation that it encounters concerns the construction of a *public issue* representing the questions the are publicly open for the group. Both the models of IDEL and those of DELQ contain in their definition only a description of the agents' private issues. Of course, *public issues* also play a crucial role in information exchange. Van Benthem and Minică are well aware of this; for instance, when discussing further research directions (p. 663), they say:

> "We need extensions of our systems to group actions of information and issue management, including common knowledge, and group issue modalities."

In the previous chapter, we saw that IDEL can deal in an elegant way with the challenge of constructing a public state map which describes public issues and allows us to interpret a public entertain modality E_*, and to define a public wondering modality W_* which talks about the public open issues. The public state map is constructed from the individual state maps, and it is fully determined by the requirement that something be publicly entertained iff it is common knowledge that everyone entertains it. This solution is not available in DELQ, since it requires the model to represent what agents know about the issues that other agents entertain, what they know about what other agents know about these issues, etcetera. This information, as we saw, is not represented in the models of DELQ.

Different notions of issues

Issues play a central role in the models of both DELQ and IDEL. However, the systems are based on two different formal notions of issues. In DELQ, an issue is modeled as an equivalence relation $\approx$ on the set of worlds. As we saw, such an equivalence relation corresponds to a partition $\pi_\approx$ of the logical space, the blocks of which correspond to the basic answers to the issue. In IDEL, on the other hand, an issue is modeled a downward closed cover I of the agent's information state, where the elements of this cover are understood as being those information states where the agent's issues are resolved.

The notion of issues adopted in IDEL is strictly more general than the one adopted in DELQ. Every issue in the sense of DELQ, modeled by an equivalence relation $\approx$, can be translated faithfully into an issue in the sense of IDEL, namely:

$$I_\approx = \{t \mid w \approx w' \text{ for all } w, w' \in t\}$$

In words, $I_\approx$ consists of those information states that are included in a block of the partition induced by $\approx$, i.e., of those information states which entail a basic answer to the issue. However, the converse does not hold: there are many issues in our sense which are not partition-like, i.e., which are not of the form $I_\approx$ for any equivalence relation $\approx$ over W.

First of all, any partition-like issue $I_\approx$ covers the set W of all worlds, whereas this need not hold for an issue I in our sense. To see that this extra generality is needed, suppose Alice is entertaining the issue of where her sister is. Alice knows she has a sister, so she can meaningfully entertain this issue. But if our model contains worlds w where Alice has no sister (which may be needed to capture the epistemic state of other agents), then Alice's issue cannot be resolved at these worlds, that is, we will have $w \notin \bigcup I$. Thus, such an issue—and, more generally, any issue with a non-trivial presupposition—cannot be represented in DELQ.

Second, if we look at the alternatives for a partition issue $I_{\approx}$, we find that these are always *mutually exclusive*. This means that at any world, there is always a unique minimal way of truthfully resolving the issue. This need not be the case for an issue I in our sense. To see why this is needed, suppose Alice entertains a mention-some issue, e.g., the issue of who has a bike she could borrow for the day. As we saw in Chapter 4, in general there will be several minimal ways of truthfully resolving such an issue. Thus, again, such an issue—and indeed, mention-some issues more generally—cannot be represented in DELQ.

We conclude that the notion of issues adopted in DELQ, while natural and formally well-behaved, is not rich enough to deal with several types of issues that play a role in ordinary information exchange scenarios.

Questions as sentences vs. questions as actions

So far we have identified two important differences between the models of DELQ and those of IDEL. One difference concerns the way issues are modeled, while the other concerns the way issues are embedded into the framework of epistemic logic. A third, crucial difference concerns the way in which *questions* are modeled.

In DELQ, the static language consists entirely of statements, whose semantics is specified in terms of truth-conditions. Questions only come into the picture in the dynamic component of the system, as a particular kind of speech act, having a statement α as its content. As we saw, the effect of a question involving a sentence α is to raise the issue of whether α holds.

In IDEL, questions enter the picture already at the more fundamental level of the static language. Just like statements, questions have a semantic value, which captures their resolution conditions. This semantic value enters the compositional interpretation process, allowing us to recursively assign meanings to sentences where questions are embedded under modal operators. It also allows us to keep the dynamic component of the system simpler: we only need a single action of uttering a sentence, rather than two distinct actions for announcing and questioning. It is the content of the sentence that is uttered which determines whether the utterance brings about a change in information or in issues.

We will argue that there are good reasons to prefer the latter approach. First, in DELQ, all questions have the effect of raising a polar issue, namely the issue whether a certain declarative sentence α holds. The same effect is obtained in IDEL by the utterance of a polar question $?\alpha$. Thus, the effect of a question action in DELQ may be simulated by the utterance of a question in IDEL. However, the converse is not the case: not all questions that may be asked in IDEL are polar questions. Consider for instance the conditional question $p \to ?q$ which is settled in a state s in case either $p \to q$ or $p \to \neg q$ is established in s. The effect of asking such a question cannot be modeled in DELQ, since as we saw in Chapter 2, the issue expressed by this question is not a partition issue. But suppose this problem were amended by moving to a more general notion of issues. DELQ would still be in trouble, since asking $p \to ?q$ does not amount to asking whether a certain statement is true or not. In order to address this problem, DELQ may be extended with an additional action of *conditional questioning*, which would involve two statements, one serving as the antecedent and one as the consequent of the question. However, considering even more complex question types would force DELQ to postulate a richer and richer repertoire of question actions.

Furthermore, the treatment of questions as speech acts faces another difficulty. In IDEL, as mentioned above, a question is assigned a semantic value, which does not only determine the effect of uttering that question, but also the meaning of more complex expressions in which the question may be *embedded*. These more complex expressions may be statements whose truth-conditions depend on the issue expressed by the embedded question. Concretely, the basic way to construct a statement from a question μ is to embed μ under a modal operator, such as K_a, E_a, W_a, or their public counterparts K_*, E_*, W_*. In this way, we can construct statements such as $K_a\mu$, which expresses the fact that a can resolve μ; $W_a\mu$, which expresses the fact that a wonders about μ; $K_*\mu$, which expresses the fact that μ is publicly settled among the agents; and $W_*\mu$, which expresses the fact that μ is a public open issue.

DELQ does not allow the construction of sentences that involve a question embedded under a modal operator. The possibility of expressing the corresponding facts depends on the possibility of analyzing claims about questions in terms of claims concerning statements. In some cases, such analyses are indeed possible. For instance, "*a knows whether* α" may be analyzed in DELQ as $K_a\alpha \vee K_a\neg\alpha$. In general, if a question μ has a finite set of predetermined "syntactic answers" $\alpha_1, \dots, \alpha_n$, then "*a knows* μ" may be analyzed in DELQ as $K_a\alpha_1 \vee \dots \vee K_a\alpha_n$. However, this strategy faces three problems.

First, it is far from clear that such a strategy is viable for "truly inquisitive" modalities such as *wonder*, which express a relation that holds between an agent and an inquisitive proposition P, a relation that cannot be reduced—as in the case of *know*—to a more basic relation holding between an agent and some particular piece of information. It is true that in DELQ we can analyze "*a wonders whether* α" as $U(Q_a\alpha \vee Q_a\neg\alpha)$. But this analysis only works if issues are treated as being world-independent, and we have argued above that this has serious drawbacks. Once issues are relativized to worlds, this account no longer works, and it seems that *wondering* is no longer expressible with the tools available in DELQ.

Second, analyzing sentences involving a question μ in this way requires knowledge of the set of answers to μ. Thus, in order to express facts about a question, DELQ needs to outsource the analysis of the question to some theory that predicts what its answers are. Our semantics, on the contrary, *includes* such a theory of questions. Equivalences such as $K_a?\alpha \;\equiv\; K_a\alpha \vee K_a\neg\alpha$, which characterize the knowledge that is needed to resolve a certain question, are obtained as *logical validities* of the theory, not merely assumed as definitions.

Finally, even supposing the 'paraphrase' strategy can be made to work for the propositional case—where questions have a finite, predetermined set of answers—the transition to a first-order setting seems problematic. As we saw in Chapter 4, it is not always possible to associate a first-order question with a set of "syntactic answers"; even when it is possible, the set of these answers is not necessarily finite. Thus, the paraphrase strategy sketched above cannot be applied. While for particular types of first-order questions a paraphrase may be found, it seems very unlikely that a *uniform* analysis exists. By contrast, our question-embedding modalities can be extended straightforwardly to the first-order setting. By combining IEL with first-order logic of Chapter 4, we obtain a system which includes, e.g., formulas such as $K_a(\forall x ?Px)$ and $W_a(\forall x ?Px)$, expressing respectively that *a knows who attended the party*, and that *a wonders who attended the party.*[10]

8.7 Further work

In this chapter, we have shown how the influential treatment of public announcements in epistemic logic generalizes smoothly in the inquisitive setting to an account of *public utterance* which encompasses both the public announcement of a statement, and the public asking of a question. By publicly uttering sentences, agents may not only provide new information—as in standard public announcement logic—but they may also raise new issues. In this way, we have obtained a minimal model of communication as a process in which agents interact by raising and resolving issues, that is to say, by requesting and providing information.

As in standard PAL, we have equipped our logic with dynamic modalities $[\varphi]$ that have the effect of shifting the point of evaluation from the current state in the current model, to the state and model that result from the public utterance of φ. We have seen that these modalities do not take us beyond the expressive power of IEL, as we can always describe exactly, with the means of IEL, what has to be the case in the current model-state pair, in order for something to hold after the utterance has taken place. We have exploited this fact to obtain an axiomatization of the resulting dynamic logic, IDEL, by enriching an axiomatization for IEL with reduction rules which enable us to paraphrase away the modalities. We have made use of the dynamic modalities in order to better understand the effects of the public utterance of "ordinary", non-epistemic sentences. And, finally, we have refined our modeling of the utterance action by means of a pragmatic principle which restricts the contexts in which the utterance of a sentence is appropriate.

Several directions for further work naturally suggest themselves. First of all,

[10]In the liguistic literature, the point that a proper treatment of questions, especially embedded questions, requires inquisitiveness to enter the picture at the semantic level, and not just at the speech act level, has been made in much detail by Groenendijk and Stokhof (1997), where it is directed mostly at the speech act treatment of questions proposed by Searle (1969) and Vanderveken (1990), and at the imperative-epistemic treatment of questions proposed by Åqvist (1965) and Hintikka (1976, 1983).

we have only axiomatized the system IDEL, which has dynamic modalities alongside individual modalities of the form K_a and E_a. While in sections 8.4 and 8.5 we have also worked with the system IDEL*, which includes the public modalities K_* and E_*, the logic of this system remains to be investigated. This is particularly interesting since, in the standard case, the system PAL* with common knowledge is not reducible to epistemic logic with common knowledge; instead, PAL* is reducible to a static language which includes a relativized common knowledge operator (van Benthem et al., 2006). It seems plausible to conjecture that the situation is analogous in our case, so that relativized versions of the public modalities K_* and E_* are needed for a reduction-style axiomatization of IDEL*. Related to this, we have also left open the issue of investigating the logic of the pragmatic modalities $[\varphi]^p$ which take into account the appropriateness conditions for the utterance of a formula.

Second, while we have axiomatized IDEL via the standard strategy of providing reduction axioms for the modalities, it would be very interesting to investigate the possibility of a direct axiomatization, like the one that Wang and Cao (2013) provided for PAL. Here, the idea would be to first show that the dynamic modalities $[\varphi]$ may be given a non-standard semantics where they are interpreted not as model transformers, but as static modal operators with respect to an inquisitive multi-modal frame satisfying certain conditions. Following Wang and Cao, we might then establish completeness with respect to this semantics by explicitly constructing a canonical model, without resorting to reduction to IEL.

More broadly, what we did in this chapter is just a first step in the development of a dynamic logic of information and issues, suitable for modeling and reasoning about information exchange. First of all, while for simplicity we have remained within a purely epistemic framework, most information exchange scenarios involve *fallible beliefs*, which are not just *enhanced* as a result of learning new information, but which may have to be *revised*. This calls for a merge of the ideas developed in this chapter with ideas from model-theoretic belief revision (van Ditmarsch, 2005; Baltag and Smets, 2006; van Benthem, 2007, among others).

Equally importantly, the public utterance of a sentence is only a particular kind of communicative action. A crucial task for further work is to generalize IDEL by providing a general framework for actions in the inquisitive case. In standard dynamic epistemic logic, classical treatments of actions are the *action models* approach of Baltag et al. (1998) and the *knowledge actions* approach of van Ditmarsch (2002); moreover, there has been recent work on integrating the benefits of the two perspectives (see, e.g., French et al., 2014; Areces et al., 2014). It would be very interesting to see how these approaches fare when generalized to the inquisitive setting.

For now, what I hope to have shown in this chapter is that the fundamental ideas and tools of dynamic epistemic logic can be extended in a simple and elegant way so as to allow for a more inclusive logical analysis of information exchange, encompassing both informative and inquisitive aspects.

Conclusions and further work

In this thesis we have laid the foundations for a uniform theory of the logic of statements and questions in various logical settings: propositional, first-order, modal, and dynamic. We will conclude here by considering the different roles that questions play in the systems that we have defined.

First, we saw that questions may be seen as names for types of information. By generalizing the notion of entailment to questions, we can capture not only the standard relation of consequence, which connects specific pieces of information, but also the relation of *dependency*, which connects information types.

Second, we saw that questions may be used in formal proofs as placeholders for generic information of a certain type, and that by manipulating such placeholders we can provide formal proofs of dependencies. Moreover, such proofs have interesting constructive content: when the question assumptions are instantiated to specific resolutions, the whole proof can be "resolved" to yield a resolution of the conclusion.

Third, we saw that questions can be used to talk about issues. By generalizing our notion of a modal operator, we can use questions embedded under suitable modalities to express facts concerning issues, such as, the fact that a certain issue is settled for an agent, the fact that an agent *entertains* a certain issue, or the fact that an agent's actions fully determine the way a certain issue will turn out.

Finally, questions can be *asked*: we showed how a language equipped with questions provides the means for a logical dynamics which can model not only the way in which making a statement changes a communication scenario by providing new information, but also the way in which asking a question changes a given scenario by raising a new issue.

Clearly, the investigations initiated here can be taken further in a multitude of different directions. Throughout the thesis, specific tasks for further work have been pointed out. In this concluding section, I just wish to outline a few broad research directions that have not been touched upon in the thesis.

Computational aspects

One of the main findings of this thesis was that the relation of dependency is a facet of the relation of entailment, once the latter is extended to cover questions. We saw that for a number of logical languages, dependencies may be proved to hold by using a complete deduction system, and that from a proof, a program that computes the relevant dependency can be automatically extracted.

In view of the relevance of the dependency relation in computer science, it is important to find out more about the computational properties of our logics. One range of issues concerns the decidability, or semi-decidability, of entailment for the various logical systems. Another interesting question concerns the complexity of the various resolution algorithms given for our logics: how costly is it to use logical proofs to compute dependencies in the way we described?

Finally, we saw that our logics allow us to go beyond the standard dependence patterns, encompassing cases of dependence that do not appear to be captured by existing logical tools. One instance is given by *mention-some questions*, the logic of which we explored in Section 4.7. It seems that these questions may be useful in query languages for databases, and thus, that techniques to handle them may be relevant—for instance, in order to decide whether a given query can be answered based on the stored answers to previous queries, without being forced to re-consult the database.

Independence

We have been concerned with the notion of dependence, and with the relation to the logical framework of dependence logic. However, starting with the work of Grädel and Väänänen (2013), this framework has been employed to investigate not just the relation of dependence, but also the relation of *in*dependence. Independence is a much more demanding relation than the mere absence of dependence: essentially, a variable y is independent of another variable x in a team if learning the value of x would not rule out any possible value for y.

Does our question-based perspective have anything to say about this relation? The answer seems positive. For, the above definition in terms of variables can be reformulated as a relation between the questions λx and λy about the values of these variables: λy is independent of λx relative to X in case settling the question λx in X does not rule out any way of settling λy; that is, in case relative to X, any alternative for λx is consistent with any alternative for λy.

This suggests the following conjecture: just like dependency arises by generalizing to questions the classical notion of *entailment*, so *in*dependence arises by generalizing to questions the classical notion of *consistency*.

A richer framework for modal logic

In Chapter 7, we saw that our support-based semantic foundation suggests a generalization of the standard framework for modal logic. Here, we have focused on two specific modal operators within this framework, $\Box$ and $\boxplus$, and on one concrete interpretation of these operators in an epistemic setting. However, the shift to this richer view of modalities stands to have much broader consequences.

First, just like standard Kripke modalities, our modalities allow for different interpretations, depending on what we take the state map of our models to encode. One important task for the future is to explore what other notions, besides *wondering*, become expressible in this richer modal logic.

Second, the set of modalities $\Box$ and $\boxplus$ that we have considered is in no sense exhaustive. In fact, for any relation R defined between two inquisitive propositions, we have a corresponding modality $[R]$ with the following truth-conditions:

$$M, w \models [R]\varphi \iff R(\Sigma(w), [\varphi]_M)$$

Since inquisitive propositions have more structure to them than classical propositions, there are a number of interesting ways in which two such objects may be related—which opens the way for a range of inquisitive modalities, each with its own specific logical features.

Finally, there is a whole new theory of modal logic to be built, of which our axiomatization results are only the beginning. Analogues of standard notions of modal logic such as bisimulation, the standard translation, and filtrations need to be devised; the correspondence between modal formulas and classes of inquisitive modal frames needs to be investigated; and so on. Overall, it seems that the move from Kripke frames to inquisitive modal frames opens up a broad range of questions, both purely theoretical and directed towards applications.

Abstracting away from possible worlds

The crucial move that allowed us to bring questions into play in logic was to shift the points of evaluation for our semantics from possible worlds to information states. Throughout this thesis, we modeled information states as sets of worlds. This choice presents several advantages, including an explicit representation of the information available in a state, and an immediate way of recovering the usual truth-conditional perspective on statements. However, it also seems interesting to abstract away from possible worlds, and ask ourselves which structural features of the space of information states really play a role for our logics. Besides being interesting in itself, this might lead us to identify a broader class of semantic structures for our logics, which in turn would make it easier to produce countermodels to disprove invalid entailments.

Questions in non-classical logics

In this thesis, we have been concerned with enriching various systems of classical logic with questions. But there is no reason why one should not be interested in considering questions, while assuming a non-classical logic for statements. An interesting research project is to explore how the construction described here may be adapted to other base logics. For instance, it seems interesting to investigate how questions can be added on top of an *intuitionistic* logic for statements.

This project would not just broaden our understanding of inquisitive logics: it would also *sharpen* it. For, it would allow us to tease apart aspects of our logics that depend crucially on the underlying logic of statements from aspects which depend exclusively on the way in which questions and statements are related.

While this project requires a careful re-consideration of our semantic approach, an interesting first step in this direction has been taken by Punčochář (2015a), who may be seen as describing, among other things, how to extend an arbitrary disjunction-free intermediate logic with an inquisitive disjunction connective.

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Samenvatting

Vragen in de Logica

In dit proefschrift worden twee doelen nagestreefd die nauw met elkaar zijn verweven: het naar voren brengen wat de relevantie van vragen voor de logica is, en het tot stand brengen van een welgefundeerde theorie van de logica van vragen in een klassiek logisch kader. Deze twee ondernemingen voeden elkaar: aan de ene kant wordt de ontwikkeling van onze formele systemen gemotiveerd door onze overwegingen betreffende de rol die vragen kunnen spelen; aan de andere kant wordt via de ontwikkeling van concrete handzame logische systemen helder en voelbaar gemaakt wat de bijdrage van vragen aan de logica kan zijn.

We beginnen met te laten zien dat als we de overgang maken van de standaardnotie van waarheid in een mogelijke wereld naar een notie van ondersteund worden in een informatietoestand, dit een semantische fundering oplevert voor een uniforme manier om beweringen en vragen te interpreteren. Deze transitie leidt tot een substantiële generalisatie van de klassieke notie van logische inclusie die niet alleen de standaard relatie van logisch gevolg omvat, maar ook de relatie van *afhankelijkheid*—een relatie die een belangrijke rol speelt in velerlei contexten, van natuurkunde tot computerwetenschap. We laten zien dat zodra een logica van vragen wordt voorzien, afhankelijkheid zich ontpopt als een contextuele relatie van logische inclusie tussen vragen.

In Hoofdstuk 2 wordt onze benadering geconcretiseerd voor het meest eenvoudige geval, dat van de propositielogica. We beschrijven hoe klassieke propositielogica kan worden verrijkt met vragen, bespreken de kenmerken van deze logica, en laten zien hoe propositionele afhankelijkheden kunnen worden opgevat als gevallen van logische vraag-inclusie.

In Hoofdstuk 3 wordt een natuurlijk deductie systeem gegeven voor deze propositielogica. We laten zien dat in dit systeem bewijzen waarin vragen voorkomen aanleiding geven tot een interessante computationele interpretatie, die doet denken aan de bewijs-als-programma interpretatie van intuïtionistische logica: namelijk, zodra een bewijs getuigt van een bepaalde afhankelijkheid, codeert het feitelijk ook een *methode* voor het berekenen van die afhankelijkheid. Tenslotte abstraheren we van de details van het gegeven bewijssysteem, en concentreren ons op de rol die vragen in afleidingen spelen. In essentie is onze bevinding dat een vraag kan worden gebruikt als iets dat in de plaats staat voor een generiek stuk informatie van een bepaald type, op een manier die lijkt op hoe individuele constanten soms worden gebruikt in bewijzen in de eerste orde logica als staand voor een generiek, willekeurig individu. Door dergelijke plaatsvervangers te manipuleren is het mogelijk formele bewijzen te geven van de geldigheid van bepaalde afhankelijkheden.

In Hoofdstuk 4 brengen we onze benadering over naar het terrein van de eerste orde logica, die een grotere uitdrukkingskracht heeft. We beschrijven een conservatieve uitbreiding van de klassieke eerste orde logica met vragen, en bespreken hoe een breed scala aan belangrijke soorten vragen uitgedrukt kan worden in dit systeem, waarbij we ook zaken aanroeren als referentie en identiteit. We karakteriseren twee fragmenten van de taal die samen de meest in het oog springende klassen van eerste-orde vragen dekken, en geven een eenvoudige axiomatisering voor elk ervan.

In Hoofdstuk 5 bespreken we in detail de overeenkomsten en verschillen tussen onze benadering van afhankelijkheid en die welke wordt gehanteerd in het kader van *dependence logic*, waarin zich de laatste jaren aanmerkelijke ontwikkelingen hebben voorgedaan. We constateren dat het fundamentele verschil ligt in het feit dat in dependence logica afhankelijkheid wordt geconstrueerd als een relatie tussen variabelen, terwijl het in dit proefschrift wordt opgevat als een relatie tussen vragen—die niet anders is dan de algemene relatie van logische inclusie. We beargumenteren dat afhankelijkheid zien als logische inclusie tussen vragen belangrijke voordelen met zich meebrengt: allereerst stelt het ons in staat om een breder bereik aan afhankelijkheden te onderkennen en te behandelen dan die welke in dependence logica in beschouwing worden genomen. Bovendien, door afhankelijkheid bloot te leggen als een facet van logische inclusie, leidt de op vragen gebaseerde benadering niet alleen tot een aantrekkelijk conceptueel beeld ervan, maar stelt het ons ook in staat om grip op deze relatie te krijgen met behulp van welbekende logische gereedschappen, zowel semantisch als bewijs-theoretisch.

Vervolgens betreden we het terrein van modale logica. In Hoofdstuk 6 laten we zien hoe Kripke modaliteiten gegeneraliseerd kunnen worden naar de context van logica's die ook vragen omvatten. Als illustratie daarvan laten we zien dat het mogelijk wordt om de modaliteit van *weten* uit de epistemische logica op een dusdanige manier te generaliseren dat ze zowel beweringen als vragen kan inbedden, wat een uniforme analyse mogelijk maakt van zinnen als "A weet dat B thuis is", "A weet of B thuis is" en, in het eerste-orde geval, "A weet waar B is". Technisch gezien is de belangrijkste bijdrage een uniform axiomatiseringsresultaat voor de inquisitieve tegenhanger van elke gegeven canonieke modale logica. Bovendien doen we in dit hoofdstuk een beargumenteerd voorstel voor een modale behandeling van afhankelijkheidsbeweringen, d.w.z., beweringen zoals "of Alice zal komen hangt af van of ze haar huiswerk al heeft gemaakt". Volgens deze benadering kan een afhankelijkheidsbewering in onze modale logica worden geformaliseerd als een conditionele modaliteit tussen vragen. Tenslotte geven we een axiomatisering voor de logica die wordt voortgebracht door deze conditionele operator als primitief te beschouwen.

In Hoofdstuk 7 exploreren we een meer genuanceerde kijk op modale operatoren, zoals die wordt gesuggereerd door onze verrijkte semantiek. We vervangen Kripke modellen door *inquisitieve modale modellen*, die elke wereld niet simpel voorzien van een informatietoestand—een verzameling opvolgers—maar van een *inquisitieve toestand*, die zowel informatie als issues codeert. Bij wijze van toepassing bestuderen we *inquistieve epistemische logica*, een verrijking van epistemische logica waarin we kunnen redeneren over personen of instanties die niet alleen over bepaalde informatie beschikken, maar ook bepaalde issues overwegen, individueel of als groep. We geven volledigheidsresultaten zowel voor de basale inquisitieve modale logica, als voor een reeks van modale logica's die resulteren door het opleggen van verschillende condities op frames die van belang zijn. We laten zien dat, terwijl inquisitieve modaliteiten een grotere uitdrukkingskracht hebben dan Kripke modaliteiten, ze een ordentelijke wiskundige theorie behouden, en worden gekarakteriseerd door eenvoudige logische kenmerken.

Tenslotte, in Hoofdstuk 8, dynamificeren we inquisitieve epistemische logica, door een generalisatie van de standaard epistemische logische behandeling van het effect van publieke mededelingen. De resulterende logica stelt ons in staat om te redeneren over de manier waarop een inquisitief-epistemische situatie zich ontwikkelt, niet alleen als nieuwe informatie wordt geboden door een mededeling met een bewering, maar ook als er een nieuwe issue wordt opgeworpen door het stellen van een vraag. Een opvallend kenmerk van deze logica is dat we niet twee aparte acties voor mededelingen en vragen hoeven te onderscheiden, maar dat één actie van *publieke uiting* volstaat; het is de betekenis van de zin die wordt geuit die bepaalt of het effect van de uiting is dat nieuwe informatie wordt gegeven of dat een nieuw issue wordt opgeworpen. We verkrijgen een volledige axiomatisering van deze dynamische logica door middel van reductieregels die ons in staat stellen elke formule in deze logica te transformeren naar een formule in inquisitieve epistemische logica.

Summary

Questions in Logic

This dissertation pursues two tightly interwoven goals: to bring out the relevance of questions for the field of logic, and to establish a solid theory of the logic of questions within a classical logical setting. These enterprises feed into each other: on the one hand, the development of our formal systems is motivated by our considerations concerning the role to be played by questions; on the other hand, it is via the development of concrete, workable logical systems that the potential of questions in logic is made clear and tangible.

We begin by showing that, if we move from the standard relation of truth at a possible world to a relation of *support at an information state*, we obtain a semantic foundation which allows us to interpret statements and questions in a uniform way. This move leads to a substantial generalization of the classical notion of entailment, which encompasses not only the standard relation of consequence, but also the relation of *dependency*—a relation that plays an important role in many contexts, from physics to computer science. We show that, once logic is extended to questions, dependency emerges as contextual question entailment.

In Chapter 2, our approach is made concrete in the simplest possible setting, that of propositional logic. We describe how classical propositional logic can be enriched with questions, discuss the features of this logic, and show how propositional dependencies may be captured as cases of question entailment.

In Chapter 3 a natural deduction system for our propositional logic is provided. We show that proofs involving questions in this system have an interesting computational interpretation, reminiscent of the proofs-as-programs interpretation of intuitionistic logic: namely, whenever a proof witnesses a certain dependency, it actually encodes a *method* for computing this dependency. Finally, we abstract away from the details of the given proof system, and focus on the role played by questions in inferences. Essentially, we find that a question can be used as a placeholder for a generic piece of information of a certain type, much like individual constants are sometimes used in first-order logical proofs as placeholders for generic individuals. By manipulating such placeholders it is possible to provide formal proofs of the validity of certain dependencies.

In Chapter 4 we take our approach to the more expressive setting of first-order logic. We describe a conservative extension of classical first-order logic with questions, and discuss how a broad range of interesting questions becomes expressible in this system, touching upon issues such as reference and identity. We identify two fragments of the language which jointly cover the most salient classes of first-order questions, and for each of them we provide a simple axiomatization.

In Chapter 5, we discuss in detail the similarities and differences between our approach to dependency and the one adopted in the framework of *dependence logic*, which has seen considerable development in recent years. We conclude that the fundamental difference lies in the fact that in dependence logic, dependency is construed as a relation between variables, while in this thesis, it is construed as a relation between questions—none other than the relation of entailment. We argue that viewing dependency as question entailment presents some important benefits: first, it allows us to recognize and handle a broader range of dependencies than those considered in dependence logic. Moreover, by uncovering dependency as a facet of entailment, the question-based approach does not only lead to a neat conceptual picture, but also allows us to handle this relation by means of familiar logical tools, both semantically and proof-theoretically.

We then turn to the setting of modal logic. In Chapter 6, we show how Kripke modalities can be generalized to the context of a logic which includes questions. As an application, this makes it possible to generalize the *knowledge* modality of epistemic logic to embed both statements and questions, allowing for a uniform analysis of sentences like “A knows that B is home”, “A knows whether B is home” and, in the first-order case, “A knows where B is”. Technically, our main result is a uniform axiomatization result for the inquisitive counterpart of any given canonical modal logic. Moreover, in this chapter we propose and defend a modal account of dependence statements, i.e., statements such as “whether Alice will come depends on whether she finishes her homework”. According to this account, a dependence statement can be formalized in our modal logic as a modal conditional among questions. Finally, we provide an axiomatization for the logic that arises from taking this conditional as our primitive modal operator.

In Chapter 7, we explore a richer view on modal operators, which is suggested by our enriched semantics. We replace Kripke models with *inquisitive modal models*, which equip each world not just with an information state—a set of successors—but with an *inquisitive state*, encoding both information and issues. As an application, we investigate *inquisitive epistemic logic*, an enrichment of epistemic logic in which we can reason about agents who do not only have certain information, but also entertain certain issues, both individually and as a group. We provide completeness results both for the basic inquisitive modal logic, and for a range of modal logics that result from imposing various interesting frame conditions. We show that, while inquisitive modalities are more expressive than standard Kripke modalities, they retain a very well-behaved mathematical theory, and they are characterized by simple logical features.

Finally, in Chapter 8 we dynamify inquisitive epistemic logic, by generalizing the standard account of public announcements in epistemic logic. The resulting logic allows us to reason about the way in which an inquisitive-epistemic situation evolves not only when new information is provided by announcing a statement, but also when a new issue is raised by asking a question. A remarkable feature of this logic is that we do not need two separate actions for announcing and asking: one action of *public utterance* suffices; it is the meaning of the sentence being uttered that determines whether the effect of the utterance is to provide new information, or to raise new issues. We establish a complete axiomatization of this dynamic logic by means of reduction rules that allow us to transform every formula of this logic into a formula of inquisitive epistemic logic.

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