

Measurement and evaluation of multi-function parallel network hierarchical DEA systems

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Abstract

Many organisations are composed of multiple departments connected either in series or in parallel, which may be further decomposed into a number of functions arranged in a hierarchical structure. Several researchers have successfully used appropriate Data Envelopment Analysis (DEA) modelling techniques to assess complex structures. However, to our knowledge, no-one has yet examined the case of measuring and evaluating a parallel network structure combined with a hierarchical one. This paper discusses the development of a multi-function parallel system with embedded hierarchical network structures. A linear additive decomposition DEA model and a non-linear multiplicative aggregation DEA model are proposed as alternatives to evaluate the operating performance of such a structure. The system, the sub-systems, and the efficiencies of their internal units, as well as their relationships, are identified. The system efficiency of the additive model is shown to be greater than or equal to that of the multiplicative model. To verify the applicability of our proposed models, we consider a hypothetical example of the measurement and evaluation of the performances of several Business Schools across a number of universities. Other envisaged areas of application of our structure could include supporting the evaluation of the supply chain management of a firm, or the determination of the most desirable ship design considering maintenance issues.

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1 Introduction

The purpose of this article is to extend the current literature on the network hierarchical DEA structures. Our study particularly considers the measurement and evaluation of the performance of several parallel processes, wherein each process integrates a multi-function hierarchical structure. Additive decomposition and multiplicative aggregation DEA models are presented and used in a higher education context to investigate the areas of weakness of the considered Business Schools.

Data envelopment analysis (DEA) has been extensively exploited as an effective performance evaluation technique to gain insight into the past accomplishments and future developments of a decision-making unit (DMU) **(Emrouznejad and Yang, 2018)**. Since the seminal work of **Charnes et al. (1978)**, DEA has been widely used in various applications, including energy and environment **(Zhai et al., 2019)**, agriculture **(Kyrgiakos et al., 2021)**, water resource efficiency **(Liang et al., 2021)**, local governments **(Amatatsu et al., 2012)**, research and development departments **(Wang et al., 2013)**, financial services and banking **(Paradi and Zhu, 2013; Tan et al., 2021; Shi et al., 2021; Li et al., 2022; Kremantzis et al., 2022b)**, insurance services **(Omrani et al., 2022b)**, supply chain management **(Azadi et al., 2014)**, sports **(Moreno and Lozano, 2014)**, international shipping **(Gan et al., 2019)**, inland transportation **(Stefaniec et al., 2020; Wang et al., 2022)**, hospital efficiency **(Dehnokhalaji et al., 2022; Omrani et al., 2022a)**, higher education **(Ekiz and Tuncer Şakar, 2020; Lee and Johnes, 2021)**, and many more. See also **Liu et al. (2013)** for a review of applications.

Traditional DEA approaches put emphasis on evaluating the most favourable efficiency measure of a DMU, only by considering its exogenous inputs and outputs. This is referred to as black-box analysis **(Kao and Hwang, 2008)**. The internal structure of a unit usually consists of several divisions with similar and/or different functions; they may be interrelated, independent, or a mixture of these, depending on the objective of the system **(Kao, 2014)**. To enable the study of internal operations, research has extended DEA models to consider network structures **(Färe and Grosskopf, 2000; Kao, 2014; Cook and Zhu, 2014; Zhu, 2020)**.

The network system differs from the black-box in that it involves more complex structures, thereby leading to a less systematic illustration **(Kao, 2017)**. In the two-stage tandem system, all inputs used by a DMU feed into a first stage, producing intermediate outputs that all feed into a second stage, producing the final outputs of the entire system. **Kao and Hwang (2008)** proposed that this system efficiency is decomposed into the product of the efficiencies of the two stages. Real-world cases, however, may extend the former structure to a general one, in which the first stage additionally generates final outputs and the second stage also produces exogenous inputs **(Yu and Shi, 2014; Jianfeng, 2015)**. Extensive research has explored two-stage network DEA structures, see for example **Chen et al. (2012), Despotis et al. (2016), Guo et al. (2017), and Kremantzis et al. (2022a)**.

The above-mentioned systems have a series structure, in that they operate interdependently. In other types of networks, the internal divisions are placed in parallel without impacting one another **(Kao, 2012)**. There are two classes of parallel systems, based on their functions. Multi-component systems involve the assessment of DMUs with multiple divisions of the same function **(Kao, 2009b)**. The second class focuses on the multi-function systems, in which the internal divisions separate their operations by consuming their own inputs, although it is a common practice to also share resources **(Kao, 2017)**. Extensive research has examined such systems in various applications, including the performance evaluation of physics and chemistry departments in UK universities **(Beasley, 1995)**, the assessment of commercial banks in Iran **(Jahanshahloo et al., 2004)**, the maximisation of sales of Portuguese retail stores **(Vaz et al., 2010)**, the evaluation of the operational efficiencies of multiple railway firms in China **(Bian et al., 2015)**, and the impact of coal-fired power plants on pollution-generating processes **(Lozano, 2015)**.

The investigation of the internal composition of a production system enables the improved use of the DEA approach **(Gan et al., 2019)**. However, treating the internal components of a system as black-boxes, continues to be widespread. This paper highlights this issue, considering the context of a parallel system. For example, the department of marketing at university X has two independent functions, teaching and research. If their internal operations are neglected, it cannot identify the potential sources of inefficiency, the way the inputs are further shared, and those layers with a beneficial impact on the respective section. To remedy these issues, each sub-system could be further split into sub-subsystems, and so on, to a reasonable level of detail. In a university department, one may want to identify sources of (in)efficiency down to the level of teaching programmes, for example.

The hierarchical structure has an eminent position in contemporary organisations. It can, inter alia, signify the organisational culture and dynamics, and coordinate the responsibilities of people across several departments and levels. Nevertheless, such a structure has to our knowledge not paid significant attention to exploring the internal operations of a network system, and in particular of a parallel system. In a university, a faculty typically operates as multiple parallel departments, each of which can be further hierarchically structured across research, teaching, and enterprise.

Some approaches in DEA have systematically examined the hierarchical structures. **Kao (2015)**, for instance, developed a relational model for a single-stage hierarchical structure to measure both the overall system and its divisions' efficiencies at the same time. He argued that this structure is identical to a parallel system **(Kao, 2009b)**, in that the system efficiency is decomposed into the weighted arithmetic average of the efficiencies of the units at the bottom level. **Kao (2015)** optimised the efficiency of the overall production system, considering only the constraints corresponding to the terminal divisions. **Li et al. (2020)** focused on the same hierarchical structure by additionally optimising the efficiencies of the terminal divisions as opposed to **Kao (2015)**. **Zhang and Chen (2019)** extended the concept of **Kao (2015)** to a generalised single-stage hierarchical structure wherein all internal units across the different levels can reflect a two-stage tandem system. **Gan et al. (2019)** suggested a general two-stage series process, in which each stage is no longer treated as a black-box, but is further elaborated into a hierarchical structure with multiple layers. They argued that a single-stage hierarchy cannot really correspond to complex production processes. A number of studies have been reported in this direction, such as **Castelli et al. (2004)**, **Cook and Green (2005)**, **Meng et al. (2008)**, **Bod'a et al. (2020)**, and **Yu et al. (2021)**. We summarize the core literature, relevant to network-hierarchical DEA structures, in Table 1.

A real-life organisation is likely to consist of several departments that could be further extended into a number of distinctive tasks, arranged either in sequence or in parallel. To better reflect the reality, we claim that these tasks can be then ordered as multi-layer hierarchical structures. These structures demonstrate that the strategic, tactical, and operational decisions cannot be made across the same level, by the same resources. The above case contributes to a more complex network system with embedded hierarchical structures. **Gan et al. (2019)** adopted such a structure, enabling the initial tasks to be interdependent (i.e., to be connected in series). The primary difference between our proposed system herein, against that of the earlier work of **Gan et al. (2019)** is that we seek to optimise the performance score of a system, in which the departments operate in parallel (i.e., they act independently from one another) incorporating as well a multi-function hierarchical structure. In the current study, we propose an additive decomposition DEA model and a multiplicative aggregation DEA model to measure and evaluate the operating performance of DMUs with a parallel multi-layer multi-function hierarchical structure. The proposed structure seeks to address the weaknesses of the traditional black-box DEA model and the parallel system of **Kao (2012)**. The black-box model evaluates the system while ignoring its internal operations. Although in **Kao (2012)** the efficiency scores of the internal parallel sub-systems are obtained, there are no computations on the efficiencies of the units within the internal parallel sub-systems. For the aforementioned reasons, this study contributes on the following points: a) it enhances the discriminatory power due to the increasing number of restrictions in the proposed DEA models **(Kao and Liu, 2019)**, and b) provides the methodology for estimating performance scores for the overall system, parallel sub-systems and their internal units arranged into a hierarchical format. Therefore, our proposed structure is shown to be a more accurate reflection of the entire production/operating process of several large scale organisations.

The remainder of the paper is organised as follows. Section 2 briefly describes the methodological background. Section 3 proposes new models to evaluate DMUs with multi-function parallel network hierarchical structure. Several properties of such a system are also analysed. Section 4 validates the proposed models with a hypothetical application in the higher education sector. Finally, Section 5 presents conclusions and further research.

Table 1: Related literature on network-hierarchical DEA systems.

	Type of network	Efficiency measurement	Area of application
Black-box DEA model	SS	n/a	n/a
Kao (2012)	MCP	D	higher-education
Kao (2015)	SSH	D	higher-education
Lozano (2015)	MFP	SBM	pollution generation
Lu et al. (2016)	G2	D	investment trust corporations
Gan et al. (2019)	G2H	D	international shipping industry
Zhang and Chen (2019)	SSHMS	D & A	high-technology
Li et al. (2020)	2LH	×	electric power generation
This Paper	MFPH	D & A	higher-education

D: decomposition, **A**: aggregation, **SBM**: slacks-based measure, **SS**: single-stage system, **MCP**: multi-component parallel system, **SSH**: single-stage hierarchical system, **MFP**: multi-function parallel system, **G2**: general two-stage system, **G2H**: general two-stage system with integrated hierarchies, **SSHMS**: single-stage hierarchical system with integrated multi-stage series processes, **2LH**: two-level hierarchical system, **MFPH**: multi-function parallel system with integrated hierarchies

2 Methodological Background

In this section, we explore the network nature of two established systems: the parallel with shared inputs, and the single-stage hierarchical structure. These will ease the presentation of the advanced structure and its mathematical models proposed in Section 3.

2.1 A parallel system with shared inputs

In a real-life application, the core of a production system may be composed of multiple divisions with distinctive functions, operating independently among themselves. Such a system tends to be a more accurate picture of the reality, once joint inputs, shared by a number of divisions, are involved, other than their own inputs. **Beasley (1995)** and **Molinero (1996)** proposed a system with p parallel processes or divisions. In this system, see also Figure 1, the X_{ij}, X_{lj}^S and Y_{rj} are the ith dedicated input value, the lth shared input value, and the rth final output value, respectively, of DMU_j $(j = 1, 2, ..., n)$. Let $M = \{1, 2, ..., m\}$, $Q = \{1, 2, ..., q\}$, and $S = \{1, 2, ..., s\}$ be the index sets associated with the dedicated inputs, the shared inputs, and the final outputs, respectively. The division k $(k = 1, 2, ..., p)$ of DMU_j utilises the division-dedicated input i with value $X_{ij}^{(k)}$, $i \in I^{(k)}$, $M = \bigcup_{k \in P} I^{(k)}$ such that $I^{(k)} \cap I^{(i)} = \emptyset$, $\forall\ k, i \in P$, and a proportion $\alpha_l^{(k)}$ of the shared input $l \in Q$ with value X_{lj}^S, to produce the final output r with value $Y_{rj}^{(k)}$, $r \in O^{(k)}$, $S = \bigcup_{k \in P} O^{(k)}$ such that $O^{(k)} \cap O^{(o)} = \emptyset$, $\forall\ k, o \in P$. In such a system, the total division-specific and shared inputs consumed, and the total outputs produced by the p divisions of DMU_j are $X_{ij} = \sum_{k=1}^{p} X_{ij}^{(k)}$, $X_{lj}^S = \sum_{k=1}^{p} \alpha_l^{(k)} X_{lj}^S$, and $Y_{rj} = \sum_{k=1}^{p} Y_{rj}^{(k)}$, respectively.

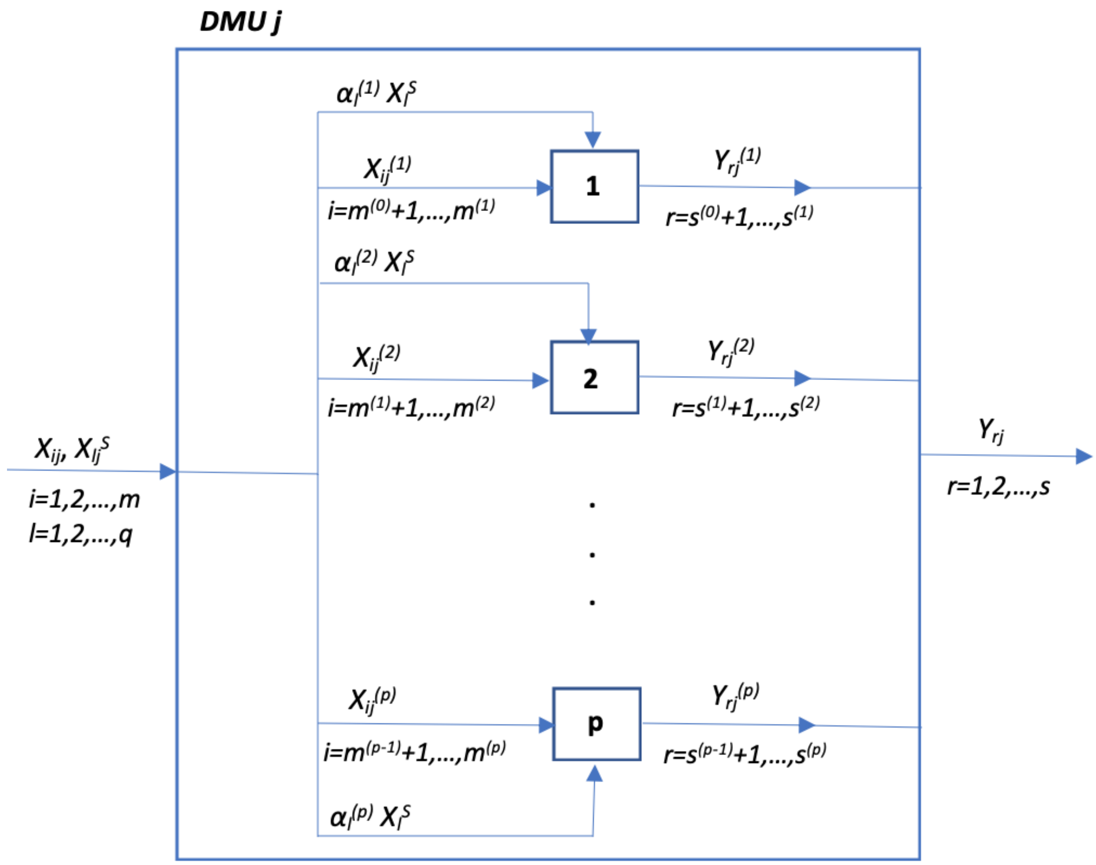

Figure 1: Parallel system with shared inputs

To measure the performance of the overall system of the target DMU_o, **Beasley (1995)** introduced and later **Kao (2012)** and **Kao (2017)** validated the following model under constant returns to scale:

$$
\begin{aligned}
& E_o = Max\ \sum_{r=1}^{s} \mu_{ro} Y_{ro} \\
& \text{subject to } \sum_{i=1}^{m} \nu_{io} X_{io} + \sum_{l=1}^{q} t_{lo} X_{lo}^S = 1, \\
& \sum_{r \in O^{(k)}} \mu_{ro} Y_{rj}^{(k)} - (\sum_{i \in I^{(k)}} \nu_{io} X_{ij}^{(k)} + \sum_{l=1}^{q} t_{lo} \alpha_l^{(k)} X_{lj}^S) \leq 0,\ \forall\ j, k, \\
& t_{lo}, \nu_{io}, \mu_{ro} \geq \epsilon,\ \forall\ l, i, r,
\end{aligned}
\tag{1}
$$

where t_{lo}, ν_{io}, and μ_{ro} are the positive optimal multipliers, and ϵ is an infinitesimal non-Archimedean number. According to the relational model **(1)**, the overall performance score of the evaluated DMU to be maximised is the ratio of the total outputs to that of inputs. It is also required that the aggregation of outputs should not exceed the aggregation of inputs, for every internal division. In such a model, $\alpha_l^{(k)}$ is a parameter that is objectively designated by the decision maker (possibly based on historical data) prior to solving the corresponding mathematical model. It is also ensured that the sum of the proportions of the lth shared input is 1. However, if it is treated as the most favourable value, reflected from the data, then it is additionally essential to involve the following constraints: $L_{lo}^{(k)} \leq \alpha_l^{(k)} \leq U_{lo}^{(k)}$, $\sum_{k=1}^{p} \alpha_l^{(k)} = 1$, and $\alpha_l^{(k)} \geq 0$, $\forall\, l, k$. At optimality, the system efficiency is calculated as $E_o = \sum_{r=1}^{s} \mu_{ro}^* Y_{ro} / (\sum_{i=1}^{m} \nu_{io}^* X_{io} + \sum_{l=1}^{q} t_{lo}^* X_{lo}^S)$, and the division efficiencies as $E_o^{(k)} = \sum_{r \in O^{(k)}} \mu_{ro}^* Y_{ro}^{(k)} / (\sum_{i \in I^{(k)}} \nu_{io}^* X_{io}^{(k)} + \sum_{l=1}^{q} t_{lo}^* \alpha_l^{(k)} X_{lj}^S)$. A property of this structure is that the system efficiency equals to the weighted average of its division efficiencies **(Kao, 2009b)**.

To the best of our knowledge, the internal divisions of such a commonly used structure are still treated as black-boxes. This may hinder our efforts to gain further insight on more complex and realistic cases, regarding the activities of a department and the mechanisms behind a core business task.

2.2 A single-stage hierarchical structure

A relatively recent network system is that of a hierarchical structure, embedded either in a single-stage or in a general two-stage series network. Its adoption may help the investigation of the operational procedures. As discussed in Section 1, **Kao (2015)** proposed a relational model to evaluate the performance of the overall system and its internal units, reflecting a single-stage hierarchical structure with three levels.

Consider a system with the general hierarchical structure shown in Figure 2 **(Kao, 2017)**. The system has q levels and is an extension of the three-level system of **Kao (2015)**. The first level, for example, consists of $p^{(1)}$ divisions, each of which is decomposed into several divisions at the follower level. The kth level ($k = 2, 3, ..., q$) has a total of $p^{(k)} - p^{(k-1)}$ divisions subordinated to the $p^{(k-1)} - p^{(k-2)}$ divisions at the $(k-1)$th level. Denote $P^{[1]} = \{1, 2, ..., p^{(1)}\}$ and $P^{[k]} = \{p^{(k-1)} + 1, ..., p^{(k)}\}$, as the sets of the divisions in the first and the kth level ($k = 2, 3, ..., q$), respectively. Moreover, let $S(l)$ be the set of divisions viewed as subordinates to division l. If $S(l) = \emptyset$, then l is referred to as terminal. Let T denote the set of the terminal divisions. They are enabled to generate the outputs, while receiving inputs allocated from their parent unit (the immediate predecessor). On the other hand, the intermediate units i.e. the non-terminal divisions cannot produce outputs themselves, but they can distribute their inputs to their subordinate divisions at the next level.

In such a single-stage system, let X_{ij} and Y_{rj} be the ith input ($i = 1, 2, ..., m$) and rth output ($r = 1, 2, ..., s$) for the DMU_j ($j = 1, 2, ..., n$). Division l distributes its inputs $X_{ij}^{(l)}$, $i \in I^{(l)}$, received by its parent unit, to its subordinate divisions $\xi \in S(l)$, and collects the outputs $Y_{rj}^{(l)}$, $r \in O^{(l)}$ received from its subordinate divisions. Hence, in mathematical terms we have $X_{ij}^{(l)} = \sum_{\xi \in S(l)} X_{ij}^{(\xi)}$ and $Y_{rj}^{(l)} = \sum_{\xi \in S(l)} Y_{rj}^{(\xi)}$.

Kao (2015) developed the relational input-oriented model **(4.2)** to optimise the efficiency score of the target DMU_o arranged as a multi-layer hierarchical structure within a single-stage system.

$$
\begin{aligned}
&E'_o = Max \quad \sum_{r=1}^{s} \mu_{ro} Y_{ro} \\
&\text{subject to} \quad \sum_{i=1}^{m} \nu_{io} X_{io} = 1, \\
&\sum_{r \in O^{(l)}} \mu_{ro} Y_{rj}^{(l)} - \sum_{i \in I^{(l)}} \nu_{io} X_{ij}^{(l)} \leq 0, \ l \in T, \ \forall\, j, \\
&\nu_{io}, \mu_{ro} \geq \epsilon, \ \forall\, i, r.
\end{aligned}
\tag{2}
$$

To avoid redundancy, Kao highlighted that only the terminal divisions need to be taken into account. In mathematical symbols, $l \in T$. One of the properties of the system is that its overall efficiency is decomposed into the weighted average of those of the terminal divisions. To apply model **(2)**, we should ensure that all DMUs have the same hierarchical structure. In particular, for every DMU, a unit at the leader level should have the same number of subordinate units at the follower level, operating different functions.

In Section 3.1, the scenario of the integration of such a hierarchical structure into the internal divisions of a parallel system with shared inputs will be thoroughly discussed. This direction can successfully enhance the performance measurement in more complex systems within the production and operations management.

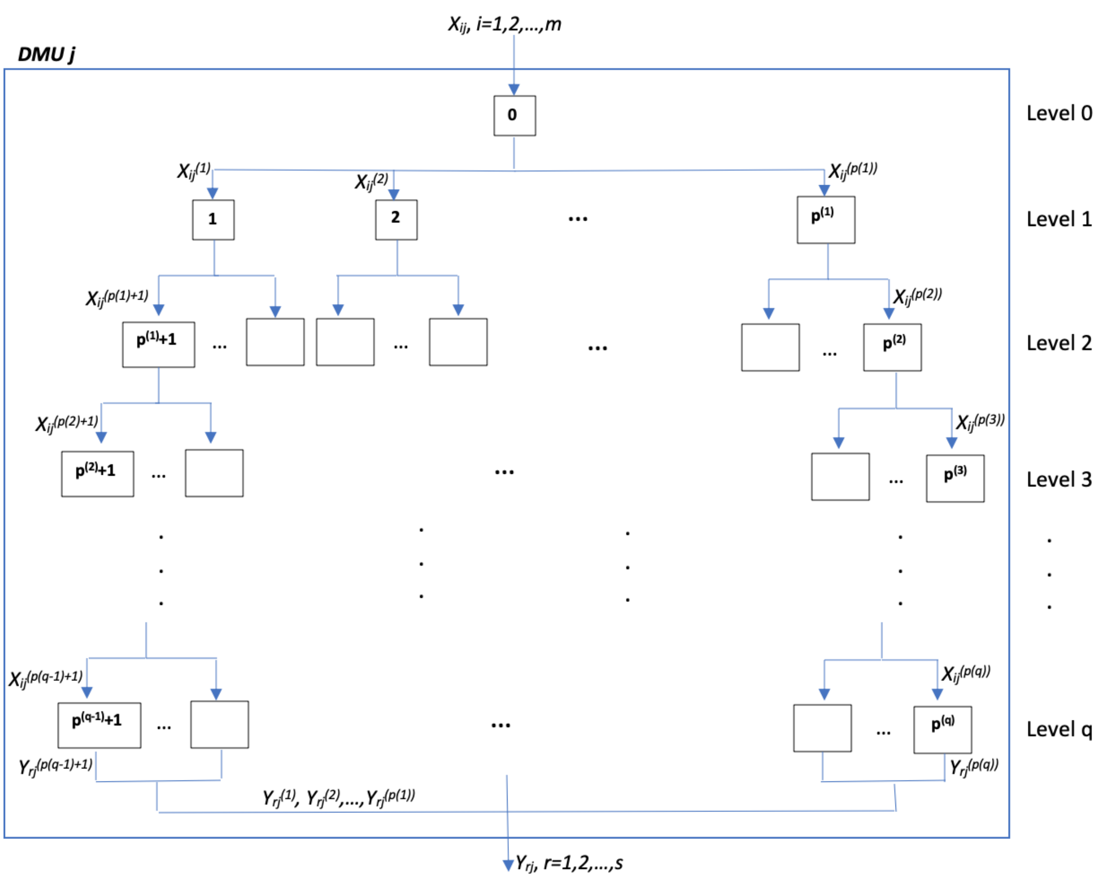

Figure 2: General single-stage hierarchical structure

3 Models Development

Real-life companies can have a complex corporate structure. The complexity corresponds to their numerous (tangible and intangible) resources, either being interactive or entirely independent, in any department. The utilities of such a structure are to successfully adapt to the constant changes of the internal and external environment, to comply with customers' requirements, and to minimise fixed and variable costs.

There are at least three separate (traditional single-stage and network) production systems, proposed in the DEA-literature, that have intertwined with multi-layer hierarchical structures: *(i)* a three-level multi-function hierarchical structure embedded in the core of a single-stage system **(Kao, 2015)**, *(ii)* a three-level with two-stage processes hierarchical structure embedded in a single-stage system **(Zhang and Chen, 2019)**, and *(iii)* a multi-level hierarchical structure integrated into an operating stage of a general two-stage series system **(Gan et al., 2019)**. In the current study, we extend the above list by considering the case of several parallel processes, wherein each sub-system integrates a multi-function hierarchical structure. The new system is introduced in Figure 3.

3.1 Parallel-hierarchical network DEA model

Based on the consolidated idea of **Kao (2015)**, the evaluated DMUs should have the same network-hierarchical structure; this can set the basis for a less demanding comparison amongst them. We have combined the ideas developed in Sections 2.1 and 2.2 into a situation like in Figure 3. From the perspective of Figure 3, the sub-systems of a system (DMU) must execute different operations, and each sub-system is obliged to have the same function with its counterpart in each of the other DMUs. In addition, the hierarchical structures of the different sub-systems of a DMU may vary in terms of the number and the arrangement of their internal units. However, the hierarchical structure of a certain sub-system of DMU_j $(j = 1, 2, ..., n)$ should be identical with the counterpart structure of the sub-system in each of the remaining DMUs.

On a macro level, the proposed system consists of two successive layers. The external one is associated with the action of retrieving managerial data from the entire system. This examines the overall performance of the DMU under consideration. The system applies m sub-system specific inputs and q shared inputs to generate s final outputs. Subsequently, in the interior part of the system, we detect p sub-systems connected in parallel, that is they are independent among each other and they cannot typically exchange information. A sub-system k $(k = 1, 2, ..., p)$ consumes the dedicated inputs $X_{ij}^{(k)}$, $i \in I^{(k)} \subseteq \{1, 2, ..., m\}$, and the shared inputs $\alpha_l^{(k)} X_{lj}^S$ $(l = 1, 2, ..., q)$ to generate the final outputs $Y_{rj}^{(k)}$, $r \in O^{(k)} \subseteq \{1, 2, ..., s\}$. The internal parallel divisions neither utilise endogenous inputs nor produce endogenous outputs. This layer evaluates the performance of each department/task, which is an integral part of the whole system.

On a micro level, in the interior of a sub-system, we identify a three-level multi-function hierarchical structure. The top level 0 (sub-system k) has two subordinate units, labelled (1) and (2), performing distinctive functions, at level 1. Functions (1) and (2) have in rotation three subordinate units (1.1), (1.2), and (1.3), and two subordinate units, (2.1) and (2.2), respectively, at level 2. Only unit (2.2) has two sub-units (2.2.1) and (2.2.2) at the bottom level 3. The internal units (1.1), (1.2), (1.3), (2.1), (2.2.1), and (2.2.2) are characterised as terminal, since they cannot be further broken down into several subordinate units. Note that the hierarchical structure presented herein is indicative and may be subject to modifications, reflecting the respective business environment. The internal unit u ($u = 1, 2, 1.1, 1.2, 1.3, 2.1, 2.2, 2.2.1, 2.2.2$) of sub-system k of DMU_j allocates the sub-system specific inputs $X_{ij}^{(k)_u}$, $i \in I^{(k)}$, and a proportion $\theta_l^{(k)_u}$ of the lth shared input X_{lj}^S, received by its parent unit, to its subordinate units at the follower level, and collects the outputs $Y_{rj}^{(k)_u}$, $r \in O^{(k)}$, received from its subordinate units.

Taking the structure of the above system into account, we obtain the following equalities:

(i) $X_{ij} = \sum_{k=1}^{p} X_{ij}^{(k)} = \sum_{k=1}^{p}(X_{ij}^{(k)_1} + X_{ij}^{(k)_2}) = \sum_{k=1}^{p}(X_{ij}^{(k)_{1.1}} + X_{ij}^{(k)_{1.2}} + X_{ij}^{(k)_{1.3}} + X_{ij}^{(k)_{2.1}} + X_{ij}^{(k)_{2.2}}) = \sum_{k=1}^{p}(X_{ij}^{(k)_{1.1}} + X_{ij}^{(k)_{1.2}} + X_{ij}^{(k)_{1.3}} + X_{ij}^{(k)_{2.1}} + X_{ij}^{(k)_{2.2.1}} + X_{ij}^{(k)_{2.2.2}}),\ \forall\, i, j,$

(ii) $X_{lj}^S = \sum_{k=1}^{p} \alpha_l^{(k)_0} X_{lj}^S = \sum_{k=1}^{p}(\sum_{k_1=1}^{2} \beta_l^{(k)_{k_1}} \alpha_l^{(k)_0} X_{lj}^S) =$
$\sum_{k=1}^{p}(\sum_{k_2=1.1}^{1.3} \gamma_l^{(k)_{k_2}} \beta_l^{(k)_1} \alpha_l^{(k)_0} X_{lj}^S \ + \ \sum_{k_3=2.1}^{2.2} \gamma_l^{(k)_{k_3}} \beta_l^{(k)_2} \alpha_l^{(k)_0} X_{lj}^S) =$
$\sum_{k=1}^{p}(\sum_{k_2=1.1}^{1.3} \gamma_l^{(k)_{k_2}} \beta_l^{(k)_1} \alpha_l^{(k)_0} X_{lj}^S \ + \ \gamma_l^{(k)_{2.1}} \beta_l^{(k)_2} \alpha_l^{(k)_0} X_{lj}^S \ + \ \sum_{k_4=2.2.1}^{2.2.2} \delta_l^{(k)_{k_4}} \gamma_l^{(k)_{2.2}} \beta_l^{(k)_2} \alpha_l^{(k)_0} X_{lj}^S),\ \forall\, l, k, j,$

(iii) $Y_{rj} = \sum_{k=1}^{p} Y_{rj}^{(k)} = \sum_{k=1}^{p}(Y_{rj}^{(k)_1} + Y_{rj}^{(k)_2}) = \sum_{k=1}^{p}(Y_{rj}^{(k)_{1.1}} + Y_{rj}^{(k)_{1.2}} + Y_{rj}^{(k)_{1.3}} + Y_{rj}^{(k)_{2.1}} + Y_{rj}^{(k)_{2.2}}) = \sum_{k=1}^{p}(Y_{rj}^{(k)_{1.1}} + Y_{rj}^{(k)_{1.2}} + Y_{rj}^{(k)_{1.3}} + Y_{rj}^{(k)_{2.1}} + Y_{rj}^{(k)_{2.2.1}} + Y_{rj}^{(k)_{2.2.2}}),\ \forall\, r, j.$

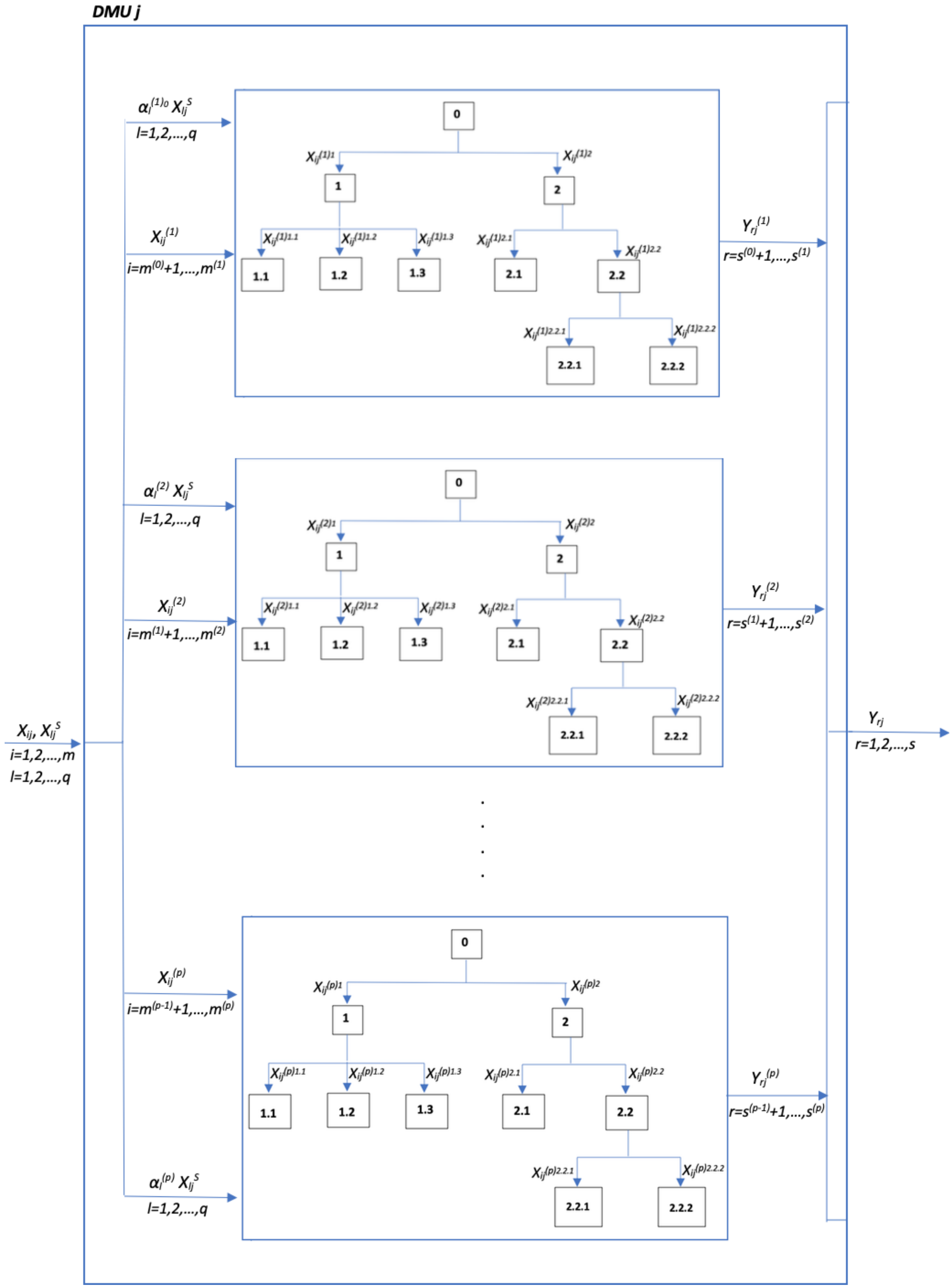

Figure 3: An embedded hierarchical network structure within a multi-function parallel system

To model the proposed network-hierarchical structure, we adopt two main properties relevant to the network relational model conceptualised by **Kao (2009a)** and **Kao (2015)**. We, firstly, assume that the same factor, either the sub-system dedicated inputs X_{ij}, the shared inputs X_{lj}^S or the outputs Y_{rj}, has the same weight ν_{io}, t_{lo}, and μ_{ro}, respectively, no matter which process (system, sub-system or internal unit of the integrated hierarchy) it corresponds to. This is a common assumption of a relational model within network DEA **(Kao, 2009a)**. Furthermore, the system cannot be handled anymore as a whole unit, but rather as a network with three successive layers, whose operations should be taken into consideration. Therefore, the aggregate output should be less than or equal to the aggregate input for each internal (sub-system or hierarchy) or external (system) process, for each DMU. Our objective function aims to maximise the ratio of the aggregated amount of final outputs to that of the inputs (both the sub-system dedicated and the shared inputs) for the system, visible from the outside.

The typical ratio-form input-oriented network-hierarchical DEA model under constant returns to scale for DMU_o can be described as follows:

$$
\begin{aligned}
&E_o^{HN} = Max \ \frac{\sum_{r=1}^{s} \mu_{ro} Y_{ro}}{\sum_{i=1}^{m} \nu_{io} X_{io} + \sum_{l=1}^{q} t_{lo} X_{lo}^S} \\
&\text{subject to} \quad \sum_{r=1}^{s} \mu_{ro} Y_{rj} - (\sum_{i=1}^{m} \nu_{io} X_{ij} + \sum_{l=1}^{q} t_{lo} X_{lj}^S) \leq 0, \ \ \forall j, \\
&\sum_{r \in O^{(k)}} \mu_{ro} Y_{rj}^{(k)} - (\sum_{i \in I^{(k)}} \nu_{io} X_{ij}^{(k)} + \sum_{l=1}^{q} t_{lo} \alpha_l^{(k)_0} X_{lj}^S) \leq 0, \ \ \forall j, k, \\
&\sum_{r \in O^{(k)}} \mu_{ro} Y_{rj}^{(k)_{k_1}} - (\sum_{i \in I^{(k)}} \nu_{io} X_{ij}^{(k)_{k_1}} + \sum_{l=1}^{q} t_{lo} \beta_l^{(k)_{k_1}} \alpha_l^{(k)_0} X_{lj}^S) \leq 0, \ \ \forall j, k, \ \ k_1 = 1, 2, \\
&\sum_{r \in O^{(k)}} \mu_{ro} Y_{rj}^{(k)_{k_2}} - (\sum_{i \in I^{(k)}} \nu_{io} X_{ij}^{(k)_{k_2}} + \sum_{l=1}^{q} t_{lo} \gamma_l^{(k)_{k_2}} \beta_l^{(k)_1} \alpha_l^{(k)_0} X_{lj}^S) \leq 0, \ \ \forall j, k, \ \ k_2 = 1.1, 1.2, 1.3, \\
&\sum_{r \in O^{(k)}} \mu_{ro} Y_{rj}^{(k)_{k_3}} - (\sum_{i \in I^{(k)}} \nu_{io} X_{ij}^{(k)_{k_3}} + \sum_{l=1}^{q} t_{lo} \gamma_l^{(k)_{k_3}} \beta_l^{(k)_2} \alpha_l^{(k)_0} X_{lj}^S) \leq 0, \ \ \forall j, k, \ \ k_3 = 2.1, 2.2, \qquad (3) \\
&\sum_{r \in O^{(k)}} \mu_{ro} Y_{rj}^{(k)_{k_4}} - (\sum_{i \in I^{(k)}} \nu_{io} X_{ij}^{(k)_{k_4}} + \sum_{l=1}^{q} t_{lo} \delta_l^{(k)_{k_4}} \gamma_l^{(k)_{2.2}} \beta_l^{(k)_2} \alpha_l^{(k)_0} X_{lj}^S) \leq 0, \ \ \forall j, k, \ \ k_4 = 2.2.1, 2.2.2, \\
&\sum_{k=1}^{p} \alpha_l^{(k)_0} = 1, \ \sum_{k_1=1}^{2} \beta_l^{(k)_{k_1}} = 1, \ \sum_{k_2=1.1}^{1.3} \gamma_l^{(k)_{k_2}} = 1, \ \sum_{k_3=2.1}^{2.2} \gamma_l^{(k)_{k_3}} = 1, \ \sum_{k_4=2.2.1}^{2.2.2} \delta_l^{(k)_{k_4}} = 1, \ \ \forall \, l, k, \\
&L_l^{(k,n)} \leq \alpha_l^{(k)_0} / \alpha_l^{(n)_0} \leq U_l^{(k,n)}, \ \forall \, l, k = 1, ..., p, \ n = 1, ..., p, \ k \neq n, \\
&L_l^{(k)_{1,2}} \leq \beta_l^{(k)_1} / \beta_l^{(k)_2} \leq U_l^{(k)_{1,2}}, \ \forall \, l, k, \\
&L_l^{(k)_{2.1,2.2}} \leq \gamma_l^{(k)_{2.1}} / \gamma_l^{(k)_{2.2}} \leq U_l^{(k)_{2.1,2.2}}, \ \forall \, l, k, \\
&L_l^{(k)_{k_2,n_2}} \leq \gamma_l^{(k)_{k_2}} / \gamma_l^{(k)_{n_2}} \leq U_l^{(k)_{k_2,n_2}}, \ \forall \, l, k_2 = 1.1, 1.2, 1.3, \ n_2 = 1.1, 1.2, 1.3, \ k_2 \neq n_2, \\
&L_l^{(k)_{2.2.1,2.2.2}} \leq \delta_l^{(k)_{2.2.1}} / \delta_l^{(k)_{2.2.2}} \leq U_l^{(k)_{2.2.1,2.2.2}}, \ \forall \, l, k, \\
&\alpha_l^{(k)_0}, \beta_l^{(k)_{k_1}}, \gamma_l^{(k)_{k_2}}, \gamma_l^{(k)_{k_3}}, \delta_l^{(k)_{k_4}} \geq 0, \quad t_{lo}, \nu_{io}, \mu_{ro} \geq \epsilon, \ \forall \, r, i, l, k, k_1, k_2, k_3, k_4,
\end{aligned}
$$

where ν_{io}, t_{lo}, and μ_{ro} are ensured to be positive, by integrating the small non-Archimedean parameter ϵ. In model **(3)**, there are four groups of constraint sets. The first group (first constraint set) reflects the entire system. The second group (second constraint set) is pertinent to the performance of each sub-system k. The third group (from third to sixth constraints sets) illustrates the operations of each of the internal units of the hierarchical structure embedded into sub-system k. For a unit at a certain level, the aggregation of outputs produced by its subordinate units at the follower level should not exceed the aggregation of inputs allocated to it by its parent unit. For example, regarding the unit (2.2) of level 2, it ought to satisfy the constraint $\sum_{r\in O^{(k)}} \mu_{ro}Y_{rj}^{(k)_{2.2}} - (\sum_{i\in I^{(k)}} \nu_{io}X_{ij}^{(k)_{2.2}} + \sum_{l=1}^{q} t_{lo}\gamma_l^{(k)_{2.2}}\beta_l^{(k)_2}\alpha_l^{(k)_0}X_{lj}^S) \leq 0, \ \ \forall j,k$. The final group (remaining constraints sets) indicates that the proportion of the shared input allocated to the respective internal unit of sub-system k should be treated as a variable. In other words, it seeks for the optimal most favourable value. For instance, with respect to the proportion variable $\delta_l^{(k)_{k_4}}$, two constraint sets are formulated to reflect its dynamics: *(i)* the $\sum_{k_4=2.2.1}^{2.2.2}\delta_l^{(k)_{k_4}} = 1, \ \forall\ l,k$, denotes that the total sum of the proportions of shared resources allocated to the internal units of the third level should be 1, and *(ii)* the $L_l^{(k)_{2.2.1,2.2.2}} \leq \delta_l^{(k)_{2.2.1}}/\delta_l^{(k)_{2.2.2}} \leq U_l^{(k)_{2.2.1,2.2.2}}, \ \forall\ l,k$, illustrates that the ratio of the proportions of shared resources in that level is bounded from below by $L_l^{(k)_{2.2.1,2.2.2}}$ and above by $U_l^{(k)_{2.2.1,2.2.2}}$. These are user-specified parameters and typically reflect the requirements of the production.

Model **(3)** is nonlinear due to its nonlinear objective function and several nonlinear terms, such as $t_{lo}\alpha_l^{(k)_0}$, $t_{lo}\beta_l^{(k)_{k_1}}\alpha_l^{(k)_0}$, and $t_{lo}\gamma_l^{(k)_{k_2}}\beta_l^{(k)_1}\alpha_l^{(k)_0}$. With respect to the objective function, we can assign a value of 1 to the denominator as a constraint, and maximise the value of the numerator. The other nonlinear terms can be linearised by variable transformations as set out below: $t_{lo}\alpha_l^{(k)_0} = \nu_{lo}^{(k)_0}$, $t_{lo}\beta_l^{(k)_{k_1}}\alpha_l^{(k)_0} = b_{lo}^{(k)_{k_1}}$, $t_{lo}\gamma_l^{(k)_{k_2}}\beta_l^{(k)_1}\alpha_l^{(k)_0} = c_{lo}^{(k)_{k_2}}$, $t_{lo}\gamma_l^{(k)_{k_3}}\beta_l^{(k)_2}\alpha_l^{(k)_0} = c_{lo}^{(k)_{k_3}}$, and $t_{lo}\delta_l^{(k)_{k_4}}\gamma_l^{(k)_{2.2}}\beta_l^{(k)_2}\alpha_l^{(k)_0} = d_{lo}^{(k)_{k_4}}, \ \forall\ l,k,k_1,k_2,k_3,k_4$. Thus, we obtain the following linear model **(4)**:

$$E_o^{HN} = Max \ \sum_{r=1}^{s}\mu_{ro}Y_{ro}$$

$$\text{subject to} \quad \sum_{i=1}^{m}\nu_{io}X_{io} + \sum_{l=1}^{q}t_{lo}X_{lo}^S = 1,$$

$$\sum_{r=1}^{s}\mu_{ro}Y_{rj} - (\sum_{i=1}^{m}\nu_{io}X_{ij} + \sum_{l=1}^{q}t_{lo}X_{lj}^S) \leq 0, \ \ \forall j,$$

$$\sum_{r\in O^{(k)}}\mu_{ro}Y_{rj}^{(k)} - (\sum_{i\in I^{(k)}}\nu_{io}X_{ij}^{(k)} + \sum_{l=1}^{q}\nu_{lo}^{(k)_0}X_{lj}^S) \leq 0, \ \ \forall j,k,$$

$$\sum_{r\in O^{(k)}}\mu_{ro}Y_{rj}^{(k)_{k_1}} - (\sum_{i\in I^{(k)}}\nu_{io}X_{ij}^{(k)_{k_1}} + \sum_{l=1}^{q}b_{lo}^{(k)_{k_1}}X_{lj}^S) \leq 0, \ \ \forall j,k, \ \ k_1 = 1,2,$$

$$\sum_{r\in O^{(k)}}\mu_{ro}Y_{rj}^{(k)_{k_2}} - (\sum_{i\in I^{(k)}}\nu_{io}X_{ij}^{(k)_{k_2}} + \sum_{l=1}^{q}c_{lo}^{(k)_{k_2}}X_{lj}^S) \leq 0, \ \ \forall j,k, \ \ k_2 = 1.1,1.2,1.3,$$

$$\sum_{r\in O^{(k)}} \mu_{ro} Y_{rj}^{(k)_{k_3}} - (\sum_{i\in I^{(k)}} \nu_{io} X_{ij}^{(k)_{k_3}} + \sum_{l=1}^{q} c_{lo}^{(k)_{k_3}} X_{lj}^S) \leq 0, \ \ \forall j, k, \ \ k_3 = 2.1, 2.2, \tag{4}$$

$$\sum_{r\in O^{(k)}} \mu_{ro} Y_{rj}^{(k)_{k_4}} - (\sum_{i\in I^{(k)}} \nu_{io} X_{ij}^{(k)_{k_4}} + \sum_{l=1}^{q} d_{lo}^{(k)_{k_4}} X_{lj}^S) \leq 0, \ \ \forall j, k, \ \ k_4 = 2.2.1, 2.2.2,$$

$$\sum_{k=1}^{p} \nu_{lo}^{(k)_0} = t_{lo}, \sum_{k_1=1}^{2} b_{lo}^{(k)_{k_1}} = \nu_{lo}^{(k)_0}, \sum_{k_2=1.1}^{1.3} c_{lo}^{(k)_{k_2}} = b_{lo}^{(k)_1}, \forall\, l, k,$$

$$\sum_{k_3=2.1}^{2.2} c_{lo}^{(k)_{k_3}} = b_{lo}^{(k)_2}, \sum_{k_4=2.2.1}^{2.2.2} d_{lo}^{(k)_{k_4}} = c_{lo}^{(k)_{2.2}}, \forall\, l, k,$$

$$\nu_{lo}^{(n)_0} L_l^{(k,n)} \leq \nu_{lo}^{(k)_0} \leq \nu_{lo}^{(n)_0} U_l^{(k,n)}, \ \forall\, l, k = 1, ..., p, \ n = 1, ..., p, \ k \neq n,$$

$$b_{lo}^{(k)_2} L_l^{(k)_{1,2}} \leq b_{lo}^{(k)_1} \leq b_{lo}^{(k)_2} U_l^{(k)_{1,2}}, \ \forall\, l, k,$$

$$c_{lo}^{(k)_{2.2}} L_l^{(k)_{2.1,2.2}} \leq c_{lo}^{(k)_{2.1}} \leq c_{lo}^{(k)_{2.2}} U_l^{(k)_{2.1,2.2}}, \ \forall\, l, k,$$

$$c_{lo}^{(k)_{n_2}} L_l^{(k)_{k_2,n_2}} \leq c_{lo}^{(k)_{k_2}} \leq c_{lo}^{(k)_{n_2}} U_l^{(k)_{k_2,n_2}}, \ \forall\, l, k_2 = 1.1, 1.2, 1.3, \ n_2 = 1.1, 1.2, 1.3, \ k_2 \neq n_2,$$

$$d_{lo}^{(k)_{2.2.2}} L_l^{(k)_{2.2.1,2.2.2}} \leq d_{lo}^{(k)_{2.2.1}} \leq d_{lo}^{(k)_{2.2.2}} U_l^{(k)_{2.2.1,2.2.2}}, \ \forall\, l, k,$$

$$t_{lo}, \nu_{io}, \mu_{ro}, \nu_{lo}^{(k)_0}, b_{lo}^{(k)_{k_1}}, c_{lo}^{(k)_{k_2}}, c_{lo}^{(k)_{k_3}}, d_{lo}^{(k)_{k_4}} \geq \epsilon \ \forall\, r, i, l, k, k_1, k_2, k_3, k_4.$$

After an optimal solution $(t_{lo^*}, \nu_{io^*}, \mu_{ro^*}, \nu_{lo}^{(k)_{0^*}}, b_{lo}^{(k)_{k_1^*}}, c_{lo}^{(k)_{k_2^*}}, c_{lo}^{(k)_{k_3^*}}, d_{lo}^{(k)_{k_4^*}})$ is obtained for DMU_o under the linear model **(4)**, the efficiencies of the overall system, its sub-systems, and its internal units at all levels of the hierarchical structure within each sub-system are calculated as follows: *(i)* $E_o^{HN} = \sum_{r=1}^{s} \mu_{ro^*} Y_{ro} / (\sum_{i=1}^{m} \nu_{io^*} X_{io} + \sum_{l=1}^{q} t_{lo^*} X_{lo}^S)$ (overall system efficiency), *(ii)* $E_o^{(k)} = \sum_{r\in O^{(k)}} \mu_{ro^*} Y_{ro}^{(k)} / (\sum_{i\in I^{(k)}} \nu_{io^*} X_{io}^{(k)} + \sum_{l=1}^{q} \nu_{lo^*}^{(k)_0} X_{lo}^S)$, $\forall\ k$ (sub-system k efficiency), *(iii)* $E_o^{(k_1,k)} = \sum_{r\in O^{(k)}} \mu_{ro^*} Y_{ro}^{(k)_{k_1}} / (\sum_{i\in I^{(k)}} \nu_{io^*} X_{io}^{(k)_{k_1}} + \sum_{l=1}^{q} b_{lo}^{(k)_{k_1^*}} X_{lo}^S)$, $\forall\ k, k_1$ (unit k_1 of level 1 efficiency), *(iv)* $E_o^{(k_2,k)} = \sum_{r\in O^{(k)}} \mu_{ro^*} Y_{ro}^{(k)_{k_2}} / (\sum_{i\in I^{(k)}} \nu_{io^*} X_{io}^{(k)_{k_2}} + \sum_{l=1}^{q} c_{lo}^{(k)_{k_2^*}} X_{lo}^S)$, $\forall\ k, k_2$ (unit k_2 of level 2 efficiency), *(v)* $E_o^{(k_3,k)} = \sum_{r\in O^{(k)}} \mu_{ro^*} Y_{ro}^{(k)_{k_3}} / (\sum_{i\in I^{(k)}} \nu_{io^*} X_{io}^{(k)_{k_3}} + \sum_{l=1}^{q} c_{lo}^{(k)_{k_3^*}} X_{lo}^S)$, $\forall\ k, k_3$ (unit k_3 of level 2 efficiency), *(vi)* $E_o^{(k_4,k)} = \sum_{r\in O^{(k)}} \mu_{ro^*} Y_{ro}^{(k)_{k_4}} / (\sum_{i\in I^{(k)}} \nu_{io^*} X_{io}^{(k)_{k_4}} + \sum_{l=1}^{q} d_{lo}^{(k)_{k_4^*}} X_{lo}^S)$, $\forall\ k, k_4$ (unit k_4 of level 3 efficiency).

3.2 Efficiency decomposition

While developing a network DEA model such as the one proposed in this paper, it could be essential to consider the concept of the efficiency decomposition. According to **Kao (2017)**, efficiency decomposition is an approach to measure the system efficiency that utilises exogenous inputs to produce exogenous outputs. It measures system-division efficiencies and then identifies a mathematical relationship that associates them. As denoted in **Kao (2015)**, when the internal divisions of a system share the available resources, then they are indispensable parts of a parallel structure. In this paper, there is a parallel hierarchical structure within each operating sub-system and a typical parallel structure among the sub-systems of such a network DEA system. From the perspective of the entire system, its efficiency is decomposed into the weighted arithmetic average of those of the sub-systems, where the weight of the sub-system k is defined as the proportion of the aggregate input consumed by this sub-system in that consumed by all sub-systems (whole system), and $\sum_{k=1}^{p} \omega^{(k)} = 1$:

$$E_o^{HN} = \sum_{k=1}^{p} \omega^{(k)} E_o^{(k)} = \sum_{k=1}^{p} \Big(\frac{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)} + \sum_{l=1}^{q} \nu_{lo}^{(k)_0} X_{lo}^{S}}{\sum_{i=1}^{m} \nu_{io} X_{io} + \sum_{l=1}^{q} t_{lo} X_{lo}^{S}} \cdot \frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)} + \sum_{l=1}^{q} \nu_{lo}^{(k)_0} X_{lo}^{S}} \Big) = \frac{\sum_{k=1}^{p} \sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)}}{\sum_{i=1}^{m} \nu_{io} X_{io} + \sum_{l=1}^{q} t_{lo} X_{lo}^{S}} = \frac{\sum_{r=1}^{s} \mu_{ro} Y_{ro}}{\sum_{i=1}^{m} \nu_{io} X_{io} + \sum_{l=1}^{q} t_{lo} X_{lo}^{S}}. \tag{5}$$

From the perspective of the hierarchical structure embedded into a sub-system, the efficiency of a unit at level ξ is a weighted average of the ones of the subordinates at level $\xi + 1$, where the respective weight is formulated in a similar approach, as before. Hence, the efficiencies of sub-system k, and the internal units (1), (2), and (2.2) are decomposed as follows:

$$E_o^{(k)} = \sum_{k_1=1}^{2} \omega^{(k_1,k)} E_o^{(k_1,k)} = \sum_{k_1=1}^{2} \Big(\frac{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_1}} + \sum_{l=1}^{q} b_{lo}^{(k)_{k_1}} X_{lo}^{S}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)} + \sum_{l=1}^{q} \nu_{lo}^{(k)_0} X_{lo}^{S}} \cdot \frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)_{k_1}}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_1}} + \sum_{l=1}^{q} b_{lo}^{(k)_{k_1}} X_{lo}^{S}} \Big) = \frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)} + \sum_{l=1}^{q} \nu_{lo}^{(k)_0} X_{lo}^{S}}, \tag{6}$$

$$E_o^{(1,k)} = \sum_{k_2=1.1}^{1.3} \omega^{(k_2,k)} E_o^{(k_2,k)} = \sum_{k_2=1.1}^{1.3} \Big(\frac{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_2}} + \sum_{l=1}^{q} c_{lo}^{(k)_{k_2}} X_{lo}^{S}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_1} + \sum_{l=1}^{q} b_{lo}^{(k)_1} X_{lo}^{S}} \cdot \frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)_{k_2}}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_2}} + \sum_{l=1}^{q} c_{lo}^{(k)_{k_2}} X_{lo}^{S}} \Big) = \frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)_1}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_1} + \sum_{l=1}^{q} b_{lo}^{(k)_1} X_{lo}^{S}}, \tag{7}$$

$$E_o^{(2,k)} = \sum_{k_3=2.1}^{2.2} \omega^{(k_3,k)} E_o^{(k_3,k)} = \sum_{k_3=2.1}^{2.2} \Big(\frac{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_3}} + \sum_{l=1}^{q} c_{lo}^{(k)_{k_3}} X_{lo}^{S}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_2} + \sum_{l=1}^{q} b_{lo}^{(k)_2} X_{lo}^{S}} \cdot \frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)_{k_3}}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_3}} + \sum_{l=1}^{q} c_{lo}^{(k)_{k_3}} X_{lo}^{S}} \Big) = \frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)_2}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)_2} + \sum_{l=1}^{q} b_{lo}^{(k)_2} X_{lo}^{S}}, \tag{8}$$

$$E_o^{(2.2,k)} = \sum_{k_4=2.2.1}^{2.2.2} \omega^{(k_4,k)} E_o^{(k_4,k)} = \sum_{k_4=2.2.1}^{2.2.2} \left(\frac{\sum_{i\in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_4}} + \sum_{l=1}^{q} d_{lo}^{(k)_{k_4}} X_{lo}^{S}}{\sum_{i\in I^{(k)}} \nu_{io} X_{io}^{(k)_{2.2}} + \sum_{l=1}^{q} c_{lo}^{(k)_{2.2}} X_{lo}^{S}} \cdot \frac{\sum_{r\in O^{(k)}} \mu_{ro} Y_{ro}^{(k)_{k_4}}}{\sum_{i\in I^{(k)}} \nu_{io} X_{io}^{(k)_{k_4}} + \sum_{l=1}^{q} d_{lo}^{(k)_{k_4}} X_{lo}^{S}} \right) = \frac{\sum_{r\in O^{(k)}} \mu_{ro} Y_{ro}^{(k)_{2.2}}}{\sum_{i\in I^{(k)}} \nu_{io} X_{io}^{(k)_{2.2}} + \sum_{l=1}^{q} c_{lo}^{(k)_{2.2}} X_{lo}^{S}}, \tag{9}$$

where $\sum_{k_1=1}^{2} \omega^{(k_1,k)} = 1$, $\sum_{k_2=1.1}^{1.3} \omega^{(k_2,k)} = 1$, $\sum_{k_3=2.1}^{2.2} \omega^{(k_3,k)} = 1$, and $\sum_{k_4=2.2.1}^{2.2.2} \omega^{(k_4,k)} = 1$.

Based on the above decompositions, the network-hierarchical system efficiency, E_o^{HN}, can in effect be decomposed as the weighted average of the ones of the terminal units, belonging to the hierarchical structure of sub-system k:

$$E_o^{HN} = \sum_{k=1}^{p} \omega^{(k)} E_o^{(k)} = \sum_{k=1}^{p} \omega^{(k)} \left(\sum_{k_1=1}^{2} \omega^{(k_1,k)} E_o^{(k_1,k)} \right) = \sum_{k=1}^{p} \omega^{(k)} \left(\omega^{(1,k)} \sum_{k_2=1.1}^{1.3} \omega^{(k_2,k)} E_o^{(k_2,k)} + \omega^{(2,k)} \sum_{k_3=2.1}^{2.2} \omega^{(k_3,k)} E_o^{(k_3,k)} \right) = \sum_{k=1}^{p} \omega^{(k)} \left[\left(\omega^{(1,k)} \sum_{k_2=1.1}^{1.3} \omega^{(k_2,k)} E_o^{(k_2,k)} \right) + \left(\omega^{(2,k)} \left(\omega^{(2.1,k)} E_o^{(2.1,k)} + \omega^{(2.2,k)} \sum_{k_4=2.2.1}^{2.2.2} \omega^{(k_4,k)} E_o^{(k_4,k)} \right) \right) \right] = \sum_{k=1}^{p} \sum_{k_2=1.1}^{1.3} w^{(k_2,k)} E_o^{(k_2,k)} + \sum_{k=1}^{p} w^{(2.1,k)} E_o^{(2.1,k)} + \sum_{k=1}^{p} \sum_{k_4=2.2.1}^{2.2.2} w^{(k_4,k)} E_o^{(k_4,k)}, \tag{10}$$

where $w^{(k_2,k)} = \omega^{(k)}\omega^{(1,k)}\omega^{(k_2,k)}$, $w^{(2.1,k)} = \omega^{(k)}\omega^{(2,k)}\omega^{(2.1,k)}$, $w^{(k_4,k)} = \omega^{(k)}\omega^{(2,k)}\omega^{(2.2,k)}\omega^{(k_4,k)}$, $\sum_{k=1}^{p} \sum_{k_2=1.1}^{1.3} w^{(k_2,k)} + \sum_{k=1}^{p} w^{(2.1,k)} + \sum_{k=1}^{p} \sum_{k_4=2.2.1}^{2.2.2} w^{(k_4,k)} = 1$.

According to **(Cook et al., 2010)**, such an additive efficiency decomposition approach enables the measurement of the performance of the system under the assumptions of both constant returns to scale and variable returns to scale.

3.3 Efficiency aggregation

Since the proposed model **(4)** firstly measures the system and its constituent processes' efficiencies and then seeks for a mathematical relationship (the additive form) that links them, it can be classified as an additive efficiency decomposition model **(Kao, 2018)**. Another known approach for measuring the performance score of a network DEA system is the efficiency aggregation **(Kao, 2017; Zhang and Chen, 2019)**. In such an approach, the internal processes are initially aggregated (either in additive or in multiplicative form) to establish the system efficiency and subsequently to address its performance measurement.

From the perspective of the additive form towards our parallel-hierarchical system, we can observe that the efficiency decomposition is identical with the concept of the efficiency aggregation. This occurs when the overall efficiency is defined as the weighted aggregation of the p parallel divisional efficiencies, where the weight is DMU-specific **(Kao, 2016)**. If the decision-maker selects to build the system efficiency by aggregating the sub-system efficiencies in a multiplicative way, then the following multiplicative efficiency aggregation model **(11)** is proposed to measure the performance of the structure in Figure 3:

$$E_o^{HN'} = Max \prod_{k=1}^{p} E_o^{(k)} = \prod_{k=1}^{p} \left(\frac{\sum_{r \in O^{(k)}} \mu_{ro} Y_{ro}^{(k)}}{\sum_{i \in I^{(k)}} \nu_{io} X_{io}^{(k)} + \sum_{l=1}^{q} \nu_{lo}^{(k)_0} X_{lo}^{S}} \right) \tag{11}$$

subject to the transformed constraints of model (3).

Model **(11)** differs from model **(4)** only in terms of its objective function. This illustrates the system efficiency as the product of those of its sub-systems.

We can additionally determine the relationship of the system efficiencies between the additive decomposition model **(4)** and the multiplicative aggregation model **(11)**, based on the inspirational ideas of **Kao (2018)** and **Zhang and Chen (2019)**:

$$E_o^{HN} = \frac{\sum_{r=1}^{s} \mu_{ro} Y_{ro}}{\sum_{i=1}^{m} \nu_{io} X_{io} + \sum_{l=1}^{q} t_{lo} X_{lo}^{S}} = \sum_{k=1}^{p} \omega^{(k)} E_o^{(k)} \geq \prod_{k=1}^{p} (E_o^{(k)})^{\omega^{(k)}} \geq \prod_{k=1}^{p} E_o^{(k)} \tag{12}$$

The first inequality from the left holds, since the weighted arithmetic mean is greater than or equal to the weighted geometric mean. The other inequality is in effect, given that $E_o^{(k)} \leq 1, \ \forall \ k \in \{1, 2, ..., p\}$ and $\sum_{k=1}^{p} \omega^{(k)} = 1$. Hence, the system efficiency of model **(4)** is always greater than or equal to that of the multiplicative model **(11)**, and this is also confirmed by the numerical application in Section 4.

4 An illustrative application to higher education

The performance evaluation of the higher education sector has been widely discussed in the literature **(Casu and Thanassoulis, 2006; Kao and Hung, 2008; Kao, 2012; Witte et al., 2013; Kao, 2015; Moncayo–Martínez et al., 2020; Ghasemi et al., 2020)**. **Kao (2012)**, for instance, explored the case of a chemistry and physics university department in UK that consists of two major parallel functions, the teaching and the research. It was said that each university department has a different proportion of resources at its disposal to allocate to teaching and research tasks. In such a parallel production system, the internal parallel divisions are still treated as black-boxes, without enabling the decision-maker to understand and identify the main sources of inefficiency within teaching and research. **Kao (2015)** suggested the measurement and evaluation of a university department in the form of a single-stage hierarchical structure. In their example, the university department under consideration is decomposed into three major functions: the enterprise, the research, and the teaching activities. The latter are further divided into work at the undergraduate and graduate levels. Although **Kao's (2015)** study successfully examined the performance of such a university department with a single-stage hierarchical structure, it did not pay attention to more complicated (parallel) network structures. In reality, a university department (e.g., Business School) could contain multiple parallel sub-departments each of which may consist of a number of internal functions arranged in a multi-layer hierarchical form. To illustrate the effectiveness of our proposed multi-function parallel network hierarchical DEA system, we expand the illustrative application presented in **Kao (2015)** by looking more closely at multiple parallel academic departments with distinctive functions, each of which is further viewed as a hierarchical form, see also Section 3.1. An embedded hierarchical structure within a multi-function parallel system has, to our knowledge, not yet been considered in the existing literature, particularly to examine the relative efficiency of the different departments and tasks of a Business School. This study illustrates the proposed models by measuring the operating performance of several Business Schools across a number of hypothetical universities.

Since our target is to better correspond to a real-life scenario, we assume that a Business School can be viewed as a more complicated network system; that is, it contains various departments (Accounting, Banking and Finance, Digital Marketing, Decision Analytics and Risk, Human Resource Management and Organisational Behaviour, Strategy Innovation and Entrepreneurship), that operate independently without affecting each other. Each of those departments performs various academic and managerial functions. For the sake of simplicity, we presume that the Business Schools to be evaluated and compared in this study, have only three departments: Accounting (A), Banking and Finance (B), and Decision Analytics and Risk (D). The internal composition of each department is no longer treated as a black-box but takes into account three main functions: teaching, research, and enterprise. Teaching is further divided into undergraduate and postgraduate teaching activities. These functions are arranged into a multi-layer hierarchical structure. Figure 4 illustrates the structure of this parallel network hierarchical system.

In determining departments' and their internal units' efficiencies for the considered Business School, the following two inputs are used: personnel (X_1) and expenses (X_2). The former represents the number of academic and administrative staff and the latter, the general expenditure (e.g., staff salaries, capital investment) and equipment expenditures. With regard to outputs, the following are generated: the number of students (at an undergraduate and postgraduate level) graduating within a year, the credit-hours taught (at an undergraduate and postgraduate level) which are derived by the total number of students attending the unit over all units taught by the department, the total number of publications published by the academic faculty of the particular department within a year, the grants received from government funding councils, and the enterprise income obtained from contractual agreements made between the department and the local businesses with respect to service provision. As for the grants and enterprise income, there is a discussion on **Cook and Zhu (2007)** which suggests that they might be either inputs or outputs. Implicitly, this paper is measuring the Business School performance from the point of view of the University. The University identifies these kinds of income as outputs produced by the Business School. Certainly, they may plough some of that income back into the Business School in the form of salaries and capital investment. However, salaries and investment are already inputs in this example. All the aforementioned outputs are dedicated, that is they are related to different functions within a specific department. In particular, students and credits are associated with teaching, publications and grants with research, and income with enterprise.

Note that the purpose of this application is to showcase whether and how the measurement and evaluation of DMUs arranged into a parallel multi-layer multi-function hierarchical structure is attained. The data follows the example of **Kao's (2015)** in spirit, and arguably still is a simplification of most real Business Schools. The study is not intended to represent real Business Schools but instead can help to illustrate how the application of this methodology may help them to identify areas that may benefit from further attention towards improving their performance.

Personnel and expenses are shared among the departments and their different functions. They can be distributed using either pre-determined (fixed) proportions or variable proportions. In our scenario, the proportions are treated as variables rather than parameters, as it is difficult to specify instances such as, the amount of time a lecturer dedicates to each function or the amount of money being collected by each of the departments.

In the spirit of **Kao's (2015)** study, we assume that each Business School allocates similar amounts of resources to its three departments; that is, $\alpha_l^{(1)_0} \cong \alpha_l^{(2)_0} \cong \alpha_l^{(3)_0}$, and $\sum_{k=1}^{3} \alpha_l^{(k)_0} = 1$, $l = 1, 2$, where $\alpha_l^{(k)_0}$ is the proportion of each resource l allocated to department k ($k = 1, 2, 3$). The proportions are expressed in ranges in the form of $0.5 \leq \alpha_l^{(1)_0}/\alpha_l^{(2)_0} \leq 2$, $0.5 \leq \alpha_l^{(1)_0}/\alpha_l^{(3)_0} \leq 2$, and $0.5 \leq \alpha_l^{(2)_0}/\alpha_l^{(3)_0} \leq 2$. Furthermore, every department allocates approximately 40%, 40%, and 20% of each input to the three major functions; that is, $\beta_l^{(k)_1} \cong \beta_l^{(k)_2} \cong 2\beta_l^{(k)_3}$, and $\sum_{k_1=1}^{3} \beta_l^{(k)_{k_1}} = 1$, $l = 1, 2$, and $k = 1, 2, 3$, where $\beta_l^{(k)_{k_1}}$ is the proportion of each resource l of department k allocated to each of these functions ($k_1 = 1, 2, 3$). The proportions are expressed in ranges in the form of $0.5 \leq \beta_l^{(k)_1}/\beta_l^{(k)_2} \leq 2$, $1 \leq \beta_l^{(k)_2}/\beta_l^{(k)_3} \leq 4$, and $1 \leq \beta_l^{(k)_1}/\beta_l^{(k)_3} \leq 4$. We also assume that the teaching function at each department allocates similar amounts of resources to both undergraduate and postgraduate levels; that is, $\gamma_l^{(k)_{1.1}} \cong \gamma_l^{(k)_{1.2}}$, and $\sum_{k_2=1.1}^{1.2} \gamma_l^{(k)_{k_2}} = 1$, $l = 1, 2$, and $k = 1, 2, 3$, where $\gamma_l^{(k)_{k_2}}$ is the proportion of each resource l of department k allocated to each of these levels ($k_2 = 1.1, 1.2$). The proportions are expressed in ranges in the form of $0.5 \leq \gamma_l^{(k)_{1.1}}/\gamma_l^{(k)_{1.2}} \leq 2$. The values of the shared inputs and the dedicated outputs for the evaluation of the Business Schools in twenty hypothetic universities are depicted in Table 2.

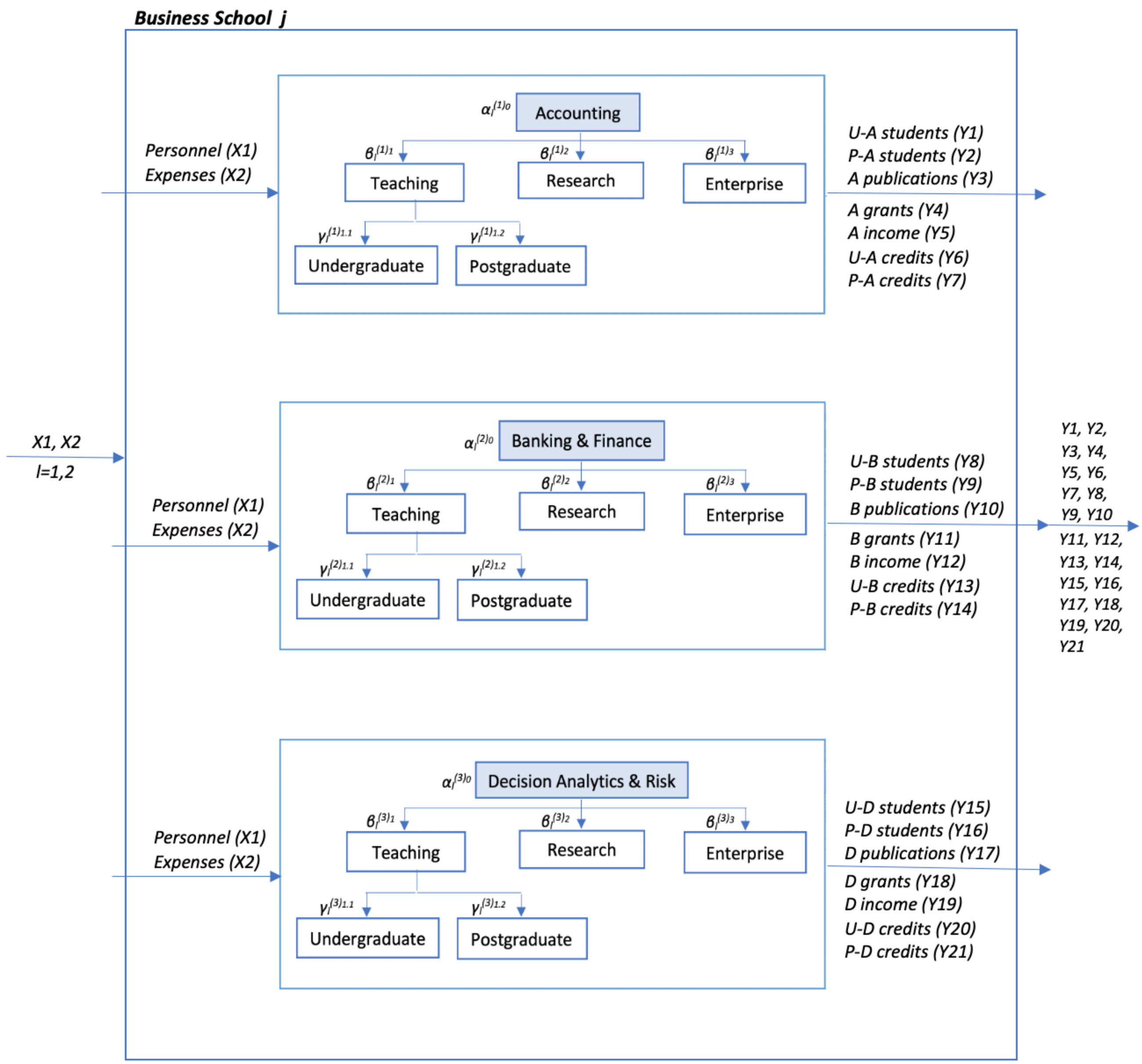

Figure 4: Embedded hierarchical network structure within a multi-function parallel system - The structure of the hypothetical Business School

Table 2: Data of the Business School in twenty hypothetic universities

DMU	X_1	X_2	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9	Y_{10}	Y_{11}	Y_{12}	Y_{13}	Y_{14}	Y_{15}	Y_{16}	Y_{17}	Y_{18}	Y_{19}	Y_{20}	Y_{21}
1	20	40	35	60	55	25	25	15	70	4	2	15	70	50	5	80	30	50	25	30	50	95	65
2	55	120	40	85	60	35	35	20	15	2	1	45	110	140	15	100	65	120	80	40	65	110	20
3	25	80	25	50	50	25	10	10	15	1	2	20	70	55	5	45	25	15	20	20	30	40	20
4	70	55	30	65	40	25	20	20	25	5	1	25	90	60	5	100	50	95	100	25	30	30	25
5	80	60	45	95	50	35	25	20	65	75	3	35	110	90	10	120	65	30	30	30	75	120	55
6	75	45	30	100	55	65	20	20	50	35	10	30	85	50	5	75	30	20	25	185	70	80	50
7	75	75	15	60	35	20	5	20	10	30	25	25	120	65	10	105	55	30	40	15	60	45	5
8	20	30	20	70	50	30	10	15	15	10	10	20	65	45	5	45	20	30	15	20	60	60	15
9	30	110	55	140	140	315	20	95	90	80	30	60	90	50	5	115	60	70	100	55	95	55	80
10	85	80	60	180	170	250	20	180	160	145	45	20	65	50	5	45	20	35	25	85	115	60	130
11	12	74	10	28	55	55	34	19	90	101	79	24	5	28	25	20	118	122	20	62	53	64	103
12	125	86	37	40	45	91	20	106	92	43	35	77	72	81	50	72	115	65	113	29	72	103	121
13	113	66	88	101	105	128	42	88	109	81	118	128	14	9	100	107	89	19	123	51	104	123	9
14	74	10	126	5	83	20	22	130	25	116	98	74	57	75	57	111	45	85	119	72	83	32	19
15	121	127	40	127	48	20	123	130	106	91	67	65	130	29	109	81	85	5	19	8	46	55	45
16	23	86	125	70	81	130	83	100	14	61	122	93	112	112	75	22	5	33	36	119	47	85	24
17	32	69	83	56	105	34	56	46	104	100	91	25	25	82	99	81	58	82	34	97	28	119	98
18	118	30	81	110	56	12	15	72	85	87	17	36	122	100	126	72	110	74	44	87	34	126	24
19	72	57	130	77	27	115	30	81	27	89	106	40	23	20	12	43	49	68	76	121	100	30	64
20	17	91	105	6	92	105	47	30	85	51	42	45	26	15	24	36	5	53	44	15	112	6	10

4.1 Models from literature

The traditional black-box model has been initially applied to evaluate the operating performance of the Business Schools, as shown in the second column of Table 3. This model is simply the typical input-oriented constant returns-to-scale DEA model **(Charnes et al., 1978)** that makes use of the two exogenous shared inputs to produce the twenty-one exogenous dedicated outputs. In other words, it entirely ignores all the internal operations and mechanisms of the system (parallel sub-systems and integrated hierarchies).

According to the second column of Table 3, we can easily notice that there are in total 14 efficient DMUs and 6 inefficient ones. The efficient DMUs cannot be readily discriminated and this especially matters, when we consider the problem as a multi-criteria decision-making case. Besides, we are not able to obtain the efficiencies of the constituent departments of the respective Business School.

The model **(1)** of **Kao (2012)**, see Section 2.1, has also been implemented to evaluate the performance of the Business Schools, as illustrated in columns 3-9 of Table 3. The simplification of such a model compared to the proposed models of this paper is that the parallel sub-divisions of the system are treated as black-boxes.

According to column 3 of Table 3, there are now 8 DMUs with a perfect efficiency score of one, that is their respective components (departments) are absolutely efficient. There is still, however, the problem with the lack of discrimination of the efficient DMUs that cannot lead us to a unique ranking order. The efficiency scores of the three departments (A, B, D) are given in columns 4, 6, and 8, respectively. The numbers next to each of the efficiency scores (in columns 5, 7, and 9) are the respective weights in the efficiency decomposition. Using the Business School 15 as an example, its efficiency scores for A (0.9997), B (0.5541), and D (0.3353) multiplied by their respective weights of 0.443, 0.309, and 0.248, will obtain the efficiency score of the whole system, 0.6973. Using the information of the efficiency score, we can conclude that DMU's 15 relatively low performance is owed to its weak Decision Analytics and Risk department, whose operations should be improved. Nevertheless, it is not clear to us which constituent functions of that department have the millstone of a heavy burden round their necks.

4.2 Results from the parallel-hierarchical network model

To obtain the information regarding the performance of the functions of each department within the considered Business School, we implement the additive decomposition model **(4)** and the multiplicative aggregation model **(11)**. For modelling, running, and analysing our data, we have utilised the programming language Python 3.7.6 and in particular the version 2.1 of PuLP as the free linear programming library for model **(4)**. As for the non-linear model **(11)**, we have implemented the GEKKO which is a Python package for machine learning and optimisation. It is combined with large-scale solvers for non-linear programming models as well. To define the type of the problem, we have used a non-dynamic mode that sets all differential terms to zero to calculate the steady-state conditions. The experiments ran on a computer with 16GB RAM.

These models do not only allow us to discriminate the efficient DMUs, but also to simultaneously calculate the efficiencies of the Business School of interest, its constituent departments, and the functions within the respective department. The results obtained by models **(4)** and **(11)** are respectively illustrated in Tables 4 and 5. The second column in each table shows the efficiency of the respective overall system along with its rank. The remaining columns provide the efficiency scores with their respective weights of each sub-system and sub-unit within the sub-system.

Table 3: Efficiency scores of black-box model and **Kao (2012)**

		Kao's (2012) Model							
DMU	Black-box [Rank]	E_o [Rank]	$E_o^{(A)}$	$\omega^{(A)}$	$E_o^{(B)}$	$\omega^{(B)}$	$E_o^{(D)}$	$\omega^{(D)}$
1	1 [1]	1 [1]	1	0.429	1	0.215	1	0.356
2	0.9556 [15]	0.7247 [15]	0.4532	0.237	0.8412	0.433	0.7665	0.330
3	0.7124 [20]	0.5499 [19]	0.4551	0.250	0.6689	0.500	0.4065	0.250
4	0.8845 [17]	0.6596 [17]	0.5015	0.320	0.6915	0.320	0.7715	0.360
5	0.8620 [18]	0.7599 [14]	0.7383	0.337	0.7654	0.283	0.7750	0.380
6	1 [1]	0.9077 [11]	0.9247	0.356	0.6951	0.215	1	0.430
7	0.7200 [19]	0.5339 [20]	0.3111	0.250	0.6903	0.500	0.4441	0.250
8	1 [1]	1 [1]	1	0.252	1	0.499	1	0.249
9	1 [1]	1 [1]	1	0.240	1	0.480	1	0.279
10	1 [1]	0.8961 [12]	1	0.400	0.7496	0.297	0.9026	0.303
11	1 [1]	1 [1]	1	0.496	1	0.248	1	0.255
12	0.8869 [16]	0.6282 [18]	0.5249	0.249	0.3353	0.296	0.8752	0.455
13	1 [1]	0.8415 [13]	0.8911	0.373	0.8199	0.355	0.8016	0.272
14	1 [1]	1 [1]	1	0.403	1	0.202	1	0.395
15	1 [1]	0.6973 [16]	0.9997	0.443	0.5541	0.309	0.3353	0.248
16	1 [1]	1 [1]	1	0.454	1	0.250	1	0.296
17	1 [1]	1 [1]	1	0.250	1	0.250	1	0.500
18	1 [1]	1 [1]	1	0.283	1	0.467	1	0.250
19	1 [1]	0.9197 [10]	0.9973	0.354	0.7184	0.225	0.9623	0.421
20	1 [1]	0.9823 [9]	1	0.400	0.9116	0.200	1	0.400

The ranks of the efficiency scores of the overall system obtained via our proposed model **(4)** are also compared with the overall systems' ranks of black-box and **Kao's (2012) models**. It can be statistically inferred that the ranks are quite similar, and this is verified by the Spearman rank-order correlation test with values 0.678 and 0.924, respectively. These are significant at the 0.01 level (two-tailed). The same situation holds even for our proposed model **(11)**. The results of the correlation analysis further validate the underpinning of our model in some way.

Using the Business School 18 as an example, its efficiency score for the Accounting (0.4553) is decomposed into the efficiencies of the teaching (0.7690), research (0.3263), and enterprise (0.2707) multiplied by their respective weights of 0.333, 0.333, and 0.333. Note that teaching efficiency is further decomposed into the efficiencies of the undergraduate (0.3069) and postgraduate (1) levels multiplied by their respective weights, 0.333 and 0.667. By the same token, the efficiency scores of the other two departments, Banking and Finance (0.8045) and Decision Analytics and Risk (0.5141), are identified. Hence, the efficiency scores of the three departments multiplied by their respective weights provide the efficiency of DMU 18, which is 0.6446.

The unsatisfactory performance of Business School 18 is mainly due to the Accounting and secondly to the Decision Analytics and Risk department. If this Business School desires to significantly improve its efficiency, then it should strengthen its contribution to society (enterprise) along with its research activities, as far as the A department is concerned. With regard to D department, particular emphasis should be placed on the enterprise. In summary, this Business School should genuinely pursue constant and long-term synergies with representatives from the public and private sector towards more impactful and effective research and educational actions.

As discussed in Section 3.3, the system efficiency of model **(4)** should be greater than or equal to the respective one in model **(11)**. By comparing the second columns of Table 4 and Table 5, we validate our initial assumption. This is further bolstered by the fact that the Business School's efficiency is the product of the ones of the three internal departments. For example, DMU's 18 efficiency (0.1946) is obtained by multiplying A's (0.4622), B's (0.8131), and D's efficiency (0.5178). With regard to DMU 18, the promising performance of its B department still has considerable potential for further improvements, through the upgrade of the teaching methods and the training of the teaching staff, to better support postgraduate taught modules. The root cause of the problem, however, is located to the A department that should better adhere to the following guidelines: *(i)* strengthen its contributions to society, and *(ii)* provide greater (financial) incentives to the academic faculty to ensure grants via more powerful research proposals.

5 Conclusions & Future Research

In the current study, we have proposed a new multi-function parallel (network) hierarchical structure to more accurately reflect the complex internal mechanisms and procedures of large organisations. These typically consist of multiple departments that could, in turn, be extended into a number of distinctive operational functions, arranged either in series or in parallel or in a hierarchical structure. These components consume and generate resources that can be interactive and/or independent.

The conventional black-box model evaluates a company (system), while ignoring its internal operations. **Kao's (2012)** model evaluates the constituent departments (sub-systems) of a company, which are independent amongst them. However, it still handles the internal structure of each department as a black-box case. **Kao's (2015)** model has successfully considered the internal processes of a single-stage system as a multi-layer hierarchical structure, yet it ignores that each department may have its own complex structure. The above models did not recognise the necessity of assessing a company, in which the network scheme might intertwine with a hierarchical structure. **Gan et al. (2019)** are one of the first to adopt such a notion, enabling the sub-systems to be interdependent. The current study presents an alternative to **Gan et al.**

Table 4: Efficiency scores of additive decomposition model **(4)**

DMU	E_o^{HN} [Rank]	$E_o^{(A-U-T)}$	$\omega^{(A-U-T)}$	$E_o^{(A-P-T)}$	$\omega^{(A-P-T)}$	$E_o^{(A-T)}$	$\omega^{(A-T)}$	$E_o^{(A-R)}$	$\omega^{(A-R)}$	$E_o^{(A-E)}$	$\omega^{(A-E)}$	$E_o^{(A)}$	$\omega^{(A)}$
1	0.7713 [4]	0.4768	0.391	1	0.609	0.7952	0.333	0.8827	0.333	0.5472	0.333	0.7417	0.236
2	0.4399 [15]	0.1475	0.333	0.3475	0.667	0.2808	0.333	0.3285	0.333	0.2616	0.333	0.2903	0.231
3	0.3723 [17]	0.1956	0.333	0.4296	0.667	0.3516	0.333	0.4488	0.333	0.1200	0.333	0.3068	0.250
4	0.4164 [16]	0.1452	0.333	0.4085	0.667	0.3207	0.333	0.3056	0.333	0.2255	0.333	0.2839	0.244
5	0.4937 [14]	0.1906	0.333	0.5908	0.667	0.4574	0.333	0.3427	0.333	0.2523	0.333	0.3508	0.244
6	0.5495 [11]	0.1380	0.333	0.7320	0.667	0.5340	0.333	0.4548	0.333	0.2714	0.333	0.4201	0.250
7	0.3459 [19]	0.0778	0.333	0.2911	0.667	0.2200	0.333	0.2197	0.333	0.0464	0.333	0.1620	0.250
8	0.7274 [7]	0.2637	0.333	1	0.667	0.7546	0.333	0.9642	0.333	0.2630	0.333	0.6606	0.250
9	0.6593 [8]	0.7211	0.333	1	0.667	0.9070	0.333	1	0.333	0.1861	0.333	0.6977	0.458
10	0.5628 [10]	0.8729	0.510	1	0.490	0.9352	0.333	1	0.333	0.1565	0.333	0.6973	0.469
11	0.8091 [3]	0.3555	0.336	1	0.664	0.7833	0.333	0.8440	0.333	0.7797	0.333	0.8023	0.250
12	0.3574 [18]	0.3129	0.480	0.4773	0.520	0.3984	0.333	0.2504	0.333	0.1544	0.333	0.2677	0.216
13	0.5081 [13]	0.2134	0.333	0.7266	0.667	0.5555	0.333	0.6052	0.333	0.4880	0.333	0.5496	0.369
14	0.9968 [1]	1	0.667	0.8563	0.333	0.9521	0.333	1	0.333	1	0.333	0.9840	0.200
15	0.3365 [20]	0.2684	0.392	0.4506	0.608	0.3792	0.333	0.1374	0.333	0.8063	0.333	0.4410	0.472
16	0.8730 [2]	1	0.667	0.6394	0.333	0.8798	0.333	0.7451	0.333	1	0.333	0.8750	0.322
17	0.7462 [5]	0.6793	0.377	0.8775	0.623	0.8029	0.333	1	0.333	0.7208	0.333	0.8412	0.476
18	0.6446 [9]	0.3069	0.333	1	0.667	0.7690	0.333	0.3263	0.333	0.2707	0.333	0.4553	0.250
19	0.5130 [12]	0.5508	0.412	0.5241	0.588	0.5351	0.333	0.5363	0.333	0.4162	0.333	0.4959	0.322
20	0.7315 [6]	1	0.667	0.6652	0.333	0.8884	0.333	1	0.333	0.7611	0.333	0.8832	0.500

Table 4 Continued:

DMU	$E_o^{(B-U-T)}$	$\omega^{(B-U-T)}$	$E_o^{(B-P-T)}$	$\omega^{(B-P-T)}$	$E_o^{(B-T)}$	$\omega^{(B-T)}$	$E_o^{(B-R)}$	$\omega^{(B-R)}$	$E_o^{(B-E)}$	$\omega^{(B-E)}$	$E_o^{(B)}$	$\omega^{(B)}$
1	0.0848	0.333	1	0.667	0.6949	0.333	0.8968	0.333	0.8115	0.333	0.8010	0.471
2	0.0849	0.333	0.4287	0.667	0.3141	0.333	0.5092	0.333	0.7754	0.333	0.5329	0.463
3	0.0535	0.333	0.3711	0.667	0.2652	0.333	0.6204	0.333	0.4892	0.333	0.4583	0.500
4	0.0429	0.333	0.5566	0.667	0.3853	0.333	0.5723	0.333	0.5014	0.333	0.4863	0.488
5	0.3577	0.333	0.5969	0.667	0.5172	0.333	0.6301	0.333	0.6731	0.333	0.6068	0.489
6	0.1755	0.333	0.3927	0.667	0.3203	0.333	0.6284	0.333	0.3974	0.333	0.4487	0.250
7	0.1325	0.333	0.4818	0.667	0.3654	0.333	0.6274	0.333	0.4470	0.333	0.4799	0.500
8	0.1453	0.333	0.6761	0.667	0.4992	0.333	1	0.333	0.8771	0.333	0.7921	0.500
9	0.3903	0.333	0.9030	0.667	0.7321	0.333	0.6186	0.333	0.3447	0.333	0.5651	0.229
10	0.6364	0.667	0.2361	0.333	0.5030	0.333	0.3011	0.333	0.2900	0.333	0.3647	0.235
11	1	0.336	1	0.664	1	0.333	0.4883	0.333	0.4758	0.333	0.6547	0.250
12	0.2623	0.607	0.2578	0.393	0.2606	0.333	0.4168	0.333	0.4037	0.333	0.3604	0.352
13	0.6235	0.518	0.5960	0.482	0.6102	0.333	0.8125	0.333	0.0481	0.333	0.4903	0.355
14	1	0.667	1	0.333	1	0.333	1	0.333	1	0.333	1	0.400
15	0.4692	0.666	0.2160	0.334	0.3847	0.333	0.4655	0.333	0.1151	0.333	0.3218	0.291
16	0.9943	0.590	1	0.410	0.9967	0.333	1	0.333	1	0.333	0.9989	0.452
17	1	0.580	1	0.420	1	0.333	0.2865	0.333	0.7822	0.333	0.6896	0.238
18	1	0.667	0.3055	0.333	0.7685	0.333	1	0.333	0.6451	0.333	0.8045	0.500
19	0.4950	0.333	0.7000	0.667	0.6316	0.333	0.3556	0.333	0.1488	0.333	0.3787	0.260
20	0.5313	0.337	0.7274	0.663	0.6613	0.333	0.6470	0.333	0.1800	0.333	0.4961	0.250

Table 4 Continued:

DMU	$E_o^{(D-U-T)}$	$\omega^{(D-U-T)}$	$E_o^{(D-P-T)}$	$\omega^{(D-P-T)}$	$E_o^{(D-T)}$	$\omega^{(D-T)}$	$E_o^{(D-R)}$	$\omega^{(D-R)}$	$E_o^{(D-E)}$	$\omega^{(D-E)}$	$E_o^{(D)}$	$\omega^{(D)}$
1	1	0.488	0.9247	0.512	0.9614	0.333	0.5586	0.333	0.7214	0.333	0.7471	0.293
2	0.4453	0.496	0.4642	0.504	0.4548	0.333	0.4694	0.333	0.3128	0.333	0.4123	0.306
3	0.3075	0.667	0.1326	0.333	0.2492	0.333	0.3056	0.333	0.2424	0.333	0.2657	0.250
4	0.3510	0.333	0.6143	0.667	0.5265	0.333	0.4900	0.333	0.2125	0.333	0.4097	0.268
5	0.7034	0.667	0.3717	0.333	0.5929	0.333	0.1837	0.333	0.4764	0.333	0.4177	0.267
6	0.5080	0.645	0.4611	0.355	0.4913	0.333	1	0.333	0.5028	0.333	0.6647	0.500
7	0.3368	0.667	0.1442	0.333	0.2726	0.333	0.1723	0.333	0.3402	0.333	0.2617	0.250
8	0.7681	0.667	0.4200	0.333	0.6521	0.333	0.3425	0.333	1	0.333	0.6649	0.250
9	0.3765	0.488	0.4541	0.512	0.4162	0.333	1	0.333	0.5994	0.333	0.6719	0.313
10	0.2493	0.333	0.7345	0.667	0.5727	0.333	0.3407	0.333	0.6064	0.333	0.5066	0.296
11	1	0.667	1	0.333	1	0.333	1	0.333	0.6688	0.333	0.8896	0.500
12	0.5873	0.353	0.6226	0.647	0.6101	0.333	0.3065	0.333	0.2825	0.333	0.3997	0.432
13	0.6513	0.667	0.0873	0.333	0.4633	0.333	0.4339	0.333	0.5291	0.333	0.4754	0.276
14	1	0.656	1	0.344	1	0.333	1	0.333	1	0.333	1	0.400
15	0.3270	0.667	0.1785	0.333	0.2775	0.333	0.0354	0.333	0.1243	0.333	0.1457	0.236
16	0.6904	0.667	0.1607	0.333	0.5138	0.333	1	0.333	0.3410	0.333	0.6183	0.226
17	0.8096	0.483	0.8220	0.517	0.8160	0.333	0.8475	0.333	0.2415	0.333	0.6350	0.286
18	1	0.667	0.4764	0.333	0.8255	0.333	0.5185	0.333	0.1982	0.333	0.5141	0.250
19	0.3354	0.333	0.4810	0.667	0.4325	0.333	0.7108	0.333	0.6858	0.333	0.6097	0.418
20	0.0643	0.337	0.3011	0.663	0.2213	0.333	0.7700	0.333	1	0.333	0.6638	0.250

Table 5: Efficiency scores of multiplicative aggregation model **(11)**

DMU	$E_o^{HN'}$ [Rank]	$E_o^{(A-U-T)}$	$\omega^{(A-U-T)}$	$E_o^{(A-P-T)}$	$\omega^{(A-P-T)}$	$E_o^{(A-T)}$	$\omega^{(A-T)}$	$E_o^{(A-R)}$	$\omega^{(A-R)}$	$E_o^{(A-E)}$	$\omega^{(A-E)}$	$E_o^{(A)}$	$\omega^{(A)}$
1	0.4452 [4]	0.4816	0.383	1	0.617	0.8016	0.333	0.8830	0.333	0.5435	0.333	0.7427	0.434
2	0.0672 [15]	0.1845	0.333	0.3580	0.667	0.3002	0.333	0.3301	0.333	0.2526	0.333	0.2943	0.409
3	0.0383 [18]	0.1961	0.333	0.4530	0.667	0.3673	0.333	0.4530	0.333	0.1181	0.333	0.3128	0.448
4	0.0589 [16]	0.0962	0.333	0.4599	0.667	0.3387	0.333	0.2363	0.333	0.3013	0.333	0.2921	0.348
5	0.0928 [14]	0.1307	0.333	0.6429	0.667	0.4721	0.333	0.2767	0.333	0.3411	0.333	0.3633	0.321
6	0.1443 [10]	0.0989	0.333	0.8113	0.667	0.5738	0.333	0.4644	0.333	0.3628	0.333	0.4670	0.374
7	0.0209 [20]	0.0825	0.333	0.2889	0.667	0.2201	0.333	0.2261	0.333	0.0452	0.333	0.1638	0.474
8	0.3538 [6]	0.2609	0.333	1	0.667	0.7536	0.333	0.9655	0.333	0.2649	0.333	0.6613	0.402
9	0.2713 [8]	0.7361	0.333	1	0.667	0.9120	0.333	1	0.333	0.1867	0.333	0.6996	0.313
10	0.1324 [11]	0.8872	0.510	1	0.490	0.9425	0.333	1	0.333	0.1565	0.333	0.6997	0.278
11	0.4724 [3]	0.3642	0.333	1	0.667	0.7881	0.333	0.8469	0.333	0.7851	0.333	0.8067	0.461
12	0.0435 [17]	0.2029	0.357	0.5062	0.643	0.3980	0.333	0.3047	0.333	0.1962	0.333	0.2996	0.362
13	0.1292 [12]	0.2148	0.333	0.7287	0.667	0.5574	0.333	0.6051	0.333	0.4870	0.333	0.5498	0.321
14	0.9869 [1]	1	0.667	0.8824	0.333	0.9608	0.333	1	0.333	1	0.333	0.9869	0.483
15	0.0222 [19]	0.1868	0.333	0.4506	0.667	0.3627	0.333	0.1186	0.333	0.8571	0.333	0.4461	0.365
16	0.5456 [2]	1	0.667	0.6477	0.333	0.8826	0.333	0.7495	0.333	1	0.333	0.8774	0.293
17	0.3705 [5]	0.6833	0.393	0.8774	0.607	0.8012	0.333	1	0.333	0.7316	0.333	0.8443	0.360
18	0.1946 [9]	0.2947	0.333	1	0.667	0.7649	0.333	0.3093	0.333	0.3125	0.333	0.4622	0.347
19	0.1178 [13]	0.4716	0.374	0.5242	0.626	0.5046	0.333	0.5671	0.333	0.4384	0.333	0.5033	0.378
20	0.2952 [7]	1	0.667	0.6666	0.333	0.8888	0.333	1	0.333	0.7660	0.333	0.8849	0.272

Table 5 Continued:

DMU	$E_o^{(B-U-T)}$	$\omega^{(B-U-T)}$	$E_o^{(B-P-T)}$	$\omega^{(B-P-T)}$	$E_o^{(B-T)}$	$\omega^{(B-T)}$	$E_o^{(B-R)}$	$\omega^{(B-R)}$	$E_o^{(B-E)}$	$\omega^{(B-E)}$	$E_o^{(B)}$	$\omega^{(B)}$
1	0.0850	0.333	1	0.667	0.6950	0.333	0.8965	0.333	0.8126	0.333	0.8014	0.236
2	0.0874	0.333	0.4284	0.667	0.3148	0.333	0.5080	0.333	0.7799	0.333	0.5342	0.288
3	0.0538	0.333	0.3669	0.667	0.2625	0.333	0.6251	0.333	0.4911	0.333	0.4596	0.283
4	0.0429	0.333	0.5646	0.667	0.3907	0.333	0.5725	0.333	0.5014	0.333	0.4882	0.273
5	0.3627	0.333	0.6064	0.667	0.5251	0.333	0.6301	0.333	0.6731	0.333	0.6095	0.324
6	0.1997	0.333	0.4472	0.667	0.3647	0.333	0.5865	0.333	0.4412	0.333	0.4641	0.329
7	0.1334	0.333	0.4879	0.667	0.3697	0.333	0.6281	0.333	0.4470	0.333	0.4816	0.250
8	0.1447	0.333	0.6757	0.667	0.4987	0.333	1	0.333	0.8786	0.333	0.7924	0.310
9	0.3654	0.333	0.9680	0.667	0.7672	0.333	0.6188	0.333	0.3438	0.333	0.5766	0.374
10	0.6871	0.667	0.2489	0.333	0.5410	0.333	0.2954	0.333	0.2754	0.333	0.3706	0.309
11	1	0.617	1	0.383	1	0.333	0.4946	0.333	0.4792	0.333	0.6579	0.309
12	0.2623	0.607	0.2579	0.393	0.2606	0.333	0.4168	0.333	0.4037	0.333	0.3604	0.305
13	0.6235	0.515	0.5965	0.485	0.6104	0.333	0.8214	0.333	0.0480	0.333	0.4932	0.310
14	1	0.493	1	0.507	1	0.333	1	0.333	1	0.333	1	0.267
15	0.4704	0.644	0.2412	0.356	0.3887	0.333	0.4580	0.333	0.1203	0.333	0.3223	0.258
16	0.9966	0.591	1	0.409	0.9980	0.333	1	0.333	1	0.333	0.9993	0.277
17	1	0.580	1	0.420	1	0.333	0.2872	0.333	0.7822	0.333	0.6898	0.381
18	1	0.667	0.3234	0.333	0.7745	0.333	1	0.333	0.6648	0.333	0.8131	0.359
19	0.5073	0.333	0.7152	0.667	0.6459	0.333	0.3574	0.333	0.1484	0.333	0.3839	0.330
20	0.5324	0.333	0.7287	0.667	0.6633	0.333	0.6545	0.333	0.1812	0.333	0.4996	0.311

Table 5 Continued:

DMU	$E_o^{(D-U-T)}$	$\omega^{(D-U-T)}$	$E_o^{(D-P-T)}$	$\omega^{(D-P-T)}$	$E_o^{(D-T)}$	$\omega^{(D-T)}$	$E_o^{(D-R)}$	$\omega^{(D-R)}$	$E_o^{(D-E)}$	$\omega^{(D-E)}$	$E_o^{(D)}$	$\omega^{(D)}$
1	1	0.488	0.9301	0.512	0.9642	0.333	0.5586	0.333	0.7214	0.333	0.7481	0.330
2	0.4370	0.530	0.4420	0.470	0.4394	0.333	0.5577	0.333	0.2859	0.333	0.4277	0.303
3	0.3075	0.667	0.1357	0.333	0.2502	0.333	0.3056	0.333	0.2424	0.333	0.2661	0.269
4	0.3510	0.333	0.6236	0.667	0.5327	0.333	0.4940	0.333	0.2125	0.333	0.4131	0.379
5	0.7034	0.667	0.3835	0.333	0.5968	0.333	0.1837	0.333	0.4764	0.333	0.4190	0.355
6	0.4934	0.667	0.4155	0.333	0.4675	0.333	1	0.333	0.5299	0.333	0.6658	0.298
7	0.3540	0.667	0.1604	0.333	0.2894	0.333	0.1609	0.333	0.3460	0.333	0.2655	0.276
8	0.7370	0.667	0.3613	0.333	0.6118	0.333	0.4138	0.333	1	0.333	0.6752	0.289
9	0.3765	0.488	0.4588	0.512	0.4186	0.333	1	0.333	0.5994	0.333	0.6727	0.313
10	0.2493	0.333	0.7526	0.667	0.5848	0.333	0.3407	0.333	0.6064	0.333	0.5106	0.413
11	1	0.572	1	0.428	1	0.333	1	0.333	0.6704	0.333	0.8901	0.230
12	0.4981	0.402	0.6163	0.598	0.5688	0.333	0.3349	0.333	0.3060	0.333	0.4032	0.332
13	0.6513	0.667	0.0896	0.333	0.4640	0.333	0.4365	0.333	0.5291	0.333	0.4765	0.369
14	1	0.403	1	0.597	1	0.333	1	0.333	1	0.333	1	0.250
15	0.3005	0.667	0.1715	0.333	0.2575	0.333	0.0460	0.333	0.1597	0.333	0.1544	0.377
16	0.7348	0.667	0.1521	0.333	0.5406	0.333	1	0.333	0.3262	0.333	0.6222	0.431
17	0.8096	0.483	0.8290	0.517	0.8196	0.333	0.8475	0.333	0.2415	0.333	0.6362	0.258
18	1	0.667	0.4758	0.333	0.8253	0.333	0.5439	0.333	0.1844	0.333	0.5178	0.294
19	0.3354	0.333	0.4818	0.667	0.4330	0.333	0.7108	0.333	0.6858	0.333	0.6099	0.291
20	0.0661	0.333	0.3066	0.667	0.2264	0.333	0.7764	0.333	1	0.333	0.6676	0.417

(2019), by proposing an embedded hierarchical network structure within a multi-function parallel system. In our proposed scheme, the constituent sub-systems act independently from one another accommodating another class of problems.

In the current study, DMUs have a parallel (network) hierarchical structure. On a macro level, the external layer is associated with the action of retrieving data from the entire system, whereas the internal layer from each of the sub-systems connected in parallel. On a micro level, that is the interior part of a sub-system, we evaluate the constituent units that form a multi-level multi-function hierarchical structure.

To measure and evaluate the performance of DMUs with such a structure, we propose an additive decomposition model **(4)** and a multiplicative aggregation model **(11)**. In both models, we obtain the system, the sub-systems, and their internal units' efficiencies as well as identify their relationship. In particular, the efficiency of a unit at a higher level is the weighted average of those of the subordinates at the immediate lower level; the weight of that unit is the proportion of the input consumed by that subordinate in that consumed by all subordinates. For the additive model **(4)**, the overall efficiency is decomposed into the weighted arithmetic average of those of the parallel sub-systems. It can also be expressed as the weighted average of the efficiencies of the terminal units that belong to the hierarchical structure of each sub-system. For the multiplicative model **(11)**, the system efficiency is defined as the product of the efficiencies of the constituent sub-systems. We have also proven that the system efficiency of model **(4)** is always greater than or equal to the respective one in model **(11)**.

The performance measurement and evaluation of several Business Schools across a number of universities illustrates the proposed models. These models allow us to not only discriminate the efficient units, but also to simultaneously calculate the efficiencies of the Business School of interest, its departments, and the functions within the respective department. Hence, decision-makers will be enabled to take certain actions by improving the areas of weakness.

Other areas of application of the proposed structure may include performance evaluation of business functions such as human resources, accounting and finance, marketing, and supply chain. The supply chain management of an organisation, for instance, ensures that goods and services get to customers in the easiest way possible. Such a department could be decomposed into several independent operations such as production, procurement, logistics, and customer service. These operations could be hierarchically divided into people's responsibilities, tasks, and values. The main target is to meaningfully compare the efficiency of several parallel (network) hierarchical supply chains of different factory branches inside and outside the country. Identifying and improving the areas of weakness of the most ineffective supply chains, could reduce operating costs, increase the quality of products, and meet customers' needs.

Another promising area could be the evaluation of the operating performance of a commercial ship. Stakeholders from the shipping industry might be interested in, for instance, exploring the most desirable ship design associated with a valid scenario, in which the maintenance policy is an integral part. The corresponding maintenance policy could be operationalised through the various input and output factors. A ship cannot operate without the effective management of its constituent sub-systems (electrical, diesel propulsion, lube oil, heavy fuel oil, deck) incorporated within its hull and deck. Because of the complex layout of the ship, its overall management system can integrate both the multi-function parallel network and the multi-layer hierarchical structures.

Our methodology could have also supported the assessment of the agricultural sector either in local or international level. For instance, the extended efforts for increased environmental protection as well as sustainable resources use, could be achieved through inputs allocation due to the fact that the current methodology does not assume that the production procedure is a black-box but assesses the internal mechanisms of the process at various levels. However, due to the fact that the proposed model assumes a constant returns to scale relationship between inputs and outputs, it is not immediately applicable in the agricultural sector, where variable returns to scale is the most common practice **(Theodoridis and Psychoudakis, 2008; Bournaris et al., 2019)**. In other words, if the amount of the fertiliser used is tripled, the final production does not increase proportionally. Modifications for embodying variable returns to scale should be made, to acquire results of greater value for agriculture related professionals (farmers, agromanagers, agriculturists) and policy makers. The aforementioned example with fertiliser provides another dimension about the resources overuse, where the amount of unused fertiliser will result in leaching, creating negative impacts for environment and local communities. By this means, unused fertilisers should be treated as undesirable outputs of the whole process.

It is also worthwhile to point out that the dataset used in Section 4 was based in part on Kao (2015), and has been extended by taking random samples for each of the additional output factors that include integer values in the range of [1 to 330]. The goal of this dataset was to indicate how the theoretical network hierarchical DEA structure is applied to an illustrative example in the higher education sector. Other methods used in the literature aim to develop multiple input-output production frontiers and bring more structure and accuracy in the generation of instances, such as the piecewise Cobb-Douglas and the cubic polynomial production functions **(Banker et al., 1993; Giraleas et al., 2012; Khezrimotlagh, 2022)**.

Among the models proposed in this study to measure the performance of the multi-function parallel network hierarchical system, the multiplicative efficiency aggregation model is the only non-linear network DEA formulation due to its non-linear objective function. Although the majority of non-linear solvers can run flexibly **(Kao, 2018)**, the model is still considered computationally complex and a global optimal solution cannot be easily guaranteed. Alternative algorithms can be used by transforming the model into either a second order cone programming or a semi-definite programming problem, following the spirit of **Chen and**

Zhu (2017) and **Kuo et al. (2020)** or **Zhang and Chen (2019)**, respectively. The aforementioned techniques lie in the field of convex optimization, see also **Boyd et al. (2004)** and **Zhu (2020)**.

The discussion of both modelling approaches in this paper is under the constant returns to scale assumption. This can be expanded to variable returns to scale situation for the additive model. Another challenge for future research could be the evaluation of a system that requires the integration of a hierarchical structure into other more complex network processes, such as assembly and disassembly, mixed, and dynamic systems **(Cook et al., 2010; Kao, 2016; Kao, 2017)**. It would also be interesting to develop appropriate DEA modelling techniques, which will acknowledge that not all competing DMUs have exactly the same internal structure.

Finally, current research studies the evaluation of the performance of DMUs with a multi-function parallel network hierarchical structure, only when the data are positive real numbers, and the DEA models are based on this condition. Future research could relax this assumption by allowing the data points (inputs, intermediate measures, and outputs) to be imprecise and lie in an interval. Other cases to be investigated concern missing data or intervals, where some values are more likely to occur over other values. In the latter case, since there is no information of the probability distributions, fuzzy numbers and mathematical operations **(Zimmermann, 2011)** could be used as an alternative option.

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Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: