UNIVERSITY OF SOUTHAMPTON

Energy methods for lossless systems using quadratic differential forms

by

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ABSTRACT

FACULTY OF ENGINEERING, SCIENCE AND MATHEMATICS
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In this thesis, we study the properties of lossless systems using the concept of quadratic differential forms (QDFs). Based on observation of physical linear lossless systems, we define a lossless system as one for which there exists a QDF known as an energy function that is positive along nonzero trajectories of the system and whose derivative along the trajectories of the system is zero if inputs to the system are made equal to zero. Using this definition, we prove that if a lossless system is autonomous, then it is oscillatory. We also give an algorithm whose output is a two-variable polynomial that induces an energy function of a lossless system and we describe a suitable way of splitting a given energy function into its potential and kinetic energy components. We further study the space of QDFs for an autonomous linear lossless system, and note that this space can be decomposed into the spaces of conserved and zero-mean quantities. We then show that there is a link between zero-mean quantities and generalized Lagrangians of an autonomous linear lossless system.

Finally, we study various methods of synthesis of lossless electric networks like Cauer and Foster methods, and come up with an abstract definition of synthesis of a positive QDF that represents the total energy of the network to be synthesized. We show that Cauer and Foster method of synthesis can be cast in the framework of our definition. We show that our definition has applications in stability tests for linear systems, and we also give a new Routh-Hurwitz like stability test.

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Declaration of Authorship

I, Shodhan Rao, declare that the thesis entitled "Energy methods for lossless systems using quadratic differential forms" and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
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- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:

  - S. Rao and P. Rapisarda. "Bases of conserved and zero-mean quadratic quantities for linear oscillatory systems". In *Proceedings of 17th International Symposium on Mathematical Theory of Networks and Systems*, pages 2524-2534, Kyoto, Japan, 2006.

  - S. Rao and P. Rapisarda. "Autonomous linear lossless systems". In *Proceedings of 17th IFAC World Congress*, pages 5891-5896, Seoul, Korea, 2008.

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Chapter 1

Introduction

The state space method and the transfer function method are the most popular methods used in the study of theory of linear dynamical systems and control. In the state space method, a state space model consisting of first order differential equations is first constructed out of the given model of the system. Most often in this method, apart from the input and output variables, another set of variables called states are introduced artificially, i.e they do not occur naturally while modelling the system. Thus the state space method is not a natural approach of dealing with linear systems. The transfer function method on the other hand is based on a *cause-effect* principle, where it is assumed that certain variables are inputs (cause) and the remaining are outputs (the effect of the inputs). In many cases, for the study of dynamical systems it is unnecessary to classify variables of the system as inputs and outputs. For example, consider Charles law in thermodynamics which states that at constant pressure, the volume (V) of a given mass of an ideal gas is directly proportional to its temperature (T). However, this law does not state which among V and T is the input and which is the output, or in other words it does not state which is the cause and which is the effect. In cases like these, the transfer function method forces us to treat some of the variables as inputs and the rest as outputs, which is clearly unnecessary. Thus the state space and transfer function methods have their own limitations owing to the fact that they do not involve a natural way of arriving at a mathematical model for a system.

The most natural way of writing a mathematical model of a linear system involves identification of subsystems within the given system and writing the governing laws which are differential equations in the system variables, for each of the subsystems using first principles. This method of mathematical modelling is often referred to as *tearing and zooming*. *Tearing* refers to the first step in modelling which is identification of subsystems within the given system and *zooming* refers to the process of writing the governing laws for each of the subsystems. This process does not involve classification of variables as inputs and outputs beforehand. For example, to construct the mathematical model of an electrical network from first principles, one way is to write Kirchoff's voltage laws for various branches and Kirchoff's current laws for various nodes. In this case, identifying the various branches, the ideal components within the branches like resistors, inductors, capacitors etc., and nodes of the given network constitutes the process of tearing. Writing the governing laws for each of the identified ideal components and using these to write Kirchoff's voltage and current laws for each of the branches and nodes respectively, constitutes the process of zooming in this case. Similarly, in order to construct the mathematical model of a mechanical spring-mass-damper system, one way is to draw the free body diagram of each of the masses involved and write Newton's laws of motion for each mass considering the forces acting on it. In this case, drawing the free body diagram of each of the masses constitutes tearing and writing the Newton's laws of motion for each of the masses constitutes zooming. After the process of tearing and zooming, in order to arrive at the model for the whole system, we combine the governing laws for the different subsystems using interconnection laws between the subsystems.

In this process, we come across two kinds of variables namely *manifest* variables and *latent* variables. Manifest variables are those variables whose evolution is of interest to us, while latent variables are the remaining ones that come up during the modelling process. For example, while modelling a complex electrical network with one voltage source, we might be interested in the evolution of only the voltage across the source and the current through it, hence these are the manifest variables. Modelling the network invariably involves the voltages across and currents through various branches, which are the latent variables. In order to describe the evolution of only the manifest variables, one needs to eliminate the latent variables. This elimination generally leads to higher order differential equations in the manifest variables. We now illustrate this using the example of a simplified mathematical model of a three-storey building.

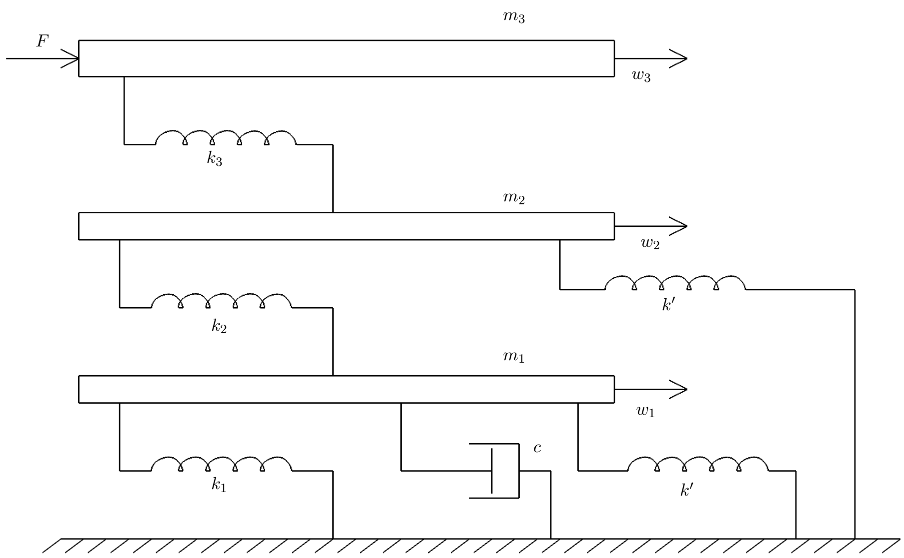

FIGURE 1.1: Model of a three-storey building

Example 1.1. Consider the mathematical model of a three-storey building, where each of the storeys is modelled as a concentrated mass. Let the masses of the three storeys be m_1, m_2 and m_3. It is assumed that the first and the second storeys are below the ground level, so that these are not affected by wind force. A force F caused by wind acts on the third storey. We model the stiffness and damping of the building and soil as shown in Figure 1.1. Assume that the stiffness of soil is k'. In addition to k', assume that a spring of stiffness k_1 and a damper of damping coefficient c_1 opposes the horizontal motion of the first storey with respect to the ground; a spring of stiffness k_2 opposes the horizontal motion of the second storey with respect to the first in addition to the soil stiffness k' opposing the motion of the second storey with respect to the ground, and a spring of stiffness k_3 opposes the horizontal motion of the third storey with respect to the second. We denote by w_1, w_2 and w_3, the horizontal displacements of the three storeys with respect to the ground. The constitutive laws for this system are obtained by writing the equations of motion for the three masses.

$$m_1\frac{d^2w_1}{dt^2} = k_2(w_2 - w_1) - c\frac{dw_1}{dt} - (k_1 + k')w_1 \tag{1.1}$$

$$m_2\frac{d^2w_2}{dt^2} = k_3(w_3 - w_2) - k_2(w_2 - w_1) - k'w_2 \tag{1.2}$$

$$m_3\frac{d^2w_3}{dt^2} = F - k_3(w_3 - w_2) \tag{1.3}$$

In this case, the model consists of four variables - w_1, w_2, w_3 and F. Let us assume that the variables of interest are F and w_3, i.e we are interested to know how wind force affects the horizontal displacement of the third storey. In this case, for the system given by equations (1.1), (1.2) and (1.3), w_3 and F can be called the manifest variables and w_1 and w_2 can be called the latent variables. We next illustrate the process of elimination of the latent variables. From equation (1.3), we obtain

$$w_2 = \frac{1}{k_3}\left(m_3\frac{d^2w_3}{dt^2} + k_3w_3 - F\right) \tag{1.4}$$

Substituting this expression for w_2 in equation (1.2), we obtain

$$\begin{aligned} w_1 = \frac{1}{k_2}&\left[\frac{m_2m_3}{k_3}\left(\frac{d^4w_3}{dt^4}\right) + \left(m_2 + m_3 + \frac{m_3(k'+k_2)}{k_3}\right)\frac{d^2w_3}{dt^2}\right] \\ &+ \frac{1}{k_2}\left[(k'+k_2)w_3 - \frac{m_2}{k_3}\left(\frac{d^2F}{dt^2}\right) - \left(\frac{k'+k_2}{k_3} + 1\right)F\right] \end{aligned} \tag{1.5}$$

Substituting expressions for w_2 and w_1 from equations (1.4) and (1.5) respectively in equation (1.1), we obtain

$$\frac{m_1m_2m_3}{k_2k_3}\left(\frac{d^6w_3}{dt^6}\right) + \frac{cm_2m_3}{k_2k_3}\left(\frac{d^5w_3}{dt^5}\right)$$

$$
\begin{aligned}
&+\left[\frac{m_1(m_2+m_3)}{k_2}+\frac{m_3(m_1+m_2)}{k_3}+\frac{m_2m_3k_1}{k_2k_3}+\frac{m_1m_3k'}{k_2k_3}\right]\frac{d^4w_3}{dt^4}\\
&+\frac{c}{k_2}\left(m_2+m_3+\frac{m_3(k'+k_2)}{k_3}\right)\frac{d^3w_3}{dt^3}\\
&+\left[m_1+m_2+m_3+\frac{k_1(m_2+m_3)}{k_2}+\frac{m_3k_1}{k_3}+\frac{k'm_3}{k_3}\left(1+\frac{k_1}{k_2}\right)+\frac{m_1k'}{k_2}\right]\frac{d^2w_3}{dt^2}\\
&+c\left(1+\frac{k'}{k_2}\right)\frac{dw_3}{dt}+\left(k_1+k'+\frac{k'k_1}{k_2}\right)w_3=\frac{m_1m_2}{k_2k_3}\left(\frac{d^4F}{dt^4}\right)+\frac{m_2c}{k_2k_3}\left(\frac{d^3F}{dt^3}\right)\\
&+\left(\frac{m_1+m_2}{k_3}+\frac{m_1}{k_2}+\frac{m_2k_1}{k_2k_3}+\frac{m_1k'}{k_2k_3}\right)\frac{d^2F}{dt^2}+c\left(\frac{1}{k_3}+\frac{1}{k_2}+\frac{k'}{k_2k_3}\right)\frac{dF}{dt}\\
&+\left[1+k_1\left(\frac{1}{k_2}+\frac{1}{k_3}\right)+\frac{k'}{k_3}\left(1+\frac{k_1}{k_2}\right)\right]F \qquad (1.6)
\end{aligned}
$$

Thus elimination of latent variables in this case leads to a differential equation of order six in w_3 and order four in F. This example shows that mathematical modelling of a system does not automatically lead to first order equations or a state space model for the system. A state space model in this case needs construction of artificial state variables from the given model.

A popular method for dealing with mechanical systems is to consider a second order model for the system. Example 1.1 shows that elimination of latent variables leads to higher order differential equations and hence there is a need to address problems at the level of higher order differential equations provided by the modeler and not force the modeler to provide a set of first or second order differential equations for the system.

The behavioural approach proposed by Jan Willems overcomes the shortcomings of state space method, transfer function method and second order approach for mechanical systems. In this approach, a system is considered as an exclusion law indicating that the system trajectories can only belong to a certain subset of the signal space. The set of admissible trajectories for the system variables is called the *behaviour* of the system. The main advantage of this method is that it is a representation-free method, meaning that it is not based on a particular representation for the system. The same system can be represented in different ways like a set of first order equations (state space model), a set of second order equations or a set of higher order differential equations, based on the applications. In the behavioural approach, an input/output classification is not presumed beforehand. It rather lets the mathematical structure of the system decide the inputs and the outputs. This will be explained in section 2.5 of this thesis. For a thorough exposition of the concepts of behavioural theory of systems, the reader is referred to Polderman and Willems (1997).

Now reconsider Example 1.1 and observe that the power (P) or the rate at which energy is supplied to the system is given by

$$P = F\frac{dw_3}{dt}$$

Part of this energy gets stored in the springs whilst the remainder is dissipated as heat energy by the damper. Thus power is equal to the sum of the rate of increase of the total energy the system, which is the summation of kinetic energies of masses m_1, m_2 and m_3, the stored energies of the springs and the rate of dissipation of energy in the damper.

$$\begin{aligned} F\frac{dw_3}{dt} &= \frac{1}{2}\frac{d}{dt}\left(m_1\left(\frac{dw_1}{dt}\right)^2 + m_2\left(\frac{dw_2}{dt}\right)^2 + m_3\left(\frac{dw_3}{dt}\right)^2\right) \\ &+ \frac{1}{2}\frac{d}{dt}\left(k_1 w_1^2 + k_2(w_2 - w_1)^2 + k_3(w_3 - w_2)^2 + k'(w_1^2 + w_2^2)\right) + c\left(\frac{dw_1}{dt}\right)^2 \end{aligned}$$

In the case of absence of the damper in the system, there is no dissipation and hence we may call the system as *lossless*. A natural question that arises here is whether it is possible to deduce whether a system is lossless or not, directly from a higher order description for the system. This is one of the problems tackled in this thesis.

Now reconsider Example 1.1, and assume that in addition to the absence of the damper, the wind force is equal to zero. In this case, the total energy stored in the springs is conserved. For this special case, namely when $F = 0$ and $c = 0$, we obtain

$$\begin{aligned} m_1\left(\frac{dw_1}{dt}\right)^2 + m_2\left(\frac{dw_2}{dt}\right)^2 + m_3\left(\frac{dw_3}{dt}\right)^2 + k_1 w_1^2 + k_2(w_2 - w_1)^2 + k_3(w_3 - w_2)^2 \\ + k'(w_1^2 + w_2^2) = \text{constant} \end{aligned}$$

This may be called as a *conservation law* for the system as the left hand side of the above equation remains constant at all times. Observe that for this special case, equations (1.1), (1.2) and (1.3) reduce to the following equations:

$$m_1\frac{d^2w_1}{dt^2} = k_2(w_2 - w_1) - (k_1 + k')w_1 \tag{1.7}$$

$$m_2\frac{d^2w_2}{dt^2} = k_3(w_3 - w_2) - k_2(w_2 - w_1) - k'w_2 \tag{1.8}$$

$$m_3\frac{d^2w_3}{dt^2} = k_3(w_2 - w_3) \tag{1.9}$$

It is well known that the Lagrangian for a system is the difference between its kinetic and potential energies. For the case of the system described by equations (1.7), (1.8) and (1.9), since the Lagrangian is a function of w_i and $\frac{dw_i}{dt}$, $i = 1, 2, 3$, let

$L(w_1, w_2, w_3, \frac{dw_1}{dt}, \frac{dw_2}{dt}, \frac{dw_3}{dt})$ denote the Lagrangian. Then

$$L(w_1, w_2, w_3, \frac{dw_1}{dt}, \frac{dw_2}{dt}, \frac{dw_3}{dt}) = \frac{1}{2}(m_1\left(\frac{dw_1}{dt}\right)^2 + m_2\left(\frac{dw_2}{dt}\right)^2 + m_3\left(\frac{dw_3}{dt}\right)^2 - k_1 w_1^2 - k_2(w_2 - w_1)^2 - k_3(w_3 - w_2)^2 - k'(w_1^2 + w_2^2)) \quad (1.10)$$

Note that the time-average of the Lagrangian over the entire time-axis is zero, i.e,

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} L(t)dt = 0$$

Quadratic functionals for a system like the Lagrangian whose time-average over the entire time axis is zero will be referred to as *zero-mean quantities.* For $i = 1, 2, 3$, define $\dot{w}_i := \frac{dw_i}{dt}$. We recall that as a consequence of *Hamilton's principle*, the system trajectories satisfy the *Euler-Lagrange equations* given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{w}_i}\right) - \frac{\partial L}{\partial w_i} = 0$$

for $i = 1, 2, 3$. Indeed by substituting the expression for L from equation (1.10) in the above equations, we get the system equations (1.7), (1.8) and (1.9).

Observe that for the system described in example 1.1, the energy, Lagrangian and power are quadratic functionals in the system variables and their derivatives. In the behavioural approach, we call such scalar functionals *quadratic differential forms* (QDFs). One of the aims of this thesis is to study in depth, the notions of *Hamilton's principle*, *conservation laws* and *zero-mean quantities* for linear lossless systems starting from a higher order description of the system using quadratic differential forms. We call the approach for studying these notions as *energy method*, because it is based on the expression for energy of the system obtained from a higher order description of the system. As a starting point, we study lossless systems or systems without any dissipative components using the behavioural approach.

In this thesis, using the behavioural approach, we also explain the underlying mechanism of synthesis methods of lossless electrical networks like Cauer and Foster methods. This also is done using *energy method* as explained below. Here we make use of the fact that both Cauer and Foster methods of synthesis of a given lossless transfer function proceed in steps, each involving the extraction of a reactance component connected in series or parallel with a network having a simpler transfer function. This simpler transfer function is the starting point for the next step of synthesis. Note that the total energy of the network to be synthesized at any step is equal to the sum of the stored energy in the reactance component that is extracted at that step and the total energy of the network to be synthesized in the next step. Also note that the total energy of the network to be synthesized at each step of synthesis is positive along nonzero trajectories of the system.

Thus the total energy of the network to be synthesized reduces at every step of synthesis. We give an example of synthesis of a simple lossless electrical network to explain the common features of Cauer and Foster methods.

Example 1.2. Consider the synthesis of a lossless one-port electrical network with an impedance transfer function given by

$$Z(s) = \frac{s^3 + 3s}{s^2 + 1}$$

One way of synthesis of Z begins by writing Z as the sum of two transfer functions Z_1 and Z_2, where $Z_1(s) = s$ and

$$Z_2(s) = \frac{2s}{s^2 + 1}$$

Note that the network corresponding to the impedance transfer function Z_1 is an inductor with unit inductance. Thus a unit inductance is extracted in the first step of synthesis, to be connected in series with a network with transfer function equal to Z_2, which is simpler than Z in the sense that the sum of the degrees of the numerator and denominator of Z_2 is lower than that of Z. The transfer function Z_2 is the starting point for the second step of synthesis of Z. In the second step of synthesis, the network corresponding to Z_2 is synthesized. Note that this network consists of an inductor of inductance equal to two units connected in parallel with a capacitor with capacitance equal to $\frac{1}{2}$ unit. The network corresponding to the impedance transfer function Z is shown in Figure 1.2.

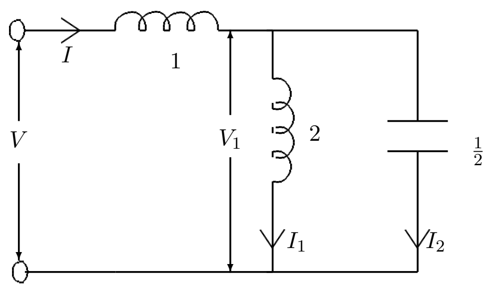

FIGURE 1.2: Example of synthesis

The total energy of the network is given by

$$E = \frac{1}{2}I^2 + I_1^2 + \frac{1}{4}V_1^2$$

Observe that the total energy of the network with impedance transfer function Z_2 is

$$E_1 = I_1^2 + \frac{1}{4}V_1^2$$

Thus E is equal to the sum of E_1 and the energy stored in the inductor component extracted in the first step of the synthesis. If V and I are not equal to zero, observe that both E and E_1 are positive. Also observe that $E \geq E_1$, and the equality occurs only when $I = 0$.

Keeping the above common features of Cauer and Foster synthesis in mind, we define the synthesis of a QDF which denotes the total energy of the network to be synthesized, as a sequence of QDFs, each of which denotes the total energy of the network to be synthesized at a particular step of the synthesis process. We show that Cauer and Foster methods of synthesis can be cast in the framework of our definition. We also show that our definition has applications in stability tests for linear systems. Since we make use of properties of the total energy of the network to be synthesized in order to extract the common features of Cauer and Foster synthesis procedures, we may call the method for studying synthesis as *energy method*.

1.1 Outline of the thesis

In chapter 2 and 3, we cover those aspects of the behavioural approach and quadratic differential forms respectively that are necessary to understand the results presented in chapters 4, 5 and 6. In chapter 2, we explain the notions of dynamical systems with and without latent variables, controllability, observability, input/output partition etc. from a behavioural point of view. Most of the material covered in this chapter can be found in Polderman and Willems (1997). Most of the material of chapter 3 has been taken from Willems and Trentelman (1998), Rapisarda and Trentelman (2004) and Rapisarda and Willems (2005). Some of the notions covered in this chapter are the notions of a QDF being zero along a behaviour, equivalence of QDFs, nonnegativity, positivity and stationarity of a QDF, conserved quantities, etc. In chapter 4, we define the notion of higher order linear lossless systems from a behavioural point of view using energy methods, and study the properties of such systems. In chapter 5, we study the QDFs associated with a special class of systems known as oscillatory systems. In this chapter, we also study the relation between the Lagrangian and zero-mean quantities for oscillatory systems. In chapter 6, we provide an abstract definition for synthesis of QDFs and show that Cauer and Foster methods of synthesis can be cast in the framework of our definition. In this chapter, we also show the application of this definition for testing of stability of linear systems. In chapter 7, we draw conclusions from chapters 4, 5 and 6.

Chapter 2

Linear differential systems

In this chapter, we give an introduction to linear differential systems, and cover the basic concepts that are required to understand the results in this thesis. Most of the material of this chapter is taken from Polderman and Willems (1997) and Rapisarda (1998).

2.1 Dynamical Systems

We begin this chapter with the study of the notion of dynamical system. As described in example 1.1, modelling a system involves writing the governing laws for the system variables and hence defining the way in which the system variables evolve. Consider a system for which w represents the vector of external variables. We denote the time axis by $\mathbb{T}$ and define the signal space $\mathbb{W}$ as the space in which the system variables w take on their values. Let $\mathbb{W}^{\mathbb{T}}$ denote the set of maps from $\mathbb{T}$ to $\mathbb{W}$. The governing laws of the system ensure that not every element of $\mathbb{W}^{\mathbb{T}}$ is allowed by the dynamical laws describing the system. The set of maps that are allowed by the system is called the *behaviour*. This leads to the following definition of a dynamical system.

Definition 2.1. *A dynamical system* Σ *is a triple* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$*, with* $\mathbb{T} \subseteq \mathbb{R}$*, the* time set*,* $\mathbb{W}$ *a set called the* signal space*, and* $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ *the* behaviour *of the system.*

In this thesis, we study dynamical systems that are linear, shift-invariant, and described by ordinary differential equations. Below, we define linearity and shift-invariance of dynamical systems.

Definition 2.2. *A dynamical system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ *is called* linear *if*

- $\mathbb{W}$ *is a vector space over* $\mathbb{R}$ *and*
- *the behaviour* $\mathfrak{B}$ *is a subspace of* $\mathbb{W}^{\mathbb{T}}$

In other words,

$$w_1, w_2 \in \mathfrak{B} \quad \text{and} \quad \alpha_1, \alpha_2 \in \mathbb{R} \Rightarrow \alpha_1 w_1 + \alpha_2 w_2 \in \mathfrak{B}$$

We call Σ *shift-invariant* if $\mathbb{T}$ is closed under addition and the following holds for all $t_1 \in \mathbb{T}$:

$$(w \in \mathfrak{B}) \Rightarrow (\sigma^{t_1}(w) \in \mathfrak{B})$$

where $\sigma^{t_1}(w)(t) := w(t + t_1)$ for all $t \in \mathbb{T}$. In this thesis, we consider only continuous time systems and hence $\mathbb{T} = \mathbb{R}$. A dynamical system whose behaviour is equal to the set of all solutions of a system of constant coefficient linear ordinary differential equations satisfies linearity and shift-invariance. We call such a system, a *linear differential system*. Assume that we have $\mathtt{g}$ linear constant coefficient differential equations in the variable w, and we are interested in the set of trajectories $w : \mathbb{R} \rightarrow \mathbb{R}^{\mathtt{w}}$ that satisfy the equations

$$R_0 w + R_1 \frac{d}{dt} w + \ldots + R_L \frac{d^L}{dt^L} w = 0 \tag{2.1}$$

where $R_i \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$ for $i = 0, 1, \ldots, L$. Define the polynomial matrix $R \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[\xi]$ as

$$R(\xi) := R_0 + R_1 \xi + \ldots + R_L \xi^L \tag{2.2}$$

A concise way of specifying the $\mathtt{g}$ equations in (2.1) is by writing

$$R(\frac{d}{dt}) w = 0$$

The space of trajectories w usually considered in problems involving linear differential systems described by equations of the form (2.1) is either the space of infinitely differentiable trajectories or the space of locally integrable trajectories which we define below.

Definition 2.3. *A Lebesgue measurable function* $w : \mathbb{R} \rightarrow \mathbb{R}^{\mathtt{w}}$ *is called* locally integrable *if for all* $a, b \in \mathbb{R}$,

$$\int_a^b \|w(t)\| \, dt < \infty$$

We denote the space of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ by $\mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

A solution w of equation (2.1) is called a *strong solution* if w is at least L times differentiable. Thus all infinitely differentiable trajectories that satisfy (2.1) are strong solutions. A *weak solution* to the system of differential equations (2.1) is a trajectory w for which all the derivatives occurring in the given set of equations may not exist at all points in $\mathbb{R}$, but which nonetheless satisfies the given set of equations in some precisely defined sense. For example, a locally integrable trajectory w that satisfies (2.1) in a distributional sense is a weak solution of (2.1). This is explained below.

Let R be defined by equation (2.2). Let $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ denote the subset of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of compact support trajectories. Let $\phi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ and $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Define

$$< w, \phi >:= \int_{-\infty}^{\infty} w(t)^\top \phi(t) dt$$

Then it can be shown that if $\psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$,

$$< R(\frac{d}{dt})w, \psi >=< w, R(-\frac{d}{dt})^\top \psi >$$

Thus $R(\frac{d}{dt})w = 0$ if and only if $< w, R(-\frac{d}{dt})^\top \psi >= 0$ for all $\psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$. This property of strong solutions can be used to define a weak solution of (2.1). Any trajectory $w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is a weak solution of the system of equations (2.1) if for all $\psi \in \mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$, there holds $< w, R(-\frac{d}{dt})^\top \psi >= 0$, since such a trajectory satisfies equations (2.1) in a distributional sense. Henceforth, whenever we say that a locally integrable trajectory satisfies a set of constant coefficient linear differential equations, we mean that it does so in a distributional sense.

In this thesis, unless otherwise specified, we are only interested in the infinitely differentiable trajectories that satisfy equation (2.1). We can define this set of solutions, namely the behaviour $\mathfrak{B}$ as

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\}$$

Thus, considering $R(\frac{d}{dt})$ as an operator from $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$, $\mathfrak{B} = \ker\left(R(\frac{d}{dt})\right)$. Linearity of the differential operator $R(\frac{d}{dt})$ results in linearity of the behaviour $\mathfrak{B}$. $\mathfrak{B}$ is shift-invariant as the coefficients of the polynomial matrix $R(\xi)$ are constant. The system of linear constant coefficient differential equations (2.1) is called a *kernel representation* of the behaviour $\mathfrak{B}$. We denote the set of linear differential systems with infinitely often differentiable manifest variable w by $\mathcal{L}^{\mathtt{w}}$ (the superscript $^{\mathtt{w}}$ in $\mathcal{L}^{\mathtt{w}}$ refers to the dimension of $w \in \mathfrak{B}$, i.e $\mathtt{w} = \dim(\mathbb{W})$).

From the following theorem, it follows that the kernel representation of a behaviour is not unique.

Theorem 2.4. *Define*

$$\begin{aligned}
\mathfrak{B}_1 &:= \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\} \\
\mathfrak{B}_2 &:= \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w = 0\}
\end{aligned}$$

with $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$. *Then*

$$\begin{aligned}
\mathfrak{B}_1 &= \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R_1(\frac{d}{dt})w = 0\} \\
\mathfrak{B}_2 &= \{w \in \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R_1(\frac{d}{dt})w = 0\}
\end{aligned}$$

if and only if there exists a unimodular matrix $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$, *such that* $R_1(\xi) = U(\xi)R(\xi)$.

Proof. See Polderman and Willems (1997), pp. 100-101, Theorem 3.6.2. ■

With reference to the above theorem, it can be inferred that the differential operators R and R_1 induce kernel representations of the same behaviour $\mathfrak{B}$ iff each row of R is a linear combination with polynomial coefficients of the rows of R_1 and likewise, each row of R_1 is a linear combination with polynomial coefficients of the rows of R. The equivalence of two behaviours with different kernel representations can be conveniently explained using the concept of *modules* (see Definition B.22, Appendix B). Observe that if a trajectory w is such that $P(\frac{d}{dt})w = 0$, with $P \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$, then $Q(\frac{d}{dt})P(\frac{d}{dt})w = 0$ for any $Q \in \mathbb{R}^{\bullet\times\mathtt{g}}[\xi]$. As a consequence, it follows that two behaviours $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$ and $\mathfrak{B}_1 = \ker\big(R_1(\frac{d}{dt})\big)$ are equivalent if and only if the rows of R and R_1 generate the same $\mathbb{R}[\xi]$-module. This point has been elaborated upon in pp. 83-84, Willems (2007).

From the next theorem, it follows that, we can always obtain a kernel representation for $\mathfrak{B}$ of the form $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, with R having full row rank (see Definition B.4, Appendix B).

Theorem 2.5. *Every behaviour* $\mathfrak{B}$ *with trajectories either infinitely differentiable or locally integrable defined by* $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$ *with* $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$, *admits an equivalent full row rank representation, i.e, there exists a kernel representation* $\mathfrak{B} = ker\big(R_1(\frac{d}{dt})\big)$, *with* $R_1 \in \mathbb{R}^{\mathtt{p}'\times\mathtt{q}}[\xi]$ *having full row rank.*

Proof. See Polderman and Willems (1997), p. 58, Theorem 2.5.23. ■

From the above Theorem, it follows that if a behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$ is such that R has linearly dependent rows, then one can obtain another kernel representation for $\mathfrak{B}$ of the form $\mathfrak{B} = \ker\big(R_1(\frac{d}{dt})\big)$, with R_1 having full row rank, or having lesser number of rows than R. This suggests that if R has linearly dependent rows, we can obtain a "simpler" kernel representation for $\mathfrak{B}$, in the sense that lesser number of equations can be used to describe $\mathfrak{B}$. This leads to the notion of *minimality of kernel representation*, which is defined below.

Definition 2.6. *A kernel representation of a behaviour* $\mathfrak{B}$ *with trajectories either infinitely differentiable or locally integrable, given by* $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$, $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ *is called* minimal *if every other kernel representation of* $\mathfrak{B}$ *has at least* $\mathtt{g}$ *rows.*

The following Theorem gives the algebraic condition on a kernel representation of a given behaviour, under which it is minimal.

Theorem 2.7. *A kernel representation of a behaviour* $\mathfrak{B}$ *with trajectories either infinitely differentiable or locally integrable, given by* $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$ *is minimal if and only if* R *has full row rank.*

Proof. See Polderman and Willems (1997), p.102, Theorem 3.6.4. ∎

Below, we define the notion of *invariant polynomials* of a behaviour.

Definition 2.8. *Let* $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$ *be a minimal kernel representation of a behaviour* $\mathfrak{B}$ *with trajectories either infinitely differentiable or locally integrable. Then the* invariant polynomials *of* $\mathfrak{B}$ *are the invariant polynomials* (*see section B.1 of Appendix B*) *of R.*

Observe that even though the minimal kernel representation of a behaviour is not unique, the invariant polynomials of a behaviour are unique.

2.2 Latent variable and image representations

When modeling a system we come across two types of variables namely the *manifest variables* (denoted by w) and the *latent variables* (denoted by ℓ). We now define *linear differential systems with latent variables.*

Definition 2.9. *A linear differential system with latent variables is a quadruple* $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$, *with* $\mathbb{T} \in \mathbb{R}$, *the time set,* $\mathbb{W}$ *the* manifest signal space, $\mathbb{L}$ *the* latent variable space, *and* $\mathfrak{B}_{\text{full}} \subseteq \mathbb{W}^{\mathbb{T}} \times \mathbb{L}^{\mathbb{T}}$ *the* full behaviour *of the system being equal to the set of all solutions of a system of constant coefficient linear ordinary differential equations.*

Hence, the pairs (w, ℓ) are the trajectories of a system with latent variables, with the vector w consisting of the manifest variables and the vector ℓ consisting of the latent variables. A linear differential system with latent variables induces a dynamical system in the sense of Definition 2.1 as follows.

Definition 2.10. *Let* $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ *be a linear differential system with latent variables. The manifest* (*or external*) *dynamical system induced by* Σ_L *is the dynamical system* $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, *with the behaviour* $\mathfrak{B}$ *defined as*

$$\mathfrak{B} = \{w : \mathbb{T} \to \mathbb{W} \mid \exists \ell : \mathbb{T} \to \mathbb{L} \textit{ such that } col(w, \ell) \in \mathfrak{B}_{\text{full}}\}$$

With respect to the above definition, $\mathfrak{B}$ is called the *manifest* (*or external*) *behaviour* of $\mathfrak{B}_{\text{full}}$. The next theorem states that the manifest dynamical system of a linear differential system with latent variables $\Sigma_L = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ is a linear differential system if $(\mathbb{W} \times \mathbb{L})^{\mathbb{T}} = \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}+l})$, where $l = \dim(\mathbb{L})$.

Theorem 2.11. *Consider a latent variable differential system whose full behaviour is* $\mathfrak{B}_{\text{full}} \in \mathcal{L}^{\mathtt{w}+l}$. *Define*

$$\mathfrak{B} := \{w \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists \ell \in \mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{l}) \textit{ such that } col(w, \ell) \in \mathfrak{B}_{\text{full}}\}.$$

Then $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$.

Proof. The proof follows from Theorem 6.2.6, pp. 206-207 of Polderman and Willems (1997). ■

With reference to the above theorem, $\mathfrak{B}$ is the behaviour obtained from $\mathfrak{B}_{\text{full}}$ by elimination of the latent variable ℓ. From Theorem 2.11, it follows that it is possible to eliminate latent variables from a linear differential behaviour with latent variables whose trajectories are infinitely differentiable, in order to obtain its manifest behaviour. This theorem is hence called *elimination theorem.*

The trajectories belonging to a linear differential system with latent variables $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{l}, \mathfrak{B}_{\text{full}})$ can be described by a set of linear constant coefficient ordinary differential equations

$$R_0 w + R_1 \frac{d}{dt} w + \ldots + R_L \frac{d^L}{dt^L} w = M_0 \ell + M_1 \frac{d}{dt} \ell + \ldots + M_{L'} \frac{d^{L'}}{dt^{L'}} \ell \tag{2.3}$$

where $M_i \in \mathbb{R}^{\mathtt{g} \times l}$ for $i = 0, 1, \ldots, L'$, and $R_k \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}$ for $k = 0, 1, \ldots, L$.

The set of equations (2.3) is called a *latent variable* or a *hybrid representation* of the latent variable system $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{l}, \mathfrak{B}_{\text{full}})$. The full behaviour $\mathfrak{B}_{\text{full}}$ consists of trajectories (w, ℓ) satisfying (2.3). A concise way of writing (2.3) is

$$R(\frac{d}{dt}) w = M(\frac{d}{dt}) \ell \tag{2.4}$$

where $R \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{g} \times l}[\xi]$ are defined as $R(\xi) = R_0 + R_1 \xi + \ldots + R_L \xi^L$ and $M(\xi) = M_0 + M_1 \xi + \ldots + M_{L'} \xi^{L'}$. If $\mathfrak{B}_{\text{full}} \in \mathcal{L}^{\mathtt{w}+l}$, then from the *elimination theorem*, it follows that there exists $R' \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$, such that the manifest behaviour of $\mathfrak{B}_{\text{full}}$ has a kernel representation given by $\mathfrak{B} = \ker\big(R'(\frac{d}{dt})\big)$.

Example 1.1 revisited: We reconsider the example of the mathematical model of a three-storey building studied in chapter 1, where we are interested in the dynamical relation between the position w_3 of the third storey and the wind force F. Equation (1.6) describes the set of trajectories belonging to a dynamical system in the sense of Definition 2.1, wherein the time set is $\mathbb{T} = \mathbb{R}$ and the signal space is $\mathbb{W} = \mathbb{R}^2$. The system can also be described as a linear differential system with latent variables, wherein the vector of latent variables is given by $\ell = \text{col}(w_1, w_2)$, and the vector of manifest variables is $w = \text{col}(w_3, F)$. In this case, the set of trajectories belonging to the system can be described by the equation $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$, where

$$\begin{aligned} R(\xi) &= \begin{bmatrix} 0 & 0 \\ k_3 & 0 \\ m_3\xi^2 + k_3 & -1 \end{bmatrix} \\ M(\xi) &= \begin{bmatrix} m_1\xi^2 + c\xi + (k_1 + k_2 + k') & -k_2 \\ -k_2 & m_2\xi^2 + (k_2 + k_3 + k') \\ 0 & k_3 \end{bmatrix} \end{aligned}$$

Furthermore, the time axis is $\mathbb{T} = \mathbb{R}$, the signal space is $\mathbb{W} = \mathbb{R}^2$, and the latent variable space is $\mathbb{L} = \mathbb{R}^2$.

A special and very important case of hybrid representation of a latent variable system is an *image representation*. Take $R(\xi) = I_{\mathtt{w}}$ in equation (2.4), yielding

$$w = M(\frac{d}{dt})\ell \tag{2.5}$$

If $\mathfrak{B}$ denotes the manifest behaviour of $\mathfrak{B}_{\text{full}}$, then another way of expressing equation (2.5) is $\mathfrak{B}$ =Im$(M(\frac{d}{dt}))$. Note that if $\mathfrak{B}_{\text{full}} \in \mathcal{L}^{\mathtt{w}+l}$, then in equation (2.5), ℓ is $\mathfrak{C}^\infty$-free, i.e it is free to take any value in the space $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l)$. Also note that not all linear differential systems have image representations. Later on in this chapter, we will give a condition on a linear differential system under which it can have an image representation.

2.3 Observability

Consider a partition $w = \text{col}(w_1, w_2)$ of the external variables of a behaviour $\mathfrak{B}$. We say that w_2 is observable from w_1 if w_1, together with the laws of the system determine w_2 uniquely. We call w_1 an *observed variable* and w_2 a *to-be-deduced* variable. The following definition formalizes the concept of observability.

Definition 2.12. *Let $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ be a linear differential system. Trajectories in $\mathfrak{B}$ are partitioned as $w = col(w_1, w_2)$ with $w_i : \mathbb{R} \to \mathbb{W}_i, i = 1, 2$. w_2 is said to be* observable *from w_1 if*

$$col(w_1, w_2), col(w_1, w_2') \in \mathfrak{B} \Rightarrow w_2 = w_2'$$

This is equivalent to the statement that there exists a polynomial differential operator $F(\frac{d}{dt}) : \mathbb{W}_1^{\mathbb{T}} \to \mathbb{W}_2^{\mathbb{T}}$ such that $w_2 = F(\frac{d}{dt})w_1$ for all $\text{col}(w_1, w_2) \in \mathfrak{B}$, i.e w_2 can be determined uniquely by observing w_1. With respect to Definition 2.12, observe that by linearity, if $\text{col}(w_1, w_2)$, $\text{col}(w_1, w_2') \in \mathfrak{B}$, then $\text{col}(0, w_2 - w_2') \in \mathfrak{B}$. This implies that if w_2 is observable from w_1, then $\text{col}(0, w_2'') \in \mathfrak{B} \Rightarrow w_2'' = 0$. The following proposition characterizes observability in terms of a kernel representation of the behaviour.

Proposition 2.13. *Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}_1+\mathtt{w}_2}$ be represented by $R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$. Then w_2 is observable from w_1 iff $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.*

Proof. See Polderman and Willems (1997), pp. 174-175. ■

From the above proposition, it can be seen that the condition for observability of w_2 from w_1 depends only on R_2.

2.4 Controllability

In the behavioural approach, controllability is a property of the system and not of a particular representation of the system, unlike what happens in the state space approach for linear systems. Below, we give the behavioural definition of controllability.

Definition 2.14. *The shift-invariant system $\Sigma = (\mathbb{R}, \mathbb{W}, \mathfrak{B})$ is said to be* controllable *if for all $w_1, w_2 \in \mathfrak{B}$, there exist $T \geq 0$ and $w \in \mathfrak{B}$ such that*

$$w(t) = \begin{cases} w_1(t) & \text{for } t < 0, \\ w_2(t-T) & \text{for } t \geq T \end{cases}$$

Thus a controllable behaviour is one which allows switching from one trajectory to another within the behaviour, provided that we allow a delay. We denote the set of controllable linear differential behaviours with infinitely differentiable manifest variable w by $\mathcal{L}^{\mathtt{w}}_{\text{cont}}$. The following proposition gives an algebraic condition on the kernel representation of a behaviour under which it is controllable. It also relates controllability to the existence of an image representation.

Proposition 2.15. *Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ have a kernel representation $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$ with $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is controllable,*
2. *$rank(R(\lambda))$ is constant for all $\lambda \in \mathbb{C}$,*
3. *there exist $l \in \mathbb{Z}^+$ and $M \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ such that $w = M(\frac{d}{dt})\ell$ is an image representation of $\mathfrak{B}$.*

Proof. See Polderman and Willems (1997), pp. 158-159, 229-230. ■

From the above Proposition, it follows that a behaviour is controllable if and only if it admits an image representation. We now show that if $\mathfrak{B}$ is controllable, it is possible to find an image representation in which the latent variable ℓ is observable from w. Such an image representation is called an *observable image representation.*

Let $\mathfrak{B} = \text{Im}\big(M(\frac{d}{dt})\big)$ with $M \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ be an image representation of a behaviour $\mathfrak{B}$, which is not observable. Consider a Smith form decomposition (see Proposition B.2, Appendix B for details) of M given by

$$M(\xi) = U(\xi)\begin{bmatrix} \Delta(\xi) & 0_{l_1\times(l-l_1)} \\ 0_{(\mathtt{w}-l_1)\times l_1} & 0_{(\mathtt{w}-l_1)\times(l-l_1)} \end{bmatrix} V(\xi) \tag{2.6}$$

where $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, $V \in \mathbb{R}^{l\times l}[\xi]$ and $\Delta \in \mathbb{R}^{l_1\times l_1}[\xi]$ has nonzero diagonal entries. Consider partitions of U and V given by

$$U(\xi) = \begin{bmatrix} U_1(\xi) & U_2(\xi) \end{bmatrix}; \qquad V(\xi) = \begin{bmatrix} V_1(\xi) \\ V_2(\xi) \end{bmatrix}$$

where $U_1 \in \mathbb{R}^{\mathtt{w}\times l_1}[\xi]$, $U_2 \in \mathbb{R}^{\mathtt{w}\times(\mathtt{w}-l_1)}[\xi]$, $V_1 \in \mathbb{R}^{l_1\times l}[\xi]$ and $V_2 \in \mathbb{R}^{(l-l_1)\times l}[\xi]$. Then, from equation (2.6), we have $M(\xi) = U_1(\xi)\Delta(\xi)V_1(\xi)$. Note that $U_1(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Define $G(\xi) := \Delta(\xi)V_1(\xi)$ and $\ell' := G(\frac{d}{dt})\ell$.

We now prove that the rows of G are linearly independent over $\mathbb{R}(\xi)$, or equivalently that the differential operator $G(\frac{d}{dt})$ is surjective (see Appendix B for a definition). For $i = 1, \dots, l_1$, let $\delta_i \in \mathbb{R}[\xi]$ denote the i^{th} diagonal element of Δ and $V_i' \in \mathbb{R}^{1\times l}[\xi]$ denote the i^{th} row of V_1. Assume by contradiction that the rows of G are linearly dependent over $\mathbb{R}(\xi)$, or that there exist nonzero $r_i \in \mathbb{R}(\xi)$ for $i = 1, \dots, l_1$ such that

$$\sum_{i=1}^{l_1} r_i(\xi)\delta_i(\xi)V_i'(\xi) = 0$$

This implies that

$$\begin{bmatrix} r_1(\xi)\delta_1(\xi) & r_2(\xi)\delta_2(\xi) & \cdots & r_{l_1}(\xi)\delta_{l_1}(\xi) & 0_{1\times(l-l_1)} \end{bmatrix} V(\xi) = 0$$

Postmultiplying both sides of the above equation with $V(\xi)^{-1}$, we obtain $r_i(\xi) = 0$ for $i = 1, \dots, l_1$. Hence the rows of G are linearly independent over $\mathbb{R}(\xi)$, which implies that $G(\frac{d}{dt})$ is surjective . This implies that ℓ' is a free trajectory. Since $M(\frac{d}{dt})\ell = U_1(\frac{d}{dt})\ell'$, we can write

$$\mathfrak{B} = \text{Im}\left(U_1\left(\frac{d}{dt}\right)\right) \tag{2.7}$$

Since $U_1(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, equation (2.7) is an observable image representation of $\mathfrak{B}$.

Below, we prove that the image of a polynomial differential operator acting on a controllable behaviour is also controllable.

Lemma 2.16. *Let $\mathfrak{B}_1 \in \mathcal{L}^{\mathtt{w}}_{\text{cont}}$ and $P \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$. Then $\mathfrak{B}_2 := Im\big(P(\frac{d}{dt})\big)_{|\mathfrak{B}_1}$ is controllable.*

Proof. Consider two trajectories $w_1, w_2 \in \mathfrak{B}_2$. Then there exist $w_1', w_2' \in \mathfrak{B}_1$, such that $w_1 = P(\frac{d}{dt})w_1'$ and $w_2 = P(\frac{d}{dt})w_2'$. Since $\mathfrak{B}_1$ is controllable, there exist $T \geq 0$ and $w' \in \mathfrak{B}_1$ such that

$$w'(t) = \begin{cases} w_1'(t) & \text{for } t < 0 \\ w_2'(t-T) & \text{for } t \geq T \end{cases}$$

Define $w := P(\frac{d}{dt})w'$, and observe that $w \in \mathfrak{B}_2$. Also observe that

$$w(t) = \begin{cases} w_1(t) & \text{for } t < 0 \\ w_2(t-T) & \text{for } t \geq T \end{cases}$$

Since the trajectories w_1 and w_2 are arbitrary trajectories of $\mathfrak{B}_2$, it follows that $\mathfrak{B}_2$ is controllable. ■

2.5 Systems with inputs and outputs

We give the behavioural definition of *input* and of *output variable.*

Definition 2.17. *Let $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ be a linear differential system with $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Partition the signal space as $\mathbb{R}^{\mathtt{w}} = \mathbb{R}^{\mathtt{w}_1} \times \mathbb{R}^{\mathtt{w}_2}$, and correspondingly any trajectory $w \in \mathfrak{B}$ as $w = col(w_1, w_2)$, with $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1})$ and $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$. This partition is called an* input/output *partition if:*

1. *w_1 is $\mathfrak{C}^\infty$-free, i.e for all $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1})$, there exists a $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$, such that $col(w_1, w_2) \in \mathfrak{B}$.*
2. *w_1 is* maximally free, *i.e given w_1, none of the components of w_2 can be chosen freely.*

If both the above conditions hold, then w_1 is called an input *variable, and w_2 is called an* output *variable.*

From Definition 2.17, it can be proved that the evolution of w_2 is completely determined by that of w_1 and by its past history. This point is elaborated upon in Willems (1989), pp. 215-221.

Note that in general for a system, there are many possible partitions of its variables into inputs and outputs. As an example, consider the electrical system $V = RI$, where V represents the voltage across a resistor whose resistance is R and I represents the current through it. Here, we can choose either V or I as the input or the free variable and the other variable will be completely determined by this choice through the relation $I = \frac{V}{R}$, respectively $V = RI$. Consequently V (or I) is a maximal set of free variables. The following proposition provides conditions under which a particular partition of $w \in \mathfrak{B}$ is an input/output partition of $\mathfrak{B}$, in terms of its kernel representation.

Proposition 2.18. *Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ induce a minimal kernel representation of a behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Then there exists a permutation matrix $\pi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ such that $R(\xi)\pi = row(P(\xi), Q(\xi))$, with $P \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}}[\xi]$, $Q \in \mathbb{R}^{\mathtt{g}\times(\mathtt{w}-\mathtt{g})}[\xi]$, $det(P) \neq 0$, and $\pi^\top w = col(u, y)$, with $u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}-\mathtt{g}})$ and $y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{g}})$, is an input/output partition of $w \in \mathfrak{B}$.*

Proof. See Polderman and Willems (1997), p. 90, Corollary 3.3.23. ■

With reference to the above Proposition, define $w' := \pi^\top w$. Observe that w' is a permutation of the external variables of $\mathfrak{B}$. Define

$$\mathfrak{B}' := \{\pi^\top w \mid w \in \mathfrak{B}\}$$

It is easy to see that $\mathfrak{B}' = \ker\big(R(\frac{d}{dt})\pi\big)$. Observe that for every $w' = \text{col}(u, y) \in \mathfrak{B}'$, $P(\frac{d}{dt})y = -Q(\frac{d}{dt})u$. We call this an *input-output representation* of the behaviour.

We now define the transfer function of a controllable behaviour with equal number of inputs and outputs. This concept will be used in chapter 6 of this thesis.

Definition 2.19. *Let $\mathfrak{B} \in \mathcal{L}^{2l}$ be a controllable behaviour with an input/output partition $col(u, y)$, such that $\mathtt{u} = \mathtt{y} = l$. Let an image representation of $\mathfrak{B}$ be $\mathfrak{B} = Im\big(M(\frac{d}{dt})\big)$, where $M = col(N_0, N_1)$ and $N_0, N_1 \in \mathbb{R}^{l\times l}[\xi]$. Then Z given by*

$$Z(\xi) = N_1(\xi)N_0(\xi)^{-1}$$

is called a transfer function *of $\mathfrak{B}$.*

With reference to the above definition, observe that N_0 is nonsingular, because if it is singular, then $N_0(\frac{d}{dt})$ is not surjective, which implies that u cannot be chosen freely, which in turn leads to a contradiction.

2.6 Autonomous systems

Autonomous behaviours are in a way the opposite of controllable behaviours. These are behaviours that have no inputs or free variables.

Definition 2.20. *A shift-invariant behaviour $\mathfrak{B}$ is called* autonomous *if for all $w_1, w_2 \in \mathfrak{B}$,*

$$w_1(t) = w_2(t) \quad \forall \quad t \leq 0 \Rightarrow w_1 = w_2$$

An autonomous behaviour is one for which the future of every trajectory is completely determined by its past. We now specialize this notion to the case of linear differential systems.

The following proposition (see Polderman and Willems (1997), Corollary 3.2.13, p. 75) gives a method for describing the set of trajectories of a scalar ($\mathtt{w} = \mathtt{1}$) autonomous behaviour starting from its kernel description.

Proposition 2.21. *Let $P \in \mathbb{R}[\xi]$ be a monic polynomial and let $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, m + 2N$, be the distinct roots of P of multiplicity n_i, i.e $P(\xi) = \prod_{k=1}^{m+2N}(\xi - \lambda_k)^{n_k}$. Assume that the first m distinct roots are real numbers and the remaining distinct roots are conjugate pairs, $\lambda_{m+1}, \bar{\lambda}_{m+1}, \lambda_{m+2}, \bar{\lambda}_{m+2}, \ldots, \lambda_{m+N}, \bar{\lambda}_{m+N}$ with nonzero imaginary parts. Then $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \mid P(\frac{d}{dt})w = 0\}$ is autonomous and has dimension n =deg $(P(\xi))$. Moreover $w \in \mathfrak{B}$ iff it is of the form*

$$w(t) = \sum_{k=1}^{m} \sum_{l=0}^{n_k - 1} r_{kl} t^l e^{\lambda_k t} + \sum_{k=m+1}^{m+N} \sum_{l=0}^{n_k - 1} t^l (r_{kl} e^{\lambda_k t} + \bar{r}_{kl} e^{\bar{\lambda}_k t})$$

where r_{kl} are arbitrary real numbers for $k = 1, 2, \ldots, m$ and arbitrary complex numbers with nonzero imaginary parts for $k = m + 1, m + 2, \ldots, m + N$.

With reference to the above proposition, $P(\xi)$ is called the *characteristic polynomial* of $\mathfrak{B}$. The roots of P are called the *characteristic frequencies* of $\mathfrak{B}$.

The following proposition relates the property of autonomy of a multivariable behaviour to the algebraic properties of a matrix R inducing a kernel representation of the behaviour.

Proposition 2.22. *Let $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$, with $R \in \mathbb{R}^{\mathtt{g} \times \mathtt{w}}[\xi]$, be a kernel representation of $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Then $\mathfrak{B}$ is autonomous iff R has full column rank. Consequently, if $\mathfrak{B}$ is autonomous, there exists $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ with $det(R) \neq 0$ such that $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$.*

Proof. (*Only if*): Assume that $\mathfrak{B}$ is autonomous. Assume by contradiction that R does not have full column rank. Let $\mathtt{w}_1$ denote the column rank of R. Then from Proposition B.6, Appendix B, it follows that there exists a unimodular matrix V such that

$$RV = \begin{bmatrix} R_1 & 0_{\mathtt{g} \times (\mathtt{w} - \mathtt{w}_1)} \end{bmatrix} \tag{2.8}$$

Define $V_1(\xi) := V(\xi)^{-1}$. Let w be a trajectory in $\mathfrak{B}$. Define $w' := V_1(\frac{d}{dt})w$ and $\mathfrak{B}' := \mathrm{Im}\big(V_1(\frac{d}{dt})\big)_{|\mathfrak{B}}$. Consider a partition of w' given by $w' = \mathrm{col}(w_1, w_2)$, where $w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1})$ and $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w} - \mathtt{w}_1})$. From equation (2.8), it follows that

$$\begin{bmatrix} R_1(\frac{d}{dt}) & 0_{\mathtt{g} \times (\mathtt{w} - \mathtt{w}_1)} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

is a kernel representation of $\mathfrak{B}'$. This implies that w_2 is $\mathfrak{C}^\infty$-free. From Definition 2.20, it follows that the behaviour $\mathfrak{B}'$ is not autonomous because every trajectory in $\mathfrak{B}'$ has some of its components free. Since V_1 is unimodular, it induces a bijective differential operator (see Definition B.16, Appendix B). This implies that $\mathfrak{B}$ is not autonomous, which is a contradiction. Consequently, R has full column rank.

(*If*): Now assume that R has full column rank. This implies that $\mathtt{w} \leq \mathtt{g}$. From Proposition B.5, Appendix B, it follows that there exists a unimodular matrix $U \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}}[\xi]$, such that $UR = \mathrm{col}(R_2, 0_{(\mathtt{g}-\mathtt{w})\times\mathtt{w}})$, where $R_2 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. This implies that $\mathfrak{B} = \ker\big(R_2(\frac{d}{dt})\big)$ is another kernel representation of $\mathfrak{B}$. Observe that $\det(R_2) \neq 0$. Hence all the invariant polynomials of R_2 are nonzero. Let $R_2 = U\Delta V$ be a Smith form decomposition of R_2. For $i = 1, \ldots, \mathtt{w}$, let $\delta_i \in \mathbb{R}[\xi]$ denote the i^{th} invariant polynomial of $\mathfrak{B}$. Consider the behaviour $\mathfrak{B}' = \ker\big(\Delta(\frac{d}{dt})\big)$ and observe that $w' \in \mathfrak{B}'$ iff $w' = V(\frac{d}{dt})w$ for some $w \in \mathfrak{B}$. For $i = 1, \ldots, \mathtt{w}$, define $\mathfrak{B}_i := \ker\big(\delta_i(\frac{d}{dt})\big)$. From Proposition 2.21, it follows that $\mathfrak{B}_i$ is autonomous for $i = 1, \ldots, \mathtt{w}$. This implies that $\mathfrak{B}'$ is autonomous. Since V is unimodular, it induces a bijective differential operator. This implies that $\mathfrak{B}$ is autonomous. This concludes the proof. ■

Below, we define the *characteristic frequencies* of an autonomous behaviour.

Definition 2.23. *Let* $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$ *with* $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, *be a kernel representation of an autonomous behaviour* $\mathfrak{B}$. *Then the roots of* $\det(R)$ *are called the* characteristic frequencies *of* $\mathfrak{B}$.

It is easy to see that the characteristic frequencies of an autonomous behaviour are the roots of its invariant polynomials.

In the following, we make use of Smith form decomposition (see Appendix B for details) in order to reduce the multivariable case ($\mathtt{w} > \mathtt{1}$) of autonomous behaviours to a set of independent scalar ($\mathtt{w} = \mathtt{1}$) behaviours. This reduction is done in order to describe the set of trajectories belonging to a multivariable autonomous behaviour starting from its kernel description. Let $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, be the given autonomous behaviour and let $R = U\Delta V$ be a Smith form decomposition of R. Consider the behaviour $\mathfrak{B}' = \ker\big(\Delta(\frac{d}{dt})\big)$ and observe that $w' \in \mathfrak{B}'$ iff $w' = V(\frac{d}{dt})w$ for some $w \in \mathfrak{B}$. Let w'_i ($i = 1, \ldots, \mathtt{w}$) be the components of w'. For $i = 1, \ldots, \mathtt{w}$, let δ_i denote the i^{th} invariant polynomial of R. Then $\mathfrak{B}'_i = \ker\big(\delta_i(\frac{d}{dt})\big)$ is a scalar autonomous behaviour for $i = 1, \ldots, \mathtt{w}$ and $w'_i \in \mathfrak{B}'_i$.

Proposition 2.21 can now be made use of in order to describe the set of trajectories $w'_i \in \mathfrak{B}'_i$, $i = 1, \ldots, \mathtt{w}$, and hence the trajectories $w' \in \mathfrak{B}'$. The trajectories $w \in \mathfrak{B}$ are then given by $w = (V_1(\frac{d}{dt}))w'$, where $V_1(\xi) = (V(\xi))^{-1}$.

2.7 State space models

We now discuss a special class of latent variables known as the state variables. State variables either occur naturally in the modelling process or can be introduced artificially. Below we provide the definition for state space models and state variables as given in Polderman and Willems (1997).

Definition 2.24. (Axiom of state) *Consider a latent variable differential system whose full behaviour is defined by*

$$\mathfrak{B}_f = \{col(w, \ell) \in \mathcal{L}_1^{loc}(\mathbb{R}, \mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{l}) \mid R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell\}$$

where $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$*, and* $M \in \mathbb{R}^{\mathtt{g}\times l}[\xi]$*. Let* $col(w_1, \ell_1), col(w_2, \ell_2) \in \mathfrak{B}_f$ *and* $t_0 \in \mathbb{R}$*. Define the* concatenation *of* $col(w_1, \ell_1)$ *and* $col(w_2, \ell_2)$ *at* t_0 *by* $col(w, \ell)$*, with*

$$w(t) = \begin{cases} w_1(t) & t < t_0, \\ w_2(t) & t \geq t_0 \end{cases} \quad \text{and} \quad \ell(t) = \begin{cases} \ell_1(t) & t < t_0, \\ \ell_2(t) & t \geq t_0 \end{cases}$$

Then $\mathfrak{B}_f$ *is said to be a* state space model*, and the latent variable* ℓ *is called a* state variable *if* $\ell_1(t_0) = \ell_2(t_0)$ *implies* $col(w, \ell) \in \mathfrak{B}_f$*.*

Note that a locally integrable trajectory need not be defined point wise. From Definition 2.24, it follows that a state space model $\mathfrak{B}_f$ with locally integrable trajectories necessarily has its state variable defined point wise, because in the definition, t_0 is an arbitrary real number.

From Definition 2.24, it also follows that the state variables "split" the past and the future of the behaviour. The values of the state variables at time t_0 contain all the information needed to decide whether or not two trajectories w_1 and w_2 can be concatenated within $\mathfrak{B}$ at time t_0.

According to the axiom of state, if ℓ denotes the state variable of a given behaviour, the only information that is required to know whether a trajectory $(w^+, \ell^+) : [0, \infty) \to (\mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{l})$ can occur as a future continuation of a trajectory $(w^-, \ell^-) : (-\infty, 0] \to (\mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{l})$ within the behaviour are the values of ℓ^- and ℓ^+ at time $t = 0$. As a consequence $\ell^-(0)$ tells us which future trajectories are admissible for the system. Hence we can say that $\ell^-(0)$ is the *memory* of the system at time $t = 0$.

The vector of state variables is usually denoted by x. In classical systems theory, state equations are of first order in x and of zeroth order in w. The following proposition shows that this property can be deduced and not postulated from the definition of state variable.

Proposition 2.25. *Let* $\Sigma_L = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathfrak{B}_f)$ *be a latent variable differential system with manifest variable* w *and latent variable* x*. Then it is a state space model if and only if there exist matrices* E, F, G *such that* $\mathfrak{B}_f$ *has the kernel representation:*

$$Gw + Fx + E\frac{dx}{dt} = 0 \tag{2.9}$$

Proof. See Rapisarda (1998), pp. 161-162. ■

Since the external variables can always be partitioned into inputs u and outputs y as described in section 2.5, we can arrive at an input-state-output model of a system from its state space model (2.9), as shown below

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

Note that for an autonomous behaviour $\mathfrak{B}$, with a state space model given by

$$\begin{aligned}\frac{dx}{dt} &= Ax \\ w &= Cx\end{aligned}$$

since $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, it follows that $x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{x}})$. We now define the concept of minimality of state space.

Definition 2.26. *Let* $\Sigma_L = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathfrak{B}_f)$ *be a state space system with manifest behaviour* $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$*.* Σ_L *is said to be* state minimal *if for every other state space system* $\Sigma_{L'} = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}'}, \mathfrak{B}_{f'})$ *with the same external behaviour* $(\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$*,* $\mathtt{x} \leq \mathtt{x}'$*.*

The dimension of the minimal state space of $\mathfrak{B}$ denoted by $\mathtt{n}(\mathfrak{B})$ is called the *McMillan degree* of $\mathfrak{B}$.

The only place where the notion of state space is used in this thesis is section 2.8, where we work with autonomous behaviours. Hence the state variables and the manifest variables of the system in this case are infinitely differentiable trajectories.

2.8 Oscillatory systems

We now turn our attention to a special case of autonomous behaviours, namely oscillatory behaviours, that play a very important role in this thesis. We begin this section with the definition of a linear bounded system.

Definition 2.27. A behaviour $\mathfrak{B}$ defines a *linear bounded system* if

- $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$;
- Every solution $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ is bounded on $(-\infty, \infty)$.

From the definition, it follows that a linear bounded system is necessarily autonomous: if there were any input variables in w, then those components of w could be chosen to be unbounded. Linear bounded systems have been called as *oscillatory systems* in Rapisarda and Willems (2005) (see Definition 1, p. 177). Indeed from the following proposition, it seems reasonable to call linear bounded systems as oscillatory systems.

Proposition 2.28. *Let $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$, with $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. Then $\mathfrak{B}$ is bounded if and only if every non-zero invariant polynomial of $\mathfrak{B}$ has distinct and purely imaginary roots.*

Proof. See proof of Proposition 2, pp. 177-178, Rapisarda and Willems (2005). ■

From the above proposition, it follows that if $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is bounded then every trajectory $w \in \mathfrak{B}$ can be written as

$$w(t) = \sum_{i=1}^{n} \big(A_i \sin(\omega_i t) + B_i \cos(\omega_i t)\big) \tag{2.10}$$

where $n \in \mathbb{N}$, $A_i, B_i \in \mathbb{R}^{\mathtt{w}}$ and ω_i are nonnegative real numbers for $i = 1, \ldots, n$. Thus trajectories belonging to a linear bounded system are linear combinations of vector sinusoidal functions, which implies that these trajectories are *almost periodic* or oscillatory. In equation (2.10), if $n = 1$, and $\omega_1 = 0$, then $w(t) = B_1$, which is a constant trajectory. Note that a constant trajectory can also be viewed as an oscillatory trajectory that oscillates with zero frequency. Therefore, in this thesis, we will henceforth call linear bounded systems as *oscillatory systems.*

In the following, the case of multivariable ($\mathtt{w} > 1$) oscillatory systems will be often reduced to the scalar case by using the Smith form of a polynomial matrix. Consequently, we now examine in more detail the properties of scalar oscillatory systems and of their representation.

From Proposition 2.28, it follows that if $r \in \mathbb{R}[\xi]$ then $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$ defines an oscillatory system if and only if all the roots of r are distinct and on the imaginary axis. This implies that r has one of the following two forms.

$$\begin{aligned} r(\xi) &= (\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2) \ldots (\xi^2 + \omega_{n-1}^2) \qquad \text{or} \\ r(\xi) &= \xi(\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2) \ldots (\xi^2 + \omega_{n-1}^2) \end{aligned}$$

where $\omega_0, \ldots, \omega_{n-1} \in \mathbb{R}^+$ and are distinct. From Proposition 2.21, it follows that the dimension of $\ker\big(r\left(\frac{d}{dt}\right)\big)$ as a linear subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$ equals the degree of the polynomial r. From Definition 2.23, it follows that the roots of r are the characteristic frequencies of $\ker\big(r(\frac{d}{dt})\big)$.

In the following, a polynomial matrix will be called oscillatory if all its invariant polynomials have distinct and purely imaginary roots. We now give a condition on the state space equation of an autonomous system under which it is oscillatory.

Lemma 2.29. *A state space model* $\mathfrak{B}_f$ *given by*

$$\mathfrak{B}_f = \{col(w, x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{2\mathtt{x}}) \mid \frac{dx}{dt} = Ax, w = x\}$$

where $A \in \mathbb{R}^{\mathtt{x}\times\mathtt{x}}$, *is oscillatory if and only if there exists an invertible matrix* $V \in \mathbb{C}^{\mathtt{x}\times\mathtt{x}}$ *such that* $VAV^{-1} = A_d$, *where* A_d *is a diagonal matrix whose diagonal entries are purely imaginary and occur in conjugate pairs.*

Proof. (*If*) Define $z := Vx \in \mathbb{C}^{\mathtt{x}\times 1}$. Consider the system $\frac{dz}{dt} = A_d z$. Each component of z is bounded because the diagonal entries of A_d are purely imaginary and occur in conjugate pairs. Since V is invertible, this implies that each component of x is also bounded. Hence, the $\mathfrak{B}_f$ is oscillatory.

(*Only if*) By contradiction, if A is not diagonalizable, it implies that A has at least one eigenvalue with geometric multiplicity less than its algebraic multiplicity, which in turn implies that the system is not bounded on $(-\infty, \infty)$. Hence A is diagonalizable. Again by contradiction, if any of the eigenvalues of A is not purely imaginary, then one of the components of $z = Vx$, is unbounded on $(-\infty, \infty)$, which implies that one or more components of x are unbounded. Hence A has purely imaginary eigenvalues. Since the characteristic polynomial of A has real coefficients, the eigenvalues of A occur in conjugate pairs. ■

Assume now that a multivariable oscillatory behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$ with $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is such that $\det(R)$ has only distinct roots. In this case, it follows from the divisibility property of the invariant polynomials (see Proposition B.2, Appendix B) that the Smith form of R is necessarily $\text{diag}(1, \ldots, 1, \det(R))$. This is the *generic* case of oscillatory systems. Below we describe the concept of genericity as introduced in Heij (1989).

A mapping $p : \mathbb{R}^n \to \mathbb{R}$ is called a *polynomial* (on $\mathbb{R}^n$) if for any $x \in \mathbb{R}^n$, $p(x)$ is a polynomial in the elements of x. A subset $V \subset \mathbb{R}^n$ is called a *proper algebraic variety* associated with a polynomial $p \neq 0$ on $\mathbb{R}^n$ if $V = p^{-1}(0)$. We call a set $\pi \in \mathbb{R}^n$ *generic* in $\mathbb{R}^n$ if there is a proper algebraic variety V in $\mathbb{R}^n$, such that $\pi \supseteq (\mathbb{R}^n \backslash V)$.

Now consider an oscillatory behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. For $i = 1, \ldots, \mathtt{w}$, define

$$\Delta_i(\xi) := \text{the greatest common divisor of all } i \times i \text{ minors of } R. \tag{2.11}$$

Define $\Delta_0(\xi) := 1$. Then (see Kailath (1980), pp. 391-392) the i^{th} invariant polynomial λ_i of R is given by

$$\lambda_i(\xi) = \frac{\Delta_i(\xi)}{\Delta_{i-1}(\xi)}$$

Take any two $i \times i$ minors of R, where $i \in \{1, \dots, \mathtt{w} - 1\}$. We show that their greatest common divisor is 1 generically. Let the two minors be denoted by $a(\xi)$ and $b(\xi)$ respectively. Let

$$\begin{aligned} a(\xi) &= \sum_{i=0}^{n_1} a_i \xi^i \\ b(\xi) &= \sum_{i=0}^{n_2} b_i \xi^i \end{aligned}$$

Assume without loss of generality that $n_1 \geq n_2$. Define $A := \text{col}(a_1, \dots, a_{n_1})$, $B := \text{col}(b_1, \dots, b_{n_2})$. Define

$$S_1(A) := \begin{bmatrix} a_0 & a_1 & \dots & a_{n_1} & 0 & 0 & \dots & 0 \\ 0 & a_0 & \dots & a_{n_1-1} & a_{n_1} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_0 & a_1 & \dots & a_{n_1} \end{bmatrix}$$

with S_1 having $n_1 + n_2$ columns. Define

$$S_2(B) := \begin{bmatrix} 0 & \dots & 0 & 0 & b_0 & \dots & b_{n_2-1} & b_{n_2} \\ 0 & \dots & 0 & b_0 & b_1 & \dots & b_{n_2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_0 & \dots & b_{n_2-1} & b_{n_2} & 0 & \dots & 0 & 0 \end{bmatrix}$$

with S_2 having $n_1 + n_2$ columns.

Define $C =: \text{col}(A, B)$. Let $S(C)$ denote the matrix obtained by stacking the matrix $S_1(A)$ over $S_2(B)$. $S(C)$ is called the *Sylvester matrix* associated with the polynomials a and b. A well known result (see Kailath (1980), p. 142) about coprimeness of two polynomials is that two polynomials are coprime iff the Sylvester matrix associated with them has zero determinant. We now show that $\det(S(C)) \neq 0$ generically. Consider two real vectors $X = \text{col}(x_0, x_1, \dots, x_{n_1})$ and $Y = \text{col}(y_0, y_1, \dots, y_{n_2})$. Define $Z := \text{col}(X, Y)$. Define the polynomial p as

$$p(Z) := \det(S(Z))$$

Observe that, by definition,

$$D := \{Z \mid p(Z) \neq 0\}$$

is a generic set. This implies that $\det(S(C)) \neq 0$ generically, which in turn implies that two polynomials are generically coprime. Consequently from equation (2.11), it follows that for $i = 1, \ldots, \mathtt{w} - 1$, $\Delta_i(\xi) = 1$ generically. This implies that the case of an oscillatory behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ having its first $\mathtt{w} - 1$ invariant polynomials equal to 1 is a generic case of oscillatory behaviours.

Chapter 3

Quadratic differential forms

Quadratic differential forms play a major role in the forthcoming chapters of this thesis. In this chapter, we discuss those properties of quadratic differential forms that are required to understand the results that are presented further on in this thesis. The material appearing in this chapter is mostly taken from Willems and Trentelman (1998). We begin with the definition of bilinear and quadratic differential forms (QDFs).

3.1 Basics

Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ be a real polynomial matrix in the indeterminates ζ and η, i.e

$$\Phi(\zeta, \eta) = \sum_{i,k=0}^{n} \Phi_{i,k} \zeta^i \eta^k$$

where n is a natural number and $\Phi_{i,k} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$ for all $i, k \in \{0, 1, \ldots, n\}$. Such a polynomial Φ induces a *bilinear differential form* on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$ as

$$\begin{aligned} L_\Phi &: \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \\ L_\Phi(w_1, w_2) &: = \sum_{i,k=0}^{n} \left(\frac{d^i w_1}{dt^i}\right)^\top \Phi_{i,k} \left(\frac{d^k w_2}{dt^k}\right) \end{aligned}$$

If $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$, then Φ induces the *quadratic differential form* on $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ as

$$\begin{aligned} Q_\Phi &: \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \\ Q_\Phi(w) &: = \sum_{i,k=0}^{n} \left(\frac{d^i w}{dt^i}\right)^\top \Phi_{i,k} \left(\frac{d^k w}{dt^k}\right) \end{aligned}$$

In the following bilinear and quadratic differential forms will be abbreviated as BDF and QDF respectively. We call Φ *symmetric* if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. In this thesis, we study only symmetric QDFs because every nonsymmetric QDF is equivalent in the obvious sense of the word, to a symmetric one. We denote the ring of $\mathtt{w} \times \mathtt{w}$ symmetric two-variable polynomial matrices by $\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$.

With every $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, is associated the infinite matrix $\text{mat}(\Phi)$ with a finite number of nonzero elements given by

$$\text{mat}(\Phi) = \begin{pmatrix} \Phi_{0,0} & \Phi_{0,1} & \dots & \Phi_{0,N} & \dots \\ \Phi_{1,0} & \Phi_{1,1} & \dots & \Phi_{1,N} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{N,0} & \Phi_{N,1} & \dots & \Phi_{N,N} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Observe that $\Phi(\zeta, \eta)$ is given by

$$\Phi(\zeta, \eta) = \begin{pmatrix} I_{\mathtt{w}} & \zeta I_{\mathtt{w}} & \dots \zeta^N I_{\mathtt{w}} \dots \end{pmatrix} \text{mat}(\Phi) \begin{pmatrix} I_{\mathtt{w}} \\ \eta I_{\mathtt{w}} \\ \vdots \\ \eta^N I_{\mathtt{w}} \\ \vdots \end{pmatrix}$$

$\text{mat}(\Phi)$ is called the *coefficient matrix* of Φ. Note that Φ is symmetric if and only if $\text{mat}(\Phi)$ is symmetric.

Also note that $\text{mat}(\Phi)$ can be factored as

$$\text{mat}(\Phi) = \text{mat}(N)^\top \text{mat}(M), \tag{3.1}$$

where $\text{mat}(N)$ and $\text{mat}(M)$ are infinite matrices having a finite number of rows and a finite number of nonzero elements. This factorization leads to the following expression for $\Phi(\zeta, \eta)$:

$$\Phi(\zeta, \eta) = N(\zeta)^\top M(\eta) \tag{3.2}$$

where $N(\xi) = \text{mat}(N)\ \text{col}(I_{\mathtt{w}}, \xi I_{\mathtt{w}}, \xi^2 I_{\mathtt{w}}, \dots)$ and $M(\xi) = \text{mat}(M)\ \text{col}(I_{\mathtt{w}}, \xi I_{\mathtt{w}}, \xi^2 I_{\mathtt{w}}, \dots)$. The factorization (3.1) of $\text{mat}(\Phi)$ is not unique. If we impose the condition that the rows of M and N are linearly independent over $\mathbb{R}$, then the factorization (3.2) is called a *canonical factorization* of Φ.

We now define the signature of a symmetric two-variable polynomial matrix.

Definition 3.1. *Let n denote the highest power of ζ or η occurring in $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Define*

$$X(\xi) := \textit{col}(I_{\mathtt{w}}, \xi I_{\mathtt{w}}, \xi^2 I_{\mathtt{w}}, \dots, \xi^n I_{\mathtt{w}})$$

Let $\tilde{\Phi} \in \mathbb{R}_s^{(n+1)\mathtt{w}\times(n+1)\mathtt{w}}$ be such that $\Phi(\zeta,\eta) = X(\zeta)^\top \tilde{\Phi} X(\eta)$. Let ϕ_+ and ϕ_- denote the number of positive and negative eigenvalues of $\tilde{\Phi}$. Then the matrix

$$\Sigma_\Phi = \begin{bmatrix} I_{\phi_+} & 0_{\phi_+\times\phi_-} \\ 0_{\phi_-\times\phi_+} & -I_{\phi_-} \end{bmatrix}$$

is called the signature *of Φ, the pair (ϕ_+, ϕ_-) is called the* inertia *of Φ and ϕ_+ and ϕ_- are respectively called the* positive *and* negative inertia *of Φ.*

With respect to the above definition, observe that $\tilde{\Phi}$ comprises of all the nonzero entries of the coefficient matrix $\mathrm{mat}(\Phi)$ of Φ and some zero entries.

If $\mathrm{mat}(\Phi)$ is symmetric, then it can be factored as $\mathrm{mat}(\Phi) = \mathrm{mat}(M)^\top \Sigma_M \mathrm{mat}(M)$, where $\mathrm{mat}(M)$ is a real matrix with a finite number of rows, infinite number of columns and a finite number of nonzero entries, Σ_M is of the form

$$\Sigma_M = \begin{bmatrix} I_{\mathtt{w}_1} & 0_{\mathtt{w}_1\times\mathtt{w}_2} \\ 0_{\mathtt{w}_2\times\mathtt{w}_1} & -I_{\mathtt{w}_2} \end{bmatrix}$$

where $\mathtt{w}_1$ and $\mathtt{w}_2$ are nonnegative integers. Thus, the two-variable polynomial matrix Φ can be written as

$$\Phi(\zeta,\eta) = M(\zeta)^\top \Sigma_M M(\eta) \tag{3.3}$$

where $M(\xi) = \mathrm{mat}(M)\,\mathrm{col}(I_\mathtt{w}, \xi I_\mathtt{w}, \xi^2 I_\mathtt{w}, \ldots)$. This decomposition of Φ is not unique, but if we impose the condition that the rows of M are linearly independent over $\mathbb{R}$, then Σ_M is unique in equation (3.3) and is equal to the signature Σ_Φ of Φ. The resulting factorization

$$\Phi(\zeta,\eta) = M(\zeta)^\top \Sigma_\Phi M(\eta) \tag{3.4}$$

is called a *symmetric canonical factorization* of Φ. A symmetric canonical factorization of a given $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is not unique. However, if we assume that (3.4) is a symmetric canonical factorization of Φ, then another symmetric canonical factorization of Φ can be obtained by replacing $M(\xi)$ in equation (3.4) with $UM(\xi)$, where U is a real square matrix of the same dimension as Σ_Φ, such that $U^\top \Sigma_\Phi U = \Sigma_\Phi$.

Example 3.1. If $\Phi(\zeta,\eta) = \zeta^3\eta^2 + \zeta^2\eta^3 + \zeta^2\eta + \zeta\eta^2 + \zeta + \eta$, then observe that with reference to Definition 3.1,

$$\tilde{\Phi} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

It can be verified that two of the eigenvalues of $\tilde{\Phi}$ are positive and the remaining two are negative. It follows that $\Sigma_\Phi = \mathrm{diag}(1, 1, -1, -1)$. It can also be verified that

$$\Phi(\zeta,\eta) = M(\zeta)^\top \Sigma_\Phi M(\eta) \tag{3.5}$$

where $M(\xi) = \frac{1}{\sqrt{2}}\text{col}(\xi^3 + \xi^2, \xi^2 + \xi + 1, \xi^3 - \xi^2, \xi^2 - \xi + 1)$. Observe that the rows of M are linearly independent over $\mathbb{R}$. Hence equation (3.5) is a symmetric canonical factorization of Φ, $(2, 2)$ is the inertia of Φ and the positive and negative inertia of Φ are both equal to 2.

We now define the McMillan degree of a symmetric two-variable polynomial matrix.

Definition 3.2. *Assume that* $\Phi(\zeta, \eta) = M(\zeta)^\top \Sigma_\Phi M(\eta)$ *is a symmetric canonical factorization of a two-variable polynomial matrix* $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. *Then the* McMillan degree *of* Φ *is the McMillan degree of the behaviour* $\mathfrak{B} = Im\big(M(\frac{d}{dt})\big)$.

The McMillan degree of a given $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ is denoted by $\mathtt{n}(\Phi)$.

The main advantage of associating two-variable polynomial matrices with QDFs is that they allow a very convenient calculus. We now illustrate this using the notion of the *derivative* of a QDF which is used extensively in this thesis.

Definition 3.3. *A QDF* Q_Ψ *is called the* derivative of a QDF Q_Φ *with* $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, *denoted by* $Q_\Psi = \frac{d}{dt}Q_\Phi$ *if* $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ *for all* $w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

In terms of the two-variable polynomial matrices associated with Q_Ψ and Q_Φ, the relationship in Definition 3.3 can be expressed as follows: Q_Ψ is the derivative of Q_Φ if and only if for the corresponding two-variable polynomial matrices, there holds $(\zeta + \eta)\Phi(\zeta, \eta) = \Psi(\zeta, \eta)$ (see Willems and Trentelman (1998), p. 1710).

3.2 Equivalence of QDFs and R-canonicity

We begin this section with the definition of equivalence of BDFs and QDFs along a behaviour.

Definition 3.4. *Two BDFs* L_{Φ_1} *and* L_{Φ_2} *are said to be* equivalent along a behaviour $\mathfrak{B}$ *denoted by* $L_{\Phi_1} \stackrel{\mathfrak{B}}{=} L_{\Phi_2}$, *if* $L_{\Phi_1}(w_1, w_2) = L_{\Phi_2}(w_1, w_2)\ \forall\ w_1, w_2 \in \mathfrak{B}$.

Definition 3.5. *Two QDFs* Q_{Φ_1} *and* Q_{Φ_2} *are said to be* equivalent along a behaviour $\mathfrak{B}$ *denoted by* $Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$, *if* $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)\ \forall\ w \in \mathfrak{B}$.

We now define the notion of a BDF and a QDF being zero along a behaviour.

Definition 3.6. *A BDF* L_Φ *is said to be* zero along a behaviour $\mathfrak{B}$ *denoted by* $L_\Phi \stackrel{\mathfrak{B}}{=} 0$, *if*

$$L_\Phi(w_1, w_2) = 0 \quad \forall \quad w_1, w_2 \in \mathfrak{B}$$

Definition 3.7. *A QDF Q_Φ is said to be* zero along a behaviour $\mathfrak{B}$ *denoted by* $Q_\Phi \stackrel{\mathfrak{B}}{=} 0$, *if*

$$Q_\Phi(w) = 0 \quad \forall \quad w \in \mathfrak{B}$$

The next Proposition gives a condition on a two-variable polynomial matrix under which the corresponding QDF is zero along a given behaviour.

Proposition 3.8. *Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ have the kernel representation $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$. Then $Q_\Phi \stackrel{\mathfrak{B}}{=} 0$ iff there exists $F \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\zeta,\eta]$ such that*

$$\Phi(\zeta,\eta) = F(\eta,\zeta)^\top R(\eta) + R^\top(\zeta)F(\zeta,\eta)$$

Proof. See Proposition 3.2 of Willems and Trentelman (1998). ∎

If $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, with $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ and $\Phi_1, \Phi_2 \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, from Proposition 3.8, it is easy to see that $Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$ iff there exists $F \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\zeta,\eta]$ such that

$$\Phi_1(\zeta,\eta) - \Phi_2(\zeta,\eta) = F(\eta,\zeta)^T R(\eta) + R^T(\zeta)F(\zeta,\eta)$$

We now define the notion of R-canonicity of polynomial operators.

Definition 3.9. *Consider a nonsingular $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. A polynomial matrix $R_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ is* R-canonical *if R_1R^{-1} is strictly proper.*

If $R \in \mathbb{R}[\xi]$ has degree equal to n, then from the above definition, it follows that the space of R-canonical polynomials is the space of polynomials of degree less than or equal to $n-1$.

We now define the concept of R-canonical representative of a polynomial matrix.

Definition 3.10. *Given a nonsingular $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $R_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$, the polynomial matrix $R_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ is called an* R-canonical representative *of R_1 if R_2 is R-canonical and $R_1(\frac{d}{dt})w = R_2(\frac{d}{dt})w$ for every trajectory $w \in ker\big(R(\frac{d}{dt})\big)$.*

If a polynomial matrix is R-canonical where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, then it is easy to see that its R-canonical representative is equal to itself. Assume that a given $R_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ is not R-canonical. Then it is easy to see that there exist $F \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ and a unique strictly proper $R_2' \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}(\xi)$, such that

$$R_1(\xi)R(\xi)^{-1} = F(\xi) + R_2'(\xi)$$

Define $R_2(\xi) := R_2'(\xi)R(\xi)$. Observe that R_2 is the R-canonical representative of R_1 because $R_2(\frac{d}{dt})w = R_1(\frac{d}{dt})w$ for every trajectory $w \in \ker\big(R(\frac{d}{dt})\big)$ and R_2 is R-canonical. It is easy to see that every $R_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ has a unique R-canonical representative. It is also easy to see that among all polynomial matrices P such that $P(\frac{d}{dt})w = R_1(\frac{d}{dt})w$ for every trajectory w of a given autonomous behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, with R square and nonsingular and a given $R_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$, the R-canonical representative of R_1 has elements of the lowest degree.

Observe that the set of equivalence classes of QDFs under $\stackrel{\mathfrak{B}}{=}$ is a vector space over $\mathbb{R}$. With every equivalence class of QDFs under $\stackrel{\mathfrak{B}}{=}$ associated with an autonomous behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, we associate a certain representative QDF whose associated two-variable polynomial matrix is known as the R-canonical representative. Both the notions of R-canonicity and R-canonical representative of two-variable polynomial matrices are similar to the ones defined for the case of one-variable polynomial matrices. Below we define the notion of R-canonicity of two-variable polynomial matrices.

Definition 3.11. *Consider a nonsingular $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. A two-variable polynomial matrix $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is* R-canonical *if $R(\zeta)^{-\top}\Phi(\zeta,\eta)R(\eta)^{-1}$ is strictly proper.*

If $R \in \mathbb{R}[\xi]$ with degree equal to n, then from the definition, it follows that the space of R-canonical two-variable polynomials is spanned by monomials $\zeta^k\eta^j$, with $k, j \leq n-1$. For example if $r(\xi) := \xi^2 + 2\xi + 3$, then the space of r-canonical two-variable polynomials has 1, ζ, η, $\zeta\eta$ as its basis elements.

Definition 3.12. *Given a nonsingular $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, a two-variable polynomial matrix $\Phi_1 \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is called an* R-canonical representative *of Φ if Φ_1 is R-canonical and $Q_\Phi(w) = Q_{\Phi_1}(w)$ for every trajectory $w \in$ ker$\big(R(\frac{d}{dt})\big)$.*

If a given two-variable polynomial matrix is R-canonical where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$, then it is easy to see that its R-canonical representative is the same as the given matrix. Now consider a two-variable polynomial matrix $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ which is not R-canonical. Let $\Phi(\zeta,\eta) = M(\zeta)^\top\Sigma_\Phi M(\eta)$ be a symmetric canonical factorization of Φ. Let $\mathtt{p}$ and $M_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$ denote the number of rows and an R-canonical representative of M respectively. Then there exists $F \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$, such that

$$M(\xi) = F(\xi)R(\xi) + M_1(\xi)$$

Define $\Phi_1(\zeta,\eta) := M_1(\zeta)^\top\Sigma_\Phi M_1(\eta)$. Observe that Φ_1 is R-canonical and $Q_\Phi \stackrel{\mathfrak{B}}{=} Q_{\Phi_1}$, where $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$. Consequently Φ_1 is an R-canonical representative of Φ. It is easy to see that every $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ has an R-canonical representative.

If $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is nonsingular and $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is not R-canonical, then observe that an R-canonical representative of Φ has elements of degree lesser than or equal to the corresponding elements of Φ. Thus among the QDFs belonging to any equivalence class under $\stackrel{\mathfrak{B}}{=}$, where $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, the two-variable polynomial matrix associated with an R-canonical representative of the two-variable polynomial matrices inducing QDFs in the equivalence class has the lowest degree elements.

3.3 Nonnegativity and positivity of a QDF

The notion of nonnegativity and positivity of a QDF is essential in applications like Lyapunov theory. Below we define the two notions.

Definition 3.13. *Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Q_Φ is said to be nonnegative, denoted by $Q_\Phi \geq 0$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$; and positive denoted by $Q_\Phi > 0$, if $Q_\Phi \geq 0$, and $Q_\Phi(w) = 0$ implies $w = 0$.*

The following Proposition gives conditions on the two-variable polynomial matrix Φ corresponding to a QDF Q_Φ under which Q_Φ is nonnegative or positive.

Proposition 3.14. *Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $Q_\Phi \geq 0$ iff there exists $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\Phi(\zeta,\eta) = D(\zeta)^\top D(\eta)$, and $Q_\Phi > 0$ iff $\Phi(\zeta,\eta) = D(\zeta)^\top D(\eta)$ and $D(\lambda)$ has full column rank $\mathtt{w}$ for all $\lambda \in \mathbb{C}$.*

Proof. See p. 1712 of Willems and Trentelman (1998). ■

We now define the notion of a QDF being nonnegative or positive along a particular behaviour.

Definition 3.15. *Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Q_Φ is said to be nonnegative along $\mathfrak{B}$, denoted by $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ if $Q_\Phi(w) \geq 0$ for all $w \in \mathfrak{B}$, and positive along $\mathfrak{B}$ denoted by $Q_\Phi \overset{\mathfrak{B}}{>} 0$, if $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $Q_\Phi(w) = 0$ implies $w = 0$.*

Below we give the condition on the two-variable polynomial matrix corresponding to a QDF such that it is either nonnegative or positive along a given behaviour.

Proposition 3.16. *Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and let $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$. Then*

1. *$Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ iff there exists $F \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta,\eta]$ and $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, such that*

$$\Phi(\zeta,\eta) = D(\zeta)^\top D(\eta) + F(\eta,\zeta)^\top R(\eta) + R^\top(\zeta)F(\zeta,\eta)$$

2. *$Q_\Phi \overset{\mathfrak{B}}{>} 0$ iff $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $col(D(\lambda), R(\lambda))$ has full column rank for all $\lambda \in \mathbb{C}$.*

Proof. See Proposition 3.5 of Willems and Trentelman (1998). ■

We now illustrate the concepts discussed so far in this chapter using the example of an RLC circuit shown below:

Example 3.2. Consider the RLC circuit depicted in Figure 3.1. Let V be the voltage drop across the source and let I be the current through the circuit as shown in the figure.

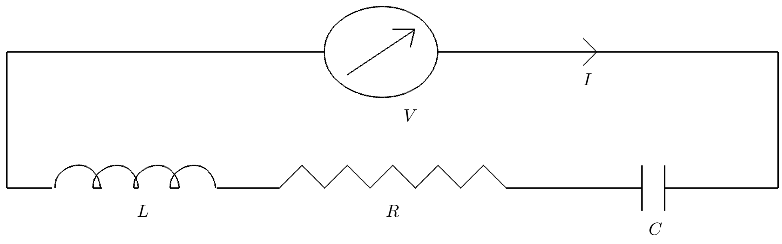

FIGURE 3.1: An electrical example

Let V_L, V_R and V_C denote the voltages across the inductor, resistor and the capacitor respectively. Assume that $L \neq 0$, $C \neq 0$ and $R \neq 0$. The following relations hold.

$$\begin{aligned} V &= V_L + V_R + V_C \\ V_L &= L\frac{dI}{dt} \\ V_R &= IR \\ I &= C\frac{dV_C}{dt} \end{aligned}$$

In this case, let the variables of interest be V and I. Eliminating the remaining variables from the above set of equations, we obtain the differential equation

$$C\frac{dV}{dt} = RC\frac{dI}{dt} + LC\frac{d^2I}{dt^2} + I \tag{3.6}$$

This equation defines a linear differential behaviour, namely

$$\mathfrak{B} = \{(V, I) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^2) \ \mid \ (3.6) \text{ is satisfied } \}$$

In the kernel form $\mathfrak{B}$ is given by $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where

$$R(\xi) = [\ -C\xi \quad LC\xi^2 + RC\xi + 1\ ]$$

The total stored energy of the system is given by

$$E = \frac{1}{2}\left[LI^2 + C\left(V - L\frac{dI}{dt} - IR\right)^2\right]$$

This can be represented by a QDF $Q_{\Phi_1}(w)$ where $w = \mathrm{col}(V, I)$ and

$$\Phi_1(\zeta, \eta) := \frac{1}{2}\begin{bmatrix} C & -C(L\eta + R) \\ -C(L\zeta + R) & L + C(L\zeta + R)(L\eta + R) \end{bmatrix}$$

Now Q_{Φ_1} is positive over $\mathfrak{B}$ because Φ_1 can be shown to be equal to $D(\zeta)^T D(\eta)$, where

$$D(\xi) = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -\sqrt{L} \\ -\sqrt{C} & \sqrt{C}(L\xi + R) \end{bmatrix}$$

and $D(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. The instantaneous power is defined as the quantity

$$P = VI = CV\frac{dV}{dt} - LCV\frac{d^2I}{dt^2} - RCV\frac{dI}{dt} \tag{3.7}$$

The power can be represented with a QDF $Q_{\Phi_2}(w)$, where

$$\Phi_2(\zeta,\eta) := \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or equivalently, by making use of equation (3.7), by the QDF Q_{Φ_3} associated with

$$\Phi_3(\zeta,\eta) := \frac{1}{2}\begin{bmatrix} C(\zeta+\eta) & -LC\eta^2 - RC\eta \\ -LC\zeta^2 - RC\zeta & 0 \end{bmatrix}$$

These two QDFs are equivalent along $\mathfrak{B}$ because

$$\Phi_3(\zeta,\eta) - \Phi_2(\zeta,\eta) = F(\eta,\zeta)^T R(\eta) + R(\zeta)^T F(\zeta,\eta)$$

where $F(\zeta,\eta) = [\ \frac{1}{2}\quad 0\ ]$. We now show using the calculus of QDFs that the rate of increase of stored energy is equal to the difference between power and the rate of dissipation through the resistor. The two-variable polynomial matrix corresponding to the rate of increase of energy is given by

$$(\zeta+\eta)\Phi_1(\zeta,\eta) = \Phi_3(\zeta,\eta) + \frac{1}{2}[\Phi_4(\zeta,\eta) + \Phi_5(\zeta,\eta)]$$

where

$$\Phi_4(\zeta,\eta) := \begin{bmatrix} 0 & 0 \\ 0 & (LC\zeta^2 + RC\zeta)(L\eta + R) + (L\zeta + R)(LC\eta^2 + RC\eta) \end{bmatrix}$$

$$\Phi_5(\zeta,\eta) := \begin{bmatrix} 0 & -C\zeta(L\eta + R) \\ -C\eta(L\zeta + R) & L(\zeta+\eta) \end{bmatrix}$$

Define

$$\Phi_6(\zeta,\eta) := \begin{bmatrix} 0 & C\zeta(L\eta + R) \\ C\eta(L\zeta + R) & -L(\zeta+\eta) - 2R \end{bmatrix}$$

Now, $Q_{\Phi_4} \overset{\mathfrak{B}}{=} Q_{\Phi_6}$, as

$$\Phi_4(\zeta,\eta) - \Phi_6(\zeta,\eta) = F_1(\eta,\zeta)^T R(\eta) + R(\zeta)^T F_1(\zeta,\eta)$$

where $F_1(\zeta,\eta) = [\ 0\quad L\eta + R\ ]$. Since $Q_{\Phi_2} \overset{\mathfrak{B}}{=} Q_{\Phi_3}$, along $\mathfrak{B}$, the following two-variable polynomial matrix corresponds to the rate of increase of energy

$$\Phi_2(\zeta,\eta) + \frac{1}{2}[\Phi_6(\zeta,\eta) + \Phi_5(\zeta,\eta)] = \Phi_2(\zeta,\eta) - \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix}$$

Notice that in the right hand side of the above equation, the first term stands for power, while the second term stands for dissipation through the resistor. Also observe that we have used only algebra of two-variable polynomial matrices for proving the result.

3.4 Stationarity with respect to a QDF

This topic has been drawn from Rapisarda and Trentelman (2004), pp. 778-779. An important concept in the calculus of two-variable polynomial matrices is the map

$$\begin{aligned} \partial \quad & : \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \longrightarrow \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi] \\ \partial\Phi(\xi) \quad & := \Phi(-\xi,\xi) \end{aligned}$$

which is used in the context of path-independent integrals and in order to describe the set of stationary trajectories associated with a QDF, as we now illustrate. Let $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ denote the subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ consisting of compact support functions. Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Consider the corresponding QDF Q_Φ on $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$. For a given w, we define the *cost degradation* of adding the compact-support function $\delta \in \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ to w as

$$J_w(\delta) = \int_{-\infty}^{+\infty} (Q_\Phi(w+\delta) - Q_\Phi(w))dt$$

It is easy to see that the cost degradation equals

$$J_w(\delta) = \int_{-\infty}^{+\infty} Q_\Phi(\delta)dt + 2\int_{-\infty}^{+\infty} L_\Phi(w,\delta)dt \tag{3.8}$$

where L_Φ is a bilinear differential form as defined in the beginning of this chapter. We call the second integral on the right hand side of equation (3.8) the *variation associated with* w denoted by $V_\Phi(w,\delta)$. It is easy to see that $V_\Phi : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \times \mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \to \mathbb{R}$. We call w a *stationary trajectory* of Q_Φ if $V_\Phi(w,\delta)$ is the zero functional.

The following proposition can be used to find the trajectories that are stationary for a given QDF.

Proposition 3.17. *Let* $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. *Then* $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ *is a stationary trajectory of the QDF* Q_Φ *if and only if* w *satisfies the system of differential equations*

$$\partial\Phi\left(\frac{d}{dt}\right)w = 0 \tag{3.9}$$

Proof. See Rapisarda and Trentelman (2004), Proposition 4.1, p. 779. ■

The equations (3.9) can be interpreted as the *Euler-Lagrange equations* of the variational problem associated with Q_Φ. For more details on Euler-Lagrange equations, the reader is referred to pp. 44-45, Goldstein et al. (2002) and pp. 40-42, Gelfand and Fomin (1963).

3.5 Conserved quantities associated with an oscillatory behaviour

The notion of a *conserved quantity* was first defined by Rapisarda and Willems (2005), and it is used for defining lossless systems in Chapter 4 of this thesis. Below we give its definition.

Definition 3.18. *Let* $\mathfrak{B}$ *be a linear autonomous behaviour. A QDF* Q_Φ *is a* conserved quantity *for* $\mathfrak{B}$ *if*

$$\frac{d}{dt}Q_\Phi \overset{\mathfrak{B}}{=} 0 \tag{3.10}$$

Thus, conserved quantities are those QDFs, whose derivative is zero along the trajectories of the behaviour. Note that the trivial QDF $Q_\Phi = 0$ is always conserved. Any conserved QDF which is identically not equal to zero will be called "nontrivial conserved quantity" in this thesis. Let $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is nonsingular. From Proposition 3.8, it follows that if Q_Φ is a conserved quantity for $\mathfrak{B}$, then there exists $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that

$$(\zeta+\eta)\Phi(\zeta,\eta) = F(\eta,\zeta)^\top R(\eta) + R^\top(\zeta)F(\zeta,\eta) \tag{3.11}$$

Let Q_{Φ_1} and Q_{Φ_2} be two conserved quantities for a behaviour $\mathfrak{B}$. Then from equation (3.11), it follows that $\alpha_1 Q_{\Phi_1} + \alpha_2 Q_{\Phi_1}$ is also a conserved quantity for $\alpha_1, \alpha_2 \in \mathbb{R}$. Thus conserved quantities for a behaviour have a linear structure and hence form a linear subspace of the space of QDFs modulo the behaviour.

Consider an oscillatory behaviour $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, where $r \in \mathbb{R}[\xi]$. If r is an even polynomial of degree $2n$, then it can be shown (see p. 188 of Rapisarda and Willems (2005)) that the two-variable polynomials $\gamma_i(\zeta,\eta)$ given by

$$\gamma_i(\zeta,\eta) = \frac{r(\zeta)\eta^{2i+1} + r(\eta)\zeta^{2i+1}}{\zeta+\eta}$$

$i = 0, 1, \ldots, n-1$, induce a basis for the space of conserved quantities modulo $\mathfrak{B}$. If r is an odd polynomial of degree $2n+1$, then it can be shown that a basis of conserved quantities modulo $\mathfrak{B}$ is induced by the set $\{\gamma_i'(\zeta,\eta)\}_{i=0,1,\ldots,n}$, where

$$\gamma_i'(\zeta,\eta) = \frac{r(\zeta)\eta^{2i} + r(\eta)\zeta^{2i}}{\zeta+\eta}$$

3.6 Zero-mean quantities associated with an oscillatory behaviour

The notion of zero-mean quantities was introduced by Rapisarda and Willems (2005).

Definition 3.19. *Let* $\mathfrak{B}$ *be a linear autonomous behaviour. A QDF* Q_Φ *is* zero-mean *over* $\mathfrak{B}$ *if*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Q_\Phi(w)(t)dt = 0 \qquad \forall w \in \mathfrak{B}$$

Thus the time-average of a zero-mean quantity over the entire real axis is zero along the trajectories of the behaviour. Rapisarda and Willems (2005) also introduced the notion of a *trivially zero-mean quantity* which is a zero-mean quantity that is zero-mean for all oscillatory systems, not just the given system.

As an example of a trivially zero mean quantity, consider the QDF $Q_\Phi(w) = 2w\frac{dw}{dt}$, where $\Phi \in \mathbb{R}[\zeta,\eta]$. Now

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Q_\Phi(w)(t)dt = \lim_{T\to\infty} \frac{1}{2T}(w^2|_{t=T} - w^2|_{t=-T}) \tag{3.12}$$

If w is a trajectory belonging to an oscillatory behaviour, then it will be linear combination of sinusoidal functions, i.e

$$w(t) = \sum_{k=1}^{n}(A_k \sin(\omega_k t) + B_k \cos(\omega_k t))$$

with $\omega_k, A_k, B_k \in \mathbb{R}$ for all k. In this case, the limit (3.12) always tends to 0, and hence Q_Φ is a trivially zero mean quantity.

Below, we give algebraic characterization of zero-mean and trivially zero-mean quantities for oscillatory systems.

Proposition 3.20. *Let* $\mathfrak{B}$ *be an oscillatory behaviour given by* $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, *where* $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. *Then*

1. $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *induces a zero-mean quantity over* $\mathfrak{B}$ *if and only if there exist* $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, $X \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *such that*

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) + R(\zeta)^T X(\zeta,\eta) + X(\eta,\zeta)^T R(\eta) \tag{3.13}$$

2. $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *induces a trivially zero-mean quantity if and only if there exists* $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *such that*

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) \tag{3.14}$$

Proof. See Proposition 15, p. 189 and Proposition 22, p. 192 of Rapisarda and Willems (2005). ■

From the above Proposition, it follows that zero-mean quantities and trivially zero-mean quantities over an oscillatory behaviour have a linear structure and hence trivially zero-mean quantities form a linear subspace of the space of zero-mean quantities which in turn is a subspace of the space of QDFs associated with the behaviour. Rapisarda and Willems (2005) also introduced the notion of *intrinsically zero-mean quantities* which form the space of zero-mean quantities complementary to that consisting of trivially zero-mean quantities. In chapter 5, we will show that an intrinsic definition can be given for intrinsically zero-mean quantities.

3.7 Lyapunov theory of stability

Lyapunov theory of stability of linear differential systems that are described by higher order differential equations has been studied in Willems and Trentelman (1998). Here we give some of the important concepts pertaining to Lyapunov theory that are used in this thesis as described in Willems and Trentelman (1998). We begin with the following definition.

Definition 3.21 (Asymptotically stable behaviour). *A behaviour* $\mathfrak{B}$ *is said to be* asymptotically stable *if* $(w \in \mathfrak{B}) \Rightarrow (\lim_{t\to\infty} w(t) = 0)$ *and* stable *if* $(w \in \mathfrak{B}) \Rightarrow$ *(*$w(t)$ *is bounded in the interval* $t \in [0, \infty)$*).*

For a behaviour to be stable, it has to be autonomous, for if there were free variables in the behaviour, then those could be chosen in such a way that $w(t)$ is unbounded in the interval $t \in [0, \infty)$. We now define the notion of a *Hurwitz polynomial matrix.*

Definition 3.22. *A polynomial matrix* $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ *is said to be* Hurwitz *if det*(R) *is a Hurwitz polynomial, i.e a polynomial with all its roots in the open left half of the complex plane.*

We now give an algebraic condition on the kernel representation of an autonomous behaviour under which it is asymptotically stable.

Lemma 3.23. *An autonomous behaviour* $\mathfrak{B}$ *with a minimal kernel representation* $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$ *is asymptotically stable iff* R *is square and Hurwitz.*

We now give a very important result related to Lyapunov theory of stability of linear differential systems. This result is used in chapter 6 of this thesis.

Theorem 3.24. *Let* $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$*. Then* $\mathfrak{B}$ *is asymptotically stable iff there exists* $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ *such that* $Q_\Psi \overset{\mathfrak{B}}{\geq} 0$ *and* $\frac{d}{dt}Q_\Psi \overset{\mathfrak{B}}{<} 0$.

Proof. See proof of Theorem 4.3, p. 1735, Willems and Trentelman (1998). ■

Example 3.3. Consider a behaviour $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, where $r(\xi) := \xi^2 + 3\xi + 4$. Define $\Psi(\zeta, \eta) := \zeta\eta + 4$. Observe that $Q_\Psi \overset{\mathfrak{B}}{>} 0$. Also observe that if $Q_\Phi := \frac{d}{dt} Q_\Psi$, then

$$\Phi(\zeta, \eta) = \zeta r(\eta) + r(\zeta)\eta - 6\zeta\eta$$

Define $\Phi_1(\zeta, \eta) := -6\zeta\eta$. Observe that $Q_\Phi \overset{\mathfrak{B}}{=} Q_{\Phi_1}$, and $Q_{\Phi_1} \overset{\mathfrak{B}}{<} 0$. This implies that $Q_\Phi \overset{\mathfrak{B}}{<} 0$. From Theorem 3.24, it follows that $\mathfrak{B}$ is asymptotically stable.

Chapter 4

Lossless systems

4.1 Introduction

What is a lossless system? This problem has occupied theoretical physicists and applied mathematicians alike since quite sometime. In theoretical physics (see Young et al. (1999)), a system is called lossless if the work done by a force is path-independent and equal to the difference between the final and initial values of an energy function that remains positive for non-zero trajectories of the system.

In most of the work done so far in the area of lossless systems, characterisation of losslessness is done assuming a given supply rate. An example for such a characterisation is the one in Willems (1972), in which losslessness is defined with respect to a given scalar function associated with the system, known as the supply rate. The system is called lossless if the supply rate is the derivative of another scalar function, known as the storage function, along the trajectories of the system. Pillai and Willems (2002) have extended the concept of losslessness introduced in Willems (1972) for the case of distributed systems.

A lot of research has been carried out in the area of characterisation of lossless systems in the state-space. For example, Hill and Moylan (see Hill and Moylan (1976) and Hill and Moylan (1980)) have characterized lossless nonlinear systems in the state space in a way similar to Willems (1972), and have proved that under certain conditions, there exists a positive definite storage function for the system. In many cases, the term "conservative" has been used instead of "lossless". Weiss et al. (2001) and Weiss and Tucsnak (2003) have given algebraic characterizations of energy preserving- and of conservative linear systems based on a state space description of the system. Here, a system is called energy preserving if the rate of change of a scalar positive definite function defined on its state space called energy, is equal to the difference between an incoming power and an outgoing power, which are respectively assumed to be the square of the norms of the input signal u and the output signal y. Note that in the sense of Willems (1972), if a system is energy preserving, then it is lossless with respect to the difference between the incoming and outgoing power. For a given energy preserving system, Weiss et al. (2001) define a related system called its dual. They call a system conservative if both the system and its dual are energy preserving. In addition, they also give results about the stability, controllability and observability of conservative systems and illustrate these with the help of a model of a controlled beam. Malinen et al. (2006) have extended the characterization of Weiss and Tucsnak (2003) for the case of infinite dimensional linear systems. Wyatt et al. (1982) have studied losslessness in the context of nonlinear network theory and proposed that a system be defined as lossless if the energy required to travel between any two points in the state space is independent of the path taken. They have also shown that under certain conditions, the interconnection of lossless n-ports is lossless.

In the following papers, special assumptions have been made in order to characterize conservative systems. Jacyno (1984) has constructed a class of nonlinear autonomous conservative systems, starting from the general class of nonlinear systems given by $\dot{x} = F(x)$, by deriving a certain condition on $F(x)$ and the total energy function $Q(x)$ for the system. Here it is assumed that the total energy function $Q(x)$ is a positive definite function of the state variables x. Van der Schaft (see van der Schaft (2000) and van der Schaft (2002)) has studied the properties of Hamiltonian and port-Hamiltonian conservative systems starting from sets of equations namely Hamiltonian, respectively port-Hamiltonian equations of motion. Here, it is assumed that the Hamiltonian (total energy) for the system is given a priori, i.e. one does not begin with the basic equations of motion.

The main aim of this chapter is to give a characterisation of higher order linear lossless systems. We now explain how using an example of a mechanical system.

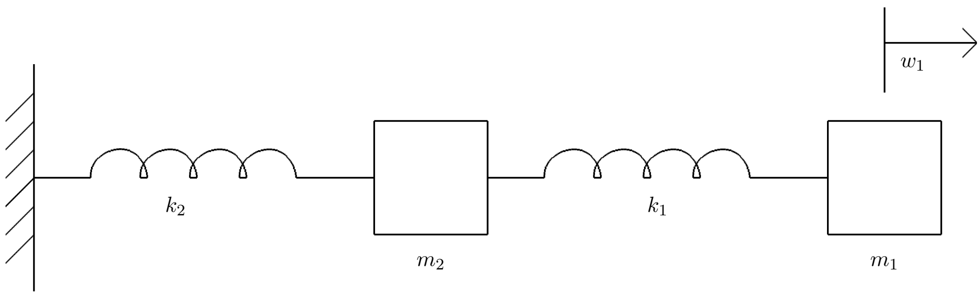

FIGURE 4.1: A mechanical example

Example 4.1. Consider two masses m_1 and m_2 attached to springs with constants k_1 and k_2. The first mass is connected to the second one via the first spring, and the second mass is connected to the wall with the second spring as shown in Figure 4.1. Denote by w_1 and w_2, the positions of the first and the second mass respectively. We first obtain the equations of motion of the two masses as

$$m_1\frac{d^2w_1}{dt^2}+k_1w_1-k_1w_2=0 \tag{4.1}$$

$$-k_1w_1+m_2\frac{d^2w_2}{dt^2}+(k_1+k_2)w_2=0 \tag{4.2}$$

Assume that in this case, we are interested only in the evolution of w_1. Via the process of elimination, we can lump equations (4.1) and (4.2) to obtain the differential equation governing w_1 as

$$r\left(\frac{d}{dt}\right)w_1=\frac{d^4}{dt^4}w_1+\left(\frac{k_1+k_2}{m_2}+\frac{k_1}{m_1}\right)\frac{d^2}{dt^2}w_1+\left(\frac{k_1k_2}{m_1m_2}\right)w_1=0$$

The above is a first principles model for the system. In this case, we can construct a state space model for the system as follows. Define $x_1:=w_1$, $x_2:=\frac{dw_1}{dt}$, $x_3:=\frac{d^2w_1}{dt^2}$, $x_4:=\frac{d^3w_1}{dt^3}$, $B:=\mathrm{col}(1,0,0,0)$ and

$$A:=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-\frac{k_1k_2}{m_1m_2}&0&-\left(\frac{k_1+k_2}{m_2}+\frac{k_1}{m_1}\right)&0\end{bmatrix}$$

Then $\frac{dx}{dt}=Ax$, $w_1=B^\top x$ is a state space model for the system. Note that mathematical modeling of a system in general, does not automatically lead to first order equations or a state space model for the system. Like the case at hand, a state space model in many cases needs artificial construction of states from the given model. This calls for a need to deduce the properties of a system using its higher order governing differential equations. In the case of our example, we obtain a fourth order governing differential equation. From physical insight, we can often deduce whether a given system is lossless or not. The natural question that arises here is whether this can be done automatically, i.e whether we can deduce that a system is lossless or not directly from a higher order description of the system. Another question that arises here is whether we can obtain expressions for the total energy of the system, and its kinetic and potential energy components directly from a higher order description of a lossless system? These are some of the questions that we answer in this chapter.

The purpose of this chapter is to give a definition of linear lossless systems which agrees with the basic intuition, derived from physics, that the external work done on such a system is equal to the difference between the final and initial values of the total energy for the system. We also make use of the fact that the total energy of such a system is a quadratic functional in the system variables and their derivatives that is positive for all infinitely differentiable non-zero trajectories of the system. In a sense, our approach is similar to the one in Jacyno (1984), as we assume positivity of energy function.

The main differences are that unlike Jacyno (1984), on the one hand, we restrict our characterisation to only linear systems but on the other hand, we do not restrict our analysis to systems described by the equation $\dot{x} = F(x)$. Unlike most of the literature on lossless systems, we do not assume a given supply rate or derivative of energy function.

We first characterise losslessness for the case of autonomous systems. We define a lossless autonomous system as one for which there exists a conserved quantity that remains positive for all infinitely differentiable non-zero trajectories of the system. We show the equivalence between linear autonomous lossless systems and oscillatory systems.

We then extend the characterization of losslessness to open systems by making use of two properties. The first property is that the total energy of such a system is always positive for all infinitely differentiable nonzero trajectories of the system. The second property is that the rate of change of total energy is zero if the inputs of the system are made equal to zero.

The work presented in this chapter has been published (see Rao and Rapisarda (2008b) and Rao and Rapisarda (2008a)).

4.2 Autonomous lossless systems

In this section, we define an autonomous lossless system as one for which there exists a positive conserved quantity. We then prove the equivalence between autonomous lossless and oscillatory systems. This is first done for the case of scalar systems and then extended to the case of multivariable systems. We also discuss a few properties of energy functions of scalar lossless systems.

We begin with the following definition for autonomous lossless systems.

Definition 4.1. A linear autonomous behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is *lossless* if there exists a conserved quantity Q_E associated with $\mathfrak{B}$, such that $Q_E \overset{\mathfrak{B}}{>} 0$. Such a Q_E is called an *energy function* for the system.

Remark 4.2. The total energy of any physical system does not have an absolute measure as such. It is always defined with respect to an arbitrary choice of a reference level, which is hence indeterminate. However this indeterminacy is not important as in any physical application, it is always the difference between the initial and final values of energy that matters, and this difference is independent of the reference level. Hence it is convenient to define the reference level for the total energy of a system as its lower bound. This point has been elaborated upon in Sears (1946), pp. 128-129. While defining lossless systems, we fix the reference level or lower bound of the energy functions for the system at zero, which leads to positivity of energy functions. We implicitly assume that an energy function of a lossless system is bounded from below.

In order to prove that lossless autonomous systems are necessarily oscillatory, we examine all the linear autonomous scalar systems, for which conserved QDFs exist. To this end, we first determine the conditions under which a linear system has conserved QDFs associated with it, and the dimension of the space of conserved QDFs for such systems. We begin with the following definition.

Definition 4.3. Let $r \in \mathbb{R}[\xi]$. The *maximal even polynomial factor* of r is its monic even factor polynomial of maximal degree.

Lemma 4.4. *Let $r \in \mathbb{R}[\xi]$. Then the maximal even polynomial factor exists for r and is unique.*

Proof. Any even polynomial $r_e(\xi)$ can be written in terms of powers of ξ^2 only. Let $r_e(\xi) =: r_e'(\xi^2)$. It can be seen that the roots of $r_e(\xi)$ occur in pairs such that the sum of roots in each pair is equal to zero. The roots in each pair are square roots of some root of $r_e'(\xi^2)$. Consider a given polynomial

$$r(\xi) = r_n(\xi - \lambda_1)(\xi - \lambda_2)\dots(\xi - \lambda_n)$$

where $r_n \in \mathbb{R}$. Let n_1 be the maximum number of subsets of the set $L = \{\lambda_i\}_{i=1,\dots,n}$ each containing two distinct elements of the set such that the two terms of each subset add up to zero and any element λ_i of the set L does not occur in more than one subset. Without loss of generality, we can consider the subsets to be

$$\{\{\lambda_1, \lambda_2\}, \{\lambda_3, \lambda_4\}, \dots, \{\lambda_{2n_1-1}, \lambda_{2n_1}\}\}$$

Let

$$r_e(\xi) = (\xi^2 - \lambda_2^2)(\xi^2 - \lambda_4^2)\dots(\xi^2 - \lambda_{2n_1}^2)$$

and

$$p(\xi) = r_n(\xi - \lambda_{2n_1+1})(\xi - \lambda_{2n_1+2})\dots(\xi - \lambda_n)$$

It can be seen that $r(\xi) = r_e(\xi)p(\xi)$ and $p(\xi)$ is not a multiple of any even polynomial under the given assumptions. This proves the existence and uniqueness of the maximal even polynomial factor. ■

In the next proposition, we examine the conditions under which a linear behaviour $\mathfrak{B}$ has conserved QDFs associated with it.

Proposition 4.5. *Let $\mathfrak{B} = ker\big(r(\frac{d}{dt})\big)$, where $r \in \mathbb{R}[\xi]$. There exists a nontrivial conserved quantity for $\mathfrak{B}$ if and only if either r has a non-unity maximal even polynomial factor r_e or $r(0) = 0$. Moreover if $p := \frac{r}{r_e}$ is such that $p(0) \neq 0$, then the dimension of the space of conserved QDFs modulo $\mathfrak{B}$ is $\frac{\deg(r_e)}{2}$, otherwise it is equal to $\frac{\deg(r_e)}{2} + 1$.*

Proof. Let the degree of r be equal to n. Assume that $\mathfrak{B}$ has a conserved QDF whose two-variable polynomial representation is $\phi(\zeta,\eta)$. Then

$$\phi(\zeta,\eta) = \frac{r(\zeta)f_1(\zeta,\eta) + r(\eta)f_1(\eta,\zeta)}{\zeta+\eta}$$

for some $f_1 \in \mathbb{R}[\zeta,\eta]$. Write $f_1(\zeta,\eta) = \sum_{k=0}^{N_1}\sum_{i=0}^{N_2} a_{ij}\zeta^i\eta^j$. It is easy to see that since ϕ is r-canonical, $N_2 = 0$ and $N_1 = n-1$. Hence

$$\phi(\zeta,\eta) = \frac{r(\zeta)f(\eta) + r(\eta)f(\zeta)}{\zeta+\eta} \tag{4.3}$$

where $f(\eta) = f_1(\zeta,\eta)$. Since $\phi \in \mathbb{R}[\zeta,\eta]$, its numerator is divisible by $\zeta+\eta$. Consequently $r(-\xi)f(\xi) + r(\xi)f(-\xi) = 0$. This implies that $g(\xi) := r(\xi)f(-\xi) = r_e(\xi)p(\xi)f(-\xi)$ is an odd function. Hence

$$p(\xi)f(-\xi) = -p(-\xi)f(\xi) \tag{4.4}$$

Two cases arise.

- Case 1: $p(\xi)$ is not divisible by ξ. In this case, the greatest common divisor of $p(\xi)$ and $p(-\xi)$ is equal to 1 and consequently for equation (4.4) to hold, it is easy to see that f should be of the form

$$f(\xi) = p(\xi)f_o(\xi)$$

where $f_o(\xi)$ is an odd function such that

$$\deg(f) \leq n-1 \tag{4.5}$$

Since

$$n = \deg(r_e) + \deg(p) \tag{4.6}$$

$$\deg(f) = \deg(p) + \deg(f_o), \tag{4.7}$$

from (4.5), (4.6) and (4.7), we obtain

$$\deg(f_o) \leq \deg(r_e) - 1 \tag{4.8}$$

We now show that there is a one-one correspondence between any odd polynomial f_o of degree less than or equal to $\deg(r_e)-1$, and ϕ given by

$$\phi(\zeta,\eta) = \frac{p(\zeta)p(\eta)(r_e(\zeta)f_o(\eta) + f_o(\zeta)r_e(\eta))}{\zeta+\eta} \tag{4.9}$$

It is easy to see that for a given odd polynomial f_o of degree less than or equal to $\deg(r_e)-1$, there exists a unique ϕ such that equation (4.9) holds. Now consider two odd polynomials f_{o1} and f_{o2} of degree less than or equal to $\deg(r_e)-1$, and assume by contradiction that

$$\phi_1(\zeta,\eta) = \frac{p(\zeta)p(\eta)(r_e(\zeta)f_{o1}(\eta) + f_{o1}(\zeta)r_e(\eta))}{\zeta+\eta} = \frac{p(\zeta)p(\eta)(r_e(\zeta)f_{o2}(\eta) + f_{o2}(\zeta)r_e(\eta))}{\zeta+\eta}$$

This implies that

$$r_e(\zeta)f_1(\eta) + f_1(\zeta)r_e(\eta) = 0 \tag{4.10}$$

where $f_1(\xi) := f_{o1}(\xi) - f_{o2}(\xi)$. From equation (4.10), it follows that

$$\frac{f_1(\zeta)}{r_e(\zeta)} = -\frac{f_1(\eta)}{r_e(\eta)} = K$$

where $K \in \mathbb{R}$. This implies that $K = -K = 0$, or that $f_{o1}(\xi) = f_{o2}(\xi)$. Hence there is a one-one correspondence between any odd polynomial f_o of degree less than or equal to $\deg(r_e) - 1$, and ϕ given by equation (4.9). It is easy to see that the space of all odd polynomials f_o of degree less than or equal to $\deg(r_e) - 1$ has dimension equal to $\frac{\deg(r_e)}{2}$. Consequently the dimension of the space of conserved QDFs in this case is $\frac{\deg(r_e)}{2}$.

- Case 2: $p(\xi)$ is divisible by ξ. This implies that r has an odd number of roots at zero. One of them is a root of p and the remaining occur as roots of r_e. Define p_1 by $p_1(\xi) = \frac{p(\xi)}{\xi}$. In this case, since p_1 does not have a root at zero, for equation (4.4) to hold, it is easy to see that f should be of the form

$$f(\xi) = p_1(\xi)f_e(\xi)$$

where $f_e(\xi)$ is an even function such that

$$\deg(f) \leq n - 1 \tag{4.11}$$

Since

$$n = \deg(r_e) + \deg(p_1) + 1 \tag{4.12}$$

$$\deg(f) = \deg(p_1) + \deg(f_e), \tag{4.13}$$

from (4.11), (4.12) and (4.13), we obtain

$$\deg(f_e) \leq \deg(r_e) \tag{4.14}$$

Following the argument used in Case 1, it can be shown that there is a one-one correspondence between any even polynomial f_e of degree less than or equal to $\deg(r_e)$ and ϕ given by

$$\phi(\zeta,\eta) = \frac{p_1(\zeta)p_1(\eta)(\zeta r_e(\zeta)f_e(\eta) + \eta f_e(\zeta)r_e(\eta))}{\zeta+\eta}$$

It is easy to see that the space of all even polynomials f_e of degree less than or equal to $\deg(r_e)$ has dimension equal to $\frac{\deg(r_e)}{2}+1$. Consequently, the dimension of the space of conserved QDFs in this case is equal to $\frac{\deg(r_e)}{2}+1$.

■

In order to prove the equivalence between oscillatory systems and autonomous lossless systems, we first consider the case of scalar behaviours.

Theorem 4.6. *An autonomous behaviour $\mathfrak{B} \in \mathcal{L}^1$ is lossless if and only if it is oscillatory.*

Proof. (*If*) We consider the two forms of scalar oscillatory behaviours mentioned in section 2.8. For each of these forms of oscillatory behaviour, we construct an energy function that is positive.

- Case 1: The oscillatory behaviour is of the form $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, where $r(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$ and $\omega_0,\dots,\omega_{n-1} \in \mathbb{R}^+$. From the discussion of section 3.5, it can be said that a general r-canonical two-variable polynomial inducing a conserved quantity for this case has the form

$$\phi(\zeta,\eta) = \frac{\eta r(\zeta) f_e(\eta) + \zeta r(\eta) f_e(\zeta)}{\zeta+\eta} \tag{4.15}$$

where f_e is an even function of degree less than or equal to $2n-2$. For $p = 0,\dots,n-1$, define $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_p^2}$. It can be seen that the set $\{v_p(\xi)\}_{p=0,\dots,n-1}$ is a basis for the set of even polynomials of degree less than or equal to $2n-2$ as the elements of the set are linearly independent and the number of elements of the set is equal to the dimension of the space. It follows that there exist $b_p \in \mathbb{R}$, $p = 0,\dots,n-1$, such that $f_e(\xi) = \sum_{p=0}^{n-1} b_p v_p(\xi)$. Now

$$\begin{aligned}\phi(\zeta,\eta) &= \sum_{p=0}^{n-1} b_p \left[\frac{\eta r(\zeta) v_p(\eta) + \zeta r(\eta) v_p(\zeta)}{\zeta+\eta}\right] \\ &= \sum_{p=0}^{n-1} b_p v_p(\zeta) v_p(\eta) \left[\frac{\eta(\zeta^2+\omega_p^2) + \zeta(\eta^2+\omega_p^2)}{\zeta+\eta}\right] \\ &= \sum_{p=0}^{n-1} b_p v_p(\zeta) v_p(\eta)(\zeta\eta+\omega_p^2)\end{aligned}$$

Define $\phi_p(\zeta,\eta) := v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$,

$$D_p(\xi) := \begin{bmatrix} \xi v_p(\xi) \\ \omega_p v_p(\xi) \end{bmatrix}$$

and observe that $\phi_p(\zeta,\eta) = D_p(\zeta)^\top D_p(\eta)$. From equation (4.15), it can be seen that linearly independent f_e's correspond to linearly independent ϕ's. Hence $\{\phi_p(\zeta,\eta)\}_{p=0,\dots,n-1}$ is a basis of the space of r-canonical two-variable polynomials that induce conserved quantities. Now consider $\sum_{p=0}^{n-1} a_p^2\phi_p(\zeta,\eta) = D(\zeta)^\top D(\eta)$, where $a_p \in \mathbb{R}\setminus\{0\}$ for $p = 0,\dots,n-1$ and

$$D(\xi) = \begin{bmatrix} a_0\xi v_0(\xi) \\ a_0\omega_0 v_0(\xi) \\ a_1\xi v_1(\xi) \\ a_1\omega_1 v_1(\xi) \\ \vdots \\ a_{n-1}\xi v_{n-1}(\xi) \\ a_{n-1}\omega_{n-1}v_{n-1}(\xi)) \end{bmatrix}$$

$\lambda v_p(\lambda) = 0$ implies that either $\lambda = 0$ or $v_p(\lambda) = 0$. Now $v_p(\lambda) = 0$ implies that λ is equal to one of $\pm j\omega_q$, $q \in \{0,1,\dots,n-1\}\setminus\{p\}$. For any one of these values of λ, not all entries of D go to zero. Hence $D(\lambda) \neq 0$ for every $\lambda \in \mathbb{C}$. This proves that a linear oscillatory behaviour of the type considered in Case 1 admits a positive conserved quantity, i.e an energy function. This concludes the proof for Case 1.

- Case 2: The oscillatory behaviour is of the form $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$ where $r(\xi) = \xi(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$ and $\omega_0,\dots,\omega_{n-1} \in \mathbb{R}^+$. Define $r_e(\xi) := \frac{r(\xi)}{\xi}$. From the discussion of section 3.5, we conclude that a general r-canonical two-variable polynomial inducing a conserved quantity for this case has the form

$$\phi(\zeta,\eta) = \frac{r(\zeta)f_e(\eta) + r(\eta)f_e(\zeta)}{\zeta+\eta} \tag{4.16}$$

where f_e is an even function of degree less than or equal to $2n$. For $p = 0,\dots,n-1$, define $v_p(\xi) := \frac{r_e(\xi)}{\xi^2+\omega_p^2}$. It can be seen that the set $\{r_e(\xi)\} \cup \{\xi^2 v_p(\xi)\}_{p=0,\dots,n-1}$ is a basis for the set of even polynomials of degree less than or equal to $2n$ as the elements of the set are linearly independent and the number of elements of the set is equal to the dimension of the space. It follows that there exist $b_p \in \mathbb{R}$, $p = 0,\dots,n$, such that $f_e(\xi) = \sum_{p=0}^{n-1} b_p\xi^2 v_p(\xi) + b_n r_e(\xi)$. Now

$$\begin{aligned} \phi(\zeta,\eta) &= \sum_{p=0}^{n-1} b_p\left[\frac{\eta^2 r(\zeta)v_p(\eta) + \zeta^2 r(\eta)v_p(\zeta)}{\zeta+\eta}\right] + b_n\left[\frac{r(\zeta)r_e(\eta) + r(\eta)r_e(\zeta)}{\zeta+\eta}\right] \\ &= \sum_{p=0}^{n-1} b_p v_p(\zeta)v_p(\eta)\left[\frac{\zeta\eta^2(\zeta^2+\omega_p^2) + \zeta^2\eta(\eta^2+\omega_p^2)}{\zeta+\eta}\right] + b_n r_e(\zeta)r_e(\eta) \\ &= \sum_{p=0}^{n-1} b_p\zeta\eta v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2) + b_n r_e(\zeta)r_e(\eta) \end{aligned}$$

For $p = 0, \ldots, n-1$, define

$$D_p(\xi) := \begin{bmatrix} \xi^2 v_p(\xi) \\ \omega_p \xi v_p(\xi) \end{bmatrix},$$

$\phi_p(\zeta, \eta) := v_p(\zeta)v_p(\eta)(\zeta\eta + \omega_p^2)$ and observe that $\zeta\eta\phi_p(\zeta,\eta) = D_p(\zeta)^\top D_p(\eta)$. From equation (4.16), it can be seen that linearly independent f_e's correspond to linearly independent ϕ's. Hence $\{r_e(\zeta)r_e(\eta)\} \cup \{\zeta\eta\phi_p(\zeta,\eta)\}_{p=0,\ldots,n-1}$ is a basis for the set of r-canonical two-variable polynomials that induce conserved quantities associated with $\mathfrak{B}$. Now consider $\sum_{p=0}^{n-1} a_p^2 \zeta\eta\phi_p(\zeta,\eta) + a_n^2 r_e(\zeta)r_e(\eta) = D(\zeta)^\top D(\eta)$, where $a_p \in \mathbb{R}^+$ for $p = 0, \ldots, n$ and

$$D(\xi) = \begin{bmatrix} a_0 \xi^2 v_0(\xi) \\ a_0 \omega_0 \xi v_0(\xi) \\ a_1 \xi^2 v_1(\xi) \\ a_1 \omega_1 \xi v_1(\xi) \\ \vdots \\ a_{n-1}\xi^2 v_{n-1}(\xi) \\ a_{n-1}\omega_{n-1}\xi v_{n-1}(\xi)) \\ a_n r_e(\xi) \end{bmatrix}$$

$\lambda^2 v_p(\lambda) = 0$ implies that either $\lambda = 0$ or $v_p(\lambda) = 0$. Now $v_p(\lambda) = 0$ implies that λ is equal to one of $\pm j\omega_q$, $q \in \{0, 1, \ldots, n-1\} \setminus \{p\}$. For any one of these values of λ, not all entries of D do not go to zero. Hence $D(\lambda) \neq 0$ for every $\lambda \in \mathbb{C}$. This proves that a linear oscillatory behaviour of the type considered in Case 2 admits a positive conserved quantity, i.e an energy function. This concludes the proof for Case 2.

(*Only if*) We consider all scalar systems for which conserved quantities exist and prove that a conserved quantity cannot be positive unless the system is oscillatory. Let $\mathfrak{B}$ be a behaviour whose kernel representation is $r(\frac{d}{dt})w = 0$. Let r_e denote the maximal even polynomial factor of r. Define $p(\xi) := \frac{r(\xi)}{r_e(\xi)}$. If $p(\xi)$ is not a constant and $p(\xi) \neq a\xi$, where $a \in \mathbb{R}$, then it has at least one real root, say $\lambda \in \mathbb{R} \setminus \{0\}$ or two complex conjugate roots, say $\lambda, \bar{\lambda} \in \mathbb{C} \setminus \mathbb{R}$. From the proof of Proposition 4.5, depending on whether $p(\xi)$ is divisible by ξ or not, any two-variable polynomial inducing conserved QDF over $\mathfrak{B}$ can either have the form

$$\phi_1(\zeta,\eta) = \frac{r(\zeta)p_1(\eta)f_e(\eta) + r(\eta)p_1(\zeta)f_e(\zeta)}{\zeta + \eta}$$

where $p_1(\xi) = \frac{p(\xi)}{\xi}$ and f_e is an even function, or the form

$$\phi_2(\zeta,\eta) = \frac{r(\zeta)p(\eta)f_o(\eta) + r(\eta)p(\zeta)f_o(\zeta)}{\zeta + \eta}$$

where $f_o(\xi)$ is an odd function. It can be seen that both ϕ_1 and ϕ_2 are divisible by $(\zeta-\lambda)(\eta-\lambda)$ if $\lambda \in \mathbb{R}$ and divisible by $(\zeta-\lambda)(\zeta-\bar{\lambda})(\eta-\lambda)(\eta-\bar{\lambda})$ if $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Hence along the trajectory $w(t)=e^{\lambda t}+e^{\bar{\lambda}t} \in \mathfrak{B}$, the QDFs induced by ϕ_1 and ϕ_2 are equal to zero. This implies that $\mathfrak{B}$ does not have a positive conserved QDF. This eliminates all scalar systems except those for which the kernel representation is $r(\frac{d}{dt})w=0$, such that either $r(\xi)$ is even, or $r(\xi)=\xi r_e(\xi)$, where $r_e(\xi)$ is an even function.

We now consider all those remaining cases except the oscillatory one, for which $\mathfrak{B}=\ker\big(r(\frac{d}{dt})\big)$ is such that r does not have repeated roots. For each of these, we will construct a general conserved quantity which is zero on at least one of the trajectories of $\mathfrak{B}$, thus proving the claim by contradiction.

- Case 1: r is even and has roots at λ_0 and $-\lambda_0$, where $\lambda_0 \in \mathbb{R}$. Define $r_1(\xi) := \frac{r(\xi)}{(\xi^2-\lambda_0^2)}$. Define r' by $r(\xi) =: r'(\xi^2)$. Assume without loss of generality that r_1 has roots at $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_{n-1}$, where $\lambda_p \in \mathbb{C}$ for $p=1,\ldots,n-1$. Any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta,\eta)=\frac{\eta r(\zeta)f(\eta^2)+\zeta r(\eta)f(\zeta^2)}{\zeta+\eta}$$

For $p=0,\ldots,n-1$, define u_p' by $u_p'(\xi^2) := \frac{r'(\xi^2)}{\xi^2-\lambda_p^2}$. We can write $f(\xi^2)$ in terms of the basis $\left\{u_p'(\xi^2)\right\}_{p=0,\ldots,n-1}$. Hence

$$\phi(\zeta,\eta)=b_0r_1(\zeta)r_1(\eta)(\zeta\eta-\lambda_0^2)+(\zeta^2-\lambda_0^2)(\eta^2-\lambda_0^2)\phi_1(\zeta,\eta)$$

Along the trajectory $w(t)=e^{\lambda_0 t}\in\mathfrak{B}$, the QDF induced by the above polynomial is zero. Hence in this case, a positive conserved QDF does not exist.

- Case 2: r is odd and has roots at λ_0 and $-\lambda_0$, where $\lambda_0 \in \mathbb{R}$. Define $r_1(\xi) := \frac{r(\xi)}{\xi(\xi^2-\lambda_0^2)}$. Define r' by $r(\xi) =: \xi r'(\xi^2)$. Assume without loss of generality that r_1 has roots at $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_{n-1}$, where $\lambda_p \in \mathbb{C}$ for $p=1,\ldots,n-1$. Any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta,\eta)=\frac{r(\zeta)f(\eta^2)+r(\eta)f(\zeta^2)}{\zeta+\eta}$$

For $p=0,\ldots,n-1$, define u_p' by $u_p'(\xi^2) := \frac{r'(\xi^2)}{\xi^2-\lambda_p^2}$. We can write $f(\xi^2)$ in terms of the basis $\left\{r'(\xi^2)\right\}\cup\left\{\xi^2u_p'(\xi^2)\right\}_{p=0,\ldots,n-1}$. Hence

$$\phi(\zeta,\eta)=b_0r_1(\zeta)r_1(\eta)\zeta\eta(\zeta\eta-\lambda_0^2)+(\zeta^2-\lambda_0^2)(\eta^2-\lambda_0^2)\phi_1(\zeta,\eta)$$

Along the trajectory $w(t)=e^{\lambda_0 t}\in\mathfrak{B}$, the QDF induced by the above polynomial is zero. Hence in this case, a positive conserved QDF does not exist.

- Case 3: r is even and has roots at λ, $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$, where λ is a point in the complex plane that is not on any of the co-ordinate axes. Define $r_1(\xi) := \frac{r(\xi)}{(\xi^2-\lambda^2)(\xi^2-\bar{\lambda}^2)}$. Assume without loss of generality that r_1 has roots at $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_{n-2}$, where $\lambda_p \in \mathbb{C}$ for $p = 1, \ldots, n-2$. Define r' by $r'(\xi^2) := r(\xi)$. Any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta, \eta) = \frac{\eta r(\zeta) f(\eta^2) + \zeta r(\eta) f(\zeta^2)}{\zeta + \eta}$$

We can write $f(\xi^2)$ in terms of a new basis as follows

$$f(\xi^2) = b_0 \frac{r'(\xi^2)}{\xi^2 - \lambda^2} + \bar{b}_0 \frac{r'(\xi^2)}{\xi^2 - \bar{\lambda}^2} + \sum_{p=1}^{n-2} b_p \frac{r'(\xi^2)}{\xi^2 - \lambda_p^2}$$

Hence

$$\begin{aligned} \phi(\zeta, \eta) &= b_0 r_1(\zeta) r_1(\eta)(\zeta^2 - \bar{\lambda}^2)(\eta^2 - \bar{\lambda^2})(\zeta\eta - \lambda^2) \\ &+ \bar{b}_0 r_1(\zeta) r_1(\eta)(\zeta^2 - \lambda^2)(\eta^2 - \lambda^2)(\zeta\eta - \bar{\lambda}^2) \\ &+ (\zeta^2 - \lambda^2)(\zeta^2 - \bar{\lambda}^2)(\eta^2 - \lambda^2)(\eta^2 - \bar{\lambda}^2)\phi_1(\zeta, \eta) \end{aligned}$$

Along the trajectory $w(t) = e^{\lambda t} + e^{\bar{\lambda} t} \in \mathfrak{B}$, the QDF induced by the above polynomial is zero. Hence in this case, a positive conserved QDF does not exist.

- Case 4: r is odd and has roots at λ, $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$, where λ is a point in the complex plane that is not on any of the co-ordinate axes. Define $r_1(\xi) := \frac{r(\xi)}{\xi(\xi^2-\lambda^2)(\xi^2-\bar{\lambda}^2)}$. Assume without loss of generality that r_1 has roots at $\pm\lambda_1, \pm\lambda_2, \ldots, \pm\lambda_{n-2}$, where $\lambda_p \in \mathbb{C}$ for $p = 1, \ldots, n-2$. Define r' by $r'(\xi^2) := \frac{r(\xi)}{\xi}$. Any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta, \eta) = \frac{r(\zeta) f(\eta^2) + r(\eta) f(\zeta^2)}{\zeta + \eta}$$

We can write $f(\xi^2)$ in terms of a new basis as follows

$$f(\xi^2) = b_0 \frac{\xi^2 r'(\xi^2)}{\xi^2 - \lambda^2} + \bar{b}_0 \frac{\xi^2 r'(\xi^2)}{\xi^2 - \bar{\lambda}^2} + \sum_{p=1}^{n-2} b_p \frac{\xi^2 r'(\xi^2)}{\xi^2 - \lambda_p^2} + b_{n-1} r'(\xi^2)$$

Hence

$$\begin{aligned} \phi(\zeta, \eta) &= b_0 r_1(\zeta) r_1(\eta)\zeta\eta(\zeta^2 - \bar{\lambda}^2)(\eta^2 - \bar{\lambda^2})(\zeta\eta - \lambda^2) \\ &+ \bar{b}_0 r_1(\zeta) r_1(\eta)\zeta\eta(\zeta^2 - \lambda^2)(\eta^2 - \lambda^2)(\zeta\eta - \bar{\lambda}^2) \\ &+ (\zeta^2 - \lambda^2)(\zeta^2 - \bar{\lambda}^2)(\eta^2 - \lambda^2)(\eta^2 - \bar{\lambda}^2)\phi_1(\zeta, \eta) \end{aligned}$$

Along the trajectory $w(t) = e^{\lambda t} + e^{\bar{\lambda} t} \in \mathfrak{B}$, the QDF induced by the above polynomial is zero. Hence in this case, a positive conserved QDF does not exist.

Finally we consider those remaining cases, for which $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$ is such that r has repeated roots.

- Case 1: r is even and has at least twice repeated roots at $\pm\lambda$, where λ is either purely real or purely imaginary. Define $r_1(\xi) := \frac{r(\xi)}{(\xi^2-\lambda^2)^2}$. Write $r_1(\xi) = \sum_{p=0}^{n-2} a_p \xi^{2p}$. In this case, any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\begin{aligned}\phi(\zeta,\eta) &= \frac{\eta r(\zeta)(\sum_{p=0}^{n-1} b_p\eta^{2p}) + \zeta r(\eta)(\sum_{p=0}^{n-1} b_p\zeta^{2p})}{\zeta+\eta} \\ &= \sum_{i=0}^{n-2}\sum_{j=0}^{n-1} a_{ij}\phi_{ij}(\zeta,\eta)\end{aligned}$$

where

$$\phi_{ij}(\zeta,\eta) = \frac{\zeta^{2i}(\zeta^2-\lambda^2)^2\eta^{2j+1} + \eta^{2i}(\eta^2-\lambda^2)^2\zeta^{2j+1}}{\zeta+\eta} \tag{4.17}$$

Observe that the numerator in the right hand side of the above equation is divisible by the denominator, because if ζ is replaced by $-\xi$ and η by ξ in the numerator, the result is zero. Hence $\phi_{ij} \in \mathbb{R}[\zeta,\eta]$. It can be verified that

$$\phi_{ij}(\lambda,\lambda) = \phi_{ij}(-\lambda,\lambda) = \phi_{ij}(-\lambda,-\lambda) = 0 \tag{4.18}$$

Hence from equation (4.18), along the trajectory $w(t) = e^{\lambda t} + e^{-\lambda t} \in \ker\big(r(\frac{d}{dt})\big)$, the QDF induced by $\phi(\zeta,\eta)$ is zero. Hence no conserved QDF is positive in this case.

- Case 2: r is even and has at least twice repeated roots at $\pm\lambda$ and $\pm\bar{\lambda}$ where λ is a point in the complex plane that does not lie on any of the co-ordinate axes. Define $r_1(\xi) := \frac{r(\xi)}{(\xi^2-\lambda^2)^2(\xi^2-\bar{\lambda}^2)^2}$. Write $r_1(\xi) = (\sum_{p=0}^{n-4} a_p\xi^{2p})$. As in the previous case, any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta,\eta) = \sum_{i=0}^{n-4}\sum_{j=0}^{n-1} a_{ij}\phi_{ij}(\zeta,\eta)$$

where

$$\phi_{ij}(\zeta,\eta) = \frac{\zeta^{2i}(\zeta^2-\lambda^2)^2(\zeta^2-\bar{\lambda}^2)^2\eta^{2j+1} + \eta^{2i}(\eta^2-\lambda^2)^2(\eta^2-\bar{\lambda}^2)^2\zeta^{2j+1}}{\zeta+\eta}$$

Observe that $\phi_{ij} \in \mathbb{R}[\zeta,\eta]$, as the numerator of the right hand side of the above equation is divisible by the denominator. It can be verified that

$$\phi_{ij}(\lambda,\lambda) = \phi_{ij}(\lambda,\bar{\lambda}) = \phi_{ij}(\bar{\lambda},\bar{\lambda}) = 0$$

Thus along the trajectory $w(t) = e^{\lambda t} + e^{\bar{\lambda} t} \in \ker\big(r(\frac{d}{dt})\big)$, the QDF induced by $\phi(\zeta,\eta)$ is zero. Hence no conserved QDF is positive in this case.

- Case 3: r is odd and has at least twice repeated roots at $\pm\lambda$, where λ is either purely real or purely imaginary. Define $r_1(\xi) := \frac{r(\xi)}{\xi(\xi^2-\lambda^2)^2}$. Write $r(\xi) = (\sum_{p=0}^{n-2} a_p \xi^{2p})$. In this case, any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\begin{aligned}\phi(\zeta,\eta) &= \frac{r(\zeta)(\sum_{p=0}^{n} b_p\eta^{2p}) + r(\eta)(\sum_{p=0}^{n} b_p\zeta^{2p})}{\zeta+\eta} \\ &= \sum_{i=0}^{n-2}\sum_{j=0}^{n} a_{ij}\phi_{ij}(\zeta,\eta)\end{aligned}$$

  where
$$\phi_{ij}(\zeta,\eta) = \frac{\zeta^{2i+1}(\zeta^2-\lambda^2)^2\eta^{2j} + \eta^{2i+1}(\eta^2-\lambda^2)^2\zeta^{2j}}{\zeta+\eta}$$

  Observe that $\phi_{ij} \in \mathbb{R}[\zeta,\eta]$, as the numerator of the right hand side of the above equation is divisible by the denominator. It can be verified that

$$\phi_{ij}(\lambda,\lambda) = \phi_{ij}(-\lambda,\lambda) = \phi_{ij}(-\lambda,-\lambda) = 0$$

  Thus along the trajectory $w(t) = e^{\lambda t} + e^{-\lambda t} \in \ker\big(r(\frac{d}{dt})\big)$, the QDF induced by $\phi(\zeta,\eta)$ is zero. Hence no conserved QDF is positive in this case.

- Case 4: r is odd and has at least twice repeated roots at $\pm\lambda$ and $\pm\bar{\lambda}$ where λ is a point in the complex plane that does not lie on any of the co-ordinate axes. Define $r_1(\xi) := \frac{r(\xi)}{\xi(\xi^2-\lambda^2)^2(\xi^2-\bar{\lambda}^2)^2}$. Write $r_1(\xi) := \sum_{p=0}^{n-4} a_p\xi^{2p}$. As in the previous case, any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta,\eta) = \sum_{i=0}^{n-4}\sum_{j=0}^{n} a_{ij}\phi_{ij}(\zeta,\eta)$$

  where

$$\phi_{ij}(\zeta,\eta) = \frac{\zeta^{2i+1}(\zeta^2-\lambda^2)^2(\zeta^2-\bar{\lambda}^2)^2\eta^{2j} + \eta^{2i+1}(\eta^2-\lambda^2)^2(\eta^2-\bar{\lambda}^2)^2\zeta^{2j}}{\zeta+\eta}$$

  Observe that $\phi_{ij} \in \mathbb{R}[\zeta,\eta]$, as the numerator of the right hand side of the above equation is divisible by the denominator. It can be verified that

$$\phi_{ij}(\zeta,\eta) = \phi_{ij}(\lambda,\bar{\lambda}) = \phi_{ij}(\bar{\lambda},\bar{\lambda}) = 0$$

  Thus along the trajectory $w(t) = e^{\lambda t} + e^{\bar{\lambda} t} \in \ker\big(r(\frac{d}{dt})\big)$, the QDF induced by $\phi(\zeta,\eta)$ is zero. Hence no conserved QDF is positive in this case.

We have considered all linear scalar systems apart from oscillatory ones for which conserved QDFs exist and we have shown that a positive conserved QDF does not exist for any of these cases. Since, we have already proved the existence of a positive conserved QDF for oscillatory systems, this concludes the proof. ■

Observe that the proof for the above theorem also holds for a stronger version of Theorem 4.6 which is given below.

Theorem 4.7. *A behaviour $\mathfrak{B} \in \mathcal{L}^1$ is oscillatory if and only if there exists a conserved quantity Q_E associated with $\mathfrak{B}$ such that $Q_E > 0$.*

This is because the energy function for an oscillatory system that we have constructed in the *if* part of the proof of Theorem 4.6 is indeed positive and not only positive along $\mathfrak{B}$.

We now discuss a few properties of energy functions for scalar oscillatory behaviours. We first present an analysis of the conditions under which a conserved quantity for a scalar oscillatory behaviour is positive. The following lemma can be used to construct an energy function for a scalar oscillatory behaviour.

Lemma 4.8. *Let $r_1 \in \mathbb{R}[\xi]$ be given by $r_1(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$, where $\omega_0, \dots, \omega_{n-1} \in \mathbb{R}^+$ and n is a positive integer. Define $v_p(\xi) := \frac{r_1(\xi)}{\xi^2+\omega_p^2}$, $p = 0, \dots, n-1$. Define $r_2(\xi) := \xi r_1(\xi)$. Then the following hold:*

1. *Let $\mathfrak{B}_1 = ker\ \left(r_1(\frac{d}{dt})\right)$. If the conserved quantity for $\mathfrak{B}_1$ induced by $\phi_1(\zeta,\eta) = \sum_{p=0}^{n-1} b_p v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$ is positive, then $b_p > 0$ for $p = 0, \dots, n-1$.*
2. *Let $\mathfrak{B}_2 = ker\ \left(r_2(\frac{d}{dt})\right)$. If the conserved quantity for $\mathfrak{B}_2$ induced by $\phi_2(\zeta,\eta) = \sum_{p=0}^{n-1} b_p\zeta\eta v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2) + b_n r_1(\zeta)r_1(\eta)$ is positive, then $b_p > 0$ for $p = 0, \dots, n$.*

Proof. Assume that $b_i \leq 0$ for some $i \in \{0, \dots, n-1\}$. Consider a trajectory $w(t) = ke^{j\omega_i t} + \bar{k}e^{-j\omega_i t} \in \mathfrak{B}_1, \mathfrak{B}_2$. Along this trajectory, $v_p(\frac{d}{dt})w = 0$ for $p \in \{0, \dots, n-1\} \setminus \{i\}$. Since $\phi_p(\zeta,\eta) = v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$ and $\zeta\eta\phi_p(\zeta,\eta)$ are non-negative, the QDFs induced by $\phi_1(\zeta,\eta)$ and $\phi_2(\zeta,\eta)$ along this trajectory turn out to be non-positive. Hence by contradiction, $b_p > 0$ for $p = 0, \dots, n-1$ in both cases.

In order to complete the proof consider now statement 2 of the Lemma and assume by contradiction that $b_n \leq 0$. Consider a trajectory $w(t) = k \in \mathfrak{B}_2$. Along this trajectory $v_p(\frac{d}{dt})w = 0$ for $p \in \{0, \dots, n-1\}$. Since $r_1(\zeta)r_1(\eta)$ is non-negative, the QDF induced by $\phi_2(\zeta,\eta)$ turns out to be non-positive. Hence, $b_n > 0$. This concludes the proof. ■

The next Theorem relates the positivity of a conserved quantity to an important property known as *interlacing property*, which also arises in applications like electrical network theory.

Theorem 4.9. *Let $r_1 \in \mathbb{R}[\xi]$ be given by $r_1(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$, where $\omega_0 < \omega_1 < \dots < \omega_{n-1} \in \mathbb{R}^+$ and n is a positive integer. Define $r'(\xi^2) := r_1(\xi)$; $r_2(\xi) := \xi r_1(\xi)$ and $\check{r}(\xi) := \xi r'(\xi)$. Then the following hold:*

1. *Let* $\mathfrak{B}_1 = ker\left(r_1(\frac{d}{dt})\right)$. *Let* $f_1(\xi)$ *be a polynomial of degree less than or equal to* $n-1$. *A conserved quantity for* $\mathfrak{B}_1$ *induced by*

$$\phi_1(\zeta,\eta) = \frac{\eta r'(\zeta^2)f_1(\eta^2) + \zeta r'(\eta^2)f_1(\zeta^2)}{\zeta+\eta} \tag{4.19}$$

   is positive if and only if $f_1(-\omega_0^2) > 0$ *and the roots of* f_1 *are interlaced between those of* r', *i.e along the real axis, exactly one root of* f_1 *occurs between any two consecutive roots of* r'.

2. *Let* $\mathfrak{B}_2 = ker\left(r_2(\frac{d}{dt})\right)$. *Let* $f_2(\xi)$ *be a polynomial of degree less than or equal to* n. *A conserved quantity associated with* $\mathfrak{B}_2$ *induced by*

$$\phi_2(\zeta,\eta) = \frac{\zeta r'(\zeta^2)f_2(\eta^2) + \eta r'(\eta^2)f_2(\zeta^2)}{\zeta+\eta}$$

   is positive if and only if $f_2(0) > 0$ *and the roots of* f_2 *are interlaced between those of* $\check{r}$.

Proof. (*Only if*) For $p = 0,\dots,n-1$, define $v_p'(\xi^2) := \frac{r'(\xi^2)}{\xi^2+\omega_p^2}$. From Lemma 4.8 and Theorem 4.6, we know that f_1 and f_2 are of the form

$$f_1(\xi^2) = \sum_{p=0}^{n-1} b_p v_p'(\xi^2) \tag{4.20}$$

$$f_2(\xi^2) = \sum_{p=0}^{n-1} b_p \xi^2 v_p'(\xi^2) + b_n r'(\xi^2) \tag{4.21}$$

where $b_p \in \mathbb{R}^+$. The roots of r' are $-\omega_0^2, -\omega_1^2, \dots, -\omega_{n-1}^2$ and the roots of $\check{r}$ are $0, -\omega_0^2, -\omega_1^2, \dots, -\omega_{n-1}^2$. From equation (4.20), it can be seen that

$$\begin{array}{rcl} f_1(-\omega_0^2) & = & b_0(\prod_{p=1}^{n-1}(\omega_p^2-\omega_0^2)) > 0 \\ f_1(-\omega_1^2) & = & b_1(\prod_{p\neq 1}(\omega_p^2-\omega_1^2)) < 0 \\ f_1(-\omega_2^2) & = & b_2(\prod_{p\neq 2}(\omega_p^2-\omega_2^2)) > 0 \\ & \vdots & \end{array}$$

Since f_1 is a continuous function and can have a maximum of $n-1$ real roots, it follows that the roots of f_1 are interlaced between those of r'.

From equation (4.21), it can be seen that

$$\begin{array}{rcl} f_2(0) & = & b_n(\prod_{p=0}^{n-1}\omega_p^2) > 0 \\ f_2(-\omega_0^2) & = & -b_0\omega_0^2(\prod_{p=1}^{n-1}(\omega_p^2-\omega_0^2)) < 0 \\ f_2(-\omega_1^2) & = & -b_1\omega_1^2(\prod_{p\neq 1}(\omega_p^2-\omega_1^2)) > 0 \\ & \vdots & \end{array}$$

Since f_2 is a continuous function and can have a maximum of n real roots, it follows that the roots of f are interlaced between those of $\check{r}$.

(*If*) Assuming that the roots of f_1 are interlaced between those of r' and $f_1(-\omega_0^2) > 0$, since f_1 is continuous and can have a maximum of $n-1$ roots, we have

$$\begin{aligned} f_1(-\omega_0^2) &= b_0(\textstyle\prod_{p=1}^{n-1}(\omega_p^2 - \omega_0^2)) > 0 \\ f_1(-\omega_1^2) &= b_1(\textstyle\prod_{p\neq 1}(\omega_p^2 - \omega_1^2)) < 0 \\ f_1(-\omega_2^2) &= b_2(\textstyle\prod_{p\neq 2}(\omega_p^2 - \omega_2^2)) > 0 \\ &\vdots \end{aligned}$$

This implies that $b_p > 0$ for $p = 0, \ldots, n-1$, which in turn implies that $\phi_1(\zeta, \eta)$ is positive.

Assuming that the roots of f_2 are interlaced between those of $\check{r}$ and $f_2(0) > 0$, since f_2 is continuous and can a have a maximum of n roots, we have

$$\begin{aligned} f_2(0) &= b_n(\textstyle\prod_{p=0}^{n-1}\omega_p^2) > 0 \\ f_2(-\omega_0^2) &= -b_0\omega_0^2(\textstyle\prod_{p=1}^{n-1}(\omega_p^2 - \omega_0^2)) < 0 \\ f_2(-\omega_1^2) &= -b_1\omega_1^2(\textstyle\prod_{p\neq 1}(\omega_p^2 - \omega_1^2)) > 0 \\ &\vdots \end{aligned}$$

This implies that $b_p > 0$ for $p = 0, \ldots, n$, which in turn implies that $\phi_2(\zeta, \eta)$ is positive.
■

Remark 4.10. The above property known as interlacing property can be deduced from Theorem 9.1.8, p. 258 of Fuhrmann (1996). This property also arises in the case of positive real transfer functions of lossless electrical networks (see Baher (1984), p. 50), wherein the transfer function is of the form

$$Z(\xi) = \left[\frac{H(\xi^2 + \omega_1^2)(\xi^2 + \omega_3^2)\ldots(\xi^2 + \omega_{2n-1}^2)}{\xi(\xi^2 + \omega_2^2)(\xi^2 + \omega_4^2)\ldots(\xi^2 + \omega_{2n-2}^2)}\right]^{\pm 1}$$

where $H \in \mathbb{R}^+$, and

$$0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \ldots$$

We show the link between interlacing property of positive real transfer functions of lossless electrical networks and the interlacing property of Theorem 4.9. Assume that the input voltage of a lossless electrical network is set to zero. Define

$$\begin{aligned} V(\xi) &= r(\xi) = r'(\xi^2) = H(\xi^2 + \omega_1^2)(\xi^2 + \omega_3^2)\ldots(\xi^2 + \omega_{2n-1}^2) \\ \frac{I(\xi)}{\xi} &= f_1(\xi^2) = (\xi^2 + \omega_2^2)(\xi^2 + \omega_4^2)\ldots(\xi^2 + \omega_{2n-2}^2) \end{aligned}$$

Observe that the behaviour $\mathfrak{B} = \ker\left(V(\frac{d}{dt})\right) = \ker\left(r(\frac{d}{dt})\right)$ corresponds to an autonomous lossless electrical network and also that r' and f_1 obey the interlacing property mentioned in Theorem 4.9. The two-variable polynomial corresponding to the power delivered to the network is given by

$$P(\zeta,\eta) = V(\zeta)I(\eta) + I(\zeta)V(\eta) = \eta r'(\zeta^2)f_1(\eta^2) + \zeta r'(\eta^2)f_1(\zeta^2)$$

If $\phi_1(\zeta,\eta)$ represents the two-variable polynomial corresponding to the energy function for the lossless network, then $(\zeta+\eta)\phi_1(\zeta,\eta) = P(\zeta,\eta)$ and this corresponds with equation (4.19).

We now introduce the notion of a modal polynomial operator of a scalar oscillatory behaviour.

Definition 4.11. *Let $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$ be an oscillatory behaviour, where $r(\xi) = (\xi^2 + \omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$ and $0 < \omega_0 < \omega_1 < \cdots < \omega_{n-1}$. Define $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_p^2}$ for $p = 0,\ldots,n-1$. Then the* modal polynomial operator $V(\frac{d}{dt}) : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \rightarrow \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^n)$ *of $\mathfrak{B}$ is defined as*

$$V := \mathit{col}(v_0, v_1, \ldots, v_{n-1})$$

We now give a property of the modal polynomial operator of a scalar oscillatory behaviour.

Lemma 4.12. *Let $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$ be an oscillatory behaviour, where $r(\xi) = (\xi^2 + \omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$ and $0 < \omega_0 < \omega_1 < \cdots < \omega_{n-1}$. Let $V(\frac{d}{dt})$ be the modal polynomial operator of $\mathfrak{B}$. Let $v_i \in \mathbb{R}[\xi]$ denote the $(i+1)^{th}$ component of V. For $i = 0,\ldots,n-1$, define $r_i(\xi) := \xi^2+\omega_i^2$, $\mathfrak{B}_i = \ker\left(r_i(\frac{d}{dt})\right)$. Then $v_i(\frac{d}{dt})\mathfrak{B} = \mathfrak{B}_i$ for $i = 0,\ldots,n-1$.*

Proof. Consider a trajectory $w \in \mathfrak{B}$, given by

$$w(t) = \sum_{p=0}^{n-1}(a_i e^{j\omega_p t} + \bar{a}_i e^{-j\omega_p t}),$$

where $a_p \in \mathbb{C}$ are arbitrary for $p = 0,\ldots,n-1$. It is easy to see that for any $i \in \{0,\ldots,n-1\}$,

$$w_i(t) := v_i(\frac{d}{dt})w(t) = a_i v_i(j\omega_i)e^{j\omega_i t} + \bar{a}_i v_i(-j\omega_i)e^{-j\omega_i t}$$

Observe that $w_i \in \mathfrak{B}_i$. Since a_p are arbitrary for $p = 0,\ldots,n-1$, it follows that for any trajectory $w \in \mathfrak{B}$, $w_i = v_i(\frac{d}{dt})w$ is a trajectory of $\mathfrak{B}_i$.

Now for some $i \in \{0,\ldots,n-1\}$, consider a trajectory $w_i \in \mathfrak{B}_i$, given by

$$w_i(t) = b_i e^{j\omega_i t} + \bar{b}_i e^{-j\omega_i t}$$

where $b_i \in \mathbb{C}$ is arbitrary. Define $a_i := \frac{b_i}{v_i(j\omega_i)}$, and observe that $\bar{a}_i = \frac{\bar{b}_i}{v_i(-j\omega_i)}$. Observe also that w given by

$$w(t) := a_i e^{j\omega_i t} + \bar{a}_i e^{-j\omega_i t}$$

is a trajectory of $\mathfrak{B}$ and $v_i(\frac{d}{dt})w = w_i$. Since b_i is arbitrary, it follows that for any trajectory $w_i \in \mathfrak{B}_i$, there exists $w \in \mathfrak{B}$, such that $v_i(\frac{d}{dt})w = w_i$. This concludes the proof. ∎

In the next corollary, we give the general expression for an energy function of a scalar lossless behaviour that has no characteristic frequency at zero.

Corollary 4.13. *Let $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$ be an oscillatory behaviour, where $r(\xi) = (\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2)\ldots(\xi^2 + \omega_{n-1}^2)$ and $0 < \omega_0 < \omega_1 < \cdots < \omega_{n-1}$. Let $V(\frac{d}{dt})$ be the modal polynomial operator of $\mathfrak{B}$. Define $\Omega :=$diag$(\omega_0, \omega_1, \ldots, \omega_{n-1})$. A two-variable polynomial E induces an energy function for $\mathfrak{B}$, if and only if there exists a diagonal matrix $C \in \mathbb{R}^{n\times n}$ with positive diagonal entries, such that*

$$E(\zeta, \eta) = \zeta\eta V(\zeta)^T C^2 V(\eta) + V(\zeta)^T C^2 \Omega^2 V(\eta) \tag{4.22}$$

Proof. For $p = 0, \ldots, n-1$, define $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_i^2}$.

(*If*): Let

$$C := \text{ diag } (a_0, a_1, \ldots, a_{n-1})$$

where $a_i \in \mathbb{R}^+$ for $i = 0, \ldots, n-1$. Then it follows from equation (4.22) that

$$E(\zeta, \eta) = \sum_{p=0}^{n-1} a_p^2 v_p(\zeta) v_p(\eta)(\zeta\eta + \omega_p^2)$$

From the (*if*) part of the proof of Theorem 4.6, it follows that E induces an energy function for $\mathfrak{B}$.

(*Only if*): From the proof of Lemma 4.8, it follows that every energy function for $\mathfrak{B}$ has an associated two-variable polynomial of the form

$$E(\zeta, \eta) = \sum_{p=0}^{n-1} b_p v_p(\zeta) v_p(\eta)(\zeta\eta + \omega_p^2)$$

where $b_p \in \mathbb{R}^+$ for $p = 0, \ldots, n-1$. Let $a_p \in \mathbb{R}^+$ be such that $a_p^2 = b_p$ for $p = 0, \ldots, n-1$. Define

$$C := \text{ diag } (a_0, a_1, \ldots, a_{n-1})$$

Then, it is easy to see that equation (4.22) holds. ∎

From Corollary 4.13, it follows that energy function of a scalar lossless system is not unique. It depends on the choice of the matrix C in equation (4.22). Later, with the help of an example, it will be shown that not all the energy functions for a scalar lossless system are physically meaningful.

We now define the notion of a complementary oscillatory behaviour of a given scalar oscillatory behaviour.

Definition 4.14. *Let* $\mathfrak{B} = ker\big(r(\frac{d}{dt})\big)$ *be an oscillatory behaviour, where* $r(\xi) := (\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2)\dots(\xi^2 + \omega_{n-1}^2)$ *and* $\omega_i \in \mathbb{R}^+$ *for* $i = 0, \dots, n-1$. *Define* $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_p^2}$ *for* $p = 0, \dots, n-1$ *and* $r'(\xi) := \sum_{i=0}^{n-1} c_p v_p(\xi)$ *where* $c_p \in \mathbb{R}^+$ *for* $p = 0, \dots, n-1$. *Define* $\mathfrak{B}' := ker\big(r'(\frac{d}{dt})\big)$. *Then* $\mathfrak{B}'$ *is called a* complementary oscillatory behaviour *of* $\mathfrak{B}$.

With reference to the above definition, from Corollary 4.13, it follows that

$$E(\zeta, \eta) = \frac{r(\zeta)\eta r'(\eta) + \zeta r'(\zeta) r(\eta)}{\zeta + \eta}$$

is an energy function for $\mathfrak{B}$. In section 5.7, we will give an interpretation for a complementary oscillatory behaviour of a given oscillatory behaviour using the example of a mechanical spring-mass system.

We now describe a suitable method of splitting a given energy function of a scalar autonomous lossless system into its kinetic and potential energy components. We show that with suitable choices for mass and stiffness matrices and a suitable choice for a "generalized position" as a function of the external variables of a scalar oscillatory system, we can obtain equations that are very similar to the equations describing a second order mechanical system. We use this idea to obtain a splitting of the total energy of a lossless system into its kinetic and potential energy components. Thus, with reference to Corollary 4.13, if we interpret $q = V(\frac{d}{dt})w$ as a generalized position, then $\frac{dq}{dt} = \frac{d}{dt}V(\frac{d}{dt})w$ is a generalized velocity. Define $M := 2C^2$ and $K := 2C^2\Omega^2$. Using these expressions the system equations can be written in a way similar to the equations describing a second order mechanical system as

$$M\frac{d^2q}{dt^2} + Kq = 0$$

$$C^2(I\frac{d^2}{dt^2} + \Omega^2)V(\frac{d}{dt})w = 0$$

which reduces to $\text{col}(r(\frac{d}{dt}), r(\frac{d}{dt}), \dots)w(t) = 0$. Thus M and K can be interpreted as the mass and the stiffness matrix respectively. This leads to the two-variable polynomials K_e and P_e corresponding to the kinetic energy $(\frac{1}{2}M(\frac{dq}{dt})^2)$ and potential energy $(\frac{1}{2}Kq^2)$ respectively being given by

$$K_e(\zeta, \eta) = \zeta\eta V(\zeta)^T C^2 V(\eta) \tag{4.23}$$

$$P_e(\zeta, \eta) = V(\zeta)^T C^2 \Omega^2 V(\eta) \tag{4.24}$$

We now illustrate the concepts discussed so far in this section using the example of a mechanical system that was introduced earlier.

Example 4.1 revisited: With reference to example 4.1, let $m_1 = m_2 = 1$, $k_1 = 2$ and $k_2 = 3$. Then $r(\xi) = \xi^4 + 7\xi^2 + 6 = (\xi^2+6)(\xi^2+1)$. This system is oscillatory and hence lossless. The characteristic frequencies of the system are given by $\omega_0 = \sqrt{6}$ and $\omega_1 = 1$. The total kinetic energy and the total potential energy for the system can be expressed as QDFs in terms of only w_1. The two variable polynomials corresponding to these are

$$K_e(\zeta,\eta) = \frac{1}{8}[\zeta^3\eta^3 + 2(\zeta\eta^3 + \zeta^3\eta) + 8\zeta\eta] \tag{4.25}$$

$$P_e(\zeta,\eta) = \frac{1}{8}[5\zeta^2\eta^2 + 6(\zeta^2+\eta^2) + 12] \tag{4.26}$$

The total energy of the system is a positive conserved quantity and hence from Lemma 4.8 will correspond to the two-variable polynomial of the form

$$E(\zeta,\eta) = a_0^2(\zeta\eta+6)(\zeta^2+1)(\eta^2+1) + a_1^2(\zeta\eta+1)(\zeta^2+6)(\eta^2+6)$$

Indeed by comparison with equations (4.25) and (4.26), we obtain real values for a_0 and a_1 as

$$a_0 = \sqrt{0.1} \qquad a_1 = \sqrt{0.025}$$

In this case, with $C = \text{diag}(a_0, a_1)$ and $\Omega = \text{diag}(\omega_0, \omega_1)$, it can be verified that equations (4.23) and (4.24) reduce to equations (4.25) and (4.26) respectively. Observe that in this case not all energy functions Q_E for the system are physically meaningful.

We now build upon the result of Theorem 4.6 and extend it to the multivariable case.

Theorem 4.15. *A linear autonomous system $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is lossless if and only if it is oscillatory.*

Proof. We proceed by reduction of the multivariable case to the scalar case by use of the Smith form. Consider a kernel representation of $\mathfrak{B}$ given by $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is nonsingular. Let $R = U\Delta V$ be the Smith form decomposition of R. Define $\mathfrak{B}' := \ker\big(\Delta(\frac{d}{dt})\big)$. Observe that $w' \in \mathfrak{B}'$ if and only if $w' = V(\frac{d}{dt})w$ for some $w \in \mathfrak{B}$.

We now prove that $\mathfrak{B}$ is lossless if and only if $\mathfrak{B}'$ is lossless. Consider two-variable polynomial matrices $\Phi, \Phi' \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ that are related by

$$V(\zeta)^\top \Phi'(\zeta,\eta) V(\eta) = \Phi(\zeta,\eta)$$

Since V is unimodular, it is easy to see that $Q_\Phi \overset{\mathfrak{B}}{>} 0 \Leftrightarrow Q_{\Phi'} \overset{\mathfrak{B}'}{>} 0$. Now assume that Q_Φ is a conserved quantity for $\mathfrak{B}$. Then there exists $F \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that

$$\Phi(\zeta,\eta) = \frac{R(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R(\eta)}{\zeta+\eta}$$

This implies that

$$\Phi'(\zeta,\eta) = \frac{\Delta(\zeta)^\top U(\zeta)^\top F(\zeta,\eta) V(\eta)^{-1} + V(\zeta)^{-\top} F(\eta,\zeta)^\top U(\zeta)\Delta(\eta)}{\zeta+\eta}$$

The above equation implies that $Q_{\Phi'}$ is a conserved quantity for $\mathfrak{B}'$. Similarly, it can be proved that if $Q_{\Phi'}$ is a conserved quantity for $\mathfrak{B}'$, then Q_Φ is a conserved quantity for $\mathfrak{B}$. Consequently $\mathfrak{B}$ is lossless iff $\mathfrak{B}'$ is lossless. Denote the number of invariant polynomials of R equal to one with $\mathtt{w_1}$ and let $\{r_i(\xi)\}_{i=\mathtt{w_1}+1,\ldots,\mathtt{w}}$ be the set consisting of the remaining invariant polynomials of R. Define $\mathfrak{B}'_i := \ker\big(r_i(\frac{d}{dt})\big)$.

(*Only If*): We assume that $\mathfrak{B}$ and hence $\mathfrak{B}'$ are lossless. Consider a trajectory $w' \in \mathfrak{B}'$. Let $\{w'_i\}_{i=1,\ldots,\mathtt{w}}$ be the components of w'. Consider an energy function Q_ϕ of $\mathfrak{B}$ acting on w. Let $\phi'(\zeta,\eta) = (V(\zeta))^{-\top}\phi(\zeta,\eta)(V(\eta))^{-1}$. Let $\phi'_i(\zeta,\eta)$ be the i^{th} diagonal entry and $\phi'_{ik}(\zeta,\eta)$ be the entry corresponding to the i^{th} row and k^{th} column of the polynomial matrix $\phi'(\zeta,\eta)$. Then

$$Q_\phi(w) = Q_{\phi'}(w') = \sum_{i=1}^{\mathtt{w}} Q_{\phi'_i}(w'_i) + \sum_{i\neq k} L_{\phi'_{ik}}(w'_i, w'_k) \tag{4.27}$$

Since Q_ϕ is an energy function, it is positive over $\mathfrak{B}$. This implies that $Q_{\phi'}$ is positive over $\mathfrak{B}'$. Assume that $Q_{\phi'_i}$ is non-positive for $i = p$. Choose a $w' \in \mathfrak{B}'$, such that all components of w' except the p^{th} are equal to zero. This can be done because each component of w' can be chosen independently of each other. For this particular w', it can be seen from equation (4.27), that $Q_{\phi'}$ is non-positive. Hence by contradiction, it turns out that $Q_{\phi'_i}$ is positive and conserved over $\mathfrak{B}'_i$ for $i = 1, 2, \ldots, \mathtt{w}$. This is possible only if each of $\mathfrak{B}'_i$ is oscillatory for $i = 1, 2, \ldots, \mathtt{w}$, which implies that $\mathfrak{B}'$ and hence $\mathfrak{B}$ is oscillatory.

(*If*): Assume that $\mathfrak{B}$ and hence $\mathfrak{B}'$ is oscillatory. We construct a QDF that is positive and conserved over $\mathfrak{B}$ and hence prove that the system is conservative. For $i = \mathtt{w_1} + \mathtt{1}, \ldots, \mathtt{w}$, let r_i have nonzero roots at $\pm j\omega_{0i}, \pm j\omega_{1i}, \pm j\omega_{2i}, \ldots$ and maximal even polynomial factor equal to s_i. Define $v_{pq}(\xi) := \frac{s_q(\xi)}{\xi^2+\omega_{pq}^2}$. Define $D_i := \operatorname{col}(a_{0i}\xi v_{0i}(\xi), a_{0i}\omega_{0i}v_{0i}(\xi),$ $a_{1i}\xi v_{1i}(\xi), a_{1i}\omega_{1i}v_{1i}(\xi), \ldots)$ if r_i is even and $D_i := \operatorname{col}(a_{0i}\xi^2 v_{0i}(\xi), a_{0i}\omega_{0i}\xi v_{0i}(\xi), a_{1i}\xi^2 v_{1i}(\xi),$ $a_{1i}\omega_{1i}\xi v_{1i}(\xi), \ldots)$ if r_i is odd, where $a_{ik} \in \mathbb{R}^+$ as in the proof of the sufficiency part of

Theorem 4.6. Define

$$D(\xi) := \begin{bmatrix} 0_{\mathtt{w}_1\times\mathtt{w}_1} & 0_{\mathtt{w}_1\times 1} & \dots & \dots & \dots & \\ 0_{\bullet\times\mathtt{w}_1} & D_{\mathtt{w}_1+1} & 0_{\bullet\times 1} & \dots & \dots & \dots \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & D_{\mathtt{w}_1+2} & 0_{\bullet\times 1} & \dots & \dots \\ \vdots & \vdots & 0_{\bullet\times 1} & \ddots & \ddots & \vdots \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & \vdots & \ddots & \ddots & 0_{\bullet\times 1} \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & \dots & \dots & \dots & D_{\mathtt{w}}(\xi) \end{bmatrix} \tag{4.28}$$

From the argument used in order to prove the scalar case, it is easy to see that $\phi'(\zeta,\eta) = D(\zeta)^\top D(\eta)$ is positive and conserved over $\mathfrak{B}'$, and hence $\phi(\zeta,\eta) = V(\zeta)^\top D(\zeta)^\top D(\eta)V(\eta)$ is positive and conserved over $\mathfrak{B}$. This concludes the proof. ∎

4.3 Open lossless systems

In this section, we consider systems that are not autonomous, i.e systems which have inputs. Here we define open lossless systems based on the observation that the total energy of a physical system of this type is positive for nonzero trajectories of the system and that the rate of change of total energy of such a system is zero if the inputs of the system are made equal to zero.

Definition 4.16. *A behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}_{\mathrm{cont}}$ is* lossless *with respect to an input/output partition col(u, y) of $\mathfrak{B}$, if ∃ a QDF $Q_E \overset{\mathfrak{B}}{>} 0$, such that*

$$\frac{d}{dt}Q_E(w) = 0 \quad \forall \;\; w = \;\; col(0, y) \in \mathfrak{B}$$

Any QDF Q_E that satisfies the properties of the above definition is called an *energy function* for the system.

Lemma 4.17. *A behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{u}+\mathtt{y}}_{\mathrm{cont}}$ is lossless with respect to an input/output partition $w =$ col(u, y) of $\mathfrak{B}$, if and only if $\mathfrak{B}_y := \{y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{y}}) \mid col(0, y) \in \mathfrak{B}\}$ is oscillatory.*

Proof. (*If*): Assume that there exists an input/output partition $w = \mathrm{col}(u, y)$ of $\mathfrak{B}$ such that $\mathfrak{B}_y$ is oscillatory and hence lossless. Let Q_{E_1} be an energy function for $\mathfrak{B}_y$ in the sense of Definition 4.1. Define

$$E(\zeta,\eta) := \begin{bmatrix} I_{\mathtt{u}} & 0 \\ 0 & E_1(\zeta,\eta) \end{bmatrix}$$

Since $Q_{E_1} \overset{\mathfrak{B}_y}{>} 0$, it follows that $Q_E \overset{\mathfrak{B}}{>} 0$. Also $\frac{d}{dt}Q_E(w) = 0 \;\forall\; w = \mathrm{col}(0, y) \in \mathfrak{B}$. Thus Q_E is an energy function for $\mathfrak{B}$ in the sense of Definition 4.16. Hence $\mathfrak{B}$ is lossless with respect to the input/output partition $w = \mathrm{col}(u, y)$.

(*Only If*): Assume that $\mathfrak{B}$ is lossless with respect to an input/output partition $w = \text{col}(u, y)$ of $\mathfrak{B}$. Hence there exists an energy function Q_E, such that $\frac{d}{dt}Q_E(w) = 0 \; \forall \; w = \text{col}(0, y) \in \mathfrak{B}$. Partition the two-variable polynomial matrix $E(\zeta, \eta)$ inducing Q_E as

$$E(\zeta,\eta) = \begin{bmatrix} E_{11}(\zeta,\eta) & E_{12}(\zeta,\eta) \\ E_{12}(\eta,\zeta)^\top & E_{22}(\zeta,\eta) \end{bmatrix}$$

where $E_{11} \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta, \eta]$, $E_{12} \in \mathbb{R}^{\mathtt{u}\times\mathtt{y}}[\zeta, \eta]$ and $E_{22} \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\zeta, \eta]$. Since $Q_E \overset{\mathfrak{B}}{>} 0$, it follows that $Q_{E_{22}} \overset{\mathfrak{B}_y}{>} 0$. Also since Q_E is an energy function for $\mathfrak{B}$ in the sense of Definition 4.16, $\frac{d}{dt}Q_{E_{22}}(w) = 0 \; \forall \; w \in \mathfrak{B}_y$. Hence $\mathfrak{B}_y$ is lossless, which in turn implies that it is oscillatory. ■

Remark 4.18. With reference to the above lemma, it is easy to see that if $P(\frac{d}{dt})y = Q(\frac{d}{dt})u$ and $u = D(\frac{d}{dt})\ell$, $y = N(\frac{d}{dt})\ell$ are respectively a minimal kernel representation and an observable image representation of a lossless behaviour $\mathfrak{B}$ then P, respectively D are oscillatory.

In the next algorithm, we show how to compute an energy function of a controllable lossless behaviour, starting from an observable image representation of the behaviour.

Algorithm 4.19. Input*: An observable image representation of* $\mathfrak{B} \in \mathcal{L}^{\mathtt{u}+\mathtt{y}}$ *of the form* $u = D(\frac{d}{dt})\ell$*,* $y = N(\frac{d}{dt})\ell$*, where* $N \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$ *and* $D \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ *is oscillatory.*

Output*: A two-variable polynomial matrix* $E \in \mathbb{R}^{(\mathtt{u}+\mathtt{y})\times(\mathtt{u}+\mathtt{y})}[\zeta, \eta]$ *that induces an energy function for* $\mathfrak{B}$ *in the sense of Definition 4.16.*

Step 1 : Compute a Smith form decomposition of D given by $D = U\Delta V$.

Step 2 : Let $\mathtt{w}_1$ = number of invariant polynomials of D equal to one.

Step 3 : Let $\{r_i(\xi)\}_{i=\mathtt{w}_1+1,\ldots,\mathtt{u}}$ be the set consisting of the nonunity invariant polynomials of D.

Step 4 : For $i = \mathtt{w}_1 + 1, \ldots, \mathtt{u}$, let $\pm j\omega_{0i}, \pm j\omega_{1i}, \pm j\omega_{2i}, \ldots$ be the nonzero roots of r_i and let s_i be the maximal even polynomial factor of r_i.

Step 5 : Assign $v_{pq}(\xi) := \frac{s_q(\xi)}{\xi^2+\omega_{pq}^2}$.

Step 6 : Construct the matrix

$$D'(\xi) = \begin{bmatrix} 0_{\mathtt{w}_1\times\mathtt{w}_1} & 0_{\mathtt{w}_1\times 1} & \cdots & \cdots & \cdots & \\ 0_{\bullet\times\mathtt{w}_1} & D_{\mathtt{w}_1+1} & 0_{\bullet\times 1} & \cdots & \cdots & \cdots \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & D_{\mathtt{w}_1+2} & 0_{\bullet\times 1} & \cdots & \cdots \\ \vdots & \vdots & 0_{\bullet\times 1} & \ddots & \ddots & \vdots \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & \vdots & \ddots & \ddots & 0_{\bullet\times 1} \\ 0_{\bullet\times\mathtt{w}_1} & 0_{\bullet\times 1} & \cdots & \cdots & \cdots & D_{\mathtt{u}}(\xi) \end{bmatrix}$$

where D_i = col $(a_{0i}\xi v_{0i}(\xi), a_{0i}\omega_{0i}v_{0i}(\xi), a_{1i}\xi v_{1i}(\xi), a_{1i}\omega_{1i}v_{1i}(\xi), \ldots)$ if r_i is even, D_i = col $(a_{0i}\xi^2 v_{0i}(\xi), a_{0i}\omega_{0i}\xi v_{0i}(\xi), a_{1i}\xi^2 v_{1i}(\xi), a_{1i}\omega_{1i}\xi v_{1i}(\xi), \ldots)$ if r_i is odd and $a_{ik} \in \mathbb{R}^+$.

Step 7 : Assign $M :=$ col (D, N). Compute a left inverse C_0 of M.

Step 8 : Compute

$$E(\zeta,\eta) = C_0(\zeta)^\top V(\zeta)^\top D'(\zeta)^\top D'(\eta)V(\eta)C_0(\eta) \tag{4.29}$$

The next lemma proves the correctness of Algorithm 4.19.

Lemma 4.20. *With reference to algorithm 4.19, the two-variable polynomial matrix E given by equation (4.29) induces an energy function for $\mathfrak{B}$.*

Proof. Let $Q_{E'}$ be a QDF, such that for any trajectory $w \in \mathfrak{B}$, $Q_E(w) = Q_{E'}(\ell)$, where $w = M(\frac{d}{dt})\ell$ is an observable image representation of $\mathfrak{B}$. Then

$$E'(\zeta,\eta) = M(\zeta)^\top E(\zeta,\eta)M(\eta) = V(\zeta)^\top D'(\zeta)^\top D'(\eta)V(\eta)$$

Consider the behaviour $\mathfrak{B}_{\rm aut} \in \mathcal{L}^{\rm u}$, defined as

$$\mathfrak{B}_{\rm aut} := \{\ell \in \mathbb{C}^\infty(\mathbb{R},\mathbb{R}^{\rm u}) \;\; | \;\; D(\frac{d}{dt})\ell = 0\}$$

From the method of construction of an energy function for an autonomous lossless system given in the proof of Theorem 4.15, it follows that $Q_{E'}$ is an energy function for the lossless autonomous behaviour $\mathfrak{B}_{\rm aut}$ in the sense of Definition 4.1. Now $w = M(\frac{d}{dt})\ell \Rightarrow \ell = C_0(\frac{d}{dt})w$. Hence from Definition 4.16, it follows that

$$E(\zeta,\eta) = C_0(\zeta)^\top V(\zeta)^\top D'(\zeta)^\top D'(\eta)V(\eta)C_0(\eta)$$

induces an energy function for $\mathfrak{B}$. ■

If we consider a mechanical system, the total power delivered to the system is equal to the sum of scalar products of various forces (inputs) acting on the system and the velocities (outputs) at the points of application of the respective forces. Similarly, the total power delivered to an electrical system is equal to the sum of the products of input voltages across various branches and the currents (outputs) through them. Hence the total power delivered to such systems can be written as a quadratic functional, each term of which involves a certain derivative of an input variable and a certain derivative of an output variable. We now investigate whether a similar property holds for the derivative of an energy function of a controllable lossless behaviour, which we may call a "power function". We begin with the following definition.

Definition 4.21. *Consider a behaviour* $\mathfrak{B}$ *with an input/output partition* $col(u, y)$*.* u *is said to have* inconsequential components *if the behaviour* $\mathfrak{B}_u = \{u \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{u}}) \mid col(u, 0) \in \mathfrak{B}\}$ *is not autonomous.*

From the above definition, we can infer that if the input of a behaviour has inconsequential components, then some its components can be freely chosen to make all outputs equal to zero. The next lemma gives the condition on the kernel and image representations of a behaviour under which its input does not have inconsequential components.

Lemma 4.22. *Consider a behaviour* $\mathfrak{B}$ *with an input/output partition* $col(u, y)$*. Let* $P(\frac{d}{dt})y = Q(\frac{d}{dt})u$ *with* $P \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\xi]$*,* $Q \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$*, be a minimal kernel representation of* $\mathfrak{B}$*.* u *does not have inconsequential components iff* $colrank(Q) = \mathtt{u}$*. Furthermore, if* $y = N(\frac{d}{dt})\ell$*,* $u = D(\frac{d}{dt})\ell$ *with* $N \in \mathbb{R}^{\mathtt{y}\times\mathtt{u}}[\xi]$*,* $D \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$*, is an observable image representation of* $\mathfrak{B}$*, then* u *does not have inconsequential components iff* $colrank(N) = \mathtt{u}$*.*

Proof. Consider the behaviour $\mathfrak{B}_u = \ker\left(Q(\frac{d}{dt})\right)$. $\mathfrak{B}_u$ is autonomous iff Q has full column rank. Hence u does not have inconsequential inputs iff $\text{colrank}(Q) = \mathtt{u}$.

Observe that $\mathfrak{B}_u$ is autonomous iff the behaviour $\mathfrak{B}_\ell = \ker\left(N(\frac{d}{dt})\right)$ is autonomous, which in turn holds iff N has full column rank. Conclude from this that u does not have inconsequential inputs iff $\text{colrank}(N) = \mathtt{u}$. ■

In the next theorem, using the concept of *inconsequential components*, we give the condition on the kernel representation of a controllable lossless system under which its power function can be written as a QDF, each term of which involves a certain derivative of an input variable and a certain derivative of an output variable.

Theorem 4.23. *Consider a controllable behaviour* $\mathfrak{B}$ *which is lossless with respect to an input/output partition* $col(u, y)$ *of* $\mathfrak{B}$*. There exists an energy function* Q_E*, such that*

$$\frac{d}{dt}Q_E\big(\text{col}(u, y)\big) \overset{\mathfrak{B}}{=} (R(\frac{d}{dt})u)^\top (S(\frac{d}{dt})y) \tag{4.30}$$

where $R \in \mathbb{R}^{\bullet\times\mathtt{u}}[\xi]$ *and* $S \in \mathbb{R}^{\bullet\times\mathtt{y}}[\xi]$*, iff the following two conditions hold*

1. *u does not have inconsequential components;*
2. *all the invariant polynomials of Q in any minimal kernel representation $P(\frac{d}{dt})y = Q(\frac{d}{dt})u$ of $\mathfrak{B}$ are oscillatory.*

Proof. (*If*): Assume that Q has full column rank and has all its invariant polynomials oscillatory. Define $\mathfrak{B}_y := \{y \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{y}}) \mid \text{col}(0, y) \in \mathfrak{B}\}$. Let Q_{E_1} be an energy function for $\mathfrak{B}_y$ in the sense of Definition 4.1, such that E_1 is P-canonical. Since Q has full column rank, $\mathtt{y} \geq \mathtt{u}$. From Proposition B.5, Appendix B, it follows that there exists a unimodular matrix $U \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\xi]$, such that

$$UQ = \begin{bmatrix} Q_1 \\ 0_{(\mathtt{y}-\mathtt{u})\times\mathtt{u}} \end{bmatrix}$$

where $Q_1 \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ is nonsingular. Now consider the behaviour $\mathfrak{B}_1 := \ker\big(Q_1(\frac{d}{dt})\big)$. Since all the invariant polynomials of Q are oscillatory, $\mathfrak{B}_1$ is an autonomous lossless behaviour. Let Q_{E_2} be an energy function for $\mathfrak{B}_1$ in the sense of Definition 4.1, such that E_2 is Q_1-canonical. Consider the energy function Q_E for $\mathfrak{B}$, where E is given by

$$E(\zeta,\eta) = \begin{bmatrix} E_2(\zeta,\eta) & 0_{\mathtt{u}\times\mathtt{y}} \\ 0_{\mathtt{y}\times\mathtt{u}} & E_1(\zeta,\eta) \end{bmatrix}$$

Define

$$E_1'(\zeta,\eta) := \begin{bmatrix} 0_{\mathtt{u}\times\mathtt{u}} & 0_{\mathtt{u}\times\mathtt{y}} \\ 0_{\mathtt{y}\times\mathtt{u}} & E_1(\zeta,\eta) \end{bmatrix}$$

Let the QDF Q_{P_1} be such that $Q_{P_1} \stackrel{\mathfrak{B}}{=} \frac{d}{dt} Q_{E_1'}$. Let

$$P_1(\zeta,\eta) = \begin{bmatrix} P_{11}(\zeta,\eta) & P_{12}(\zeta,\eta) \\ P_{12}(\eta,\zeta)^\top & P_{13}(\zeta,\eta) \end{bmatrix}$$

$P_{11} \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta,\eta]$, $P_{12} \in \mathbb{R}^{\mathtt{u}\times\mathtt{y}}[\zeta,\eta]$ and $P_{13} \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\zeta,\eta]$. Assume that P_{13} is P-canonical and P_{11} is Q_1-canonical. Since Q_{E_1} is an energy function for $\mathfrak{B}_y$, $\forall\ w = \mathrm{col}(0,y) \in \mathfrak{B}$, $Q_{P_1}(w) = 0$. Since P_{13} is P-canonical, it follows that $P_{13}(\zeta,\eta) = 0_{\mathtt{y}\times\mathtt{y}}$. Since $Q_{E_1} \stackrel{\mathfrak{B}_y}{>} 0$, $\forall$ $w = \mathrm{col}(u,0) \in \mathfrak{B}$, $Q_{E_1'}(w) = 0$ and consequently $Q_{P_1}(w) = 0$. Since P_{11} is Q_1-canonical, it follows that $P_{11}(\zeta,\eta) = 0_{\mathtt{u}\times\mathtt{u}}$. Define

$$E_2'(\zeta,\eta) := \begin{bmatrix} E_2(\zeta,\eta) & 0_{\mathtt{u}\times\mathtt{y}} \\ 0_{\mathtt{y}\times\mathtt{u}} & 0_{\mathtt{y}\times\mathtt{y}} \end{bmatrix}$$

Consider a QDF Q_{P_2}, which is equivalent to $\frac{d}{dt} Q_{E_2'}$ along $\mathfrak{B}$. Let

$$P_2(\zeta,\eta) = \begin{bmatrix} P_{21}(\zeta,\eta) & P_{22}(\zeta,\eta) \\ P_{22}(\eta,\zeta)^\top & P_{23}(\zeta,\eta) \end{bmatrix}$$

where $P_{21} \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta,\eta]$, $P_{22} \in \mathbb{R}^{\mathtt{u}\times\mathtt{y}}[\zeta,\eta]$ and $P_{23} \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\zeta,\eta]$. Assume that P_{23} is P-canonical and P_{21} is Q_1-canonical. Since $Q_{E_2} \stackrel{\mathfrak{B}_1}{>} 0$, $\forall\ w = \mathrm{col}(0,y) \in \mathfrak{B}$, $Q_{E_2'}(w) = 0$, and consequently $Q_{P_2}(w) = 0$. Since P_{23} is P-canonical, it follows that $P_{23}(\zeta,\eta) = 0_{\mathtt{y}\times\mathtt{y}}$.

Since Q_{E_2} is an energy function for $\mathfrak{B}_1$, $\forall\ w = \mathrm{col}(u,0) \in \mathfrak{B}$, $Q_{P_2}(w) = 0$. Since P_{21} is Q_1-canonical, it follows that $P_{21}(\zeta,\eta) = 0_{\mathtt{u}\times\mathtt{u}}$. It is easy to see that

$$P_1(\zeta,\eta) + P_2(\zeta,\eta) = \begin{bmatrix} 0_{\mathtt{u}\times\mathtt{u}} & Z(\zeta,\eta) \\ Z(\eta,\zeta)^\top & 0_{\mathtt{y}\times\mathtt{y}} \end{bmatrix} \overset{\mathfrak{B}}{=} (\zeta+\eta)E(\zeta,\eta)$$

where $Z(\zeta,\eta) = P_{12}(\zeta,\eta) + P_{22}(\zeta,\eta)$. This implies that

$$\frac{d}{dt}Q_E\big(\mathrm{col}(u,y)\big) \overset{\mathfrak{B}}{=} (R(\frac{d}{dt})u)^\top (S(\frac{d}{dt})y)$$

where $R(\zeta)^\top S(\eta)$ is a canonical factorization of $2Z(\zeta,\eta)$.

(*Only If*): Assume by contradiction that u has inconsequential components. Let $Q = U\Delta V$ be a Smith form decomposition of Q. Define $u' := V(\frac{d}{dt})u$, $V_1(\xi) := V(\xi)^{-1}$ and $\mathfrak{B}' := \{\mathrm{col}(u',y) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{u}+\mathtt{y}}) \mid \mathrm{col}(V_1(\frac{d}{dt})u',y) \in \mathfrak{B}\}$. Since $\mathfrak{B}$ is lossless with respect to the input/output partition $w = \mathrm{col}(u,y)$, $\mathfrak{B}'$ is also lossless with respect to the input/output partition $w' = \mathrm{col}(u',y)$. Define

$$R'(\xi) := [\ \Delta(\xi) \quad -U(\xi)^{-1}P(\xi)\ ] \tag{4.31}$$

Then $\mathfrak{B}' = \ker\big(R'(\frac{d}{dt})\big)$. Since u has inconsequential components, we can write

$$\Delta(\xi) = [\ D(\xi) \quad 0_{\mathtt{y}\times\mathtt{u}_1}\ ] \tag{4.32}$$

where $D \in \mathbb{R}^{\mathtt{y}\times(\mathtt{u}-\mathtt{u}_1)}[\xi]$. Consider an energy function $Q_{E'}$ for $\mathfrak{B}'$ given by

$$E'(\zeta,\eta) = \begin{bmatrix} E_{11}(\zeta,\eta) & E_{12}(\zeta,\eta) & E_{13}(\zeta,\eta) \\ E_{12}(\eta,\zeta)^\top & E_{22}(\zeta,\eta) & E_{23}(\zeta,\eta) \\ E_{13}(\eta,\zeta)^\top & E_{23}(\eta,\zeta)^\top & E_{33}(\zeta,\eta) \end{bmatrix}$$

where $E_{33} \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\zeta,\eta]$, $E_{12} \in \mathbb{R}^{(\mathtt{u}-\mathtt{u}_1)\times\mathtt{u}_1}[\zeta,\eta]$, $E_{13} \in \mathbb{R}^{(\mathtt{u}-\mathtt{u}_1)\times\mathtt{y}}[\zeta,\eta]$ and $E_{23} \in \mathbb{R}^{\mathtt{u}_1\times\mathtt{y}}[\zeta,\eta]$, $E_{22} \in \mathbb{R}^{\mathtt{u}_1\times\mathtt{u}_1}[\zeta,\eta]$ and $E_{11} \in \mathbb{R}^{(\mathtt{u}-\mathtt{u}_1)\times(\mathtt{u}-\mathtt{u}_1)}[\zeta,\eta]$.

Consider a trajectory $w = \mathrm{col}(0_{(\mathtt{u}-\mathtt{u}_1)\times 1}, u_1, 0_{\mathtt{y}\times 1}) \in \mathfrak{B}'$, where $u_1 \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{u}_1})$ is nonzero. Observe that $Q_{E'}(w) = Q_{E_{22}}(u_1) > 0$. Hence it is easy to see that $Q_{E_{22}} > 0$. Property (4.36) implies that the derivative of the energy function Q_E is equivalent to another QDF along the behaviour $\mathfrak{B}$ whose associated two-variable polynomial matrix P_0 has the form

$$P_0(\zeta,\eta) = \begin{bmatrix} 0_{\mathtt{u}\times\mathtt{u}} & Z(\zeta,\eta) \\ Z(\eta,\zeta)^\top & 0_{\mathtt{y}\times\mathtt{y}} \end{bmatrix} \tag{4.33}$$

where $2Z(\zeta,\eta) = R(\zeta)^\top S(\eta)$. Assume that there exists a QDF $Q_{P'}$ such that $Q_{P'} \stackrel{\mathfrak{B}'}{=} \frac{d}{dt}Q_{E'}$ and

$$P'(\zeta,\eta) = \begin{bmatrix} 0_{\mathtt{u}\times\mathtt{u}} & P_2(\zeta,\eta) \\ P_2(\eta,\zeta)^\top & 0_{\mathtt{u}\times\mathtt{u}} \end{bmatrix} \tag{4.34}$$

where $P_2 \in \mathbb{R}^{\mathtt{u}\times\mathtt{y}}[\zeta,\eta]$. Since $(\zeta+\eta)E'(\zeta,\eta) \stackrel{\mathfrak{B}'}{=} P'(\zeta,\eta)$,

$$(\zeta+\eta)E'(\zeta,\eta) = R'(\zeta)^\top F(\zeta,\eta) + F(\eta,\zeta)^\top R'(\eta) + P'(\zeta,\eta) \tag{4.35}$$

where $F \in \mathbb{R}^{\mathtt{y}\times(\mathtt{y}+\mathtt{u})}[\zeta,\eta]$. Let the right hand side of the above equation be denoted by $P''(\zeta,\eta)$. Let

$$P''(\zeta,\eta) = \begin{bmatrix} P_{11}(\zeta,\eta) & P_{12}(\zeta,\eta) & P_{13}(\zeta,\eta) \\ P_{12}(\eta,\zeta)^\top & P_{22}(\zeta,\eta) & P_{23}(\zeta,\eta) \\ P_{13}(\eta,\zeta)^\top & P_{23}(\eta,\zeta)^\top & P_{33}(\zeta,\eta) \end{bmatrix}$$

where $P_{33} \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\zeta,\eta]$, $P_{12} \in \mathbb{R}^{(\mathtt{u}-\mathtt{u}_1)\times\mathtt{u}_1}[\zeta,\eta]$, $P_{13} \in \mathbb{R}^{(\mathtt{u}-\mathtt{u}_1)\times\mathtt{y}}[\zeta,\eta]$ and $P_{23} \in \mathbb{R}^{\mathtt{u}_1\times\mathtt{y}}[\zeta,\eta]$, $P_{22} \in \mathbb{R}^{\mathtt{u}_1\times\mathtt{u}_1}[\zeta,\eta]$ and $P_{11} \in \mathbb{R}^{(\mathtt{u}-\mathtt{u}_1)\times(\mathtt{u}-\mathtt{u}_1)}[\zeta,\eta]$. Then using equations (4.31), (4.32) and (4.35), it can be verified that for any $F \in \mathbb{R}^{\mathtt{y}\times(\mathtt{y}+\mathtt{u})}[\zeta,\eta]$ and P' given by equation (4.34), we obtain $P_{22}(\zeta,\eta) = 0_{\mathtt{u}_1\times\mathtt{u}_1}$, which is a contradiction as $Q_{E_{22}} > 0$. Hence there does not exist a QDF $Q_{P'}$ which is equivalent to the derivative of an energy function along the behaviour $\mathfrak{B}'$, with P' given by equation (4.34). This implies that there does not exist a QDF Q_{P_0} which is equivalent to the derivative of an energy function along the behaviour $\mathfrak{B}$ such that equation (4.33) holds.

Now assume that $\text{colrank}(Q) = \mathtt{u}$, but at least one of the invariant polynomials of Q is not oscillatory. Hence, there exists a unimodular matrix $U \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\xi]$, such that

$$UQ = \begin{bmatrix} Q_1 \\ 0_{(\mathtt{y}-\mathtt{u})\times\mathtt{u}} \end{bmatrix}$$

where $Q_1 \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$ is such that $\det(Q_1) \neq 0$. Now consider the behaviour $\mathfrak{B}_1 := \ker\big(Q_1(\frac{d}{dt})\big)$. Since all invariant polynomials of Q are not oscillatory, $\mathfrak{B}_1$ is not lossless. Consider an energy function Q_E for $\mathfrak{B}$ given by

$$E(\zeta,\eta) = \begin{bmatrix} E_1(\zeta,\eta) & E_2(\zeta,\eta) \\ E_2(\eta,\zeta)^\top & E_3(\zeta,\eta) \end{bmatrix}$$

where $E_1 \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta,\eta]$, $E_2 \in \mathbb{R}^{\mathtt{u}\times\mathtt{y}}[\zeta,\eta]$ and $E_3 \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\zeta,\eta]$. Let the QDF Q_{P_0} be such that $Q_{P_0} \stackrel{\mathfrak{B}}{=} \frac{d}{dt}Q_E$. Let

$$P_0(\zeta,\eta) = \begin{bmatrix} P_1(\zeta,\eta) & P_2(\zeta,\eta) \\ P_2(\eta,\zeta)^\top & P_3(\zeta,\eta) \end{bmatrix}$$

where $P_1 \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\zeta,\eta]$, $P_2 \in \mathbb{R}^{\mathtt{u}\times\mathtt{y}}[\zeta,\eta]$ and $P_3 \in \mathbb{R}^{\mathtt{y}\times\mathtt{y}}[\zeta,\eta]$. Since $\mathfrak{B}_1$ is not lossless, there does not exist a positive QDF Q_{E_1} such that $\frac{d}{dt}Q_{E_1} \stackrel{\mathfrak{B}_1}{=} 0$. Hence $Q_{P_0}(w) \neq 0\ \forall$

$w = \text{col}(u, 0) \in \mathfrak{B}$. Thus $P_1(\zeta, \eta) \neq 0_{\mathtt{u}\times\mathtt{u}}$. It follows that there does not exist a QDF Q_{P_0} which is equivalent to the derivative of an energy function along the behaviour $\mathfrak{B}$ such that equation (4.33) holds. ■

We now state a result that is equivalent to the one stated in Theorem 4.23 in terms of an observable image representation of the behaviour.

Corollary 4.24. *Consider a controllable behaviour $\mathfrak{B}$ which is lossless with respect to an input/output partition col(u, y) of $\mathfrak{B}$. There exists an energy function Q_E, such that*

$$\frac{d}{dt}Q_E\big(\text{col}(u, y)\big) \overset{\mathfrak{B}}{=} (R(\frac{d}{dt})u)^\top(S(\frac{d}{dt})y) \tag{4.36}$$

where $R \in \mathbb{R}^{\bullet\times\mathtt{u}}[\xi]$ and $S \in \mathbb{R}^{\bullet\times\mathtt{y}}[\xi]$, iff the following two conditions hold

1. *u does not have inconsequential components;*
2. *all the invariant polynomials of N in any observable image representation $y = N(\frac{d}{dt})\ell$, $u = D(\frac{d}{dt})\ell$ of $\mathfrak{B}$ are oscillatory.*

Example 4.2. Consider a behaviour $\mathfrak{B}_m$ whose external variables are the generalized position vector $q \in \mathbb{R}^{\mathtt{q}}$ and generalized force vector $F \in \mathbb{R}^{\mathtt{q}}$ of a second order undamped mechanical system given by the equation

$$M\frac{d^2q}{dt^2} + Kq = F \tag{4.37}$$

where $M,\ K \in \mathbb{R}_s^{\mathtt{q}\times\mathtt{q}}$ are positive definite and denote generalized mass and stiffness matrices respectively. $\mathfrak{B}_m$ can be represented in kernel form as $\mathfrak{B}_m = \ker\big(R_m(\frac{d}{dt})\big)$, where

$$R_m(\xi) = [\ (M\xi^2 + K) \quad -I_{\mathtt{q}}\ ]$$

If q and F are considered as output and input respectively, then it easy to see that the behaviour is controllable and lossless. Also from Theorem 4.23, it follows that there exists an energy function Q_E, such that

$$\frac{d}{dt}Q_E\big(\text{col}(q, F)\big) \overset{\mathfrak{B}_m}{=} (R(\frac{d}{dt})F)^\top(S(\frac{d}{dt})q)$$

where $R, S \in \mathbb{R}^{\bullet\times\mathtt{q}}[\xi]$. Indeed the power delivered to the second order mechanical system given by equation (4.37) is $F^\top\left(\frac{dq}{dt}\right)$, which implies that $R(\xi) = I_{\mathtt{q}}$ and $S(\xi) = \xi I_{\mathtt{q}}$.

Example 4.3. Consider a behaviour $\mathfrak{B}_e$ whose external variables are the voltage V across and the current I through a one-port lossless electrical network given by the equation

$$d(\frac{d}{dt})V = n(\frac{d}{dt})I$$

where $n, d \in \mathbb{R}[\xi]$. It is well known that $Z \in \mathbb{R}(\xi)$ defined by $Z(\xi) := \frac{n(\xi)}{d(\xi)}$ is lossless positive real (see Appendix B for a definition) and hence both n and d are oscillatory.

Hence by Theorem 4.23, there exists an energy function Q_E, such that

$$\frac{d}{dt}Q_E\big(\mathrm{col}(V,I)\big) \overset{\mathfrak{B}_e}{=} (R(\frac{d}{dt})V)^\top (S(\frac{d}{dt})I)$$

where $R, S \in \mathbb{R}^\bullet[\xi]$. Indeed the power delivered to the one-port electrical network is equal to $V \cdot I$ which implies that $R(\xi) = S(\xi) = 1$.

4.4 Summary

In this chapter, we have defined an autonomous lossless behaviour as one with a conserved quantity that is positive along the trajectories of the behaviour. We have then showed that an autonomous behaviour is lossless if and only if it is oscillatory. We have studied a few properties of energy functions of scalar autonomous lossless behaviours, and also discussed a way of splitting a given energy function of an oscillatory behaviour into its kinetic and potential energy components. We have also defined open lossless systems and have given an algorithm for computing an energy function of a given open lossless system. Finally, we have investigated the conditions under which a power function or the derivative of an energy function of an open lossless system can be written as a quadratic functional, each term of which involves a certain derivative of an input variable and a certain derivative of an output variable of the given lossless system.

Chapter 5

Quadratic differential forms and oscillatory behaviours

5.1 Introduction

An oscillatory system is an autonomous system where there is no dissipation in any of its components. As seen in chapter 4, such systems exhibit the property of conservation of energy which is a QDF. Note that the Lagrangian for an oscillatory system which is another QDF, is a zero-mean quantity. Rapisarda and Willems (2005) showed that there are conserved quantities other than the total energy, and zero-mean quantities other than the Lagrangian for an oscillatory system. They proved that the space of QDFs associated with oscillatory systems can be decomposed into the spaces of conserved and zero-mean quantities.

In this chapter we study the space of QDFs on oscillatory behaviours. We also study in greater detail, its decomposition into the spaces of conserved and zero-mean QDFs. We then investigate the structure of zero-mean and conserved QDFs modulo the given behaviour. We show that for oscillatory systems, intrinsically zero-mean functionals do not need to be defined only as complementary to the trivially zero-mean ones, but they can also be given an inherent algebraic characterization. We also provide an algorithm to construct a basis for the set of conserved, trivially zero-mean and intrinsically zero-mean QDFs starting from a kernel description of the behaviour. For reasons of simplicity and in order to introduce gradually the relevant concepts and techniques, we first study the case of scalar oscillatory behaviours and then move on to the case of multivariable oscillatory behaviours. Then, using a generalization of the principle of least action, we define a class of zero-mean quantities for oscillatory systems which we call generalized Lagrangians and study their properties.

In physical examples of oscillatory systems, the system can be represented by a set of first or second order differential equations in the external variables. However, most often the variables of actual interest are just a few, and via algebraic manipulations the original description is reduced to a system of higher-order differential equations in those variables. From this situation arises the need to develop a theory of mechanical systems where the starting point of interest is what is in behavioural terms a kernel description of the behaviour. An extreme case of the above situation is that of a system which exhibits only one variable of interest and is described by a single high-order differential equation. In order to provide motivation for this chapter, we now examine an example of such a mechanical system that was also considered in the previous chapter.

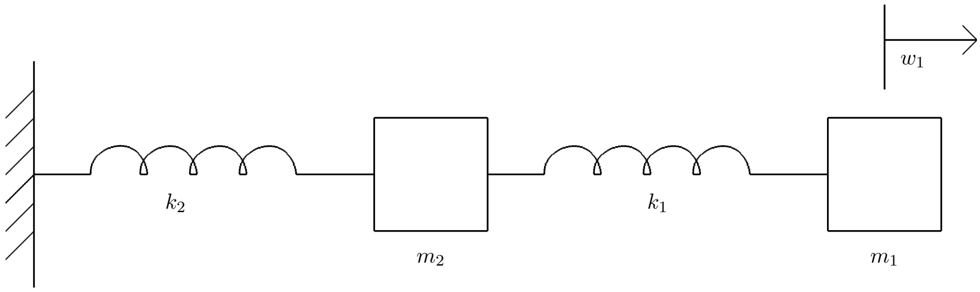

FIGURE 5.1: A mechanical example

Example 5.1. Consider two-masses m_1 and m_2 attached to springs with constants k_1 and k_2. The first mass is connected to the second one via the first spring, and the second mass is connected to the wall with the second spring as shown in Figure 5.1. Denote by w_1 and w_2 the positions of the first and the second mass respectively. Let $m_1 = m_2 = 1$, $k_1 = 2$ and $k_2 = 3$. If w_1 is the manifest variable, then its evolution can be described by the equation $r(\frac{d}{dt})w_1 = 0$, where $r(\xi) = \xi^4 + 7\xi^2 + 6 = (\xi^2 + 6)(\xi^2 + 1)$. Define the behaviour $\mathfrak{B}$ as

$$\mathfrak{B} := \{w_1 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}) \mid r(\frac{d}{dt})w_1 = 0\}$$

It is easy to see that $\mathfrak{B}$ is oscillatory. The total physical kinetic energy and the total physical potential energy for the system can be expressed as QDFs in terms of only w_1. The two-variable polynomials corresponding to these are

$$K'(\zeta, \eta) = \frac{1}{8}[\zeta^3\eta^3 + 2(\zeta\eta^3 + \zeta^3\eta) + 8\zeta\eta]$$

$$P'(\zeta, \eta) = \frac{1}{8}[5\zeta^2\eta^2 + 6(\zeta^2 + \eta^2) + 12]$$

The two-variable polynomial corresponding to the Lagrangian for the system, which is the difference between its kinetic and potential energies can be written as

$$\Lambda(\zeta, \eta) = \frac{1}{8}[\zeta^3\eta^3 - 5\zeta^2\eta^2 + 2(\zeta\eta^3 + \zeta^3\eta) - 6(\zeta^2 + \eta^2) + 8\zeta\eta - 12] \tag{5.1}$$

Recall from Definition 2.23 of chapter 2 that the roots of r are the characteristic frequencies of $\mathfrak{B}$, and that any trajectory $w_1 \in \mathfrak{B}$ can be written as

$$w_1(t) = c_1 e^{\sqrt{6}jt} + \bar{c}_1 e^{-\sqrt{6}jt} + c_2 e^{jt} + \bar{c}_2 e^{-jt} \tag{5.2}$$

where $c_i \in \mathbb{C}$ are arbitrary constants for $i = 1, 2$. From equations (5.1) and (5.2), it follows that

$$Q_\Lambda(w_1)(t) = c_3 e^{2\sqrt{6}jt} + \bar{c}_3 e^{-2\sqrt{6}jt} + c_4 e^{2jt} + \bar{c}_4 e^{-2jt} \tag{5.3}$$

where $c_3, c_4 \in \mathbb{C}$. Since the right hand side of the above equation is a linear combination of sinusoids, it is easy to see that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} Q_\Lambda(w)(t)dt = 0$$

and hence the Lagrangian for $\mathfrak{B}$ is a zero-mean quantity. Observe from equation (5.3), that the expression for $Q_\Lambda(w_1)$ is a linear combination of sinusoids of frequencies that are equal to twice the characteristic frequencies of $\mathfrak{B}$. A QDF which is a linear combination of sinusoids of frequencies that are equal to twice the characteristic frequencies of a behaviour, along the trajectories of the behaviour, will be referred to as a *single-frequency zero-mean quantity*. A formal definition for a single-frequency zero-mean quantity will be given further on in this chapter. We have shown that the Lagrangian for the given example is a single-frequency zero-mean quantity.

In this chapter, we explore further the relation between single frequency zero-mean quantities and Lagrangians for an oscillatory behaviour. We show a method of generation of the bases of conserved, trivially zero-mean and intrinsically zero-mean quantities for examples of oscillatory systems described by higher order differential equations like the one described. We also show how the Lagrangian for the system is related to the basis elements of trivially and intrinsically zero-mean quantities for the system.

Part of the work presented in this chapter has been published (see Rao and Rapisarda (2006)).

5.2 One- and two-variable polynomial shift operators

In this section, we describe the notions of one- and two-variable shift operators which will be used in order to prove the decomposition theorem for the space of QDFs associated with oscillatory behaviours. These notions were first introduced by Peters and Rapisarda (2001).

Let $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ be an autonomous behaviour represented by $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ is nonsingular. We denote the subset of $\mathbb{C}^{1\times\mathtt{w}}[\xi]$ consisting of all R-canonical polynomial matrices by $\mathfrak{r}_R^{1\times\mathtt{w}}[\xi]$ and the subset of $\mathbb{C}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ consisting of all R-canonical symmetric two-variable polynomial matrices by $\mathfrak{x}^{\mathtt{w}\times\mathtt{w}}_{R,\mathrm{sym}}[\zeta,\eta]$. The following definitions and propositions from Peters and Rapisarda (2001) will be used to study the space of conserved and zero-mean quantities for oscillatory systems.

Definition 5.1. The *one-variable shift operator* $S_R : \mathfrak{x}^{1\times\mathtt{w}}_R[\xi] \to \mathfrak{x}^{1\times\mathtt{w}}_R[\xi]$ is defined as

$$S_R : D(\xi) \mapsto \xi D(\xi) \bmod R(\xi)$$

Definition 5.2. The *two-variable shift operator* $\mathfrak{L}_R : \mathfrak{x}^{\mathtt{w}\times\mathtt{w}}_{R,\mathrm{sym}}[\zeta,\eta] \to \mathfrak{x}^{\mathtt{w}\times\mathtt{w}}_{R,\mathrm{sym}}[\zeta,\eta]$ is defined as

$$\mathfrak{L}_R : \Phi(\zeta,\eta) \mapsto (\zeta+\eta)\Phi(\zeta,\eta) \bmod R(\xi)$$

The following proposition gives the expression for the characteristic polynomial of the operator S_R.

Proposition 5.3. *The characteristic polynomial of S_R is equal to det$(R(\xi))/r_n$, where r_n denotes the leading coefficient of det$(R(\xi))$.*

Proof. See proof of Proposition 2.4, p. 123, Peters and Rapisarda (2001). ■

From the above Proposition, it follows that the eigenvalues of the linear operator S are the roots of $\det(R)$. The next Proposition gives a similar result about the two-variable shift operator $\mathfrak{L}_R$.

Proposition 5.4. *The characteristic polynomial of the two-variable shift operator $\mathfrak{L}_R$ is $\prod_{1\le i\le k\le n}(\xi-(\lambda_i+\lambda_k))$, where $n =$ deg(det(R)) and $\lambda_1,\ldots,\lambda_n$ are the eigenvalues of the operator S_R. If $v_i(\xi)$ represents the eigenvector corresponding to the eigenvalue λ_i of S_R, then $v_i(\zeta)^\top v_k(\eta)+v_k(\zeta)^\top v_i(\eta)$ is the eigenvector of $\mathfrak{L}_R$ corresponding to the eigenvalue $(\lambda_i+\lambda_k)$.*

Proof. See proof of Proposition 3.4, p. 124, Peters and Rapisarda (2001) ■

5.3 Space of QDFs modulo an oscillatory behaviour

In this section we study the space of QDFs modulo an oscillatory behaviour with no characteristic frequencies at zero. The next theorem has been already proved (see Proposition 4 of Rapisarda and Willems (2005)). However, we prove the same theorem using a different approach, which will be useful also in the construction of basis elements of conserved, trivially zero-mean and intrinsically zero-mean quantities for oscillatory behaviours.

Theorem 5.5. *Let $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ and $\det(R) \neq 0$, be an oscillatory behaviour that has no characteristic frequency at zero. The dimension of QDFs modulo $\mathfrak{B}$ is equal to $n(2n+1)$, where $2n = \deg(\det(R))$.*

Proof. Let $R = U\Delta V$ be a Smith form decomposition of R. Now consider the behaviour $\mathfrak{B}' = \ker\left(\Delta(\frac{d}{dt})\right)$ and observe that $w' \in \mathfrak{B}'$ if and only if $w' = V\left(\frac{d}{dt}\right)w$ for some $w \in \mathfrak{B}$. Define $R_1(\xi) := \Delta(\xi)V(\xi)$, and note that $\mathfrak{B} = \ker\bigl(R_1(\frac{d}{dt})\bigr)$ is another kernel representation for $\mathfrak{B}$. We now show that there is a one-one correspondence between the spaces of Δ-canonical and R_1-canonical two-variable polynomial matrices. Consider a Δ-canonical $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Define $\Phi_1(\zeta,\eta) := V(\zeta)^\top\Phi(\zeta,\eta)V(\eta)$. Observe that $R_1(\zeta)^{-\top}\Phi_1(\zeta,\eta)R_1(\eta)^{-1} = \Delta(\zeta)^{-\top}\Phi(\zeta,\eta)\Delta(\eta)^{-1}$ is strictly proper, which implies that Φ_1 is R_1-canonical. Also observe that there is a one-one correspondence between Φ and Φ_1, i.e for any given $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, there exists a unique $\Phi_1 \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that $\Phi_1(\zeta,\eta) = V(\zeta)^\top\Phi(\zeta,\eta)V(\eta)$ and vice versa. This implies that there is a one-one correspondence between the spaces of R_1-canonical and Δ-canonical symmetric two-variable polynomial matrices and consequently both these spaces have the same dimension. We now construct a basis for the space of Δ-canonical symmetric two-variable polynomial matrices.

We show that for the space of QDFs whose associated two-variable polynomial matrices are Δ-canonical, we can choose a basis consisting of diagonal QDFs and zero-diagonal QDFs. Diagonal QDFs are those QDFs that are induced by polynomial matrices having all the nondiagonal entries equal to zero and zero-diagonal QDFs are those induced by polynomial matrices that have all diagonal entries equal to zero. Let K_{pq} denote a square matrix of size $\mathtt{w}$ whose entry in the (p,q) position is 1 and the remaining entries are equal to 0. Let $\mathtt{w}_1 + 1$ be the index corresponding to the first nonunity invariant polynomial of $\mathfrak{B}$. Note that each of the non-unity invariant polynomials has distinct and purely imaginary roots.

We now define the following set.

$\mathcal{Q}_{pq}$:= A basis of Δ-canonical symmetric two-variable polynomial matrices of the form

$$K_{pq}f(\zeta,\eta) + K_{qp}f(\eta,\zeta) \tag{5.4}$$

where $f \in \mathbb{R}[\zeta,\eta]$. Denote by $\delta_i \in \mathbb{R}[\xi]$, the i^{th} diagonal entry of Δ. Define $n_i := \deg(\delta_i(\xi))/2$ and $\mathfrak{B}_i := \ker\left(\delta_i(\frac{d}{dt})\right)$. We make use of the following lemma to obtain $\mathcal{Q}_{ii}$ for $i = 1,\dots,\mathtt{w}$.

Lemma 5.6. *Consider a behaviour $\mathfrak{B} = ker(r(\frac{d}{dt}))$ $(r \in \mathbb{R}[\xi])$ of dimension $2n$. The set $\left\{\zeta^i\eta^j + \zeta^j\eta^i\right\}_{0\le i\le j\le 2n-1}$ induces a basis for the space of r-canonical symmetric two-variable polynomials.*

Proof. Consider the set $\left\{\zeta^i\eta^j + \zeta^j\eta^i\right\}_{0\le i\le j\le 2n-1}$. Observe that each element of the set is r-canonical, because each term $\zeta^k\eta^l$ appearing in it is such that k,$l \le 2n-1$. The elements of the set are linearly independent of each other because a term $\zeta^k\eta^l$ that appears in one of them does not appear in any other element of the set. Finally it is easy to see that any r-canonical symmetric two-variable polynomial can be written as a linear combination of the elements of the set. Hence this set forms a basis for the space of r-canonical symmetric two-variable polynomials. This concludes the proof. ■

It is easy to see that the set $\mathcal{Q}_{ii}$ can be obtained from the basis of QDFs modulo $\mathfrak{B}_i$, using the result of Lemma 5.6, as follows.

$$\mathcal{Q}_{ii} = \left\{K_{ii}(\zeta^l\eta^m + \zeta^m\eta^l)\right\}_{0\leq l\leq m\leq 2n_i-1}$$

for $i = \mathtt{w}_1 + 1, \ldots, \mathtt{w}$ and $\mathcal{Q}_{ii}$ is empty for $i = 1, \ldots, \mathtt{w}_1$. Observe that each element of the above set is Δ-canonical.

For the construction of the bases of zero-diagonal QDFs, we make use of the following Lemma.

Lemma 5.7. *Define $r_1(\xi) := (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{(n_1-1)}^2)$ and $r_2(\xi) := (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{(n_2-1)}^2)$ where $n_1, n_2 \in \mathbb{Z}^+$, $n_2 \geq n_1$ and $\omega_i \in \mathbb{R}^+$ are distinct for $i = 0, 1, \ldots, n_2-1$. Define $\mathfrak{B} := \ker\big(R(\frac{d}{dt})\big)$ where $R(\xi) :=$ diag$(r_1(\xi), r_2(\xi))$. Consider the space of R-canonical symmetric two-variable polynomial matrices of the form*

$$\begin{bmatrix} 0 & f(\zeta,\eta) \\ f(\eta,\zeta) & 0 \end{bmatrix} \tag{5.5}$$

where $f \in \mathbb{R}[\zeta,\eta]$. Then a basis for this space is

$$\left\{\begin{bmatrix} 0 & \zeta^l\eta^m \\ \zeta^m\eta^l & 0 \end{bmatrix}\right\}_{0\leq l\leq 2n_1-1, 0\leq m\leq 2n_2-1} \tag{5.6}$$

Proof. For Φ_{12} to be R-canonical, the highest power of ζ and η in $f(\zeta,\eta)$ must be less than or equal to $(2n_1-1)$ and $(2n_2-1)$ respectively. It can be seen that each element of the set (5.6) is R-canonical. To prove that the elements of the set are linearly independent, notice that a term of the form

$$\begin{bmatrix} 0 & \zeta^k\eta^i \\ \zeta^i\eta^k & 0 \end{bmatrix}$$

that occurs in one of the elements does not occur in any other element of the set. Observe that linear combinations of the elements of the set can produce any R-canonical symmetric two-variable polynomial matrix of the form (5.5). Hence the set forms the required basis. ■

For integers i, k such that $\mathtt{w}_1 + 1 \leq i < k \leq \mathtt{w}$, it is now easy to see that

$$\mathcal{Q}_{ik} = \left\{K_{ik}\zeta^l\eta^m + K_{ki}\zeta^m\eta^l\right\}_{0\leq l\leq 2n_i-1, 0\leq m\leq 2n_k-1}$$

Also observe that $\mathcal{Q}_{ik}$ is empty for all the remaining combinations of i and k.

Now observe that the spaces of zero-diagonal and diagonal QDFs are linearly independent. Also observe that any Δ-canonical two-variable polynomial matrix can be written as a sum of two matrices, one diagonal and the other zero-diagonal. Hence it is easy to see that the union of bases of diagonal and zero-diagonal Δ-canonical symmetric two-variable polynomial matrices forms a basis of Δ-canonical symmetric two-variable polynomial matrices, i.e the set $\cup_{i=\mathtt{w}_1+1}^{\mathtt{w}} \cup_{k=i}^{\mathtt{w}} \mathcal{Q}_{ik}$ is a basis of Δ-canonical symmetric two-variable polynomial matrices.

Note that the corresponding basis of two-variable polynomial matrices inducing QDFs modulo $\mathfrak{B}$ is obtained by premultiplication with $V(\zeta)^T$ and postmultiplication with $V(\eta)$ of the basis elements mentioned above. Hence the dimension of QDFs modulo $\mathfrak{B}$ is the total number of elements in the basis which is given by

$$\begin{aligned}\dim(\text{QDFs}) &= \sum_{i=\mathtt{w}_1+1}^{\mathtt{w}} (2n_i^2 + n_i) + \sum_{(\mathtt{w}_1+1)\leq i<k\leq \mathtt{w}} 4n_i n_k \\ &= 2\left(\sum_{i=\mathtt{w}_1+1}^{\mathtt{w}} n_i\right)^2 + \sum_{i=\mathtt{w}_1+1}^{\mathtt{w}} n_i\end{aligned}$$

Observe that $\deg(\det(R)) = 2\sum_{i=\mathtt{w}_1+1}^{\mathtt{w}} n_i$. This concludes the proof. ■

5.4 A decomposition theorem for QDFs: The scalar case

In this section, we show that if $r \in \mathbb{R}[\xi]$ is oscillatory and $\mathfrak{B} := \ker(r(\frac{d}{dt}))$, then for any r-canonical $\Phi \in \mathbb{R}_s[\zeta, \eta]$, Q_Φ is a linear combination of a conserved quantity and a zero-mean quantity for $\mathfrak{B}$. Part of these results have been presented already in section 4 of Rapisarda and Willems (2005). However, we derive them following a novel approach which will be instrumental in proving further results of this chapter.

In the following analyses, we consider only those oscillatory behaviours that have no characteristic frequency at zero. This is because physically a zero characteristic frequency implies the presence of a rigid body motion in the case of a mechanical system and part of the output voltage or current being constant in the case of an electrical system, and neglecting the rigid body motion in the case of a mechanical system (for example, by placing the center of mass of the system in a fixed position) or the constant current or voltage in the case of an electrical system does not impair the analysis of the most relevant dynamical features of the system. The following theorem is the decomposition theorem for the space of QDFs associated with scalar oscillatory behaviours.

Theorem 5.8. *Let $\mathfrak{B}$ be an oscillatory behaviour given by $\mathfrak{B} = \ker\left(r(\frac{d}{dt})\right)$, where $r(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$ and $\omega_i \in \mathbb{R}^+$ are distinct for $i = 0, 1, \ldots, n-1$. The space $\mathcal{Q}$ of r-canonical symmetric two-variable polynomials can be decomposed into* *the spaces $\mathcal{Q}_{\mathcal{C}}$ of r-canonical polynomials inducing conserved quantities and $\mathcal{Q}_{\mathcal{Z}}$ of r-canonical polynomials inducing zero-mean quantities for $\mathfrak{B}$ as follows:*

$$\mathcal{Q} = \mathcal{Q}_{\mathcal{C}} \oplus \mathcal{Q}_{\mathcal{Z}}$$

The dimensions of the spaces $\mathcal{Q}_{\mathcal{C}}$ and $\mathcal{Q}_{\mathcal{Z}}$ are n and $2n^2$ respectively.

Proof. As in section 5.2, consider the one-variable shift operator S_r and the two-variable shift operator $\mathfrak{L}_r$ for $\mathfrak{B}$. Since r is real and has distinct and purely imaginary roots, from the discussion of section 5.2, it is easy to see that the eigenvectors v_i of S_r occur in conjugate pairs.

Now consider the set $\mathcal{Q}_c = \{v_i(\zeta)v_k(\eta) + v_k(\zeta)v_i(\eta)\}_{1\leq i\leq k\leq 2n}$ of eigenvectors of the linear operator $\mathfrak{L}_r$. Define $v_{(\lambda_i,\lambda_k)}(\zeta,\eta) := v_i(\zeta)v_k(\eta) + v_k(\zeta)v_i(\eta)$. Let j represent the imaginary unit. Consider $v_{(+j\omega_k,-j\omega_k)}(\zeta,\eta) = v_k(\zeta)\bar{v}_k(\eta) + \bar{v}_k(\zeta)v_k(\eta)$, where $\bar{v}$ denotes complex conjugation of v. It is easy to see that $v_{(+j\omega_k,-j\omega_k)}$ is purely real. Consider the set $\mathcal{Q}'_c := \mathcal{Q}_c \setminus \left\{v_{(+j\omega_k,-j\omega_k)}\right\}_{k=0,\ldots,n-1}$. It is easy to see that this set consists of $2n^2$ elements that occur in complex conjugates. From every conjugate pair of this set, choose one element. Call the resulting set $\mathcal{X}_c$.

Let $\mathcal{Z}$ be the set consisting of all the real and imaginary parts of the elements of the set $\mathcal{Q}'_c$. We claim that the set

$$\mathcal{X} = \mathcal{Z} \cup \left\{v_{(+j\omega_k,-j\omega_k)}\right\}_{k=0,\ldots,n-1} \tag{5.7}$$

is a basis for the set of r-canonical symmetric two-variable polynomials. Firstly observe that the number of elements in the set is equal to $2n^2 + n$ which according to Lemma 5.6 is equal to the dimension of r-canonical symmetric two-variable polynomials. To complete the proof of the claim, it suffices to show that the set consists of linearly independent elements. Assume that there exist $a_{(\bullet,\bullet)}$, $b_{(\bullet,\bullet)}$ and $c_\bullet \in \mathbb{R}$ such that

$$\sum_{v_{(\lambda_i,\lambda_k)}\in\mathcal{X}_c} \left[a_{(\lambda_i,\lambda_k)}\mathrm{Re}(v_{(\lambda_i,\lambda_k)}) + b_{(\lambda_i,\lambda_k)}\mathrm{Im}(v_{(\lambda_i,\lambda_k)})\right] + \sum_{p=0}^{n-1} c_p v_{(+j\omega_p,-j\omega_p)} = 0$$

Consider an element $v_{(\lambda_i,\lambda_k)}$ of the set $\mathcal{X}_c$. It is easy to see that both λ_i and λ_k are purely imaginary and $\lambda_i + \lambda_k \neq 0$. Hence

$$\begin{aligned} v_{(\lambda_i,\lambda_k)}(\zeta,\eta) &= v_i(\zeta)v_k(\eta) + v_k(\zeta)v_i(\eta) \quad \text{and} \\ v_{(-\lambda_i,-\lambda_k)}(\zeta,\eta) &= \bar{v}_i(\zeta)\bar{v}_k(\eta) + \bar{v}_k(\zeta)\bar{v}_i(\eta) \end{aligned}$$

are complex conjugates. It follows that

$$\sum_{v_{(\lambda_i,\lambda_k)}\in\mathcal{X}_c} \left[a_{(\lambda_i,\lambda_k)}\left(v_{(\lambda_i,\lambda_k)}+v_{(-\lambda_i,-\lambda_k)}\right)+jb_{(\lambda_i,\lambda_k)}\left(v_{(-\lambda_i,-\lambda_k)}-v_{(\lambda_i,\lambda_k)}\right)\right] + \sum_{p=0}^{n-1} 2c_p v_{(+j\omega_p,-j\omega_p)} = 0$$

$$\implies \sum_{v_{(\lambda_i,\lambda_k)}\in\mathcal{X}_c} \left[v_{(\lambda_i,\lambda_k)}\left(a_{(\lambda_i,\lambda_k)}-jb_{(\lambda_i,\lambda_k)}\right)+v_{(-\lambda_i,-\lambda_k)}\left(a_{(\lambda_i,\lambda_k)}+jb_{(\lambda_i,\lambda_k)}\right)\right] + \sum_{p=0}^{n-1} 2c_p v_{(+j\omega_p,-j\omega_p)} = 0$$

The left hand side of the above is a linear combination of the elements of the set $\mathcal{Q}_c$ which are linearly independent. Hence we obtain $a_{(\lambda_i,\lambda_k)} = b_{(\lambda_i,\lambda_k)} = c_p = 0$. Thus $\mathcal{X}$ forms a basis of r-canonical symmetric two-variable polynomials.

From the discussion of section 5.2 and the characterization of equation (3.10), we conclude that the set $\left\{v_{(+j\omega_k,-j\omega_k)}\right\}_{k=0,\dots,n-1}$ induces linearly independent conserved quantities for $\mathfrak{B}$ and is a basis of the kernel of the linear operator $\mathfrak{L}_r$.

From the characterization of zero-mean quantities (see equation (3.13)), it can be seen that the image of the linear operator $\mathfrak{L}_r$ is a space inducing complex zero-mean quantities. Hence the set $\mathcal{Q}'_c$ consists of polynomials inducing symmetric complex zero-mean quantities. We now make use of the following lemma to prove that the elements of the set $\mathcal{Z}$ induce linearly independent zero-mean quantities for $\mathfrak{B}$.

Lemma 5.9. *The real and imaginary parts of a complex symmetric two-variable polynomial matrix inducing a zero-mean quantity for a behaviour $\mathfrak{B}$ each induce zero-mean quantities for $\mathfrak{B}$.*

Proof. Let j represent the imaginary unit. Assume that $\Phi \in \mathbb{C}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ induces a zero-mean quantity for $\mathfrak{B}$. Write $\Phi(\zeta,\eta) = \Phi_1(\zeta,\eta) + j\Phi_2(\zeta,\eta)$, where $\Phi_1, \Phi_2 \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Then for any trajectory $w \in \mathfrak{B}$

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Q_\Phi(w)(t)dt = 0$$

This implies that

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left(Q_{\Phi_1}(w)(t) + jQ_{\Phi_2}(w)(t)\right)dt = 0$$

Equating the real and imaginary parts to zero, we obtain

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Q_{\Phi_1}(w)(t)dt = 0 \qquad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} Q_{\Phi_2}(w)(t)dt = 0$$

which implies that Q_{Φ_1} and Q_{Φ_2} are zero-mean quantities for $\mathfrak{B}$. This concludes the proof. ■

Since the set $\mathcal{Q}'_c$ consists of polynomials inducing complex symmetric zero-mean quantities, from the above Lemma, it follows that the elements of the set $\mathcal{Z}$ induce linearly independent zero-mean quantities for $\mathfrak{B}$. The proof of the decomposition part of the theorem now follows from equation (5.7). Also the dimensions of the respective bases given in the statement of the theorem can be easily verified to be true. ■

5.5 Bases of intrinsically- and trivially zero-mean, and of conserved quantities

In this section, we illustrate how to compute bases of the spaces of trivially zero-mean, intrinsically zero-mean and conserved quantities for oscillatory systems. We begin by examining the case of systems with one external variable.

Proposition 5.10. *Let $\mathfrak{B} = ker\big(r(\frac{d}{dt})\big)$, where $r \in \mathbb{R}[\xi]$, be an oscillatory behaviour of dimension $2n$. Then $\big\{(\zeta+\eta)(\zeta^i\eta^j + \zeta^j\eta^i)\big\}_{0\leq i\leq j\leq 2n-2}$ is a basis of two-variable symmetric polynomials that induce trivially zero-mean quantities modulo $\mathfrak{B}$.*

Proof. Observe that each element of the set is r-canonical because each term $\zeta^k\eta^l$ appearing in it is such that $k,l \leq 2n-1$. From equation (3.14), it follows that each element of the set induces a trivially zero-mean quantity. Following the proof of Lemma 5.6, we conclude that $\big\{\zeta^i\eta^j + \zeta^j\eta^i)\big\}_{0\leq i\leq j\leq 2n-2}$ consists of linearly independent elements and consequently also $\big\{(\zeta+\eta)(\zeta^i\eta^j + \zeta^j\eta^i)\big\}_{0\leq i\leq j\leq 2n-2}$ consists of linearly independent elements. Observe also that any r-canonical multiple of $(\zeta+\eta)$ can be expressed as a linear combination of the elements of the set. Hence the set induces trivially zero-mean quantities that span the entire space of trivially zero-mean quantities modulo $\mathfrak{B}$. This concludes the proof. ■

From the above, it can be inferred that the dimension of the space of trivially zero-mean quantities modulo $\mathfrak{B}$ is $n(2n-1)$ and consequently that the dimension of the space of intrinsically zero-mean quantities modulo $\mathfrak{B}$ is n.

Proposition 5.11. *Let $\mathfrak{B} = ker\ \big(r(\frac{d}{dt})\big)$ be an oscillatory behaviour, where $r(\xi) = r_0 + r_2\xi^2 + \ldots + r_{2n-2}\xi^{2n-2} + \xi^{2n}$ and $r_0 \neq 0$. Then the set $\{\varphi_i\}_{i=0,1,\ldots,n-1}$ where*

$$\varphi_i(\zeta,\eta) := r(\zeta)\eta^{2i} + r(\eta)\zeta^{2i} - (\zeta+\eta)(\zeta^{2n-1}\eta^{2i} + \zeta^{2i}\eta^{2n-1}) \tag{5.8}$$

induces a basis for the set of intrinsically zero-mean quantities modulo $\mathfrak{B}$.

Proof. Observe that $\varphi_i(\zeta,\eta)$ is r-canonical, because each of its terms $\zeta^k\eta^l$ is such that $k,l \leq 2n-1$. That Q_{φ_i} is zero-mean over $\mathfrak{B}$ can be concluded from the fact that it is of the form of a general zero-mean quantity as in equation (3.13). We now show that the polynomials φ_i are linearly independent. Assume by contradiction that there exist $a_i \in \mathbb{R}$, $i = 0, \ldots, n-1$, such that

$$\begin{aligned}\sum_{i=0}^{n-1} a_i\varphi_i(\zeta,\eta) &= \sum_{i=0}^{n-1} a_i[(r_0 + r_2\zeta^2 + \ldots + r_{2n-2}\zeta^{2n-2})\eta^{2i} \\ &+ (r_0 + r_2\eta^2 + \ldots + r_{2n-2}\eta^{2n-2})\zeta^{2i} \\ &- (\zeta^{2i+1}\eta^{2n-1} + \zeta^{2n-1}\eta^{2i+1})] = 0\end{aligned}$$

From this equation it follows that for $i = 0, 1, \ldots, n-1$, the coefficient a_i of each term $(\zeta^{2i+1}\eta^{2n-1} + \zeta^{2n-1}\eta^{2i+1})$ is zero. This implies that the set $\{\varphi_i\}_{i=0,1,\ldots,n-1}$ consists of elements inducing linearly independent zero-mean quantities. We also prove that no trivially zero mean quantity can be induced by a linear combination of the elements of the set. Assume by contradiction that $\sum_{i=0}^{n-1} a_i\varphi_i(\zeta,\eta)$ induces a trivially zero-mean quantity. Then $\partial\big(\sum_{i=0}^{n-1} a_i\varphi_i(\zeta,\eta)\big) = 0$, or

$$\sum_{i=0}^{n-1} a_i r(\xi)\xi^{2i} = r(\xi)\sum_{i=0}^{n-1} a_i\xi^{2i} = 0$$

which is possible if and only if $a_i = 0$ for $i = 0, \ldots, n-1$. This shows that the space of QDFs induced by the set $\{\varphi_i\}_{i=0,\ldots,n-1}$ is complementary to the space of trivially zero-mean quantities and hence consists of linearly independent intrinsically zero-mean quantities. Also note that the number of elements in the set is equal to n. This concludes the proof. ■

Let $\mathfrak{B}$ be a scalar oscillatory behaviour that has no characteristic frequency at zero. Now consider an oscillatory behaviour $\mathfrak{B}_1$, such that $\mathfrak{B}_1 \subset \mathfrak{B}$ and $\mathfrak{B}_1$ has no characteristic frequency at zero. Let Q_φ be an element of the basis of intrinsically zero mean quantities for $\mathfrak{B}$ as described in Proposition 5.11. Since $\mathfrak{B}_1 \subset \mathfrak{B}$, it is easy to see that Q_φ is a zero-mean quantity for $\mathfrak{B}_1$. We now prove that if Q_φ is a zero-mean quantity for another oscillatory behaviour $\mathfrak{B}_2$ that has no characteristic frequency at zero, then $\mathfrak{B}_2 \subset \mathfrak{B}$.

Proposition 5.12. *Let $\mathfrak{B} = ker\big(r(\frac{d}{dt})\big)$ ($r \in \mathbb{R}[\xi]$) be a kernel representation of an oscillatory behaviour. Assume that r is even and has degree equal to $2n$. For $i = 0, \ldots, n-1$, define*

$$\varphi_i(\zeta,\eta) := r(\zeta)\eta^{2i} + r(\eta)\zeta^{2i} - (\zeta+\eta)(\zeta^{2n-1}\eta^{2i} + \zeta^{2i}\eta^{2n-1}) \tag{5.9}$$

Then for $i = 0, \ldots, n-1$, if Q_{φ_i} is a zero-mean quantity for another scalar oscillatory behaviour $\mathfrak{B}_{2i}$ that has no characteristic frequency at zero, then $\mathfrak{B}_{2i} \subset \mathfrak{B}$.

Proof. For $i = 0, \ldots, n-1$, let $\mathfrak{B}_{2i} = \ker\big(r_{2i}(\frac{d}{dt})\big)$, where $r_{2i} \in \mathbb{R}[\xi]$. Since $\mathfrak{B}_{2i}$ has no characteristic frequency at zero, r_{2i} is even. Assume that for $i = 0, \ldots, n-1$, Q_{φ_i} is a zero-mean quantity for $\mathfrak{B}_{2i}$. This implies that

$$\varphi_i(\zeta,\eta) = r_{2i}(\zeta)f_i(\zeta,\eta) + f_i(\eta,\zeta)r_{2i}(\eta) + (\zeta+\eta)\psi_i(\zeta,\eta) \tag{5.10}$$

where $f_i \in \mathbb{R}[\zeta,\eta]$, and $\psi_i \in \mathbb{R}_s[\zeta,\eta]$ for $i = 0,\ldots,n-1$. By replacing ζ with $-\xi$ and η with ξ in the above equation, we obtain

$$\partial\varphi_i(\xi) = r_{2i}(\xi)(\partial f_i(\xi) + \partial f_i(-\xi))$$

From equation (5.9), it follows that

$$2\xi^{2i}r(\xi) = r_{2i}(\xi)(\partial f_i(\xi) + \partial f_i(-\xi))$$

Since r_{2i} is even and oscillatory, from the above equation, it follows that r is divisible by r_{2i}. Define

$$r_{1i}(\xi) := \frac{r(\xi)}{r_{2i}(\xi)}$$

Then

$$2\xi^{2i}r_{1i}(\xi) = (\partial f_i(\xi) + \partial f_i(-\xi)) \tag{5.11}$$

Let $r_{ik} \in \mathbb{R}^+$ be the coefficient of ξ^{2k} in the expression for $r_{1i}(\xi)$. Assume without loss of generality that r_{1i} has degree equal to $2p_i$, where p_i is a nonnegative integer less than n. This implies that

$$r_{1i}(\xi) = \sum_{k=0}^{p_i} r_{ik}\xi^{2k}$$

Observe that

$$f_i(\zeta,\eta) = \zeta^{2i}\sum_{k=0}^{p_i} r_{ik}(-1)^k\zeta^k\eta^k$$

is a solution of equation (5.11). It is now easy to see that there exists a $\psi_i \in \mathbb{R}[\zeta,\eta]$, such that equation (5.10) holds. Since r is divisible by r_{2i} for $i = 0,\ldots,n-1$, it follows that $\mathfrak{B}_{2i} \subset \mathfrak{B}$. ■

Example 5.2. We now illustrate Proposition 5.12 with an example. Define $r(\xi) := \xi^4 + 3\xi^2 + 2$, and $r_2(\xi) := \xi^2 + 1$. Define $\mathfrak{B} := \ker\big(r(\frac{d}{dt})\big)$ and $\mathfrak{B}_2 := \ker\big(r_2(\frac{d}{dt})\big)$ and observe that $\mathfrak{B}_2 \subset \mathfrak{B}$. Consider an element of the basis of intrinsically zero-mean quantities for $\mathfrak{B}$ induced by

$$\varphi(\zeta,\eta) = r(\zeta) + r(\eta) - (\zeta+\eta)(\zeta^3+\eta^3)$$

It can be verified that

$$\varphi(\zeta,\eta) = r_2(\zeta)(2-\zeta\eta) + (2-\zeta\eta)r_2(\eta) + (\zeta+\eta)^2$$

This implies that Q_φ is a zero-mean quantity for $\mathfrak{B}_2$.

Remark 5.13. From Proposition 5.12, it follows that the basis elements of the basis of intrinsically zero-mean quantities for a given oscillatory behaviour as described in Proposition 5.11 are zero-mean only along the given behaviour and all oscillatory behaviours that are subspaces of the given behaviour. However an intrinsically zero-mean quantity for an oscillatory behaviour in general can be zero-mean along oscillatory behaviours that are not subspaces of the given behaviour. For example, consider an oscillatory behaviour $\mathfrak{B} := \ker\big(r(\frac{d}{dt})\big)$, where $r(\xi) := \xi^4 + 3\xi^2 + 2$. Define $r'(\xi) := \xi^4 + 4\xi^2 + 3$, and $\mathfrak{B}' := \ker\big(r'(\frac{d}{dt})\big)$. Define

$$\begin{aligned}\varphi_0(\zeta,\eta) &:= r(\zeta) + r(\eta) - (\zeta+\eta)(\zeta^3+\eta^3)\\ \varphi_1(\zeta,\eta) &:= r(\zeta)\eta^2 + \zeta^2 r(\eta) - (\zeta+\eta)(\zeta^3\eta^2+\zeta^2\eta^3)\end{aligned}$$

From Proposition 5.11, it follows that $\varphi(\zeta,\eta) := \varphi_1(\zeta,\eta) + 3\varphi_0(\zeta,\eta)$ induces an intrinsically zero-mean quantity for $\mathfrak{B}$. It can be verified that

$$\varphi(\zeta,\eta) = r'(\zeta)(\eta^2+2) + (\zeta^2+2)r'(\eta) - (\zeta+\eta)(\zeta^2\eta^3 + \zeta^3\eta^2 + \zeta^2\eta + \zeta\eta^2 + 2\zeta^3 + 2\eta^3)$$

This implies that Q_φ is a zero-mean quantity also for $\mathfrak{B}'$. Observe that φ is r'-canonical and that $\mathfrak{B}'$ is not a subspace of $\mathfrak{B}$.

We now show the construction of bases of conserved, trivially zero-mean and intrinsically zero-mean quantities for the multivariable ($\mathtt{w} > 1$) case of oscillatory behaviours. In order to facilitate better understanding of the results, we first show the construction of the bases for the generic multivariable case, and then move on to the nongeneric multivariable case. The results of propositions 5.10 and 5.11 are instrumental in dealing with the generic multivariable case, which we shall now examine.

Theorem 5.14. *Let $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, where $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$. Assume that $det(R) \neq 0$ and $det(R)$ has distinct roots on the imaginary axis, none of which is equal to zero. Let $R = U\Delta V$ be a Smith form decomposition of R. Let $F := diag(0,\dots,0,1) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and let $n = \frac{deg(detR)}{2}$. Then the QDFs induced by $\{\Phi_i(\zeta,\eta)\}_{i=0,1,\dots,n-1}$, $\{\Gamma_i(\zeta,\eta)\}_{i=0,1,\dots,n-1}$ and $\{\Theta_{kj}\}_{0\le k\le j\le 2n-2}$ where*

$$\begin{aligned}\Phi_i(\zeta,\eta) :=& V(\zeta)^T(\Delta(\zeta)F\eta^{2i} + F\zeta^{2i}\Delta(\eta)\\ &-(\zeta+\eta)F(\zeta^{2n-1}\eta^{2i} + \zeta^{2i}\eta^{2n-1}))V(\eta) \qquad (5.12)\\ \Gamma_i(\zeta,\eta) :=& \frac{V(\zeta)^T(\Delta(\zeta)F\eta^{2i+1} + F\zeta^{2i+1}\Delta(\eta))V(\eta)}{\zeta+\eta} \qquad (5.13)\\ \Theta_{kj}(\zeta,\eta) :=& (\zeta+\eta)(\zeta^k\eta^j + \zeta^j\eta^k)V(\zeta)^T F V(\eta) \qquad (5.14)\end{aligned}$$

respectively, form bases for intrinsically zero-mean, conserved and trivially zero-mean quantities modulo $\mathfrak{B}$.

Proof. From the divisibility property of invariant polynomials it follows that $\mathtt{w}-1$ of the $\mathtt{w}$ invariant polynomials of R are equal to one, and that the last invariant polynomial, which we denote with $\delta_{\mathtt{w}}$, is equal to $(\det(R))/r_n$, where r_n denotes the leading coefficient of $\det(R)$. Since zero is not a characteristic frequency of $\mathfrak{B}$, $\delta_{\mathtt{w}}$ is an even polynomial. Consider now the behaviour $\mathfrak{B}' := \ker\big(\Delta(\frac{d}{dt})\big)$ and observe that $w' \in \mathfrak{B}'$ if and only if $w' = V\left(\frac{d}{dt}\right) w$ for some $w \in \mathfrak{B}$. For $i = 0, 1, \ldots, n-1$, define

$$\begin{aligned}\varphi_i(\zeta,\eta) &:= \delta_{\mathtt{w}}(\zeta)\eta^{2i} + \delta_{\mathtt{w}}(\eta)\zeta^{2i} - (\zeta+\eta)(\zeta^{2n-1}\eta^{2i} + \zeta^{2i}\eta^{2n-1})\\ \gamma_i(\zeta,\eta) &:= \frac{(\delta_{\mathtt{w}}(\zeta)\eta^{2i+1} + \delta_{\mathtt{w}}(\eta)\zeta^{2i+1})}{\zeta+\eta}\end{aligned}$$

For all possible integers j, k such that $0 \leq k \leq j \leq 2n-2$, define

$$\theta_{kj}(\zeta,\eta) := (\zeta+\eta)(\zeta^k\eta^j + \zeta^j\eta^k)$$

It follows from the material of sections 3.5, 3.6, and from Propositions 5.11 and 5.10 that $\{\gamma_i\}_{i=0,1,\ldots,n-1}$, $\{\varphi_i\}_{i=0,1,\ldots,n-1}$ and $\{\theta_{kj}\}_{0\leq k\leq j\leq 2n-2}$ are $\delta_{\mathtt{w}}$-canonical and induce linearly independent conserved, intrinsically zero-mean and trivially zero-mean quantities over $\ker\left(\delta_{\mathtt{w}}(\frac{d}{dt})\right)$. We now show how to obtain from them conserved, intrinsically zero-mean and trivially zero-mean quantities for $\ker\big(R(\frac{d}{dt})\big)$. Observe first that $w' \in \ker\left(\Delta(\frac{d}{dt})\right)$ if and only if $w' = \mathrm{col}(0, \ldots, 0, w'_{\mathtt{w}})$, with $w'_{\mathtt{w}} \in \ker\left(\delta_{\mathtt{w}}(\frac{d}{dt})\right)$. Now consider

$$\Phi'_i(\zeta,\eta) = \Delta(\zeta)F\eta^{2i} + F\Delta(\eta)\zeta^{2i} - (\zeta+\eta)F(\zeta^{2n-1}\eta^{2i} + \zeta^{2i}\eta^{2n-1})$$

$$\Gamma'_i(\zeta,\eta) = \frac{(\Delta(\zeta)F\eta^{2i+1} + F\Delta(\eta)\zeta^{2i+1})}{\zeta+\eta}$$

for $i = 0, 1, \ldots, n-1$ and consider

$$\Theta'_{kj}(\zeta,\eta) = (\zeta+\eta)(\zeta^k\eta^j + \zeta^j\eta^k)F$$

for integers j, k such that $0 \leq k \leq j \leq 2n-2$. It is immediate to see that $Q_{\Phi'_i}(w') = Q_{\varphi_i}(w'_{\mathtt{w}})$, $Q_{\Gamma'_i}(w') = Q_{\gamma_i}(w'_{\mathtt{w}})$ and $Q_{\Theta'_{kj}}(w') = Q_{\theta_{kj}}(w'_{\mathtt{w}})$. The linear independence of the sets $\{\Phi'_i\}_{i=0,1,\ldots,n-1}$, $\{\Gamma'_i\}_{i=0,1,\ldots,n-1}$ and $\left\{\Theta'_{kj}\right\}_{0\leq k\leq j\leq 2n-2}$ follows from the linear independence of $\{\varphi_i\}_{i=0,1,\ldots,n-1}$, $\{\gamma_i\}_{i=0,1,\ldots,n-1}$ and of $\{\theta_{kj}\}_{0\leq k\leq j\leq 2n-2}$ respectively. We now prove that these two-variable polynomial matrices are also Δ-canonical. Write

$$\begin{aligned}\Delta(\zeta)^{-T}\Phi'_i(\zeta,\eta)\Delta(\eta)^{-1} &= F\eta^{2i}\Delta(\eta)^{-1} + F\zeta^{2i}\Delta(\zeta)^{-1}\\ &- (\zeta+\eta)\Delta(\zeta)^{-1}F(\zeta^{2n-1}\eta^{2i} + \zeta^{2i}\eta^{2n-1})\Delta(\eta)^{-1}\\ \Delta(\zeta)^{-T}\Gamma'_i(\zeta,\eta)\Delta(\eta)^{-1} &= \frac{F\eta^{2i+1}\Delta(\eta)^{-1} + F\zeta^{2i+1}\Delta(\zeta)^{-1}}{\zeta+\eta}\\ \Delta(\zeta)^{-T}\Theta'_{kj}(\zeta,\eta)\Delta(\eta)^{-1} &= (\zeta+\eta)(\zeta^k\eta^j + \zeta^j\eta^k)\Delta(\zeta)^{-1}F\Delta(\eta)^{-1}\end{aligned}$$

Observe that the matrices appearing in the last three equations have only strictly proper entries. It follows from this and from the linear independence of the two-variable polynomial matrices Φ_i', Γ_i' and Θ_{kj}' that $\Phi_i(\zeta,\eta)$, $\Gamma_i(\zeta,\eta)$, and $\Theta_{kj}(\zeta,\eta)$ defined in the statement of the Theorem induce linearly independent intrinsically zero-mean, conserved and trivially zero-mean QDFs respectively for $\mathfrak{B}$. An argument based on dimensionality shows that they form bases for the respective spaces. This concludes the proof. ∎

5.6 The nongeneric multivariable case

In this section, we consider the case where an oscillatory behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is such that more than one of the invariant polynomials of $\mathfrak{B}$ are not equal to 1. Further on in this chapter, it will be shown that for a generic oscillatory behaviour, the expression for the Lagrangian can be written in terms of basis elements of intrinsically zero-mean and trivially zero-mean quantities. Since we have explained in detail the construction of basis elements of zero-mean and conserved quantities for generic oscillatory behaviours, for the sake of completeness, we do so even for the case of nongeneric oscillatory behaviours.

Consider a nongeneric oscillatory behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$. Let $R = U\Delta V$ be a Smith form decomposition of R. Define $R_1(\xi) := \Delta(\xi)V(\xi)$. Note that $\mathfrak{B} = \ker\big(R_1(\frac{d}{dt})\big)$ is another kernel representation of $\mathfrak{B}$. Consider the behaviour $\mathfrak{B}' = \ker\big(\Delta(\frac{d}{dt})\big)$. From the proof of Theorem 5.5, it follows that there is a one-one correspondence between the spaces of Δ-canonical and R_1-canonical symmetric two-variable polynomial matrices and if $\mathcal{Q}_\Delta$ denotes a basis for the space of Δ-canonical two-variable polynomial matrices, then $V(\zeta)^\top \mathcal{Q}_\Delta V(\eta)$ is a basis for the space of R_1-canonical two-variable polynomial matrices. Thus bases for the two-variable polynomial matrices inducing zero-mean and conserved quantities for $\mathfrak{B}$ can be constructed out of the corresponding bases for the behaviour $\mathfrak{B}'$.

Now as in the proof of Theorem 5.5, we can obtain a basis of QDFs consisting of diagonal QDFs and zero-diagonal QDFs. Diagonal QDFs are those that are induced by diagonal polynomial matrices, and zero-diagonal QDFs are those that are induced by polynomial matrices having zero-diagonal entries. Since the spaces of diagonal and zero-diagonal QDFs are linearly independent and since any Δ-canonical QDF can be written as a sum of a diagonal and a zero-diagonal QDF, it is easy to see that the union of the bases of diagonal and zero-diagonal zero-mean and conserved quantities for $\mathfrak{B}'$ respectively form bases for the spaces of zero-mean and conserved quantities for $\mathfrak{B}'$. The bases for diagonal zero-mean and conserved quantities for $\mathfrak{B}'$ can be easily constructed using the material in section 5.4. In order to construct bases of zero-diagonal zero-mean and conserved quantities for $\mathfrak{B}'$, we make use of the following theorem, which deals with the case of $\mathfrak{B}$ having two nonunity invariant polynomials.

Theorem 5.15. *Define* $r_1(\xi) := (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{(n_1-1)}^2)$ *and* $r_2(\xi) := (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{(n_2-1)}^2)$ *where* $n_1, n_2 \in Z^+$, $n_2 \geq n_1$ *and* $\omega_i \in \mathbb{R}^+$ *are distinct for* $i = 0, 1, \dots, n_2-1$. *Define* $\mathfrak{B} := \ker\big(R(\frac{d}{dt})\big)$ *where* $R(\xi) := diag(r_1(\xi), r_2(\xi))$. *Define* $u_i(\xi) := \frac{r_1(\xi)}{\xi - j\omega_i}$ *and* $y_i(\xi) := \frac{r_2(\xi)}{\xi - j\omega_i}$ *where* j *is the imaginary square root of* -1. *Consider the space of QDFs induced by R-canonical symmetric two-variable polynomial matrices of the form*

$$\Phi_{12}(\zeta,\eta) = \begin{bmatrix} 0 & f(\zeta,\eta) \\ f(\eta,\zeta) & 0 \end{bmatrix} \tag{5.15}$$

where $f \in \mathbb{R}[\zeta,\eta]$.

(i) This space can be decomposed into the spaces $\mathcal{Q}_{\mathcal{C}}$ *of conserved quantities and* $\mathcal{Q}_{\mathcal{Z}}$ *of zero-mean quantities respectively modulo* $\mathfrak{B}$ *as follows.*

$$\mathcal{Q} = \mathcal{Q}_{\mathcal{C}} \oplus \mathcal{Q}_{\mathcal{Z}}$$

(ii) For $i = 0, 1, \dots, n_1 - 1$, *the real and imaginary parts of*

$$\begin{bmatrix} 0 & u_i(\zeta)\bar{y}_i(\eta) \\ \bar{y}_i(\zeta)u_i(\eta) & 0 \end{bmatrix}$$

form a basis of the space of R-canonical symmetric two-variable polynomial matrices of the form (5.15) that induce conserved quantities. The dimensions of the spaces $\mathcal{Q}_{\mathcal{C}}$ *and* $\mathcal{Q}_{\mathcal{Z}}$ *are* $2n_1$ *and* $2n_1(2n_2-1)$ *respectively.*

(iii) The space of trivially zero-mean quantities induced by R-canonical two-variable polynomial matrices of the form (5.15) has dimension equal to $(2n_1-1)(2n_2-1)$.

(iv) The set

$$\left\{\begin{bmatrix} 0 & \big(r_1(\zeta)\eta^i - (\zeta+\eta)\zeta^{2n_1-1}\eta^i\big) \\ \big(r_1(\eta)\zeta^i - (\zeta+\eta)\zeta^i\eta^{2n_1-1}\big) & 0 \end{bmatrix}\right\}_{i=0,\dots,2n_2-2}$$

is a basis of the space of R-canonical symmetric two-variable polynomial matrices of the form (5.15) that induce intrinsically zero-mean quantities over $\mathfrak{B}$.

Proof. From Lemma 5.7, it is easy to see that the dimension of the space of QDFs of the form (5.15) is $4n_1n_2$. Define

$$K_{12} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad K_{21} := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

(i) As in section 5.2, consider the one-variable shift operator S_R and the two-variable shift operator $\mathfrak{L}_R$ for $\mathfrak{B}$. Observe that the map S_R has twice repeated eigenvalues at the characteristic frequencies of r_1, i.e $\pm j\omega_i$, $i = 0, 1, \ldots, n_1 - 1$. The remaining eigenvalues are distinct and occur at $\pm j\omega_i$, $i = n_1, \ldots, n_2 - 1$ provided $n_2 > n_1$.

Let λ_k and v_k, $(k = 0, 1, \ldots, n_1 + n_2 - 1)$ denote the eigenvalues and corresponding eigenvectors of the map S_R. It can be verified that for $i = 0, 1, \ldots, n_1 - 1$, the two eigenvectors corresponding to the repeated eigenvalues at $j\omega_i$ are $v_{i1}(\xi) = u_i(\xi)[1 \quad 0]$ and $v_{i2}(\xi) = y_i(\xi)[0 \quad 1]$. Similarly the eigenvectors corresponding to the repeated eigenvalues at $-j\omega_i$ are $v_{i3}(\xi) = \bar{u}_i(\xi)[1 \quad 0]$ and $v_{i4}(\xi) = \bar{y}_i(\xi)[0 \quad 1]$. For $i = n_1, \ldots, n_2 - 1$, the eigenvector corresponding to the eigenvalue $j\omega_i$ is $v_{i2}(\xi) = y_i(\xi)[0 \quad 1]$ and the one corresponding to $-j\omega_i$ is $v_{i4}(\xi) = \bar{y}_i(\xi)[0 \quad 1]$.

The eigenvalues of the map $\mathfrak{L}_R$ are all possible combinations $(\lambda_p + \lambda_q)$ and the corresponding eigenvectors are $v_{(\lambda_p,\lambda_q)}(\zeta, \eta) = v_p(\zeta)^\top v_q(\eta) + v_q(\zeta)^\top v_p(\eta)$. It can be seen that $4n_1n_2$ eigenvectors of $\mathfrak{L}_R$ are of the form (5.15) with $f \in \mathbb{C}[\zeta, \eta]$, and these occur in complex conjugates. Denote the set of these eigenvectors by $\mathcal{E}$. From every conjugate pair of this set, choose one element. Call the resulting set $\mathcal{X}$. Now consider a subset of $\mathcal{E}$ given by $w_1 \cup w_2$, where

$$
\begin{aligned}
w_1 &:= \{v_{i1,i4}(\zeta, \eta)\}_{i=0,\ldots,n_1-1} = \left\{v_{i1}(\zeta)^\top v_{i4}(\eta) + v_{i4}(\zeta)^\top v_{i1}(\eta)\right\}_{i=0,\ldots,n_1-1} \\
w_2 &:= \{v_{i2,i3}(\zeta, \eta)\}_{i=0,\ldots,n_1-1} = \left\{v_{i2}(\zeta)^\top v_{i3}(\eta) + v_{i3}(\zeta)^\top v_{i2}(\eta)\right\}_{i=0,\ldots,n_1-1}
\end{aligned}
$$

It is easy to see that the eigenvectors belonging to this subset also belong to the kernel space of $\mathfrak{L}_R$ and hence induce complex conserved QDFs. The remaining eigenvectors of $\mathcal{E}$ belong to the image space of $\mathfrak{L}_R$ and hence induce complex zero-mean QDFs.

Assume that there exist $a_{(\bullet,\bullet)}$, $b_{(\bullet,\bullet)} \in \mathbb{R}$ such that

$$
\sum_{v_{(i_1k_1,i_2k_2)} \in \mathcal{X}} \left[a_{(i_1k_1,i_2k_2)} \mathrm{Re}(v_{(i_1k_1,i_2k_2)}) + b_{(i_1k_1,i_2k_2)} \mathrm{Im}(v_{(i_1k_1,i_2k_2)})\right] = 0
$$

where $i_1, i_2 \in \{0, \ldots, n_2\}$, $k_1, k_2 \in \{1, \ldots, 4\}$. Let $\bar{v}_{(i_1k_1,i_2k_2)} \in \mathcal{E}$ denote the complex conjugate of $v_{(i_1k_1,i_2k_2)} \in \mathcal{X}$. It follows that

$$
\sum_{v_{(i_1k_1,i_2k_2)} \in \mathcal{X}} \left[a_{(i_1k_1,i_2k_2)} \left(v_{(i_1k_1,i_2k_2)} + \bar{v}_{(i_1k_1,i_2k_2)}\right) + jb_{(i_1k_1,i_2k_2)} \left(\bar{v}_{(i_1k_1,i_2k_2)} - v_{(i_1k_1,i_2k_2)}\right)\right] = 0
$$

This implies that

$$
\sum_{v_{(i_1k_1,i_2k_2)} \in \mathcal{X}} \left[v_{(i_1k_1,i_2k_2)} \left(a_{(i_1k_1,i_2k_2)} - jb_{(i_1k_1,i_2k_2)}\right) + \bar{v}_{(i_1k_1,i_2k_2)} \left(a_{(i_1k_1,i_2k_2)} + jb_{(i_1k_1,i_2k_2)}\right)\right] = 0
$$

The left hand side of the above is a linear combination of the elements of the set $\mathcal{E}$ which are linearly independent. Hence we obtain $a_{(i_1k_1,i_2k_2)} = b_{(i_1k_1,i_2k_2)} = 0$. This implies that the real and imaginary parts of the set $\mathcal{E}$ are linearly independent. From Lemma 5.7, it follows that the dimension of the space of R-canonical symmetric two-variable polynomial matrices of the form (5.15) is equal to $4n_1n_2$. Hence it can be inferred that the real and imaginary parts of eigenvectors of the set $\mathcal{E}$ form a basis of the space of R-canonical symmetric two-variable polynomial matrices of the form (5.15). Since the real and imaginary parts of a conserved QDF are conserved and similarly by Proposition 5.9, the real and imaginary parts of a complex zero-mean QDF are zero-mean, the basis that we have constructed consists of polynomials inducing either conserved or zero-mean QDFs. This concludes the decomposition part of the proof.

(ii) For $i = 0, \ldots, n_1 - 1$, observe that

$$\begin{aligned} v_{(i1,i4)}(\zeta,\eta) &= K_{12}u_i(\zeta)\bar{y}_i(\eta) + K_{21}\bar{y}_i(\zeta)u_i(\eta) \\ v_{(i2,i3)}(\zeta,\eta) &= K_{12}\bar{u}_i(\zeta)y_i(\eta) + K_{21}y_i(\zeta)\bar{u}_i(\eta) \end{aligned}$$

are complex conjugates. Hence the real and imaginary parts of $v_{(i1,i4)}(\zeta,\eta)$ form a basis of R-canonical symmetric two-variable polynomial matrices of the form (5.15) that induce conserved quantities. The dimension of the space of such conserved QDFs is the number of elements in the basis which is equal to $2n_1$. Hence the dimension of the space of zero-mean QDFs of the form given in the statement of the theorem can be obtained by subtracting $2n_1$ from the total dimension $4n_1n_2$ of such QDFs. This concludes the proof.

(iii) We assert that the set

$$\left\{(\zeta+\eta)(K_{12}\zeta^l\eta^m + K_{21}\zeta^m\eta^l)\right\}_{0\leq l\leq 2n_1-2,\, 0\leq m\leq 2n_2-2} \tag{5.16}$$

forms a basis of R-canonical two-variable polynomial matrices of the form (5.15) that induce trivially zero-mean quantities. Observe that each element of the set is R-canonical and induces a trivially zero-mean quantity because each element is divisible by $(\zeta+\eta)$. The elements of the set are linearly independent because a term of the form $(K_{12}\zeta^k\eta^i + K_{21}\zeta^i\eta^k)$ that occurs in one of the elements does not occur in any other element of the set. Finally observe that linear combinations of the elements of the set can produce any R-canonical symmetric two-variable polynomial matrix of the form (5.15) that is divisible by $(\zeta+\eta)$. Hence the set forms a basis of R-canonical symmetric two-variable polynomial matrices of the form (5.15) that induce trivially zero mean quantities. The dimension of the space of trivially zero-mean QDFs induced by this basis is the number of elements in the set which is equal to $(2n_1-1)(2n_2-1)$.

(iv) Now consider the set

$$Z = \left\{ \begin{bmatrix} 0 & \left(r_1(\zeta)\eta^i - (\zeta+\eta)\zeta^{2n_1-1}\eta^i\right) \\ \left(r_1(\eta)\zeta^i - (\zeta+\eta)\zeta^i\eta^{2n_1-1}\right) & 0 \end{bmatrix} \right\}_{i=0,\ldots,2n_2-2}$$

For $k = 0, \ldots, n_1$, let g_k denote the coefficient of ξ^{2k} in the polynomial $r_1(\xi)$. The i^{th} element of the set Z can be written as

$$\Phi_i(\zeta, \eta) = K_{12}\left[\eta^i \sum_{k=0}^{n_1-1} g_k \zeta^{2k} - \zeta^{2n_1-1}\eta^{i+1}\right] + K_{21}\left[\zeta^i \sum_{k=0}^{n_1-1} g_k \eta^{2k} - \zeta^{i+1}\eta^{2n_1-1}\right] \quad (5.17)$$

It can be seen that Φ_i is R-canonical. To show that the different elements of the set are linearly independent, assume that there exist real numbers a_i such that

$$\sum_{i=0}^{2n_2-2} a_i \Phi_i(\zeta, \eta) = 0$$

Now, equating the coefficients of $(K_{12}\zeta^{2n_1-1}\eta^{i+1} + K_{21}\zeta^{i+1}\eta^{2n_1-1})$ to zero for $i = 0, \ldots, 2n_2 - 2$ gives $a_i = 0$. Hence the set consists of linearly independent elements. Observe that $\Phi_i(\zeta, \eta)$ can be written as

$$\Phi_i(\zeta, \eta) = R(\zeta)H(\zeta, \eta) + H(\eta, \zeta)^T R(\eta) - (\zeta + \eta)(K_{12}\zeta^{2n_1-1}\eta^i + K_{21}\zeta^i\eta^{2n_1-1})$$

where $H(\zeta, \eta) = K_{12}\eta^i$. From the characterisation of zero-mean quantities for oscillatory behaviours (see equation (3.13)), it follows that Φ_i induces a zero mean quantity. Observe from equation (5.17) that Φ_i does not contain any element of the basis of two-variable polynomial matrices inducing trivially zero mean quantities mentioned in equation (5.16). We also prove that no trivially zero-mean quantity can be induced by a linear combination of the elements of the set. By contradiction, assume that $\sum_{i=0}^{2n_2-2} a_i \Phi_i(\zeta, \eta)$ induces a trivially zero-mean quantity. Then $\partial\big(\sum_{i=0}^{2n_2-2} a_i \Phi_i\big) = 0$, which implies that

$$\sum_{i=0}^{2n_2-2} a_i [K_{12} + (-1)^i K_{21}] r_1(\xi)\xi^i = 0$$

It can be verified that equating the coefficients of ξ^i to zero for $i = 0, 1, \ldots, 2n_2 - 2$ in the resulting expression yields $a_i = 0$. This implies that the set induces linearly independent zero-mean quantities that are complementary to the space of trivially zero-mean quantities and hence induce intrinsically zero-mean quantities. Observe that the number of elements in the set is equal to $2n_2 - 1$. This concludes the proof. ■

5.6.1 Construction of bases of zero-mean and conserved quantities

We now present a method for the construction of bases of trivially zero-mean, intrinsically zero-mean and conserved quantities for the non-generic case of oscillatory systems. We use the result of Theorem 5.15 for this purpose. Consider an oscillatory behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$ $(R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi], \det(R) \neq 0)$. Let $R = U\Delta V$ be a Smith form decomposition for R. Define $\mathfrak{B}' := \ker\big(\Delta(\frac{d}{dt})\big)$. Let $\mathtt{w}_1 + 1$ be the index corresponding to the first nonunity invariant polynomial of $\mathfrak{B}$. Let $\delta_i \in \mathbb{R}[\xi]$ denote the i^{th} invariant polynomial of $\mathfrak{B}$. For $i = \mathtt{w}_1 + 1, \ldots, \mathtt{w}$, denote $\delta_i(\xi) = \prod_{i=0}^{n_i-1}(\xi^2 + \omega_i^2)$. Let K_{pq} denote a square matrix of size $\mathtt{w}$ whose entry in the (p, q) position is 1 and the remaining entries are equal to 0. As explained earlier, we first construct bases of trivially zero-mean, intrinsically zero-mean and conserved quantities for the behaviour $\mathfrak{B}'$.

We define the following sets.

$\mathcal{T}_{pq}$:= A basis of Δ-canonical symmetric two-variable polynomial matrices of the form

$$K_{pq} f(\zeta, \eta) + K_{qp} f(\eta, \zeta) \tag{5.18}$$

with $f \in \mathbb{R}[\zeta, \eta]$, inducing trivially zero-mean quantities.

$\mathcal{C}_{pq}$:= A basis of Δ-canonical symmetric two-variable polynomial matrices of the form (5.18) inducing conserved quantities for $\mathfrak{B}'$.

$\mathcal{I}_{pq}$:= A basis of Δ-canonical symmetric two-variable polynomial matrices of the form (5.18) inducing intrinsically zero-mean quantities for $\mathfrak{B}'$.

For the construction of bases of diagonal conserved and zero-mean QDFs, we adopt a technique similar to the one used in the proof of Theorem 5.14. For $i = \mathtt{w}_1 + 1, \ldots, \mathtt{w}$, it is easy to see that we can choose

$$\begin{aligned}
\mathcal{T}_{ii} &= \left\{ K_{ii}(\zeta + \eta)(\zeta^l \eta^m + \zeta^m \eta^l) \right\}_{0 \le l \le m \le 2n_i - 2} \\
\mathcal{C}_{ii} &= \left\{ \frac{\Delta(\zeta) K_{ii} \eta^{2l+1} + K_{ii} \zeta^{2l+1} \Delta(\eta)}{\zeta + \eta} \right\}_{0 \le l \le n_i - 1} \\
\mathcal{I}_{ii} &= \left\{ \Delta(\zeta) K_{ii} \eta^{2l} + K_{ii} \zeta^{2l} \Delta(\eta) - (\zeta + \eta) K_{ii} (\zeta^{2n_i - 1} \eta^{2l} + \zeta^{2l} \eta^{2n_i - 1}) \right\}_{0 \le l \le n_i - 1}
\end{aligned}$$

For the construction of the bases of zero-diagonal conserved and zero-mean QDFs, we adopt the technique used in the proof of Theorem 5.15. For integers i, k such that $\mathtt{w}_1 + 1 \le i < k \le \mathtt{w}$, it is easy to see that

$$\begin{aligned}
\mathcal{T}_{ik} &= \left\{ (\zeta + \eta)(K_{ik} \zeta^l \eta^m + K_{ki} \zeta^m \eta^l) \right\}_{0 \le l \le 2n_i - 2, 0 \le m \le 2n_k - 2} \\
\mathcal{I}_{ik} &= \left\{ K_{ik} [\delta_i(\zeta) - (\zeta + \eta) \zeta^{2n_i - 1}] \eta^l + K_{ki} [\delta_i(\eta) - (\zeta + \eta) \eta^{2n_i - 1}] \zeta^l \right\}_{l = 0, \ldots, 2n_k - 2}
\end{aligned}$$

Define $u_{li}(\xi) := \frac{\delta_i(\xi)}{\xi - j\omega_l}$ where j is the imaginary unit. From the argument used in the proof of Theorem 5.15, we can conclude that for $l = 0, 1, \ldots, n_i - 1$, the real and imaginary parts of $K_{ik} u_{li}(\zeta) \bar{u}_{lk}(\eta) + K_{ki} \bar{u}_{lk}(\zeta) u_{li}(\eta)$ form the set $\mathcal{C}_{ik}$.

Now it is easy to see that the union of diagonal and zero-diagonal bases form respective bases for the behaviour $\mathfrak{B}'$, i.e the set $\cup_{i=\mathtt{w}_1+1}^{\mathtt{w}} \cup_{k=i}^{\mathtt{w}} \mathcal{T}_{ik}$ is a basis of Δ-canonical symmetric two-variable polynomial matrices inducing trivially zero-mean quantities, the set $\cup_{i=\mathtt{w}_1+1}^{\mathtt{w}} \cup_{k=i}^{\mathtt{w}} \mathcal{I}_{ik}$ is a basis of Δ-canonical symmetric two-variable polynomial matrices inducing intrinsically zero-mean quantities for $\mathfrak{B}'$, and the set $\cup_{i=\mathtt{w}_1+1}^{\mathtt{w}} \cup_{k=i}^{\mathtt{w}} \mathcal{C}_{ik}$ is a basis of Δ-canonical symmetric two-variable polynomial matrices inducing conserved quantities for $\mathfrak{B}'$.

As in the proof of Theorem 5.14, the corresponding bases of two-variable polynomial matrices inducing QDFs modulo $\mathfrak{B}$ are obtained by premultiplication by $V(\zeta)^T$ and postmultiplication by $V(\eta)$ of the basis elements mentioned above.

Example 5.3. Let

$$R(\xi) = \begin{bmatrix} \xi^2+1 & 0 \\ 0 & (\xi^2+1)(\xi^2+4) \end{bmatrix}$$

It is easy to see that $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$ is an oscillatory behaviour. We use the result of section 5.6.1 to construct bases for zero-diagonal trivially zero-mean, intrinsically zero-mean and conserved quantities modulo $\mathfrak{B}$. With reference to Theorem 5.15, $n_1 = 1$ and $n_2 = 2$. The dimension of zero-diagonal trivially zero-mean quantities modulo $\mathfrak{B}$ is $(2n_1-1)(2n_2-1) = 3$, the dimension of zero-diagonal conserved quantities modulo $\mathfrak{B}$ is $2n_1 = 2$ and the dimension of zero-diagonal intrinsically zero-mean quantities modulo $\mathfrak{B}$ is $2n_2 - 1 = 3$. Using the result of section 5.6.1,

$$\mathcal{T}_{12} = \left\{ \begin{bmatrix} 0 & \zeta+\eta \\ \zeta+\eta & 0 \end{bmatrix}, \begin{bmatrix} 0 & (\zeta+\eta)\eta \\ (\zeta+\eta)\zeta & 0 \end{bmatrix}, \begin{bmatrix} 0 & (\zeta+\eta)\eta^2 \\ (\zeta+\eta)\zeta^2 & 0 \end{bmatrix} \right\}$$

$$\mathcal{I}_{12} = \left\{ \begin{bmatrix} 0 & \zeta\eta-1 \\ \zeta\eta-1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \eta-\zeta\eta^2 \\ \zeta-\zeta^2\eta & 0 \end{bmatrix}, \begin{bmatrix} 0 & \eta^2-\zeta\eta^3 \\ \zeta^2-\zeta^3\eta & 0 \end{bmatrix} \right\}$$

$$\mathcal{C}_{12} = \{\mathrm{Re}(M), \mathrm{Im}(M)\}$$

where

$$M = \begin{bmatrix} 0 & (\zeta+j)(\eta-j)(\eta^2+4) \\ (\eta+j)(\zeta-j)(\zeta^2+4) & 0 \end{bmatrix}$$

This implies that

$$\mathcal{C}_{12} = \left\{ (\zeta\eta+1) \begin{bmatrix} 0 & \eta^2+4 \\ \zeta^2+4 & 0 \end{bmatrix}, (\zeta-\eta) \begin{bmatrix} 0 & -(\eta^2+4) \\ +(\zeta^2+4) & 0 \end{bmatrix} \right\}$$

5.7 Lagrangian of an oscillatory behaviour

In this section, we give an interpretation for the set of stationary trajectories of the Lagrangian of a high-order oscillatory system. For this interpretation, we make use of *Hamilton's principle* which states that the trajectories of a given system are stationary for the Lagrangian of the system. In order to facilitate the interpretation, consider a mechanical spring-mass system consisting of n springs with spring constants k_1, k_2, ..., k_n and n masses m_1, m_2, ..., m_n interconnected to each other and to the wall as shown in Figure 5.2.

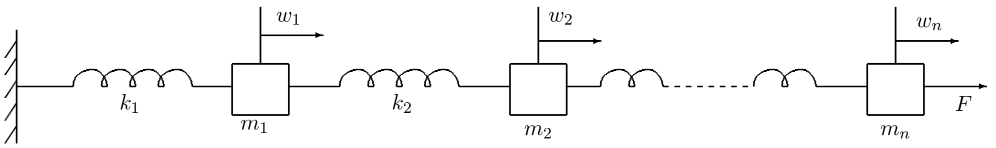

FIGURE 5.2: A spring-mass system

Assume that the springs and masses are constrained to move in a horizontal plane in a particular direction. In this direction, let w_i denote the horizontal displacement of the i^{th} mass. Assume that a force F acts on the n^{th} mass as shown in Figure 5.2. Define $w_0 := 0$. The equations of motion for the system can be written as

$$m_1 \frac{d^2 w_n}{dt^2} + k_n(w_n - w_{n-1}) = F \tag{5.19}$$

$$k_{i+1}(w_{i+1} - w_i) - k_i(w_i - w_{i-1}) = m_i \frac{d^2 w_i}{dt^2} \tag{5.20}$$

for $i = 1, \ldots, n-1$. Putting $i = 1$ in equation (5.20), we obtain

$$k_2(w_2 - w_1) - k_1 w_1 = m_1 \frac{d^2 w_1}{dt^2} \tag{5.21}$$

Define

$$f_2(\xi) := \frac{m_1}{k_2}\xi^2 + \frac{k_1}{k_2} + 1$$

From equation (5.21), we obtain $w_2 = f_2(\frac{d}{dt})w_1$. From equation (5.20), it follows that we can obtain $f_i \in \mathbb{R}[\xi]$ such that $w_i = f_i(\frac{d}{dt})w_1$ for $i = 3, \ldots, n$ by recursively using the following equations

$$f_{i+1}(\xi) = \left(\frac{m_i\xi^2 + k_i - k_{i-1}}{k_{i-1}}\right) f_i(\xi) - \left(\frac{k_i}{k_{i-1}}\right) f_{i-1}(\xi)$$

where $f_1(\xi) := 1$. Now from equation (5.19), it follows that $F = r(\frac{d}{dt})w_1$, where

$$r(\xi) := (m_1\xi^2 + k_n)f_n(\xi) - k_n f_{n-1}(\xi)$$

Let $\mathfrak{B}$ denote a behaviour whose external variables are w_n and F. Define $r'(\xi) := f_n(\xi)$. Define

$$M(\xi) := \operatorname{col}(r'(\xi), r(\xi))$$

It is easy to see that an image representation for $\mathfrak{B}$ is

$$\begin{bmatrix} w_n \\ F \end{bmatrix} = M(\frac{d}{dt})w_1 \tag{5.22}$$

Since the kinetic and potential energies of the system are quadratic functionals in w_i and $\frac{dw_i}{dt}$, $i = 1, \ldots, n$, and since $w_i = f_i(\frac{d}{dt})w_1$ for $i = 2, \ldots, n$, it follows that there exist $E, K_e, P_e \in \mathbb{R}_s[\zeta,\eta]$ such that $Q_E(w_1)$, $Q_{K_e}(w_1)$ and $Q_{P_e}(w_1)$ are the total energy, kinetic and potential energies for the system.

It is easy to see that

$$E(\zeta,\eta) = K_e(\zeta,\eta) + P_e(\zeta,\eta)$$

Define

$$L(\zeta,\eta) := K_e(\zeta,\eta) - P_e(\zeta,\eta)$$

Since power delivered to the system is $F\frac{dw_n}{dt}$, we have

$$\frac{d}{dt}Q_E(w_1) = \left(r(\frac{d}{dt})w_1\right)\left(\frac{d}{dt}r'(\frac{d}{dt})w_1\right)$$

This implies that

$$P(\zeta,\eta) := (\zeta+\eta)E(\zeta,\eta) = \frac{1}{2}\big(r(\zeta)\eta r'(\eta) + \zeta r'(\zeta)r(\eta)\big) \tag{5.23}$$

Also observe that since $Q_E(w_1)$ represents the total energy of the system, $Q_E(w_1) \geq 0$, and if $Q_E(w_1) = 0$, then $w_1 = 0$. Consequently, $Q_E > 0$. Define

$$\begin{aligned}
\mathfrak{B}_{\ell_1} &:= \{\ell_1 \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid M(\frac{d}{dt})\ell_1 = \text{col}(w_n, 0) \in \mathfrak{B}\} \\
\mathfrak{B}_{\ell_2} &:= \{\ell_2 \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}) \mid M(\frac{d}{dt})\ell_2 = \text{col}(0, F) \in \mathfrak{B}\}
\end{aligned}$$

Observe that $\mathfrak{B}_{\ell_1} = \ker\big(r(\frac{d}{dt})\big)$ and $\mathfrak{B}_{\ell_2} = \ker\big(r'(\frac{d}{dt})\big)$. From equation (5.23), it follows that Q_E is a conserved quantity for both $\mathfrak{B}_{\ell_1}$ and $\mathfrak{B}_{\ell_2}$. Since $Q_E > 0$, it follows that both $\mathfrak{B}_{\ell_1}$ and $\mathfrak{B}_{\ell_2}$ are oscillatory and Q_E is an energy function for both the behaviours. Since $\partial P(\xi) = 0$, it follows that either r and r' are both even or both odd. If r' is odd, then r'' defined by $r''(\xi) := \xi r'(\xi)$ is not oscillatory, which implies that there cannot exist a conserved quantity for $\mathfrak{B}' := \ker\big(r''(\frac{d}{dt})\big)$ that is positive. But Q_E is both conserved and positive along $\mathfrak{B}'$ which is a contradiction. Hence it follows that r and r' are both even. Let $\pm j\omega_0, \ldots, \pm j\omega_{n-1}$ be the characteristic frequencies of $\mathfrak{B}_{\ell_1}$. For $p = 0, \ldots, n-1$, define $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_p^2}$. From Lemma 4.8, it follows that r' is of the form

$$r'(\xi) = \sum_{i=0}^{n-1} c_p^2 v_p(\xi)$$

where $c_p \in \mathbb{R}^+$ for $p = 0, \ldots, n-1$.

Observe that $\mathfrak{B}_{\ell_1}$ is the scalar autonomous behaviour whose external variable ℓ_1 is such that $r'(\frac{d}{dt})\ell_1$ is the position of the n^{th} mass of the original mechanical system, when the force F acting on the mass is put equal to zero. This is shown in Figure 5.3. Also observe that $\mathfrak{B}_{\ell_2}$ is the scalar autonomous behaviour whose external variable ℓ_2 is such that $r(\frac{d}{dt})\ell_2$ is the force acting on the n^{th} mass of the original mechanical system, which is required to keep the n^{th} mass stationary, with the other masses not necessarily being stationary. This is shown in Figure 5.4.

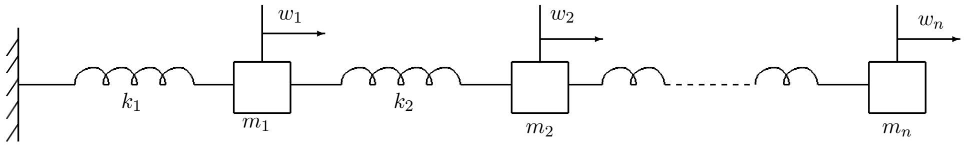

FIGURE 5.3: Autonomous mechanical system with $F = 0$

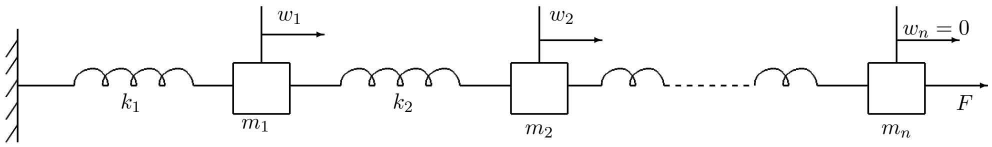

FIGURE 5.4: Autonomous mechanical system with $w_n = 0$

Observe that $\mathfrak{B}_{\ell_2}$ is a complementary oscillatory behaviour of $\mathfrak{B}_{\ell_1}$. It is easy to see that the two-variable polynomials $K_e(\zeta,\eta)$, $P_e(\zeta,\eta)$, $E(\zeta,\eta)$ and $L(\zeta,\eta)$ respectively induce the kinetic energy, potential energy, total energy and the Lagrangian for both the behaviors $\mathfrak{B}_{\ell_1}$ and $\mathfrak{B}_{\ell_2}$. From Hamilton's principle, it follows that $\mathfrak{B}_{\ell_1}$ and $\mathfrak{B}_{\ell_2}$ are both contained in the set of stationary trajectories with respect to Q_L. This implies that $\partial L(\xi)$ is divisible by $r(\xi)r'(\xi)$.

Thus the set of stationary trajectories with respect to the difference between the kinetic and potential energies of the original mechanical system consists of trajectories belonging to two oscillatory behaviours. One of these consists of trajectories w_1 of the autonomous system obtained from the original system by putting $F = 0$. The other behaviour corresponds to trajectories w_1 of the original system, such that w_n is constrained to be equal to zero. Using the expressions obtained for kinetic and potential energies of a scalar oscillatory behaviour obtained in section 4.2 of chapter 4, we now show that the set of stationary trajectories of a Lagrangian of a given scalar oscillatory behaviour is the direct sum of the given behaviour and a complementary oscillatory behaviour of the given behaviour.

Consider a scalar oscillatory behaviour $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, where $r(\xi) = \prod_{i=0}^{n-1}(\xi^2+\omega_i^2)$ and $\omega_0 < \omega_1 < \cdots < \omega_{n-1}$. Let $V(\frac{d}{dt})$ be the modal polynomial operator of $\mathfrak{B}$ and let $v_i \in \mathbb{R}[\xi]$ denote the $(i+1)^{\text{th}}$ component of V. Define $\Omega := \text{diag}(\omega_0, \omega_1, \ldots, \omega_{n-1})$. Let C be a diagonal real matrix with nonzero diagonal entries of size n. Let c_i denote the $(i+1)^{\text{th}}$ diagonal element of C. From the result of Chapter 4, recall that the general expression for the two-variable polynomial inducing an energy function for $\mathfrak{B}$ is given by

$$E(\zeta,\eta) = \zeta\eta V(\zeta)^T C^2 V(\eta) + V(\zeta)^T C^2 \Omega^2 V(\eta) \tag{5.24}$$

It can be verified that

$$(\zeta+\eta)E(\zeta,\eta)=r(\zeta)\eta\sum_{i=0}^{n-1}c_p^2v_p(\eta)+\zeta\sum_{i=0}^{n-1}c_p^2v_p(\zeta)r(\eta)$$

Define $r'(\xi):=\sum_{i=0}^{n-1}c_p^2v_p(\xi)$. Observe that $\mathfrak{B}':=\ker\big(r'(\frac{d}{dt})\big)$ is a complementary oscillatory behaviour of $\mathfrak{B}$. It was also shown in section 4.2 that we can interpret Q_{P_e} and Q_{K_e} as the kinetic and potential energy components respectively for a given energy function Q_E, where E is given by equation (5.24), and

$$\begin{aligned}K_e(\zeta,\eta)&=\zeta\eta V(\zeta)^TC^2V(\eta)\\ P_e(\zeta,\eta)&=V(\zeta)^TC^2\Omega^2V(\eta)\end{aligned}$$

Corresponding to the energy function Q_E, the two-variable polynomial associated with the Lagrangian which is the difference between kinetic and potential energy is

$$L(\zeta,\eta)=\zeta\eta V(\zeta)^TC^2V(\eta)-V(\zeta)^TC^2\Omega^2V(\eta)=\sum_{p=0}^{n-1}c_p^2v_p(\zeta)v_p(\eta)(\zeta\eta-\omega_p^2)$$

It is a matter of straightforward verification to see that

$$\partial L(\xi)=-r(\xi)\sum_{p=0}^{n-1}c_p^2v_p(\xi)$$

Observe that if $\mathfrak{B}_s:=\ker\big(\partial L(\frac{d}{dt})\big)$, then $\mathfrak{B}_s=\mathfrak{B}'\oplus\mathfrak{B}$. Recall from Proposition 3.17 of chapter 3 that $w\in\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is a stationary trajectory of a QDF Q_Φ ($\Phi\in\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$) iff w satisfies the differential equation $\partial\Phi(\frac{d}{dt})w=0$. Hence it follows that the set of stationary trajectories of a Lagrangian of a scalar oscillatory behaviour is the direct sum of the given behaviour and a complementary oscillatory behaviour of the given behaviour.

5.8 Generalized Lagrangians

Motivated by the discussion of stationary trajectories of a Lagrangian of a high-order oscillatory behaviour in the previous section, we now introduce the notion of "generalized Lagrangian" as a QDF whose stationary trajectories include a given autonomous behaviour. In this way, we make contact with the point of view adopted in classical mechanics, where Hamilton's principle states that the system trajectories are stationary for the Lagrangian.

The starting point of our investigation is the following definition.

Definition 5.16. *A* generalized Lagrangian *for an autonomous behaviour* $\mathfrak{B} = ker\big(R(\frac{d}{dt})\big)$ *(*$R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$*,* $det(R) \neq 0$*) is a QDF induced by an R-canonical* $\Lambda \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$*, such that* $\partial\Lambda(\xi) \neq 0$ *and* $\mathfrak{B}$ *is a subspace of the behaviour* $\mathfrak{B}_s = \ker\big(\partial\Lambda(\frac{d}{dt})\big)$

With respect to the above definition, observe that $\mathfrak{B}_s$ is the space of stationary trajectories with respect to Q_Λ. Hence Q_Λ is a generalized Lagrangian for an autonomous behaviour $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, if Λ is R-canonical and $\mathfrak{B}$ is a subspace of the space of stationary trajectories with respect to Q_Λ.

We now proceed to give an algebraic characterization of generalized Lagrangians. In the following, we prove that for a generic oscillatory behaviour, a generalized Lagrangian is a zero-mean quantity that is not trivially zero-mean.

Theorem 5.17. *Let* $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$ *be an oscillatory behaviour such that* $det(R)$ *has distinct roots, none of which is equal to zero. Let* $R = U\Delta V$ *be a Smith form decomposition of R. Define* $R_1 := \Delta V$*. A two-variable symmetric polynomial* $\Lambda(\zeta,\eta)$ *induces a generalized Lagrangian for* $\mathfrak{B}$ *if and only if it is of the form*

$$\Lambda(\zeta,\eta) = \Phi(\zeta,\eta) + \Theta(\zeta,\eta) \tag{5.25}$$

where $\Phi \neq 0$ *induces an intrinsically zero-mean quantity and* $\Theta(\zeta,\eta)$ *induces a trivially zero-mean quantity over* $\mathfrak{B}$*, and* Φ *and* Θ *are* R_1*-canonical.*

Proof. (*If*) Denote $\dim(\mathfrak{B}) = 2n$. Let

$$\Phi = a_0\Phi_0 + a_1\Phi_1 + \ldots + a_{n-1}\Phi_{n-1}$$

and

$$\Theta = b_1\Theta_1 + b_2\Theta_2 + \ldots + b_{(2n^2-n)}\Theta_{(2n^2-n)}$$

where for $i = 0,\ldots,n-1$ and $j = 1,\ldots,2n^2-n$, $a_i \in \mathbb{R}$, $b_j \in \mathbb{R}$, and Φ_i and Θ_j are the basis elements of the two-variable polynomials inducing intrinsically and trivially zero-mean quantities as in equations (5.12) and (5.14) of Theorem 5.14. Observe that since $\Phi \neq 0$, at least one of the coefficients a_i is not equal to zero. It follows from the algebraic characterization of Theorem 5.14 that $w \in \ker\big(\partial\Lambda(\frac{d}{dt})\big)$ if and only if

$$2\sum_{i=0}^{n-1} a_i \left(\frac{d^{2i}}{dt^{2i}}\right) V^T\left(-\frac{d}{dt}\right) F\Delta\left(\frac{d}{dt}\right) V\left(\frac{d}{dt}\right) w = 0 \tag{5.26}$$

Observe that any trajectory of $\mathfrak{B}$ satisfies equation (5.26). It follows that $\mathfrak{B}$ is a subspace of the space of stationary trajectories of the QDF Q_Λ and consequently $\Lambda(\zeta,\eta)$ induces a generalized Lagrangian for $\mathfrak{B}$.

(*Only If*) Since any QDF can be written as a linear combination of conserved, trivially zero-mean and intrinsically zero-mean quantities, it follows that

$$\Lambda = c_0\Gamma_0 + \ldots + c_{n-1}\Gamma_{n-1} + a_0\Phi_0 + \ldots + a_{n-1}\Phi_{n-1} + b_1\Theta_1 + \ldots + b_{(2n^2-n)}\Theta_{(2n^2-n)}$$

where for $i = 0, \ldots, n-1$ and $k = 1, \ldots, 2n^2 - n$, $c_i, a_i, b_k \in \mathbb{R}$, Φ_i and Θ_k are the basis elements of the two-variable polynomials inducing intrinsically and trivially zero-mean quantities as in equations (5.12) and (5.14) of Theorem 5.14 and the set $\{\Gamma_0, \ldots, \Gamma_{n-1}\}$ is as in equation (5.13) of Theorem 5.14. We aim to prove that all the coefficients c_i are zero. In order to do this, we apply the map ∂ to the expression above. Observe that ∂ is linear; now applying ∂ to the two-variable polynomial matrices Θ_j inducing the trivially zero-mean functionals yields zero. Also, observe that

$$\partial\left(\sum_{i=0}^{n-1} a_i\Phi_i\right)(\xi) = 2\sum_{i=0}^{n-1} a_i(\xi^{2i})V(-\xi)^T F\Delta(\xi)V(\xi)$$

We now consider the result of the application of ∂ to $\sum_{i=0}^{n-1} c_i\Gamma_i$. Observe that

$$(\partial\Gamma_i)(\xi) = \lim_{\eta\to\xi} \frac{V(-\xi)^T(\Delta(-\xi)F\eta^{2i+1} - F\Delta(\eta)\xi^{2i+1})V(\eta)}{\eta - \xi}$$

Using the formula of L'Hôpital we conclude that

$$(\partial\Gamma_i)(\xi) = \xi^{2i}V(-\xi)^T\left\{(2i+1)\Delta(\xi)F - F\xi\Delta'(\xi)\right\}V(\xi)$$

where

$$\Delta'(\xi) := \frac{d\Delta(\xi)}{d\xi}$$

From these computations we conclude that

$$\partial\Lambda(\xi) = V(-\xi)^T\left(\sum_{i=0}^{n-1} c_i\xi^{2i}\left[(2i+1)\Delta(\xi)F - F\xi\Delta'(\xi)\right] + 2\sum_{i=0}^{n-1} a_i\xi^{2i}F\Delta(\xi)\right)V(\xi)$$

We now prove that in order for $\ker\big(\partial\Lambda(\frac{d}{dt})\big)$ to contain $\mathfrak{B}$ it must be true that $c_i = 0$ for $i = 0.\ldots, n-1$.

Observe that the only nonzero term in

$$\sum_{i=0}^{n-1} c_i\xi^{2i}\left[(2i+1)\Delta(\xi)F - F\xi\Delta'(\xi)\right] + 2\sum_{i=0}^{n-1} a_i\xi^{2i}F\Delta(\xi)$$

is the last diagonal element, which equals

$$\sum_{i=0}^{n-1} c_i\xi^{2i}\left[(2i+1)\delta_{\mathtt{w}}(\xi) - \xi\frac{d\delta_{\mathtt{w}}(\xi)}{d\xi}\right] + 2\sum_{i=0}^{n-1} a_i\xi^{2i}\delta_{\mathtt{w}}(\xi)$$

where $\delta_{\mathtt{w}}$ is the last entry of Δ. Now define $w' := V(\frac{d}{dt})w$ and $\delta'_{\mathtt{w}} := \frac{d\delta_{\mathtt{w}}}{d\xi}$, and observe that

$$0 = \partial\Lambda\left(\frac{d}{dt}\right)w = \sum_{i=0}^{n-1} c_i \frac{d^{2i+1}}{dt^{2i+1}}\left(\delta'_{\mathtt{w}}\left(\frac{d}{dt}\right)w'_{\mathtt{w}}\right)$$

Observe that in order for this expression to be zero, it must be true that $\sum_{i=0}^{n-1} c_i\xi^{2i+1}\delta'_{\mathtt{w}}(\xi)$ is divisible by $\delta_{\mathtt{w}}(\xi)$. Now write $\delta_{\mathtt{w}}(\xi) = \prod_{k=0}^{n-1}(\xi^2+\omega_k^2)$, and consequently $\delta'_{\mathtt{w}}(\xi) = 2\xi\sum_{k=0}^{n-1}\frac{\delta_{\mathtt{w}}(\xi)}{\xi^2+\omega_k^2}$. Now consider that

$$\left(\sum_{i=0}^{n-1} c_i\xi^{2i+1}\right)\delta'_{\mathtt{w}}(\xi) = 2\left(\sum_{i=0}^{n-1} c_i\xi^{2i+2}\right)\sum_{k=0}^{n-1}\frac{\delta_{\mathtt{w}}(\xi)}{\xi^2+\omega_k^2} = 2\xi^2\left(\sum_{i=0}^{n-1} c_i\xi^{2i}\right)\left(\sum_{k=0}^{n-1}\frac{\delta_{\mathtt{w}}(\xi)}{\xi^2+\omega_k^2}\right)$$

If this polynomial were divisible by $\delta_{\mathtt{w}}$, it would annihilate on $\pm j\omega_p$, $p = 0,\ldots,n-1$. It is easy to see that in order for this to be true, the polynomial $\sum_{i=0}^{n-1} c_i\xi^{2i}$ must have roots in each of those points. Since $\deg(\sum_{i=0}^{n-1} c_i\xi^{2i}) = 2n-2$, this is possible if and only if $\sum_{i=0}^{n-1} c_i\xi^{2i} = 0$, which yields $c_i = 0$ for $i = 0,\ldots,n-1$.

Observe also that if $a_i = 0$ for $i = 0,1,\ldots,n-1$, then $\partial\Lambda(\xi) = 0$ which implies that $\Lambda(\zeta,\eta)$ does not induce a generalized Lagrangian. From this observation it follows that at least one of the coefficients a_i is nonzero. This concludes the proof. ■

The following corollary is a straightforward consequence of the above Proposition.

Corollary 5.18. *For a generic oscillatory behaviour, the sets of nontrivial zero-mean quantities and generalized Lagrangians are the same.*

Example 5.1 revisited: We reconsider the example of the oscillatory mechanical system considered in the beginning of this chapter. For this example, we show the construction of the bases of trivially and intrinsically zero-mean quantities and also show the relation between the Lagrangian and the basis elements of the zero-mean quantities. With reference to this example, define

$$r_0 := \frac{k_1k_2}{m_1m_2}$$

$$r_2 := \frac{k_1+k_2}{m_2} + \frac{k_1}{m_1}$$

In this case, $n = 2$. It follows from Proposition 5.11 that we can construct two linearly independent intrinsically zero-mean quantities. Equation (5.8) yields

$$\varphi_0(\zeta,\eta) = 2r_0 + r_2(\zeta^2+\eta^2) - (\zeta\eta^3+\zeta^3\eta)$$

$$\varphi_1(\zeta,\eta) = r_0(\zeta^2+\eta^2) + 2r_2\zeta^2\eta^2 - 2\zeta^3\eta^3$$

The space of trivially zero-mean quantities in this case has dimension equal to 6. A basis for this space can be computed from Proposition 5.10:

$$\begin{aligned}
\theta_1(\zeta,\eta) &= 2(\zeta+\eta)\\
\theta_2(\zeta,\eta) &= 2(\zeta+\eta)\zeta\eta\\
\theta_3(\zeta,\eta) &= (\zeta+\eta)^2\\
\theta_4(\zeta,\eta) &= (\zeta+\eta)(\zeta^2+\eta^2)\\
\theta_5(\zeta,\eta) &= 2(\zeta+\eta)\zeta^2\eta^2\\
\theta_6(\zeta,\eta) &= (\zeta+\eta)^2\zeta\eta
\end{aligned}$$

As in Example 5.1, assume that $m_1 = m_2 = 1$, $k_1 = 2$ and $k_2 = 3$. It was shown in Example 5.1 that the two-variable polynomial Λ corresponding to the Lagrangian for the system can be written as

$$\Lambda(\zeta,\eta) = \frac{1}{8}[\zeta^3\eta^3 - 5\zeta^2\eta^2 + 2(\zeta\eta^3+\zeta^3\eta) - 6(\zeta^2+\eta^2) + 8\zeta\eta - 12]$$

In this case, it can be verified that

$$\Lambda(\zeta,\eta) = \frac{1}{8}[\theta_6(\zeta,\eta) + 4\theta_3(\zeta,\eta) - \varphi_0(\zeta,\eta) - \frac{1}{2}\varphi_1(\zeta,\eta)]$$

which is of the form of a generalized Lagrangian mentioned in Theorem 5.17.

Remark 5.19. Theorem 5.17 is not in general true for the case of nongeneric oscillatory behaviours. We now provide an example of a nongeneric oscillatory behaviour and an intrinsically zero-mean quantity for this behaviour which is not a generalized Lagrangian. Consider the nongeneric oscillatory behaviour of Example 5.3, i.e $\mathfrak{B} = \ker\big(R(\frac{d}{dt})\big)$, with

$$R(\xi) = \begin{bmatrix} \xi^2+1 & 0 \\ 0 & (\xi^2+1)(\xi^2+4) \end{bmatrix}$$

We showed in Example 5.3, that

$$\Psi(\zeta,\eta) = \begin{bmatrix} 0 & \zeta\eta-1 \\ \zeta\eta-1 & 0 \end{bmatrix}$$

induces an intrinsically zero-mean quantity for $\mathfrak{B}$. Observe that

$$\partial\Psi(\xi) = \begin{bmatrix} 0 & -(\xi^2+1) \\ -(\xi^2+1) & 0 \end{bmatrix}$$

Consider a trajectory $\text{col}(w_1, w_2) \in \mathfrak{B}$. We have $r_2(\frac{d}{dt})w_2 = 0$, where $r_2(\xi) := (\xi^2+1)(\xi^2+4)$. Define $r_1(\xi) := (\xi^2+1)$. Now $r_1(\frac{d}{dt})w_2$ is not necessarily equal to zero. This implies that $\mathfrak{B}$ is not a subspace of $\mathfrak{B}_s = \ker\big(\partial\Psi(\frac{d}{dt})\big)$. Hence for the case of a nongeneric oscillatory behaviour, it is not true that a QDF Q_Λ is a generalized Lagrangian iff it is of the form

$$Q_\Lambda = Q_\Phi + Q_\Theta$$

where $Q_\Phi \neq 0$ is an intrinsically zero-mean quantity and Q_Θ is a trivially zero-mean quantity.

5.9 Zero-mean quantities for autonomous systems

In the remaining part of this chapter, we study the space of zero-mean quantities for autonomous systems, define the notion of single-frequency zero-mean quantities that was briefly touched upon in the introductory section of this chapter, and study its relation with the set of generalized Lagrangians for an oscillatory behaviour.

In this section, we study the space of zero-mean quantities for autonomous systems. In the following proposition, we show that along any trajectory of an autonomous behaviour, a zero-mean quantity is a linear combination of sinusoids.

Proposition 5.20. *Let $A \subset \mathbb{C}$ denote the set of all the distinct characteristic frequencies of an autonomous behaviour $\mathfrak{B}$. Let D_i denote the i^{th} element of set $A \times A$. Let H denote the set consisting of the same number of elements as the set $A \times A$, and whose i^{th} element H_i is given by*

$$H_i = \{d_1 + d_2 \mid (d_1, d_2) = D_i\}$$

Let G be the subset of H consisting of all its distinct purely imaginary elements that have positive imaginary parts. A QDF Q_Φ is zero-mean for $\mathfrak{B}$ iff for any trajectory $w \in \mathfrak{B}$, $Q_\Phi(w)$ is of the form

$$Q_\Phi(w)(t) = \sum_{i=1}^{N} (b_i \cos(\omega_i t) + c_i \sin(\omega_i t)) \tag{5.27}$$

where N is equal to the number of elements of the set G, for $i = 1, \ldots, N$, b_i, c_i are arbitrary real numbers and $j\omega_i$ is the i^{th} element of the set G.

Proof. (*If*): Since $\omega_i \neq 0$ for $i = 1, \ldots, N$, observe that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (b_i \cos(\omega_i t) + c_i \sin(\omega_i t))dt = 0$$

This implies that Q_Φ is a zero-mean quantity for $\mathfrak{B}$.

(*Only If*): Let R, V and $\mathfrak{B}'$ be related to $\mathfrak{B}$ as in the proof of Theorem 4.15 of chapter 4. Also let Φ and Φ' be as in the proof of Theorem 4.15. Let w and w' be two trajectories that are related by the equation $w' = V(\frac{d}{dt})w$. Observe that

$$Q_\Phi(w) = Q_{\Phi'}(w')$$

Let δ_i ($i = 1, \ldots, \mathtt{w}$) be the i^{th} invariant polynomial of R. Define $\mathfrak{B}_i := \ker\big(\delta_i(\frac{d}{dt})\big)$ for $i = 1, \ldots, \mathtt{w}$. Let $\Phi'(\zeta, \eta) = M(\zeta)^\top \Sigma M(\eta)$ ($\Sigma \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}, M \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}[\xi]$) be a symmetric canonical factorization of Φ'. Let $M_{ki} \in \mathbb{R}[\xi]$ be the entry in the (k, i) position of M. Then for any trajectory $w' \in \mathfrak{B}'$,

$$M(\frac{d}{dt})w' = \text{col}_{k=1}^{\mathtt{p}} \left(\sum_{i=1}^{\mathtt{w}} M_{ki} \left(\frac{d}{dt} \right) w_i' \right)$$

where w_i' is the i^{th} component of w'. Observe that $w_i' \in \mathfrak{B}_i$ and $M_{ki}(\frac{d}{dt})w_i' \in \mathfrak{B}_i$ for all $i \in \{1, \ldots, \mathtt{w}\}$, and $k \in \{1, \ldots, \mathtt{p}\}$. Denote with $\mathtt{q}$ the positive inertia of Φ'. Then

$$Q_\Phi(w) = Q_{\Phi'}(w') = \sum_{k=1}^{\mathtt{q}} \left(\sum_{i=1}^{\mathtt{w}} M_{ki} \left(\frac{d}{dt} \right) w_i' \right)^2 - \sum_{k=\mathtt{q}+1}^{\mathtt{p}} \left(\sum_{i=1}^{\mathtt{w}} M_{ki} \left(\frac{d}{dt} \right) w_i' \right)^2 \quad (5.28)$$

Let $\lambda_{ik} \in \mathbb{C}$, $k = 1, \ldots, m_i$, be the distinct roots of δ_i of multiplicity n_{ik}. Then from Proposition 2.21 of chapter 2, it follows that any trajectory $w_i' \in \mathfrak{B}_i$ is of the form

$$w_i'(t) = \sum_{k=1}^{m_i} \sum_{l=0}^{n_{ik}-1} t^l (r_{ikl} e^{\lambda_{ik} t} + \bar{r}_{ikl} e^{\bar{\lambda}_{ik} t}) \quad (5.29)$$

where r_{ikl} are arbitrary complex numbers for $k = 1, \ldots, m_i$ and $i = 1, \ldots, \mathtt{w}$. For any trajectory $w_i' \in \mathfrak{B}_i$, since $M_{ki}(\frac{d}{dt})w_i' \in \mathfrak{B}_i$, from equations (5.28) and (5.29), it follows that $Q_\Phi(w)$ is of the form

$$Q_\Phi(w)(t) = \sum_{i,\alpha=1}^{\mathtt{w}} \sum_{k=1}^{m_i} \sum_{\beta=1}^{m_\alpha} \sum_{l=0}^{n_{ik}+n_{\alpha\beta}-2} t^l (a_{ik\alpha\beta} e^{(\lambda_{ik}+\lambda_{\alpha\beta})t} + \bar{a}_{ik\alpha\beta} e^{(\bar{\lambda}_{ik}+\bar{\lambda}_{\alpha\beta})t}) \quad (5.30)$$

where $a_{ik\alpha\beta}$ are arbitrary complex numbers for $i, \alpha \in \{1, \ldots, \mathtt{w}\}$, $k \in \{1, \ldots, m_i\}$ and $\beta \in \{1, \ldots, m_\alpha\}$. Observe that in the right hand side of equation (5.30), $(\lambda_{ik}, \lambda_{\alpha\beta}) \in A \times A$. Now assume that Q_Φ is a zero-mean quantity for $\mathfrak{B}$. Then from Definition 3.19, it follows that

$$\sum_{i,\alpha=1}^{\mathtt{w}} \sum_{k=1}^{m_i} \sum_{\beta=1}^{m_\alpha} \sum_{l=0}^{n_{ik}+n_{\alpha\beta}-2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} t^l (a_{ik\alpha\beta} e^{(\lambda_{ik}+\lambda_{\alpha\beta})t} + \bar{a}_{ik\alpha\beta} e^{(\bar{\lambda}_{ik}+\bar{\lambda}_{\alpha\beta})t}) = 0 \quad (5.31)$$

For the limit in the left hand side of equation (5.31) to exist, observe that for each term within the summation in the right hand side of equation (5.30), l should be equal to 0, and $(\lambda_{ik} + \lambda_{\alpha\beta})$ should be purely imaginary. This implies that

$$Q_\Phi(w)(t) = C + \sum_{i=1}^{N} (b_i \cos(\omega_i t) + c_i \sin(\omega_i t))$$

where $C \in \mathbb{R}$, N is a positive finite integer and

$$\{j\omega_1, j\omega_2, \ldots, j\omega_N\} = G$$

Since

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} C dt = C,$$

$C = 0$. This concludes the proof. ■

We now show that for some autonomous scalar behaviours, every zero-mean quantity is a generalized Lagrangian.

Lemma 5.21. *Consider an autonomous behaviour* $\mathfrak{B} = ker\big(r(\frac{d}{dt})\big)$, *where* $r \in \mathbb{R}[\xi]$ *is even. Every zero-mean quantity for* $\mathfrak{B}$ *induced by an* r*-canonical polynomial that is not trivially zero-mean, is a generalized Lagrangian.*

Proof. Let n denote the degree of r. Let A, H and G be as defined in the statement of Proposition 5.20. Assume without loss of generality that the set G is given by

$$G = \{j\omega_1, j\omega_2, \ldots, j\omega_N\}$$

Assume that $z \in \mathbb{R}[\zeta, \eta]$ is r-canonical and induces a zero-mean quantity for $\mathfrak{B}$ that is not trivially zero-mean and w is a nonzero trajectory of $\mathfrak{B}$. From Proposition 5.20, it follows that

$$Q_z(w) = \sum_{i=1}^{N} (Q_{z_i} + Q_{\bar{z}_i})(w) \tag{5.32}$$

where, for $i = 1, \ldots, N$,

$$Q_{z_i}(w)(t) = a_i e^{j\omega_i t}, \tag{5.33}$$

$a_i \in \mathbb{C}$, and $z_i \in \mathbb{C}[\zeta, \eta]$ is r-canonical. Differentiating both sides of equation (5.33), we get

$$\frac{d}{dt} Q_{z_i}(w)(t) = a_i j\omega_i e^{j\omega_i t}$$

This implies that

$$\frac{d}{dt} Q_{z_i} \stackrel{\mathfrak{B}}{=} j\omega_i Q_{z_i}$$

In terms of the two-variable polynomial corresponding to Q_{z_i}, the above equation is equivalent with

$$(\zeta + \eta) z_i(\zeta, \eta) = r(\zeta) f_{1i}(\zeta, \eta) + r(\eta) f_{1i}(\eta, \zeta) + j\omega_i z_i(\zeta, \eta)$$

$$\Rightarrow z_i(\zeta, \eta) = \frac{r(\zeta) f_{1i}(\zeta, \eta) + r(\eta) f_{1i}(\eta, \zeta)}{\zeta + \eta - j\omega_i}$$

for some $f_{1i} \in \mathbb{C}[\zeta, \eta]$. Write $f_{1i}(\zeta, \eta) = \sum_{k=0}^{N_1} \sum_{l=0}^{N_2} b_{ilk} \zeta^l \eta^k$, where $b_{ilk} \in \mathbb{C}$. It is easy to see that since z_i is r-canonical, $N_2 = 0$ and $N_1 = n - 1$. Thus

$$z_i(\zeta, \eta) = \frac{r(\zeta) f_i(\eta) + r(\eta) f_i(\zeta)}{\zeta + \eta - j\omega_i}$$

where $f_i(\eta) = f_{1i}(\zeta, \eta)$. Hence

$$\partial z_i(\xi) = r(\xi) \left(\frac{f_i(\xi) + f_i(-\xi)}{-j\omega_i} \right) \tag{5.34}$$

is divisible by $r(\xi)$. From equation (5.32), it follows that

$$z(\zeta, \eta) = \sum_{i=1}^{N} (z_i(\zeta, \eta) + \bar{z}_i(\zeta, \eta))$$

Consequently,

$$\partial z(\xi) = \sum_{i=1}^{N} (\partial z_i(\xi) + \partial \bar{z}_i(\xi))$$

From equation (5.34), it follows that the right hand side of the above equation is divisible by $r(\xi)$. Since Q_z is not trivially zero-mean, $\partial z(\xi) \neq 0$. From Definition 5.16, conclude that Q_z is a generalized Lagrangian for $\mathfrak{B}$ or that every nontrivial zero-mean quantity induced by an r-canonical polynomial is a generalized Lagrangian for $\mathfrak{B}$. ■

We know from Corollary 5.18 that for a scalar oscillatory behaviour, the sets of nontrivial zero-mean quantities and generalized Lagrangians are the same which implies that every generalized Lagrangian is a non-trivial zero-mean quantity. However in general, the converse of Proposition 5.21 is not true. For example, consider the autonomous behaviour $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, where $r(\xi) = \xi^2 - 1$. Now consider the QDF Q_Φ, where $\Phi(\zeta, \eta) = \zeta\eta + 1$. It is easy to see that Q_Φ is a generalized Lagrangian for $\mathfrak{B}$ as Φ is r-canonical and $\partial\Phi(\xi) = -\xi^2 + 1$ is divisible by $r(\xi)$. However Q_Φ is not a zero-mean quantity for $\mathfrak{B}$, as for any trajectory $w \in \mathfrak{B}$, where $w(t) = ae^t + be^{-t}$, $a, b \in \mathbb{R}$, $Q_\Phi(w)(t) = 2(a^2 e^{2t} + b^2 e^{-2t})$ is not of the form mentioned in equation (5.27) in Proposition 5.20.

5.10 Single- and mixed-frequency zero-mean quantities

We now divide the space of zero-mean quantities for an autonomous behaviour into two complementary subspaces, namely the space of single-frequency zero-mean quantities and the space of mixed-frequency zero-mean quantities. Later on in this section, we will discuss a property linking single-frequency zero-mean quantities and generalized Lagrangians for a scalar oscillatory behaviour. Below we define both single-frequency and mixed-frequency zero-mean quantities.

Definition 5.22 (Single-frequency zero-mean quantity)**.** *Let λ_i, $i = 1, \ldots, n$ denote the distinct characteristic frequencies of a given autonomous behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ with nonzero imaginary parts $Im(\lambda_i)$. A zero-mean quantity Q_z for $\mathfrak{B}$ is called a* single-frequency zero-mean quantity *if for every nonzero trajectory $w \in \mathfrak{B}$,*

$$Q_z(w)(t) = \sum_{i=1}^{n} \left[a_i e^{2jIm(\lambda_i)t} + \bar{a}_i e^{-2jIm(\lambda_i)t} \right] \tag{5.35}$$

where $a_i \in \mathbb{C}$ for $i = 1, \ldots, n$.

Definition 5.23 (Mixed-frequency zero-mean quantity)**.** *Let λ_i, $i = 1, \ldots, n$ denote the distinct characteristic frequencies of a given autonomous behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ with nonzero imaginary parts $Im(\lambda_i)$. A zero-mean quantity Q_z for $\mathfrak{B}$ is called a* mixed-frequency zero-mean quantity *if for every nonzero trajectory $w \in \mathfrak{B}$,*

$$Q_z(w)(t) = \sum_{i,k \in \{1,\ldots,n\}}^{i \neq k, \lambda_i \neq -\bar{\lambda}_k} \left[b_{ik} e^{j(Im(\lambda_i+\lambda_k))t} + \bar{b}_{ik} e^{-j(Im(\lambda_i+\lambda_k))t} \right] \tag{5.36}$$

where $b_{ik} \in \mathbb{C}$ for $i, k \in \{1, \ldots, n\}$, such that $i \neq k$ and $\lambda_i + \bar{\lambda}_k \neq 0$.

We now give an example of a single-frequency zero-mean quantity.

Example 5.4. Now consider a scalar oscillatory behaviour $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, where $r(\xi) = \prod_{i=0}^{n-1}(\xi^2 + \omega_i^2)$. Define $v_p(\xi) := \frac{r(\xi)}{\xi^2+\omega_p^2}$. In section 5.7, we showed that the two-variable polynomial associated with the Lagrangian of the system is

$$L(\zeta, \eta) = \sum_{p=0}^{n-1} c_p^2 v_p(\zeta) v_p(\eta)(\zeta\eta - \omega_p^2)$$

where $c_p \in \mathbb{R}^+$ for $p = 0, \ldots, n-1$. We now show that Q_L is a single frequency zero-mean quantity. In other words, we prove that the Lagrangian of a scalar oscillatory system is always a single-frequency zero-mean quantity. Any trajectory $w \in \mathfrak{B}$ is of the form

$$w(t) = \sum_{p=0}^{n-1} (k_p e^{j\omega_p t} + \bar{k}_p e^{-j\omega_p t})$$

$$\Rightarrow v_p(\frac{d}{dt}) w(t) = v_p(j\omega_p)[k_p e^{j\omega_p t} + \bar{k}_p e^{-j\omega_p t}]$$

where $k_p \in \mathbb{C}$ for $p = 0, \ldots, n-1$. Hence

$$\frac{d}{dt} v_p(\frac{d}{dt}) w(t) = v_p(j\omega_p) \frac{d}{dt}(k_p e^{j\omega_p t} + \bar{k}_p e^{-j\omega_p t}) = j\omega_p v_p(j\omega_p)[k_p e^{j\omega_p t} - \bar{k}_p e^{-j\omega_p t}]$$

From these, the expressions for $Q_L(w(t))$ can be written as follows

$$Q_L(w)(t) = -2\sum_{p=0}^{n-1} c_p^2\omega_p^2 v_p^2(j\omega_p)(k_p^2 e^{2j\omega_p t} + \bar{k}_p^2 e^{-2j\omega_p t})$$

which is of the form mentioned in equation (5.35). Hence $L(\zeta,\eta)$ induces a single-frequency zero-mean quantity.

From equations (5.35) and (5.36), it is easy to see that any linear combination of single-frequency zero-mean quantities is a single-frequency zero-mean quantity and any linear combination of mixed-frequency zero-mean quantities is a mixed frequency zero-mean quantity. Hence the spaces of single and mixed frequency zero-mean quantities are linear subspaces of the space of zero-mean quantities. In the next lemma, we prove that the space of zero-mean quantities is the direct sum of these two subspaces.

Lemma 5.24. *Let S and M denote the spaces of single and mixed frequency zero-mean quantities for an autonomous behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$. Let Z denote the space of zero-mean quantities for $\mathfrak{B}$. Then $Z = S \oplus M$.*

Proof. Let A denote the set of all the distinct characteristic frequencies of $\mathfrak{B}$. Let $D \subset (A \times A)$ denote the set of all ordered pairs (d_1, d_2) in $A \times A$ for which $d_1 + d_2$ is nonzero and purely imaginary. Let $S \subset D$ denote the set of all ordered pairs (s_1, s_2) in D for which s_1 and s_2 have the same imaginary parts, i.e, $\mathrm{Im}(s_1) = \mathrm{Im}(s_2)$. Define $M := D\backslash S$. Let S_i denote the i^{th} element of S. Let S_1 be a set consisting of the same number of elements as S, and whose i^{th} element S_{1i} is given by

$$S_{1i} = \{s_1 + s_2 \mid (s_1, s_2) = S_i\}$$

Note that the imaginary part of each element of S_1 is equal to twice the imaginary part of a characteristic frequency of $\mathfrak{B}$. Let M_i denote the i^{th} element of M. Let M_1 be a set consisting of the same number of elements as M, and whose i^{th} element M_{1i} is given by

$$M_{1i} = \{m_1 + m_2 \mid (m_1, m_2) = M_i\}$$

Note that the imaginary part of each element of M_1 is equal to the sum of imaginary parts of two distinct characteristic frequencies of $\mathfrak{B}$.

Let N_s and N_m denote the number of elements in the set S and M respectively. For $i = 1, \ldots, N_s$, let $j\alpha_i$ denote the i^{th} element of S_1 and for $i = 1, \ldots, N_m$, let $j\beta_i$ denote the i^{th} element of the set M_1. From Proposition 5.20, it follows that if Q_z is any zero-mean quantity for $\mathfrak{B}$ and w is a nonzero trajectory of $\mathfrak{B}$, then

$$Q_z(w) = Q_{z_1}(w) + Q_{z_2}(w)$$

where

$$Q_{z_1}(w)(t) = \sum_{i=1}^{N_s} (a_i e^{j\alpha_i t} + \bar{a}_i e^{-j\alpha_i t})$$

and

$$Q_{z_2}(w)(t) = \sum_{k=1}^{N_m} (b_k e^{j\beta_k t} + \bar{b}_k e^{-j\beta_k t})$$

where $a_i \in \mathbb{C}$ for $i = 1, \dots, N_s$ and $b_k \in \mathbb{C}$ for $k = 1, \dots, N_m$. Since for $i = 1, \dots, N_s$, α_i is equal to twice the imaginary part of a characteristic frequency of $\mathfrak{B}$, from Definition 5.22, it follows that Q_{z_1} is a single-frequency zero-mean quantity for $\mathfrak{B}$. Since for $i = 1, \dots, N_m$, β_i is equal to the sum of imaginary parts of two distinct characteristic frequencies of $\mathfrak{B}$, from Definition 5.23, it follows that Q_{z_2} is a mixed-frequency zero-mean quantity for $\mathfrak{B}$.

From definitions 5.22 and 5.23, it is easy to see that no single-frequency zero-mean quantity can be obtained by a linear combination of mixed-frequency zero-mean quantities and no mixed-frequency zero-mean quantity can be obtained by a linear combination of single-frequency zero-mean quantities. Hence the spaces of single-frequency and mixed-frequency zero-mean quantities are complementary to each other. For given z and w, the parameters a_i for $i = 1, \dots, N_s$, and b_k for $k = 1, \dots, N_m$ are unique. Hence the proof. ■

We now give a result that links single-frequency zero-mean quantities and generalized Lagrangians for a scalar oscillatory behaviour.

Theorem 5.25. *Let L denote the set of generalized Lagrangians for an oscillatory behaviour $\mathfrak{B} \in \mathcal{L}^1$. Consider an equivalence relation in L defined by $Q_{\phi_1} \sim Q_{\phi_2}$ iff $\phi_1(\zeta, \eta) - \phi_2(\zeta, \eta) = (\zeta + \eta)\psi(\zeta, \eta)$ for some $\psi \in \mathbb{R}[\zeta, \eta]$. In every equivalence class under $\sim$, there exists a single-frequency zero-mean quantity for $\mathfrak{B}$.*

Proof. We consider two cases, namely the one in which $\mathfrak{B}$ does not have a characteristic frequency at 0, and the one in which $\mathfrak{B}$ has a characteristic frequency at 0.

Case 1: $\mathfrak{B}$ does not have a characteristic frequency at 0. Let $2n$ denote the dimension of $\mathfrak{B}$. Without loss of generality, we can assume that $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$ is a kernel representation of $\mathfrak{B}$, with

$$r(\xi) = (\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2) \dots (\xi^2 + \omega_{n-1}^2)$$

and $\omega_i \in \mathbb{R}^+$ being distinct for $i = 0, \dots, n-1$. Define $r_i(\xi) := (\xi^2 + \omega_i^2)$, $\mathfrak{B}_i := \ker\big(r_i(\frac{d}{dt})\big)$ and $v_i(\xi) := \frac{r(\xi)}{r_i(\xi)}$. Define

$$z_i'(\zeta, \eta) := c_i(\zeta\eta - \omega_i^2)$$

where $c_i \in \mathbb{R}$ for $i = 0, \ldots, n-1$. Observe that for any trajectory $w_i \in \mathfrak{B}_i$ of the form

$$w_i(t) = a_i e^{j\omega_i t} + \bar{a}_i e^{-j\omega_i t},$$

where $a_i \in \mathbb{C}$, we have

$$Q_{z_i'}(w_i(t)) = -2c_i\omega_i^2(a_i^2 e^{2j\omega_i t} + \bar{a}_i^2 e^{-2j\omega_i t})$$

From Definition 5.22, it follows that $Q_{z_i'}$ is a single-frequency zero-mean quantity for $\mathfrak{B}_i$. Define $\phi_i(\zeta,\eta) := v_i(\zeta)v_i(\eta)z_i'(\zeta,\eta)$. We now prove that Q_{ϕ_i} is a single frequency zero-mean quantity for $\mathfrak{B}$. Observe that any trajectory $w \in \mathfrak{B}$ can be written as

$$w(t) = w_i'(t) + w_i(t)$$

where $w_i' \in \ker\big(v_i(\frac{d}{dt})\big)$ and $w_i \in \mathfrak{B}_i$. We have

$$Q_{\phi_i}(w)(t) = Q_{z_i'}\left(v_i\left(\frac{d}{dt}\right)w\right)(t) = Q_{z_i'}\left(v_i\left(\frac{d}{dt}\right)w_i\right)(t)$$

It is easy to see that $v_i(\frac{d}{dt})w_i \in \mathfrak{B}_i$. We know that $Q_{z_i'}(w_i)$ is of the form

$$Q_{z_i'}(w_i)(t) = b_i e^{2j\omega_i t} + \bar{b}_i e^{-2j\omega_i t} \tag{5.37}$$

where $b_i \in \mathbb{C}$ for $i = 0, \ldots, n-1$. Since $v_i(\frac{d}{dt})w_i \in \mathfrak{B}_i$, $Q_{z_i'}(v_i(\frac{d}{dt})w_i)(t)$ also has the same form as the right hand side of equation (5.37). This implies that Q_{ϕ_i} is a single-frequency zero-mean quantity for $\mathfrak{B}$. It is easy to see that $Q_\Phi := \sum_{i=0}^{n-1} Q_{\phi_i}$ is a single frequency zero-mean quantity for $\mathfrak{B}$. Also observe that

$$\Phi(\zeta,\eta) = \sum_{i=0}^{n-1}\phi_i(\zeta,\eta) = \sum_{i=0}^{n-1} v_i(\zeta)v_i(\eta)z_i'(\zeta,\eta)$$

This implies that

$$\begin{aligned}\partial\Phi(\xi) &= \sum_{i=0}^{n-1} v_i(\xi)^2\partial z_i'(\xi) = -\sum_{i=0}^{n-1} c_i v_i^2(\xi)(\xi^2+\omega_i^2)\\ &= -r(\xi)\sum_{i=0}^{n-1} c_i v_i(\xi)\end{aligned}$$

Since for $i = 0, \ldots, n-1$, $v_i(\xi)$ are linearly independent polynomials of degree $2n-2$, $f_e(\xi) := \sum_{i=0}^{n-1} c_i v_i(\xi)$ spans the entire space of even polynomials of degree less than or equal to $2n-2$.

Now consider a generalized Lagrangian for $\mathfrak{B}$ induced by $\Lambda \in \mathbb{R}_s[\zeta,\eta]$. Define

$$r_1(\xi) := \frac{\partial\Lambda(\xi)}{r(\xi)}$$

Since Λ is symmetric, $\Lambda(\zeta,\eta) = \Lambda(\eta,\zeta)$. This implies that $\partial\Lambda(\xi)$ is even. Since Λ is r-canonical, $r(\zeta)^{-1}\Lambda(\zeta,\eta)r(\eta)^{-1}$ is strictly proper. This implies that $r_1(\xi)r(-\xi)^{-1}$ is strictly proper. Since the sum of degrees of the polynomials r and r_1 is even, the difference deg(r)-deg(r_1) is at least 2. Since $\partial\Lambda(\xi) = r(\xi)r_1(\xi)$ is even, $r_1(\xi)$ is also even and of maximum degree $2n-2$. Observe that there is a one-one correspondence between r_1 and the equivalence classes under $\sim$. Since the space of single frequency zero-mean quantities Q_Φ for $\mathfrak{B}$ is such that $\frac{\partial\Phi(\xi)}{r(\xi)}$ spans the entire space of even polynomials of degree less than or equal to $2n-2$, there exists a single-frequency zero-mean quantity in every equivalence class under $\sim$. This concludes the proof for Case 1.

Case 2: $\mathfrak{B}$ has a characteristic frequency at 0. In this case, without loss of generality, it can be assumed that $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$ is a kernel representation of $\mathfrak{B}$, with

$$r(\xi) = \xi(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$$

and $\omega_i \in \mathbb{R}^+$ being distinct for $i = 0,\dots,n-1$. Define $r_i(\xi) := (\xi^2+\omega_i^2)$, $\mathfrak{B}_i := \ker\big(r_i(\frac{d}{dt})\big)$ and $v_i(\xi) := \frac{r(\xi)}{\xi r_i(\xi)}$. Let $Q_{z_i'}$ ($z_i' \in \mathbb{R}[\zeta,\eta]$) be a single-frequency zero-mean quantity for $\mathfrak{B}_i$ as in Case 1. Define $\phi_i(\zeta,\eta) := \zeta\eta v_i(\zeta)v_i(\eta)z_i'(\zeta,\eta)$. We now prove that Q_{ϕ_i} is a single frequency zero-mean quantity for $\mathfrak{B}$. Observe that any trajectory $w \in \mathfrak{B}$ can be written as

$$w(t) = w_i'(t) + w_i(t) + c$$

where $c \in \mathbb{R}$, $w_i' \in \ker\big(v_i(\frac{d}{dt})\big)$ and $w_i \in \mathfrak{B}_i$. We have

$$Q_{\phi_i}(w)(t) = Q_{z_i'}\left(v_i\left(\frac{d}{dt}\right)\left(\frac{dw}{dt}\right)\right)(t) = Q_{z_i'}\left(v_i\left(\frac{d}{dt}\right)\left(\frac{dw_i}{dt}\right)\right)(t)$$

It is easy to see that $v_i(\frac{d}{dt})(\frac{dw_i}{dt}) \in \mathfrak{B}_i$. From Definition 5.22, it follows that

$$Q_{z_i'}(w_i)(t) = b_i e^{2j\omega_i t} + \bar{b}_i e^{-2j\omega_i t} \tag{5.38}$$

where $b_i \in \mathbb{C}$ for $i = 0,\dots,n-1$. Since $v_i(\frac{d}{dt})(\frac{dw_i}{dt}) \in \mathfrak{B}_i$, $Q_{z_i'}\big(v_i(\frac{d}{dt})(\frac{dw_i}{dt})\big)(t)$ also has the same form as the right hand side of equation (5.38). This implies that Q_{ϕ_i} is a single-frequency zero-mean quantity for $\mathfrak{B}$. As in the proof for Case 1, take

$$z_i'(\zeta,\eta) = c_i(\zeta\eta - \omega_i^2)$$

where $c_i \in \mathbb{R}$ for $i = 0,\dots,n-1$. Define $\Phi(\zeta,\eta) := \sum_{i=0}^{n-1}\phi_i(\zeta,\eta)$. It is easy to see that Q_Φ is a single-frequency zero-mean quantity for $\mathfrak{B}$. Now observe that

$$\Phi(\zeta,\eta) = \sum_{i=0}^{n-1}\zeta\eta v_i(\zeta)v_i(\eta)z_i'(\zeta,\eta)$$

This implies that

$$\begin{aligned}\partial\Phi(\xi) &= \sum_{i=0}^{n-1}\xi^2 v_i(\xi)^2\partial z_i'(\xi) = -\sum_{i=0}^{n-1} c_i\xi^2 v_i^2(\xi)(\xi^2+\omega_i^2)\\ &= -\xi r(\xi)\sum_{i=0}^{n-1} c_i v_i(\xi)\end{aligned}$$

Since for $i = 0, \ldots, n-1$, $v_i(\xi)$ are linearly independent polynomials of degree $2n-2$, $f_e(\xi) := \sum_{i=0}^{n-1} c_i v_i(\xi)$ spans the entire space of even polynomials of degree less than or equal to $2n-2$.

Now consider a generalized Lagrangian for $\mathfrak{B}$ induced by $\Lambda \in \mathbb{R}_s[\zeta,\eta]$. Define

$$r_1(\xi) := \frac{\partial\Lambda(\xi)}{r(\xi)}$$

Since Λ is symmetric, $\Lambda(\zeta,\eta) = \Lambda(\eta,\zeta)$. This implies that $\partial\Lambda(\xi)$ is even. Since Λ is r-canonical, $r(\zeta)^{-1}\Lambda(\zeta,\eta)r(\eta)^{-1}$ is strictly proper. This implies that $r_1(\xi)r(-\xi)^{-1}$ is strictly proper. Since the sum of degrees of the polynomials r and r_1 is even, the difference deg(r)-deg(r_1) is at least 2. Since $\partial\Lambda(\xi) = r(\xi)r_1(\xi)$ is even, $r_1(\xi)$ is odd and of maximum degree $2n-1$. Define $r_2(\xi) := \frac{r_1(\xi)}{\xi}$. Observe that r_2 is even of maximum degree $2n-2$ and there is a one-one correspondence between r_2 and the equivalence classes of $\sim$. Since the space of single frequency zero-mean quantities Q_Φ for $\mathfrak{B}$ is such that $\frac{\partial\Phi(\xi)}{\xi r(\xi)}$ spans the entire space of even polynomials of degree less than or equal to $2n-2$, there exists a single-frequency zero-mean quantity in every equivalence class of $\sim$. This concludes the proof for Case 2. ∎

One interpretation of the above theorem is that given a scalar oscillatory behaviour $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, and another scalar behaviour $\mathfrak{B}_1 = \ker\big(r'(\frac{d}{dt})\big)$, with $r'(\xi) = r_1(\xi)r(\xi)$ being even and deg(r)− deg(r_1) ≥ 2, we can always find a single-frequency zero-mean quantity for $\mathfrak{B}$ such that the set of its stationary trajectories is equal to $\mathfrak{B}_1$.

5.11 Summary

In this chapter, we have studied the space of QDFs modulo an oscillatory behaviour and then studied in detail its decomposition into the spaces of conserved and zero-mean quantities. We have then given a method of construction of bases of conserved, trivially zero-mean and intrinsically zero-mean quantities for oscillatory behaviours. Next, we have given an interpretation for the space of stationary trajectories of the Lagrangian of a scalar oscillatory behaviour with the help of an example of a mechanical spring-mass system. We have then defined a generalized Lagrangian for an autonomous behaviour as a QDF whose stationary trajectories include the given behaviour. It has been proved that for a generic oscillatory behaviour, the sets of nontrivial zero-mean quantities and generalized Lagrangians are the same. We have showed that along any trajectory of an autonomous behaviour, a zero-mean quantity is a linear combination of sinusoids. Finally, we have defined single-frequency and mixed-frequency zero-mean quantities and have given a result that links single-frequency zero-mean quantities and generalized Lagrangians of a scalar oscillatory behaviour.

Chapter 6

Synthesis of positive QDFs and interconnection of J-lossless behaviours

6.1 Motivation and aim

Synthesis of electrical networks is a well studied topic (see Baher (1984), Karni (1966), Anderson and Vongpanitlerd (1973), Belevitch (1968), Balabanian (1958), Newcomb (1966), Chen (1964)). A typical problem of synthesis of an electrical network involves construction of an electrical network consisting of resistors, capacitors, inductors and transformers, that realizes a given transfer function. A fundamental result that was proved by Brune (1931), is that synthesis using these components is possible if and only if the given transfer function is positive real (see Definition B.18, Appendix B). Some of the well known methods of synthesis are Cauer, Foster, Brune, Darlington, Miyata, Bott-Duffin, etc. Of these, the first two are methods of synthesis of lossless positive real transfer functions (see Definition B.19, Appendix B).

In this chapter, we are mainly concerned with the problem of synthesis of lossless positive real transfer functions. Both Cauer and Foster methods which are the two most well-known methods of synthesis of lossless positive real transfer functions, proceed in steps. Each of these steps involves extraction of a reactance component and simplification of the given transfer function, in a sense that the degrees of the numerator and denominator of some of the elements of the transfer function matrix get reduced. In other words, in every step, what we obtain is a network which comprises of the extracted reactive component connected either in series or parallel with another network, whose transfer function is the simplified transfer function obtained in the given step. The simplified transfer function matrix obtained in every step becomes the starting point for synthesis in the next step. Figure 6.1 shows the i^{th} step of a synthesis process. With reference to

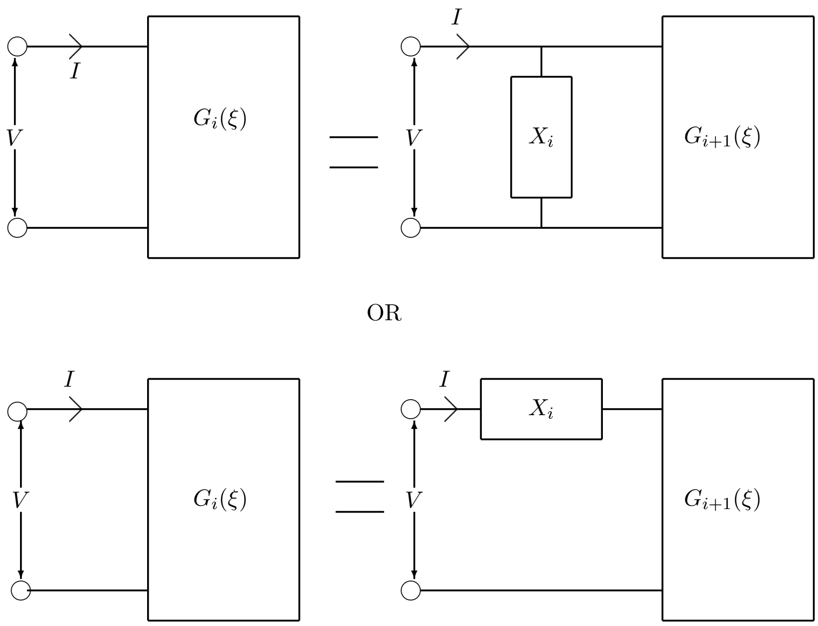

Figure 6.1: i^{th} step of a synthesis

this figure, $G_i, G_{i+1} \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ represent the given transfer function in the i^{th} step and simplified transfer function obtained in the i^{th} step respectively, and X_i represents the reactance component extracted in the i^{th} step.

In lossless systems, since there is no dissipation, at each step of Cauer or Foster synthesis, the total energy of the given system is the sum of the energy stored in the extracted reactance component and the total energy of the network with the simplified transfer function obtained in that step. Hence the problem of synthesis of a given transfer function matrix can be viewed as that of synthesizing the total energy function of the network, wherein we obtain a sequence of energy functions associated with every step of the synthesis, each corresponding to the total energy of the network to be synthesized in that particular step. It should be noted that the total energy of a network is positive for all nonzero values of the external variables for the network, namely the voltage across and current through it. With this as the point of view, we present an abstract definition for synthesis of positive QDFs. We then show that synthesis of a lossless positive real transfer function involves the synthesis of the total energy function of the network corresponding to the given transfer function. The main aim of this chapter is to give an abstract definition for synthesis of positive QDFs that encompasses Cauer and Foster methods of synthesis. Later on in this chapter, we will show that this abstract definition of synthesis also has applications in stability tests of autonomous behaviours. We will also show that there is a close connection between the idea of displacement structure, discussed in Kailath and Sayed (1995) and our abstract definition of synthesis.

6.2 Synthesis of positive QDFs

We begin with the definition of diagonalization and synthesis of a positive QDF.

Definition 6.1 (Diagonalization of a positive QDF)**.** *A* diagonalization *of a positive QDF* Q_{Φ_0} ($\Phi_0 \in \mathbb{R}_s^{l\times l}[\zeta,\eta]$) *is a sequence of QDFs* $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$*, such that,*

1. *For* $i = 0, \ldots, m-1$, $\Phi_{i+1} \in \mathbb{R}_s^{l\times l}[\zeta,\eta]$ *and* $\Phi_m(\zeta,\eta) = 0$
2. *For* $i = 0, \ldots, m-1$,

$$Q_{\Phi_i} \geq Q_{\Phi_{i+1}} \tag{6.1}$$

Definition 6.2 (Synthesis of a positive QDF)**.** *A* synthesis *of a positive QDF* Q_{Φ_0} ($\Phi_0 \in \mathbb{R}_s^{l\times l}[\zeta,\eta]$) *is a sequence of QDFs* $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$*, such that,*

1. $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$ *is a diagonalization of* Q_{Φ_0}.
2. *For the sequence* $\{\Psi_0, \Psi_1, \ldots, \Psi_m\}$*, where* $\Psi_i(\zeta,\eta) := (\zeta+\eta)\Phi_i(\zeta,\eta)$*, the following properties hold:*
   - $\mathtt{n}(\Psi_i) > \mathtt{n}(\Psi_{i+1}) > 0$ *for* $i = 0, \ldots, m-2$.
   - *For* $i = 0, \ldots, m-2$, Ψ_i *and* Ψ_{i+1} *have the same signature.*

The first subitem under item 2 of Definition 6.2 says that the McMillan degree of Ψ_i reduces as i increases. This is analogous to saying that the given QDF $Q_{\Phi_{i+1}}$ at the $(i+1)^{\text{th}}$ step of synthesis is "simpler" than the QDF Q_{Φ_i} at the i^{th} step of synthesis in the sense that the degree of the highest degree element of Φ_{i+1} is less than that of Φ_i. Later, we will show that equation (6.1) can be interpreted as an energy balance equation at the i^{th} step of synthesis. The second subitem under item 2 of Definition 6.2 ensures that the inertia of Ψ_i remains the same for any i.

In the next Lemma, we prove that item 2 of Definition 6.1 leads to a diagonalization of the two-variable polynomial matrix corresponding to the QDF to be synthesized.

Lemma 6.3. *Let* $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$ *be a synthesis of a given positive QDF* Q_{Φ_0} ($\Phi_0 \in \mathbb{R}^{l\times l}[\zeta,\eta]$)*. For* $i = 0, \ldots, m-1$*, define*

$$Q_{\Lambda_i} := Q_{\Phi_i} - Q_{\Phi_{i+1}}. \tag{6.2}$$

Let $D_i \in \mathbb{R}^{\bullet\times l}[\xi]$ *be such that* $\Lambda_i(\zeta,\eta) = D_i(\zeta)^\top D_i(\eta)$*. Then for* $i = 0, \ldots, m-1$, $Q_{\Phi_i} \geq 0$*. Further, for* $i = 0, \ldots, m-1$, $Q_{\Phi_i} > 0$ *if and only if* $col_{k=i}^{m-1}\big(D_k(\lambda)\big)$ *has full column rank for all* $\lambda \in \mathbb{C}$*.*

Proof. From equation (6.2), for $i \in \{0, \dots, m-1\}$, we have

$$\begin{aligned} Q_{\Phi_i} - Q_{\Phi_{i+1}} &= Q_{\Lambda_i} \\ Q_{\Phi_{i+1}} - Q_{\Phi_{i+2}} &= Q_{\Lambda_{i+1}} \\ &\vdots \\ Q_{\Phi_{m-1}} - Q_{\Phi_m} &= Q_{\Lambda_{m-1}} \end{aligned}$$

We have $\Phi_m(\zeta, \eta) = 0$. Adding the above equations, we get

$$Q_{\Phi_i} = \sum_{k=i}^{m-1} Q_{\Lambda_k}$$

Hence $\Phi_i(\zeta, \eta) = B_i(\zeta)^\top B_i(\eta)$, where

$$B_i = \mathrm{col}_{k=i}^{m-1}(D_k)$$

Consequently $Q_{\Phi_i} \geq 0$ for $i = 0, \dots, m-1$. From Proposition 3.14, it follows that $Q_{\Phi_i} > 0$ iff $\mathrm{col}_{k=i}^{m-1}\big(D_k(\lambda)\big)$ has full column rank for all $\lambda \in \mathbb{C}$. ∎

Remark 6.4. It should be noted that there are several methods of diagonalization of positive QDFs. For example, by Cholesky decomposition of the coefficient matrix of a positive QDF Q_{Φ_0} ($\Phi_0 \in \mathbb{R}_s^{l \times l}[\zeta, \eta]$), one can obtain a diagonalization of Q_{Φ_0}. However not all methods of diagonalization lead to synthesis of the given positive QDF. For example, consider the two-variable polynomial $\Phi_0(\zeta, \eta) = \zeta^2\eta^2 + \zeta\eta + 1$. It is easy to see that $Q_{\Phi_0} > 0$. Now consider the sequence of QDFs $\{Q_{\Phi_1}, Q_{\Phi_2}, Q_{\Phi_3}\}$, where $\Phi_1(\zeta, \eta) = \zeta\eta + 1$, $\Phi_2(\zeta, \eta) = 1$, and $\Phi_3(\zeta, \eta) = 0$. This sequence is obtained by a diagonalization of Φ_0 and is a diagonalization of Q_{Φ_0}. However, it is not a synthesis of Q_{Φ_0} according to Definition 6.2, because Ψ_0 and Ψ_1 given by $\Psi_0(\zeta, \eta) := (\zeta + \eta)\Phi_0(\zeta, \eta)$ and $\Psi_1(\zeta, \eta) := (\zeta + \eta)\Phi_1(\zeta, \eta)$ have different signatures.

Remark 6.5. Observe that synthesis of a QDF Q_Φ leads to the diagonalization of the coefficient matrix $\mathrm{mat}(\Phi)$ of Φ. Kailath and Sayed (1995) have showed that triangular factorization of $\mathrm{mat}(\Phi)$, which is one of the methods of diagonalization of $\mathrm{mat}(\Phi)$, also leads to synthesis of Q_Φ under certain conditions. Specifically, in Lemma 7.6, p. 345, they show that if F denotes a lower triangular matrix consisting of real entries, F_1 denotes the submatrix obtained after deleting the first row and first column of F, and $\tilde{\Phi}_1$ denotes the first Schur complement of $\mathrm{mat}(\Phi)$, then the matrices $F\mathrm{mat}(\Phi) + \mathrm{mat}(\Phi)F^\top$, and $F_1\tilde{\Phi}_1 + \tilde{\Phi}_1 F_1^\top$ have the same inertia and signature. Below we explain why this implies that triangular factorization of $\mathrm{mat}(\Phi)$ is a synthesis of Q_Φ under certain conditions.

We first describe the procedure of triangular factorization of a real symmetric positive definite matrix A. Let l_0 and d_0 denote the first column and the first diagonal entry of

A. Assuming that $d_0 \neq 0$, it can be easily verified that $A - l_0 d_0^{-1} l_0^\top$ has its first column and first row full of zeroes. The matrix obtained after deleting the first row and first column of $A - l_0 d_0^{-1} l_0^\top$ is called the *first Schur complement* of A. Denote this matrix by A_1. The first step of triangular factorization of A involves finding the first Schur complement of A. Let l_1 and d_1 denote the first column and the first diagonal entry of A_1. Using l_1 and d_1, the next step of triangular factorization of A involves finding the first Schur complement of A_1, which will be referred to as the *second Schur complement* of A. The i^{th} step of this procedure corresponds to the computation of the i^{th} Schur complement A_i of A using the formula

$$\text{diag}(0, A_i) = A_{i-1} - l_{i-1} d_{i-1}^{-1} l_{i-1}^\top$$

where l_{i-1} and d_{i-1} denote the first column and the first diagonal entry of A_{i-1}. This procedure is continued for n steps, where n is defined as the smallest index such that $A_n = 0$. Using this procedure, we obtain a triangular factorization of A as

$$A = l_0 d_0^{-1} l_0^\top + \text{col}(0, l_1) d_1^{-1} \text{row}(0, l_1^\top) + \text{col}(0, 0, l_2) d_2^{-1} \text{row}(0, 0, l_2^\top) + \ldots = L D^{-1} L^\top$$

where $D = \text{diag}(d_0, d_1, \ldots, d_{n-1})$ and the nonzero parts of the columns of the lower triangular matrix L are $\{l_0, l_1, \ldots, l_{n-1}\}$. The positive definiteness of A ensures that $d_i \neq 0$ for $i = 0, \ldots, n-1$.

Now consider a positive QDF Q_{Φ_0} ($\Phi \in \mathbb{R}_s^{l\times l}[\zeta, \eta]$), and let $D \in \mathbb{R}^{\bullet \times l}[\xi]$ be such that $\Phi_0(\zeta, \eta) = D(\zeta)^\top D(\eta)$. Let n denote the degree of the highest degree polynomial in D. Then we can write $\Phi_0(\zeta, \eta) = X(\zeta)^\top \tilde{\Phi}_0 X(\eta)$, where

$$X(\xi) = \text{col}(I_l, \xi I_l, \xi^2 I_l, \ldots, \xi^n I_l)$$

and $\tilde{\Phi}_0 \in \mathbb{R}_s^{(n+1)l\times(n+1)l}$ comprises of all the nonzero entries of the coefficient matrix of Φ and some zero entries. Assume that $\tilde{\Phi}_0$ is positive definite. Then, we can carry out a triangular factorization of $\tilde{\Phi}_0$. Define $m := (n+1)l$. Let $\{\tilde{\Phi}_1', \tilde{\Phi}_2', \ldots, \tilde{\Phi}_{m-1}'\}$ denote the set of Schur complements of $\tilde{\Phi}_0$. For $i = 1, \ldots, m-1$, define $\tilde{\Phi}_i = \text{diag}(0_{i\times i}, \tilde{\Phi}_i')$. Define $\tilde{\Phi}_m := 0_{l\times l}$. For $i = 0, \ldots, m$, define $\Phi_i(\zeta, \eta) = X(\zeta)^\top \tilde{\Phi}_i X(\eta)$. It is easy to see that $Q_{\Phi_i} - Q_{\Phi_{i+1}} \geq 0$ for $i = 0, \ldots, m-1$.

For $i = 0, \ldots, m-1$, define $\Psi_i(\zeta, \eta) = (\zeta + \eta)\Phi_i(\zeta, \eta)$. Let $F \in \mathbb{R}^{(n+2)l\times(n+2)l}$ be a lower diagonal matrix with 1's on the l^{th} subdiagonal and zeroes elsewhere. Define $\tilde{\Psi}_i := F \text{diag}(\tilde{\Phi}_i, 0_{l\times l}) + \text{diag}(\tilde{\Phi}_i, 0_{l\times l}) F^\top$,

$$X_1(\xi) := \text{col}(I_l, \xi I_l, \xi^2 I_l, \ldots, \xi^{n+1} I_l)$$

Now observe that $\Psi_i(\zeta, \eta) = X_1(\zeta)^\top \tilde{\Psi}_i X_1(\eta)$. From Lemma 7.6, p. 345 of Kailath and Sayed (1995), it follows that for $i = 0, \ldots, m-2$, the signatures of Ψ_i and Ψ_{i+1} are the same. If in addition for $i = 0, \ldots, m-2$, we have $\mathtt{n}(\Psi_i) > \mathtt{n}(\Psi_{i+1})$, then all conditions of Definition 6.2 are obeyed. This implies that under certain conditions, the triangular factorization of the coefficient matrix of a positive QDF leads to its synthesis. Specifically, when $l = 1$, observe that the triangular factorization of the coefficient matrix of a positive QDF leads to its synthesis.

In their paper, Kailath and Sayed (1995) refer to $FA + AF^\top$ as a *displacement* of the matrix A and denote it with ∇A. The rank of the matrix ∇A is referred to the *displacement rank* of A with respect to the displacement $FA+AF^\top$. If the displacement rank of A is less than the dimension of A, then A is said to be *structured* with respect to the given displacement. Kailath and Sayed (1995) also apply the theory of displacement structure to arrive at a Routh-Hurwitz type of stability test in pp. 364-365 of their paper.

We will now show that synthesis of a positive QDF Q_{Φ_0} leads to the generation of a sequence of behaviours $\{\mathfrak{B}_i\}_{i=0,\dots,m}$ with some special set of properties. We also show that Cauer and Foster methods of synthesis of electrical networks also lead to a sequence of behaviours with the same special set of properties.

Theorem 6.6. *Let $\{Q_{\Phi_1}, Q_{\Phi_2}, \dots, Q_{\Phi_m}\}$ be a synthesis of a given positive QDF Q_{Φ_0} ($\Phi_0 \in \mathbb{R}_s^{l\times l}[\zeta,\eta]$). For $i = 0,\dots,m-1$, define $\Psi_i(\zeta,\eta) := (\zeta+\eta)\Phi_i(\zeta,\eta)$, $\Sigma := \Sigma_{\Psi_i}$ and let $M_i(\zeta)^\top \Sigma M_i(\eta)$ with $M_i \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$, be a canonical factorization of $\Psi_i(\zeta,\eta)$. Define $\mathfrak{B}_i := Im\big(M_i(\frac{d}{dt})\big)$ for $i = 0,\dots,m-1$. Let $\mathfrak{B}_m$ be any behaviour such that $Q_\Sigma(w_m) = 0$ for any trajectory $w_m \in \mathfrak{B}_m$. For $i = 0,\dots,m-1$, define*

$$\mathcal{B}_i := \{v_i \mid \exists w_i \in \mathfrak{B}_i, \text{ such that } w_i(0) = v_i\}.$$

Then for $i = 0,\dots,m-1$,

1. *$\mathcal{B}_i$ is a linear space and has dimension equal to $\mathtt{w}$.*
2. *There exists $P_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ such that $\mathfrak{B}_{i+1} = Im\big(P_i(\frac{d}{dt})\big)_{|\mathfrak{B}_i}$.*
3. *If $i \neq m-1$, then $\mathtt{n}(\mathfrak{B}_i) > \mathtt{n}(\mathfrak{B}_{i+1}) > 0$.*
4. *There exists a nonnegative QDF Q_{Ω_i} ($\Omega_i \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\xi]$), such that any two trajectories $w_i \in \mathfrak{B}_i$ and $w_{i+1} \in \mathfrak{B}_{i+1}$ such that $w_{i+1} = P_i(\frac{d}{dt})w_i$ obey*

$$Q_\Sigma(w_i) - \frac{d}{dt}Q_{\Omega_i}(w_i) = Q_\Sigma(w_{i+1}) \tag{6.3}$$

Proof. By definition of $\mathfrak{B}_i$, for $i = 0,\dots,m-1$, we have $\mathtt{n}(\Psi_i) = \mathtt{n}(\mathfrak{B}_i)$. Therefore, from Definition 6.2, for $i = 0,\dots,m-2$, it follows that $\mathtt{n}(\mathfrak{B}_i) > \mathtt{n}(\mathfrak{B}_{i+1}) > 0$. Since $M_i(\zeta)^\top \Sigma M_i(\eta)$ is a canonical factorization of $\Psi_i(\zeta,\eta)$ for $i = 0,\dots,m-1$, the rows of M_i are linearly independent over $\mathbb{R}$. We now prove using the following lemma that for $i = 0,\dots,m-1$, $\mathfrak{B}_i$ has dimension equal to $\mathtt{w}$.

Lemma 6.7. *Let* $\mathfrak{B} = Im\big(M(\frac{d}{dt})\big)$, *where* $M \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$. *Define*

$$\mathcal{B} := \{v \mid \exists w \in \mathfrak{B},\ \textit{such that}\ w(0) = v\}.$$

Then

1. $\mathcal{B}$ *is a linear space.*
2. $\mathcal{B}$ *has dimension equal to* $\mathtt{w}$ *iff the rows of* $M(\xi)$ *are linearly independent over* $\mathbb{R}$.

Proof. We first prove that $\mathcal{B}$ is a linear space. Let $v_1, v_2 \in \mathcal{B}$. Then there exist trajectories $w_1, w_2 \in \mathfrak{B}$, such that $w_1(0) = v_1$ and $w_2(0) = v_2$. By definition, for any $\alpha, \beta \in \mathbb{R}$, $v_3 := \alpha v_1 + \beta v_2 = \alpha w_1(0) + \beta w_2(0) \in \mathcal{B}$, because $\alpha w_1 + \beta w_2 \in \mathfrak{B}$. This proves that $\mathfrak{B}$ is a linear space over $\mathbb{R}$.

We now prove item 2 of the lemma. From the statement of the lemma, for any trajectory $w \in \mathfrak{B}$,

$$w = M(\frac{d}{dt})\ell$$

for some trajectory $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l)$. Write M as

$$M(\xi) = M_0 + M_1\xi + M_2\xi^2 + \ldots + M_k\xi^k$$

where $M_q \in \mathbb{R}^{\mathtt{w}\times l}$ for $q = 0, \ldots, k$, and k denotes the degree of the highest degree polynomial in M. We have

$$w(0) = M_0\ell(0) + M_1\ell^{(1)}(0) + M_2\ell^{(2)}(0) + \ldots + M_k\ell^{(k)}(0)$$

where for $q = 1, \ldots, k$, $\ell^{(q)}(0)$ denotes the q^{th} derivative of ℓ at time $t = 0$.

(*If*): Assume that the rows of M are linearly independent over $\mathbb{R}$. Assume by contradiction that $\mathcal{B}$ has dimension less than $\mathtt{w}$. This implies that there exists a nonzero $C \in \mathbb{R}^{\mathtt{w}}$, such that $C^\top w(0) = 0$ for all $w \in \mathfrak{B}$. Since ℓ can be chosen arbitrarily, this implies that $C^\top M_q = 0$ for $q = 0, \ldots, k$. Consequently $C^\top M(\xi) = 0$, which implies that the rows of M are linearly dependent over $\mathbb{R}$. This is a contradiction. Hence, $\mathcal{B}$ has dimension equal to $\mathtt{w}$.

(*Only if*): Assume that $\mathcal{B}$ has dimension equal to $\mathtt{w}$. Now assume by contradiction that the rows of M are linearly dependent over $\mathbb{R}$. Then there exists a nonzero $C \in \mathbb{R}^{\mathtt{w}}$, such that $C^\top M(\xi) = 0$. Since ξ is indeterminate, this implies that $C^\top M_q = 0$ for $q = 0, \ldots, k$, or $C^\top w(0) = 0$. Consequently $\mathcal{B}$ has dimension less than $\mathtt{w}$, which is a contradiction. Hence the rows of M are linearly independent over $\mathbb{R}$. ∎

Since for $i = 0, \ldots, m-1$, the rows of M_i are linearly independent over $\mathbb{R}$, from the above lemma, it follows that $\mathcal{B}_i$ is a linear space and has dimension equal to $\mathtt{w}$.

We now prove items 2 and 4 of the theorem. In order to do this, for $i = 0, \ldots, m-1$, we need to obtain an observable image representation of $\mathfrak{B}_i$. For $i = 0, \ldots, m-1$, define $Q_{\Lambda_i} := Q_{\Phi_i} - Q_{\Phi_{i+1}}$. Let $D_i \in \mathbb{R}^{\bullet \times l}[\xi]$ be such that $\Lambda_i(\zeta, \eta) = D_i(\zeta)^\top D_i(\eta)$. From the proof of Lemma 6.3, it follows that for $i = 0, \ldots, m-1$, $\Phi_i(\zeta, \eta) = B_i(\zeta)^\top B_i(\eta)$, where

$$B_i = \text{col}_{k=i}^{m-1}(D_k) = \text{col}(D_i, B_{i-1})$$

Denote by $\mathtt{b}_i$, the number of rows of B_i. Consider a Smith form decomposition of B_i given by

$$B_i = U_i \begin{bmatrix} \Delta_i & 0_{l_i \times (l-l_i)} \\ 0_{(\mathtt{b}_i - l_i) \times l_i} & 0_{(\mathtt{b}_i - l_i) \times (l-l_i)} \end{bmatrix} V_i \tag{6.4}$$

where $U_i \in \mathbb{R}^{\mathtt{b}_i \times \mathtt{b}_i}[\xi]$, $V_i \in \mathbb{R}^{l \times l}[\xi]$ and $\Delta_i \in \mathbb{R}^{l_i \times l_i}[\xi]$ has nonzero diagonal entries. Consider partitions of U_i and V_i given by

$$U_i = \begin{bmatrix} U_{i1} & U_{i2} \end{bmatrix}; \qquad V_i = \begin{bmatrix} V_{i1} \\ V_{i2} \end{bmatrix}$$

where $U_{i1} \in \mathbb{R}^{\mathtt{b}_i \times l_i}[\xi]$, $U_{i2} \in \mathbb{R}^{\mathtt{b}_i \times (\mathtt{b}_i - l_i)}[\xi]$, $V_{i1} \in \mathbb{R}^{l_i \times l}[\xi]$ and $V_{i2} \in \mathbb{R}^{(l-l_i) \times l}[\xi]$. Then, from equation (6.4), we have $B_i = U_{i1} \Delta_i V_{i1}$. Note that $U_{i1}(\lambda)$ has full column rank and $V_{i1}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Define $G_i(\xi) := \Delta_i(\xi) V_{i1}(\xi)$. Consider a partition of U_{i1} given by

$$U_{i1} = \begin{bmatrix} U_{ti} \\ U_{di} \end{bmatrix}$$

where $U_{ti} \in \mathbb{R}^{(\mathtt{b}_i - \mathtt{b}_{i+1}) \times l_i}[\xi]$ and $U_{di} \in \mathbb{R}^{\mathtt{b}_{i+1} \times l_i}[\xi]$. Then

$$B_i = \begin{bmatrix} U_{ti} G_i \\ U_{di} G_i \end{bmatrix} = \begin{bmatrix} D_i \\ B_{i+1} \end{bmatrix}$$

Consequently $D_i = U_{ti} G_i$ and $B_{i+1} = U_{di} G_i$. Let $M_i'(\zeta)^\top \Sigma_1 M_i'(\eta)$ be a canonical factorization of the two-variable polynomial matrix $(\zeta + \eta) U_{i1}(\zeta)^\top U_{i1}(\eta)$, where $M_i' \in \mathbb{R}^{\mathtt{m} \times l_i}[\xi]$. Then

$$\Psi_i(\zeta, \eta) = G_i(\zeta)^\top M_i'(\zeta)^\top \Sigma_1 M_i'(\eta) G_i(\eta) \tag{6.5}$$

We know that the rows of M_i' are linearly independent over $\mathbb{R}$. We now prove that the rows of $M_i' G_i$ are also linearly independent over $\mathbb{R}$. From this fact, we will be able to conclude that the right side of equation (6.5) is a canonical factorization of $\Psi_i(\zeta, \eta)$. Note that

$$M_i' G_i = M_i' \Delta_i V_{i1}$$

Define $M_i''(\xi) := M_i'(\xi) \Delta_i(\xi)$. We first prove that the rows of M_i'' are linearly independent over $\mathbb{R}$. Let $\delta_{ik}(\xi)$ be the k^{th} diagonal entry of $\Delta_i(\xi)$. Write

$$M_i' = \text{row}_{k=1}^{l_i}(M_{ik})$$

where $M_{ik} \in \mathbb{R}^{\mathtt{m}\times 1}[\xi]$ is the k^{th} column of M_i'. Then

$$M_i'' = \text{row}_{k=1}^{l_i}\left(M_{ik}\delta_{ik}\right)$$

Let $M_{ikp}(\xi)$ denote the p^{th} entry of $M_{ik}(\xi)$. For $p = 1,\ldots,\mathtt{m}$, assume by contradiction that there exist nonzero $a_p \in \mathbb{R}$, such that

$$\sum_{p=1}^{\mathtt{m}} a_p M_{ikp}(\xi)\delta_{ik}(\xi) = \left(\sum_{p=1}^{\mathtt{m}} a_p M_{ikp}(\xi)\right)\delta_{ik}(\xi) = 0$$

for $k = 1,\ldots,l_i$. Since $\delta_{ik}(\xi) \neq 0$, the above implies that

$$\sum_{p=1}^{\mathtt{m}} a_p M_{ikp}(\xi) = 0$$

for $k = 1,\ldots,l_i$. This in turn implies that $a_p = 0$ for $p = 1,\ldots,\mathtt{m}$, since the rows of M_i' are linearly independent over $\mathbb{R}$. This is a contradiction. Consequently, it follows that the rows of M_i'' are linearly independent over $\mathbb{R}$. We now prove that the rows of $M_i''V_{1i}$ are also linearly independent over $\mathbb{R}$. Let $M_{ik}'(\xi)$ denote the k^{th} row of $M_i''(\xi)$. For $p = 1,\ldots,\mathtt{m}$, assume by contradiction that there exist nonzero $b_p \in \mathbb{R}$, such that

$$\sum_{p=1}^{\mathtt{m}} b_p M_{ip}'(\xi)V_{i1}(\xi) = \left(\sum_{p=1}^{\mathtt{m}} b_p M_{ip}'(\xi)\right)V_{i1}(\xi) = 0$$

The above implies that $\sum_{p=1}^{\mathtt{m}} b_p M_{ip}'(\xi)$ is in the left annihilator space of $V_{i1}(\xi)$, which is zero as $V_{i1}(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Thus

$$\sum_{p=1}^{\mathtt{m}} b_p M_{ip}'(\xi) = 0$$

which implies that $b_p = 0$ for $p = 1,\ldots,\mathtt{m}$ as the rows of M_i'' are linearly independent over $\mathbb{R}$. This is a contradiction. Consequently, the rows of $M_{i1}(\xi) := M_i'(\xi)G_i(\xi)$ are linearly independent over $\mathbb{R}$, which implies that its coefficient matrix is surjective. Thus $M_{i1}(\zeta)^\top\Sigma_1 M_{i1}(\eta)$ is a canonical factorization of $\Psi_i(\zeta,\eta)$. Since $M_i(\zeta)^\top\Sigma M_i(\eta)$ is also a canonical factorization of $\Psi_i(\zeta,\eta)$, it follows that $\Sigma_1 = \Sigma$ and $M_i(\xi) = UM_{i1}(\xi)$, where $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is such that $U^\top\Sigma U = \Sigma$. Taking $U = I_{\mathtt{w}}$, we get $M_i(\xi) = M_{i1}(\xi)$.

Assuming that $i \neq m-1$, let $S_i(\zeta)^\top\Sigma_2 S_i(\eta)$ be a canonical factorization of the two-variable polynomial matrix $(\zeta+\eta)U_{di}(\zeta)^\top U_{di}(\eta)$, where $S_i \in \mathbb{R}^{\mathtt{s}\times l_i}[\xi]$. Then

$$\Psi_{i+1}(\zeta,\eta) = (\zeta+\eta)G_i(\zeta)^\top U_{di}(\zeta)^\top U_{di}(\eta)G_i(\eta) = G_i(\zeta)^\top S_i(\zeta)^\top\Sigma_2 S_i(\eta)G_i(\eta) \qquad (6.6)$$

Since the rows of S_i are linearly independent over $\mathbb{R}$, it can be proved as was proved in the case of M_i', that the rows of S_iG_i are also linearly independent over $\mathbb{R}$. Hence the right hand side of equation (6.6) is a canonical factorization of $\Psi_{i+1}(\zeta,\eta)$, which implies that $\Sigma_2 = \Sigma$ since $M_{i+1}(\zeta)^\top \Sigma M_{i+1}(\eta)$ is also a canonical factorization of $\Psi_{i+1}(\zeta,\eta)$. This also implies that $M_{i+1}(\xi) = US_i(\xi)G_i(\xi)$, where $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is such that $U^\top \Sigma U = \Sigma$. Taking $U = I_{\mathtt{w}}$, we get $M_{i+1} = S_i G_i$. For the case, $i = m-1$, choose $S_i \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$, such that $S_i(\zeta)^\top \Sigma S_i(\eta) = 0$. Define $M_m(\xi) := S_{m-1}(\xi)G_{m-1}(\xi)$. Observe that $M_m(\zeta)^\top \Sigma M_m(\eta) = 0$.

We now make use of the following lemma to prove that $M_i'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

Lemma 6.8. *Consider a positive QDF Q_Φ $(\Phi \in \mathbb{R}^{l\times l})[\zeta,\eta]$. Let $M(\zeta)^\top \Sigma M(\eta)$ $(M \in \mathbb{R}^{\mathtt{w}\times l}[\xi], \Sigma \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}})$ be a factorization of $\Psi(\zeta,\eta) := (\zeta+\eta)\Phi(\zeta,\eta)$. Then $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.*

Proof. Let $D \in \mathbb{R}^{\bullet\times l}[\xi]$ be such that $\Phi(\zeta,\eta) = D(\zeta)^\top D(\eta)$. Since $Q_\Phi > 0$, $D(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Observe that

$$D(\zeta)^\top D(\eta) = \frac{M(\zeta)^\top \Sigma M(\eta)}{\zeta+\eta}$$

Now assume by contradiction that $M(\lambda)$ loses column rank for some $\lambda \in \mathbb{C}$. Then there exists a $v_\lambda \in \mathbb{C}^l$, such that $M(\lambda)v_\lambda = 0$ which implies that $M(\xi)v_\lambda$ is divisible by $(\xi-\lambda)$. Hence $v_\lambda^\top \left(\frac{M(\zeta)^\top \Sigma M(\eta)}{\zeta+\eta}\right) v_\lambda$ is divisible by $(\zeta-\lambda)(\eta-\lambda)$, which implies that $v_\lambda^\top D(\zeta)^\top D(\eta)v_\lambda$ is divisible by $(\zeta-\lambda)(\eta-\lambda)$. This implies that $D(\lambda)$ loses rank for some $\lambda \in \mathbb{C}$. Hence by contradiction, we get that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. ∎

Since $Q_{\Gamma_i} > 0$, where $\Gamma_i(\zeta,\eta) = U_{i1}(\zeta)^\top U_{i1}(\eta)$, from the above Lemma it follows that $M_i'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Now, we know that

$$\begin{aligned} M_i &= M_i' G_i \\ M_{i+1} &= S_i G_i \end{aligned}$$

For a trajectory $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^l)$, if we define

$$\begin{aligned} \ell' : &= G_i(\frac{d}{dt})\ell \\ w_i : &= M_i(\frac{d}{dt})\ell = M_i'(\frac{d}{dt})G_i(\frac{d}{dt})\ell \\ w_{i+1} : &= M_{i+1}(\frac{d}{dt})\ell = S_i(\frac{d}{dt})G_i(\frac{d}{dt})\ell, \end{aligned}$$

then by definition, $w_i \in \mathfrak{B}_i$ and $w_{i+1} \in \mathfrak{B}_{i+1}$. Since $M_i'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, ℓ' is observable from w_i. Hence there exists a $F_i \in \mathbb{R}^{l_i\times\mathtt{w}}[\xi]$, such that

$\ell' = F_i(\frac{d}{dt})w_i$. Define $P_i(\xi) := S_i(\xi)F_i(\xi)$. We have

$$w_{i+1} = S_i(\frac{d}{dt})\ell' = S_i(\frac{d}{dt})F_i(\frac{d}{dt})w_i = P_i(\frac{d}{dt})w_i$$

Now from Definition 6.2, it follows that there exists a nonnegative Q_{Λ_i} ($\Lambda_i \in \mathbb{R}_s^{l\times l}[\zeta, \eta]$), such that

$$Q_{\Phi_i}(\ell) - Q_{\Phi_{i+1}}(\ell) = Q_{\Lambda_i}(\ell)$$

Differentiating the above equation with respect to time, we get

$$Q_{\Psi_i}(\ell) - Q_{\Psi_{i+1}}(\ell) = \frac{d}{dt}Q_{\Lambda_i}(\ell) \tag{6.7}$$

Define $T_i(\zeta, \eta) := U_{ti}(\zeta)^\top U_{ti}(\eta)$. It is easy to see that equation (6.7) implies

$$Q_\Sigma\left(M_i'\left(\frac{d}{dt}\right)\ell'\right) - Q_\Sigma\left(S_i\left(\frac{d}{dt}\right)\ell'\right) = \frac{d}{dt}Q_{T_i}(\ell')$$

Hence

$$Q_\Sigma(w_i) - Q_\Sigma(w_{i+1}) = \frac{d}{dt}Q_{T_i}\left(F_i\left(\frac{d}{dt}\right)w_i\right)$$

Define $\Omega_i(\zeta, \eta) := F_i(\zeta)^\top T_i(\zeta, \eta)F_i(\eta)$. Then

$$Q_\Sigma(w_i) - Q_\Sigma(w_{i+1}) = \frac{d}{dt}Q_{\Omega_i}(w_i) \tag{6.8}$$

Since G_i has full row rank, from Lemma B.14, Appendix B, it follows that the mapping $\ell \rightarrow G_i(\frac{d}{dt})\ell$ is surjective, which in turn implies that $\ell' = G_i(\frac{d}{dt})\ell$ is a free trajectory. Now we have $w_{i+1} = P_i(\frac{d}{dt})w_i$. Observe that for any $w_{i+1} \in \mathfrak{B}_{i+1}$, we have a trajectory ℓ', such that $w_{i+1} = S_i(\frac{d}{dt})\ell'$, and for any given ℓ', we have a trajectory $w_i \in \mathfrak{B}_i$, such that $w_i = M_i'(\frac{d}{dt})\ell'$. Hence $\mathfrak{B}_{i+1} = \mathrm{Im}\big(P_i(\frac{d}{dt})\big)_{|\mathfrak{B}_i}$. For any two trajectories $w_{i+1} \in \mathfrak{B}_{i+1}$ and $w_i \in \mathfrak{B}_i$ that are related by $w_{i+1} = P_i(\frac{d}{dt})w_i$, we have already proved that equation (6.8) holds. It is easy to see that in this equation, Q_{Ω_i} is nonnegative. This concludes the proof. ■

Remark 6.9. It can be seen from Theorem 6.6 that synthesis of a positive QDF leads to the generation of a sequence of behaviours that obey certain interesting properties. We now discuss the similarities between these properties and the properties of the networks that are synthesized at every step of Cauer and Foster synthesis. One of these properties is that the McMillan degree of the behaviours decreases along the sequence. This suggests that the behaviours become "simpler" in a certain sense along the sequence. As discussed at the beginning of this chapter, in Cauer and Foster methods of synthesis, at each step a reactive component is extracted, and a simpler network is left to be synthesized in the next step. A second important property of these behaviours corresponds to equation (6.3) of Theorem 6.6, which is given below:

$$Q_\Sigma(w_i) - \frac{d}{dt}Q_{\Omega_i}(w_i) = Q_\Sigma(w_{i+1})$$

With reference to the above equation if $Q_\Sigma(w_i)$ and $\frac{d}{dt}Q_{\Omega_i}(w_i)$ are interpreted respectively as the power or energy supply rate to the network to be synthesized at the i^{th} step and the supply rate to the reactive element that is extracted at the i^{th} step of the synthesis procedure, then equation (6.3) says that the energy supply rates to the network to be synthesized at the $(i+1)^{\text{th}}$ step of synthesis is equal to the difference between energy supply rate to the network to be synthesized at the i^{th} step and the reactive element that is extracted at the i^{th} step of synthesis. Thus equation (6.3) can be interpreted as an energy balance equation at the i^{th} step of the synthesis procedure.

6.3 Synthesis of J-lossless behaviours

Motivated by Theorem 6.6, we now give a definition for the synthesis of behaviours. In order to do this, we need to first define the concept of J-nonnegative losslessness and J-losslessness.

Definition 6.10 (J-nonnegative lossless behaviour)**.** *Let $J \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}$ be a matrix whose entries are either 1 or 0. A controllable behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ is said to be* J-nonnegative lossless *if*

$$\mathcal{B} := \{v \mid \exists w \in \mathfrak{B}, \text{ such that } w(0) = v\}$$

has dimension equal to w *and there exists a QDF $Q_E \overset{\mathfrak{B}}{\geq} 0$ with $E \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that for every trajectory $w \in \mathfrak{B}$, $Q_J(w) = \frac{d}{dt}Q_E(w)$. We call Q_E the* energy function *of $\mathfrak{B}$.*

Definition 6.11 (J-lossless behaviour)**.** *A J-nonnegative lossless behaviour $\mathfrak{B}$ is called* J-lossless *if its energy function is positive over $\mathfrak{B}$.*

We now give an example of a J-lossless behaviour.

Example 6.1. Consider a lossless electrical network consisting of an inductance L, a capacitance C connected in series with a voltage source V as shown in Figure 6.2. Let I denote the current through the circuit. Define $\mathfrak{B}_e$ as the space of all admissible trajectories $(V, I) : \mathbb{R} \to \mathbb{R}^2$ of the system. The governing differential equation for the system is given by

$$C\frac{dV}{dt} = I + LC\frac{d^2I}{dt^2} \tag{6.9}$$

Define

$$\mathcal{B}_e := \{v \mid \exists w \in \mathfrak{B}_e \text{ such that } w(0) = v\}$$

Define $M_e(\xi) := \text{col}(LC\xi^2 + 1, C\xi)$. Observe that $\mathfrak{B}_e = \text{Im}\big(M_e(\frac{d}{dt})\big)$. Since the rows of M_e are linearly independent over $\mathbb{R}$, from Lemma 6.7, it follows that $\mathcal{B}_e$ has dimension equal to 2. It can be verified that equation (6.9) implies

$$2VI = \frac{d}{dt}\big(C(V - L\frac{dI}{dt})^2 + LI^2\big) \tag{6.10}$$

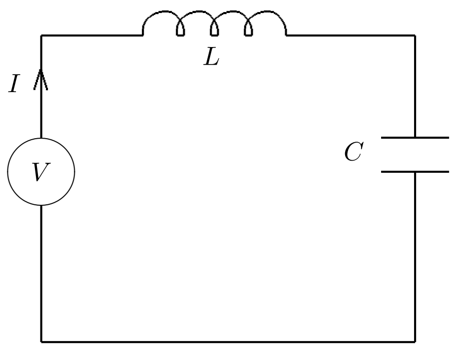

FIGURE 6.2: Example 6.1

It can also be verified that $\mathfrak{B}_e$ is controllable. Now define

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Define the QDF Q_E ($E \in \mathbb{R}_s^{2\times 2}[\zeta,\eta]$) as

$$Q_E(w) := \frac{1}{2}\big(C(V - L\frac{dI}{dt})^2 + LI^2\big)$$

where $w = \mathrm{col}(V, I)$. It is easy to see that $Q_E(w) > 0$ for all nonzero trajectories of the behaviour $\mathfrak{B}_e$, and $Q_E(w) = 0 \Rightarrow w = 0$. Hence from equation (6.10), it follows that $\mathfrak{B}_e$ is J-lossless.

In the next Lemma, we give algebraic conditions on the image representation of a controllable behaviour $\mathfrak{B}$ for it to be J-nonnegative lossless and J-lossless respectively.

Lemma 6.12. *Consider a controllable behaviour $\mathfrak{B} \in \mathcal{L}^{\mathtt{w}}$ for which an observable image representation is $\mathfrak{B} = Im\ \big(M(\frac{d}{dt})\big)$ ($M \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$). $\mathfrak{B}$ is J-nonnegative lossless if and only if the following hold:*

1. *The rows of M are linearly independent over $\mathbb{R}$.*
2. *$M(-\xi)^\top JM(\xi) = 0$.*
3. *$\Phi(\zeta,\eta) := \frac{M(\zeta)^\top JM(\eta)}{\zeta+\eta}$ is such that $Q_\Phi \geq 0$.*

$\mathfrak{B}$ is J-lossless if and only if items (1) and (2) above hold and $Q_\Phi > 0$.

Proof. (*If*): Assume that $M(-\xi)^\top JM(\xi) = 0$ and that the rows of $M(\xi)$ are linearly independent over $\mathbb{R}$. Define

$$\mathcal{B} := \{v \mid \exists w \in \mathfrak{B} \text{ such that } w(0) = v\}$$

From Lemma 6.7, it follows that $\mathcal{B}$ has dimension equal to $\mathtt{w}$. Consider a trajectory $w = M(\frac{d}{dt})\ell$. By assumption, ℓ is observable from w. Consequently, there exists $F \in \mathbb{R}^{l\times \mathtt{w}}[\xi]$, such that $\ell = F(\frac{d}{dt})w$. Now define

$$Q_E(w) := Q_\Phi\left(F\left(\frac{d}{dt}\right)w\right) = Q_\Phi(\ell)$$

Observe that $E(\zeta,\eta) = F(\zeta)^\top \Phi(\zeta,\eta)F(\eta)$. It can be easily verified that $\frac{d}{dt}Q_E(w) = Q_J(w)$. It is easy to see that $Q_\Phi \geq 0$ implies that $Q_E(w) \geq 0$ for all $w \in \mathfrak{B}$. Hence $\mathfrak{B}$ is J-nonnegative lossless if $Q_\Phi \geq 0$. Now assume that $Q_\Phi > 0$. Then it is easy to see that $Q_E(w) > 0$ for any nonzero trajectory $w \in \mathfrak{B}$, and $Q_E(w) = 0$ implies that $w = 0$. Hence $\mathfrak{B}$ is J-lossless if $Q_\Phi > 0$.

(*Only If*): Consider a trajectory $w \in \mathfrak{B}$ given by $w = M(\frac{d}{dt})\ell$. Let $E \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be such that $Q_J(w) = \frac{d}{dt}Q_E(w)$. Define $J'(\zeta,\eta) := M(\zeta)^\top JM(\eta)$ and $\Phi(\zeta,\eta) := M(\zeta)^\top E(\zeta,\eta)M(\eta)$ and observe that

$$Q_{J'}(\ell) = \frac{d}{dt}Q_\Phi(\ell) \tag{6.11}$$

We have from equation (6.11),

$$M(\zeta)^\top JM(\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$$

From the above equation, it follows that $M(-\xi)^\top JM(\xi) = 0$ and $\Phi(\zeta,\eta) = \frac{M(\zeta)^\top JM(\eta)}{\zeta+\eta}$. We have $Q_E(w) = Q_\Phi(\ell)$ for all (w,ℓ), such that $w = M(\frac{d}{dt})\ell$. Now assume that $\mathfrak{B}$ is J-nonnegative lossless with $Q_E \overset{\mathfrak{B}}{\geq} 0$. This implies that $Q_\Phi \geq 0$. From Lemma 6.7, it follows that the rows of M are linearly independent over $\mathbb{R}$. If we assume that $\mathfrak{B}$ is J-lossless with $Q_E \overset{\mathfrak{B}}{>} 0$, then it is easy to see that $Q_\Phi > 0$. Hence the claim. ■

Equipped with the definition of J-losslessness, we now give a definition for synthesis of behaviours.

Definition 6.13 (Synthesis of a J-lossless behaviour). *A sequence of behaviours $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_m\}$ is a synthesis of a J-lossless behaviour $\mathfrak{B}_0 \in \mathcal{L}^{\mathtt{w}}$ if for $i = 0,\ldots,m-1$,*

1. *$\mathcal{B}_i := \{v_i \mid \exists w_i \in \mathfrak{B}_i$ such that $w_i(0) = v_i\}$ has dimension equal to $\mathtt{w}$.*
2. *$\mathfrak{B}_{i+1} = Im\big(P_i(\frac{d}{dt})\big)_{|\mathfrak{B}_i}$, for some $P_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$.*
3. *If $i \neq m-1$, then $\mathtt{n}(\mathfrak{B}_i) > \mathtt{n}(\mathfrak{B}_{i+1}) > 0$; and $Q_J(w_m) = 0$ for any trajectory $w_m \in \mathfrak{B}_m$.*
4. *There exists a nonnegative Q_{Ω_i} with $\Omega_i \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\xi]$, such that any two trajectories $w_i \in \mathfrak{B}_i$ and $w_{i+1} \in \mathfrak{B}_{i+1}$ that are related by $w_{i+1} = P_i(\frac{d}{dt})w_i$ obey*

$$Q_J(w_i) - \frac{d}{dt}Q_{\Omega_i}(w_i) = Q_J(w_{i+1})$$

In the next proposition, we show that synthesis of a J-lossless behaviour is a sequence of J-nonnegative lossless behaviours.

Proposition 6.14. *Let $\{\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_m\}$ be a synthesis of a given J-lossless behaviour $\mathfrak{B}_0 \in \mathcal{L}^{\mathtt{w}}$. Then for $i = 1, \dots, m-1$, $\mathfrak{B}_i$ is J-nonnegative lossless.*

Proof. We prove the Proposition by induction. We first prove that for $i = 1, \dots, m$, $\mathfrak{B}_i$ is controllable. $\mathfrak{B}_0$ is controllable by assumption. Assume that for some $i \in \{0, \dots, m-1\}$, $\mathfrak{B}_i$ is controllable. From Lemma 2.16, it follows that $\mathfrak{B}_{i+1}$ is also controllable. Hence by induction, it follows that $\mathfrak{B}_i$ is controllable for $i = 1, \dots, m$.

For $i = 0, \dots, m-1$, since $\mathfrak{B}_i$ is controllable, we can write an image representation of $\mathfrak{B}_i$ as $\mathfrak{B}_i = \operatorname{Im}(M_i(\frac{d}{dt}))$, where $M_i \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$. From the statement of the theorem, $\mathfrak{B}_0$ is J-lossless, which implies that $M_0(\zeta)^\top J M_0(\eta)$ is divisible by $(\zeta+\eta)$. Assume that $\mathfrak{B}_i$ is J-nonnegative lossless for some $i \in \{0, \dots, m-1\}$. From Definition 6.13, for any trajectory $w_{i+1} \in \mathfrak{B}_{i+1}$, there exists a trajectory $w_i \in \mathfrak{B}_i$ such that $w_{i+1} = P_i(\frac{d}{dt})w_i$, ($P_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$) and

$$Q_J(w_i) - \frac{d}{dt}Q_{\Omega_i}(w_i) = Q_J(w_{i+1})$$

for some nonnegative Q_{Ω_i} ($\Omega_i \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$). Thus

$$\frac{M_{i+1}(\zeta)^\top J M_{i+1}(\eta)}{\zeta+\eta} = \frac{M_i(\zeta)^\top J M_i(\eta)}{\zeta+\eta} - M_i(\zeta)^\top \Omega_i M_i(\eta) \tag{6.12}$$

where $M_{i+1}(\xi) = P_i(\xi)M_i(\xi)$. Observe that since $\mathfrak{B}_i$ is J-nonnegative lossless, the right hand side of equation (6.12) is a two-variable polynomial, hence the numerator of the term on the left hand side of this equation is divisible by $(\zeta+\eta)$. By induction it follows that $M_k(\zeta)^\top J M_k(\eta)$ is divisible by $(\zeta+\eta)$ and consequently $M_k(-\xi)^\top J M_k(\xi) = 0$ for $k = 1, \dots, m$.

Now consider a trajectory $w_i \in \mathfrak{B}_i$ for some $i \in \{1, \dots, m\}$. Define $N_{i,p-1}(\xi) := P_{p-1}(\xi) \dots P_{i+1}(\xi)P_i(\xi)$ for $p \in \{i+1, \dots, m\}$ and $N_{i,i-1}(\xi) := I_{\mathtt{w}}$. For $p = \{i+1, \dots, m\}$, define trajectories $w_p := N_{i,p-1}(\frac{d}{dt})w_i$. Observe that $w_p \in \mathfrak{B}_p$ for $p = \{i+1, \dots, m\}$. From Definition 6.13, it follows that there exist $\Omega_i \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ for $i = 0, \dots, m-1$, such that

$$\begin{aligned}
Q_J(w_i) - Q_J(w_{i+1}) &= \frac{d}{dt}Q_{\Omega_i}(w_i)\\
Q_J(w_{i+1}) - Q_J(w_{i+2}) &= \frac{d}{dt}Q_{\Omega_{i+1}}(w_{i+1})\\
&\vdots\\
Q_J(w_{k-1}) - Q_J(w_m) &= \frac{d}{dt}Q_{\Omega_{m-1}}(w_{m-1})
\end{aligned}$$

and $\Omega_i(\zeta,\eta) = H_i(\zeta)^\top H_i(\eta)$ for some $H_i \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$. Adding the above equations, since $Q_J(w_m) = 0$ for all $w_m \in \mathfrak{B}_m$, we get

$$Q_J(w_i) = \frac{d}{dt}\left(\sum_{p=i}^{m-1} Q_{\Omega_p}(w_p)\right) = \frac{d}{dt}\left(\sum_{p=i}^{m-1} Q_{\Omega_p}(N_{i,p-1}(\frac{d}{dt})w_i)\right) \tag{6.13}$$

Define

$$D_i'(\xi) := \mathrm{col}_{p=1}^{m-1}\big(H_i(\xi)N_{i,p-1}(\xi)\big)$$

and $E_i(\zeta,\eta) := D_i'(\zeta)^\top D_i'(\eta)$. From equation (6.13), it follows that

$$Q_J(w_i) = \frac{d}{dt}Q_{E_i}(w_i)$$

This implies that $Q_{E_i} \overset{\mathfrak{B}_i}{\geq} 0$ for $i = 1,\dots,m-1$, which in turn implies that $\mathfrak{B}_i$ is J-nonnegative lossless for $i = 1,\dots,m-1$. ∎

From the above proposition, it follows that synthesis of a J-lossless behaviour $\mathfrak{B}_0$ gives rise to a sequence $\{\mathfrak{B}_i\}_{i=1,\dots,m}$ of J-nonnegative lossless behaviours. We now give an algebraic condition for the elements of this sequence to be J-lossless.

Corollary 6.15. *Let $\{\mathfrak{B}_1,\mathfrak{B}_2,\dots,\mathfrak{B}_m\}$ be a synthesis of a given J-lossless behaviour $\mathfrak{B}_0 \in \mathcal{L}^{\mathtt{w}}$. Let $P_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ be such that for $i = 0,\dots,m-1$, $\mathfrak{B}_{i+1} = Im\big(P_i(\frac{d}{dt})\big)_{|\mathfrak{B}_i}$. Define $N_{i,p-1}(\xi) := P_{p-1}(\xi)\dots P_{i+1}(\xi)P_i(\xi)$ for $p \in \{i+1,\dots,m\}$ and $N_{i,i-1}(\xi) := I_{\mathtt{w}}$. For $i = 0,\dots,m-1$, let $\Omega_i \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be such that for any trajectory $w_i \in \mathfrak{B}_i$*

$$Q_J(w_i) - Q_J\left(P_i\left(\frac{d}{dt}\right)w_i\right) = \frac{d}{dt}Q_{\Omega_i}(w_i)$$

and let $H_i \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ be such that $\Omega_i(\zeta,\eta) = H_i(\zeta)^\top H_i(\eta)$. For $i = 1,\dots,m-1$, define

$$D_i'(\xi) := col_{p=1}^{m-1}\big(H_i(\xi)N_{i,p-1}(\xi)\big)$$

Let $\mathfrak{B}_i = ker\big(R_i(\frac{d}{dt})\big)$ be a minimal kernel representation of $\mathfrak{B}_i$. Then for $i = 1,\dots,m-1$, $\mathfrak{B}_i$ is J-lossless if and only if $col\big(D_i'(\lambda),R_i(\lambda)\big)$ has full column rank for all $\lambda \in \mathbb{C}$.

Remark 6.16. From Proposition 6.14, we know that the behaviour $\mathfrak{B}_i$ at the i^{th} step of synthesis is J-nonnegative lossless if the behaviour $\mathfrak{B}_{i-1}$ at the previous step is J-nonnegative lossless. Note the similarity with physical synthesis methods like Cauer and Foster synthesis methods, where the network to be synthesized at every step of the process is ensured to be lossless.

We now prove that synthesis of a J-lossless behaviour leads to the synthesis of a positive QDF, namely the one that is associated with the energy function of the J-lossless behaviour.

Theorem 6.17. *Let* $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_m\}$ *be a synthesis of a* Σ*-lossless behaviour* $\mathfrak{B}_0 \in \mathcal{L}^{\mathtt{w}}$*, where*

$$\Sigma := \begin{bmatrix} I_{\mathtt{w}_1} & 0_{\mathtt{w}_1 \times \mathtt{w}_2} \\ 0_{\mathtt{w}_2 \times \mathtt{w}_1} & -I_{\mathtt{w}_2} \end{bmatrix}$$

and $\mathtt{w}_1$ *and* $\mathtt{w}_2$ *are nonnegative integers such that* $\mathtt{w}_1 + \mathtt{w}_2 = \mathtt{w}$*. Let* $\mathfrak{B}_0 = Im\big(M_0(\frac{d}{dt})\big)$ *denote an observable image representation of* $\mathfrak{B}_0$*. Let* $P_i \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi]$ *be such that for* $i = 0, \ldots, m-1$*,* $\mathfrak{B}_{i+1} = Im\big(P_i(\frac{d}{dt})\big)_{|\mathfrak{B}_i}$*. Define* $N_{i,p-1}(\xi) := P_{p-1}(\xi) \ldots P_{i+1}(\xi) P_i(\xi)$ *for* $p > i$ *and* $N_{i,i-1}(\xi) := I_{\mathtt{w}}$*. Define* $M_i(\xi) := N_{0,i-1}(\xi) M_0(\xi)$*. Define*

$$\Phi_i(\zeta, \eta) := \frac{M_i(\zeta)^\top \Sigma M_i(\eta)}{\zeta + \eta}$$

Then $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$ *is a synthesis of* Q_{Φ_0}*.*

Proof. It is easy to see that for $i = 1, \ldots, m-1$, $\mathfrak{B}_i = \mathrm{Im}\big(M_i(\frac{d}{dt})\big)$. From Lemma 6.7, it follows that the rows of M_i are linearly independent over $\mathbb{R}$. Hence $M_i(\zeta)^\top \Sigma M_i(\eta)$ is a canonical factorization of $\Psi_i(\zeta, \eta) := (\zeta + \eta)\Phi_i(\zeta, \eta)$. Since $\mathtt{n}(\mathfrak{B}_i) > \mathtt{n}(\mathfrak{B}_{i+1}) > 0$ for $i = 0, \ldots, m-2$, it also follows that $\mathtt{n}(\Psi_i) > \mathtt{n}(\Psi_{i+1}) > 0$ for $i = 0, \ldots, m-2$. Also observe that the signatures of Ψ_i and Ψ_{i+1} are the same for $i = 0, \ldots, m-2$. Since $Q_\Sigma(w_m) = 0$ for all $w_m \in \mathfrak{B}_m$, we have $M_m(\zeta)^\top \Sigma M_m(\eta) = 0$. Hence $\Phi_m(\zeta, \eta) = \frac{M_m(\zeta)^\top \Sigma M_m(\eta)}{\zeta + \eta} = 0$.

From the fourth item of Definition 6.13, we have

$$Q_J(w_i) - Q_J(w_{i+1}) = \frac{d}{dt} Q_{\Omega_i}(w_i)$$

This implies that

$$Q_J\left(M_i\left(\frac{d}{dt}\right)\ell\right) - Q_J\left(M_{i+1}\left(\frac{d}{dt}\right)\ell\right) = \frac{d}{dt} Q_{\Omega_i}\left(M_i\left(\frac{d}{dt}\right)\ell\right)$$

The above equation is equivalent with

$$(\zeta + \eta)\left(\Phi_i(\zeta, \eta) - \Phi_{i+1}(\zeta, \eta)\right) = (\zeta + \eta) M_i(\zeta)^\top \Omega_i(\zeta, \eta) M_i(\eta)$$

or

$$Q_{\Phi_i} - Q_{\Phi_{i+1}} = Q_{\Lambda_i},$$

where $\Lambda_i(\zeta, \eta) = M_i(\zeta)^\top \Omega_i(\zeta, \eta) M_i(\eta)$. Since $Q_{\Omega_i} \geq 0$, it follows that also $Q_{\Lambda_i} \geq 0$. This concludes the proof. ■

We have showed through Theorems 6.6 and 6.17 that synthesis of a positive QDF Q_{Φ_0} leads to the synthesis of a Σ-lossless behaviour and vice versa, where Σ denotes the signature of $\Psi_0(\zeta, \eta) := (\zeta + \eta)\Phi_0(\zeta, \eta)$. In the next section, we show that both Cauer and Foster methods of synthesis can be cast in our framework of synthesis of behaviours and of positive QDFs. We prove that Cauer and Foster methods of synthesis involve both the synthesis of the behaviour corresponding to the given transfer function and also the synthesis of the energy function of the network to be synthesized.

6.4 Cauer and Foster synthesis

Both Cauer and Foster methods of synthesis start with the assumption that a lossless positive real transfer function matrix is given. Hence, in order to understand the relation between synthesis of positive QDFs and synthesis of electrical networks using Cauer and Foster methods, we first show that systems which are J-lossless with respect to a certain J have lossless positive real transfer function matrices. Throughout this section, we denote

$$\begin{aligned} J &= \begin{bmatrix} 0_{l\times l} & I_l \\ I_l & 0_{l\times l} \end{bmatrix}, \\ \Xi &= \frac{1}{\sqrt{2}} \begin{bmatrix} I_l & I_l \\ I_l & -I_l \end{bmatrix}, \\ \Sigma &= \Xi^\top J \Xi = \begin{bmatrix} I_l & 0_{l\times l} \\ 0_{l\times l} & -I_l \end{bmatrix} \end{aligned}$$

Theorem 6.18. *Consider a J-lossless behaviour $\mathfrak{B}$ and an observable image representation $\mathfrak{B} = Im(M(\frac{d}{dt}))$, where $M = col(N, D)$ with $N, D \in \mathbb{R}^{l\times l}[\xi]$. Then*

1. *N and D are oscillatory, i.e all the invariant polynomials of N and D have distinct roots on the imaginary axis.*
2. *$H(\xi) := N(\xi)+D(\xi)$ is Hurwitz, i.e $det(H)$ has all roots in the open left half plane.*
3. *ND^{-1} and DN^{-1} are both lossless positive real.*

Proof. Since $\mathfrak{B}$ is J-lossless, we have from Lemma 6.12 that $M(-\xi)^\top JM(\xi) = 0$ and

$$\Phi(\zeta, \eta) = \frac{N(\zeta)^\top D(\eta) + D(\zeta)^\top N(\eta)}{\zeta + \eta}$$

induces a positive QDF.

(*1*) Consider the behaviour $\mathfrak{B}_1 := \ker\big(N(\frac{d}{dt})\big)$. $\mathfrak{B}_1$ has a positive QDF Q_Φ such that $\frac{d}{dt}Q_\Phi(w) = 0\ \forall\ w \in \mathfrak{B}_1$, i.e Q_Φ is a positive conserved quantity for $\mathfrak{B}_1$. From Theorem 4.15 of Chapter 4, it follows that $\mathfrak{B}_1$ is oscillatory. Similarly $\mathfrak{B}_2 := \ker\big(D(\frac{d}{dt})\big)$ is also oscillatory. Therefore N and D are both oscillatory.

(*2*) Consider the behaviour $\mathfrak{B}_3 := \ker\ H(\frac{d}{dt})$. Now

$$(\zeta + \eta)\Phi(\zeta, \eta) = N(\zeta)^\top D(\eta) + D(\zeta)^\top N(\eta)$$

Consider the following equations

$$H(\zeta)^\top H(\eta) = N(\zeta)^\top N(\eta) + D(\zeta)^\top D(\eta) + (\zeta+\eta)\Phi(\zeta,\eta) \quad (6.14)$$

$$(N(\zeta) - D(\zeta))^\top (N(\eta) - D(\eta)) = N(\zeta)^\top N(\eta) + D(\zeta)^\top D(\eta) - (\zeta+\eta)\Phi(\zeta,\eta) \quad (6.15)$$

Subtracting equation (6.15) from (6.14) gives

$$2(\zeta+\eta)\Phi(\zeta,\eta) = H(\zeta)^\top H(\eta) - (N(\zeta) - D(\zeta))^\top (N(\eta) - D(\eta))$$

We now show that $Q_{\overset{\bullet}{\Phi}} \overset{\mathfrak{B}_3}{<} 0$, where $Q_{\overset{\bullet}{\Phi}} := \frac{d}{dt} Q_\Phi$. Define

$$M'(\lambda) := \begin{bmatrix} N(\lambda) - D(\lambda) \\ N(\lambda) + D(\lambda) \end{bmatrix} = \begin{bmatrix} I_l & I_l \\ I_l & -I_l \end{bmatrix} M(\lambda)$$

Since $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, also $M'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. This implies that $Q_{\overset{\bullet}{\Phi}}$ is a negative QDF along the behaviour $\mathfrak{B}_3$. Hence from Lyapunov theory of stability, it follows that $\mathfrak{B}_3$ is asymptotically stable or equivalently that H is Hurwitz.

(*3*) Since H is Hurwitz, it is invertible. Define

$$S(\xi) := (N(\xi) - D(\xi))(N(\xi) + D(\xi))^{-1}$$

Then it is easy to verify that

$$I - S(\xi)^\top S(-\xi) = I - S(-\xi)^\top S(\xi) = 0$$

Consequently $I - |S(j\omega)|^2 = 0$ for all $\omega \in \mathbb{R}$ such that $j\omega$ is not a pole of any element of S. Note that for any $\lambda \in \mathbb{C}$, $|S(\lambda)|^2 = S(\lambda)^* S(\lambda)$. Therefore, we have $|S(j\omega)|^2 = I$. For any arbitrary complex constant vector $x \in \mathbb{C}^{\mathtt{w}}$, we have $x^*(I - |S(j\omega)|^2)x = 0$. Now since $H(\xi) = N(\xi) + D(\xi)$ is Hurwitz, $S(\xi)$ is analytic in the right half plane. Hence by maximum modulus theorem, since x is arbitrary, for $\mathrm{Re}(\lambda) > 0$, $I - |S(\lambda)|^2$ is positive definite, i.e

$$2(N(\lambda)^* + D(\lambda)^*)^{-1}(N(\lambda)^* D(\lambda) + D(\lambda)^* N(\lambda))(N(\lambda) + D(\lambda))^{-1}$$

is positive definite. This implies that for $\mathrm{Re}(\lambda) > 0$, $N(\lambda)^* D(\lambda) + D(\lambda)^* N(\lambda)$ is positive definite, which further implies that

$$(D(\lambda)^*)^{-1}(N(\lambda)^* D(\lambda) + D(\lambda)^* N(\lambda)) D(\lambda)^{-1}$$

is positive definite, as D is oscillatory. This implies that for $\mathrm{Re}(\lambda) > 0$, $Z(\lambda)^* + Z(\lambda)$ is positive definite, where $Z(\xi) := N(\xi) D(\xi)^{-1}$. We also have $Z(j\omega) + Z(-j\omega) = 0$ for all $\omega \in \mathbb{R}$, except those for which $j\omega$ is a pole of one of the elements of Z. This proves that

Z is lossless positive real. In a similar way, it can be proved that Z^{-1} is also lossless positive real. ∎

6.4.1 Cauer synthesis

The aim of this section is to show that the Cauer method of synthesis can be cast in the general framework of synthesis of a J-lossless behaviour in the sense of Definition 6.13, and consequently involves the synthesis of a positive QDF that is associated with the energy function of the J-lossless behaviour in the sense of Definition 6.2.

Consider a J-lossless behaviour $\mathfrak{B}_0$ for which $\mathfrak{B}_0 = \text{Im}\big(M_0(\frac{d}{dt})\big)$ is an image representation such that M_0 is column reduced (see Definition B.11, Appendix B), and $M_0 = \text{col}(N_0, N_1)$, with $N_0, N_1 \in \mathbb{R}^{l\times l}[\xi]$. Assume that $N_1N_0^{-1}$ is strictly proper or that the transfer function to be synthesized by Cauer method is strictly proper. Without this condition, we cannot prove that Cauer synthesis can be cast in the framework of our definition of synthesis of behaviours and of positive QDFs. This may be considered as a limitation of our definition.

We know from Theorem 6.18 that $Z(\xi) := N_0(\xi)N_1(\xi)^{-1}$ is lossless positive real. In Anderson and Vongpanitlerd (1973), pp. 53-54, 215-218, it has been proved that any transfer function matrix $Z \in \mathbb{R}^{l\times l}(\xi)$ is lossless positive real if and only if it has a Foster partial fraction expansion given by

$$Z(\xi) = J_0 + \xi L_0 + \frac{C}{\xi} + \sum_i \left(\frac{\xi A_i + B_i}{\xi^2 + \omega_i^2}\right) \tag{6.16}$$

where $L_0, C, A_i \in \mathbb{R}_s^{l\times l}$ are nonnegative definite and $J_0, B_i \in \mathbb{R}^{l\times l}$ are skew-symmetric. Below, we explain the steps involved in Cauer synthesis of a lossless electrical circuit with transfer function matrix equal to Z. Define

$$Z_1(\xi) := \frac{C}{\xi} + \sum_i \left(\frac{\xi A_i + B_i}{\xi^2 + \omega_i^2}\right)$$

Observe that

$$\begin{aligned} N_0(\xi) &= (\xi L_0 + J_0)N_1(\xi) + Z_1(\xi)N_1(\xi) \quad \text{or} \\ Z(\xi) &= \xi L_0 + J_0 + Z_1(\xi) \end{aligned}$$

In the above equation, $(\xi L_0 + J_0)$ corresponds to the extracted reactance components in the first step of Cauer synthesis of Z. Define $N_2 := Z_1N_1$. From the partial fraction expansion for Z_1, it follows that Z_1 is lossless positive real. Hence $N_1N_2^{-1} = Z_1^{-1}$ is also lossless positive real. The next step of Cauer synthesis involves synthesis of Z_1^{-1}, whose corresponding behaviour is $\mathfrak{B}_1 = \text{Im}\big(M_1(\frac{d}{dt})\big)$, where $M_1 := \text{col}(N_1, N_2)$. Since

$\lim_{\xi \to \infty} Z_1(\xi) = 0$, Z_1 is strictly proper. Hence there exist a nonnegative definite matrix $L_1 \in \mathbb{R}_s^{l\times l}$, a skew symmetric matrix $J_1 \in \mathbb{R}^{l\times l}$ and $N_3 \in \mathbb{R}^{l\times l}[\xi]$, such that $L_1 \neq 0$,

$$Z_1(\xi)^{-1} = (\xi L_1 + J_1) + \frac{C'}{\xi} + \sum_i \left(\frac{\xi A_i' + B_i'}{\xi^2 + \omega_i^2} \right),$$

and

$$N_3(\xi) = \left(\frac{C'}{\xi} + \sum_i \left(\frac{\xi A_i' + B_i'}{\xi^2 + \omega_i^2} \right) \right) N_2(\xi)$$

Define

$$Z_2(\xi) := \frac{C'}{\xi} + \sum_i \left(\frac{\xi A_i' + B_i'}{\xi^2 + \omega_i^2} \right)$$

It follows that

$$N_1(\xi) = (\xi L_1 + J_1) N_2(\xi) + N_3(\xi)$$

and $N_2 N_3^{-1}$ is lossless positive real. Observe that

$$Z_1(\xi)^{-1} = \xi L_1 + J_1 + Z_2(\xi)$$

In the above equation $(\xi L_1 + J_1)$ corresponds to the extracted reactance components in the second step of Cauer synthesis of Z. The next step of Cauer synthesis involves synthesis of Z_2^{-1}, whose corresponding behaviour is $\mathfrak{B}_2 = \operatorname{Im}\big(M_2(\frac{d}{dt})\big)$, where $M_2 := \operatorname{col}(N_2, N_3)$. We can continue this procedure to obtain the following set of equations.

$$N_i(\xi) = (\xi L_i + J_i) N_{i+1}(\xi) + N_{i+2}(\xi) \tag{6.17}$$

for $i = 0, 1, \ldots, m-1$, $N_{m+1} = 0$, with $L_i \in \mathbb{R}_s^{l\times l}$ being nonnegative definite and not equal to zero, $J_i \in \mathbb{R}^{l\times l}$ skew-symmetric and $N_i N_{i+1}^{-1}$ lossless positive real for $i = 0, 1, \ldots, m-1$.

We now show that if $l = 1$, then Cauer synthesis of Z leads to a continued fraction expansion of Z as in the standard description of Cauer synthesis of a SISO lossless positive real transfer function. In this case, observe that $J_0, J_1 = 0$, $B_i = 0$, $B_i' = 0$, and $L_0, L_1, C, C', A_i, A_i'$ are positive real numbers. Therefore, we have

$$Z(\xi) = \xi L_0 + \frac{1}{Z_1(\xi)^{-1}}$$

Note that in the right hand side of the above equation, the term ξL_0 corresponds to a reactive component extracted in the first step of Cauer synthesis of Z, and Z_1^{-1} is the transfer function to be synthesized in the next step. The second step of Cauer synthesis involves writing Z as

$$Z(\xi) = \xi L_0 + \frac{1}{\xi L_1 + \frac{1}{Z_2(\xi)^{-1}}}$$

In the above equation, the term ξL_1 corresponds to the reactance component extracted in the second step of Cauer synthesis, and Z_2^{-1} is the transfer function to be synthesized in the third step. Continuing this way, we obtain a continued fraction expansion of Z as shown below:

$$Z(\xi) = \xi L_0 + \cfrac{1}{\xi L_1 + \cfrac{1}{\xi L_2 + \cfrac{1}{\ddots}}}$$

We now turn our attention back to the case when l is not necessarily equal to 1. Observe that equations (6.17) can be written in another form as shown below.

$$\begin{bmatrix} N_{i-1}(\xi) \\ N_i(\xi) \end{bmatrix} = \begin{bmatrix} S_{i-1}(\xi) & I_l \\ I_l & 0_{l\times l} \end{bmatrix} \begin{bmatrix} N_i(\xi) \\ N_{i+1}(\xi) \end{bmatrix} = Q_{i-1}(\xi) \begin{bmatrix} N_i(\xi) \\ N_{i+1}(\xi) \end{bmatrix} \tag{6.18}$$

for $i = 1, \ldots, m$, where $S_i(\xi) = \xi L_i + J_i$. Observe that

$$\begin{aligned} \Phi_0(\zeta,\eta) :&= \frac{[\; N_0(\zeta)^\top \quad N_1(\zeta)^\top \;] J \begin{bmatrix} N_0(\eta) \\ N_1(\eta) \end{bmatrix}}{\zeta+\eta} \\ &= \frac{[\; N_1(\zeta)^\top \quad N_2(\zeta)^\top \;] Q_0(\zeta)^\top J Q_0(\eta) \begin{bmatrix} N_1(\eta) \\ N_2(\eta) \end{bmatrix}}{\zeta+\eta} \end{aligned}$$

It can be easily verified that

$$Q_i(\zeta)^\top J Q_i(\eta) = 2(\zeta+\eta) \begin{bmatrix} L_i & 0_{l\times l} \\ 0_{l\times l} & 0_{l\times l} \end{bmatrix} + J$$

Hence

$$\Phi_0(\zeta,\eta) = \frac{[\; N_1(\zeta)^\top \quad N_2(\zeta)^\top \;] J \begin{bmatrix} N_1(\eta) \\ N_2(\eta) \end{bmatrix}}{\zeta+\eta} + 2N_1(\zeta)^\top L_0 N_1(\eta)$$

Proceeding this way, using equations (6.18) we obtain

$$\Phi_0(\zeta,\eta) = 2\sum_{i=0}^{m-1} N_{i+1}(\zeta)^\top L_i N_{i+1}(\eta) \tag{6.19}$$

which is a diagonalization of $\Phi_0(\zeta,\eta)$. Each term $2N_{i+1}(\zeta)^\top L_i N_{i+1}(\eta)$ in the diagonalization corresponds to energy of a particular component in the synthesis. We illustrate this diagonalization using the following example.

Example 6.2. Let

$$N_0(\xi) = \begin{bmatrix} 3\xi^3 + 14\xi & \xi^3 + 2\xi^2 + 6\xi + 11 \\ \xi^3 - 2\xi^2 + 6\xi - 11 & 2\xi^3 + 12\xi \end{bmatrix} \qquad N_1(\xi) = \begin{bmatrix} \xi^2 + 4 & 0 \\ 0 & \xi^2 + 4 \end{bmatrix}$$

i	$N_{i+1}(\xi)$	$S_i(\xi)$
0	$\begin{bmatrix} \xi^2+4 & 0 \\ 0 & \xi^2+4 \end{bmatrix}$	$\begin{bmatrix} 3\xi & \xi+2 \\ \xi-2 & 2\xi \end{bmatrix}$
1	$\begin{bmatrix} 2\xi & 2\xi+3 \\ 2\xi-3 & 4\xi \end{bmatrix}$	$\begin{bmatrix} \xi & -\frac{\xi}{2}-\frac{3}{4} \\ -\frac{\xi}{2}+\frac{3}{4} & \frac{\xi}{2} \end{bmatrix}$
2	$\begin{bmatrix} \frac{7}{4} & 0 \\ 0 & \frac{7}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{8\xi}{7} & \frac{8\xi}{7}+\frac{12}{7} \\ \frac{8\xi}{7}-\frac{12}{7} & \frac{16\xi}{7} \end{bmatrix}$

TABLE 6.1: Example for Cauer synthesis

Define $M_0 := \text{col}(N_0, N_1)$ and

$$J' := \begin{bmatrix} 0_{2\times 2} & I_2 \\ I_2 & 0_{2\times 2} \end{bmatrix}$$

It can be verified that $\mathfrak{B}_0 = \text{Im}\big(M_0(\frac{d}{dt})\big)$ is J'-lossless. The values of $N_{i+1}(\xi)$ and $S_i(\xi)$ for $i = 0, 1, 2$ obtained through Cauer diagonalization procedure are given in Table 6.1.

Remark 6.19. Consider the synthesis of an electrical network whose behaviour $\mathfrak{B}_e$ is J-lossless. Assume that $\mathfrak{B}_e$ is given in image form as

$$w = \begin{bmatrix} V \\ I \end{bmatrix} = \begin{bmatrix} N_0(\frac{d}{dt}) \\ N_1(\frac{d}{dt}) \end{bmatrix} \ell \tag{6.20}$$

where $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ is a free trajectory, $V \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ is a vector of voltages across the branches of the network, $I \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l)$ is a vector of currents through the respective branches and $N_0, N_1 \in \mathbb{R}^{l\times l}[\xi]$. In this case, for any trajectory $w = \text{col}(V, I) \in \mathfrak{B}_e$, we have the power supply $Q_\Psi(w)$ to the network given by

$$Q_\Psi(w) = \frac{1}{2}Q_J(w) = \frac{1}{2}(V^\top I + I^\top V)$$

If Q_E denotes the QDF that represents the energy of the network, then we have

$$\frac{d}{dt}Q_E(w) = Q_\Psi(w) = \frac{1}{2}Q_J(w) \tag{6.21}$$

Let Q_Φ be a QDF such that

$$Q_\Phi(\ell) = Q_E(w) \tag{6.22}$$

for every w, ℓ related by equation (6.20). It is easy to see from equation (6.20) that

$$Q_J(w) = Q_{\Psi'}(\ell) \tag{6.23}$$

where $\Psi'(\zeta,\eta) = N_0(\zeta)^\top N_1(\eta) + N_1(\zeta)^\top N_0(\eta)$. From equations (6.21), (6.22) and (6.23), it follows that

$$\Phi(\zeta,\eta) = \frac{N_0(\zeta)^\top N_1(\eta) + N_1(\zeta)^\top N_0(\eta)}{2(\zeta+\eta)}$$

Observe that the QDF Q_Φ is related to the energy function of the network. We have showed that Cauer synthesis of a network with transfer function equal to $N_0 N_1^{-1}$ involves the diagonalization of the QDF Q_Φ. Thus Cauer synthesis of a network with lossless positive real transfer function involves the diagonalization of a QDF that is related to the energy function of the network.

We have proved that Cauer synthesis of a J-lossless behaviour leads to diagonalization of a positive QDF associated with its energy function. We now prove that Cauer synthesis of a J-lossless behaviour $\mathfrak{B}_0$ leads to a synthesis of behaviours in the sense of Definition 6.13, and consequently leads to a synthesis of a positive QDF that is associated with the energy function of $\mathfrak{B}_0$.

Theorem 6.20. *Consider a J-lossless behaviour $\mathfrak{B}_0$ for which an observable image representation is $\mathfrak{B}_0 = Im\big(M_0(\frac{d}{dt})\big)$, where $M_0 = col(N_0, N_1)$, $N_0, N_1 \in \mathbb{R}^{l\times l}[\xi]$. Assume that M_0 is column reduced, and $N_1 N_0^{-1}$ is strictly proper. For $i = 1,\dots,m$, let $N_{i+1} \in \mathbb{R}^{l\times l}[\xi]$ be such that*

$$N_{i-1}(\xi) = (\xi L_{i-1} + J_{i-1})N_i(\xi) + N_{i+1}(\xi) \tag{6.24}$$

where $N_{m+1} = 0$,

$$Z_k(\xi) = J_k + \xi L_k + \frac{C_k}{\xi} + \sum_i \left(\frac{\xi A_{ki} + B_{ki}}{\xi^2 + \omega_{ki}^2}\right)$$

is a Foster series expansion of $Z_k(\xi) := N_k(\xi)\big(N_{k+1}(\xi)\big)^{-1}$ for $k = 0,\dots,m-1$, with $A_{ki}, L_k, C_k \in \mathbb{R}_s^{l\times l}$ being nonnegative definite and $J_k, B_{ki} \in \mathbb{R}_s^{l\times l}$ being skew-symmetric. Now define $M_i := col(N_i, N_{i+1})$, $\mathfrak{B}_i := Im\big(M_i(\frac{d}{dt})\big)$,

$$\Phi_i(\zeta,\eta) := \frac{M_i(\zeta)^\top J M_i(\eta)}{\zeta+\eta},$$

$\mathfrak{B}'_i = Im(\Xi)_{|\mathfrak{B}_i}$ for $i = 0,\dots,m$. Then $\mathfrak{B}'_0$ is Σ-lossless, $\{\mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_m\}$ is a synthesis of the behaviour $\mathfrak{B}'_0$ and $\{Q_{\Phi_1}, Q_{\Phi_2}, \dots, Q_{\Phi_m}\}$ is a synthesis of the QDF Q_{Φ_0}.

Proof. We first prove that $\{\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_m\}$ is a synthesis of the J-lossless behaviour $\mathfrak{B}_0$. This will be helpful in proving that $\{\mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_m\}$ is a synthesis of the behaviour $\mathfrak{B}'_0$.

We now prove by induction that $\mathfrak{B}_i = \mathrm{Im}\big(M_i(\frac{d}{dt})\big)$ is an observable image representation for $i = 1,\dots,m-1$. We know that $M_0(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Now assume that for some $i \in \{0,\dots,m-1\}$, $M_i(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

We now prove that $M_{i+1}(\lambda)$ also has full column rank for all $\lambda \in \mathbb{C}$. By contradiction, assume that there exists a $\lambda \in \mathbb{C}$, such that $M_{i+1}(\lambda)u_\lambda = 0$ for some nonzero $u_\lambda \in \mathbb{C}^l$. This implies that $N_{i+1}(\lambda)u_\lambda = N_{i+2}(\lambda)u_\lambda = 0$. Now, from equation (6.24), it follows that

$$N_i(\lambda)u_\lambda = (\lambda L_i + J_i)N_{i+1}(\lambda)u_\lambda + N_{i+2}(\lambda)u_\lambda = 0$$

This implies that $M_i(\lambda)u_\lambda = 0$, which is a contradiction. This proves that $M_{i+1}(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. By induction, it follows that $\mathfrak{B}_i = \operatorname{Im}\big(M_i(\frac{d}{dt})\big)$ is an observable image representation for $i = 1, \ldots, m-1$.

Observe that

$$\begin{aligned} Z_k(\xi)^{-1} &= N_{k+1}(\xi)N_k(\xi)^{-1} = N_{k-1}(\xi)N_k(\xi)^{-1} - \xi L_{k-1} - J_{k-1} \\ &= \frac{C_{k-1}}{\xi} + \sum_i \left(\frac{\xi A_{k-1,i} + B_{k-1,i}}{\xi^2 + \omega_{k-1,i}^2} \right) \end{aligned}$$

Since $\lim_{\xi \to \infty} Z_k(\xi)^{-1} = 0$, Z_k^{-1} is strictly proper for $k = 0, \ldots, m-1$. Hence from Lemma B.10, Appendix B, it follows that the column degrees of N_i are greater that those of N_{i+1} for $i = 0, \ldots, m-2$. This implies that $\deg(\det(N_i)) > \deg(\det(N_{i+1}))$ for $i = 0, \ldots, m-2$. Observe that for $i = 0, \ldots, m-1$, $\mathtt{n}(\mathfrak{B}_i) = \deg(\det(N_i))$. Hence $\mathtt{n}(\mathfrak{B}_i) > \mathtt{n}(\mathfrak{B}_{i+1})$ for $i = 0, \ldots, m-2$. Also $\mathtt{n}(\mathfrak{B}_i) > 0$ for $i = 1, \ldots, m-1$, because, if $\mathtt{n}(\mathfrak{B}_i) = 0$, then M_i consists of constant entries and there does not exist N_{i+2} such that equation

$$N_i(\xi) = (\xi L_i + J_i)N_{i+1}(\xi) + N_{i+2}(\xi)$$

is obeyed, which will imply that M_m does not exist. Also since $N_{m+1} = 0$, it can be inferred that $M_m(\zeta)^\top J M_m(\eta) = 0$, which implies that $Q_J(w_m) = 0$ for any trajectory $w_m \in \mathfrak{B}_m$.

We now make use of the following lemma to prove that the rows of M_i are linearly independent over $\mathbb{R}$ for $i = 0, \ldots, m-1$.

Lemma 6.21. *Let $N, D \in \mathbb{R}^{l \times l}[\xi]$ be such that*

$$N(\xi)\big(D(\xi)\big)^{-1} = \frac{C}{\xi} + \sum_{i=1}^{N} \left(\frac{\xi A_i + B_i}{\xi^2 + \omega_i^2} \right) \tag{6.25}$$

where $C, A_i \in \mathbb{R}_s^{l \times l}$ are nonnegative definite and $B_i \in \mathbb{R}^{l \times l}$ are skew-symmetric. Define $M := col(D, N)$. Then the rows of M are linearly independent over $\mathbb{R}$.

Proof. It is easy to see that $Z(\xi) := N(\xi)\big(D(\xi)\big)^{-1}$ is lossless positive real. Therefore Z^{-1} exists and is also lossless positive real. Note that Z has strictly proper entries. Now assume by contradiction that the rows of M are linearly dependent over $\mathbb{R}$. Then $\exists\ E, F \in \mathbb{R}^l$, such that at least one of E, F is nonzero and $E^\top N(\xi) = -F^\top D(\xi)$. This implies that

$$E^\top N(\xi)D(\xi)^{-1} = -F^\top \tag{6.26}$$

Note that ND^{-1} has nonzero rows, because otherwise Z^{-1} will not exist. Also note that $E \neq 0$, because this implies that also $F = 0$. Since

$$F^\top D(\xi)N(\xi)^{-1} = -E^\top,$$

$F \neq 0$, because this implies that $E = 0$. Hence the left hand side of equation (6.26) consists of strictly proper entries, while the right hand side does not, which is a contradiction. Consequently the rows of M are linearly independent over $\mathbb{R}$. ■

For $i = 0, \ldots, m-1$, since $N_{i+1}N_i^{-1} = Z_i^{-1}$ has same form of partial fraction expansion as the right hand side of equation (6.25), it follows from the above lemma that the rows of M_i are linearly independent over $\mathbb{R}$. Define

$$\mathcal{B}_i := \{v_i \mid \exists w_i \in \mathfrak{B}_i \text{ such that } w_i(0) = v_i\}$$

From Lemma 6.7, it follows that for $i = 0, \ldots, m-1$, $\mathcal{B}_i$ has dimension equal to $2l$. For $i = 0, \ldots, m-1$, define

$$Q_i(\xi) := \begin{bmatrix} \xi L_i + J_i & I_l \\ I_l & 0_{l\times l} \end{bmatrix},$$

$P_i(\xi) := Q_i(\xi)^{-1}$. Observe that for $i = 0, \ldots, m-1$, $\mathfrak{B}_{i+1} = \operatorname{Im}\big(P_i(\frac{d}{dt})\big)_{|\mathfrak{B}_i}$. Note that

$$Q_i(\zeta)^\top J Q_i(\eta) = 2(\zeta+\eta)\begin{bmatrix} L_i & 0_{l\times l} \\ 0_{l\times l} & 0_{l\times l} \end{bmatrix} + J$$

Hence

$$P_i(\zeta)^\top J P_i(\eta) = J - 2(\zeta+\eta)\begin{bmatrix} L_i & 0_{l\times l} \\ 0_{l\times l} & 0_{l\times l} \end{bmatrix} = J - (\zeta+\eta)\Omega_i \tag{6.27}$$

where

$$\Omega_i = 2\begin{bmatrix} L_i & 0_{l\times l} \\ 0_{l\times l} & 0_{l\times l} \end{bmatrix}$$

Note also that the QDF Q_{Ω_i} is nonnegative. Now for some $i \in \{0, \ldots, m-1\}$, consider two trajectories $w_i \in \mathfrak{B}_i$ and $w_{i+1} \in \mathfrak{B}_{i+1}$ that are related by the equation $w_{i+1} = P_i(\frac{d}{dt})w_i$. From equation (6.27),

$$Q_J(w_i) - Q_J(w_{i+1}) = Q_J(w_i) - Q_J\left(P_i\left(\frac{d}{dt}\right)w_i\right) = \frac{d}{dt}Q_{\Omega_i}(w_i) \tag{6.28}$$

Thus all conditions of Definition 6.13 are obeyed which implies that $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_m\}$ is a synthesis of $\mathfrak{B}_0$.

For $i = 1, \dots, m$, define $M'_i(\xi) := \Xi M_i(\xi)$. We have

$$\Phi_i(\zeta, \eta) = \frac{M_i(\zeta)^\top J M_i(\eta)}{\zeta + \eta} = \frac{M'_i(\zeta)^\top \Sigma M'_i(\eta)}{\zeta + \eta}$$

for $i = 1, \dots, m$. Observe that $\Xi^\top = \Xi^{-1} = \Xi$. In order to prove that $\mathfrak{B}'_0$ is Σ-lossless and $\{\mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_m\}$ is a synthesis of the behaviour $\mathfrak{B}'_0$, we first need to prove that $\mathtt{n}(\mathfrak{B}'_i) = \mathtt{n}(\mathfrak{B}_i)$ for $i = 0, \dots, m$, for which we make use of the following lemma.

Lemma 6.22. *Consider a controllable behaviour $\mathfrak{B} \in \mathcal{L}^{2l}$. Define $\mathfrak{B}' := Im(\Xi)_{|\mathfrak{B}}$. Then $\mathfrak{B}'$ is controllable and $\mathtt{n}(\mathfrak{B}') = \mathtt{n}(\mathfrak{B})$.*

Proof. Let $\mathfrak{B} = \operatorname{Im}\big(M(\frac{d}{dt})\big)$ denote an observable image representation of $\mathfrak{B}$, where $M \in \mathbb{R}^{2l \times l_1}[\xi]$ and M is column reduced. Write

$$M(\xi) = A \operatorname{diag}(\xi^{n_1}, \xi^{n_2}, \dots, \xi^{n_{l_1}}) + B(\xi)$$

where n_i denotes the column degree of the i^{th} column of M, A denotes the coefficient matrix of the polynomial matrix formed by the highest degree terms of M and $B \in \mathbb{R}^{2l \times l_1}[\xi]$ consists of all the lower degree terms of M. Since M is column reduced, A has full column rank, and $\mathtt{n}(\mathfrak{B}) = \sum_{i=1}^{l_1} n_i$. Define $M'(\xi) := \Xi M(\xi)$. Since $\det(\Xi) \neq 0$, $\mathfrak{B}' = \operatorname{Im}\big(M'(\frac{d}{dt})\big)$ is an observable image representation of $\mathfrak{B}'$. The existence of an image representation implies that $\mathfrak{B}'$ is controllable. We have

$$M'(\xi) = \Xi M(\xi) = \Xi A \operatorname{diag}(\xi^{n_1}, \xi^{n_2}, \dots, \xi^{n_{l_1}}) + \Xi B(\xi)$$

Let A'_i and A_i denote the i^{th} columns of ΞA and A respectively. Assume that there exist $a_i \in \mathbb{R}$ for $i = 1, \dots, l_1$, such that

$$\sum_{i=1}^{l_1} a_i A'_i = 0 \tag{6.29}$$

Since $\Xi^{-1} = \Xi$, we have $\Xi A'_i = A_i$. Premultiplying both sides of equation (6.29) with Ξ, we get

$$\sum_{i=1}^{l_1} a_i A_i = 0$$

Since A has full column rank, the above implies that $a_i = 0$ for $i = 1, \dots, l_1$. This in turn implies that ΞA has full column rank, or that M' is column reduced. Hence $\mathtt{n}(\mathfrak{B}') = \sum_{i=1}^{l_1} n_i = \mathtt{n}(\mathfrak{B})$. ∎

We now prove that $\{\mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_m\}$ is a synthesis of the behaviour $\mathfrak{B}'_0$.

Lemma 6.23. *Let $\{\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_m\}$ be a synthesis of a J-lossless behaviour $\mathfrak{B}_0 \in \mathcal{L}^{2l}$. For $i = 0, \dots, m$, define $\mathfrak{B}'_i := Im(\Xi)_{|\mathfrak{B}}$. Then $\mathfrak{B}'_0$ is Σ-lossless and $\{\mathfrak{B}'_1, \mathfrak{B}'_2, \dots, \mathfrak{B}'_m\}$ is a synthesis of $\mathfrak{B}'_0$.*

Proof. Consider a trajectory $w'_m \in \mathfrak{B}'_m$. We have $Q_\Sigma(w'_m) = Q_J(w_m)$, where $w_m = \Xi w'_m$. This implies that $w_m \in \mathfrak{B}_m$ and hence $Q_J(w_m) = 0$. Thus $Q_\Sigma(w'_m) = 0$ for any trajectory $w'_m \in \mathfrak{B}'_m$. From Lemma 6.22, it is easy to see that $\mathtt{n}(\mathfrak{B}'_i) = \mathtt{n}(\mathfrak{B}_i)$ for $i = 0, \ldots, m$. Therefore $\mathtt{n}(\mathfrak{B}'_i) > \mathtt{n}(\mathfrak{B}'_{i+1}) > 0$ for $i = 0, \ldots, m-2$.

Let $\mathfrak{B}_0 = \mathrm{Im}\big(M_0(\frac{d}{dt})\big)$ be an observable image representation of $\mathfrak{B}_0$, and $\mathfrak{B}_{i+1} = \mathrm{Im}(P_i(\frac{d}{dt}))_{|\mathfrak{B}_i}$ for $i = 0, \ldots, m-1$. For $i = 1, \ldots, m$, define

$$M_i(\xi) := P_{i-1}(\xi)P_{i-2}(\xi)\ldots P_0(\xi)M_0(\xi)$$

Then it is easy to see that $\mathfrak{B}_i = \mathrm{Im}\big(M_i(\frac{d}{dt})\big)$ for $i = 1, \ldots, m$. For $i = 0, \ldots, m$, define $M'_i(\xi) := \Xi M_i(\xi)$. Observe that $\mathfrak{B}'_i = \mathrm{Im}\big(M'_i(\frac{d}{dt})\big)$ for $i = 0, \ldots, m$. Since $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_m\}$ is a synthesis of $\mathfrak{B}_0$, it follows from Lemma 6.7 that for $i = 0, \ldots, m-1$, the rows of M_i are linearly independent over $\mathbb{R}$. Since $\det(\Xi) \neq 0$, it follows that the rows of M'_i are also linearly independent over $\mathbb{R}$ for $i = 0, \ldots, m-1$. Define

$$\mathcal{B}'_i := \{v_i \mid \exists w'_i \in \mathfrak{B}'_i \text{ such that } w'_i(0) = v_i\}$$

From Lemma 6.7, it follows that $\mathcal{B}'_i$ has dimension equal to $2l$ for $i = 0, \ldots, m-1$.

Observe that $\mathfrak{B}'_0 = \mathrm{Im}\big(M'_0(\frac{d}{dt})\big)$ is an observable image representation for $\mathfrak{B}'_0$. If we define

$$\Phi_0(\zeta, \eta) := \frac{M_0(\zeta)^\top J M_0(\eta)}{\zeta + \eta} = \frac{M'_0(\zeta)^\top \Sigma M'_0(\eta)}{\zeta + \eta}$$

then from Lemma 6.12, it follows that $Q_{\Phi_0} > 0$. From the same Lemma, it follows that $\mathfrak{B}'_0$ is Σ-lossless.

For $i = 0, \ldots, m-1$, define $P'_i(\xi) := \Xi P_i(\xi)$. It is easy to see that $\mathfrak{B}'_{i+1} = \mathrm{Im}(P'_i(\frac{d}{dt}))_{|\mathfrak{B}'_i}$ for $i = 0, \ldots, m-1$. For any two trajectories $w'_i \in \mathfrak{B}'_i$ and $w'_{i+1} \in \mathfrak{B}'_{i+1}$ that are related by $w'_{i+1} = P'_i(\frac{d}{dt})w'_i$, where $i \in \{0, \ldots, m-1\}$, consider related trajectories $w_i \in \mathfrak{B}_i$, and $w_{i+1} \in \mathfrak{B}_{i+1}$, such that $w_i = \Xi w'_i$ and $w_{i+1} = \Xi w'_{i+1}$. Since $w_{i+1} = P_i(\frac{d}{dt})w_i$, from Definition 6.13, it follows that $\exists$ a nonnegative Q_{Ω_i} $(\Omega_i \in \mathbb{R}_s^{2l\times 2l})[\zeta, \eta]$, such that

$$Q_\Sigma(w'_i) - Q_\Sigma(w'_{i+1}) = \frac{d}{dt}Q_{\Omega'_i}(w'_i)$$

where $\Omega'_i(\zeta, \eta) = \Xi\Omega_i(\zeta, \eta)\Xi$. Observe that $Q_{\Omega'_i}$ is nonnegative for $i = 0, \ldots, m-1$. Hence from Definition 6.13, it follows that $\{\mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_m\}$ is a synthesis of the Σ-lossless behaviour $\mathfrak{B}'_0$. ■

From Theorem 6.17 and the above Lemma, conclude that $\mathfrak{B}'_0$ is Σ-lossless, $\{\mathfrak{B}'_1, \mathfrak{B}'_2, \ldots, \mathfrak{B}'_m\}$ is a synthesis of the behaviour $\mathfrak{B}'_0$ and $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$ is a synthesis of the positive QDF Q_{Φ_0}. ■

6.4.2 Foster synthesis

Next for the purpose of illustration, we explain the steps involved in the Foster synthesis of a special class of lossless positive real transfer functions of the form

$$Z_0(\xi) = \sum_{i=0}^{m-1} \left(\frac{\xi A_i}{\xi^2 + \omega_i^2} \right)$$

where $A_i \in \mathbb{R}_s^{l\times l}$ is positive definite for $i = 0, \dots, m-1$. Consider $N_0, D_0 \in \mathbb{R}^{l\times l}[\xi]$, such that $D_0(\xi) = \prod_{i=0}^{m-1}(\xi^2 + \omega_i^2)I_l$ and $Z_0 = N_0 D_0^{-1}$. For $k = 1, \dots, m-1$, define

$$Z_k(\xi) := \sum_{i=k}^{m-1} \left(\frac{\xi A_i}{\xi^2 + \omega_i^2} \right)$$

Assume that N_0, D_0 are known, and the values of A_i for $i = 0, \dots, m-1$ are unknown. The behaviour corresponding to the transfer function Z_0 has the image representation $\mathfrak{B}_0 = \mathrm{Im}\big(M_0(\frac{d}{dt})\big)$, where $M_0 := \mathrm{col}(D_0, N_0)$. Foster's synthesis for this case consists of obtaining the values of A_i and expressions for polynomial matrices $N_k(\xi) := Z_k(\xi)\left(\prod_{i=k}^{m-1}(\xi^2 + \omega_i^2)\right)$ for $k = 1, \dots, m-1$ from N_0 and D_0. Define $D_k(\xi) := \prod_{i=k}^{m-1}(\xi^2 + \omega_i^2)I_l$ for $k = 1, \dots, m-1$. The first step in the Foster synthesis of Z_0 is to obtain A_0 and N_1 from the known expressions of N_0 and D_0 and to write Z_0 as

$$Z_0(\xi) = \frac{\xi A_0}{\xi^2 + \omega_0^2} + N_1(\xi) D_1(\xi)^{-1}$$

In the right hand side of the above equation, the first term corresponds to a pair of reactive components extracted in the first step, and the second term corresponds to the transfer function to be synthesized in the next step. The behaviour corresponding to the transfer function $N_1 D_1^{-1}$ is given by $\mathfrak{B}_1 = \mathrm{Im}\big(M_1(\frac{d}{dt})\big)$, where $M_1 := \mathrm{col}(D_1, N_1)$. The next step in the Foster synthesis of Z_0 is to obtain A_1 and N_2 from the known expressions of N_1 and D_1 and to write Z_0 as

$$Z_0(\xi) = \frac{\xi A_0}{\xi^2 + \omega_0^2} + \frac{\xi A_1}{\xi^2 + \omega_1^2} + N_2(\xi) D_2(\xi)^{-1}$$

The second term in the right hand side of the above equation corresponds to a pair of reactive components extracted in the second step, and the third term corresponds to the transfer function to be synthesized in the third step, whose corresponding behaviour is $\mathfrak{B}_2 = \mathrm{Im}\big(M_2(\frac{d}{dt})\big)$, where $M_2 := \mathrm{col}(D_2, N_2)$. Continuing this way, it is easy to see that Foster synthesis of Z_0 actually leads to a Foster partial fraction expansion of Z_0 as in the standard description of Foster synthesis of a lossless positive real transfer function.

We now prove that Foster's synthesis for this case leads to a synthesis of behaviours in the sense of Definition 6.13 and consequently a synthesis of a QDF associated with the behaviours.

Theorem 6.24. *Define* $r(\xi) := \prod_{k=0}^{m-1}(\xi^2 + \omega_k^2)$*,* $v_q(\xi) := \frac{r(\xi)}{(\xi^2+\omega_q^2)}$*, where* $\omega_q \in \mathbb{R}^+$ *are distinct for* $q = 0, \ldots, m-1$*. For* $i = 0, \ldots, m-1$*, define*

$$M_i(\xi) := \begin{bmatrix} r(\xi)I_l \\ \sum_{q=i}^{m-1} \xi v_q(\xi) A_q \end{bmatrix},$$

Define $M_m(\xi) := col\big(r(\xi)I_l, 0_{l\times l}\big)$*. For* $i = 0, \ldots, m$*, define* $\mathfrak{B}_i := Im\big(M_i(\frac{d}{dt})\big)$*,*

$$\Phi_i(\zeta, \eta) := \frac{M_i(\zeta)^\top J M_i(\eta)}{\zeta + \eta}.$$

Then $\mathfrak{B}_0$ *is J-lossless,* $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_m\}$ *is a synthesis of* $\mathfrak{B}_0$ *and* $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$ *is a synthesis of the QDF* Q_{Φ_0}*.*

Proof. It is a matter of straightforward verification to see that

$$\Phi_0(\zeta, \eta) = \sum_{i=0}^{m-1} (\zeta\eta + \omega_i^2) v_i(\zeta) v_i(\eta) A_i$$

Let $B_i \in \mathbb{R}^{l\times l}$ be such that $A_i = B_i^\top B_i$ for $i = 0, \ldots, m-1$. Since A_i is positive definite, $\det(B_i) \neq 0$ for $i = 0, \ldots, m-1$. Define

$$F_0(\xi) := \text{col}_{i=0}^{m-1} \begin{bmatrix} \xi v_i(\xi) B_i \\ \omega_i v_i(\xi) B_i \end{bmatrix}$$

Observe that $\Phi_0(\zeta, \eta) = F_0(\zeta)^\top F_0(\eta)$. We now prove that $F_0(\lambda)$ has full column rank $\forall$ $\lambda \in \mathbb{C}$. Assume by contradiction that $F_0(\lambda)$ loses column rank for some $\lambda \in \mathbb{C}$. Then $\exists$ a nonzero $u_\lambda \in \mathbb{C}^l$ such that $\lambda v_i(\lambda) B_i u_\lambda = 0$ and $v_i(\lambda) B_i u_\lambda = 0$ for $i = 0, \ldots, m-1$. Since $\det(B_i) \neq 0$, this implies that $v_i(\lambda) = 0$ for $i = 0, \ldots, m-1$. But there does not exist a $\lambda \in \mathbb{C}$ for which this is true. This implies that $Q_{\Phi_0} > 0$.

Define $r_q(\xi) := \prod_{i=q}^{m-1}(\xi^2 + \omega_i^2)$, $v_{iq}(\xi) := \frac{r_q(\xi)}{\xi^2+\omega_i^2}$ for $q = 0, \ldots, m-1$, $r_m(\xi) := 1$, $v_{im}(\xi) := 0$ and

$$S_q(\xi) := \begin{bmatrix} r_q(\xi) I_l \\ \sum_{i=q}^{m-1} \xi v_{iq}(\xi) A_i \end{bmatrix}.$$

It is easy to see that $\mathfrak{B}_i = \text{Im}\big(S_i(\frac{d}{dt})\big)$ is an alternative image representation of $\mathfrak{B}_i$ for $i = 0, \ldots, m$. We now prove that for $q = 0, \ldots, m-1$, $S_q(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Assume by contradiction that $S_q(\lambda)$ loses column rank for some $\lambda \in \mathbb{C}$. Then there exists a nonzero $y_\lambda \in \mathbb{C}^l$, such that $S_q(\lambda) y_\lambda = 0$. This implies that $r_q(\lambda) y_\lambda = 0$, and $\sum_{i=q}^{m-1} \lambda v_{iq}(\lambda) A_i y_\lambda = 0$. Now $r_q(\lambda) y_\lambda = 0 \Rightarrow r_q(\lambda) = 0$, because $y_\lambda \neq 0$. This implies that $\lambda = \pm j\omega_p$, where $p \in \{q, \ldots, m-1\}$. For this value of λ, observe that

$$\sum_{i=q}^{m-1} v_{iq}(\lambda) A_i = v_{pq}(\lambda) A_p \neq 0$$

since A_p is positive definite. Thus $\sum_{i=q}^{m-1} \lambda v_{iq}(\lambda) A_i y_\lambda \neq 0$ for $\lambda = \pm j\omega_p$, where $p \in \{q, \ldots, m-1\}$. This implies that for $q = 0, \ldots, m-1$, $S_q(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. It is easy to see that $S_m(\lambda)$ has full column rank $\forall\ \lambda \in \mathbb{C}$. Observe that $S_0(\xi) = M_0(\xi)$. This implies that $\mathfrak{B}_0 = \operatorname{Im}\left(M_0(\frac{d}{dt})\right)$ is an observable image representation of $\mathfrak{B}_0$.

For $i = 0, \ldots, m-1$, define

$$Z_i(\xi) := \frac{\sum_{q=i}^{m-1} \xi v_q(\xi) A_q}{r(\xi)} = \sum_{q=i}^{m-1} \left(\frac{\xi A_q}{\xi^2 + \omega_q^2} \right)$$

Since Z_i has the same form of partial fraction expansion as the right hand side of equation (6.25), from Lemma 6.21, it follows that the rows of M_i are linearly independent over $\mathbb{R}$. For $i = 0, \ldots, m-1$, define

$$\mathcal{B}_i = \{v_i \mid \exists w_i \in \mathfrak{B}_i \text{ such that } w_i(0) = v_i\}$$

From Lemma 6.7, it follows that for $i = 0, \ldots, m-1$, $\mathcal{B}_i$ has dimension equal to $2l$. It now follows from Lemma 6.12 that $\mathfrak{B}_0$ is J-lossless.

It is easy to see that $\mathfrak{B}_i = \operatorname{Im}\left(S_i(\frac{d}{dt})\right)$ is an observable image representation of $\mathfrak{B}_i$ for $i = 0, \ldots, m$. Observe that the coefficient matrix A of the polynomial matrix formed by the highest degree terms in S_i is the same for $i = 0, \ldots, m-1$ and is given by $A = \operatorname{col}(I_l, 0_{l\times l})$. Observe also that A has full column rank. Hence S_i is column reduced for $i = 0, \ldots, m-1$. It follows that for $i = 0, \ldots, m-1$, $\mathtt{n}(\mathfrak{B}_i) = 2(m-i)l$, which implies that $\mathtt{n}(\mathfrak{B}_i) > \mathtt{n}(\mathfrak{B}_{i+1}) > 0$ for $i = 0, \ldots, m-2$. It is easy to see that $Q_J(w_m) = 0$ for any trajectory $w_m \in \mathfrak{B}_m$. Let $Y_i \in \mathbb{R}^{l\times 2l}[\xi]$ denote a left inverse of S_i for $i = 0, \ldots, m$.

For $q = 0, \ldots, m-2$, define

$$P_q(\xi) := \begin{bmatrix} \prod_{i=q}^{m-1} (\xi^2 + \omega_i^2) I_l \\ \sum_{i=q+1}^{m-1} \xi A_i v_{iq}(\xi) \end{bmatrix} Y_q(\xi). \tag{6.30}$$

Define

$$P_{m-1}(\xi) := \begin{bmatrix} (\xi^2 + \omega_{m-1}^2) I_l \\ 0_{l\times l} \end{bmatrix} Y_{m-1}(\xi) \tag{6.31}$$

It can be verified that $M_{i+1} = P_i M_i$ for $i = 0, \ldots, m-1$. Hence $\mathfrak{B}_{i+1} = \operatorname{Im}\left(P_i(\frac{d}{dt})\right)_{|\mathfrak{B}_i}$ for $i = 0, \ldots, m-1$. For $i = 0, \ldots, m-1$, define

$$Q_i(\xi) := \begin{bmatrix} (\xi^2 + \omega_i^2) I_l & 0_{l\times l} \\ \xi A_i & (\xi^2 + \omega_i^2) I_l \end{bmatrix}$$

and observe that

$$(\xi^2 + \omega_i^2) M_i(\xi) = Q_i(\xi) M_{i+1}(\xi) \tag{6.32}$$

Note that

$$Q_i(\zeta)^\top J Q_i(\eta) = (\zeta^2+\omega_i^2)(\eta^2+\omega_i^2)J + (\zeta+\eta)(\zeta\eta+\omega_i^2)\begin{bmatrix} A_i & 0_{l\times l} \\ 0_{l\times l} & 0_{l\times l}\end{bmatrix}$$

Pre- and postmultiplying both sides of the above equation with $M_{i+1}(\zeta)^\top$ and $M_{i+1}(\eta)$ respectively, using equation (6.32), we get

$$\begin{aligned}(\zeta^2+\omega_i^2)(\eta^2+\omega_i^2)M_i(\zeta)^\top J M_i(\eta) &= (\zeta^2+\omega_i^2)(\eta^2+\omega_i^2)M_{i+1}(\zeta)^\top J M_{i+1}(\eta) \\ &+(\zeta+\eta)(\zeta\eta+\omega_i^2)M_i(\zeta)^\top P_i(\zeta)^\top \begin{bmatrix} A_i & 0_{l\times l} \\ 0_{l\times l} & 0_{l\times l}\end{bmatrix} P_i(\eta)M_i(\eta)\end{aligned}$$

Using equations (6.30) and (6.31), it follows that

$$\begin{aligned}M_i(\zeta)^\top J M_i(\eta) &= (\zeta+\eta)(\zeta\eta+\omega_i^2)r_{i+1}(\zeta)r_{i+1}(\eta)M_i(\zeta)^\top Y_i(\zeta)^\top Y_i(\eta)M_i(\eta) \\ &+ M_{i+1}(\zeta)^\top J M_{i+1}(\eta)\end{aligned}$$

Define

$$\Omega_i(\zeta,\eta) := (\zeta\eta+\omega_i^2)r_{i+1}(\zeta)r_{i+1}(\eta)Y_i(\zeta)^\top Y_i(\eta)$$

For any two trajectories $w_i \in \mathfrak{B}_i$ and $w_{i+1} \in \mathfrak{B}_{i+1}$, that are related by $w_{i+1} = P_i(\frac{d}{dt})w_i$, observe that

$$Q_J(w_i) - Q_J(w_{i+1}) = \frac{d}{dt}Q_{\Omega_i}(w_i)$$

Also observe that $Q_{\Omega_i} \geq 0$ for $i = 0,\ldots,m-1$. Thus all conditions of Definition 6.13 are obeyed. Hence $\{\mathfrak{B}_1,\mathfrak{B}_2,\ldots,\mathfrak{B}_m\}$ is a synthesis of the J-lossless behaviour $\mathfrak{B}_0$. For $i = 1,\ldots,m$, define $M_i'(\xi) := \Xi M_i(\xi)$, and observe that

$$\Phi_i(\zeta,\eta) = \frac{M_i(\zeta)^\top J M_i(\eta)}{\zeta+\eta} = \frac{M_i'(\zeta)^\top \Sigma M_i'(\eta)}{\zeta+\eta}$$

For $i = 0,\ldots,m$, define $\mathfrak{B}_i' := \mathrm{Im}\big(M_i'(\frac{d}{dt})\big) = \mathrm{Im}(\Xi)_{|\mathfrak{B}_i}$. From Lemma 6.23, it follows that $\mathfrak{B}_0'$ is Σ-lossless and $\{\mathfrak{B}_1',\mathfrak{B}_2',\ldots,\mathfrak{B}_m'\}$ is a synthesis of $\mathfrak{B}_0'$. From Theorem 6.17, conclude that $\{Q_{\Phi_1},Q_{\Phi_2},\ldots,Q_{\Phi_m}\}$ is a synthesis of the positive QDF Q_{Φ_0}. ■

6.5 Nevanlinna diagonalization

We now introduce a new method of diagonalization of positive QDFs, which we call Nevanlinna diagonalization because this method is based on the method used for solving the subspace Nevanlinna interpolation problem in Rapisarda and Willems (1997). This method of diagonalization works only for a special category of positive QDFs. It will be shown that under certain conditions, Nevanlinna diagonalization of a positive QDF leads to a synthesis of the given QDF. In our algorithm for Nevanlinna diagonalization, we assume that we are given a two-variable polynomial matrix $\Phi_0 \in \mathbb{R}_s^{l\times l}[\zeta,\eta]$ which obeys the conditions $Q_{\Phi_0} > 0$ and $\Sigma_{\Phi_0} = I_{\mathtt{n}(\Psi_0)}$, where $\Psi_0(\zeta,\eta) := (\zeta+\eta)\Phi_0(\zeta,\eta)$ and Σ_{Φ_0} and $\mathtt{n}(\Psi_0)$ denote the signature of Φ_0 and McMillan degree of Ψ_0 respectively. The output of our algorithm is a diagonalization of Q_{Φ_0}. Throughout this section, $\mathtt{w}$, $\mathtt{w}_1$ and $\mathtt{w}_2$ denote nonnegative integers such that $\mathtt{w}_1 + \mathtt{w}_2 = \mathtt{w}$, and Σ denotes the following matrix:

$$\begin{bmatrix} I_{\mathtt{w}_1} & 0_{\mathtt{w}_1\times\mathtt{w}_2} \\ 0_{\mathtt{w}_2\times\mathtt{w}_1} & -I_{\mathtt{w}_2} \end{bmatrix}$$

The first step of our algorithm involves obtaining a canonical factorization of $\Psi_0(\zeta,\eta) := (\zeta+\eta)\Phi_0(\zeta+\eta)$ of the form $\Psi_0(\zeta,\eta) = M_0(\zeta)^\top \Sigma M_0(\eta)$, where $M_0 \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$. The next step involves column reduction of M_0 to obtain a matrix $V_1 \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ such that V_1 has nondecreasing order of column degrees. We then choose a real, nonzero number λ_1 and compute $T_1 = \frac{V_1(\lambda_1)^\top \Sigma V_1(\lambda_1)}{2\lambda_1}$.

Since $\Gamma_1(\zeta,\eta) = \frac{V_1(\zeta)^\top \Sigma V_1(\eta)}{\zeta+\eta}$ is such that $Q_{\Gamma_1} > 0$, it follows that T_1 is positive definite and hence $\det(T_1) \neq 0$. The next step of the algorithm involves computation of the following

$$S_1(\xi) = (\xi+\lambda_1)I_{\mathtt{w}} - V_1(\lambda_i)T_1^{-1}V_1(\lambda_i)^\top \Sigma$$

$$V_2(\xi) = \frac{S_1(\xi)V_1(\xi)}{\xi^2-\lambda_1^2} \qquad T_2 = \frac{V_2(\lambda_2)^\top \Sigma V_2(\lambda_2)}{2\lambda_2}$$

where λ_2 is another nonzero real number. We now prove that V_2 is column reduced, has polynomial entries and $V_2(\xi)^\top \Sigma V_2(-\xi) = V_2(-\xi)^\top \Sigma V_2(\xi) = 0$. We also prove that the column degree of every column of V_2 is one less than that of the corresponding column of V_1.

Lemma 6.25. *Consider a column reduced $V_1 \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$, such that $V_1(-\xi)^\top \Sigma V_1(\xi) = 0$. Define $T_1 := \frac{V_1(\lambda_1)^\top \Sigma V_1(\lambda_1)}{2\lambda_1}$ where $\lambda_1 \in \mathbb{R}$ is nonzero. Assume that $\det(T_1) \neq 0$. Define*

$$S_1(\xi) := (\xi+\lambda_1)I_{\mathtt{w}} - V_1(\lambda_1)T_1^{-1}V_1(\lambda_1)^\top \Sigma$$

Define $V_2(\xi) := \frac{S_1(\xi)V_1(\xi)}{\xi^2-\lambda_1^2}$. Then

1. *V_2 has polynomial entries.*
2. *$V_2(\xi)^\top \Sigma V_2(-\xi) = V_2(-\xi)^\top \Sigma V_2(\xi) = 0$.*
3. *V_2 is column reduced.*
4. *The column degree of every column of V_2 is one less than that of the corresponding column of V_1.*

Proof. It is easy to see that

$$S_1(\lambda_1)V_1(\lambda_1) = 0 \tag{6.33}$$

and

$$S_1(-\lambda_1)V_1(-\lambda_1) = 0 \tag{6.34}$$

From equations (6.33) and (6.34), it follows that $S_1(\xi)V_1(\xi)$ is divisible by $(\xi^2 - \lambda_1^2)$, and consequently $V_2(\xi)$ consists of polynomial entries. Now observe that $S_1(\zeta)^\top \Sigma S_1(\eta)$

$$\begin{aligned} &= [(\zeta+\lambda_1)I_{\mathtt{w}} - \Sigma V_1(\lambda_1)T_1^{-1}V_1(\lambda_1)^\top]\Sigma[(\eta+\lambda_1)I_{\mathtt{w}} - V_1(\lambda_1)T_1^{-1}V_1(\lambda_1)^\top\Sigma] \\ &= (\zeta+\lambda_1)(\eta+\lambda_1)I_{\mathtt{w}} - (\zeta+\eta)\Sigma V_1(\lambda_1)T_1^{-1}V_1(\lambda_1)^\top\Sigma \end{aligned}$$

Also

$$V_1(\zeta)^\top S_1(\zeta)^\top \Sigma S_1(\eta)V_1(\eta) = (\zeta^2-\lambda_1^2)(\eta^2-\lambda_1^2)V_2(\zeta)^\top\Sigma V_2(\eta)$$

Hence it follows that

$$V_1(\zeta)^\top\Sigma V_1(\eta)(\zeta+\lambda_1)(\eta+\lambda_1) = (\zeta^2-\lambda_1^2)(\eta^2-\lambda_1^2)V_2(\zeta)^\top\Sigma V_2(\eta)$$

$$+(\zeta+\eta)V_1(\zeta)^\top\Sigma V_1(\lambda_1)T_1^{-1}V_1(\lambda_1)^\top\Sigma V_1(\eta)$$

Hence $V_2(-\xi)^\top\Sigma V_2(\xi) = V_2(\xi)^\top\Sigma V_2(-\xi) = 0$.

Let $n_1, n_2, \ldots, n_l$ denote the column degrees of columns $1, 2, \ldots, l$ of V_1. We can write

$$V_1(\xi) = A \text{ diag}(\xi^{n_1}, \xi^{n_2}, \ldots, \xi^{n_{l-1}}, \xi^{n_l}) + B(\xi)$$

where $A \in \mathbb{R}^{\mathtt{w}\times l}$ is the coefficient matrix of the matrix formed by the highest degree terms in every column of V_1. $B \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ is the matrix containing the remaining lower degree terms of V_1. Since V_1 is column reduced, it follows that A has full column rank. Now observe that

$$S(\xi) = \begin{bmatrix} \xi + c_{1,1} & c_{1,2} & \cdots & \cdots & c_{1,\mathtt{w}} \\ c_{2,1} & \xi + c_{2,2} & c_{2,3} & \cdots & c_{2,\mathtt{w}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ c_{\mathtt{w}-1,1} & \cdots & c_{\mathtt{w}-1,\mathtt{w}-2} & \xi + c_{\mathtt{w}-1,\mathtt{w}-1} & c_{\mathtt{w}-1,\mathtt{w}} \\ c_{\mathtt{w},1} & \cdots & \cdots & c_{\mathtt{w},\mathtt{w}-1} & \xi + c_{\mathtt{w},\mathtt{w}} \end{bmatrix}$$

where $c_{i,k} \in \mathbb{R}$ for integral values of i and k. Since $V_2(\xi) = \frac{S(\xi)V_1(\xi)}{\xi^2-\lambda^2}$ has polynomial entries, it can be inferred that the position and coefficients of the highest degree terms in V_2 are the same as in V_1, the only difference being that the degree of these terms is one less than those in V_1. Hence, we can write

$$V_2(\xi) = A \text{ diag}(\xi^{n_1-1}, \xi^{n_2-1}, \ldots, \xi^{n_l-2}, \xi^{n_l-1}) + C(\xi)$$

where $C \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ consists of lower degree terms of V_2. Since A has full column rank, it can be inferred that V_2 is column reduced, and has column rank of each column, one less than that of the corresponding column of V_1. ■

Now assume that the last $l-\mathtt{p}$ ($l \geq \mathtt{p}$) columns of V_1 have column degree equal to 1. We prove that this will lead to $\det(T_2)=0$.

Lemma 6.26. *Assume that $V_1 \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ is column reduced in such a way that if n_i denotes the column degree of the i^{th} column of V_1, then $n_1 \geq n_2 \geq \cdots \geq n_{\mathtt{p}} > n_{\mathtt{p}+1} = n_{\mathtt{p}+2} = \cdots = n_l = 1$. Also assume that $V_1(-\xi)^\top \Sigma V_1(\xi) = 0$. Define $T_1 := \frac{V_1(\lambda_1)^\top \Sigma V_1(\lambda_1)}{2\lambda_1}$, and assume that $\det(T_1) \neq 0$. Define $S_1 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ as*

$$S_1(\xi) := (\xi+\lambda_1)I_{\mathtt{w}} - V_1(\lambda_1)T_1^{-1}V_1(\lambda_1)^\top\Sigma$$

where $\lambda_1 \in \mathbb{R}$ is nonzero, and $V_2(\xi) := \frac{S_1(\xi)V_1(\xi)}{\xi^2-\lambda_1^2}$. Then $(V_2(\zeta)^\top\Sigma V_2(\eta))$ has the last $l-\mathtt{p}$ rows and columns full of zeroes.

Proof. From Lemma 6.25, it follows that $V_2(\xi)^\top\Sigma V_2(-\xi) = V_2(-\xi)^\top\Sigma V_2(\xi) = 0$ and $S_1(-\lambda_1)V_1(-\lambda_1) = S_1(\lambda_1)V_1(\lambda_1) = 0$. Consequently $V_2 \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$. From the proof of Lemma 6.25, it follows that we can write V_2 as

$$V_2(\xi) = A\text{diag}(\xi^{n_1-1}, \xi^{n_2-1}, \ldots, \xi^{n_{\mathtt{p}}-1}, 1, \ldots, 1) + C'(\xi)$$

where $C' \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ consists of lower degree terms of V_2, and $A \in \mathbb{R}^{\mathtt{w}\times l}$ has full column rank. It can be inferred that the last $(l-\mathtt{p})$ columns of V_2 consist of constants. Partition V_2 as

$$V_2 = \begin{bmatrix} D \\ N \end{bmatrix}$$

where $D \in \mathbb{R}^{\mathtt{w}_1\times l}, N \in \mathbb{R}^{\mathtt{w}_2\times l}[\xi]$. We have

$$V_2(\zeta)^\top\Sigma V_2(\eta) = D(\zeta)^\top D(\eta) - N(\zeta)^\top N(\eta)$$

Consider the two-variable polynomial matrix $D(\zeta)^\top D(\eta)$. Since we know that the last $(l-\mathtt{p})$ columns of D consist of constants, it can be verified that $D(\zeta)^\top D(\eta)$ can be partitioned as

$$D(\zeta)^\top D(\eta) = \begin{bmatrix} P_1(\zeta,\eta) & G_1(\zeta) \\ F_1(\eta) & K_1 \end{bmatrix} \tag{6.35}$$

where $P_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\zeta,\eta]$, $G_1 \in \mathbb{R}^{\mathtt{p}\times(l-\mathtt{p})}[\xi]$, $F_1 \in \mathbb{R}^{(l-\mathtt{p})\times\mathtt{p}}[\xi]$ and $K_1 \in \mathbb{R}^{(l-\mathtt{p})\times(l-\mathtt{p})}$. Also observe that since the last $(l-\mathtt{p})$ columns of N consist of constants $N(\zeta)^\top N(\eta)$ can be partitioned as

$$N(\zeta)^\top N(\eta) = \begin{bmatrix} P_2(\zeta,\eta) & G_2(\zeta) \\ F_2(\eta) & K_2 \end{bmatrix} \tag{6.36}$$

where $P_2 \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\zeta,\eta]$, $G_2 \in \mathbb{R}^{\mathtt{p}\times(l-\mathtt{p})}[\xi]$, $F_2 \in \mathbb{R}^{(l-\mathtt{p})\times\mathtt{p}}[\xi]$ and $K_2 \in \mathbb{R}^{(l-\mathtt{p})\times(l-\mathtt{p})}$. From Lemma 6.25, it follows that $V_2(-\xi)^\top\Sigma V_2(\xi) = 0$. Hence $D(-\xi)^\top D(\xi) = N(-\xi)^\top N(\xi)$. Hence, we get $G_1(\xi) = G_2(\xi)$, $F_1(\xi) = F_2(\xi)$ and $K_1 = K_2$. It now follows from equations (6.35) and (6.36) that $V_2(\zeta)^\top \Sigma V_2(\eta) = D(\zeta)^\top D(\eta) - N(\zeta)^\top N(\eta)$ has the last $(l-\mathtt{p})$ rows and columns consisting of zeroes. ■

Now if V_2 has its last $(l-\mathtt{p})$ columns constants, then we assign a new expression for V_2 as follows. Write V_2 as $V_2(\xi) = \text{row}(V_2'(\xi), K)$, where $V_2' \in \mathbb{R}^{\mathtt{w}\times\mathtt{p}}[\xi]$ and $K \in \mathbb{R}^{\mathtt{w}\times(l-\mathtt{p})}$. Assign $V_2(\xi) := V_2'(\xi)$. Note that the new expression for V_2 has been obtained by just deleting its constant columns. With this new expression for V_2, compute T_2 as

$$T_2 = \frac{V_2(\lambda_2)^\top \Sigma V_2(\lambda_2)}{2\lambda_2}$$

If all columns of V_2 have column degrees greater than zero, then we continue with the same expression for V_2, and the same value for T_2 as before. We now prove that T_2 is positive definite, and $V_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

Theorem 6.27. *Assume that $V_1 \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$ is column reduced in such a way that if $n_1, n_2, \ldots, n_l$ represent the column degrees of columns $1, 2, \ldots, l$ of V_1, then $n_1 \geq n_2 \geq \cdots \geq n_l$. Also assume that $V_1(-\xi)^\top \Sigma V_1(\xi) = 0$. Define*

$$\Gamma_1(\zeta, \eta) := \frac{V_1(\zeta)^\top \Sigma V_1(\eta)}{\zeta + \eta}$$

Let n_0 denote the McMillan degree of $\mathfrak{B}_0 = Im\big(V_1(\frac{d}{dt})\big)$. Assume that $Q_{\Gamma_1} > 0$ and $\Sigma_{\Gamma_1} = I_{n_0}$. Define $T_1 := \Gamma_1(\lambda_1, \lambda_1)$, where $\lambda_1 \in \mathbb{R}$ is nonzero. Define $S_1 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ as

$$S_1(\xi) := (\xi + \lambda_1) I_{\mathtt{w}} - V_1(\lambda_1) T_1^{-1} V_1(\lambda_1)^\top \Sigma$$

Define $V_2(\xi) := \frac{S_1(\xi)V_1(\xi)}{\xi^2 - \lambda_1^2}$. Assume that $\mathtt{p}$ columns of V_2 have degree greater than or equal to 1. Assume also that $\mathtt{p} > 0$. Define

$$V_2'(\xi) := V_2(\xi) \begin{bmatrix} I_{\mathtt{p}} \\ 0_{l-\mathtt{p}} \end{bmatrix}$$

Define

$$\Gamma_2(\zeta, \eta) := \frac{V_2'(\zeta)^\top \Sigma V_2'(\eta)}{\zeta + \eta}$$

Then $Q_{\Gamma_2} > 0$, $\Sigma_{\Gamma_2} = I_{n_1}$, where n_1 denotes the McMillan degree of $\mathfrak{B}_1 = Im\big(V_2'(\frac{d}{dt})\big)$ and $V_2'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$.

Proof. Since $Q_{\Gamma_1} > 0$, it follows that T_1 is positive definite and hence $\det(T_1) \neq 0$. From Lemma 6.25, it follows that V_2 has polynomial entries. Now observe that $S_1(\zeta)^\top \Sigma S_1(\eta)$

$$\begin{aligned} &= [(\zeta + \lambda_1) I_{\mathtt{w}} - \Sigma V_1(\lambda_1) T_1^{-1} V_1(\lambda_1)^\top] \Sigma [(\eta + \lambda_1) I_{\mathtt{w}} - V_1(\lambda_1) T_1^{-1} V_1(\lambda_1)^\top \Sigma] \\ &= (\zeta + \lambda_1)(\eta + \lambda_1) I_{\mathtt{w}} - (\zeta + \eta) \Sigma V_1(\lambda_1) T_1^{-1} V_1(\lambda_1)^\top \Sigma \end{aligned}$$

Also

$$V_1(\zeta)^\top S_1(\zeta)^\top \Sigma S_1(\eta) V_1(\eta) = (\zeta^2 - \lambda_1^2)(\eta^2 - \lambda_1^2) V_2(\zeta)^\top \Sigma V_2(\eta)$$

Hence it follows that

$$\begin{aligned} V_1(\zeta)^\top \Sigma V_1(\eta)(\zeta+\lambda_1)(\eta+\lambda_1) &= (\zeta^2-\lambda_1^2)(\eta^2-\lambda_1^2)V_2(\zeta)^\top \Sigma V_2(\eta) \\ &+ (\zeta+\eta)V_1(\zeta)^\top \Sigma V_1(\lambda_1) T_1^{-1} V_1(\lambda_1)^\top \Sigma V_1(\eta) \end{aligned}$$

Dividing the above equation by $(\zeta+\eta)(\zeta+\lambda_1)(\eta+\lambda_1)$, we obtain

$$\frac{V_1(\zeta)^\top \Sigma V_1(\eta)}{\zeta+\eta} = (\zeta-\lambda_1)(\eta-\lambda_1)\frac{V_2(\zeta)^\top \Sigma V_2(\eta)}{\zeta+\eta} + \frac{V_1(\zeta)^\top \Sigma V_1(\lambda_1) T_1^{-1} V_1(\lambda_1)^\top \Sigma V_1(\eta)}{(\zeta+\lambda_1)(\eta+\lambda_1)} \tag{6.37}$$

Observe that the second term of the right hand side of the above equation is a matrix with polynomial entries because $V_1(\lambda_1)^\top \Sigma V_1(\xi)$ is divisible by $(\xi+\lambda_1)$. From Lemma 6.25, it follows that the column degree of every column of V_2 is one less than that of the corresponding column of V_1. Define

$$\Phi_2(\zeta,\eta) := \frac{V_2(\zeta)^\top \Sigma V_2(\eta)}{\zeta+\eta}$$

Since the last $(l-\mathtt{p})$ columns of V_2 have constant entries, from Lemma 6.26, it follows that

$$\Phi_2(\zeta,\eta) = \begin{bmatrix} \Gamma_2(\zeta,\eta) & 0_{\mathtt{p}\times(l-\mathtt{p})} \\ 0_{(l-\mathtt{p})\times\mathtt{p}} & 0_{(l-\mathtt{p})\times(l-\mathtt{p})} \end{bmatrix}$$

Let $X_2(\xi) := \operatorname{col}_{i=0}^{n_1-2}\big(\operatorname{row}(\xi^i I_{y_i}, 0_{y_i\times(l-y_i)})\big)$, where y_i denotes the number of columns of V_2 with column degree greater than or equal to $i+1$. Let $\mathtt{x}_2$ denote the number of rows of X_2. Partition X_2 as $X_2(\xi) = \operatorname{row}(X_2'(\xi), 0_{\mathtt{x}_2\times(l-\mathtt{p})})$. Observe that

$$\Gamma_2(\zeta,\eta) = \frac{V_2'(\zeta)\Sigma V_2'(\eta)}{\zeta+\eta} = X_2'(\zeta)\tilde{\Gamma}_2 X_2'(\eta)$$

where $\tilde{\Gamma}_2 \in \mathbb{R}_s^{\mathtt{x}_2\times\mathtt{x}_2}$ is related to the coefficient matrix of Γ_2, in the sense that it has finite dimensions and it consists of all the nonzero terms of the coefficient matrix of Γ_2. Hence we can write

$$\Phi_2(\zeta,\eta) = X_2(\zeta)^\top \tilde{\Gamma}_2 X_2(\eta), \tag{6.38}$$

Let $\Sigma_1 \in \mathbb{R}_s^{\mathtt{q}\times\mathtt{q}}$ denote the signature matrix of $\tilde{\Gamma}_2$. Then we can write a symmetric canonical factorization of $\tilde{\Gamma}_2$ as $\tilde{\Gamma}_2 = L^\top \Sigma_1 L$, where $L \in \mathbb{R}^{\mathtt{q}\times\mathtt{x}_2}$ has full row rank. It is easy to see that the rows of X_2 are linearly independent over $\mathbb{R}$. We now prove that the rows of $LX_2(\xi)$ are also linearly independent over $\mathbb{R}$. Write L as

$$L = \operatorname{col}(L_1, L_2, \ldots, L_{\mathtt{q}})$$

where for $i = 1, \ldots, \mathtt{q}$, L_i denotes the i^{th} row of L. Assume by contradiction that the rows of $LX_2(\xi)$ are linearly dependent over $\mathbb{R}$. Then there exist nonzero real numbers $a_1, \ldots, a_{\mathtt{q}}$, such that

$$a_1 L_1 X_2(\xi) + a_2 L_2 X_2(\xi) + \ldots + a_{\mathtt{q}} L_{\mathtt{q}} X_2(\xi) = 0$$

This implies that

$$a_1 L_1 + a_2 L_2 + \ldots + a_{\mathtt{q}} L_{\mathtt{q}} = 0$$

This in turn implies that $a_i = 0$ for $i = 1, \ldots, \mathtt{q}$ as L has linearly independent rows, which is a contradiction. Thus the rows of $LX_2(\xi)$ are linearly independent over $\mathbb{R}$. Since T_1^{-1} is positive definite, we can factorize it as $T_1^{-1} = D^\top D$, where $D \in \mathbb{R}^{l \times l}$ is nonsingular. Define

$$X_1(\xi) := \frac{V_1(\lambda_1)^\top \Sigma V_1(\xi)}{\xi + \lambda_1}$$

Then from equations (6.37) and (6.38), it follows that $\Gamma_1(\zeta, \eta) = X(\zeta)^\top \Delta X(\eta)$, where $X(\xi) = \text{col}(DX_1(\xi), (\xi - \lambda_1)LX_2(\xi))$ and $\Delta = \text{diag}(I_l, \Sigma_1)$. Let $\mathtt{x}$ denote the number of rows of X. We first prove that the rows of X are linearly independent over $\mathbb{R}$. Assume by contradiction that there exists a nonzero $G \in \mathbb{R}^{\mathtt{x}}$, such that

$$G^\top X(\xi) = 0$$

Partition G as $G = \text{col}(G_1, G_2)$, where $G_1 \in \mathbb{R}^l$ and $G_2 \in \mathbb{R}^{\mathtt{x}-l}$. Now $G^\top X(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. Putting $\lambda = \lambda_1$, we get $G_1^\top T_1 = 0$. Since $\det(T_1) \neq 0$, this implies that $G_1 = 0$. Hence we have $G_2^\top (\xi - \lambda) LX_2(\xi) = 0$. Since the rows of $LX_2(\xi)$ are linearly independent over $\mathbb{R}$, we have $G_2 = 0$, which is a contradiction. Hence the rows of X are linearly independent over $\mathbb{R}$. Thus

$$\Gamma_1(\zeta, \eta) = X(\zeta)^\top \Delta X(\eta)$$

is a symmetric canonical factorization of Γ_1, which implies that Δ is the signature of Γ_1. It can be verified that $\mathtt{x} = \sum_{i=1}^{l} n_i$, which is equal to the McMillan degree of $\mathfrak{B}_0 = \text{Im}\big(V_1(\frac{d}{dt})\big)$. Consequently, $\Delta = I_{\mathtt{x}}$, which implies that $\Sigma_1 = I_{\mathtt{x}_2}$. Observe that $\mathtt{x}_2 = \mathtt{x} - l = n_1 = \mathtt{n}(\mathfrak{B}_1)$. This implies that $\Sigma_{\Gamma_2} = I_{n_1}$, $\mathtt{q} = \mathtt{x}_2$, and L is square and nonsingular. Since $y_0 = \mathtt{p}$, it follows that $X_2'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. We have

$$\Gamma_2(\zeta, \eta) = X_2'(\zeta) \tilde{\Gamma}_2 X_2'(\eta)$$

Since $\tilde{\Gamma}_2 = L^\top \Sigma_1 L = L^\top I_{\mathtt{x}_2} L$, with L square and nonsingular, $\tilde{\Gamma}_2$ is positive definite. Consequently, $Q_{\Gamma_2} > 0$. Since

$$\Gamma_2(\zeta, \eta) = \frac{V_2'(\zeta) \Sigma V_2'(\eta)}{\zeta + \eta}$$

from Lemma 6.8, it follows that $V_2'(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. ■

Since $Q_{\Gamma_2} > 0$, it follows that $\det(T_2) \neq 0$. Let N denote the degree of the highest degree polynomial of V_1. The next step of the algorithm is to iteratively carry out the following operations for $i = 2, \ldots, N$:

(1) Compute

$$S_i(\xi) = (\xi + \lambda_i)I_{\mathtt{w}} - V_i(\lambda_i)T_i^{-1}V_i(\lambda_i)^\top \Sigma$$

$$V_{i+1}(\xi) = \frac{S_i(\xi)V_i(\xi)}{\xi^2 - \lambda_i^2}$$

where λ_i are arbitrarily chosen nonzero real numbers for $i = 2, \ldots, N$.

(2) In case V_{i+1} has columns with column degree 0, delete such columns from V_{i+1} to obtain V'_{i+1}. Then assign $V_{i+1}(\xi) := V'_{i+1}(\xi)$, and compute

$$T_{i+1} = \frac{V_{i+1}(\lambda_{i+1})^\top \Sigma V_{i+1}(\lambda_{i+1})}{2\lambda_{i+1}}$$

We now prove by induction that $\det(T_i) \neq 0$ for $i = 2, \ldots, N$. For $i = 3, \ldots, N$, define

$$\Gamma_i(\zeta, \eta) := \frac{V_i(\zeta)^\top \Sigma V_i(\eta)}{\zeta + \eta}$$

We know from Theorem 6.27 that $Q_{\Gamma_2} > 0$, and $\Sigma_{\Gamma_2} = I_{n_1}$, where n_1 denotes the McMillan degree of $\mathfrak{B}_1 = \mathrm{Im}\big(V_2(\frac{d}{dt})\big)$. Assume that for some $i \in \{2, \ldots, N-1\}$, $Q_{\Gamma_i} > 0$, and $\Sigma_{\Gamma_i} = I_{n_{i-1}}$, where n_{i-1} denotes the McMillan degree of $\mathfrak{B}_{i-1} = \mathrm{Im}\big(V_i(\frac{d}{dt})\big)$. Then from Theorem 6.27, it follows that $Q_{\Gamma_{i+1}} > 0$, and $\Sigma_{\Gamma_{i+1}} = I_{n_i}$, where n_i denotes the McMillan degree of $\mathfrak{B}_i = \mathrm{Im}\big(V_{i+1}(\frac{d}{dt})\big)$. Consequently $\det(T_{i+1}) \neq 0$. Hence by induction, it follows that $\det(T_i) \neq 0$ for $i = 2, \ldots, N$.

We now give the formal algorithm for Nevanlinna diagonalization of a positive QDF, and also prove that under certain conditions it leads to a synthesis of the given QDF.

Algorithm 6.28. Data*:* $\Phi_0 \in \mathbb{R}_s^{l \times l}[\zeta, \eta]$, *such that* $Q_{\Phi_0} > 0$ *and* $\Sigma_{\Phi_0} = I_{\mathtt{n}(\Psi_0)}$, *where* $\Psi_0(\zeta, \eta) := (\zeta + \eta)\Phi_0(\zeta, \eta)$ *and* Σ_{Φ_0} *and* $\mathtt{n}(\Psi_0)$ *denote the signature of* Φ_0 *and McMillan degree of* Ψ_0 *respectively.*

Output*: A sequence* $\{\Phi_1(\zeta, \eta), \Phi_2(\zeta, \eta), \ldots, \Phi_m(\zeta, \eta)\}$, *such that* $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_m}\}$ *is a Nevanlinna diagonalization of* Q_{Φ_0}.

Step 1 : Obtain a canonical factorization of $\Psi_0(\zeta, \eta) := (\zeta + \eta)\Phi_0(\zeta + \eta)$ of the form $\Psi_0(\zeta, \eta) = M_0(\zeta)^\top \Sigma M_0(\eta)$, where Σ denotes the signature of Ψ_0 and $M_0 \in \mathbb{R}^{\mathtt{w} \times l}[\xi]$.

Step 2 : Find a unimodular matrix $V \in \mathbb{R}^{l \times l}[\xi]$, such that $V_1(\xi) := M_0(\xi)V(\xi)$ is column reduced with $n_1 \geq n_2 \geq \cdots \geq n_l$, where $n_1, n_2, \ldots, n_l$ denote the column degrees of columns $1, 2, \ldots, l$ of V_1.

Step 3 : Choose real, nonzero numbers $\lambda_1, \lambda_2, \ldots, \lambda_N$, where N denotes the degree of the highest degree polynomial in V_1. Assign $T_1 = \frac{V_1(\lambda_1)^\top \Sigma V_1(\lambda_1)}{2\lambda_1}$

Step 4 : Assign $i := 1$.

Step 5 : Assign

$$\begin{aligned}
S_i(\xi) &= (\xi+\lambda_i)I_{\mathtt{w}} - V_i(\lambda_i)T_i^{-1}V_i(\lambda_i)^\top\Sigma \\
V_{i+1}(\xi) &= \frac{S_i(\xi)V_i(\xi)}{\xi^2-\lambda_i^2} \\
T_{i+1} &= \frac{V_{i+1}(\lambda_{i+1})^\top \Sigma V_{i+1}(\lambda_{i+1})}{2\lambda_{i+1}} \\
\Phi_i'(\zeta,\eta) &= \left(\frac{V_{i+1}(\zeta)^\top \Sigma V_{i+1}(\eta)}{\zeta+\eta}\right)\prod_{k=1}^{i}(\zeta-\lambda_k)(\eta-\lambda_k) \\
\Phi_i(\zeta,\eta) &= V(\zeta)^{-\top}\begin{bmatrix} \Phi_i'(\zeta,\eta) & 0_{l_{i+1}\times(l-l_{i+1})} \\ 0_{(l-l_{i+1})\times l_{i+1}} & 0_{(l-l_{i+1})\times(l-l_{i+1})}\end{bmatrix}V(\eta)^{-1}
\end{aligned}$$

where l_{i+1} denotes the number of columns of V_{i+1}.

Step 6 : If $T_{i+1} = 0$, output "$\{\Phi_1(\zeta,\eta), \Phi_2(\zeta,\eta), \ldots, \Phi_i(\zeta,\eta)\}$" and stop. Else if $\det(T_{i+1}) \neq 0$, go to step 7, else go to step 8.

Step 7 : Put $i = i+1$ and go to step 5.

Step 8 : Verify that T_{i+1} has at least one row and one column full of zeroes. Delete the corresponding columns of V_{i+1} to obtain V_{i+1}'.

Step 9 : Assign $V_{i+1}(\xi) := V_{i+1}'(\xi)$.

Step 10 : Evaluate $T_{i+1} = \frac{V_{i+1}(\lambda_{i+1})^\top \Sigma V_{i+1}(\lambda_{i+1})}{2\lambda_{i+1}}$. Go to step 7.

We now prove the correctness of Algorithm 6.28.

Theorem 6.29. *With reference to Algorithm 6.28, $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_N}\}$ is a diagonalization of Q_{Φ_0}. Further if the rows of V_i are linearly independent over $\mathbb{R}$ for $i = 2, \ldots, m$, where $m \leq N$ is an integer, then $\{Q_{\Phi_1}, Q_{\Phi_2}, \ldots, Q_{\Phi_{m-1}}, Q_{\Phi_N}\}$ is a synthesis of Q_{Φ_0}.*

Proof. $Q_{\Phi_0} > 0$ implies that also $Q_{\Gamma_1} > 0$, where $\Gamma_1(\zeta,\eta) = V(\zeta)^\top \Phi_0(\zeta,\eta)V(\eta) = \frac{V_1(\zeta)^\top \Sigma V_1(\eta)}{\zeta+\eta}$, since V is unimodular. From Lemma 6.8, it follows that $V_1(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. We now prove by induction that $\det(T_i) \neq 0$ for $i = 2, \ldots, N$. For $i = 3, \ldots, N$, define

$$\Gamma_i(\zeta,\eta) := \frac{V_i(\zeta)^\top \Sigma V_i(\eta)}{\zeta+\eta}$$

We know from Theorem 6.27 that $Q_{\Gamma_2} > 0$, and $\Sigma_{\Gamma_2} = I_{n_1}$, where n_1 denotes the McMillan degree of $\mathfrak{B}_1 = \mathrm{Im}\big(V_2(\frac{d}{dt})\big)$. Assume that for some $i \in \{2, \ldots, N-1\}$, $Q_{\Gamma_i} > 0$, and $\Sigma_{\Gamma_i} = I_{n_{i-1}}$, where n_{i-1} denotes the McMillan degree of $\mathfrak{B}_{i-1} = \mathrm{Im}\big(V_i(\frac{d}{dt})\big)$. Then from Theorem 6.27, it follows that $Q_{\Gamma_{i+1}} > 0$, and $\Sigma_{\Gamma_{i+1}} = I_{n_i}$, where n_i denotes the McMillan degree of $\mathfrak{B}_i = \operatorname{Im}\big(V_{i+1}(\frac{d}{dt})\big)$. Consequently, $\det(T_{i+1}) \neq 0$. Hence by induction, it follows that $\det(T_i) \neq 0$ for $i = 2, \dots, N$. From Lemmas 6.25 and 6.26, it is easy to see that $\Phi_N(\zeta, \eta) = 0$.

Let l_i denote the number of columns of V_i for $i = 2, \dots, N$. We have for every $i \in \{1, \dots, N\}$,

$$S_i(\zeta)^\top \Sigma S_i(\eta) = (\zeta + \lambda_i)(\eta + \lambda_i) I_{\mathtt{w}} - (\zeta + \eta)\Sigma V_i(\lambda_i) T_i^{-1} V_i(\lambda_i)^\top \Sigma$$

Also

$$V_i(\zeta)^\top S_i(\zeta)^\top \Sigma S_i(\eta) V_i(\eta) = (\zeta^2 - \lambda_i^2)(\eta^2 - \lambda_i^2) \begin{bmatrix} V_{i+1}(\zeta)^\top \Sigma V_{i+1}(\eta) & 0_{l_{i+1}\times(l_i - l_{i+1})} \\ 0_{(l_i - l_{i+1})\times l_{i+1}} & 0_{(l_i - l_{i+1})\times(l_i - l_{i+1})} \end{bmatrix}$$

Hence it follows that $V_i(\zeta)^\top \Sigma V_i(\eta)(\zeta + \lambda_i)(\eta + \lambda_i)$

$$\begin{aligned} &= (\zeta^2 - \lambda_i^2)(\eta^2 - \lambda_i^2) \begin{bmatrix} V_{i+1}(\zeta)^\top \Sigma V_{i+1}(\eta) & 0_{l_{i+1}\times(l_i - l_{i+1})} \\ 0_{(l_i - l_{i+1})\times l_{i+1}} & 0_{(l_i - l_{i+1})\times(l_i - l_{i+1})} \end{bmatrix} \\ &+ (\zeta + \eta) V_i(\zeta)^\top \Sigma V_i(\lambda_i) T_i^{-1} V_i(\lambda_i)^\top \Sigma V_i(\eta) \end{aligned}$$

Dividing the above equation by $(\zeta + \eta)(\zeta + \lambda_i)(\eta + \lambda_i)$, we obtain

$$\begin{aligned} \frac{V_i(\zeta)^\top \Sigma V_i(\eta)}{\zeta + \eta} &= (\zeta - \lambda_i)(\eta - \lambda_i) \begin{bmatrix} \frac{V_{i+1}(\zeta)^\top \Sigma V_{i+1}(\eta)}{\zeta + \eta} & 0_{l_{i+1}\times(l_i - l_{i+1})} \\ 0_{(l_i - l_{i+1})\times l_{i+1}} & 0_{(l_i - l_{i+1})\times(l_i - l_{i+1})} \end{bmatrix} \\ &+ \frac{V_i(\zeta)^\top \Sigma V_i(\lambda_1) T_i^{-1} V_i(\lambda_i)^\top \Sigma V_i(\eta)}{(\zeta + \lambda_i)(\eta + \lambda_i)} \end{aligned}$$

Observe that the second term in the right hand side of the above equation is a matrix with polynomial entries because $V_i(\lambda_i)^\top \Sigma V_i(\xi)$ is divisible by $(\xi + \lambda_i)$. Pre- and postmultiplying both sides of the equation with $V(\zeta)^{-\top} \prod_{k=1}^{i-1}(\zeta - \lambda_k)$ and $V(\eta)^{-1} \prod_{k=1}^{i-1}(\eta - \lambda_k)$ respectively, we get

$$\Phi_{i-1}(\zeta, \eta) = \Phi_i(\zeta, \eta) + \Lambda_{i-1}(\zeta, \eta)$$

where

$$\Lambda_{i-1}(\zeta, \eta) = V(\zeta)^{-\top} \Lambda'_{i-1}(\zeta, \eta) V(\eta)^{-1}$$

$$\Lambda'_{i-1}(\zeta, \eta) = \begin{bmatrix} \left(\frac{V_i(\zeta)^\top \Sigma V_i(\lambda_1) T_i^{-1} V_i(\lambda_i)^\top \Sigma V_i(\eta)}{(\zeta + \lambda_i)(\eta + \lambda_i)}\right) \prod_{k=1}^{i-1}(\zeta - \lambda_k)(\eta - \lambda_k) & 0_{l_i\times(l - l_i)} \\ 0_{(l - l_i)\times l_i} & 0_{(l - l_i)\times(l - l_i)} \end{bmatrix}$$

Since T_i is positive definite for $i = 1, \dots, N$, it is easy to see that $Q_{\Lambda_i} \geq 0$. Hence all conditions of Definition 6.1 are obeyed. This implies that $\{Q_{\Phi_1}, Q_{\Phi_2}, \dots, Q_{\Phi_N}\}$ is a diagonalization of Q_{Φ_0}.

Now assume that for $i = 2, \dots, m$, the rows of V_i are linearly independent over $\mathbb{R}$. Define $V'_i(\xi) := \operatorname{row}\big(V_i(\xi), 0_{\mathtt{w}\times(l - l_i)}\big)$. It is easy to see that for $i = 2, \dots, m$, the rows

i	$V_i(\xi)$	λ_i	T_i
1	$\begin{bmatrix} 2\xi^3+\xi^2+11\xi+4 & 2\xi+3 \\ 2\xi-3 & \xi^2+3\xi+4 \\ 2\xi^3-\xi^2+11\xi-4 & 2\xi+3 \\ 2\xi-3 & -\xi^2+3\xi-4 \end{bmatrix}$	1	$\begin{bmatrix} 130 & 20 \\ 20 & 30 \end{bmatrix}$
2	$\begin{bmatrix} 2\xi^2+\frac{17\xi}{35}+\frac{1733}{175} & \frac{307}{175} \\ \frac{38\xi}{35}+\frac{67}{175} & \xi+\frac{363}{175} \\ 2\xi^2+\frac{\xi}{5}+\frac{239}{25} & \frac{31}{25} \\ -\frac{2\xi}{35}+\frac{457}{175} & -\xi+\frac{423}{175} \end{bmatrix}$	-1	$\begin{bmatrix} \frac{466}{175} & \frac{344}{175} \\ \frac{344}{175} & \frac{786}{175} \end{bmatrix}$
3	$\begin{bmatrix} 2\xi+\frac{11}{161} & 0 \\ \frac{100}{161} & 1 \\ 2\xi-\frac{11}{161} & 0 \\ -\frac{100}{161} & -1 \end{bmatrix}$	2	$\begin{bmatrix} \frac{44}{161} & 0 \\ 0 & 0 \end{bmatrix}$

TABLE 6.2: Example for Nevanlinna diagonalization

of V_i' are also linearly independent over $\mathbb{R}$. Observe that a canonical factorization of $\Psi_i(\zeta,\eta) := (\zeta+\eta)\Phi_i(\zeta,\eta)$ is

$$\Psi_i(\zeta,\eta) = V(\zeta)^{-\top}\left(\prod_{k=1}^{i}(\zeta-\lambda_k)\right)V_{i+1}'(\zeta)^\top \Sigma V_{i+1}'(\eta)\left(\prod_{k=1}^{i}(\eta-\lambda_k)\right)V(\eta)^{-1}$$

This implies that Ψ_i has the same signature for $i = 0,\ldots,m-1$. Define $V_i''(\xi) := V_i'(\xi)V(\xi)^{-1}\prod_{k=1}^{i-1}(\xi-\lambda_k)$ and $\mathfrak{B}_i := \mathrm{Im}\big(V_{i+1}''(\frac{d}{dt})\big)$ for $i = 0,\ldots,m-1$. From Theorem 6.27, it follows that for $i = 2,\ldots,N$, $V_i(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Consequently, an observable image representation for $\mathfrak{B}_i$ is $\mathfrak{B}_i = \mathrm{Im}\big(V_{i+1}(\frac{d}{dt})\big)$. From Lemma 6.25, we know that for every $i \in \{1,\ldots,N\}$, the column degrees of V_{i+1} are less than that of the corresponding columns of V_i. Since V_i is column reduced for $i = 0,\ldots,N$, we get $\mathtt{n}(\mathfrak{B}_i) > \mathtt{n}(\mathfrak{B}_{i+1}) > 0$ for $i = 0,\ldots,N-1$. This implies that for $i = 0,\ldots,m-2$, $\mathtt{n}(\Psi_i) > \mathtt{n}(\Psi_{i+1}) > 0$. Thus the sequence $\{Q_{\Phi_1}, Q_{\Phi_2},\ldots,Q_{\Phi_{m-1}},Q_{\Phi_N}\}$ obeys all conditions of Definition 6.2. Consequently this sequence is a synthesis of Q_{Φ_0}. ■

We now illustrate Nevanlinna diagonalization with an example.

Example 6.3. Consider

$$M_0(\xi) = \begin{bmatrix} 2\xi^3+\xi^2+11\xi+4 & 2\xi+3 \\ 2\xi-3 & \xi^2+3\xi+4 \\ 2\xi^3-\xi^2+11\xi-4 & 2\xi+3 \\ 2\xi-3 & -\xi^2+3\xi-4 \end{bmatrix}$$

Let $\Phi_0(\zeta,\eta) = \frac{M_0(\zeta)^\top \Sigma M_0(\eta)}{\zeta+\eta}$, where

$$\Sigma = \begin{bmatrix} I_2 & 0_{2\times 2} \\ 0_{2\times 2} & -I_2 \end{bmatrix}$$

It can be verified that $Q_{\Phi_0} > 0$, and $\Sigma_{\Phi_0} = I_5$. Observe that the McMillan degree of $\mathfrak{B}_0 = \text{Im}\big(M_0(\frac{d}{dt})\big)$ is 5. For this example, the chosen values of λ_i, and the resulting expressions of $V_i(\xi)$ and values of T_i obtained from Algorithm 6.28 are given in Table 6.2.

In this case, it can be verified that T_1^{-1} and T_2^{-1} are both positive definite. Observe that $\det(T_3) = 0$. Hence for this example, we have to go through steps 8 to 10 of Algorithm 6.28. This gives $T_3 = \frac{44}{161}$ and $V_3(\xi) = \text{col}(2\xi + \frac{11}{161}, \frac{100}{161}, 2\xi - \frac{11}{161}, -\frac{100}{161})$. In the next stage, we obtain $T_4 = 0$ and hence stop. For this example, define $\Phi_1(\zeta,\eta) := \frac{(\zeta-1)(\eta-1)V_2(\zeta)^\top \Sigma V_2(\eta)}{\zeta+\eta}$, $\Phi_2(\zeta,\eta) := \text{diag}\big(\frac{44}{161}(\zeta^2-1)(\eta^2-1), 0\big)$ and $\Phi_3(\zeta,\eta) := 0_{2\times 2}$. Then $\{Q_{\Phi_1}, Q_{\Phi_2}, Q_{\Phi_3}\}$ is a diagonalization of Q_{Φ_0}. It can be verified that the rows of V_2 are linearly independent over $\mathbb{R}$. From Theorem 6.29, it follows that $\{Q_{\Phi_1}, Q_{\Phi_3}\}$ is a synthesis of Q_{Φ_0}.

6.6 Application to stability tests

We now show that the procedure for synthesis of behaviours can be used to check stability of scalar autonomous behaviours. We show that stability of a scalar autonomous behaviour can be checked using the steps for synthesis of a related QDF.

We make use of the following theorem in order to establish a link between stability tests for a scalar autonomous behaviour and synthesis of QDFs.

Theorem 6.30. *Consider a polynomial $r(\xi) = r'(\xi^2) + \xi r''(\xi^2)$, where $r', r'' \in \mathbb{R}[\xi]$. Let ω denote the root of r' with the lowest absolute value. Define $r_1(\xi^2) := \xi^2 r''(\xi^2)$. r is Hurwitz iff r' and r'' have distinct roots on the negative real axis and*

- *the roots of r'' are interlaced between those of r' and $r''(\omega) > 0$, if the degree of r' is greater than that of r''*
- *the roots of r' are interlaced between those of r_1 and $r'(0) > 0$ if the degree of r' is less than or equal to that of r''.*

Proof. See proof of Theorem 1, p. 107 of Holtz (1989). ∎

In the above Theorem, note that $r'(\xi^2)$ and $\xi r''(\xi^2)$, denote the even and odd parts of $r(\xi)$. From Theorem 6.30 and Theorem 4.9 of chapter 4, it follows that a polynomial r

is Hurwitz iff $Q_{\Phi_0} > 0$, where

$$\Phi_0(\zeta, \eta) = \frac{r_1(\zeta)r_0(\eta) + r_0(\zeta)r_1(\eta)}{\zeta + \eta} \tag{6.39}$$

and one of r_1 and r_0 denotes the even part and other denotes the odd part of r. Thus in any stability test of the behaviour $\mathfrak{B} = \ker\big(r(\frac{d}{dt})\big)$, we first find r_0 and r_1 and check if $Q_{\Phi_0} > 0$, where Φ_0 is given by equation (6.39). Note that one way of checking whether a given QDF is positive or not is to diagonalize the two variable polynomial matrix associated with it, and this can be done via synthesis. In this section, we show that Cauer synthesis and Nevanlinna diagonalization of Q_{Φ_0} lead to stability tests for $\mathfrak{B}$. Throughout this section, we denote

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

6.6.1 Cauer synthesis and Routh-Hurwitz test

Assume that one of r_0 and r_1 is the even part and the other is the odd part of a given polynomial $r \in \mathbb{R}[\xi]$. Assume that $\deg(r_0) > \deg(r_1)$ and $\Phi_0 \in \mathbb{R}[\zeta, \eta]$ is given by equation (6.39). Then from previous discussion, $Q_{\Phi_0} > 0$ iff r is Hurwitz. Now consider the behaviour $\mathfrak{B}_0 = \mathrm{Im}\big(M_0(\frac{d}{dt})\big)$, where $M_0 = \mathrm{col}(r_0, r_1)$. Observe that

$$\Phi_0(\zeta, \eta) = \frac{M_0(\zeta)^\top J M_0(\eta)}{\zeta + \eta}$$

From Lemma 6.12, it follows that $\mathfrak{B}_0$ is J-lossless iff r is Hurwitz. Now assume that $\mathfrak{B}_0$ is J-lossless. Consider Cauer synthesis of $\mathfrak{B}_0$. From Theorem 6.20, this leads to the synthesis of the QDF Q_{Φ_0}. From the discussion of Cauer synthesis in section 6.4.1, it follows that Cauer synthesis of $\mathfrak{B}_0$ leads to a sequence of polynomials $r_2, r_3, \ldots, r_m$, with $\deg(r_i) - \deg(r_{i+1}) = 1$ for $i = 0, \ldots, m-1$ such that the following equations hold:

$$\begin{aligned}
r_0(\xi) &= a_0\xi r_1(\xi) + r_2(\xi) \\
r_1(\xi) &= a_1\xi r_2(\xi) + r_3(\xi) \\
r_2(\xi) &= a_2\xi r_3(\xi) + r_4(\xi) \\
&\vdots \\
r_{m-2}(\xi) &= a_{m-2}\xi r_{m-1}(\xi) + r_m(\xi) \\
r_{m-1}(\xi) &= a_{m-1}\xi r_m(\xi)
\end{aligned}$$

b_n	b_{n-2}	b_{n-4}	$\ldots$
b_{n-1}	b_{n-3}	b_{n-5}	$\ldots$
$c_{3,n}$	$c_{3,n-2}$	$c_{3,n-4}$	$\ldots$
$c_{4,n}$	$c_{4,n-2}$	$c_{4,n-4}$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$c_{n+1,n}$			

TABLE 6.3: Routh table for r

where $a_0, a_1, \ldots, a_{m-1} \in \mathbb{R}^+$ and $r_m(\xi)$ is a constant. Observe that the above equations can be obtained by a repeated division process similar to the one in Euclid's algorithm for finding the greatest common divisor of two given polynomials. We also get from equation (6.19) of section 6.4.1 that

$$\Phi_0(\zeta, \eta) = \sum_{i=0}^{m-1} a_i r_{i+1}(\zeta) r_{i+1}(\eta) \tag{6.40}$$

Observe that the steps followed in the Cauer synthesis of $\mathfrak{B}_0$ are the same as those followed in the Routh-Hurwitz test for the polynomial r. We now give the method for construction of Routh table (see Routh (1892), pp. 192-201) for r. Let

$$r(\xi) = \sum_{i=0}^{n} b_i \xi^i$$

Then Table 6.3 is a Routh table for r. With reference to this table, the entries from row 3 onwards are given by

$$c_{k,n-2p} = \frac{-1}{c_{k-1,n}} \begin{vmatrix} c_{k-2,n} & c_{k-2,n-2p} \\ c_{k-1,n} & c_{k-1,n-2p} \end{vmatrix}$$

where $c_{1,i} = b_i$ and $c_{2,i} = b_{i-1}$ for $i = n, n-2, n-4, \ldots$. It can be verified that the entries of the i^{th} row of the Routh table for r, are the coefficients of r_{i-1}. Hence the first column entries obtained in the Routh table construction for r are the leading coefficients of r_i for $i = 0, \ldots, m$. It follows that these entries are positive iff $a_i > 0$ for $i = 0, \ldots, m-1$.

The following algorithm which is based on Cauer synthesis and Routh-Hurwitz test can be used to check if a given polynomial is Hurwitz or not.

Algorithm 6.31. Data: *A given polynomial* $r \in \mathbb{R}[\xi]$.

Output: *Whether* r *is Hurwitz or not.*

Step 1 : Split r into its even and odd parts. Let one of r_0 and r_1 denote the even part and the other denote the odd part in such a way that $\deg(r_0) > \deg(r_1)$.

Step 2 : Assign $R := r_1$.

Step 3 : Assign $i := 0$.

Step 4 : Find the remainder r_{i+2} and the quotient f_i when r_i is divided by r_{i+1}.

Step 5 : Assign $R := \text{col}(R, r_{i+2})$.

Step 6 : If $f_i(\xi) \neq a_i\xi$, where $a_i \in \mathbb{R}^+$, output "r is not Hurwitz" and stop. Else if $r_{i+2}(\xi) = 0$, go to step 7. Else put $i = i + 1$, and go to step 4.

Step 7 : If $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, output "r is Hurwitz" and stop. Else output "r is not Hurwitz" and stop.

Note that in step 7 of the above Algorithm, we are checking if $Q_{\Phi_0} > 0$, where Φ_0 is given by equation (6.40).

6.6.2 Nevanlinna test of stability

We now explain the steps of an algorithm which is based on Nevanlinna diagonalization for checking whether a given $r \in \mathbb{R}[\xi]$ is Hurwitz or not. We call this test as *Nevanlinna test of stability*. Let m denote the degree of r. Let one of r_0 and r_1 denote the even part and the other denote the odd part of r in such a way that $\deg(r_0) > \deg(r_1)$. The first step of the algorithm is to assign

$$\begin{aligned} v_1 &:= \text{col}(r_0, r_1) \\ T_1 &:= \frac{v_1(\lambda_1)^\top J v_1(\lambda_1)}{2\lambda_1}, \end{aligned}$$

where λ_1 is a nonzero real number. If r is Hurwitz, it is easy to see that $T_1 > 0$. At this step if $T_1 \leq 0$, the algorithm stops and the output is "r is not Hurwitz". Assuming that $T_1 > 0$, the next step of the algorithm is to compute the following:

$$S_1(\xi) := (\xi + \lambda_1)I_2 - v_1(\lambda_1)T_1^{-1}v_1(\lambda_1)^\top J$$

$$v_2(\xi) = \frac{S_1(\xi)v_1(\xi)}{\xi^2 - \lambda_1^2} \qquad T_2 = \frac{v_2(\lambda_2)^\top J v_2(\lambda_2)}{2\lambda_2}$$

where λ_2 is a nonzero real number. Note that r is Hurwitz iff $Q_\Phi > 0$, where

$$\Phi(\zeta, \eta) = \frac{v_1(\zeta)^\top J v_1(\eta)}{\zeta + \eta}$$

We now prove that if r is Hurwitz, then $Q_{\Phi_2} > 0$, where

$$\Phi_2(\zeta, \eta) = \frac{v_2(\zeta)^\top J v_2(\eta)}{\zeta + \eta}$$

Proposition 6.32. *Consider* $v_1 \in \mathbb{R}^2[\xi]$*, such that* $v_1(-\xi)^\top J v_1(\xi) = 0$ *and* $\Phi(\zeta,\eta) = \frac{v_1(\zeta)^\top J v_1(\eta)}{\zeta+\eta}$ *induces a positive QDF. Consider* $S \in \mathbb{R}^{2\times 2}[\xi]$*, given by*

$$S(\xi) = (\xi+\lambda)I_2 - \frac{2\lambda v_1(\lambda)v_1(\lambda)^\top J}{v_1(\lambda)^\top J v_1(\lambda)}$$

where $\lambda \in \mathbb{R}$ *is nonzero. Define* $v_2(\xi) := \frac{S(\xi)v_1(\xi)}{\xi^2-\lambda^2}$*. Then* $\Phi_2(\zeta,\eta) = \frac{v_2(\zeta)^\top J v_2(\eta)}{\zeta+\eta}$ *also induces a positive QDF.*

Proof. Let r_0 and r_1 denote the first and second components of v_1. Then

$$\Phi(\zeta,\eta) = \frac{v_1(\zeta)^\top J v_1(\eta)}{\zeta+\eta} = \frac{r_1(\zeta)r_0(\eta) + r_1(\eta)r_0(\zeta)}{\zeta+\eta}$$

induces a positive QDF. Consider a behaviour $\mathfrak{B} = \ker\big(r_1(\frac{d}{dt})\big)$ for which Q_Φ turns out to be a positive conserved quantity. Hence from Theorem 4.6 of Chapter 4, we can say that $\mathfrak{B}$ is oscillatory, and that one of r_1 and r_2 is even and the other is odd, and also that their roots are interlaced. We will now assume that r_0 is even and r_1 is odd, and the degree of r_0 is higher than that of r_1. The proof of the case when the odd component has degree higher than that of the even component is similar, and we will not prove that explicitly.

Denote r_0 by r_e and r_1 by r_o. It can be verified that

$$S(\xi) = \begin{bmatrix} \xi & -\frac{\lambda r_e(\lambda)}{r_o(\lambda)} \\ -\frac{\lambda r_o(\lambda)}{r_e(\lambda)} & \xi \end{bmatrix}$$

Hence

$$(\xi^2-\lambda^2)v_2(\xi) = \begin{bmatrix} \xi r_e(\xi) - \frac{\lambda r_e(\lambda) r_o(\xi)}{r_o(\lambda)} \\ \xi r_o(\xi) - \frac{\lambda r_o(\lambda) r_e(\xi)}{r_e(\lambda)} \end{bmatrix}$$

Let r_2 and r_3 denote the first and second components of v_2. We can assume without loss of generality from Theorem 4.9 of Chapter 4 that

$$\begin{aligned} r_e(\xi) &= (\xi^2+\omega_0^2)(\xi^2+\omega_2^2)\dots(\xi^2+\omega_{2n}^2) \\ r_o(\xi) &= \xi(\xi^2+\omega_1^2)(\xi^2+\omega_3^2)\dots(\xi^2+\omega_{2n-1}^2) \end{aligned}$$

with $\omega_0 < \omega_1 < \omega_2 < \omega_3 < \cdots < \omega_{2n-1} < \omega_{2n}$. Note that $\frac{\lambda r_e(\lambda)}{r_o(\lambda)}$ and $\frac{\lambda r_o(\lambda)}{r_e(\lambda)}$ are positive for all values of λ. Now define

$$s(\alpha^2) := (\omega_0^2-\alpha^2)(\omega_2^2-\alpha^2)\dots(\omega_{2n}^2-\alpha^2) - \frac{\lambda r_e(\lambda)}{r_o(\lambda)}(\omega_1^2-\alpha^2)(\omega_3^2-\alpha^2)\dots(\omega_{2n-1}^2-\alpha^2)$$

It is easy to verify that

$$\left\{\begin{array}{c} s(\omega_1^2) < 0 \\ s(\omega_2^2) > 0 \end{array}\right\} \Rightarrow \text{At least 1 real root of } s \text{ between } \omega_1^2 \text{ and } \omega_2^2$$
$$\left\{\begin{array}{c} s(\omega_3^2) > 0 \\ s(\omega_4^2) < 0 \end{array}\right\} \Rightarrow \text{At least 1 real root of } s \text{ between } \omega_3^2 \text{ and } \omega_4^2$$
$$\vdots$$
$$\left\{\begin{array}{c} s(\omega_{2n-1}^2)\dots \\ s(\omega_{2n}^2)\dots \end{array}\right\} \Rightarrow \text{At least 1 real root of } s \text{ between } \omega_{2n-1}^2 \text{ and } \omega_{2n}^2$$

It is easy to see that r_2 has one root at 0, and hence is odd. The remaining roots of r_2 are the roots of the polynomial $s(-\xi^2)$ other than $\pm\lambda$. These are $2n$ in number. Therefore there is exactly one root of s between ω_1^2 and ω_2^2, one root between ω_3^2 and ω_4^2, and so on.

Now define

$$\begin{aligned} p(\alpha^2) := & - \alpha^2(\omega_1^2-\alpha^2)(\omega_3^2-\alpha^2)\dots(\omega_{2n-1}^2-\alpha^2) \\ & - \frac{\lambda r_o(\lambda)}{r_e(\lambda)}(\omega_0^2-\alpha^2)(\omega_2^2-\alpha^2)\dots(\omega_{2n}^2-\alpha^2) \end{aligned}$$

It is easy to verify that

$$\left\{\begin{array}{c} p(\omega_0^2) < 0 \\ p(\omega_1^2) > 0 \end{array}\right\} \Rightarrow \text{At least 1 real root of } s \text{ between } \omega_0^2 \text{ and } \omega_1^2$$
$$\left\{\begin{array}{c} p(\omega_2^2) > 0 \\ p(\omega_3^2) < 0 \end{array}\right\} \Rightarrow \text{At least 1 real root of } s \text{ between } \omega_2^2 \text{ and } \omega_3^2$$
$$\vdots$$
$$\left\{\begin{array}{c} p(\omega_{2n-2}^2)\dots \\ p(\omega_{2n-1}^2)\dots \end{array}\right\} \Rightarrow \text{At least 1 real root of } s \text{ between } \omega_{2n-2}^2 \text{ and } \omega_{2n-1}^2$$

It is easy to see that r_3 is even and its roots are those of the polynomial $p(-\xi^2)$ other than $\pm\lambda$. These are $2n$ in number. Therefore there is exactly one root of p between ω_0^2 and ω_1^2, one root between ω_2^2 and ω_3^2, and so on.

From the above discussion, it turns out that the real roots of s and p are interlaced. Consequently the roots of r_2 and r_3 are also interlaced and also are purely imaginary and distinct. Hence it follows from Theorem 4.9 of Chapter 4 that $\Phi_2(\zeta,\eta) = \frac{v_2(\zeta)^\top J v_2(\eta)}{\zeta+\eta}$ induces a positive QDF. ■

The next step of the algorithm is to iteratively compute the following for $i = 2,\dots,m-1$.

$$S_i(\xi) = (\xi+\lambda_i)I_2 - v_i(\lambda_i)T_i^{-1}v_i(\lambda_i)^\top J$$

$$v_{i+1}(\xi) = \frac{S_i(\xi)v_i(\xi)}{\xi^2 - \lambda_i^2} \qquad T_{i+1} = \frac{v_{i+1}(\lambda_{i+1})^\top J v_{i+1}(\lambda_{i+1})}{2\lambda_{i+1}}$$

where $\lambda_i \in \mathbb{R}$ are nonzero for $i = 3, \dots, m$ and check at every step if $T_{i+1} > 0$. If not, the algorithm stops and the output is "r is not Hurwitz". Else the algorithm proceeds to the next iteration. If the algorithm proceeds upto the last iteration, and $T_m > 0$, then the output of the algorithm is "r is Hurwitz". We now prove that r is Hurwitz iff T_i is positive for $i = 1, \dots, m$. We make use of Proposition 6.32 in order to do so.

Proposition 6.33. *Consider $v_1 \in \mathbb{R}^{2\times 1}[\xi]$, such that $v_1(-\xi)^\top J v_1(\xi) = 0$ and $\Phi_0(\zeta,\eta) = \frac{v_1(\zeta)^\top J v_1(\eta)}{\zeta+\eta}$ induces a positive QDF. Let m denote the column degree of v_1. For $i = 1, \dots, m$, define $S_i \in \mathbb{R}^{2\times 2}[\xi]$ as*

$$S_i(\xi) := (\xi + \lambda_i)I_2 - \frac{2\lambda_i v_i(\lambda_i)v_i(\lambda_i)^\top J}{v_i(\lambda_i)^\top J v_i(\lambda_i)}$$

where $\lambda_i \in \mathbb{R}$, and $v_{i+1}(\xi) := \frac{S_i(\xi)v_i(\xi)}{\xi^2-\lambda_i^2}$. Then $Q_{\Phi_0} > 0$ iff $T_i := \frac{v_i(\lambda_i)^\top J v_i(\lambda_i)}{2\lambda_i} > 0$ for $i = 1, \dots, m$.

Proof. (*If*): Assume that $T_i > 0$ for $i = 1, \dots, m$. Observe that $S_1(\zeta)^\top \Sigma S_1(\eta)$

$$\begin{aligned} &= [(\zeta+\lambda_1)I_{\mathtt{w}} - \Sigma v_1(\lambda_1)T_1^{-1}v_1(\lambda_1)^\top]\Sigma[(\eta+\lambda_1)I_{\mathtt{w}} - v_1(\lambda_1)T_1^{-1}v_1(\lambda_1)^\top\Sigma] \\ &= (\zeta+\lambda_1)(\eta+\lambda_1)I_{\mathtt{w}} - (\zeta+\eta)\Sigma v_1(\lambda_1)T_1^{-1}v_1(\lambda_1)^\top\Sigma \end{aligned}$$

Also

$$v_1(\zeta)^\top S_1(\zeta)^\top \Sigma S_1(\eta)v_1(\eta) = (\zeta^2-\lambda_1^2)(\eta^2-\lambda_1^2)v_2(\zeta)^\top\Sigma v_2(\eta)$$

Hence it follows that

$$\begin{aligned} v_1(\zeta)^\top \Sigma v_1(\eta)(\zeta+\lambda_1)(\eta+\lambda_1) &= (\zeta^2-\lambda_1^2)(\eta^2-\lambda_1^2)v_2(\zeta)^\top\Sigma v_2(\eta) \\ &+ (\zeta+\eta)v_1(\zeta)^\top\Sigma v_1(\lambda_1)T_1^{-1}v_1(\lambda_1)^\top\Sigma v_1(\eta) \end{aligned}$$

Dividing the above equation by $(\zeta+\eta)(\zeta+\lambda_1)(\eta+\lambda_1)$, we obtain

$$\frac{v_1(\zeta)^\top\Sigma v_1(\eta)}{\zeta+\eta} = (\zeta-\lambda_1)(\eta-\lambda_1)\frac{v_2(\zeta)^\top\Sigma v_2(\eta)}{\zeta+\eta} + \frac{v_1(\zeta)^\top\Sigma v_1(\lambda_1)T_1^{-1}v_1(\lambda_1)^\top\Sigma v_1(\eta)}{(\zeta+\lambda_1)(\eta+\lambda_1)}$$

Observe that the second term of the right hand side of the above equation is a polynomial because $v_1(\lambda_1)^\top\Sigma v_1(\xi)$ is divisible by $(\xi+\lambda_1)$. In a similar way, we can write an expression for $\frac{v_2(\zeta)^\top J v_2(\eta)}{\zeta+\eta}$ in terms of $\frac{v_3(\zeta)^\top J v_3(\eta)}{\zeta+\eta}$. If this process is continued, we obtain the following expression for $\Phi_0(\zeta,\eta)$:

$$\Phi_0(\zeta,\eta) = X(\zeta)^\top \Delta X(\eta) \tag{6.41}$$

where $\Delta = \text{diag}(T_1, T_2, \ldots, T_m)$ and

$$X(\xi) = \text{col}_{i=1}^m \left\{ \left(\frac{v_i'(\lambda_i)^\top J v_i'(\xi)}{\xi + \lambda_i} \right) \prod_{k=1}^{i-1} (\xi - \lambda_k) \right\},$$

$\prod_{k=1}^{0}(\xi - \lambda_k) := 1$. We now prove that $X(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Let $x_i(\xi)$ denote the i^{th} element of $X(\xi)$. Assume by contradiction that $X(\lambda)$ loses column rank for some $\lambda \in \mathbb{C}$. Then $X(\lambda) = 0$, which implies that λ is one of the roots of x_m, which implies that $\lambda \in \{\lambda_1, \ldots, \lambda_{m-1}\}$. Now $\lambda \neq \lambda_1$, because $x_1(\lambda_1) \neq 0$ owing to T_1 being positive. Thus $X(\lambda_1)$ has full column rank. Now assume that $X(\lambda)$ has full column rank for $\lambda \in \{\lambda_i\}_{i=1,\ldots,i}$, where $i < m-1$. We now prove that $X(\lambda_{i+1})$ also has full column rank if $\lambda_{i+1} \neq \lambda_k$ for $k = 1, \ldots, i$. It is easy to see that in this case $x_{i+1}(\lambda_{i+1}) \neq 0$, because $T_{i+1} > 0$. Hence by induction, it follows that $X(\lambda)$ has full column rank for $\lambda \in \{\lambda_1, \ldots, \lambda_{m-1}\}$, which implies that $X(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. From equation (6.41), it follows that $Q_{\Phi_0} > 0$. This concludes the proof.

(*Only if*): Assume that $Q_{\Phi_0} > 0$. Define $\Phi_{i-1}(\zeta, \eta) = \frac{v_i(\zeta)^\top J v_i(\eta)}{\zeta + \eta}$. From Proposition 6.32, it follows that if $\Phi_i(\zeta, \eta)$ induces a positive QDF, then so does $\Phi_{i+1}(\zeta, \eta)$. Consider trajectories $w_i(t) = e^{\lambda_i t}$. We have $Q_{\Phi_{i-1}}(w_i)(t) = T_i e^{2\lambda_i t}$. If $Q_{\Phi_{i-1}} > 0$, then it follows that $T_i > 0$. We know that $Q_{\Phi_0} > 0$. Hence by induction, it follows that $Q_{\Phi_{i-1}} > 0$, and consequently $T_i > 0$ for $i = 1, \ldots, m$. ■

We now give the formal algorithm for Nevanlinna test of stability.

Algorithm 6.34. Data: *A polynomial* $r \in \mathbb{R}[\xi]$.

Output: *Whether* r *is Hurwitz or not.*

Step 1 : Let m denote the degree of r. Let one of r_0 and r_1 denote the even part and the other denote the odd part in such a way that $\deg(r_0) > \deg(r_1)$.

Step 2 : Assign $v_1 := \text{col}(r_0, r_1)$.

Step 3 : Choose real, nonzero numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$. Assign $T_1 = \frac{v_1(\lambda_1)^\top J v_1(\lambda_1)}{2\lambda_1}$.

Step 4 : Assign $i = 1$.

Step 5 : Assign

$$S_i(\xi) = (\xi + \lambda_i) I_2 - v_i(\lambda_i) T_i^{-1} v_i(\lambda_i)^\top J$$

$$v_{i+1}(\xi) = \frac{S_i(\xi) v_i(\xi)}{\xi^2 - \lambda_i^2} \qquad T_{i+1} = \frac{v_{i+1}(\lambda_{i+1})^\top J v_{i+1}(\lambda_{i+1})}{2\lambda_{i+1}}$$

Step 6 : Check sign of T_{i+1}. If $T_{i+1} \notin \mathbb{R}^+$, output "$r$ is not Hurwitz" and stop. Else if $i = m-1$, output "r is Hurwitz" and stop. Else put $i = i+1$ and go to step 5.

We now show the application of Algorithm 6.34 to an example of a Hurwitz polynomial.

i	$v_i(\xi)$	λ_i	T_i
1	$\begin{bmatrix} \xi^5+6\xi^3+8\xi \\ 2\xi^4+8\xi^2+6 \end{bmatrix}$	2	3360
2	$\begin{bmatrix} \xi^4+\frac{158\xi^2}{35}+\frac{144}{35} \\ \frac{13\xi^3}{24}+\frac{17\xi}{12} \end{bmatrix}$	1	$\frac{15839}{840}$
3	$\begin{bmatrix} \xi^3+\frac{134\xi}{47} \\ \frac{114\xi^2}{337}+\frac{282}{337} \end{bmatrix}$	-1	$\frac{71676}{15839}$
4	$\begin{bmatrix} \xi^2+\frac{181}{66} \\ \frac{6\xi}{181} \end{bmatrix}$	-2	$\frac{445}{1991}$
5	$\begin{bmatrix} \xi \\ \frac{6}{445} \end{bmatrix}$	1	$\frac{6}{445}$

TABLE 6.4: An example for Nevanlinna test for stability

Example 6.4. Consider the Hurwitz polynomial $r(\xi) = \xi^5 + 2\xi^4 + 6\xi^3 + 8\xi^2 + 8\xi + 6$. In this case $n = 5$. The chosen values of λ_i and the obtained expressions for $v_i(\xi)$ and values of T_i for $i = 1, \dots, 5$ are given in Table 6.4.

Observe that $T_i > 0$ for $i = 1, \dots, 5$. This implies that r is Hurwitz.

6.7 Interconnection of J-lossless behaviours

Consider a lossless one-port electrical network, for which the system equation is

$$d(\frac{d}{dt})V = n(\frac{d}{dt})I \tag{6.42}$$

where V and I denote the voltage across the port and the current through the network respectively. From the theory of electrical networks (see Baher (1984), p. 50), it follows that Z defined by $Z(\xi) := \frac{n(\xi)}{d(\xi)}$ is lossless positive real. Hence both d and n are oscillatory, one of them is even and the other is odd and the purely imaginary roots of one are interlaced between those of the other. From Theorem 6.30, it follows that $n + d$ is Hurwitz.

Define $\mathfrak{B}$ as the set of all admissible trajectories $(I, V) : \mathbb{R} \to \mathbb{R}^2$ that obey equation (6.42). It is easy to see that $\mathfrak{B} = \text{Im}\big(M(\frac{d}{dt})\big)$, where $M(\xi) := \text{col}\big(d(\xi), n(\xi)\big)$. From

Theorem 4.9 of Chapter 4, it follows that $Q_\Phi > 0$, where

$$\Phi(\zeta,\eta) := \frac{n(\zeta)d(\eta) + d(\zeta)n(\eta)}{\zeta+\eta}$$

It is easy to see that d and n are co-prime. Hence from Lemma 6.12, it follows that $\mathfrak{B}$ is J-lossless, where

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

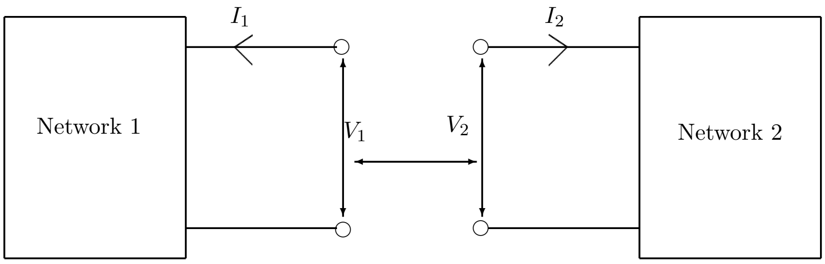

FIGURE 6.3: Interconnection of one-port electrical networks

Now consider the interconnection of two lossless one-port electrical networks as depicted in Figure 6.3. Let the system equations for the two networks be given by

$$\begin{aligned} d_1(\frac{d}{dt})V_1 &= n_1(\frac{d}{dt})I_1 \\ d_2(\frac{d}{dt})V_2 &= n_2(\frac{d}{dt})I_2 \end{aligned}$$

When we interconnect the two networks as depicted in the figure, from Kirchoff's voltage and current laws, we obtain $V_1 = V_2$ and $I_1 = -I_2$. Hence the characteristic equation for the resulting autonomous system is $r = n_1d_2 + n_2d_1$. We now prove that r is oscillatory which implies that the resulting autonomous system is lossless.

We know that $(n_1 + d_1)$ and $(n_2 + d_2)$ are both Hurwitz. Hence their product

$$\begin{aligned} p &= (n_1+d_1)(n_2+d_2) \\ &= (n_1n_2 + d_1d_2) + (n_1d_2 + n_2d_1) \end{aligned}$$

is also Hurwitz. We consider four cases.

- Case 1: n_1 and n_2 are even and d_1 and d_2 are odd. In this case, r is the odd part of p and hence from Theorem 6.30, it is oscillatory.

- Case 2: n_1 and d_2 are even and n_2 and d_1 are odd. In this case, r is the even part of p and hence from Theorem 6.30, it is oscillatory.

- Case 3: d_1 and n_2 are even and n_1 and d_2 are odd. In this case, r is the even part of p and hence from Theorem 6.30, it is oscillatory.
- Case 4: d_1 and d_2 are even and n_1 and n_2 are odd. In this case, r is the odd part of p and hence from Theorem 6.30, it is oscillatory.

This proves that the interconnection of two lossless one-port networks of the type depicted in Figure 6.3 always results in a lossless autonomous system.

Below, we give the generalization of the above phenomenon for the multivariable case of J-lossless behaviours.

Lemma 6.35. *Define*

$$J := \begin{bmatrix} 0_{l\times l} & I_l \\ I_l & 0_{l\times l} \end{bmatrix}; \qquad \Sigma := \begin{bmatrix} I_l & 0_{l\times l} \\ 0_{l\times l} & -I_l \end{bmatrix}$$

Consider two J-lossless behaviours $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^{2l}$. Define $\mathfrak{B}_i' := Im(\Sigma)_{|\mathfrak{B}_i}$ for $i = 1, 2$. Then the behaviours $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2'$ and $\mathfrak{B}' = \mathfrak{B}_1' \cap \mathfrak{B}_2$ are oscillatory.

Proof. We first prove that $\mathfrak{B}$ is oscillatory. Consider a trajectory $w = \mathrm{col}(w_1, w_2) \in \mathfrak{B}$, where $w_1, w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^l)$. It is easy to see that $w \in \mathfrak{B}_1$ and $\Sigma w \in \mathfrak{B}_2$, or $\mathrm{col}(w_1, -w_2) \in \mathfrak{B}_2$. Since $\mathfrak{B}_1$ is J-lossless, there exists $E_1 \in \mathbb{R}_s^{2l\times 2l}[\zeta, \eta]$, such that $Q_{E_1} \overset{\mathfrak{B}_1}{>} 0$, and

$$Q_J(w) = w_1^\top w_2 + w_2^\top w_1 = \frac{d}{dt} Q_{E_1}(w) \tag{6.43}$$

Since $\mathfrak{B}_2$ is also J-lossless, there exists $E_2 \in \mathbb{R}_s^{2l\times 2l}[\zeta, \eta]$, such that $Q_{E_2} \overset{\mathfrak{B}_2}{>} 0$, and

$$Q_J(\Sigma w) = -w_1^\top w_2 - w_2^\top w_1 = \frac{d}{dt} Q_{E_2}(\Sigma w) = \frac{d}{dt} Q_{E_2'}(w) \tag{6.44}$$

where $E_2'(\zeta, \eta) := \Sigma^\top E_2(\zeta, \eta)\Sigma$. Define $E(\zeta, \eta) := E_1(\zeta, \eta) + E_2'(\zeta, \eta)$. Adding equations (6.43) and (6.44), we get

$$\frac{d}{dt} Q_E(w) = 0 \tag{6.45}$$

Note that $\Sigma^\top = \Sigma^{-1} = \Sigma$. Let $D_2 \in \mathbb{R}^{\bullet\times 2l}[\xi]$, be such that $E_2(\zeta, \eta) = D_2(\zeta)^\top D_2(\eta)$. Let $\mathfrak{B}_2 = \ker\big(R_2(\frac{d}{dt})\big)$ denote a minimal kernel representation for $\mathfrak{B}_2$. Since $Q_{E_2} \overset{\mathfrak{B}_2}{>} 0$, $\mathrm{col}(D_2(\lambda), R_2(\lambda))$ has full column rank for all $\lambda \in \mathbb{C}$. Observe that $\mathfrak{B}_2' = \ker\big(R_2(\frac{d}{dt})\Sigma\big)$ is a minimal kernel representation for $\mathfrak{B}_2'$. Since $\mathrm{col}(D_2(\lambda)\Sigma, R_2(\lambda)\Sigma)$ has full column rank for all $\lambda \in \mathbb{C}$, $Q_{E_2'} \overset{\mathfrak{B}_2'}{>} 0$. Now observe that

$$Q_E(w) = Q_{E_1}(w) + Q_{E_2'}(w)$$

for any trajectory $w \in \mathfrak{B}$. We know that for any nonzero trajectory $w \in \mathfrak{B}$, the right hand side of the above equation is positive. If $Q_E(w) = 0$, then $Q_{E_1}(w) = Q_{E_2'}(w) = 0$, which imply that $w = 0$. Hence $Q_E(w) \overset{\mathfrak{B}}{>} 0$. Since equation (6.45) holds for every trajectory $w \in \mathfrak{B}$, $\mathfrak{B}$ is lossless and hence oscillatory.

Along similar lines, it can be proved that $\mathfrak{B}'$ is oscillatory. This proof will not be given explicitly. ■

The following is a corollary of the above Lemma.

Corollary 6.36. *Define*

$$\Sigma := \begin{bmatrix} I_l & 0_{l\times l} \\ 0_{l\times l} & -I_l \end{bmatrix}; \qquad J := \begin{bmatrix} 0_{l\times l} & I_l \\ I_l & 0_{l\times l} \end{bmatrix}$$

Consider two Σ-lossless behaviours $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{L}^{2l}$. Define $\mathfrak{B}_i' := Im(J)_{|\mathfrak{B}_i}$ for $i = 1, 2$. Then the behaviours $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2'$ and $\mathfrak{B}' = \mathfrak{B}_1' \cap \mathfrak{B}_2$ are oscillatory.

Proof. Define $\tilde{\mathfrak{B}}_i := \mathrm{Im}(\Xi)_{|\mathfrak{B}_i}$ for $i = 1, 2$, where

$$\Xi := \frac{1}{\sqrt{2}} \begin{bmatrix} I_l & I_l \\ I_l & -I_l \end{bmatrix}$$

From Lemma 6.23, it follows that $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_2$ are J-lossless. Define $\tilde{\mathfrak{B}} := \mathrm{Im}(\Xi)_{|\mathfrak{B}}$, $\tilde{\mathfrak{B}}_2' := \mathrm{Im}(\Xi)_{|\mathfrak{B}_2'}$. It is easy to see that $\tilde{\mathfrak{B}} = \tilde{\mathfrak{B}}_1 \cap \tilde{\mathfrak{B}}_2'$. Now observe that

$$\tilde{\mathfrak{B}}_2' = \mathrm{Im}(\Xi)_{|\mathfrak{B}_2'} = \mathrm{Im}(\Xi J)_{|\mathfrak{B}_2} = \mathrm{Im}(\Xi J \Xi)_{|\tilde{\mathfrak{B}}_2} = \mathrm{Im}(\Sigma)_{|\tilde{\mathfrak{B}}_2}$$

It now follows from Lemma 6.35 that $\tilde{\mathfrak{B}}$ is oscillatory. This implies that $\mathfrak{B}$ is also oscillatory. It can be similarly proved that $\mathfrak{B}'$ is also oscillatory. ■

We now consider the problem of decomposition of an oscillatory behaviour with a given characteristic polynomial as an interconnection of two SISO behaviours, such that one has a lossless positive real transfer function and the other has a lossless negative real transfer function and provide an algorithm for the same. Note that this problem can be considered as an inverse problem to the one where we analyse the autonomous behaviour which is an interconnection of two SISO behaviours such that one has a positive real transfer function and the other has a negative real transfer function. The solution to this problem also provides ways of decomposing an autonomous electrical lossless circuit with a given characteristic polynomial as an interconnection of two one-port lossless electrical circuits.

Algorithm 6.37. Data: *An oscillatory even polynomial $r \in \mathbb{R}[\xi]$ of degree 2m.*
Output: *Two J-lossless behaviours $\mathfrak{B}_1$ and $\mathfrak{B}_2$, such that $\mathfrak{B} = \mathfrak{B}_1 \cap \mathfrak{B}_2'$ has its characteristic polynomial equal to r, where*

$$J := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$\mathfrak{B}_2' := Im(\Sigma)_{|\mathfrak{B}_2}$, *with*

$$\Sigma := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Step 1 Either choose an odd polynomial $r_1 \in \mathbb{R}[\xi]$ of degree $2m+1$ in such way that the roots of r are interlaced between those of r_1 or choose an odd polynomial of degree $2m-1$ in such a way that the roots of r_1 are interlaced between those of r.

Step 2 Factorize the polynomial $r + r_1$ into two factors $p, q \in \mathbb{R}[\xi]$, i.e $r + r_1 = pq$. Let p_e and p_o be the even and odd parts of p and let q_e and q_o be the even and odd parts of q.

Step 3 Output:

$$\begin{aligned} \mathfrak{B}_1 &= \ker \left[\; p_e(\tfrac{d}{dt}) \quad -p_o(\tfrac{d}{dt}) \; \right] \\ \mathfrak{B}_2 &= \ker \left[\; q_o(\tfrac{d}{dt}) \quad -q_e(\tfrac{d}{dt}) \; \right] \end{aligned}$$

With reference to the above algorithm, observe that

$$r + r_1 = (p_e q_e + p_o q_o) + (p_e q_o + p_o q_e)$$

Define $s_1(\xi) := p_e(\xi)q_e(\xi) + p_o(\xi)q_o(\xi)$ and $s_2(\xi) := p_e(\xi)q_o(\xi) + p_o(\xi)q_e(\xi)$. Then it is easy to see that s_1 is even and s_2 is odd, and hence $s_1 = r$ which is the characteristic polynomial of $\mathfrak{B}_1 \cap \mathfrak{B}_2'$. Since r and r_1 obey interlacing property, from Theorem 6.30 it follows that $r + r_1$ is Hurwitz, and consequently both p and q are Hurwitz. Conclude from Theorem 6.30 that both the pairs (p_e, p_o) and (q_e, q_o) obey interlacing property. Define $M_1(\xi) := \text{col}(p_e(\xi), p_o(\xi))$ and $M_2(\xi) := \text{col}(q_e(\xi), q_o(\xi))$. It is easy to see that $\mathfrak{B}_1 = \text{Im}\big(M_1(\frac{d}{dt})\big)$, and $\mathfrak{B}_2 = \text{Im}\big(M_2(\frac{d}{dt})\big)$. Define

$$\begin{aligned} \Phi_1(\zeta, \eta) &:= \frac{p_e(\zeta)p_o(\eta) + p_o(\zeta)p_e(\eta)}{\zeta + \eta} \\ \Phi_2(\zeta, \eta) &:= \frac{q_e(\zeta)q_o(\eta) + q_o(\zeta)q_e(\eta)}{\zeta + \eta} \end{aligned}$$

From Theorem 4.9 of Chapter 4, it follows that Q_{Φ_1} and Q_{Φ_2} are both positive. Consequently by definition, $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are J-lossless. This proves the correctness of Algorithm 6.37.

Observe that in step 1 of the above algorithm, there are infinite number of ways of choosing r_1 such that $\frac{r}{r_1}$ is lossless positive real. For each of these ways, in step 2, there are a finite number of ways of factorizing $(r + r_1)$. Hence, there are infinite number of ways of choosing two behaviours with lossless positive real and lossless negative real transfer functions respectively, such that their intersection has for its characteristic polynomial, a given oscillatory polynomial.

6.8 Summary

In this chapter, motivated by synthesis of lossless electrical networks, we have given abstract definitions for diagonalization and synthesis of positive QDFs, followed by a definition for synthesis of J-lossless behaviours. We have then showed that Cauer and Foster methods of synthesis of a given lossless positive real transfer function matrix involve the synthesis of an associated J-lossless behaviour and the synthesis of a positive QDF that is related to the energy function of the J-lossless behaviour. We have also introduced a new method of diagonalization of positive QDFs, which we have called Nevanlinna diagonalization. We have showed the application of Cauer synthesis and Nevanlinna diagonalization for checking whether a given polynomial is Hurwitz or not. Finally, motivated by the problem of interconnection of lossless electrical networks, we have studied the problem of interconnection of J-lossless behaviours.

Chapter 7

Conclusions

The main contributions of this thesis are the study of properties of higher order linear lossless systems using *energy method* as explained in the first chapter of this thesis and a definition of synthesis of positive QDFs that encompasses the mechanisms of Cauer and Foster methods of synthesis. Throughout this thesis, we have extensively made use of algebra of two-variable polynomial matrices to represent operations of QDFs like differentiation and to characterize properties of QDFs like positivity, equivalence of QDFs along a behaviour and stationarity with respect to a QDF. Below, we give the significant contributions and conclusions drawn from Chapters 4, 5 and 6 of this thesis.

In chapter 4, the main focus has been to give a characterisation for higher order linear lossless systems as opposed to the characterisation of first order systems using state space method (see Malinen et al. (2006), Weiss et al. (2001) and Weiss and Tucsnak (2003)). Using the material covered in this chapter, one can easily implement a computer program wherein the input is a higher order description of a scalar oscillatory system and the outputs are its energy functions and the kinetic and potential energy components of a given energy function for the system. Given a multivariable oscillatory system, using the material in this chapter, one can implement a program to compute an energy function for the system. Similarly one can also implement a computer program for open lossless systems wherein the inputs are either a kernel or an image description of a controllable system and a given input/output partition of the system and the outputs of the program are the following:

- whether the system is lossless with respect to the given input/output partition or not.
- if the answer to the previous question is yes, then an energy function for the system.
- whether the power delivered to the system is of the form mentioned in equation (4.36).

Observe that our definition of an autonomous lossless system is based on Lyapunov stability theory in the following sense. From a behavioural point of view, Lyapunov theory can be stated as follows: an autonomous system is stable if and only if there exists a QDF such that the QDF and its derivative are nonnegative and nonpositive respectively along the behaviour; and it is asymptotically stable if and only if there exists a QDF such that the QDF and its derivative are nonnegative and negative respectively along the behaviour. A lossless autonomous system is a stable autonomous system which is not asymptotically stable. Hence in order to define an autonomous lossless system, we have restricted the conditions of Lyapunov theory and defined it as an autonomous system for which there exists a QDF, such that the QDF and its derivative are positive and zero respectively along the behaviour.

From the method of construction of an energy function for a lossless system described in Algorithm 4.19, it follows that there is no unique energy function for a given lossless system. Since the total energy for a physical lossless system is unique, it follows that not all energy functions for a given lossless system are physically meaningful. The parametrization of energy functions for a multivariable autonomous lossless system is one possible direction of future research.

In chapter 5, we have dealt separately with the generic and nongeneric cases of multivariable oscillatory behaviours. For the generic case of oscillatory behaviours, we have given the relation between the bases of intrinsically and trivially zero-mean quantities and generalized Lagrangians. The same has not been done for the case of nongeneric oscillatory systems. One reason for this is that physical autonomous systems are rarely nongeneric, since nongeneric systems consist of subsystems that function independently of each other. Note that the zero-diagonal QDFs for such systems involve coupling of variables that belong to two different subsystems. The physical significance of such QDFs as well as that of some of these being conserved and zero mean QDFs is an issue that needs to be addressed. We remark here that the characterization of generalized Lagrangians for the case of open lossless systems is a possible direction for future research.

In chapter 6, the main contribution is an abstract definition of synthesis of positive QDFs and a definition of synthesis of J-lossless behaviours. The definition of synthesis of positive QDFs (Definition 6.2) helps us to understand the common features of Cauer and Foster methods of synthesis. Note that although Nevanlinna diagonalization sometimes leads to synthesis of positive QDFs according to Definition 6.2, it cannot be used for synthesis of lossless electrical networks.

One possible extension of the work done in chapter 6 is to give an abstract definition for synthesis of behaviours that encompasses the underlying mechanisms of not just Cauer and Foster methods of synthesis, but all the methods of synthesis of passive electrical networks. We briefly outline the points about methods of synthesis of passive electrical networks that can be made use of for this extension. It should be noted that the first step in all the methods of synthesis of transfer functions of passive electrical networks that are not lossless is to write the given transfer function as a series or parallel connection of a lossless transfer function and a second transfer function which is not lossless. We may call these transfer functions as the *lossless* and *lossy* part of the given transfer function. In the next step, the lossless transfer function obtained in the first step is synthesized either by Cauer or by Foster method. Next the lossy part of the given transfer function is synthesized in steps, that are similar to the steps followed in Foster and Cauer methods of synthesis, in a sense that the given transfer function is simplified in every step. In every step of synthesis of the lossy part, either a reactive component like a capacitor, inductor or a transformer is extracted or a resistor which is a dissipative component is extracted in order to simplify the given transfer function. Let E_i denote the total energy of the network that is synthesized in the i^{th} step of synthesis of the lossy part. If a reactive component is extracted in the i^{th} step of synthesis of the lossy part, then E_i is equal to the sum of E_{i+1} and the energy stored in the reactive component which is positive, else $\frac{dE_i}{dt}$ is equal to the sum of $\frac{dE_{i+1}}{dt}$ and the rate of dissipation of energy through the resistor extracted in the i^{th} step, which is positive. We believe that by making note of these points, it is possible to come up with a suitable definition of synthesis of behaviours and of QDFs that encompasses the underlying mechanism of all the methods of synthesis of passive electrical networks.

Appendices

Appendix A

Notation

$\mathbb{N}$	natural numbers
$\mathbb{C}$	complex numbers
$\mathbb{R}^+$	positive real numbers
$\mathbb{Z}^+$	positive integers
$\mathbb{T}$	time axis
$\mathbb{W}$	space in which variables w of a system take on their values
$\mathbb{W}^{\mathbb{T}}$	set of maps from $\mathbb{T}$ to $\mathbb{W}$
j	imaginary square root of -1
$\bar{a}$	complex conjugation of a
$\mathrm{Re}(s)$	real part of s
$\mathrm{Im}(s)$	imaginary part of s
$\mathbb{R}[\xi]$	space of polynomials with real coefficients in the indeterminate ξ
$\mathbb{R}^{\mathtt{n}}$	space of $\mathtt{n}$ dimensional real vectors
$\mathbb{C}^{\mathtt{n}}$	space of $\mathtt{n}$ dimensional complex vectors
$\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$	space of $\mathtt{m}\times\mathtt{n}$ dimensional real matrices
$\mathbb{C}^{\mathtt{m}\times\mathtt{n}}$	space of $\mathtt{m}\times\mathtt{n}$ dimensional complex matrices
$\mathbb{R}_s^{\mathtt{m}\times\mathtt{m}}$	space of $\mathtt{m}\times\mathtt{m}$ dimensional real symmetric matrices
$\mathbb{C}_s^{\mathtt{m}\times\mathtt{m}}$	space of $\mathtt{m}\times\mathtt{m}$ dimensional complex symmetric matrices
$\mathbb{R}^{\bullet\times\mathtt{n}}$	space of real matrices with $\mathtt{n}$ columns and an unspecified finite number of rows
$\mathrm{diag}(a_1,\ldots,a_n)$	diagonal matrix whose diagonal entries are $a_1,\ldots,a_n$ in the given order if $a_1,\ldots,a_n\in\mathbb{R}$ and block diagonal matrix with entries $a_1,\ldots,a_n$ along the diagonal in the given order if $a_1,\ldots,a_n$ are square matrices
$\mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$	space of $\mathtt{m}\times\mathtt{n}$ dimensional real polynomial matrices in the indeterminate ξ
$\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$	set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$
$\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{q}})$	subset of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{q}})$ consisting of compact support functions
$\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{q}})$	Set of locally integrable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$
$\mathrm{col}(L_1,L_2)$	matrix obtained by stacking the matrix L_1 over L_2, which has the same number of columns as L_1
$\mathrm{row}(R_1,R_2)$	matrix obtained by stacking the matrix R_2 to the right of R_1, which has the same number of rows as R_2
$I_{\mathtt{p}}$	identity matrix of size $\mathtt{p}$

$0_{\mathtt{p}\times\mathtt{q}}$	matrix of size $\mathtt{p}\times\mathtt{q}$ consisiting of zeroes
$A^\top$	transpose of a matrix A
A^*	matrix obtained by transposing the complex conjugate of a matrix A
$\mathcal{L}^{\mathtt{w}}$	class of linear differential behaviours with infinitely differentiable manifest variable w
$\mathcal{L}^{\mathtt{w}}_{\text{cont}}$	class of controllable linear differential behaviours with infinitely differentiable manifest variable w
$\text{rank}(R)$	row rank of a matrix R
$\text{colrank}(R)$	column rank of a matrix R
$\deg(r)$	degree of a polynomial r
$\mathbb{R}[\zeta,\eta]$	two-variable polynomials with real coefficients in the indeterminates ζ and η
$\mathbb{R}(\xi)$	space of real rational functions in the indeterminate ξ
$\mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\zeta,\eta]$	$\mathtt{m}\times\mathtt{n}$ dimensional real polynomial matrices in the indeterminates ζ and η
$\mathbb{R}^{\mathtt{p}\times\mathtt{q}}(\xi)$	Space of all matrices of size $\mathtt{p}\times\mathtt{q}$, whose entries are real rational functions of the indeterminate ξ
$\mathtt{n}(\mathfrak{B})$	McMillan degree of a given behavior $\mathfrak{B}$
$\text{mat}(\Phi)$	coefficient matrix of a two-variable polynomial matrix Φ
Σ_Φ	signature of a two-variable polynomial matrix Φ
$\ker(M)$	kernel of a linear map M
$\text{Im}(M)$	image of a linear map M
$\det(A)$	determinant of a matrix A
$\dim(S)$	dimension of a vector space S
$\text{Im}(M)_{\mid\mathfrak{B}}$	image of a linear map M with domain restricted to the behaviour $\mathfrak{B}$

Appendix B

Background material

Here, we give algebraic concepts and other background material that is required to understand the thesis. Most of the material presented here has been taken from Polderman and Willems (1997), Kailath (1980), Anderson and Vongpanitlerd (1973) and Willems (2007).

B.1 Polynomial Matrices

Definition B.1 (Unimodular matrix). *Let $U \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}}[\xi]$. Then U is said to be a* unimodular *polynomial matrix if there exists a polynomial matrix $V \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}}[\xi]$ such that $V(\xi)U(\xi) = I_{\mathtt{g}}$.*

From the above definition, it follows that if U is unimodular, then $\det(U)$ is a nonzero constant.

Proposition B.2. (Smith form decomposition): *Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{q}}[\xi]$. Define $\mathtt{p} := \mathtt{g}$, if $\mathtt{q} \geq \mathtt{g}$, and $\mathtt{p} := \mathtt{q}$ otherwise. Then there exist unimodular matrices $U_1 \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}}[\xi]$, $V_1 \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$, and polynomials δ_i for $i = 1, \ldots, \mathtt{p}$, such that*

1. *$U_1 R V_1 = \Delta$, where*
$$\Delta = row(diag(\delta_1, \ldots, \delta_{\mathtt{g}}), 0_{\mathtt{g}\times(\mathtt{q}-\mathtt{g})})$$
if $\mathtt{q} \geq \mathtt{g}$ and
$$\Delta = col(diag(\delta_1, \ldots, \delta_{\mathtt{q}}), 0_{(\mathtt{g}-\mathtt{q})\times\mathtt{q}})$$
otherwise.

2. *There exist $q_i \in \mathbb{R}[\xi]$ such that $\delta_{i+1} = q_i \delta_i$, $i = 1, \ldots, \mathtt{p} - 1$.*

Proof. See Polderman and Willems (1997), pp. 404-405, Appendix B, Theorem B.1.4 and Remark B.1.5. ∎

We call the nonzero δ_i ($i = 1, \dots, \mathtt{p}$), the *invariant polynomials* of R. With reference to the above proposition, define $U(\xi) := (U_1(\xi))^{-1}$ and $V(\xi) := (V_1(\xi))^{-1}$. From the first part of the above theorem, we have

$$R = U\Delta V \tag{B.1}$$

We refer to equation (B.1) as the *Smith form decomposition* of a polynomial matrix R. We remark here that the unimodular matrices U and V involved in the Smith form decomposition are not unique, whereas the invariant polynomials of R are unique.

Definition B.3 (Row rank and column rank)**.** *Let* $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$. *The* row (column) rank *of* R *is defined as the maximal number of linearly independent rows (columns) of* R *over* $\mathbb{R}(\xi)$.

Definition B.4 (Full row (column) rank)**.** $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ *is said to have* full row (column) rank *if all the rows (columns) of* R *are linearly independent over* $\mathbb{R}(\xi)$.

Proposition B.5. *If* $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ *does not have full row rank, then there exists a unimodular matrix* $U \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$, *such that* R_1 *defined by* $R_1(\xi) := U(\xi)R(\xi)$ *has its last* $\mathtt{p} - \mathtt{g}$ *rows full of zeroes, where* $\mathtt{g}$ *denotes the row rank of* R.

Proof. The proof follows from the proof of Theorem 2.5.23, p. 58, Polderman and Willems (1997). ∎

Proposition B.6. *If* $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ *does not have full column rank, then there exists a unimodular matrix* $V \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$, *such that* R_2 *defined by* $R_2(\xi) := R(\xi)V(\xi)$ *has its last* $\mathtt{q} - \mathtt{g}$ *columns full of zeroes, where* $\mathtt{g}$ *denotes the column rank of* R.

Proof. The proof can be deduced from Proposition B.5. ∎

We now define left prime and right prime matrices.

Definition B.7. $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ *is* left prime *if there exists* $S \in \mathbb{R}^{\mathtt{q}\times\mathtt{p}}[\xi]$, *such that* $R(\xi)S(\xi) = I_{\mathtt{p}}$. R *is* right prime *if there exists* $L \in \mathbb{R}^{\mathtt{q}\times\mathtt{p}}[\xi]$, *such that* $L(\xi)R(\xi) = I_{\mathtt{q}}$.

In the following proposition, we give the condition under which a given polynomial matrix is right prime.

Proposition B.8. $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ *is right prime iff* $R(\lambda)$ *has full column rank for all* $\lambda \in \mathbb{C}$.

Proof. (*If*): Assume that $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. This implies that $\mathtt{p} \geq \mathtt{q}$. If $R = U\Delta V$ is a Smith form decomposition of R, then necessarily Δ is of the form $\Delta = \text{col}(I_{\mathtt{q}}, 0_{(\mathtt{p}-\mathtt{q})\times\mathtt{q}})$. Consider a partition of U, given by

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

where $U_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ and $U_2 \in \mathbb{R}^{\mathtt{p}\times(\mathtt{p}-\mathtt{q})}[\xi]$. Then it is easy to see that $R = U_1 V$. Let U' be such that $U'(\xi) = U(\xi)^{-1}$. Consider a partition of U' given by

$$U' = \begin{bmatrix} U'_1 \\ U'_2 \end{bmatrix}$$

where $U'_1 \in \mathbb{R}^{\mathtt{q}\times\mathtt{p}}[\xi]$ and $U'_2 \in \mathbb{R}^{(\mathtt{p}-\mathtt{q})\times\mathtt{p}}[\xi]$. It is easy to see that $U'_1(\xi)U_1(\xi) = I_{\mathtt{q}}$. Define $L(\xi) := V(\xi)^{-1}U'_1(\xi)$. Then it is easy to see that $L(\xi)R(\xi) = I_{\mathtt{q}}$. Consequently R is right prime.

(*Only If*): Assume that R is right prime. Let $L \in \mathbb{R}^{\mathtt{q}\times\mathtt{p}}[\xi]$ be such that $L(\xi)R(\xi) = I_{\mathtt{q}}$. Assume by contradiction, that $R(\lambda)$ loses column rank for some $\lambda \in \mathbb{C}$. Then there exists a nonzero $v_\lambda \in \mathbb{C}^{\mathtt{q}}$, such that $R(\lambda)v_\lambda = 0$. Premultiplying by $L(\lambda)$, we obtain

$$L(\lambda)R(\lambda)v_\lambda = I_{\mathtt{q}}v_\lambda = v_\lambda = 0$$

Consequently $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. ∎

Note that $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ is right prime iff $R^\top$ is left prime. If $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ is right prime, then any matrix $L \in \mathbb{R}^{\mathtt{q}\times\mathtt{p}}[\xi]$ for which $L(\xi)R(\xi) = I_{\mathtt{q}}$, is called a *left inverse* of R. If $R \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ is left prime, then any matrix $S \in \mathbb{R}^{\mathtt{q}\times\mathtt{p}}[\xi]$ for which $R(\xi)S(\xi) = I_{\mathtt{p}}$, is called a *right inverse* of R.

We now define the notions of properness and strict properness of a rational matrix.

Definition B.9. *A matrix $H \in \mathbb{R}^{\mathtt{p}\times\mathtt{u}}(\xi)$ is said to be* proper *if*

$$\lim_{\xi\to\infty} H(\xi) < \infty$$

and strictly proper *if*

$$\lim_{\xi\to\infty} H(\xi) = 0$$

We define the degree of a polynomial vector as the highest degree of all the entries of the vector.

Lemma B.10. *If $H \in \mathbb{R}^{\mathtt{p}\times\mathtt{u}}(\xi)$ is strictly proper (proper) and is given by $H(\xi) := N(\xi)D(\xi)^{-1}$, where $N \in \mathbb{R}^{\mathtt{p}\times\mathtt{u}}[\xi]$, $D \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}[\xi]$, then every column of N has degree strictly less than (less than or equal to) that of the corresponding column of D.*

Proof. See Lemma 6.3-10, p. 383, Kailath (1980). ■

The converse of the above Lemma however is not always true. The crucial algebraic condition for the converse of Lemma B.10 to hold true is related to the property of column reducedness of a polynomial matrix, which we now define.

Definition B.11 (Column reduced matrix). *A full column rank matrix $H \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ is said to be* column reduced *if its leading column coefficient matrix has full column rank.*

The following Lemma gives the condition under which a rational matrix is proper.

Lemma B.12. *If $D \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$ is column reduced, then $H \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}(\xi)$ defined as $H(\xi) := N(\xi)D(\xi)^{-1}$, with $N \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$, is strictly proper (proper) iff each column of N has degree less than (less than or equal to) the degree of the corresponding column of D.*

Proof. See proof of Lemma 6.3-11, p. 385, Kailath (1980). ■

We remark here that if a matrix $M \in \mathbb{R}^{\mathtt{w}\times 1}[\xi]$ is not column reduced, then there exists a unimodular matrix $V \in \mathbb{R}^{1\times 1}[\xi]$, such that M' defined by $M'(\xi) := M(\xi)V(\xi)$ is column reduced. This has been explained in Kailath (1980), p. 386, with the help of an example.

B.2 Polynomial differential operators

Definition B.13. *A differential operator $G(\frac{d}{dt}) : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ is said to be* surjective *if the mapping $\ell \to G(\frac{d}{dt})\ell$ ($\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$) is surjective.*

Below, we give the condition for a differential operator to be surjective.

Lemma B.14. *A differential operator $G(\frac{d}{dt}) : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ is surjective iff the rows of G are linearly independent over $\mathbb{R}(\xi)$.*

Proof. Let $G = U\Delta V$ be a Smith form decomposition of G. For any trajectory $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$, define

$$\begin{aligned} h &:= G(\frac{d}{dt})\ell \\ \ell' &:= V(\frac{d}{dt})\ell \end{aligned}$$

Define $h' := U_1(\frac{d}{dt})h$, where $U_1(\xi) := U(\xi)^{-1}$.

(*If*): Assume that the rows of G are linearly independent over $\mathbb{R}(\xi)$. Then Δ cannot have zero rows. This implies that $\mathtt{p} \leq \mathtt{q}$. Hence Δ has the form

$$\Delta = \text{row}(\Delta_1, 0_{\mathtt{p}\times(\mathtt{q}-\mathtt{p})})$$

where $\Delta_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ has nonzero diagonal entries. Let δ_i denote the i^{th} diagonal entry of Δ_1. Let h_i and ℓ_i denote the i^{th} components of h' and ℓ' respectively. We have $h' = \Delta(\frac{d}{dt})\ell'$. Hence for $i = 1, \ldots, \mathtt{p}$, $h_i = \delta_i(\frac{d}{dt})\ell_i$. This implies that for $i = 1, \ldots, \mathtt{p}$, given h_i, we can solve for ℓ_i by integration, which in turn implies that given h, we can solve for ℓ, or that the mapping $\ell \to G(\frac{d}{dt})\ell$ is surjective. Hence $G(\frac{d}{dt})$ is surjective.

(*Only if*): Assume that $G(\frac{d}{dt})$ is surjective. Assume by contradiction that the rows of G are linearly dependent over $\mathbb{R}(\xi)$. Then at least one of the rows of Δ is full of zeroes. Since $h' = \Delta(\frac{d}{dt})\ell'$, at least one of the components of h' is equal to zero, which implies that the mapping $\ell' \to \Delta(\frac{d}{dt})\ell'$ is not surjective, which in turn implies that the mapping $\ell \to G(\frac{d}{dt})\ell$ is not surjective, which is a contradiction. Hence the proof. ∎

We now define injectivity of a differential operator.

Definition B.15. *A differential operator* $G(\frac{d}{dt}) : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ *is said to be* injective *if the mapping* $\ell \to G(\frac{d}{dt})\ell$ $(\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}))$ *is injective.*

For any trajectory $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$, define $\ell' := G(\frac{d}{dt})\ell$. Then it is easy to see that $G(\frac{d}{dt})$ being injective is equivalent with ℓ being observable from ℓ'. Hence the differential operator G is injective iff $G(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. We now define the concept of bijectivity of a differential operator.

Definition B.16. *A differential operator is* bijective *if it is both surjective and injective.*

Note that a unimodular differential operator $G(\frac{d}{dt})$ $(G \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi])$ is bijective as the rows of G are linearly independent over $\mathbb{R}(\xi)$ and $G(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. We now prove the converse, i.e if a differential operator is bijective then it is unimodular.

Lemma B.17. *If a differential operator* $G(\frac{d}{dt}) : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{p}})$ *is bijective, then G is unimodular.*

Proof. Let $G = U\Delta V$ be a Smith form decomposition of G. Since $G(\frac{d}{dt})$ is bijective, $G(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. This implies that $\mathtt{p} \geq \mathtt{q}$, and

$$\Delta = \begin{bmatrix} I_{\mathtt{q}} \\ 0_{\mathtt{p}-\mathtt{q}} \end{bmatrix}$$

Assume by contradiction that G is not unimodular, which implies that $\mathtt{p} > \mathtt{q}$. Then the last $\mathtt{p} - \mathtt{q}$ rows of ΔV consist of zeroes. Consider a partition of U given by

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$$

where $U_1 \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$, and $U_2 \in \mathbb{R}^{\mathtt{p}\times(\mathtt{p}-\mathtt{q})}[\xi]$. Then $G = U_1V$. It is easy to see that the rows of U_1 are not linearly independent over $\mathbb{R}(\xi)$. Let $U_{1i} \in \mathbb{R}^{1\times\mathtt{q}}[\xi]$ denote the i^{th} row of U_1. Observe that $U_{1i}V$ is the i^{th} row of G. Since the rows of U_1 are linearly dependent over $\mathbb{R}(\xi)$, there exist nonzero $r_i \in \mathbb{R}(\xi)$ for $i = 1, \ldots, \mathtt{p}$, such that

$$\left(\sum_{i=1}^{\mathtt{p}} r_i(\xi) U_{1i}(\xi)\right) V(\xi) = \sum_{i=1}^{\mathtt{p}} r_i(\xi)\big(U_{1i}(\xi)V(\xi)\big) = 0$$

which implies that the rows of G are linearly dependent over $\mathbb{R}(\xi)$. This is a contradiction. Hence it follows that $\mathtt{p} = \mathtt{q}$, or that G is unimodular. ■

The conditions for surjectivity, injectivity and bijectivity of polynomial operators in the shift for discrete-time linear shift-invariant systems are similar to the ones for the case of polynomial differential operators for continuous-time systems. These conditions are given and proved in Lemma 4.1.3, pp. 231-232 of Willems (1989).

B.3 Positive real transfer functions

Definition B.18. *A rational matrix $B \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ is called* positive real *if the following conditions hold*

1. *All elements of B are analytic in the open right half plane.*
2. *$B^*(\lambda) + B(\lambda) \geq 0$ for $Re(\lambda) > 0$.*

Definition B.19. *A rational matrix $B \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ is called* lossless positive real *if the following conditions hold*

1. *B is positive real.*
2. *$B^*(j\omega) + B(j\omega) = 0$ for all $\omega \in \mathbb{R}$, with $j\omega$ not a pole of any element of B.*

A positive real matrix $B \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ is said to have a *pole at infinity* if

$$\lim_{\xi\to\infty} \frac{B(\xi)}{\xi} \neq 0$$

If a positive real matrix $B \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ has a pole at infinity, its residue (L_∞) at that pole can be calculated using the formula

$$L_\infty = \lim_{\xi\to\infty} \frac{B(\xi)}{\xi}$$

If a positive real matrix $B \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ has a pole at $j\omega_0$, where $\omega_0 \in \mathbb{R}$, its residue ($L_{j\omega_0}$) at that pole can be calculated using the formula

$$(L_{j\omega_0}) = \lim_{\xi\to j\omega_0} (\xi - j\omega_0) B(\xi)$$

Below, we give an important property of positive real matrices.

Theorem B.20. *A matrix $B \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ is positive real if and only if*

1. *No element of B has a pole in $Re(\xi) > 0$.*
2. *$B^*(j\omega) + B(j\omega) \geq 0$ for all $\omega \in \mathbb{R}$, with $j\omega$ not a pole of any element of B.*
3. *If $j\omega$ is a pole of any element of B, it is at most a simple pole, and the residue matrix, $K = \lim_{\xi \to j\omega}(\xi - j\omega)B(\xi)$ in case ω is finite, and $K = \lim_{\xi \to \infty} B(\xi)/\xi$ in case ω is infinite, is nonnegative definite Hermitian.*

Proof. See p. 53, Theorem 2.7.2 of Anderson and Vongpanitlerd (1973). ■

The next Theorem gives the Foster partial fraction expansion of a lossless positive real transfer function.

Theorem B.21. *$Z \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ is lossless positive real if and only if it has a Foster partial fraction expansion given by*

$$Z(\xi) = J_0 + \xi L_0 + \frac{C}{\xi} + \sum_i \left(\frac{\xi A_i + B_i}{\xi^2 + \omega_i^2} \right)$$

where $L_0, C, A_i \in \mathbb{R}_s^{\mathtt{u}\times\mathtt{u}}$ are nonnegative definite and $J_0, B_i \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}$ are skew-symmetric.

Proof. See pp. 215-218, Anderson and Vongpanitlerd (1973). ■

Some other properties of positive real functions are

1. The sum of two positive real functions is also positive real, the difference however may not be positive real.
2. If $B \in \mathbb{R}^{\mathtt{u}\times\mathtt{u}}(\xi)$ is positive real, then $\det(B(\xi)) \neq 0$, i.e $B(\xi)^{-1}$ exists and is also positive real.

B.4 Module

Definition B.22. *A module over a ring $\mathcal{R}$ or an $\mathcal{R}$-module is an abelian group $(\mathcal{M}, +)$ such that if $m_1, m_2 \in \mathcal{M}$ and $r \in \mathcal{R}$, then $m_1 + m_2 \in \mathcal{M}$ and $rm_1 \in \mathcal{M}$.*

We can think of the concept of a module as a generalization of the notion of vector space where scalars, instead of being required to belong to a field, can belong to a ring. An $\mathcal{R}$-module $\mathcal{M}$ is said to be *finitely generated* if there exist finite number of elements $g_1, g_2, \ldots, g_n \in \mathcal{M}$ called generators, such that for each element $m \in \mathcal{M}$, there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{R}$, such that $m = \alpha_1 g_1 + \alpha_2 g_2 + \ldots + \alpha_n g_n$. If the generators can be chosen to be linearly independent, then $\mathcal{M}$ is said to be *free*.

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