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Generalizing Single Variable Functions to Two-variable Functions, Function Machine and APOS

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Abstract

The focus of this study in which the theoretical framework of APOS was used is students' generalizing function notion from single variable to two-variable function concepts in Analysis II course in the elementary mathematics education program. In the teaching process, teaching activities that support generalizing the function notion with multiple representations and relations between them and the function machine were used. For data collection, two tests on each of single and two-variable function concepts and interviews with six students were used. It was concluded that the students' understanding level of function concept and students' schema of three-dimensional space is fundamental for their construction of two-variable function. Moreover, significant results that can support prospective studies for students' conceptual levels of two-variable functions were obtained.

Key Words

Function Notion, Two-variable Function Concept, APOS, Function Machine.

Mathematics educators have paid attention to the function concept, which in fact deserves this attention, by supporting the literature with their studies on this concept. There are many different studies on the concept of function; however, all of them are on the notion of general function or the concept of single variable function. The studies on the concept of two-variable function, which requires transferring the notion of general function and many aspects of single variable function are few. Nonetheless, promising development is that this concept has started to be studied recently. Two-variable function concept, which has quite importance for science, engineering, mathematics and mathematics education students, is one of the fundamental concepts of advanced mathematics. Since understanding two-variable functions requires transferring not only general function notion but also key aspects of single variable functions, this causes difficulty for most of the students (Montiel, Vidakovich, & Kabael, 2008). Moreover, generalizing geometric properties of functions to two-variable functions also requires sufficient knowledge of three-dimensional geometry and visualization, which makes difficult to understand geometric properties of two-variable functions for students. Trigueros and Martinez-Planell (2009) emphasized the necessity of relating different representations of functions to understand geometric properties of function related to visualization. Yerushalmy (1997) stated the complexity of generalizing single variable functions to two-variable function and emphasized that this complexity depends on both the concept and its representation. He insisted on the importance of multiple representations and relation between them in the process of generalizing function notion from single variable to two-variables. The significance of multiple representations and relations between them has been highlighted in most of the studies (Bower, & Lobato, 2000; Breidenbach, Hawks, Nichol, & Dubinsky, 1992; Carlson, Oehrtman, Thompson, 2007; Christou,

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Elia, & Gagatsis, 2004; Ferrini-Mundy & Graham, 1990; Janvier, 1987; National Council of Teachers of Mathematics [NCTM], 1989; Sierpinska, 1992; Yerushalmy, 1997). Trigueros and Planell (2009) who have studied on geometrical aspects related to students' notion of the two-variable function emphasized that relating treatments of representations with converting different representations should be paid attention before abstract concepts related to functions are given. Meanwhile, they stated that the table representations of functions were not analyzed in their study, the reason of this was not that table representation is less important. Conversely, they insisted that different representations of functions should also include table representation.

This study is on students' generalizations about the concept of function from single variable to two-variables through two-variable calculus course in which function machine was used as a cognitive root. After review of general function concept at the beginning of the course, multiple representations (table, algebraic, graph) and relation between them were used in the teaching process of two-variable functions. For each representation, generalizing the function notion from single variable to two-variables was emphasized during this process.

Our research questions are:

1. What is the relationship between the students' construction of two-variable function concept and their conceptual levels of general function concept?
2. What is the relationship between students' construction of two-variable function concept and their schema of three-dimensional space?
3. What is the relationship between students' concept definitions for the single variable and two-variable function concepts?
4. What is the effect of function machine on students' understanding of single and two-variable function concepts?

Function Machine as a Cognitive Root

Tall, McGowen and DeMarois (2000) defined the notion of cognitive root as follows:

A *cognitive root* is a concept that:

- is a meaningful cognitive unit of core knowledge for the student at the beginning

of the learning sequence,

- allows initial development through a strategy of cognitive expansion rather than significant cognitive reconstruction,
- contains the possibility of long-term meaning in later developments,
- is robust enough to remain useful as more sophisticated understanding develops.

(Tall et al., 2000, p. 3)

Tall et al. (2000) suggested using the function machine (input-output box) as a cognitive root while teaching the function concept.

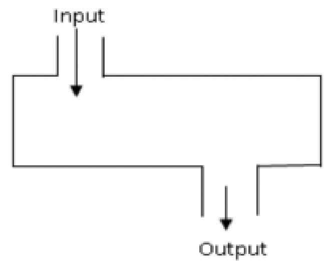


Figure2.
Function Machine

Theoretical Framework

APOS is a specific theoretical framework for research and curriculum development in collegiate mathematics education. We will briefly describe APOS in this paper, but for more details reader can refer to Asiala et al. study in 1996. According to the APOS theory, a learner's specific mental constructions are called *action, process*, *object*, and *schema*. An action is any transformation of objects to obtain other objects. The individual perceives an action as an explicit algorithm so as an externally driven. When the individual reflects on the action and constructs an internal operation, s/he *interiorizes* the action to a process. When the individual performs actions on a process, s/he *encapsulates* the process as a mathematical entity, or an object. Furthermore, a *schema* is a collection of actions, processes, objects and other schemas linked consciously or unconsciously in a coherent manner in individual's mind.

Conceptual Levels of the Function Concept

Dubinsky and his colleagues (Breidenbach et al., 1992; Dubinsky, 1991; Dubinsky & Harel, 1992) studied the conceptions of function concept. According to these studies, a subject who is at the level of action is able to calculate the value of the function for a function formula and a point. The subject, whose understanding of function concept is at the action level, has difficulties while interpreting a situation as a function unless a formula for computing values is given. Besides, the inverses of functions and the notion that the derivative of a function is a function are difficulties for such a subject. According to Dubinsky (1991), most students' idea of function is completely related with "*formula*", and a typical example of such a student for a function is an algebraic expression like x^2+3. Furthermore, Dubinsky added that such a student does not have the notions of domain and range, and s/he cannot relate the graphs with the functions.

When the subject's conceptual level reaches the process conception of function, s/he can think of a function as receiving one or more inputs that are independent variables, performing one or more operations on the inputs and returning the results as outputs that are dependent variables. According to Dubinsky, perceiving a situation that can be related with function means to view the situation as an action on objects. That is, action is interiorized. Regarding the graph of a function, Dubinsky stated that the subject, whose conceptual level is process, can coordinate the process of a function and its graph. In other words, s/he can comprehend that the height of the graph at a point x on the horizontal axis is precisely the value $f(x)$. This means that the subject relates the physical shape of the graph with the behavior of the function. Moreover, Dubinsky and Harel (1992) put forth that the process conception of function is very complex. They found that process conception of function contains the following four factors.

1. Restrictions students possess about what a function is. The three main restrictions observed are:

(a) the *manipulation restriction* (you must be able to perform explicit manipulations or you do not have a function), (b) the *quantity restriction* (inputs and outputs must be numbers), (c) the *continuity restriction* (a graph representing a function must be continuous)

2. The severity of the restriction. Some students feel, for example, that before they are willing to refer to a situation as a function, they personally have to know how to manipulate an explicit expression to get the output for a given input. Other students are satisfied with the presence of an expression even though they admit that they don't know how to deal with it.

3. Ability to construct a process when none is explicit in the situation, and students' autonomy in such a construction.

4. Uniqueness to the right condition; confusion with 1-1. We argue here that this issue is related to a process conception. According to our theoretical perspective, the confusion that is prevalent among students can only be resolved in terms of the process conception. The process notion entails a unique finishing point, whereas the idea of 1-1 is about uniqueness of starting point (Dubinsky and Harel, 1992, pp. 86-87).

The individual, whose conceptual level reaches object level, realizes that transformations can act on process. That is, a subject, who perceives manipulations of functions such as adding or multiplying, encapsulates the process conception of function to an object.

Method

The context of the study is two-variable calculus course given in the mathematics education program in education faculty of a university. The study was conducted with 23 students, whose instructor was author of the study through the spring semester of 2007-2008 academic year. At the beginning of the course, a test on the single variable functions was prepared and after validity and reliability study of the test (Miles & Huberman, 1994; Yıldırım & Şimşek, 2003) applied to the students to evaluate their conceptions of general function concept. Since students must generalize the function notion from single variable to two-variables in this course, single variable function concept was reviewed after the first test. Function machine was used as a cognitive root in this revision. To see whether the students' understandings of function concept was changed or not, another test on single variable function concept was prepared as similar to the previous test and administrated to the students after revision. Then, the process of instruction of two-variable function concept started. Representations and converting between different representations of not only single variable functions but also two-variable functions were paid more attention in the course. Furthermore, a lot of tasks were designed to gain the students' generalizing the different representations from single variable to two-variables. Function machine was used as a cognitive root also in treatment process of two-variable functions. After two-variable function notion, different representations (algebraic, geometric, set of triples, table) and drawings of some special surfaces were given; a test on two-variable functions was prepared and conducted. In the teaching process, a lot of tasks, which required perceiving function situations in the algebraic or graph representations, were posed to the students. Beside these tasks, finding domain and ranges of two-variables functions in the both algebraic and graph representations were emphasized. Then, teaching process of limit, continuity, directed derivative and double integral notions for functions of two-variable started. At the end of the course, another test on two-variable functions was prepared and applied. It is indisputable that comprehension of two-variable function concept needs to be developed, but we thought that we could estimate students' understanding levels of two-variable function concepts in a similar manner to the conception levels of general function concept in order to see the developments from single variable to two-variables. After analyzing the final test, for clinical interview (Clement, 2000; Ginsburg, 1981), six students were selected among the students, whose conceptual levels were determined as (at least) process for both single and two-variable function concepts because the tests were prepared to assess whether the subject's conceptual levels are in process or at lower level. It was emphasized to select the students in various conceptual developments through the tests.

Analysis

Since a student's explanations about function notion might not be consistent in different situations or items, we attempted to perceive student's function conception, considering her/his all responses as a whole. An individual, whose function conception is process, should perceive a function situation and convert between different representations (Dubinsky, 1991), thus, when the student was able to perceive function situations only in table representations, also if s/he was able to convert between graph, algebraic, and table representations, her/his conceptual level was interpreted as relatively weak process conception. At the same time, if such a student was unsuccessful about converting between representations, it was thought that such a student's conception of function must be transference from action to process level. Furthermore, a student, who was not able to perceive function situations in any representation, was determined to be at action level for function concept.

When we examined students' definitions in the first test, we found that most of them had manipulation restriction and some of them had also quantity restriction. Moreover, most of the students struggled to convert given graph representation to algebraic representation firstly, thus it was interpreted that they were to know how to manipulate an explicit expression to investigate whether a situation was a function or not. So, these students should have had severity restriction (Dubinsky & Harel, 1992) On the other hand, if a student perceived all function situations in all representations, and converted between algebraic, graph and table representations, the conceptual level should have been at least process level in this case. Since the second test was prepared in a similar way to the first test, the second test was also analyzed in the way mentioned above.

As indicated, we analyzed the students' conceptual levels for two-variable functions in a similar way to the conception levels of general function concept to facilitate following students' generalizations from single variable to two-variables through the tests. A student, whose conceptual level for function concept is process should perceive function situations (Dubinsky, 1991) on the basis of this argument, if a student had perceived two-variable function situations in graph and algebraic representations, we interpreted that her/his conceptual level should have been at least process. Once such a student perceived function situations in graph representations with algebraic approach, that is, converting given graph to algebraic representation firstly in order to perceive situation, we interpreted that her/his conceptual level as process with severity restriction.

After analyzing the students' responses to the third item, we saw that almost all students had difficulties with finding domain and range of the function, whose graph was given. Some of them were aware that the elements of domain and range of a two-variable function are pairs and real numbers respectively, even if they were not able to project the surface successfully. We considered although they had the notions of domain and range of two-variable function, they could not project the surface to find the domain and range exactly, due to their difficulties with three-dimensional space. Thus, we interpreted the conceptual level of such a student as process, if s/he was also able to perceive the function situations. Conceptual levels of such students were determined as process level with severity restriction when they had algebraic approach in graph representations. Some other students had dimensional difficulties with domain and range. For instance, some of them gave the element of domain or range as (x,y,z). In other words, they had serious difficulties with the notions of domain and/or range of a two-variable function. If such a student had perceived the functions in graph and algebraic cases successfully, it seemed to us that s/he had weak process conception of two-variable function. When such a student also had difficulty with perceiving functions, then we thought that her/his conceptual level must be action.

On the other hand, two students were successful in all items, but with algebraic approach. They were able to obtain domain and range in the third item apart from perceiving the function situations in the first and second items successfully in the first test on two-variable functions. Their difficulties were only with graph representations. They were able to perceive function situations by algebraic way also in graph representations as most students did. That is, they were able to convert between graph and algebraic representations, but due to their difficulties with three-dimensional geometrical knowledge and visualization, they were not able to coordinate the function process and the surface in the space. We interpreted their conceptual level as at least process.

The findings of the last test on two-variable functions were analyzed in a similar way. Since the students had developed their knowledge about two-variable functions with the treatments of two-variable calculus concepts, we had the students, who were successful about both perceiving functions in graph and algebraic cases by vertical line test and algebraic analysis respectively, and about determining the domain and range of the function whose graph was given. Their conceptual levels seemed to us as at least relatively strong process.

Findings

Findings of Tests on Single Variable Functions

Most students' responses to the tasks of the first test, in which the function situations, whose independent variable is x, are required in graph and algebraic representations, provided evidence of focusing on the independent variable notion by overriding the function notion. They investigated the situations, whose independent variable is x, with the algebraic formula not only in algebraic but also in graph representations by determining firstly the algebraic formulas of the graphs. Table representations were the only cases in which most students (16 out of 23) could perceive the function situations in the first test. All students could find the input corresponding to the given output for the function given by algebraic formula in both tests. Seventeen students out of 23 could convert the given graph representation to table representation. The students who had mistakes in this task could not perceive the output corresponding to the input, in which the function is discontinuous in the first test. Almost all students (except two students) could draw the graph of partial function given by algebraic formula, while 12 of them drew a whole circle instead of semi-circle regardless of range of the function in the second item of the sixth task. Similar to the first test, the graph representation of the function given by algebraic formula, which is the lower semi-circle, is required. Most students, who drew whole circle instead of upper semi-circle in the first test, considered the range of function and drew lower semi-circle in the second test. Only four students drew whole circle instead of semi-circle in the second test. For the definition of function, various types of definitions related to the function concept were given in the first test. Definitions given in the first test were similar to the definitions that Vinner and Dreyfus (1989) had. Only four students could give an acceptable definition. These students defined the concept of function as a relation such that each elements of domain corresponds to only one element of range. Other students defined the concept of function variously: They determined the class in the definition like "*term*", "*rule*", "*sequence of operations*", "*system of variables*" and "*expression*". When the definitions of function in the first and second tests were compared, it was seen that the students' development was surprising. Eighteen subjects gave correct definition of function concept, which included the notions of input-output and transference with the uniqueness to the right condition. Most of them gave the class as a special relation, and the rest of them gave the class as a correspondence in the definition. Also three of remaining five students had input-output and transference notions, but one of them gave the class as a rule and two of them gave the class as a machine in their definitions.

Findings of Tests on Functions of Two-Variables

The most conspicuous result obtained from the first test on functions of two-variables is that almost all students (21 out of 23) had algebraic approach. That is, they struggled to determine the algebraic representation of the graph first, and then made algebraic analysis to investigate whether the situation is a function or not. Then, they had indications in the second test on single variable functions that they developed to investigate the function situation in the graph representations without algebraic formula. In other words, they abandoned algebraic approach in the second test on single variable function notion, but they again had algebraic approach at the beginning of teaching process of two-variable functions. After treatments of two-variable calculus concepts through the semester, four of them again abandoned algebraic approach, and gained to investigate the function situations in the graphs by coordinating the function process and shape of graph. They perceived function situations by using vertical line test in graph representations in the last test at the end of the semester. Four of the remaining students had algebraic approach in the graph representations in all tests. It was a considerable development that 13 students could perceive the function situations in the graph representations by using vertical line test in the last test on functions of two-variable.

In the algebraic representations of the first test on functions of two-variables, except one student, who could not abandon the complexity about independent variable notion, all students made algebraic analysis by expressing the algebraic representation as *z=f(x,y)*, and they chose correct items. However, after analysis, it was seen that some of them (four students) made analysis without perceiving input-output and transference notions in the function situations. These students had indications in their expressions that they had memorized to make algebraic analysis.

Algebraic representation of a partial function was given in the second item in the last test on functions of two-variable. It was observed that the students abandoned making memorized algebraic analysis in this test after treatments of two-variable calculus concepts. Most students (19 out of 23) could perceive that this could not be a function. Furthermore, five students drew the graph of surface firstly, and then perceived the situation on graph by using vertical line test. It was seen that these students used vertical line test also in the first item, which includes graph representations. Also another four students drew the graph of surface correctly, but they perceived the situation in algebraic way, not by using the vertical line test.

The item, in which most of the students had difficulties, is the third item of the first test on two-variable functions. Graph of a surface of elliptic paraboloid was given, and domain and range of the function were required. Only two students were able to give both domain and range correctly. Some students demonstrated that they were not able to coordinate the function's process and its graph. Two students indicated that they did not understand the height of the graph at a point *(x,y)* on the horizontal axis is the value of *f(x,y)=z* because they gave the elements of range as *(x,y,z)*. Similarly, three students, who did not have the notion that elements of domain of a two-variable function should always be pairs, gave the elements of domain as *(x,y,z)*, where two students gave elements of both domain and range as *(x,y,z)*. Other students demonstrated they had the notions that the elements of domain and range of a two-variable function should be always ordered pairs and a real number respectively. They were able to give the elements of domain or range correctly, but they could not determine the region or interval respectively. It was seen that most students' difficulties with domain were about projecting since they gave the domain as

$$\left\{(x,y)\middle|\frac{x^2}{9}+\frac{y^2}{25}=1\right\}$$

instead

$$\left\{(x,y)\middle|\frac{x^2}{9}+\frac{y^2}{25}\leq 1\right\}$$

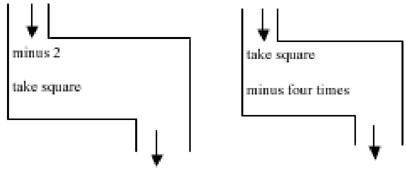
of . As can be seen, the domain and range of the function, whose graph was given, were required also in the last test. It was seen that the students' difficulties with projecting continued also in this test. For instance, some students (three) projected the surface to the coordinate axis separately instead of projecting to the plane. However, the students did not generally give the elements of domain or range wrongly as in the previous test. Only one or two students used triple *(x,y,z)* as the element of domain and range respectively. In the last test, the number of students, who were successful about determining the domain increased from three to 10.

Findings of Interviews

The interviews, which were also designed for data triangulation, enabled us to comprehend whether the subjects' conceptual levels, which were determined at least project level according to the results of the tests were actually higher than the process or not. The interview tasks were prepared as follows:

1. According to you what is a function?

2.



What do the figures on the up remind you?
What are the functions the figures on the up represent?

3. Is there a relation between the functions, which are represented with above machines?

4. According to you, what do following notations mean?

$$* f : D_f \subset \Re \to \Re \qquad * f^{-1} \qquad * f \circ g$$

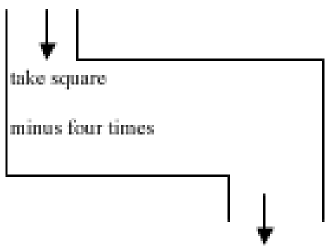
5. Could you give me a function whose inputs are functions?

Prompt if necessary: *Let's remember what the function machine is?

* Is there any restriction about the inputs of a function machine?
* Can the inputs of a function machine are also functions?
* Do you know such a machine?

6. What is two-variable function?

7.



What are the functions the figures on the up represent?

8. Is the algebraic expression of $z=f(x,y)=5$ a function situation? Why? Could you represent this algebraic expression graphically?

9. Could you represent the algebraic expression of $z=x^2+y^2$ graphically? Could you find the domain and range of the function? How do you find?

10.

(x,y)	*(2,1)*	*(3,2)*	*3,1)*	*4,4)*
f(x,y)	*1*	*3*	*6*	*4*

x	*1*	*3*	*4*	*6*
g(x)	*2*	*6*	*8*	*12*

According to the tables given above, can the algebraic expression of g(f(x,y)) be possible?

What does this algebraic expression mean? Could you find the following values?

**g(f(3,2)*   **g(f(4,4)*

We present some indications of process level, which have been revealed from literature, (Carlson et al., 2007; Dubinsky, 1991; Dubinsky & Harel, 1992) and related tasks of interview are presented in the following:

- Defining or explaining a function as a input-output process that transforms each input to only one output (1,6)
- Perceiving a function situation (8)
- Converting a function situation to another representation (2, 7, 8, 9)
- Understanding equality of two functions (3)
- Finding domain and range of a function (9)
- Combining a function process with another function process (10)

Apart from above indications, we considered the subjects' all responses as a whole to see their conceptual levels. With the fourth and fifth tasks, we questioned whether the subject was able to see a function as an object as seen in the following excerpts.

We gave numbers to the interviewed students from one to six. We will explain some students' results of the interview analysis briefly and more detailed results for other students.

Results of the First Student: According to the analysis of interviews, first student's conceptual level for function notion was determined as at least process, while her conceptual level was estimated as relatively weak process conception with severity and manipulation restrictions at the beginning of the course. After review of function notion with function machine, first student, who were not able to perceive function situations neither in graph nor in algebraic representations in the first test, gained to perceive function situations successfully in graph and algebraic cases with vertical line test and algebraic manipulations respectively. In the interview, she perceived the same functions rapidly. She demonstrated her knowledge about input-output and transference of function notions, while the interviewer and she were talking about the given notations. Furthermore, she was able to give the inverse and composition operations as examples of functions, whose inputs are also function, so her conceptual level for function concept seemed to be object in the interview.

In the first test on two-variable functions, it seemed that her difficulties were about coordinating function notion with three-dimensional space schema. She was able to perceive function situations in both algebraic and graph cases, but perceived function situations with algebraic manipulations also in graph representations by converting the graphs to algebraic representations firstly. She had difficulties also with domain and range. Her conception seemed to us weak process conception with severity and maybe manipulation restriction. At the end of the course, she had capability of perceiving function situations with vertical line test in some graph cases, and she gained dimensional notion of domain of a two-variable function. Consequently, we estimated her conceptual level of two-variable function as process conception. When the two-variable function notion was questioned, the manipulation restriction that seemed to be in the responses of the first student in the test items on two-variable functions was not observed in the interviews. In addition to defining two-variable function correctly, not only was she able to draw the graph of given function but also obtain the domain and range of function by projecting accurately. That is, it was seen that she coordinated the process of function and three dimensional space notion. These capabilities of her indicated that her conceptual level of two-variable function was *at least* process level.

Results of the Second Student: The second student demonstrated a good understanding of single and two-variable function notions in the tests, but she had a lot of difficulties in the interview. It seemed that the second student's conceptual level was relatively weak process conception with severity, manipulation, and quantitative restrictions in the first test. Then, she gained to perceive function situations in both graph and algebraic cases in the second test, whereas she was not able to perceive functions geometrically in graph representations. At the end of the analysis of this test, her conceptual level of single variable function was determined to be process conception with severity and manipulation restrictions. In the first test on two-variable functions, she was able to respond all items correctly, but again with algebraic approach. That is, her severity restriction continued for the two-variable functions and her conceptual level estimated as strong process conception with severity restriction. In the last test, it seemed that she perceived function situations in some graph cases geometrically, so her conceptual level was considered to be relatively strong process conception. However, the results of the interview about two-variable functions are not consistent with the results of the tests. Regarding single variable functions, the qualitative restriction that she had in the tests was obvious also in the interview. Moreover, qualitative restriction prevented her from seeing a function as an object. We estimated her conceptual level of two-variable function notion as weak process conception at the end of the study.

Results of the Third Student: The third student's conceptual development of function notion was different from the previous two students. This student was not able to abandon algebraic approach neither in tests nor in the interview. That is, she had severity restriction in all tests and interview. Her conceptual levels were determined as relatively weak process conception with severity restriction, process conception with severity and manipulation restrictions, process conception with severity and manipulation restrictions, and lastly process conception with severity restriction. She had manipulation restriction in her general function notion also in the interview, but according to all responses in the interview, her conceptual level seemed to be object level. Also in the interview, it was observed that she had severity restriction, which means to have algebraic approach. She was able to determine domain of a two-variable function from its algebraic representation, but she was unsuccessful about determining domain from graph. On the other hand, she was able to draw graphs of surfaces correctly, yet she did not have the capability of coordinating her schema of three-dimensional space and general function concept schema. Thus, we concluded that the lack of this capability prevented her from developing her conceptual level of two-variable function notion. After the interview, her conceptual level of level of two-variable function notion seemed to be process.

Results of the Fourth Student: The fourth student's conceptual capabilities were similar to the third student's. On the other hand, there were differences between their conceptions that the fourth student abandoned algebraic approach after review of function concept, but she gained severity restriction again in the two-variable function concept. That is, this student gained more capabilities about coordinating function notion with geometrical knowledge than the third student gained in the revision. Although the third student always had algebraic approach, both of the third and fourth students' conceptual levels of function concept were determined as object level at the end of the interviews. In the aspect of two-variable functions, the fourth student demonstrated some knowledge about coordination between schemas of three-dimensional space and function notion. For instance, her knowledge about coordination between the schemas was obvious while she was drawing the surface, on the contrary to the third student's drawing in the interviews. Moreover, the third student attempted to determine domain of two-variable function from algebraic representation, while the fourth student was able to find the domain graphically by taking projection even if she was not able to find the domain exactly. She projected the surface to the axis separately. On the other hand, the fourth student' generalizing was not based on independent variable. She generalized general function notion as in the following:

> *"Two-variable function is based on definition of function concept, this time domain consisted of AxB that is, domain consisted of pairs, range is a set, range should already be a set, because it is two-variable function, it has two independent variables and one dependent variable"*

However, her definition and applications were not consistent. She demonstrated a good understanding of two-variable function notion. Her conceptual level of two-variable functions was estimated as strong process.

Results of the Fifth Student: This is the most successful student not only in the interview but also in the tests. In the first test on single variable function notion, she was able to give correct answers except first and second questions in which function situations are required respectively in graph and algebraic representations. She had difficulties about independent variable of a single variable function like most of other students, but she *did not have any restriction*. Her conceptual level seemed to be weak process conception in the first test. In the following test on single variable functions, she was able to perceive function situations in graph and algebraic cases by gaining the notion of variables of a single variable function. Furthermore, she did not have any restriction also in this test, contrary to the most of other students, who had severity restrictions. That is, she was able to perceive function situations of given graphs by using vertical line test without determining algebraic formula. Her conceptual level was determined as relatively strong process conception in the second test. She was able to perceive two-variable function situations also in the first test on two-variable functions, but it was seen that she had severity restriction like most of other students in this test. In the second test on two-variable functions, she abandoned severity restriction. The fifth student's conceptual levels for two-variable function concept were determined as "process conception with severity restriction" and "process conception" in the first and second tests on two-variable function notion respectively. The only difficulty she encountered was about finding domain by projecting graph of surface to the plane in both tests. She projected the surface to axis separately, but she was aware that elements of domain must always be pairs. When we focused on the interview with this student, we concluded that her conceptions were at least object (can be schema) and at least process (can be object) for single variable and two-variable function concepts respectively. She demonstrated her conceptual level for single variable function concept in all tasks with her expressions in the interview.

Results of Sixth Student: The sixth student is the weakest one both in the interview and tests. At the beginning of the course, her conceptual level was determined as action conception in the first test. She struggled to determine the algebraic formulas in the graph representations to perceive function situations like most of the other students, but then she chose all items except the fix as function situation. The only case in which she was able to perceive function situations was table representations. Moreover, she demonstrated that she did not have input-output and transfer notions in the definition item. She gave complicated expressions instead of definition of function. After review by function machine, she gained input-output, transfer notions and unique right condition in definition. She was able to perceive function situations with algebraic approach both in graph and algebraic representations. So, it was interpreted that she had process conception with severity restriction in the second test. In the first test on two-variable function notion, she was able to perceive function situations only in algebraic representations. Furthermore, it was observed that she could not project surface to the plane, even worse she gave elements of domain of two-variable function given by graph as triples, so it could be interpreted she did not have the notion that elements of domain of a two-variable function must be pairs. Besides, she was able to determine two-variable function, but she demonstrated that she had manipulation restriction. Her conceptual level for two-variable function seemed weak process conception with manipulation restriction. She had a little development in the last test on two-variable function notion, so her conceptual level was again weak process. She only abandoned dimensional complexity about elements of domain of a two-variable function, but she projected the surface to the axis separately at that time. That is, again she could not obtain domain of the two-variable function given by graph correctly. This student's weak function conception was also obvious in the interview.

Conclusion

Two-variable function concept is fundamental for most advanced mathematics concepts. Either this concept is given in theoretical manner as in science faculties or in applied manner as in engineering faculties; understanding of this concept needs to generalize a lot of key aspects of single variable functions to two independent variables. As Martinez-Planell and Trigueros-Gaismann (2009) indicated, one of the objectives of an instructor who gives the concept of two-variable function should be helping students develop a deep understanding of general function notion. Also being aware of this necessity, we reviewed general function notion at the beginning of the course. Multiple representations and relation between them are recommended not only in the teaching processes of single variable functions but also in instruction of two-variable functions (Martinez-Planell & Trigueros-Gaismann, 2009). Our instructional approach is consistent with these recommendations. We gave attention to different representations and converting between them. Furthermore, we posed such tasks to students that help them to generalize key aspects of function notion from single independent variable to two independent variables in each representation. Also the students' responses on representations and converting between them supported our approach. Generally, the students did not have much difficulty about converting different representations not only in single variable functions but also two-variable functions.

The relationship between the students' understanding level of function concept and their construction of two-variable function concept is obvious from the data of this study. Thus, we have concluded that students' understanding level of function concept plays a fundamental role in understanding two-variable functions. It has seen that especially being at the object level for the function concept is crucial to reach at least process conception of two-variable function concept. Like Martinez-Planell and Trigueros-Gaismann (2009), we have concluded that most of the students have difficulties with graph representations of two-variable functions. Moreover, we have observed that a student who has object conception of function may have process conception of two-variable function concept even if s/he cannot coordinate the process of two-variable function and the graph. When such student begins to understand the graph of a two-variable function, s/he begins to encapsulate the two-variable function process as an object. If a student's conceptual level for function concept is at most process, s/he may have weak understanding of two-variable function concept. Especially, restrictions prevent students from understanding two-variable function. For instance, quantitative and/or manipulative restrictions prevent students from not only seeing function as an object, but also generalizing function process from single independent variable to two independent variables. Furthermore, if a student has restrictions on general function concept, s/he will transform restrictions to two-variable function concept. We have seen that most students have algebraic approach at the beginning of the understanding process of two-variable function concept. That is, they struggle to determine algebraic formula, and then make algebraic analysis to perceive function situations in the graph representations. Some students can have indications in their algebraic analysis that they transform input-output and function process notions to two-variable functions, while some of others can make algebraic analysis by memorizing. For instance, the student, who has object conception of function concept, can abandon algebraic approach rapidly, and s/he begins to coordinate graph and process of two-variable function. Such a student perceives function situations by vertical line test in graph representations of two-variable functions. Moreover, the student, who has object conception of function concept can reach process level of understanding two-variable function concept even if s/he is not able to abandon algebraic approach due to her/his difficulties about three dimensional geometry. Besides, the student, who has any restriction on function concept, but has coordination of graph and process of single variable functions, can abandon algebraic approach rapidly even if s/he is not at object level. To sum up, we can say that there is a direct relationship between students' construction of two-variable function concept and their conceptual levels of general function concept. Particularly, having object conception of function and restrictions have respectively positive and negative effects on students' understanding two-variable function concept.

In the aspect of graph representations of two-variable functions, we have concluded that students' most problematic point in understanding process of two-variable functions is graph representations. Trigueros and Martinez-Planell (2009) indicated that students' understanding of graphs of two-variable functions is not easy and it is related with their three dimensional space schema. We have seen that generally students can draw graphs of some special surfaces, but some of them can make this drawing by using memorized facts. The students, who have memorized drawing way, may use statements like "lift", as a memorized fact without awareness. Trigueros and Martinez-Planell (2009) emphasized about such words that the use of certain words such as "cutting a surface" or "lifting a curve" enable students to visualize, but the meaning of this kind of sentences should be explicitly discussed in class. We have seen otherwise some students can memorize this kind of sentence as drawing way of surfaces. Such students attempt to perceive function situation algebraically, even if they draw graph correctly by using memorized facts. That is, they cannot coordinate the function process with their three-dimensional space schema. Due to the lack of coordinating capability, they cannot find domain or range from graph by projecting. Moreover, some of such students may not be aware that the elements of domain and range of a two-variable function are always pair and a real number, respectively. On the other hand, the students, who can coordinate the function process with their three-dimensional space schema, reflect these capabilities by their drawing the graph and perceiving function situations in graph representations by using vertical line test. We have observed that such students can perceive function situations in the graph by using vertical line test even if they have some difficulties with three-dimensional space, but they cannot project the graph to the plane to find the domain correctly. Consequently, students' schema of three-dimensional space is fundamental for their construction of two-variable function concept also. However, it is not as much prerequisite as general function conception, because a student, whose function conception is object level, can reach process conception of two-variable functions by her/his algebraic approach even if s/he has weak schema for three-dimensional space.

When we regard students' concept definitions for the single variable and two-variable function concepts, we can conclude that some students transform their knowledge about general function notion to two-variable functions, whereas all of the students who have restrictions on general function notion transform their restrictions from single variable functions to two-variable functions. That is, the student who has correct concept definition of function concept including input-output and transference notions does not mean to have such a concept definition of two-variable function concept. Once we consider students' generalizations of definition of function notion from single independent variable to two independent variables, we have seen that some students generalize concept definition of single variable function notion to two independent variables, while some of others may generalize the concept definition of general function notion to a relation from Cartesian product of any nonempty two sets to any nonempty set, which ensures the unique right condition. On the other hand, we have concluded that a student might demonstrate good understanding of two-variable function concept even if s/he has such a wrong generalization.

We concluded that function machine is effective as a cognitive root on understanding the function concept. We saw that after revision of general function notion with function machine, students gained the notion of independent variable of function, which almost all of them did not have at the beginning of the course. We have been sure that students' achievement about perceiving function situations would have increased, if we had asked the questions like "which are functions?" instead of "which are functions whose independent variable is x?" in the first test. They became confused with these questions at the beginning, but they were aware what was required from them with such questions after they reviewed the function notion with function machine. Moreover, emphasizing with various examples that function machine transforms input of the machine, which is independent variable of function to output, which is dependent variable of function gained the students function process. We used function machine not only at the beginning of the course as a cognitive root, but also through the course. We introduced calculus concepts like derivative or integral also as a function machine whose inputs and outputs are functions. Generally students can make some operations on the set of functions, but they see neither these operations as functions nor the functions as elements of a set. To give such operations on the sets of functions as an example of function machine seemed helpful to gain the students to see a function as an object. Moreover, such examples are helpful to remove the restrictions like quantitative or manipulation. In the interviews, subjects agreed with us about function machine's positive effects on understanding general function notion or two-variable function concept. For instance, the third student's idea about function machine is as following:

Interviewer: What is the effect of function machine on your understanding of function concept?

The third student: It is very helpful. Even if this concept won't be given to me any other time, I know what it means. It has showed to me that there is only one output for every input. Actually, I can apply this machine to other mathematical concepts. I am sure also my friends won't have any difficulty when asked what function is…because we usually do it from memory without awareness if it is not explained concretely, but this machine is really good.

Although this paper has focused on students' generalizing the function notion from single variable function to two-variable function, the obtained results have also informed us about students' understanding of two-variable function concept. In this context, we have seen that the students fell into different groups according to their knowledge and capabilities related to two-variable function concept. Considering the conceptual levels of general function concept, our experiences and students' conceptual groups revealed in this study, we have attained some indications of conceptual levels of two-variable function concept. Accordingly, a student at action level for two-variable function concept can make manipulative calculations like finding the output for given input and algebraic formula of two-variable function. In terms of function situations, s/he may perceive function situations in table representations. Maybe s/he makes some algebraic manipulations by memorizing facts to perceive function situations in algebraic or graph representations, even s/he is able to give correct answer about whether the situation is function or not, but s/he cannot perceive function process. Such a student may draw graphs of some special surfaces by using memorized facts, but s/he does not have coordination between three-dimensional space schema and function schema. The students who are at the action level cannot coordinate function process and three-dimensional space schema anytime. Moreover, they can give triples as elements of domain or range.

Process conceptual level of two-variable function concept seems very complicated as process conceptual level of general function concept. Students transfer general function process to two-variable functions and begin to generalize key aspects of single variable functions to two independent variables. Furthermore, coordinating function process and schema of three-dimensional space begins at this level. At the beginning of process conception, most students may have algebraic approach. That is, they attempt to perceive function situations with algebraic manipulations either in graph or algebraic representations. After their experiences with graphs of two-variable functions, in which they coordinate function process and the shape of graph, they begin to perceive function situations in graph representations by using vertical line test. Moreover, such students should have begun to draw some special surfaces regarding the coordination between function process and three-dimensional space schema. These students should have domain and range notions of two-variable functions and they are aware that the elements of domain and range of a two-variable function are always pairs and real numbers respectively even if they are not able to obtain domain or range from graph representation by projecting.

When students reach object level of two-variable function concept, they should have transfer key aspects of general function concept to two-variable function concepts and they can perceive two-variable function situations in any representations. Since students should have good understanding of three-dimensional space and capability of coordinating their knowledge about three-dimensional space with their two-variable function schema in this level, they can perceive two-variable function situations in graph representations by using vertical line test and obtain domain or range from graph representations by projecting. In other words, whether a student has good understanding of three-dimensional space or not is effective to reach this level.

The studies on understanding of two-variable functions, which have been recently included to literature, have focused on representations of two-variable functions (Yerushalmy, 1997) and in particular, geometric representations of two-variable functions and students' understandings of two-variable functions in graphical representations (Trigueros & Martinez-Planell, 2009). Considering the results obtained in this study with these previous studies, further studies can be designed to clarify more aspects of understanding levels of two-variable functions. Such further studies can contribute to many studies on students' understandings of multivariate calculus concepts.

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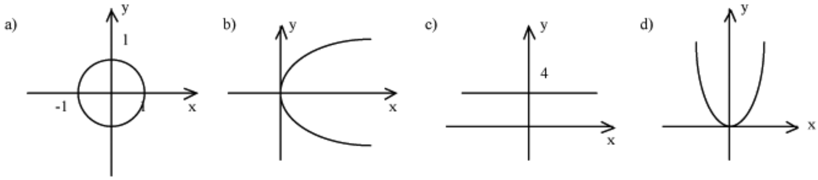
Ek 1: Tek Değişkenli Fonksiyonlar Konusundaki İlk Test:

1. Aşağıdaki grafiklerden hangileri bağımsız değişkeni x olan bir fonksiyon grafiğidir? Neden?



2. Aşağıda verilen tablolardan hangileri bağımsız değişkeni x olan bir fonksiyon temsil eder? Neden?

a)

Girdi	*0*	*1*	*2*	*3*	*4*
Çıktı	*2*	*2*	*2*	*2*	*2*

b)

Girdi	*-1*	*0*	*1*	*2*	*3*
Çıktı	*0*	*2*	*2*	*4*	*4*

c)

Girdi	*0*	*1*	*-1*	*0*	*2*
Çıktı	*1*	*2*	*0*	*-1*	*1*

3. Aşağıdaki eşitliklerden hangileri bağımsız değişkeni x olan bir fonksiyon temsil eder? Neden?

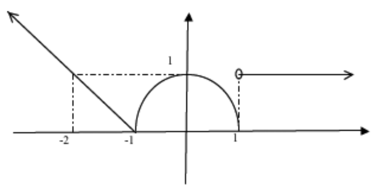
a) $x^2 + y^2 = 1$ b) $y = 4$ c) $x = y^2$ d) $y = 2x^2$

4. $y=5x^2-6$ cebirsel eşitliği ile verilen fonksiyonun aşağıda verilen çıktı değerlerine karşılık gelen girdi değerlerini bulunuz.

a) -1 b) –11

5. Aşağıda verilen grafiği kullanarak tabloyu tamamlayınız.

girdi	çıktı
0	
1	
½	
-1	
-3	



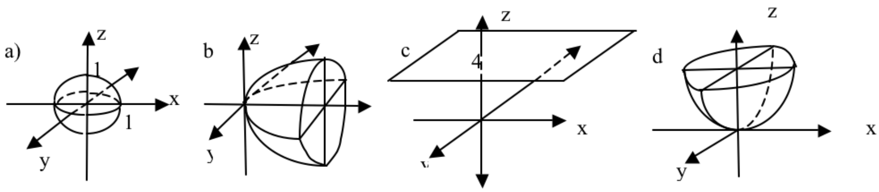
6. Aşağıdaki fonksiyonların grafiğini çiziniz.

a) $f(x) = \begin{cases} y = -x & ; \quad x < -2 \\ 1 & ; \quad -2 \le x \le 2 \\ 3 & ; \quad x > 2 \end{cases}$ b) $\begin{array}{lcl} f:[-1,1] & \to & \Re^+ \\ x & \to & y \end{array}$, $x^2 + y^2 = 1$

7. Fonksiyon kavramını tanımlayınız.

Ek 2: İki değişkenli fonksiyonlar konusundaki ilk test:

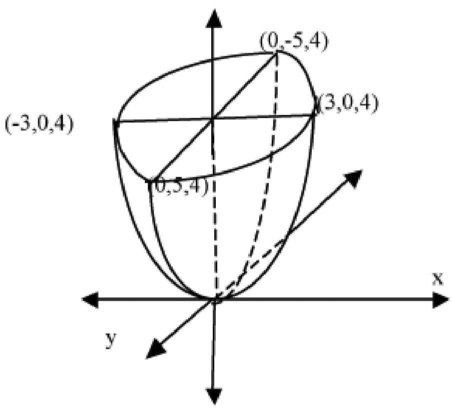
1.Aşağıdaki grafiklerden hangileri bağımsız değişkenleri x ve y olan iki değişkenli fonksiyon grafiğidir? Neden?



2. Aşağıdaki eşitliklerden hangileri bağımsız değişkenleri x ve y olan iki değişkenli fonksiyon temsilidir? Neden?

a) $x^2 + y^2 + z^2 = 1$ b) $z = 4$ c) $x = y^2 + z^2$ d) $z = x^2 + y^2$

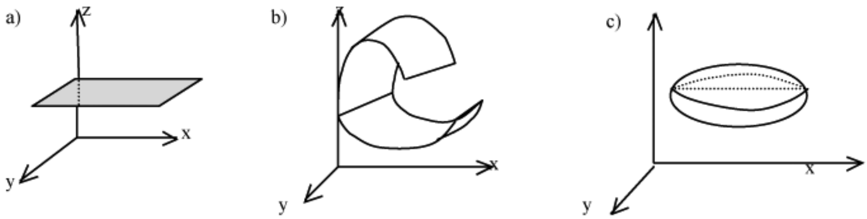
3. Aşağıda grafiği verilen iki değişkenli fonksiyonun tanım ve değer kümelerini bulunuz.



4.İki değişkenli fonksiyon kavramını tanımlayınız.

Ek 3: İki değişkenli fonksiyonlar konusundaki ikinci test:

1.Aşağıdaki yüzeylerden hangileri iki değişkenli fonksiyon temsilidir? Neden?



2. Aşağıda verilen cebirsel ifade bir iki değişkenli fonksiyon temsili midir? Neden

$$f(x,y)=\begin{cases}4-x^2-y^2 & ; \quad 0\le x^2+y^2\le 3\\ 1 & ; \quad 2\le x^2+y^2\le 4\end{cases}$$

3. $A=(0,\sqrt{2},3)$, $B=(0,-\sqrt{2},3)$, $C=(-\sqrt{2},0,3)$, $D=(\sqrt{2},0,3)$, $E=(0,\sqrt{2},1)$, $F=(0,-\sqrt{2},1)$, $G=(0,2,1)$
$H=(0,-2,1)$, $I=(-2,0,1)$, $J=(2,0,1)$ olsun. Aşağıdaki grafik iki değişkenli bir fonksiyon grafiği midir? Neden?

Aşağıdaki grafik iki değişkenli bir fonksiyon