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Comparative study of projection schemes for stochastic finite element analysis

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Abstract

We present a comparison of subspace projection schemes for stochastic finite element analysis in terms of accuracy and computational efficiency. More specifically, we compare the polynomial chaos projection scheme with reduced basis projection schemes based on the preconditioned stochastic Krylov subspace. Numerical studies are presented for two problems: (1) static analysis of a plate with random Young's modulus and (2) settlement of a foundation supported on a randomly heterogeneous soil. Monte Carlo simulation results based on exact structural analysis are used to generate benchmark results against which the projection schemes are compared. We show that stochastic reduced basis methods require significantly less computer memory and execution time compared to the polynomial chaos approach, particularly for large-scale problems with many random variables. For the class of problems considered, we find that stochastic reduced basis methods can be up to orders of magnitude faster, while providing results of comparable or better accuracy.

Keywords: Stochastic finite element analysis; Projection schemes; Polynomial chaos; Stochastic reduced basis methods

1. Introduction

The finite element method (FEM) has emerged as an effective and versatile numerical tool for analyzing engineering systems governed by partial differential equations (PDEs). More recently, much research has focused on extending the FEM to tackle uncertain systems governed by stochastic PDEs; see, for example,

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the research monographs by Ghanem and Spanos [1] and Kleiber and Hien [2], and a comprehensive review of the state-of-the-art edited by Schuëller [3]. In the stochastic FEM, uncertainties are represented by random variables or fields. To enable a computational treatment of this problem, the random fields are first discretized to represent them in terms of a finite number of random variables. Subsequent application of spatial and temporal discretization schemes leads to a system of random algebraic equations to be solved for the response process. In order for the stochastic FEM to be practical for large-scale systems, it is essential that efficient numerical schemes be available for solving random algebraic equations arising from discretization of stochastic PDEs in space, time and the random dimension of the problem.

Given a system of random algebraic equations, the Monte Carlo simulation technique can be readily applied to compute the response statistics to an arbitrary degree of accuracy [4,5]. In practice, this is the method of last resort since the attendant computational cost can be prohibitive for systems modeled using a large number of degrees of freedom (dof). However, in recent years, researchers have applied MCS combined with high performance solution procedures to efficiently analyze finely discretized real world structures (such as 3D frames and shells); see, for example, [6,7]. The perturbation method and the Neumann series offer computationally efficient alternatives and have been popularly applied to compute the first two statistical moments of the response quantities; see, for example, [2,3,8–10]. The major drawback of such local approximation techniques is that the results become highly inaccurate when the coefficients of variation of the input random variables are increased. An overview of some alternatives to perturbation methods for analyzing structural systems can be found in [11]. More recently, a series expansion technique based on eigensolutions was proposed by Impollonia and his colleagues [12,13].

In 1990, Ghanem and Spanos [14] proposed a spectral stochastic FEM, in which the random algebraic equations arising from spatial discretization of stochastic PDEs are solved using a Wiener polynomial chaos (PC) decomposition approach [15,16]. The basic idea is to represent the response process by a linear combination of multidimensional Hermite polynomials with undetermined deterministic coefficients. The theoretical justification for this representation is provided by Cameron and Martin's theorem [17], which essentially states that any second-order stochastic process can be represented by a PC expansion to an arbitrary degree of accuracy. Ghanem and Spanos showed that the coefficients in the expansion can be uniquely computed using the Galerkin scheme, which involves the solution of a deterministic system of equations with increased dimensionality. Over the last decade, the PC projection scheme has been successfully applied to solve a wide range of problems in stochastic mechanics, including elasticity problems [1], random vibration [18], soil mechanics [19], transport process in heterogeneous media [20], plasticity problems [21], fluid dynamics [22], mid-frequency structural dynamics [23] and wave propagation in random media [24]. Recent reviews of the theoretical foundations of the spectral stochastic FEM can be found in the literature; see, for example, [25–27].

More recently, Xiu and Karniadakis [28] proposed a generalized PC representation which employs orthogonal functions from the Askey family of hypergeometric polynomials. It was shown that by choosing an appropriate basis from the Wiener–Askey chaos, exponential convergence can be ensured for a wide variety of stochastic processes. The Wiener–Askey chaos has subsequently been applied to solve diffusion problems [29], fluid–structure interaction problems [30], the Navier–Stokes equations [31], and heat transfer problems [32] in the presence of parameter uncertainty.

In the context of reliability analysis, Sudret and Der Kiureghian [33,34] presented a detailed comparison between the PC projection scheme and the first-order reliability method (FORM). It was shown that the PC approach becomes computationally much more expensive than the FORM, particularly with increase in the number of dof and random variables. Based on extensive numerical studies, it was suggested that the chaos expansion scheme may not be practical for reliability analysis of systems with large number of random variables.

More recently, stochastic reduced basis methods (SRBMs) were introduced in the literature [35,36] for solving large-scale linear random algebraic system of equations obtained by discretizing stochastic PDEs in

space and the random dimension of the problem. In contrast to the PC approach, the solution process is represented using basis vectors spanning the preconditioned stochastic Krylov subspace. Subsequent application of the Galerkin scheme leads to a reduced-order deterministic system of equations to be solved for the undetermined coefficients in the stochastic reduced basis representation. A more detailed account of the theoretical foundations of SRBMs and connections to the PC projection scheme can be found in [37]. A key advantage of SRBMs is that they are computationally very efficient compared to the PC projection scheme since a reduced-order system of equations is solved in contrast to one with increased dimensionality. However, a detailed comparison between SRBMs and the PC projection scheme is necessary to investigate whether this improvement in computational efficiency is achieved at the cost of lower accuracy. The objective of the present paper is to investigate this issue by extensive numerical studies.

In this paper, we present a detailed comparison of SRBMs with the PC projection scheme in terms of their relative accuracy and computational efficiency. Results are presented for the original formulation of SRBMs [35,36] and a new scheme which uses an augmented set of basis vectors to improve accuracy. Numerical studies are presented for a two-dimensional elasticity problem and a geotechnical engineering problem with parameter uncertainty modeled using stochastic finite elements. Results are presented for the first two statistical moments of the displacements and its probability distribution functions (pdfs). For the cases considered, we observed that the accuracy of SRBMs is comparable to that obtained using the PC expansion, particularly for the dof where the displacement levels are high. We also compare the computational efficiency of the projection schemes when the number of dof and random variables are increased. The studies suggest that SRBMs can be up to orders of magnitude faster than PC projection schemes.

The remainder of this paper is organized as follows. In the next section we outline the steps involved in stochastic finite element analysis of random media. Section 3 summarizes the PC projection scheme for solving random algebraic equations arising from discretization of the governing stochastic PDEs in space and the random dimension. In Section 4, we present SRBMs based on the preconditioned stochastic Krylov subspace as an alternative to the PC projection approach and outline some of its theoretical properties. A new formulation which employs the PC expansion scheme to arrive at an augmented set of basis vectors is also proposed. Comparison of the relative accuracies of the projection schemes for two example problems is presented in Section 5. Section 6 presents a study on the computational efficiency of PC projection schemes and SRBMs when the number of dof and random variables are increased. Section 7 concludes the paper and outlines some areas for further investigation.

2. Stochastic finite element formulation

In order to apply the FEM to problems wherein one or more of the physical parameters are modeled as random fields, we need to represent them first by an enumerable set of random variables. Subsequently, standard spatial and temporal discretization schemes can be applied to arrive at a system of random algebraic equations. To illustrate the basic steps involved in stochastic finite element analysis, consider a two-dimensional isotropic solid whose Young's modulus is modeled as a random field, say $h(\mathbf{x};\omega)$. In other words, for each $\mathbf{x} \in \mathbb{R}^2, h : \Omega \to \mathbb{R}$ is a random variable on a suitable probability space $(\Omega, \mathscr{F}, \Gamma)$, where Ω is the set of elementary events, $\mathscr{F}$ is the σ-algebra associated with Ω and Γ is a probability measure. We use the symbol ω to indicate the dependence of a quantity on the random dimension of the problem.

Since the Young's modulus is represented by a random field, the elasticity matrix becomes a function of the spatial coordinates and a random dimension (ω), and can be represented as follows:

$$\mathbf{D}(\mathbf{x};\omega) = h(\mathbf{x};\omega)\mathbf{D}_0, \tag{1}$$

where $\mathbf{D}_0$ is the deterministic part of the elasticity matrix.

Random field discretization is a key step in the numerical solution of stochastic PDEs. Various discretization techniques are available in the literature for approximating random fields including the mid-point method, shape function methods, optimal linear estimation, weighted integral methods, orthogonal series expansion and the Karhunen–Loéve (KL) expansion scheme. For a detailed overview and comparison of these methods the reader is referred to [33,38]. In the present study, we have chosen the KL expansion technique to discretize random fields.

Let the autocovariance function of the random field $h(\mathbf{x};\omega)$ be $R_h(\mathbf{x}_1, \mathbf{x}_2)$. Then, the KL expansion of the random field $h(\mathbf{x};\omega)$ can be written as

$$h(\mathbf{x};\omega) = \langle h(\mathbf{x};\omega)\rangle + \sum_{i=1}^{\infty} \sqrt{\lambda_i}\theta_i h_i(x), \tag{2}$$

where λ_i and $h_i(x)$ are the eigenvalues and eigenfunctions of $R_h(\mathbf{x}_1, \mathbf{x}_2)$, respectively. θ_i, $i = 1, 2, \ldots, \infty$, are a set of uncorrelated random variables and $\langle\cdot\rangle$ denotes the expectation operator.

The eigenvalues and eigenfunctions of the autocovariance function can be computed by solving a Fredholm integral equation of the second kind, i.e.,

$$\int_{\mathscr{D}} R_h(\mathbf{x}_1, \mathbf{x}_2) h_i(\mathbf{x}_1)\,\mathrm{d}\mathbf{x}_1 = \lambda_i h_i(\mathbf{x}_2). \tag{3}$$

Analytical solutions of the above integral eigenvalue problem can be obtained only for a special class of functions (e.g., the exponential covariance function) defined on geometrically simple domains. For more general cases, numerical discretization schemes have to be employed to compute the eigenvalues and eigenfunctions of $R_h(\mathbf{x}_1, \mathbf{x}_2)$; see, for example, [1,39,40].

Truncating Eq. (2) at the Mth term gives a finite-dimensional approximation of the random field as

$$\hat{h}(x;\omega) = \langle h(\mathbf{x};\omega)\rangle + \sum_{i=1}^{M} \sqrt{\lambda_i}\theta_i h_i(x). \tag{4}$$

Substitution of the discretized random field representation into Eq. (1) results in a representation of the elasticity matrix in terms of a finite number of random variables. Now, the element stiffness matrix for a two-dimensional elastic solid with random Young's modulus can be written as

$$k^e = \int_{\mathscr{D}_e} \mathbf{B}^{\mathrm{T}}\mathbf{D}(\mathbf{x};\omega)\mathbf{B}\,\mathrm{d}\mathbf{x}, \tag{5}$$

where $\mathbf{B}$ is the strain–displacement matrix and $\mathscr{D}_e$ denotes the domain of the element.

Substituting the discretized version of the random elasticity matrix in the preceding equation gives the following expression for the stochastic element stiffness matrix:

$$k^e(\boldsymbol{\theta}) = k_0^e + \sum_{j=1}^{M} k_j^e \theta_j, \tag{6}$$

where

$$k_0^e = \int_{\mathscr{D}_e} \langle h(\mathbf{x},\omega)\rangle \mathbf{B}^{\mathrm{T}}\mathbf{D}_0\mathbf{B}\,\mathrm{d}\mathbf{x}, \tag{7}$$

$$k_j^e = \sqrt{\lambda_j}\int_{\mathscr{D}_e} h_j(\mathbf{x})\mathbf{B}^{\mathrm{T}}\mathbf{D}_0\mathbf{B}\,\mathrm{d}\mathbf{x} \tag{8}$$

and $\boldsymbol{\theta} = \{\theta_1, \theta_2, \ldots, \theta_M\} \in \mathbb{R}^M$ denotes the set of random variables arising from discretization of the random field representing uncertainty in Young's modulus.

Assembly of the element stiffness matrices and application of suitable boundary conditions result in the following system of linear random algebraic equations

$$\left(\mathbf{K}_0 + \sum_{i=1}^{M} \mathbf{K}_i \theta_i\right)\mathbf{u}(\theta) = \mathbf{f} \quad \text{or} \quad \left(\sum_{i=0}^{M} \mathbf{K}_i \theta_i\right)\mathbf{u}(\theta) = \mathbf{f}, \tag{9}$$

where $\mathbf{K}_0 \in \mathbb{R}^{n\times n}$ and $\mathbf{K}_i \in \mathbb{R}^{n\times n}$ are deterministic matrices, $\mathbf{u}(\theta) \in \mathbb{R}^n$ is the random displacement vector, and $\theta_0 = 1$. $\mathbf{f} \in \mathbb{R}^n$ denotes the force vector which we assume to be deterministic for simplicity of presentation and n is the total number of dof. Note that the matrix $\mathbf{K}_0$ is obtained by assembling the first term of the element stiffness matrix, k_0^e, defined earlier in Eqs. (6) and (7).

In the next two sections we discuss subspace projection schemes based on PC chaos expansions and the preconditioned stochastic Krylov subspace for computing the statistics of the response process $\mathbf{u}(\theta)$.

3. Polynomial chaos projection scheme

In the PC projection scheme of Ghanem and Spanos [1], the random nodal displacements are expanded using a set of multidimensional Hermite polynomials. This results in the following expansion for the response process:

$$\mathbf{u}(\theta) = \sum_{i=0}^{P-1} \mathbf{u}_i \varphi_i(\theta), \tag{10}$$

where $\mathbf{u}_i \in \mathbb{R}^n$, $i = 0, 1, 2, \ldots, P-1$, are sets of vectors formed from the undetermined coefficients in the PC expansions for each nodal displacement. $\varphi_j(\theta)$ is a set of orthogonal Hermite polynomials in θ_k. The number of terms in the expansion, P, is given by

$$P = \sum_{k=0}^{p} \frac{(M+k-1)!}{k!(M-1)!}, \tag{11}$$

where p is called the order of the PC expansion, i.e., the highest order of the set of Hermite polynomials φ_j.

Substitution of the PC expansion for $\mathbf{u}(\theta)$ into the governing random algebraic equations given in Eq. (9) gives

$$\left(\sum_{i=0}^{M} \mathbf{K}_i \theta_i\right)\left(\sum_{j=0}^{P-1} \mathbf{u}_j \varphi_j(\theta)\right) = \mathbf{f}. \tag{12}$$

As shown by Ghanem and Spanos [1], the undetermined terms in the PC expansion can be uniquely computed by imposing the Galerkin condition, which involves orthogonalizing the stochastic residual error to the approximating subspace as shown below

$$\langle \epsilon(\theta), \varphi_k(\theta)\rangle = 0, \quad k = 0, 1, 2, \ldots, P-1, \tag{13}$$

where the stochastic residual $\epsilon(\theta)$ is given by

$$\epsilon(\theta) = \left(\sum_{i=0}^{M} \mathbf{K}_i \theta_i\right)\left(\sum_{j=0}^{P-1} \mathbf{u}_j \varphi_j(\theta)\right) - \mathbf{f}. \tag{14}$$

Substitution of Eq. (14) in Eq. (13) results in the following system of deterministic equations:

$$\sum_{i=0}^{M}\sum_{j=0}^{P-1}\mathbf{K}_i\mathbf{u}_j\langle\theta_i\varphi_j\varphi_k\rangle = \langle\varphi_k\mathbf{f}\rangle, \quad k=0,1,2,\ldots,P-1. \tag{15}$$

The above equation can be rewritten in a more compact fashion as

$$\sum_{j=0}^{P-1}\mathbf{K}_{jk}\mathbf{u}_j = \mathbf{f}_k, \quad k=0,\ldots,P-1, \tag{16}$$

where

$$\mathbf{K}_{jk} = \sum_{i=0}^{M}\langle\theta_i\varphi_j\varphi_k\rangle\mathbf{K}_i \in \mathbb{R}^{n\times n} \tag{17}$$

and

$$\mathbf{f}_k = \langle\varphi_k\mathbf{f}\rangle \in \mathbb{R}^n. \tag{18}$$

The expectation operations in Eqs. (17) and (18) can be readily carried out using the properties of Wiener–Hermite chaos; see, for example, [1,33]. Now, expanding the above equation about the subscripts j and k, we arrive at the following system of linear algebraic equations:

$$\begin{bmatrix} \mathbf{K}_{0,0} & \mathbf{K}_{0,1} & \cdot & \mathbf{K}_{0,P-1} \\ \mathbf{K}_{1,0} & \mathbf{K}_{1,1} & \cdot & \mathbf{K}_{1,P-1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \mathbf{K}_{P-1,0} & \mathbf{K}_{P-1,1} & \cdot & \mathbf{K}_{P-1,P-1} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \cdot \\ \cdot \\ \mathbf{u}_{P-1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_0 \\ \mathbf{f}_1 \\ \cdot \\ \cdot \\ \mathbf{f}_{P-1} \end{bmatrix}, \tag{19}$$

which is of the form $\widetilde{\mathbf{K}}\tilde{\mathbf{u}} = \tilde{\mathbf{f}}$, where $\widetilde{\mathbf{K}} \in \mathbb{R}^{nP\times nP}$ and $\tilde{\mathbf{u}}, \tilde{\mathbf{f}} \in \mathbb{R}^{nP}$.

Table 1 shows the values of P for different values of p (order of the polynomial chaos) and M (number of terms in the KL expansion). It can be noted from the table that the computational complexity and memory requirements of the PC projection scheme increase significantly when M and p are increased. The memory requirements can be reduced by precomputing and storing the ensemble average terms of the form $\langle\theta_i\varphi_j\varphi_k\rangle$ instead of storing the matrices $\mathbf{K}_{jk}$ given in Eq. (17). Further, the sparsity of the tensor products $\langle\theta_i\varphi_j\varphi_k\rangle$ can also be exploited to accelerate the computations. A detailed overview of numerical schemes which exploit the peculiar structure of Eq. (19) can be found in the literature [41,42].

After solving Eq. (19) and substituting the results in Eq. (10), we arrive at an explicit expression for the response process. This enables the statistics of the displacements as well as other response quantities of interest to be efficiently computed in the post-processing phase [1].

Table 1
Values of P for different values of M and p

M	Order of PC(p)				
	0	1	2	3	4
2	1	3	6	10	15
4	1	5	15	35	70
6	1	7	28	83	210

4. Stochastic reduced basis methods (SRBMs)

In contrast to the PC approach, SRBMs approximate the response process by representing $\mathbf{u}(\theta)$ using a set of basis vectors spanning the stochastic Krylov subspace defined below:

$$\mathscr{K}_m(\mathbf{K}(\theta),\mathbf{f}) = \text{span}\left\{\mathbf{f},\mathbf{K}(\theta)\mathbf{f},\mathbf{K}(\theta)^2\mathbf{f},\ldots,\mathbf{K}(\theta)^{m-1}\mathbf{f}\right\}. \tag{20}$$

This representation of the response process can be justified by the following theorem which establishes that the solution of Eq. (9) lies in the stochastic Krylov subspace [35,36].

Theorem 1. *If the minimal random polynomial of a nonsingular random square matrix* $\mathbf{K}(\theta)$ *has degree* m*, then the solution to* $\mathbf{K}(\theta)\mathbf{u}(\theta) = \mathbf{f}$ *lies in the stochastic Krylov subspace* $\mathscr{K}_m(\mathbf{K}(\theta),\mathbf{f})$.

It is to be noted here that the number of basis vectors required to compute accurate approximations depends on the degree of overlap of the pdfs of the eigenvalues of the coefficient matrix $\mathbf{K}(\theta)$ [36]. To ensure good approximations using a small number of basis vectors, it is preferable to use a preconditioner. The key idea here is to transform the coefficient matrix such that the pdfs of the modified eigenvalues numerically tend to have a high degree of overlap. Following [35,36], we use the deterministic matrix $\langle\mathbf{K}(\theta)\rangle^{-1} = \mathbf{K}_0^{-1}$ as the preconditioner.[1] This choice is motivated by the observation that $\mathbf{K}_0^{-1}\mathbf{K}(\theta)$ will numerically behave like a matrix with a small number of distinct eigenvalues, particularly for small randomness. Note here that, in theory, convergence can be guaranteed as long as the preconditioner is invertible. However, by using the preconditioner suggested here, convergence can be significantly accelerated—in other words, it becomes possible to achieve high accuracy using a few number of basis vectors (usually 3–4).

The stochastic reduced basis representation for the response process can be written as

$$\hat{\mathbf{u}}(\theta) = \xi_1\psi_1(\theta) + \xi_2\psi_2(\theta) + \cdots + \xi_m\psi_m(\theta) = \boldsymbol{\Psi}(\theta)\boldsymbol{\xi}, \tag{21}$$

where $\boldsymbol{\Psi}(\theta) = \{\psi_1(\theta),\psi_2(\theta),\ldots,\psi_m(\theta)\} \in \mathbb{R}^{n\times m}$ is a set of basis vectors spanning the preconditioned stochastic Krylov subspace $\mathscr{K}_m(\mathbf{K}_0^{-1}\mathbf{K}(\theta),\mathbf{K}_0^{-1}\mathbf{f})$ and $\boldsymbol{\xi} = \{\xi_1,\xi_2,\ldots,\xi_m\}^{\mathrm{T}} \in \mathbb{R}^m$ is a vector of undetermined coefficients.

The numerical studies conducted by Nair and Keane [36] suggest that using the first three basis vectors spanning the preconditioned stochastic Krylov subspace, highly accurate results can be obtained. These basis vectors can be written as

$$\psi_1(\theta) = \mathbf{K}_0^{-1}\mathbf{f}, \tag{22}$$

$$\psi_2(\theta) = \mathbf{K}_0^{-1}\mathbf{K}(\theta)\psi_1(\theta), \tag{23}$$

$$\psi_3(\theta) = \mathbf{K}_0^{-1}\mathbf{K}(\theta)\psi_2(\theta). \tag{24}$$

Since $\mathbf{K}(\theta) = \mathbf{K}_0 + \sum_{i=1}^{M}\theta_i\mathbf{K}_i$, the basis vectors can be compactly rewritten as follows:

$$\psi_1(\theta) = \mathbf{u}_0, \tag{25}$$

$$\psi_2(\theta) = \sum_{i=1}^{M}\mathbf{d}_i\theta_i, \tag{26}$$

$$\psi_3(\theta) = \sum_{i=1}^{M}\sum_{j=1}^{M}\mathbf{e}_{ij}\theta_i\theta_j, \tag{27}$$

where $\mathbf{u}_0 = \mathbf{K}_0^{-1}\mathbf{f}$, $\mathbf{d}_i = \mathbf{K}_0^{-1}\mathbf{K}_i\mathbf{u}_0$ and $\mathbf{e}_{ij} = \mathbf{K}_0^{-1}\mathbf{K}_i\mathbf{d}_j$.

[1] If the matrix $[\mathbf{K}(\theta_0)]^{-1}$ is used as the preconditioner, then the error in the stochastic reduced basis representation will converge faster to zero near the point θ_0. This feature of SRBMs can be exploited in practice to accurately estimate the statistics of the extremes. For example, in reliability analysis problems, θ_0 can be chosen to be the most probable point of failure.

It can clearly seen from the above expressions that the basis vectors are random polynomials which can be written as explicit functions of θ. Because of the recursive representation of the basis vectors, they can be efficiently computed given the factored form of the preconditioner $\mathbf{K}_0^{-1}$, which is readily available as a byproduct of deterministic analysis of the problem. Another point worth noting is that the basis vectors coincide with the Neumann series when the matrix $\mathbf{K}_0^{-1}$ is chosen to be the preconditioner. However, when a general preconditioner $\mathbf{K}(\theta_0)^{-1}$ is chosen, this observation does not hold true.

To compute ξ, we first substitute Eq. (21) in the governing random algebraic equations given in Eq. (9) to arrive at the following stochastic residual error vector:

$$\epsilon(\theta) = \left(\mathbf{K}_0 + \sum_{i=1}^{M} \mathbf{K}_i \theta_i\right) \Psi(\theta)\xi - \mathbf{f}. \tag{28}$$

If we restrict our attention to self-adjoint stochastic PDEs, the matrices $\mathbf{K}_i$, $i = 0, 1, 2, \ldots, M$, are guaranteed to be symmetric. Hence, the undetermined coefficients in Eq. (21) can be computed by enforcing the Galerkin condition[2]

$$\left[\mathbf{K}_0 + \sum_{i=1}^{M} \mathbf{K}_i \theta_i\right] \Psi(\theta)\xi - \mathbf{f} \perp \Psi(\theta). \tag{29}$$

Enforcing this condition using the definition of inner products in the Hilbert space of random variables, we obtain the following deterministic system of equations for ξ:

$$\left\langle \Psi(\theta)^{\mathrm{T}}\mathbf{K}_0\Psi(\theta) + \sum_{i=1}^{M} \theta_i \Psi(\theta)^{\mathrm{T}}\mathbf{K}_i\Psi(\theta) \right\rangle \xi = \left\langle \Psi(\theta)^{\mathrm{T}}\mathbf{f} \right\rangle. \tag{30}$$

Since explicit expressions for the stochastic basis vectors are available, the expectation operations required to compute the elements of the reduced-order terms in Eq. (30) can be readily carried out. Expressions for the terms of the reduced-order coefficient matrices and the right-hand side of Eq. (30) for the case when $m = 3$ are given in Appendix. As we shall show in the numerical studies section, three basis vectors are usually sufficient to ensure high accuracy. Solving the preceding reduced-order system of equations and substituting the computed value of ξ in the stochastic subspace approximation, we arrive at an explicit expression for $\mathbf{u}(\theta)$. Hence, similar to the PC representation, this enables the application of efficient post-processing techniques to compute the complete statistics of the response quantities of interest.

Note that when two basis vectors are used, the stochastic reduced basis approximation is of first-order since the second basis vector is a linear function of random variables. Similarly, when three basis vectors are used, the approximation is of second-order. It is to be noted here that the undetermined coefficients in Eq. (21) can also be computed by enforcing a strong Galerkin condition $P[\,\Psi(\theta)^{\mathrm{T}}\epsilon(\theta) = 0] = 1$. In other words, ξ is computed such that the random functions $\psi_i(\theta)^{\mathrm{T}}\epsilon(\theta)$, $i = 1, 2, \ldots, m$, are zero with probability one. In order to meet this condition, the undetermined coefficients $\xi_1, \xi_2, \ldots, \xi_m$ need to be modeled as functions of θ [36,37]. Initial studies presented by Nair and Keane [36] suggest that by enforcing the strong Galerkin condition, it is possible to derive higher-order approximations that are significantly more accurate than standard SRBMs and the PC projection scheme.

In the present work, we propose to enhance the original formulation of SRBMs by representing each term in the vector of undetermined coefficients ξ using a PC expansion as follows (as we shall show later this is equivalent to using an augmented set of basis vectors):

$$\xi_i = \sum_{j=0}^{P_1-1} \xi_j^i \varphi_j(\theta). \tag{31}$$

[2] For the case of nonsymmetric coefficient matrices, it is preferable to employ the stochastic Petrov–Galerkin scheme presented by Nair [35,37], which guarantees mean-square convergence of the L_2-norm of the stochastic residual error.

The stochastic reduced basis approximation for the response process given earlier in Eq. (21) can hence be rewritten as

$$\hat{\mathbf{u}}(\theta) = \left(\sum_{j=0}^{P_1-1} \xi_j^1 \varphi_j(\theta)\right)\psi_1(\theta) + \left(\sum_{j=0}^{P_1-1} \xi_j^2 \varphi_j(\theta)\right)\psi_2(\theta) + \cdots + \left(\sum_{j=0}^{P_1-1} \xi_j^m \varphi_j(\theta)\right)\psi_m(\theta). \tag{32}$$

Consider the case when we use the first-order SRBM to represent the response process (i.e., $m = 2$) and four terms are retained in the KL expansion of the random field (i.e., $M = 4$). Further, let a first-order PC expansion be employed in Eq. (31), i.e., $P_1 = 5$. Then the preceding equation can be written as

$$\hat{\mathbf{u}}(\theta) = \left(\sum_{j=0}^{4} \xi_j^1 \theta_j\right)\psi_1(\theta) + \left(\sum_{j=0}^{4} \xi_j^2 \theta_j\right)\psi_2(\theta). \tag{33}$$

Rearranging the terms in the preceding equation we obtain the following compact expression for the response process in matrix notation:

$$\hat{\mathbf{u}}(\theta) = \boldsymbol{\Psi}_{\text{new}}(\theta)\boldsymbol{\xi}_{\text{new}}, \tag{34}$$

where $\boldsymbol{\Psi}_{\text{new}}(\theta) = \{\psi_1, \psi_1\theta_1, \psi_1\theta_2, \psi_1\theta_3, \psi_1\theta_4, \psi_2, \psi_2\theta_1, \psi_2\theta_2, \psi_2\theta_3, \psi_2\theta_4\} \in \mathbb{R}^{n\times 10}$ and $\boldsymbol{\xi}_{\text{new}} = \{\xi_0^1, \xi_1^1, \xi_2^1, \xi_3^1, \xi_4^1, \xi_0^2, \xi_1^2, \xi_2^2, \xi_3^2, \xi_4^2\}^{\mathrm{T}} \in \mathbb{R}^{10}$.

It can be clearly seen from Eq. (34) that expanding the undetermined coefficients using a PC expansion is equivalent to employing an augmented set of basis vectors to represent the solution process. It is also of interest to note that the new formulation can be viewed as a generalization of SRBMs since we recover the original approximation in Eq. (21) when we set $P_1 = 1$ in Eq. (31). We can now substitute Eq. (34) in Eq. (9) to obtain an expression for stochastic residual error similar to Eq. (28), i.e.,

$$\epsilon(\theta) = \left(\mathbf{K}_0 + \sum_{i=1}^{M} \mathbf{K}_i\theta_i\right)\boldsymbol{\Psi}_{\text{new}}(\theta)\boldsymbol{\xi}_{\text{new}} - \mathbf{f}. \tag{35}$$

Subsequent application of the Galerkin projection scheme results in a reduced-order deterministic system of equations which can be solved for $\boldsymbol{\xi}_{\text{new}}$. Clearly, the size of the reduced-order system of equations is a function of the order of PC expansion employed in Eq. (31) and the original number of basis vectors, m. The formulation outlined here can be extended to the case when we use higher-order SRBMs to approximate the solution process and higher-order PC expansions to represent the undetermined coefficients. In the numerical studies outlined in the next section, we present results for first- and second-order SRBMs when the undetermined coefficients are expanded using a first-order PC expansion.

Before delving into detailed numerical studies, we briefly outline some theoretical properties of SRBMs which employ the weak Galerkin condition outlined earlier. The following theorem [36] guarantees the convergence of SRBMs when the number of basis vectors is increased.

Theorem 2. *Let $\hat{\mathbf{u}}(\theta) = \boldsymbol{\Psi}(\theta)\boldsymbol{\xi}$ be a stochastic reduced basis approximation to the solution of $\mathbf{K}(\theta)\mathbf{u}(\theta) = \mathbf{f}$, where $\mathbf{K}(\theta) \in \mathbb{R}^{n\times n}$ is a random symmetric positive definite matrix, $\mathbf{u}(\theta), \mathbf{f} \in \mathbb{R}^n$ are random vectors, $\boldsymbol{\Psi}(\theta) \in \mathbb{R}^{n\times m}$ is a matrix of stochastic basis vectors, and $\boldsymbol{\xi} \in \mathbb{R}^m$ is a vector of undetermined coefficients. If the coefficient vector $\boldsymbol{\xi}$ is computed by imposing the condition $\mathbf{K}(\theta)\boldsymbol{\Psi}(\theta)\boldsymbol{\xi} - \mathbf{f} \perp \boldsymbol{\Psi}(\theta)$, then the following deterministic error function is minimized:*

$$\Delta_m = \left\langle \{\mathbf{u}(\theta) - \hat{\mathbf{u}}(\theta)\}^{\mathrm{T}}\mathbf{K}(\theta)\{\mathbf{u}(\theta) - \hat{\mathbf{u}}(\theta)\} \right\rangle, \tag{36}$$

where Δ_m denotes the $\mathbf{K}$-norm of the error.

A corollary of Theorem 2 is that the Galerkin projection scheme used here ensures that Δ_m converges in a mean-square sense when the number basis vectors is increased. Since the new formulation presented in this paper uses an augmented set of stochastic basis vectors, it can be shown that this yields a lower value of Δ_m compared to the original SRBM formulation. It is also worth noting that the **K**-norm of the error can be interpreted as an energy norm. This has an important practical ramification for stochastic structural systems. Since SRBMs employing the Galerkin scheme minimize the **K**-norm of the error, the results are bound to be more accurate for those dof which contain most strain energy. This is a very useful property since in many practical applications such as reliability analysis, we are primarily interested in ensuring good approximations for the highly strained regions of the structure.

5. Numerical studies

In this section, we apply SRBMs and the PC projection schemes to compute the response statistics and pdfs for two example problems. The results obtained are benchmarked against Monte Carlo simulations (MCS) based on deterministic structural analysis. For all the problems considered, we use the following two-dimensional exponential covariance function to represent the stochastic field describing the uncertainty in Young's modulus:

$$R(\mathbf{x},\mathbf{y}) = \exp\left(-\frac{|x_1 - x_2|}{b_1} - \frac{|y_1 - y_2|}{b_2}\right), \tag{37}$$

where b_1 and b_2 are the correlation lengths of the random field. This random field is discretized using the KL expansion scheme.

We compare the accuracy of the results obtained using first- and second-order PC schemes and SRBMs when the standard deviation of the random field is increased. Recollect that in first- and second-order SRBMs, we use two and three basis vectors, respectively. Henceforth, we refer to the first-order SRBM, second-order SRBM, first-order PC and second-order PC schemes by SRBMI, SRBMII, PCI and PCII, respectively. The abbreviations SRIPCI and SRIIPCI are used to denote the new formulation outlined in the earlier section where the unknown coefficients in SRBMI and SRBMII are expanded using PC expansions of order one.

The numerical studies presented here were conducted using the 'SSFEM toolbox' developed by Sudret and Der Kiureghian [33,34]. This toolbox contains routines for discretizing random fields, stochastic finite element formulation based on four-noded quadrilateral elements and an implementation of the PC expansion scheme for solving random algebraic equations. Routines implementing SRBMs were linked with this toolbox to enable a systematic comparison of the various projection schemes considered here. Note that all the problems considered here were spatially discretized using four-noded quadrilateral elements with two dof per node.

5.1. Problem 1: Two-dimensional static plate analysis

The first problem considered is a thin square plate of unit length clamped at one edge and subjected to uniform inplane tension at the opposite edge (from [1]). The domain of the plate is discretized into 16 square elements as shown in Fig. 1, which leads to a total of 50 dof. The external loads are assumed to be deterministic and of unit magnitude. The modulus of elasticity of the plate is modeled as a two-dimensional Gaussian random field with the exponential covariance model given in Eq. (37) with $b_1 = b_2 = 1$. The random field is discretized using the KL expansion scheme and four terms are retained, i.e., $M = 4$. Numer-

ical studies were conducted to analyze the performance of the projection schemes when the standard deviation of the random Young's modulus (σ) is increased.

SRBMs and PC projection schemes are applied to compute the statistics of the displacements when σ is kept at 0.05, 0.10, 0.15, 0.20 and 0.25. For each case, MCS with a sample size of 250,000 is used to generate the reference results. The percentage errors in the mean and standard deviation of the vertical displacement at the free corner 'E' of the square plate for increasing standard deviation of Young's modulus are shown in Figs. 2 and 3, respectively. It can be observed that for very low values of σ, the accuracy of the first-order

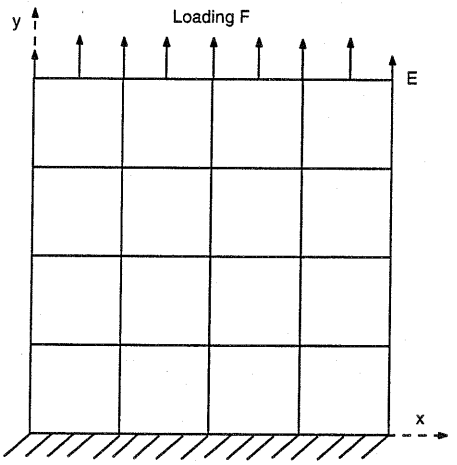

Fig. 1. Schematic of static plate problem.

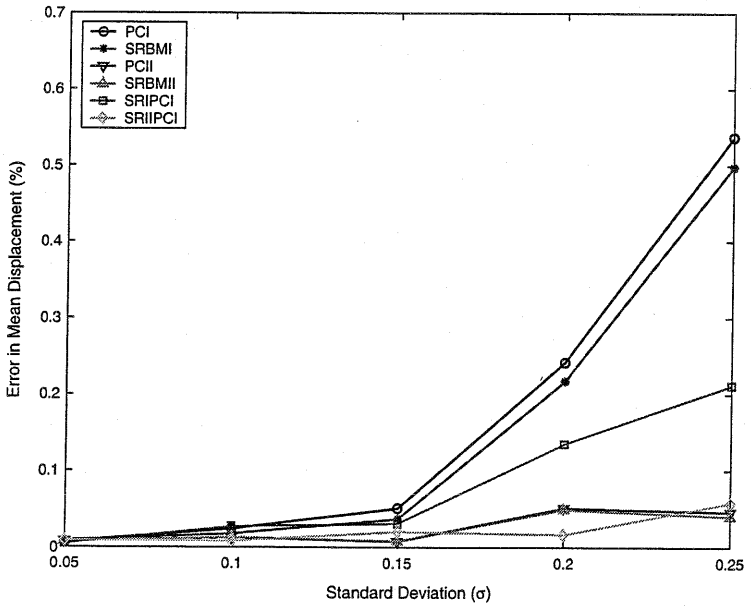

Fig. 2. Percentage error in mean displacement at point 'E' for various projection schemes as a function of the standard deviation of the random Young's modulus for Problem 1.

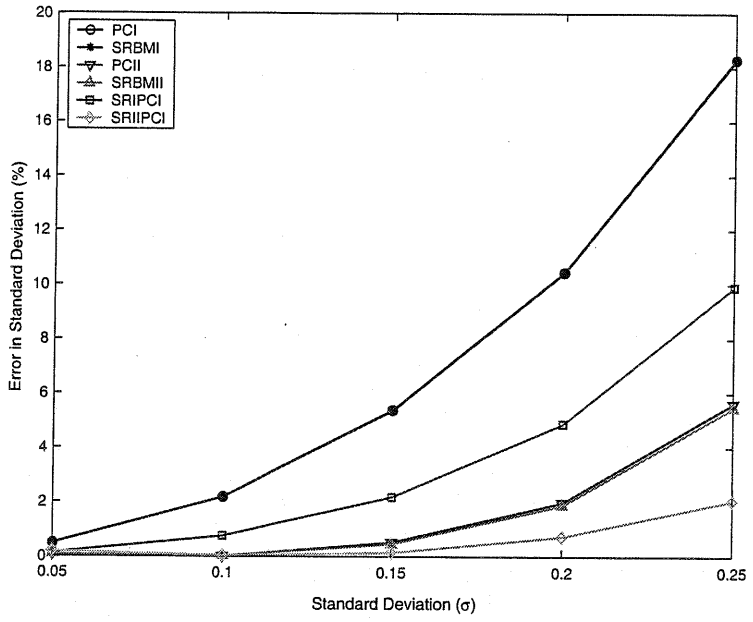

Fig. 3. Percentage error in standard deviation of displacement at point 'E' for various projection schemes as a function of the standard deviation of the random Young's modulus for Problem 1.

approximations are comparable to the second-order approximations. Hence, first-order methods are preferable for such cases from the viewpoint of computational efficiency.

It can be seen from the results that there is virtually no difference in the accuracies of the PC projection schemes and SRBMs as far as the mean and standard deviation of the displacement is concerned. As expected, schemes employing an augmented set of basis vectors (SRIPCI and SRIIPCI) give better results compared to SRBMI and SRBMII. In particular, SRIIPCI is more accurate than SRBMII and PCII.

Fig. 4 shows the pdfs of the vertical displacement at the corner 'E' of the plate computed using MCS, PCII and SRBMII when $\sigma = 0.2$. It can be noted that SRIIPCI approximates the pdf better than all the other schemes considered. The pdf obtained using SRBMII is very much in consonance with that obtained using MCS and PCII. Fig. 5 shows how well the second-order projection schemes approximate the tail of the pdf. It can be observed from this figure that the accuracy of SRBMII is comparable to PCII. At first, this trend may appear to be counter-intuitive since SRBMII solves a 3×3 reduced-order system of equations whereas PCII solves a system of equations with increased dimensionality (750×750). In other words, even though both PCII and SRBMII are second-order approximations, PCII has more unknowns compared to SRBMII. However, the good performance of SRBMII can be attributed to the fact that for most realizations of the random Young's modulus, the point 'E' has high displacement levels compared to other dofs. Hence, due to Theorem 2, the accuracy of SRBMs will be high for the dofs close to 'E'. As we mentioned earlier, this is a very desirable property since in many engineering problems we are interested in accurately predicting the statistics of the response in the highly strained regions of the structure under consideration.

However, for the dofs where the displacement levels are low, we expect PCII to be more accurate than SRBMII. For illustration, consider a dof where the mean and the standard deviation of the displacement predicted using MCS are 1.71×10^{-4} and 1.1×10^{-3}, respectively. Note that the displacement of this dof is much lower compared to the point 'E'. Using PCII, the mean and standard deviation of the displacement is

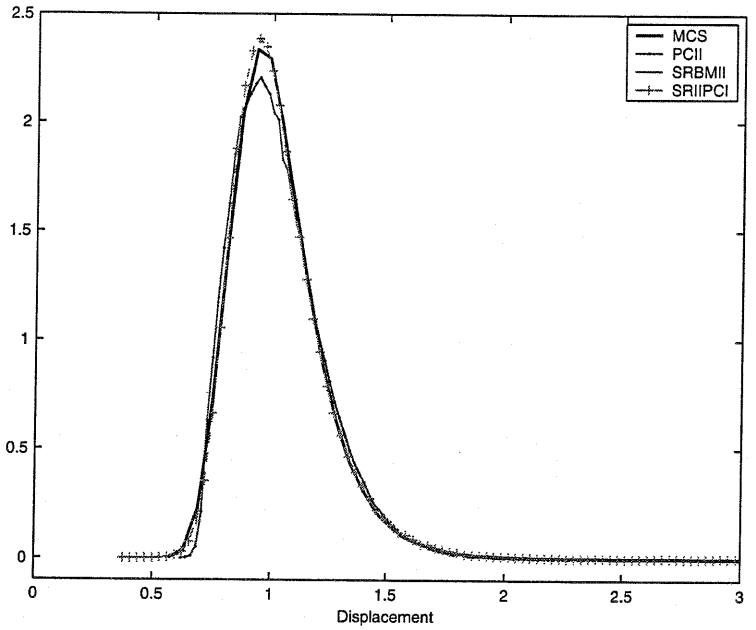

Fig. 4. Probability density function of displacement at point 'E' computed using MCS and second-order projection schemes for Problem 1.

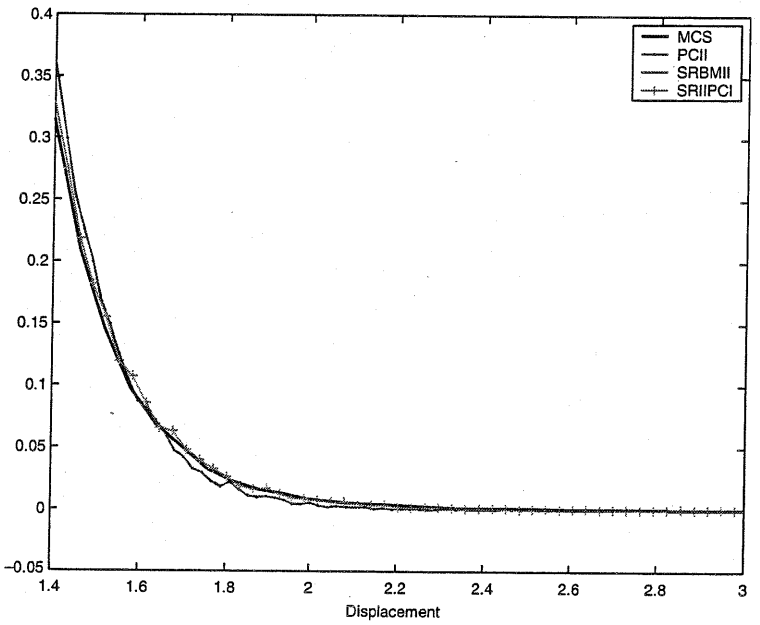

Fig. 5. Tail of probability density function of displacement at point 'E' computed using MCS and second-order projection schemes for Problem 1.

1.69×10^{-4} and 0.99×10^{-3}, respectively. In comparison, SRBMII gives marginally less accurate approximations, namely 1.54×10^{-4} and 0.97×10^{-3} for the mean and standard deviation, respectively.

5.2. Problem 2: Foundation on a heterogeneous soil layer

Next, we consider a geotechnical problem which involves settlement analysis of a foundation on a random heterogeneous soil. This problem has been studied earlier by Sudret and Der Kiureghian [33,34] in the context of reliability analysis. Consider an elastic soil layer of thickness t lying on a rigid stratum as shown in Fig. 6. A uniform pressure P is applied over a length $2B$ of the free surface. The soil is modeled as an elastic linear isotropic material and plane strain analysis is carried out. Exploiting symmetry considerations, only one half of the structure is modeled by finite elements. The values of the soil parameters used in the computations are given in Fig. 6.

Young's modulus of the soil is assumed to be a homogeneous Gaussian random field. Young's modulus is considered to vary only in the vertical direction and hence a one-dimensional random field with exponential covariance function (see Eq. (37)) is employed. The correlation length is set at $b = 30$ m. An optimal mesh containing 99 nodes and 80 elements developed by Sudret and Der Kiureghian [34] shown in Fig. 7 was used for analysis. Fig. 8 shows the deformed shape of the soil structure after application of the specified load for the mean Young's modulus distribution. It can be seen from Fig. 8 that point 'A' has maximum displacement when the external pressure load is applied. Results are generated using PCI, PCII, SRBMI, SRBMII, SRIPCI and SRIIPCII schemes in conjunction with the fourth-order KL expansion to discretize the input random field. MCS with a sample size of 250,000 is carried out to obtain the reference results against which all methods are compared. The statistics of the vertical displacement at point 'A' are computed using various methods for increasing coefficient of variation of the input random field.

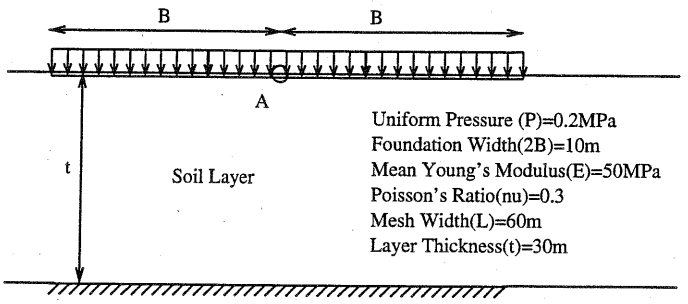

Fig. 6. Schematic for soil foundation problem.

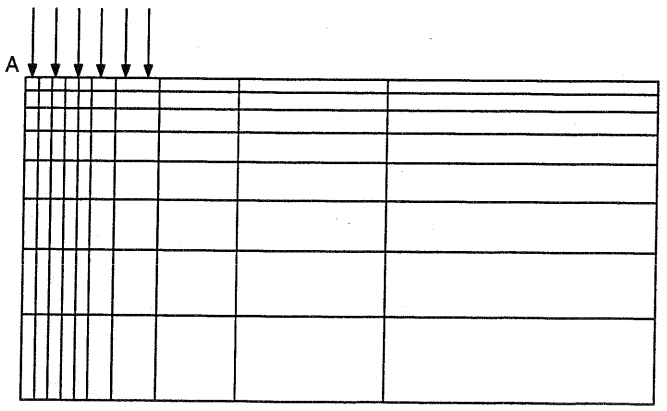

Fig. 7. Optimal mesh for soil foundation problem.

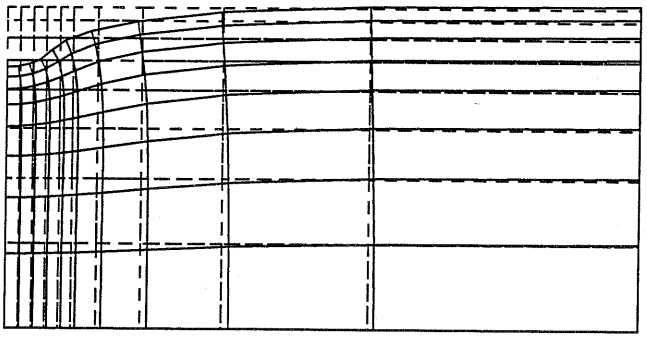

Fig. 8. Deformed mesh.

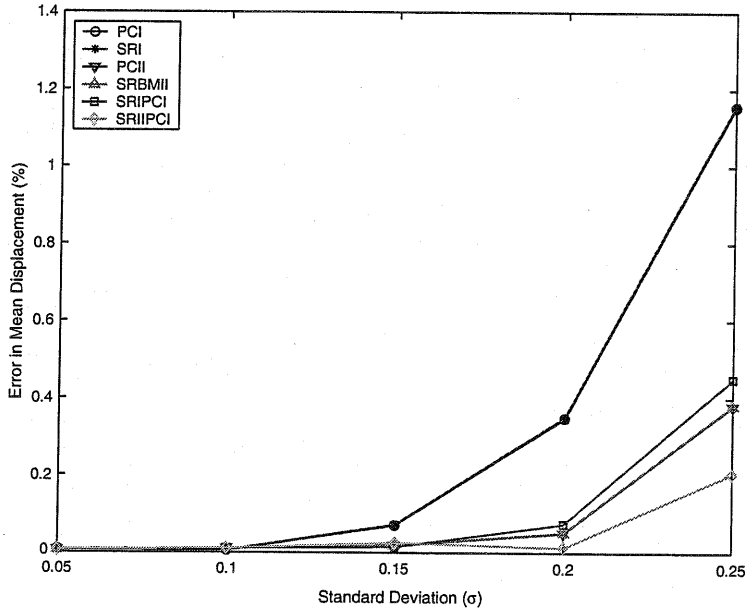

Fig. 9. Percentage error in mean displacement at point 'A' for various projection schemes as a function of the standard deviation of the random Young's modulus for Problem 2.

Fig. 9 shows the percentage error in estimating the mean of the response at point 'A' versus standard deviation of the input random field (σ) for SRBMI, SRBMII, SRIPCI, SRIIPCI, PCI, PCI and PCII schemes. For lower input standard deviation ($\sigma < 0.1$) all the schemes produce minimal error in computing the mean nodal displacement. However, when $\sigma > 0.2$, SRBMII and PCII perform significantly better than SRBMI and PCI. Fig. 10 shows the percentage error in the standard deviation of the vertical displacement at point 'A' for various methods when σ is increased. This figure shows that the error increases monotonously for all the schemes. It can also be seen from the figure that all schemes of same order produce equally accurate results and exhibit a similar convergence trend. The trends for the mean and standard deviation of the response are similar to those obtained earlier for the plate problem. In other words, SRIIPCI gives better accuracy compared to all other schemes considered here.

As no obvious distinction between PC projection schemes and SRBMs was observed from the plots of first two statistical moments, we now examine how well the projection scheme approximate the response

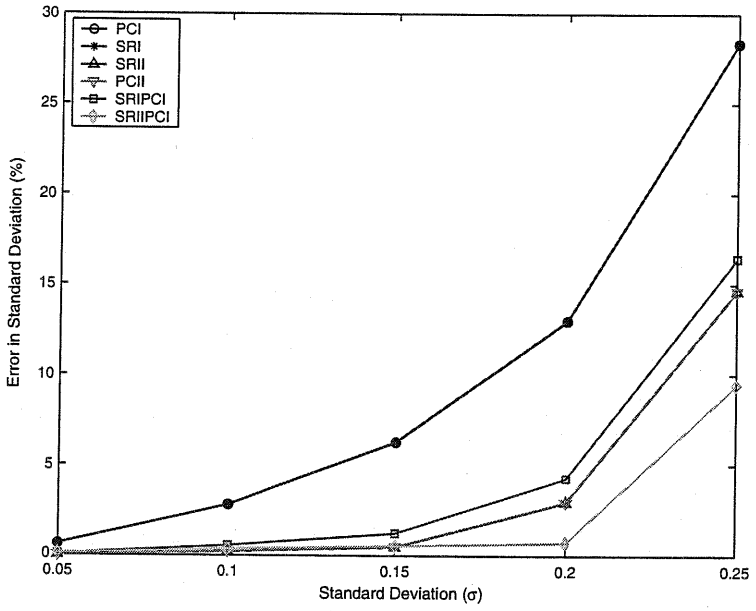

Fig. 10. Percentage error in standard deviation of displacement at point 'E' for various projection schemes as a function of the standard deviation of the random Young's modulus for Problem 2.

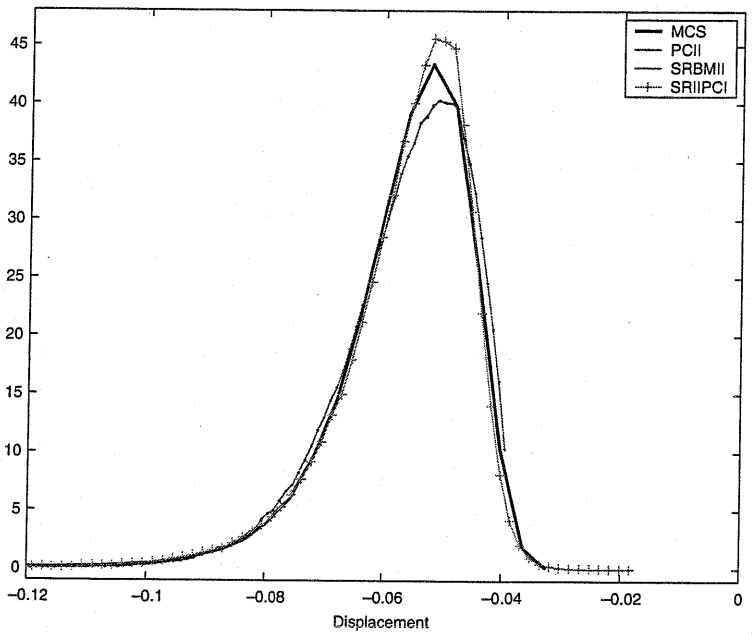

Fig. 11. Probability density function of displacement at point 'A' computed using MCS and second-order projection schemes for Problem 2.

pdfs. Fig. 11 shows the pdfs of displacement at point 'A' obtained using MCS, SRBMII, SRIIPCI and PCII schemes. The tails of the pdf computed using the projection schemes are shown in greater detail in Fig. 12.

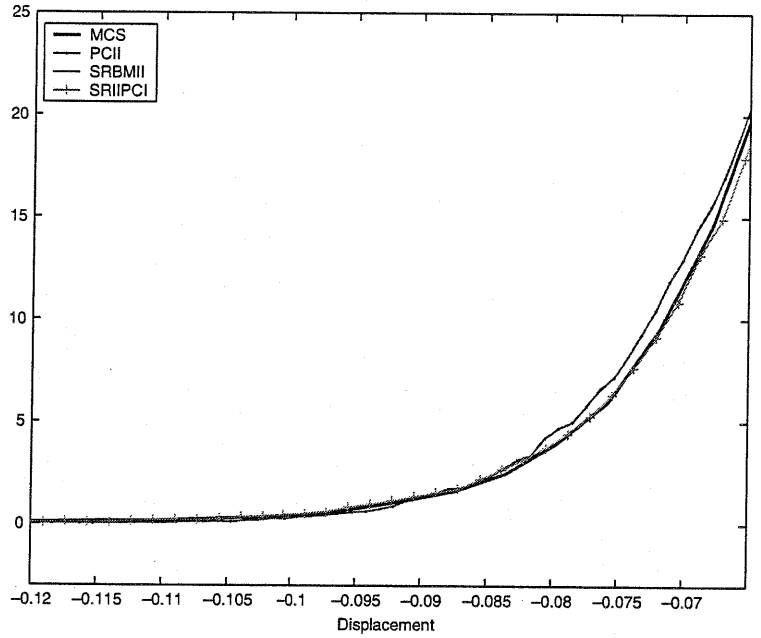

Fig. 12. Tail of probability density function of displacement at point 'A' computed using MCS and second-order projection schemes for Problem 2.

It can be observed that both SRBMII and PCII give virtually identical results for the dof under consideration. The results show that SRIIPCI approximates the pdf and its tail better than other schemes considered.

6. Comparison of computational efficiency

In this section we study the computational aspects of the PC and stochastic reduced basis projection schemes. Numerical studies were conducted on the plate problem to investigate the computational efficiency of the two projection schemes when the number of dof and terms in the KL expansion are increased. A consistent comparison was possible since both the PC projection scheme and SRBMs are coded in the MATLAB environment. The PC projection scheme implementation of Sudret and Der Kiureghian [33] was used here. All the runs were conducted on a Pentium IV 2 GHz processor running Linux.

Fig. 13 shows the total wall time taken by each scheme to solve the problem as a function of the total number of dof when four terms are retained in the KL expansion. Note that the time axis is plotted on log scale. It can be seen that for small-scale problems (with around 20 dof), the total computational time is almost the same for all the schemes. However, for practical problems (with >1000 dof) SRBMs can be orders of magnitude faster than PC schemes. For example, consider the case when the spatial discretization of the plate problem leads to 1250 dof. Here, PCI and PCII schemes took 137 and 2735 s, respectively. In comparison, SRBMI and SRBMII took only 13.2 and 17.4 s, respectively.

Clearly, all the schemes require equal amount of time to discretize the random field. Computation of the mean and weighted stiffness matrices (see Eq. (9)) also requires equal amount of time when the number of terms in the KL expansion is kept fixed. The PC projection scheme takes huge amount of time in computing

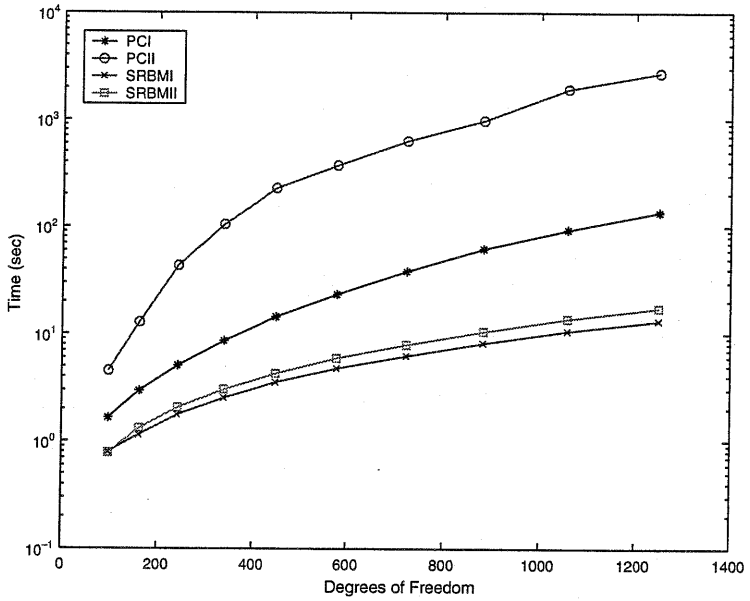

Fig. 13. Comparison of wall time as a function of total number of dof for various projection schemes, $M = 4$.

and assembling the matrices $\mathbf{K}_{jk}$ resulting from application of the Galerkin condition (see Eq. (16)). For a system with n degrees of freedom, the PC projection schemes result in a $nP \times nP$ deterministic system of equations. For example, consider the plate problem with 1250 dof and four terms in the KL expansion. Here, the PCI and PCII schemes result in 6250×6250 and $18{,}750 \times 18{,}750$ deterministic system of equations, respectively. In comparison, SRBMs require only the mean and weighted stiffness matrices, which after application of the Galerkin scheme results in a 2×2 or 3×3 system of deterministic equations for SRBMI and SRBMII, respectively. The most time consuming step in SRBMs involves evaluation of the mean and weighted stiffness matrices.

It is worth pointing out that the PC projection scheme can be speeded up if the matrices $\mathbf{K}_{jk}$ in Eq. (16) are not explicitly assembled. Further, instead of using a direct solver as was done in this paper, an iterative scheme can be employed to reduce computational cost [42]. Even after these modifications are made, PC projection schemes will be significantly slower than SRBMs, particularly for large-scale problems with many random variables.

It is interesting to note that schemes employing the augmented set of basis vectors are also computationally more efficient compared to PC schemes. For example, SRIPCI and SRIIPCI took 4.2 and 5.6 s, respectively, to solve a system with 450 dof whereas PCI and PCII schemes took 14.4 and 228.8 s, respectively. For the same system, SRBMI and SRBMII took 3.5 and 4.2 s, respectively. This suggests that schemes employing an augmented set of basis vectors can offer efficient alternatives to the original formulation of SRBMs which is capable of being more accurate than PC projection schemes.

Fig. 14 shows the wall time taken by PC projection schemes and SRBMs to compute the solution when six terms are retained in the KL expansion of the random field. It can be seen that there is a significant increase in time taken by PCII as compared to PCI and the time required by both PC projection schemes for the case when $M = 4$. In comparison, the execution time of SRBMI and SRBMII increase only marginally when M is increased. These studies suggest that for large-scale problems, SRBMs can be expected to be many orders of magnitude faster than PC projection schemes.

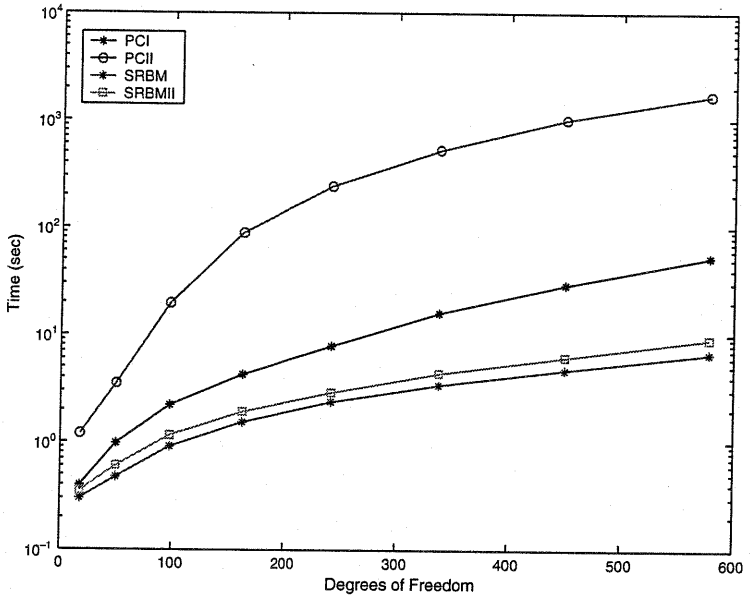

Fig. 14. Comparison of wall time as a function of total number of dof for various projection schemes, $M = 6$.

7. Concluding remarks

In this paper, we presented a detailed comparison of the accuracy and computational cost of stochastic reduced basis methods and polynomial chaos projection schemes. A new formulation which uses an augmented set of basis vectors was proposed to improve the accuracy of SRBMs. Numerical studies were presented for two example problems involving stochastic finite element analysis of random media. Based on the numerical studies, we conclude that the accuracy of stochastic reduced basis projection schemes is comparable to PC projection schemes for the first two statistical moments. Further, our results also suggest that the accuracy of SRBMs is comparable to PC projection schemes even for the tail of response pdfs. The numerical studies suggest that SRBMs employing an augmented set of basis vectors can be more accurate than PC projection schemes, while incurring much lower computational cost.

Studies on the computational complexity of the projection schemes suggest that SRBMs can be orders of magnitude faster than PC projection schemes. It is to be noted here that the present implementation of PC projection schemes does not fully exploit the structure of the system of equations resulting from Galerkin projection. The computational cost involved in PC projection can be reduced further by using solvers such as that presented in [41,42] or by employing a Gauss–Seidel iterative procedure [21]. Even after such modifications are made, we still expect SRBMs to be significantly faster than PC projection schemes, particularly for large-scale problems with many random variables.

Based on the numerical studies conducted in this paper, we are not recommending SRBMs as a replacement for PC projection schemes. We must point out here that the PC expansion scheme is a very general approach which can be used to represent the response process of a wide class of stochastic systems. In contrast, SRBMs is a specific approach which is directly applicable only to systems governed by linear stochastic PDEs. Due to the assumed linearity of the problem, we were able to motivate our formulations using theoretical results established earlier for the applicability of the stochastic Krylov subspace to represent the solution of a system of linear random algebraic equations [35,36]. Since the theoretical foundations of SRBMs for general nonlinear stochastic PDEs have not yet been established, the PC expansion is

currently the only choice for such general problems. We would like to point out here that it may be possible to apply SRBMs to nonlinear stochastic systems by invoking simplifying assumptions such as those made by Anders and Hori [21] in the context of the PC projection scheme applied to plasticity problems.

The numerical studies presented in this paper suggest that for linear stochastic PDEs, it makes sense from the viewpoint of computational efficiency to employ SRBMs. Further, since SRBMs make use of a preconditioner, it is possible to improve the accuracy locally, i.e., a sequence of approximations can be constructed by changing the preconditioner. Our initial numerical studies on such adaptive schemes indicate that the accuracy of SRBMs can be significantly improved. Further work is also required to apply SRBMs to problems with multiple random fields.

Acknowledgment

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Appendix

The deterministic reduced-order 3×3 system of equations for the second-order SRBM ($m = 3$) are given below for the case θ_i, $i = 1, 2, \ldots, M$, are uncorrelated Gaussian random variables. Note that for the sake of compactness, we have used the Einstein repeated index notation; for example, a repeated index i indicates summation with respect to that index over the range $1, 2, \ldots, M$:

$$\begin{bmatrix} \mathbf{u}_0^{\mathrm{T}}\mathbf{K}_0\mathbf{u}_0 & \langle\theta_i^2\rangle\mathbf{u}_0^{\mathrm{T}}\mathbf{K}_i\mathbf{d}_i & \langle\theta_i^2\rangle\mathbf{u}_0^{\mathrm{T}}\mathbf{K}_0\mathbf{e}_{ii} \\ & \langle\theta_i^2\rangle\mathbf{d}_i^{\mathrm{T}}\mathbf{K}_0\mathbf{d}_i & \langle\theta_i\theta_j\theta_k\theta_l\rangle\mathbf{d}_i^{\mathrm{T}}\mathbf{K}_j\mathbf{e}_{kl} \\ \text{sym} & & \langle\theta_i\theta_j\theta_k\theta_l\rangle\mathbf{e}_{ij}^{\mathrm{T}}\mathbf{K}_0\mathbf{e}_{kl} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_0^{\mathrm{T}}\mathbf{f} \\ 0 \\ \langle\theta_i^2\rangle\mathbf{e}_{ii}^{\mathrm{T}}\mathbf{f} \end{bmatrix}. \tag{38}$$

Note that the terms involving fourth-order products of the form $\langle\theta_i\theta_j\theta_k\theta_l\rangle$ can be readily computed using the identity

$$\langle\theta_i\theta_j\theta_k\theta_l\rangle = \delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}, \tag{39}$$

where δ is the Kronecker delta function.

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