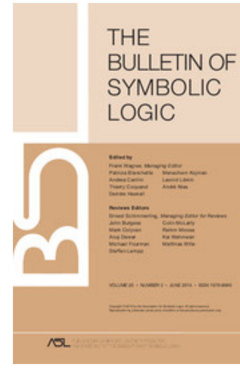

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Abstract. We try to recast in modern terms a choice principle conceived by Beppo Levi, who called it the Approximation Principle (AP). Up to now, there was almost no discussion about Levi's contribution, due to the quite obscure formulation of AP the author has chosen. After briefly reviewing the historical and philosophical surroundings of Levi's proposal, we undertake our own attempt at interpreting AP. The idea underlying the principle, as well as the supposed faithfulness of our version to Levi's original intention, are then discussed. Finally, an application of AP to a property of metric spaces is presented, with the aim of showing how AP may work in contexts where other forms of choice are commonly at use.

§1. Introduction.

1.1. The Approximation Principle in the context of the Axiom of Choice. In one of the best sources for the history of the Axiom of Choice (AC), G.H. Moore's monograph [30], several members of the "Italian school" are acknowledged to have provided, between the end of the 19th and the beginning of the 20th century, some strong criticisms against allowing arbitrary choices in mathematics. Among those who were most involved into this critique, Moore mentions Giuseppe Peano, Rodolfo Bettazzi, and Beppo Levi. In the present paper we focus on a principle that is characteristic of Levi's contribution to this debate, which has remained neglected so far.

Beppo Levi's role in the history of AC can roughly be summarized as follows. In 1902 Levi wrote a short note [18] inspired by Bernstein's 1901 doctoral dissertation at Göttingen [2], which contains a proof of what today is known as the Partition Principle. Levi noticed that this argument requires to freely choose an element from every set of any given family of sets. This is usually regarded as one of the very first explicit references to AC in the literature, for example by H. Rubin and J.E. Rubin [38, p. 7]:

> Apparently, the first specific reference to the axiom of choice was given in a paper by G. Peano [31]. In proving an existence theorem for ordinary differential equations [which is now named after him], he ran across a situation in which such a statement is needed. Beppo Levi [18] , while discussing the statement that the union of a disjoint set t of nonempty sets has a cardinal number greater than, or equal to

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> the cardinal number of t (the Partition Principle), remarked that its proof depended on the possibility of selecting a single member from each element of t. It is not known whether the partition principle implies the axiom of choice. Others, including Cantor, had used the principle earlier, but did not mention it specifically.[1]

Beppo Levi's remark rather is an observation made in the course of a proof than an attempt to single out the underlying principle in the form of an axiom. The note of Levi's has nonetheless prompted the wrong belief that he had preceded Zermelo in this respect (see, for instance, Abraham Lévy [27, p. 159]). On the other hand, Moore [30, p. 80], Lolli [28, p. 156], and already Zermelo himself [49, p. 113] rightly ascribe to Levi the partition principle only.

Later on, after AC was made explicit by Zermelo in his 1904 paper [48] containing the proof of the Well-Ordering Theorem, Levi went back to this topic in a series of publications between 1918 and 1934. As quite a few other mathematicians of his time, Levi was a fierce critic of Zermelo's proposal to use AC in mathematics. Levi even pushed his opposition as far as to proposing a new principle, the so-called Principle of Approximation (AP), which he intended *expressis verbis* as an alternative to Zermelo's AC; this is clear, for instance, from Levi's 1923 letter to Hilbert [22, pp. 169–170]. Levi's AP was then studied, also in connection with some mathematical applications, by himself and some of his followers up to the early 1930s, but has hardly reappeared since.

As a matter of fact, very little is known of this part of Levi's work, both in and outside Italy. G. Lolli's note introducing Levi's contributions to logic and metamathematics [25, vol. I, pp. LXVII–LXXVI] contains some reference to AP, but is more or less limited to an historical reconstruction of the context in which that proposal was made. Moore [30, p. 244] does refer to Levi's principle, and actually presents a formulation of AP, which, however, we cannot consider as a satisfactory. An excuse for this certainly is Levi's cumbersome way of presenting AP, plus the fact that all his writings about AP—including his letter to Hilbert in 1923 [22]—were written in a somewhat old-fashioned Italian (see Section 2.3 for a sample). Moore, moreover, pays little attention to Levi's somewhat philosophical remarks around the formulation of AP, which actually help to gain a better understanding of it.

Presumably because of the unclear status of AP in relation to AC, Levi's name is missing from the name index of today's most comprehensive list of forms of AC by Howard and Rubin [15]. Apart from Levi's work on the partition principle, there is no further mention of his name by H. Rubin and J.E. Rubin [38]. The situation is analogous for the earlier editions of the latter monograph [36, 37], and there is no talk of Levi at all by Jech [16]. One would not even expect to find Levi's name in Herrlich's more mathematical monograph [11], from which he is absent indeed.

[1] In this quote the obvious typographical slips were corrected, and the references were redirected to the bibliography of the present paper.

The main character of the historical debate on AC is metamathematical, as we would call it today. How far metamathematical issues do have an impact on proper mathematical ones is sometimes measured by referring to the viewpoint of the "working mathematician". The limitation of such a measure is that the concept in question is bound to change historically. It is hard to deny that Levi was, when dealing with AP, a working mathematician of his time, though certainly one who was not indifferent to the debate on the foundations of his discipline. And Levi was pretty clear about the impact that he thought this issue could have on the mathematics as it was practiced in his days [21, p. 310]:

> *The assumption of Zermelo's* [*choice*] *postulate contradicts the very nature of analysis* and must thus be rejected as deprived of any sense.[2]

Nobody can nowadays be expected to address the issue of choice in a similar manner. Quite on the contrary, if only vegetarianism had not become such a common practice, one would rather accept the view according to which, from the perspective of a (meat-eater) working mathematician, doing mathematics without AC looks like avoiding steaks.

Even today, however, there are areas of mathematical practice which do recommend paying attention to what one would better (not) "eat", mathematically speaking. Where an interplay with computer science is at issue, for example, as is the case for automated theorem proving and proof checking, particular attention is required toward axiomatic resources (see, for instance, [10]). This is the kind of attitude we are concerned with here, one which aims at displaying, for the purpose of mutual comparison, the axiomatic variety of degrees of freedom in picking elements from a given collection of mathematical objects. Hence discussing Levi's AP is an undertaking certainly not void of implications for today's research.

1.2. A first approximation to the Approximation Principle. Due to the lack of consideration outside the circle of Levi and his scholars, AP has arrived to us in a form in which it cannot possibly be understood by the present-day scholar, who may even doubt that Levi is describing a mathematical principle at all. This was one of our reasons for undertaking the effort to interpret AP in modern terms. The interpretation of AP we focus on in Section 2.2 is not easy to grasp either, but it represents a considerable step forward with respect to the principle as originally given. In particular, we use the language of modern mathematics, with which nowadays everyone working in formal logic is acquainted.

We thus are able to relate our interpretation of AP with AC (Section 2.2), and to study this relation in connection with an application of AP by Levi himself (Section 3). In a first and rough approximation, our interpretation of AP can be put as follows:

[2]Since all of Levi's works we will make reference to are in Italian, their translation to English is our own. Emphasis in quotations goes back to their author's own choice, unless specified otherwise.

> Let X, Y be sets. Assume that $F \subseteq X \times Y$ is a total relation, i.e., for every $a \in X$ there is $b \in Y$ with $(a,b) \in F$. Then there is a *partial* function $f : X \to Y$, defined on those $a \in X$ for which
>
> $$F(a) = \{b \in Y \mid (a,b) \in F\}$$
>
> is "arbitrarily approximable", such that $(a, f(a)) \in F$ for every such $a \in X$.

No matter what "arbitrarily approximable" actually means, the principle above can readily be recognized as a conditional form of AC, simply because the latter asserts straightaway, for any given total relation $F \subseteq X \times Y$, the existence of a subfunction with the same domain X: that is, a *total* function $f : X \to Y$, i.e., one that is defined on any $a \in X$ whatsoever, such that $f(a) \in F(a)$ for every $a \in X$. In other words, with AC at hand there is no need to restrict the domain of the choice function f to those arguments $a \in X$ for which the possible range $F(a)$ of the value $f(a)$, i.e., the set from which $f(a)$ is to be chosen, is "arbitrarily approximable".

What, however, does "arbitrarily approximable" mean? Referring the reader to Section 2.2 for the precise picture, we try to give a rough idea as follows. Typically X is a family of sets; whence from now we denote the elements of X by capital letters A, A', A''. Moreover, Y is endowed with a certain notion of distance between subsets of Y: that is, a function

$$d : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{Q}_0^+ .$$

On this d Levi imposes the condition

$$(*) \qquad d(Z, Z') = 0 \iff Z = Z'$$

for subsets Z, Z' of Y, which is familiar as part of the definition of a metric.

In an instance that is relevant for the mathematical application of AP we will study eventually (Section 3), we will have $Y = \mathbb{Q}^+$, and

$$d(Z, Z') = \sup(\{0\} \cup (Z \triangle Z'))$$

for subsets Z, Z' of Y where $\triangle$ denotes the symmetric difference: that is,

$$Z \triangle Z' = (Z \cup Z') \setminus (Z \cap Z') .$$

For this d, clearly, the intended meaning of a "measure of difference" is met, and $(*)$ is satisfied.

In general, let X be a family of sets and Y a set with a rational-valued distance function d between subsets of Y, satisfying $(*)$. Given any total relation $F \subseteq X \times Y$ and $A \in X$, following Levi we will call $F(A)$ "arbitrarily approximable" if for every $\delta > 0$ there is a finite subset A_0 of A such that if $A' \in X$ and $A'' \in X$ both contain A_0, then $F(A')$ and $F(A'')$ are within δ of one another: that is, more formally,

$$A_0 \subseteq A' \cap A'' \Longrightarrow d(F(A'), F(A'')) < \delta .$$

This is Levi's definition as interpreted in modern terms, no less and no more.

The reader wondering what it actually means is in numerous company, including ourselves when we started our work on this topic, but may now understand why we found it worthwhile to dig further down. We hope that with the present paper we do justice to Levi's proposal of AP—not only from a historical or purely philosophical perspective, but also by highlighting some important logical and mathematical aspects of Levi's work on AP.

§2. The route toward Levi's principle.

2.1. On "deductive domains". One of the most important papers of Levi's on the issue of choice opens up as follows [21, p. 305]:

> *Any given mathematical argument presupposes* [. . .] *one or more aggregates, for each of which IT IS POSTULATED the possibility of picking an arbitrary element, as a prime and irreducible act of thought.*

This statement indeed addresses a crucial feature of what should be referred to as Levi's philosophical view of mathematics and its foundations. Related ideas had already guided a lengthy note of his [19] in which he aimed at explaining the source of logical paradoxes in mathematical reasoning (see also [6]).

The conceptual bulk of the quote above can be summarized as follows: mathematical reasoning requires that a domain of primitive objects be specified, to which one refers either directly, or indirectly by means of objects which are defined in terms of the primitive ones. It is allowed then (it is *postulated*, as Levi says), to pick an *arbitrary* element from each aggregate belonging to the chosen domain.

Levi might have had in mind a very common aspect of mathematical reasoning: if we want to prove a statement about a class of mathematical objects, then an attempted proof would typically start by "Let x be any given object of that class. . . ". For the sake of thus picking an *arbitrary* element, the class of objects must be assumed to be given as primitive, independently from any possible way of defining some of its elements in terms of objects outside the class.[3]

The logical counterpart of that situation is particularly interesting inasmuch as the issue of choice is concerned. The above kind of argumentation can be seen as just a special case of what is known as the *witnessing*, or *existence property* in formal axiomatics. In the simplest terms, this means the problem, for a given formal system of axioms T and a statement $\exists x\varphi(x)$ provable in T, i.e., $\mathsf{T} \vdash \exists x\varphi(x)$, to find a closed term t which, provably in T, is a witness for this existential statement, i.e., $\mathsf{T} \vdash \varphi(t)$.

[3] Levi speaks of the concept of real number as of a primitive concept in the above sense, which is to be compared to the well-known fact that the collection of reals can be defined in terms of the rationals, e.g., by means of Dedekind cuts. Levi notices that the act of taking the collection of Dedekind cuts as a whole, or to pick an *arbitrary* cut from it, would remain independent from that definition, in the sense that the reals would not be "describable by means of operations on the natural numbers, or in any case reducible to them" ([21, p. 306]). Hence, as far as the above situation is concerned, the collection of reals (either as such, or in the form of Dedekind cuts) must be assumed as given.

In particular, this is known to be a requirement for theories endorsing a constructive approach to mathematics (the most popular examples for these being P. Aczel's CZF, P. Martin-Löf's type theory ITT, and J. Myhill's system CST).[4] As a matter of fact, in the case of this group of theories, the above problem naturally connects with the Brouwer–Heyting–Kolmogorov interpretation of existential statements. Incidentally, Levi himself referred to this connection, in a later paper of his [23, p. 68], trying to clarify the meaning behind his own terminology.

Let us go back to Levi's own words. He introduces the notion of *prime aggregate*, which refers to all those collections of objects having the property that an arbitrary element can be singled out by an independent act of thought. In turn, aggregates of this sort give rise to a *deductive domain*, which is the collection of all the prime aggregates that have to be specified prior to performing a given mathematical argument. In a later note of his [23] (the last one, in chronological order, among those dealing with AP), Levi himself, as we have said above, presented deductive domains as providing an answer to "the demand of rigor of the so-called intuitionism". It is unclear, and beyond the scope of the present paper, whether this sentence was made on the basis of a proper understanding of the foundational view in question.[5] In order to justify the above connection, it seems enough, however, to refer to the concise conclusion of the passage in question, according to which "*every existential proof is also a constructive one in a deductive domain conveniently made precise*".

These comments have prompted the reformulation of Levi's principle that we are going to present next (Section 2.2). When trying to give it a modern and readable form we have realized that there are alternative interpretations of what Levi wrote. Hence we found it necessary to study their mutual relationships (see Section 2.2 again), as well as to discuss their faithfulness to Levi's original intention (see Section 2.3).

2.2. Functions vs. relations. As we said, in this part of the section we are going to recast Levi's notions in modern terms, to eventually make his principle more intelligible. Notice that in the definitions and results below we use the time-honored term "family of sets" for a set, class, or collection of sets.

[4]A systematic treatment of (special cases of) the existence property in systems for constructive set theory, as well as details regarding the theories we have made reference to here, can be found in the nowadays classical book by M.J. Beeson [1]. On this topic, the reader should also see [33, 34].

[5]What follows the sentence we have just quoted may lean towards the second alternative, as it seems that Levi had in mind a restricted interpretation of what intuitionism might be like. Indeed, he speaks of it as the view according to which the only arguments allowed in mathematics should be those "depending upon a finite number of elements, obtained by means of a finite number of really performable mental operations" ([23, p. 68]).

We begin by means of the following:

DEFINITION 2.1.

(1) A *prime aggregate* is an inhabited set.[6]
(2) For any given sets X, Y, a *function* from X to Y (in symbols: $f : X \to Y$) assigns to every $a \in X$ an element $f(a) \in Y$ such that, for any $a, b \in X$, if $a = b$, then $f(a) = f(b)$.
(3) A *deductive domain* is a pair $(\mathcal{E}, f)$, where $\mathcal{E}$ is a family of prime aggregates and $f : \mathcal{E} \to \bigcup \mathcal{E}$ is a function such that $f(A) \in A$ for every $A \in \mathcal{E}$.
(4) For a given deductive domain $(\mathcal{E}, f)$, a *natural extension* is a deductive domain $(\mathcal{E}', f')$ where (i) $\mathcal{E} \subseteq \mathcal{E}'$, and (ii) the restriction $f' \restriction \mathcal{E}$ of f' to $\mathcal{E}$ is equal to f (that is, $f'(A) = f(A)$ for all $A \in \mathcal{E}$). We use $(\mathcal{E}, f) \leq (\mathcal{E}', f')$ as a shorthand for "$(\mathcal{E}', f')$ is a natural extension of $(\mathcal{E}, f)$".

REMARK 2.2.

(i) A natural extension of a deductive domain is again a deductive domain.
(ii) If a family of prime aggregates only consists of singletons, then it automatically is a deductive domain.
(iii) The preceding remark extends to any family of sets, of arbitrary size, the elements of which are well-ordered sets.
(iv) For a given a deductive domain $(\mathcal{E}, f)$, if $\mathcal{E}' = \mathcal{E} \cup \{B\}$ (that is, $\mathcal{E}'$ extends $\mathcal{E}$ by a single set), then to give a natural extension $(\mathcal{E}', f')$ of $(\mathcal{E}, f)$ is tantamount to give an element $b \in B$ such that if $B \in \mathcal{E}$, then $b = f(B)$.

Although we have defined deductive domains as pairs, in the statements of AP below we may still refer to them by means of their domain, if only for the sake of simplicity.

The connection of these notions with AC should now turn out clearly to the well-versed reader. Let us make it more explicit. To this end, we first need another definition introducing the concept of many-valued function, as opposed to the usual, single-valued one. This distinction is indeed fundamental for our formulation of AP, as well as for AC in one of its equivalent presentations.

DEFINITION 2.3. Given X, Y sets, a *many-valued function* F from X to Y (for short, an MV-function; in symbols: $F : X \rightsquigarrow Y$), assigns to every $a \in X$ an inhabited subset $F(a)$ of Y.

[6] A set is inhabited if one can exhibit an element of it, which is the constructive interpretation of a nonempty set. We have preferred this notion for the present paper, and understand "nonempty" as synonymous with "inhabited", because the latter is closer to Levi's concept of a prime aggregate. In view of what follows, however, one must not confuse an inhabited set with a pair consisting of a set and a distinguished element of this set: from the latter, AC would readily follow.

Note that if X is inhabited and Y empty, then there is no MV-function of type $X \rightsquigarrow Y$. In set-theoretic terms, an MV-function from X to Y is nothing but a *total relation*: that is, a subset R of $X \times Y$ such that for every $a \in X$ there is $b \in Y$ with $(a, b) \in R$. Equivalently, any $F : X \rightsquigarrow Y$ is a function of type $X \to \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ denotes the power set of Y, such that $F(a)$ is inhabited for every $a \in X$.

Now, let us first look at the impact of this definition on AC. It is well known that among the different, albeit equivalent, formulations of this principle, we have the following:[7]

AC For all sets X, Y, and $F : X \rightsquigarrow Y$, there exists $f : X \to Y$ such that $f(a) \in F(a)$ for all $a \in X$.

As to the distinction between the two notions of function we have made, there are foundational and conceptual reasons supporting the need for it. In particular, the concept of an MV-function can be used to encapsulate the idea of a procedure to be performed according to a finite routine — though not necessarily one leading to a unique output for the same choice of the input — and would thus be the counterpart of the concept of nondeterministic algorithm of what is taken, under the name of "operation", as a primitive notion in, e.g., E. Bishop's approach to constructive mathematics [3, 4].[8]

However, the most pressing reason for proceeding this way comes from Levi himself, who has made use of this distinction in the statement of AP. He usually did not give details, with one notable exception: the formulation of the principle in [21, p. 312] (see Section 2.3 below for a full quote of this passage). There, in fact, footnote 12 reads: "To the word 'function' we attach here the most general meaning". The reference for it is Levi's own [20, Section III], where the above distinction (in the form of "polydrome" vs. "monodrome" functions) is clearly made. As will turn out from looking at our versions of AP and the subsequent results, the above form of AC seems—at least to us—to be the most suitable one for a comparison with Levi's principle.

Now, we return to AP. We proceed in a perhaps unexpected manner. Rather than presenting AP through Levi's own words, we offer various formulations of AP and discuss their mutual relationships. As will turn out, this is a way of getting closer and closer to Levi's own phrasing of AP—as far as this is possible. Only then (Section 2.3), we return to Levi's AP, and discuss the relation to our versions. So, we are basically showing the reader the way to AP step by step, in the spirit of our paper. Nonetheless, we make some observations along the way which might be of interest on their own, and

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[7]Due to the remark we have just made on MV-functions, it is easy to see that this form is easily proved to be equivalent to one formulation of AC which maybe is more familiar: For all sets X, Y, and $S \subseteq X \times Y$ such that for every $a \in X$, there is some $y \in Y$ with $(x, y) \in S$, there exists a function $f : X \to Y$ such that, for all $a \in X$, $(a, f(a)) \in S$. Again, if we understood an inhabited set as a pair in the sense of footnote 6, then this form of AC would be provable.

[8]The interested reader will find a detailed discussion of this topic in [39], together with a useful consideration of how it may affect the role of choice principles in constructive mathematics.

may help to gradually get accustomed to Levi's principle and its intricacies, which we think will let the reader be in a better position to fully appreciate it eventually.

In the following, let $(\mathcal{E}, f)$ be a deductive domain, Y a prime aggregate and $F : \mathcal{E} \rightsquigarrow Y$ an MV-function. Let further $d : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{Q}_0^+$ be a function such that $d(X, X') = 0$ if and only if $X = X'$ for every $X, X' \subseteq Y$. We write $\psi(A, F, d)$ whenever $A \in \mathcal{E}$ and

$$(\forall \delta > 0)(\exists x_1, \dots, x_n \in A)(\forall A', A'' \in \mathcal{E})$$
$$(x_1, \dots, x_n \in A' \cap A'' \to d(F(A'), F(A'')) < \delta),$$

Let further $\mathcal{D} = \{A \in \mathcal{E} | \psi(A, F, d)\}$. Now we can formulate AP in three different forms:

- **AP**^{1} For every $A \in \mathcal{E}$, if $A \in \mathcal{D}$, then $\mathcal{E} \cup \{F(A)\}$ is a natural extension of $(\mathcal{E}, f)$.
- **AP**$^+$ $\mathcal{E} \cup \{F(A) \mid A \in \mathcal{D}\}$ is a natural extension of $(\mathcal{E}, f)$.
- **AP**^{C} There is $f^* : \mathcal{D} \to Y$ such that $f^*(A) \in F(A)$ for every $A \in \mathcal{D}$, and $f^*(A) = f(F(A))$ whenever $F(A) \in \mathcal{E}$.

Clearly, the difference between AP^1 and AP$^+$ is that they allow for the construction of an extension of the given deductive domain in two different ways: "one by one" and "once and for ever". Since a deductive domain is a family of nonempty sets together with a choice function, it may seem unnecessary to make explicit the connection with AC, but we have preferred to do so nonetheless by giving AP also the specific choice version AP^C.

The last condition on f^* in AP^C ensures that the functions we introduce along the proof of Proposition 2.4 below are well defined. Such care is not required in AP$^+$ or AP^1, owing to the fact that one has $f' \restriction \mathcal{E} = f$, where f' is the choice function for the extended deductive domain both AP^1 and AP$^+$ state to exist. The assumption in question could be avoided by imposing the extra condition on Y that $\mathcal{P}(Y) \cap \mathcal{E} = \emptyset$. Although a similar condition seems to be presupposed in some of the formulations of AP which appear in the literature, we have preferred not to include it in our version of AP, for the sake of generality.

In the following, as throughout the paper, we work in ZF, possibly in a suitable fragment.

PROPOSITION 2.4. *The three versions AP*^{1}, *AP*$^+$, *and AP*^{C} *are equivalent.*

PROOF. We prove:

1. AP$^+$ $\Rightarrow$ AP^C.
2. AP^C $\Rightarrow$ AP^1.
3. AP^1 $\Rightarrow$ AP$^+$.

Let $(\mathcal{E}, f)$, Y, F, d and $\mathcal{D}$ be given as before.

1. By AP$^+$ we have a natural extension $(\mathcal{E}', f')$ with $\mathcal{E}' = \mathcal{E} \cup \{F(A) \mid A \in D\}$ of the deductive domain $(\mathcal{E}, f)$. Let $A \in \mathcal{D}$. Set $f^*(A) = f'(F(A))$. Then, $f^*(A) \in F(A)$. Further, $f^*(A) = f(F(A))$ if $F(A) \in \mathcal{E}$, since $f' \restriction \mathcal{E} = f$.

2. Let f^* be as in AP^C and assume that $A \in \mathcal{D}$. Set $\mathcal{E}' = \mathcal{E} \cup \{F(A)\}$. Define $f' : \mathcal{E}' \to \bigcup \mathcal{E}'$ by $f'(A) = f(A)$ if $A \in \mathcal{E}$, and $f'(F(A)) = f^*(A)$. Clearly, f' is well defined (since if $F(A) \in \mathcal{E}$, then $f(F(A)) = f^*(A)$). Also, $(\mathcal{E}, f) \leq (\mathcal{E}', f')$ as required.
3. Set $\mathcal{E}' = \mathcal{E} \cup \{F(A) \mid A \in \mathcal{D}\}$. Define $f' : \mathcal{E}' \to \bigcup \mathcal{E}'$ by:
   i. $f'(A) = f(A)$ if $A \in \mathcal{E}$;
   ii. $f'(F(A)) = f''(F(A))$ if $A \in \mathcal{D}$, where $(\mathcal{E} \cup \{F(A)\}, f'')$ is the natural extension of $(\mathcal{E}, f)$ which exists by AP^1.

Then, clearly $(\mathcal{E}, f) \leq (\mathcal{E}', f')$. It remains to show that f' is well defined. Let $F(A) \in \mathcal{E}$ for some $A \in \mathcal{D}$. By definition of f' we have $f'(F(A)) = f''(F(A))$, and $f'' \restriction \mathcal{E} = f$ yields $f''(F(A)) = f(F(A))$. ⊣

The result does not come as a surprise, except maybe for implication 3 of the proof. Indeed, it is not at all trivial that one can define a choice function for the whole $\mathcal{E} \cup \{F(A) | A \in \mathcal{D}\}$ out of the pieces which are given for each $\mathcal{E} \cup \{F(A)\}$, whenever $A \in \mathcal{D}$. This is made possible by the definition of deductive domains as pairs of a family of sets *and* a choice function. By doing this, if a given domain $\mathcal{E}$ is extended by two sets B_1 and B_2 which happen to be equal, then the choice functions associated with $\mathcal{E} \cup \{B_1\}$ and $\mathcal{E} \cup \{B_2\}$ are equal as well (as this is already made clear by Remark 1, part (iv)). This is equivalent to having a functional which, for a given extension $\mathcal{E}'$ of the deductive domain $\mathcal{E}$, selects one choice function out of the possibly many. In other words, the proof of $\mathrm{AP}^1 \Rightarrow \mathrm{AP}^+$ corresponds, in a general situation, to the proof of the following lemma:

LEMMA 2.5. *Let E, D be sets. If $f : E \to \bigcup E$ such that $f(C) \in C$ for all $C \in E$, and*

$$G : D \to \{g : E \cup \{B\} \to \bigcup(E \cup \{B\}) \mid B \in D, g \restriction E = f, g(B) \in B\},$$

then there exists $h : E \cup D \to \bigcup(E \cup D)$ such that $h \restriction E = f$ and $h(B) = G(B)(B)$ for $B \in D$. In particular, $h(C) \in C$ for every $C \in E \cup D$.

The use of the distance function d stands out among the notable aspects of AP. Our choice is to view d as yielding the distances between subsets of the given set Y, i.e., d as a function on the Cartesian product of $\mathcal{P}(Y)$ with itself. This choice was made in view of Levi's formulation of the condition $\psi(A, F, d)$, where the distance is taken between sets $F(A)$ with $A \in \mathcal{E}$, which in turn are subsets of Y. Levi has nonetheless defined d as acting on *points*, that is on the Cartesian product of Y itself (see Section 2.3). This possible ambiguity has leads us to some more possible interpretations of AP in "pointwise" form, as follows.

Let $(\mathcal{E}, f)$ be a deductive domain, Y a prime aggregate and $F : \mathcal{E} \rightsquigarrow Y$ an MV-function. Let further $\tilde{d} : Y \times Y \to \mathbb{Q}_0^+$ be such that $\tilde{d}(x, x') = 0$ if and only if $x = x'$ for every $x, x' \in Y$. We write $\tilde{\psi}(A, F, \tilde{d})$ whenever $A \in \mathcal{E}$ and

$$(\forall \delta > 0)(\exists x_1, \ldots, x_n \in A)(\forall A', A'' \in \mathcal{E})$$
$$(x_1, \ldots, x_n \in A' \cap A'' \to \tilde{d}(F(A'), F(A'')) < \delta),$$

where $\tilde{d}(F(A'), F(A'')) < \delta$ is a shorthand for $(\forall y \in F(A'))(\forall z \in F(A''))(\tilde{d}(y,z) < \delta)$. Let finally $\tilde{\mathcal{D}} = \{A \in \mathcal{E} | \tilde{\psi}(A, F, \tilde{d})\}$. As for AP itself, we introduce the modified variant of AP in three versions:

$\mathbf{AP}^1_P$ For every $A \in \mathcal{E}$, if $A \in \tilde{\mathcal{D}}$, then $\mathcal{E} \cup \{F(A)\}$ is a natural extension of $(\mathcal{E}, f)$.

$\mathbf{AP}^+_P$ $\mathcal{E} \cup \{F(A) \mid A \in \tilde{\mathcal{D}}\}$ is a natural extension of $(\mathcal{E}, f)$.

$\mathbf{AP}^C_P$ There exists a function $f^* : \tilde{\mathcal{D}} \to Y$ such that $f^*(A) \in F(A)$ for every $A \in \tilde{\mathcal{D}}$, and $f^*(A) = f(F(A))$ if $F(A) \in \mathcal{E}$.

Inspection of the proof of Proposition 2.4 shows that the choice of the distance function plays no role. Hence, it is clear that, by the very same argument, also the following holds:

Proposition 2.6. *The three pointwise versions AP^1_P, AP^+_P, and AP^C_P are all equivalent.*

One interesting fact to notice is that AP in this pointwise version seems weaker than in the original one. As a matter of fact, one can prove:

Proposition 2.7. *AP^1 implies AP^1_P.*

Proof. Let $(\mathcal{E}, f)$, F, Y and $\tilde{d} : Y \times Y \to \mathbb{Q}^+_0$ be as in the statement of AP^1_P (hence, $\tilde{d}$ is such that $\tilde{d}(y, y') = 0$ iff $y = y'$ for every $y, y' \in Y$). Let $\mathcal{P}^+(Y)$ be the collection of nonempty subsets of Y (that is, $X \in \mathcal{P}^+(Y)$ iff $X \subseteq Y$ and $X \neq \emptyset$).[9] Define $d : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \mathbb{Q}^+_0$ by

$$d(X, X') = \begin{cases} 0, & \text{if } X = X' \\ \sup\{\tilde{d}(x, x') \mid x \in X, x' \in X'\}, & \text{otherwise} \end{cases}$$

for every $X, X' \in \mathcal{P}^+(Y)$. Then, $d(X, X') = 0$ if and only if $X = X'$.

Observe that, for every $A \in \mathcal{E}$, if A satisfies condition $\tilde{\psi}(A, F, \tilde{d})$ from AP^1_P, then

$$(\forall \delta > 0)(\exists x_1, \ldots, x_n \in A)(\forall A', A'' \in \mathcal{E})(x_1, \ldots, x_n \in A' \cap A'' \\ \to d(F(A'), F(A'')) < \delta).$$

(This holds trivially, by definition of d, in case $F(A') = F(A'')$, and follows, in the other case and for an arbitrary $\delta \in \mathbb{R}^+$, by choosing those $x_1, \ldots, x_n \in A$ for which $\tilde{\psi}(A, F, \tilde{d})$ holds for, say, $\delta/2$.) By AP^1, $\mathcal{E} \cup \{F(A)\}$ is a natural extension of $\mathcal{E}$ as required by AP^1_P. ⊣

Although there are ways to approach the converse direction in the case of metric functions, that is to define a pointwise metric starting from one between sets of elements, no one of them seems suitable in order to fulfil Levi's more strict requirement on d to be such that $d(X, X') = 0$ iff $X = X'$. This makes it likely that no reverse implication can be proved *directly*. As we shall see, however, an equivalence between AP in its pointwise and non-pointwise versions is possible in the general case through AC. In order to show this, we first need to point out some general observations regarding AP.

[9]Remember that the distance function d we are seeking for, needs to be defined on sets $F(A)$ for $A \in \mathcal{E}$ which are nonempty subsets of Y.

Let in the following $\mathcal{P}_1(Z) = \{\{z\} : z \in Z\}$ denote the set of singleton subsets of Z, for any given set Z.

LEMMA 2.8.

(a) *If* $\mathcal{E} = \mathcal{P}_1(Z)$, *then* $\psi(A, F, d)$ *and* $\tilde{\psi}(A, F, \tilde{d})$ *hold automatically for every* $A \in \mathcal{E}$, *and independently of the choice of* F, d *and* $\tilde{d}$.
(b) *If* $\psi(A, F, d)$ *holds and* $A', A'' \in \mathcal{E}$ *are such that* $A \subseteq A' \cap A''$, *then* $F(A') = F(A'')$.
(c) *If* $\psi(A, F, d)$ *holds and* $B \in \mathcal{E}$ *is such that* $A \subseteq B$, *then* $F(A) = F(B)$.
(d) *Let* $\mathcal{E}$ *be a family of sets which is inhabited and totally ordered by* $\subseteq$. *Let* A_0 *be the least element of* $\mathcal{E}$. *If* $\psi(A_0, F, d)$, *then* $\psi(A, F, d)$ *for every* $A \in \mathcal{E}$.

PROOF.

(a) Let, for some given set Z, $\mathcal{E} = \mathcal{P}_1(Z)$ be the case. Let Y, F, d be as in the statement of AP^1. Now, for every $A \in \mathcal{E}$ on has $A = \{a\}$ for some $a \in Z$. Hence, given $A \in \mathcal{E}$ with $A = \{a\}$, for every $A', A'' \in \mathcal{E}$ one has $a \in A' \cap A''$ if and only if $A = A' = A''$. This entails $F(A') = F(A'')$, that is $d(F(A'), F(A'')) = 0 < \delta$ for every $\delta \in \mathbb{R}^+$. One reasons in a similar manner for $\tilde{\psi}(A, F, \tilde{d})$.
(b) It suffices to show that $d(F(A'), F(A'')) = 0$, that is, $d(F(A'), F(A'')) < \delta$ for every $\delta \in \mathbb{R}^+$. Let $\delta > 0$, pick $x_1, \ldots, x_n \in A$ as in $\psi(A, F, d)$. Since $A \subseteq A' \cap A''$, we have $x_1, \ldots, x_n \in A' \cap A''$ in particular, and then $d(F(A'), F(A'')) < \delta$ by $\psi(A, F, d)$ itself.
(c) Follows from (b), with $A' = A$ and $A'' = B$.
(d) An immediate consequence of (b). ⊣

It is important to notice that only part (a) of this lemma extends to $\tilde{\psi}(A, F, \tilde{d})$. This has to do with the different choice of the distance function. As a matter of fact, if $\tilde{d}$ is as in the statement of AP^1_P, it is in general possible that $\tilde{d}(X, X') > 0$ even if $X = X'$, for $X, X' \subseteq Y$. By part (a) of Lemma 2.8 one nonetheless proves the following:

PROPOSITION 2.9.

1. AP^C_P *implies AC*.
2. *AC implies* AP^C.

PROOF.

1. Let X, Y, F be as in the statement of AC. Set $\mathcal{E} = \mathcal{P}_1(X)$ and $f : \mathcal{E} \to \bigcup \mathcal{E}$ be defined by $f(\{a\}) = a$ for every $a \in X$. Set $F' : \mathcal{E} \rightsquigarrow Y$ by $F'(\{a\}) = F(a)$, for every $a \in X$. Let $\tilde{d} : Y \times Y \to \mathbb{Q}^+_0$ be *any* function satisfying $\tilde{d}(z, z') = 0$ if, and only if $z = z'$, for every $z, z' \in Y$ (one can, e.g., define $\tilde{d}$ by $\tilde{d}(z, z') = 0$ if $z = z'$, and $\tilde{d}(z, z') = 1$ otherwise, for every $z, z' \in Y$). By Lemma 2.8, part (a) and AP^C_P one has that there exists a function $f' : \mathcal{E} \to Y$ such that $f'(\{a\}) \in F'(\{a\}) = F(a)$ for every $a \in X$. Define $f : X \to Y$ by $f(a) = f'(\{a\})$ for every $a \in X$.
2. Let $(\mathcal{E}, f), F, d$, and $\mathcal{D}$ as in the statement of AP^C. By AC, there exists a function $g : \mathcal{E} \to Y$ such that $g(A) \in F(A)$ for every $A \in \mathcal{E}$.

Define $f^* : \mathcal{D} \to Y$ by

$$f^*(A) = \begin{cases} f(F(A)), & \text{if } F(A) \in \mathcal{E} \\ g(A), & \text{otherwise} \end{cases}$$

for every $A \in \mathcal{D}$. $\dashv$

By Propositions 2.4, 2.6 and 2.7 we have:

Corollary 2.10.

1. $AP^*_{(P)}$ *and AC are equivalent, for* $* \in \{1, +, C\}$.
2. AP^* *is equivalent to* AP^*_P, *for* $* \in \{1, +C\}$.

Since Levi was criticizing AC, this last result would allow one to say that the possibility of taking deductive domains made out of singletons trivializes AP, or even betrays the spirit of Levi's proposal entirely. Vice versa, one notices that the usage of AP is nontrivial (or, nontrivially equivalent to AC) whenever it is applied to deductive domains which contain at least one set consisting of two or more elements. In particular, this holds whenever a deductive domain contains at least one infinite set. As it will turn out more clearly from Section 2.3, it is likely that Levi was inclined to accept only those deductive domains whose elements are infinite sets. For the sake of the issue of choice, relativizations of AP to deductive domains in this form are particularly interesting, as allowing choice over families of infinite sets is most problematic. By elaborating on a remark by Levi himself [21, pp. 323–324], one obtains a very interesting reduction to the countable case for AP applied to deductive domains which contain infinite sets.

Let $\mathrm{AP}^C_{(P)}(\mathbb{N})$ stay for AP applied to a deductive domain which contains only *countable sets*. For the sake of the foregoing result, as well as for later on in the section, we would like to remark that a proof of the fact that a countable union of finite sets is countable is already possible in ZF alone, hence it *does not* require $\mathrm{AC}_{\mathbb{N}}$. On the contrary, the generalization of it to the countable union of *countable* sets is known to require $\mathrm{AC}_{\mathbb{N}}$ (see, e.g., [17]). Then we have:

Proposition 2.11. *The axiom of countable choice* $AC_{\mathbb{N}}$ *proves that* $AP^C(\mathbb{N})$ *implies* AP^C.

Proof. Let $(\mathcal{E}, f)$, F, Y, d and $\mathcal{D}$ as in the statement of AP^C. Fix a descending sequence of positive rational numbers $(r_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} r_i = 0$ (for example, $r_n = 2^{-n}$). Let $A \in \mathcal{D}$. Then, fix, for every $i \in \mathbb{N}$, $a_{i1}, \dots, a_{in_i} \in A$ such that, if $A', A'' \in \mathcal{E}$ and $a_{i1}, \dots, a_{in_i} \in A' \cap A''$, then $d(F(A'), F(A'')) < r_i$ (notice that this requires $\mathrm{AC}_{\mathbb{N}}$).

Let C_A be the union of the countable family of the finite sets $\{a_{i1}, \dots, a_{in_i}\}$, with $i \in \mathbb{N}$. Notice that C_A is countable. It follows that if $A' \in \mathcal{E}$ is such that $C_A \subseteq A'$, we have $d(F(A'), F(A)) < r_i$ for every $i \in \mathbb{N}$, that is, $d(F(A'), F(A)) = 0$. Hence, $F(A) = F(A')$.

Set $\mathcal{E}' = \{C_A | A \in \mathcal{D}\}$. Now, every finite set $\{a_{i1}, \dots, a_{in_i}\}$ has an element carrying the lowest index for every $i \in \mathbb{N}$. More than that, every set C_A has an element carrying the lowest index, being it the union of all of these finite sets. Call it a_ι.[10] Set $f'(C_A) = a_\iota$. Then, $(\mathcal{E}', f')$ is a deductive domain. Moreover, we are entitled to apply $\mathrm{AP}^C(\mathbb{N})$ since $\mathcal{E}'$ contains only countable sets. Then, there exists a function $f'' : \mathcal{E}' \to Y$ such that $f''(C_A) \in F(C_A)$, where $F(C_A) = F(A)$ by the preceding argument.

Define $f^* : \mathcal{E} \to Y$ by $f^*(A) = f''(C_A)$. ⊣

Notice that Proposition 2.11 does not give any clue concerning the relationship between $\mathrm{AC}_\mathbb{N}$ and AP. In particular, it does not allow to build an argument showing that the latter is implied by the former. Incidentally, the impression that such a result may hold seems legitimated by parts of the literature dealing with AP. Viola, for instance, seems to be suggesting this in [44, p. 289]. Moore [30, p. 80, 244] is even more explicit, and speaks of AP as being a principle that is a weak form of $\mathrm{AC}_\mathbb{N}$. He gives no proof, and presumably is referring to Levi's own remark in this respect. It should be made clear, however, that what we have just said does not suffice to conclude anything of that sort, for we have no information about the size of the set $\{C_A \mid A \in \mathcal{D}\}$ which may well exceed the power of $\mathrm{AC}_\mathbb{N}$ to produce a choice function.

Levi was quite sober in this respect, and declared that AP is “determined” by always countably many choices. He thought this could be important for the general appreciation of AP owing to an “unconscious tendency” of many a mathematician to “consent infinitely–many arbitrary choices, *in case* they are denumerably many”. As a matter of fact, the result is quite notable: for, it shows that the approximation principle restricted to a deductive domain $\mathcal{E}$ which contains a *countable* set yields a choice function for *any* domain containing arbitrary supersets of the elements of $\mathcal{E}$. This is something peculiar to AP, and is hardly conceivable for the usual choice principles.

Let us now leave these remarks behind, and summarize this section instead. We have here presented two different interpretations of Levi's AP. On the one hand, by noticing that Levi defined the distance function d as applying to sets $F(A)$ $(A \in \mathcal{D})$, we have viewed d as defined on subsets of the given set Y. On the other, we have retained d to be defined on points (that is, elements of Y) as Levi did, and showed how to modify the approximation condition instead. This gives rise to AP in its pointwise versions AP^*_P. In both cases, we obtained three forms of AP, and showed that they are equivalent (over ZF). However, both versions of the principle turned out to be fairly strong if applied to the general definition of deductive domain, for which they are equivalent to AC. This suggests to take into account Levi's remark that deductive domains should include infinite sets. In every case, Proposition 2.11 shows that AP in its countable form AP($\mathbb{N}$) is enough to ensure the full strength of its general form.

In the end, we do have a reasonable version $\mathrm{AP}^*_P(\mathbb{N})$ of Levi's principle to work with. In order to let the reader reconcile all the passages we have hinted at with Levi's actual statement of AP, we first reproduce and discuss the latter before testing AP in the mathematical setting.

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[10]Notice that ι has always the form 01, since every set $\{a_{i1}, ..., a_{in_i}\}$ is nonempty.

2.3. Some historical remarks. Since the versions of AP we presented before are the result of our own interpretation of Levi's wordings of AP, we owe a couple of comments to those who aim at being faithful to prime sources. One might even think that we should instead have proceeded the other way round, starting from Levi's way to put AP. We are convinced, however, that this would rather have confused the reader than being of any help, and hope that the reader will agree with us after reading Levi's formulation of AP as reproduced below. For this we stick to the source [21, pp. 312–313], which is the one we have already referred to.[11]

> **Approximation principle**. *Let a deductive domain* Ω *be given, in which the collection of real numbers (or the set of rational numbers at least) be a prime aggregate; let* A,B,C,. . . *be sets defined in* Ω *and contained as subsets in prime aggregates of* Ω *(such that we are entitled to consider arbitrary elements of them). Let* E *be a set which is a prime aggregate not contained in the deductive domain* Ω *instead, the elements of which could be regarded as containing infinite elements arbitrarily chosen from the aggregates* A,B,C,. . . *(possibly, only from some of them): let a function* f(x)[12] *be given, with respect to which the domain* D *of* x *be contained in* E*, while the corresponding domain* F *of the function be contained in a set* G*, which is either prime aggregate or not, and either belongs to* Ω *or not; finally let a numerical function* d(y,z) *be defined on pairs of elements of* G*, which be null always and only when* y=z*; let us suppose that* a *is an element of* D *such that, given an arbitrary number* δ*, among the elements of* A,B,C,. . . *constituting* a*, it is possible to fix a finite number* n *of them such that, for any two elements* a′,a″ *of* D *having in common with* a *the said* n *elements, it is always* d(f(a′),f(a″))< δ.
>
> *Then, we consider the statement "*f(a) *exists" as belonging to the* NATURAL EXTENSION OF THE DEDUCTIVE DOMAIN Ω.
>
> [12]To the word "function" we assign here the most general meaning (see my *Introduction to mathematical analysis*, vol. I, Section III, Parma, 1916).

Our presentation of Levi's principle looks *prima facie* quite different from Levi's own formulation. For, given a deductive domain Ω, Levi speaks of a collection of infinite sets E the elements of which are "chosen" among the elements of the sets in Ω, and further restricts the domain of the MV-function f to a subset D of this E. In other formulations of his principle [23, p. 72], the elements of D are said to "correspond" to infinite elements of $\bigcup \Omega$, this correspondence being identified with the "simultaneous conception of these infinite elements, sometimes with 'assigning a name' to their collection".

Since the principle is complicated already, we thought it inappropriate to follow Levi's formulation word by word. So, we have set $\Omega = E$ (and renamed it $\mathcal{E}$, at the same time) for the sake of simplicity, the extension of the principle to Levi's general case being recoverable from that. In our opinion, this is further justified by the fact that the principle appears as a way

[11]Small capital letters and emphasis are already in the source [21].

to pass from the originally given deductive domain Ω to a natural extension of it. In this respect, our definition of "natural extension" of a given deductive domain, made out of Levi's remark that "$f(a)$ exists" be part of this concept, corresponds to Levi's conviction that the act of picking elements from a given collection of mathematical objects is an essential aspect of the existence of that collection.

We would like to emphasize what we have already referred to as Levi's quite involved way of writing. This is made worse here by what seems to be a lack of sensibility on his behalf. For instance, he decided to use the word "chosen" in the preamble of what should be a choice principle, to express the fact that E is defined out of the given elements of the deductive domain (which can be stated in simple terms by $E \subseteq \bigcup \Omega$). This certainly explains Moore's interpretation of AP [30, p. 244], which we consequently regard as a little inadequate.

This said, the rest of the principle is easier to reconcile to what we have said above. When introducing the function f, one notices the footnote with which Levi refers the reader to his introduction to analysis. This source contains a distinction between "monodrome", or single-valued functions, and "polydrome", or many–valued ones (the latter being clearly the "most general" meaning of the word that Levi mentions).

Furthermore, notice that Levi's distance function d is actually defined on points, that is, elements of the codomain F of f, which in Levi's terminology is the "domain" over which the expression $f(x)$ varies (note that this and the next occurrence of F mean Levi's F, which is Y in our variants of AP). We thus have also considered the pointwise versions AP^*_P. On the other hand, the decision to initially provide the versions AP^* of AP was motivated by the fact that, in the course of explaining what we have called the approximation condition, Levi speaks of d as applying to values of the function f, which, due to the preceding remark, are *subsets* of (Levi's) F. Since Levi gives no clue as to how one should interpret this, we thought it beneficial to explore two natural ways of explaining what he might have meant, by studying the conditions $\psi(A, F, d)$ and $\tilde{\psi}(A, F, \tilde{d})$ from Section 2.2.

Finally, it should now be clear that Levi thought of AP as applying to sets containing infinite sets, since the counterpart of a deductive domain $\mathcal{E}$ as this concept occurs in our formulation of AP is the set E, which Levi defines as a family of infinite sets containing elements "chosen" from sets belonging to the deductive domain Ω. This was the reason for considering the reduction of AP to the countable case $\mathrm{AP}^*_P(\mathbb{N})$.

Needless to say, other interpretations of the principle are possible. In particular, one could think of ways to relax the condition $\tilde{\psi}(A, F, \tilde{d})$ even further. This is why we now turn to study an application of AP that was proposed by Levi himself. We consider the fact that we can make it intelligible to be an essential support for the claim that our versions of AP embody the mathematical content of Levi's.

§3. AP at work: a case study. Levi's own papers, especially [21, 22], contain a discussion of AP together with samples of applications. Other mathematicians coming from the Italian school got involved in this project of testing AP around the early and mid 1930's. Levi's student T. Viola in particular [26,44,45,46], and G. Scorza–Dragoni [41,42] (the latter contribution being the last ever, chronologically speaking), took part in this project. Further references to Levi's principle can be found, as reported by Moore [30, p. 246], in a few other works by U. Cassina [8] and A. Faedo [9], without, however, any further study of AP. "Outside Italy", as Moore [30, p. 246] puts it, "the Principle of Approximation generated no interest at all". This means in particular that Levi's attempt [22] to attract Hilbert's attention to his own proposal, as well as Levi's efforts to convince Hilbert that the spirit of AP could be seen to agree with the bulk of Hilbert's program for the foundations of mathematics [12, 13], presumably turned out to be a failure. This cannot be said for sure, however, since there is no trace of Hilbert replying to Levi's letter.[12]

For the sake of illustrating how the principle works, as well as of contributing a little bit to the range of applications and critical evaluations of it, we have selected an example from the above literature. The example we have chosen clearly reflects the authors' taste and expertise, but in our opinion is also among the most interesting invocations of AP.

The example in question, which Levi has discussed in [21, pp. 318–320], concerns a property of metric spaces E. Borel has dealt with before [5, p. 12–13] in what seems to be the attempt of carrying over to open sets something like the property for closed sets reflected by the theorem named after H.E. Heine and Borel himself. Having been unable to find a name for this property in the literature, we have decided to baptize it as in part (ii) of Definition 3.1 below,[13]

Throughout we are using the standard notation for open balls

$$B_r(a) = \{x \in X \mid d_X(a, x) < r\}$$

with center $a \in A$, for A subset of a metric space (X, d_X), and radius $r \in \mathbb{Q}^+$.

DEFINITION 3.1.

(i) In a metric space (X, d_X), a subset $A \subseteq X$ is *open* if for every $a \in A$ there exists $r_a > 0$ such that $B_{r_a}(a) \subseteq A$.

(ii) An open subset A of X is *countably coverable*, if there exists a countable subset C of A such that

$$A = \bigcup_{a \in C} B_{r_a}(a).$$

[12] V.M. Abrusci [14, p. 32] states that Hilbert had, among others, Levi's viewpoint in mind, when arguing against some approaches to the foundations. However, no further mention of Levi can be found in C. Reid's biography of Hilbert [35], to which Abrusci refers for further information.

[13] The property is reminiscent of the Lindelöf Covering Lemma. Among the scholars working on AP, Viola [7, p. 46] explicitly related Levi's remark to Lindelöf's *Überdeckungssatz* part (i) of which is a variant of standard terminology. However, the reader should notice that the property is here made specific to covers of a certain form (that is, consisting of open balls).

It should be clear that the property of being countably coverable does not trivialize in the case of a bounded set, since the property requires a family of balls that covers the set *precisely*: that is, does not include any space outside the given set. Note that in computable analysis [47] an open subset of a (separable) metric space is just *defined* as a countable union of open balls.

The reader may also notice that in part (i) of this definition, we have, by assigning a to r_a following Borel [5], tacitly invoked AC. Part (ii) of Definition 3.1 is instead the natural consequence of such a definition of openness. By forming pairs of centers and radii, or, equivalently, by considering balls as a whole, it is of course possible to hide any reference to AC. In fact, one can equivalently define "A is countably coverable" in the following manner: there exists a countable set $\mathcal{F}$ consisting of open balls such that $A = \bigcup \mathcal{F}$.

Incidentally, the latter is Borel's very definition in [5], apart from the fact that he uses symbols only for the open set A and its points a. Therefore, his definition of "A is countably coverable" is not in complete accordance with his definition of openness, which is part (i) of Definition 3.1 above. Let us stress once more that the property Definition 3.1 refers to does not coincide with the generic property of having a covering for a subset in metric space, since here we are seeking for sets with a certain size, and made out of sets of a given form.

Borel's original theorem is about the special case $X = \mathbb{R}^2$, though it can be generalized to the case $X = \mathbb{R}^n$. We now modify the argument by Borel to encapsulate an observation by Levi, which makes it easier to verify Theorem 3.4 below when AP is at use. For convenience, we divide the proof of Theorem 3.3 into its components, to make it clear which assumptions the argument rests upon. One first needs the following lemma:

LEMMA 3.2 (E. Borel 1917, B. Levi 1918). *Let A be an open and bounded subset of $\mathbb{R}^2$. Fix $r^* > 0$, and let $A^* = \{a \in A \mid B_{r^*}(a) \subseteq A\}$.*

1. *There exists a finite sequence $a_1, \ldots, a_n \in A^*$ such that, for $1 \leq i \leq n$, there is $r_i \geq r^*$ such that*

$$A^* \subseteq \bigcup_{1 \leq i \leq n} B_{r_i}(a_i) \subseteq A, \tag{1}$$

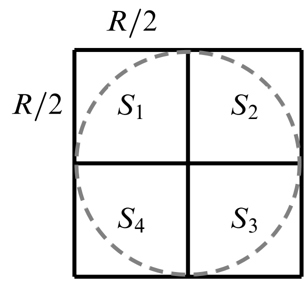
2. *In part 1 one can achieve that, for $1 \leq i \leq n$ and $1 \leq j \leq n$, if $i \neq j$, then*

$$a_i \notin B_{r_j}(a_j) \wedge a_j \notin B_{r_i}(a_i)\,.$$

Part 2 of Lemma 3.2 means that any two balls of the finite cover of A^* are "distant enough", or, more literally, that their centers are such that either lies outside the other ball.

PROOF.

1. Since A is bounded, A^* is bounded as well. We may assume A^* to be bounded by a square $S = [R, -R]^2$. The proof is by contradiction. Hence, the property (1) must fail to hold, in particular, for the intersection of A^* with at least one of the subsquares $S_1, \ldots, S_4$ of S, whose edges have length $R/2$ and which are as in the picture below:



Now, by continuing the process of halving the edges, one can build a sequence of squares $(S_i)_{i\in\mathbb{N}}$ such that $S_i \cap A^* \neq \emptyset$ and (1) fails (with $S_i \cap A^*$ in place of A^*), for every $i \in \mathbb{N}$. In particular, among the elements of that sequence, one can find a square S_j, for some $j \in \mathbb{N}$, with diagonal $d < r^*$. Let $p \in S_j \cap A^*$. Then, one has $S_j \subseteq B_{r^*}(p) \subseteq A$ in contradiction to the assumption.

2. Let $a_i, a_j \in A^*$ be among the elements of the finite sequence of elements of A obtained by 1, with $i \neq j$. If, for instance, $a_i \in B_{r_j}(a_j)$ is the case (the argument for the other case being obviously symmetric), we remove $B_{r_i}(a_i)$ from the cover and replace $B_{r_j}(a_j)$ by $B_{r_i+r_j}(a_j)$ which covers both $B_{r_i}(a_i)$ and $B_{r_j}(a_j)$. ⊣

This proof, and the one of the next result, is carried out along the lines of Borel [5, pp. 12–13]. It has been rewritten here including Levi's remark [21, p. 318] as part 2.

THEOREM 3.3 (Borel 1917). *$AC_{\mathbb{N}}$ proves that every open and bounded subset of $\mathbb{R}^2$ is countably coverable.*

PROOF. Let $A \subseteq \mathbb{R}^2$ be an open, bounded set. Take a descending sequence of positive rationals $(r_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} r_n = 0$. For every $n \in \mathbb{N}$ set

$$A_n = \{a \in A \mid B_{r_n}(a) \subseteq A\},$$

and note that $A = \bigcup_{n\in\mathbb{N}} A_n$. For every $n \in \mathbb{N}$ choose a finite cover C_n of A_n as in Lemma 3.2, part 1 such that C_n consists of open balls with rational radii $\geq r_n$ and centers in A_n, and $\bigcup C_n \subseteq A_n$. Note that this move requires an instance of $\mathrm{AC}_{\mathbb{N}}$. Now we have

$$A = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \bigcup C_n \subseteq A$$

and thus $A = \bigcup \mathcal{F}$ where $\mathcal{F} = \bigcup_{n\in\mathbb{N}} C_n$. ⊣

Levi [21, p. 320] remarks that the proof above still depends upon "infinitely many arbitrary choices", and looks for a "more deterministic" version of it which could make precise the law according to which one assigns the points of any open subset A of the given space to the corresponding balls. Notably, he leaves unnoticed that Borel's proof requires $\mathrm{AC}_{\mathbb{N}}$. He thus acts differently from how one may expect, metamathematically speaking. In particular, he does not seek a proof showing that $\mathrm{AC}_{\mathbb{N}}$ can be dispensed with, in favor of

AP alone. This is probably related to the "unconscious tendency" of mathematicians we referred to at the end of Section 2.2, to believe that one could make it legitimate to perform *countably* many choices. Instead, Levi shows how, by means of AP and $AC_{\mathbb{N}}$, one can actually address what bothered him about Borel's argument: that is, how to find a *function* assigning points to balls in such a way that Borel's proof of Theorem 3.3 would go through.

This is why the theorem below, which we ascribe to Levi, is just the same as Theorem 3.3 except for the additional hypothesis concerning AP. The latter makes Borel's original proof "more deterministic" in the sense we have just explained, as we shall notice again at the end of the argument.

THEOREM 3.4 (Levi 1918). *$AP^C_P(\mathbb{N})$ and $AC_{\mathbb{N}}$ prove that every open and bounded subset of $\mathbb{R}^2$ is countably coverable.*

PROOF. Assume that A is an open and bounded subset of $\mathbb{R}^2$. Let $(r_n)_{n\in\mathbb{N}}$ be a descending sequence of positive rational numbers with $\lim_{n\to\infty} r_n = 0$. For every $n \in \mathbb{N}$ set again

$$A_n = \{a \in A \mid B_{r_n}(a) \subseteq A\}\,,$$

for which $A = \bigcup_{n\in\mathbb{N}} A_n$. By Lemma 3.2 and $AC_{\mathbb{N}}$ we can choose, for every $n \in \mathbb{N}$, a finite set $C_n \subseteq A$ of centers and a finite set $R_n \subseteq \mathbb{Q}^+$ of radii such that, for every $c \in C_n$, there exists $r(c) \in R_n$ with $r(c) \geq r_n$ and

$$A_n \subseteq \bigcup_{c\in C_n} B_{r(c)}(c) \subseteq A\,.$$

For every $Z \subseteq A$ we consider the following properties of any given $Y \subseteq \mathbb{Q}^+$:

(A) For every $r \in Y$ there exists $c \in Z$ such that $B_r(c) \subseteq A$.
(B) If $C_n \subseteq Z$ for some $n \in \mathbb{N}$, then $R_n \subseteq Y$.
(C) For every $r, r' \in Y$ and $c, c' \in Z$, if $c \neq c' \vee r \neq r'$, then

$$B_r(c) \cup B_{r'}(c') \subseteq A \to c \notin B_{r'}(c') \wedge c' \notin B_r(c)\,.$$

Now set, as Levi [21, p. 320] suggests, $\mathcal{E} = \mathcal{P}_{\aleph_0}(A)$, where $\mathcal{P}_{\aleph_0}(A)$ is the set of countable subsets of A. By Lemma 3.2 we know that we can define an MV-function $F : \mathcal{E} \rightsquigarrow \mathcal{P}(\mathbb{Q}^+)$ such that, for every $Z \in \mathcal{E}$, $Y \in F(Z)$ if and only if $Y \subseteq \mathbb{Q}^+$, and (A)+(B)+(C) hold for Y. In particular, notice that condition (C) above comes from Levi's own remark on Borel's original proof.

Let $\tilde{d} : \mathcal{P}(\mathbb{Q}^+) \times \mathcal{P}(\mathbb{Q}^+) \to \mathbb{Q}^+_0$ be defined by setting, for every $Y, Y' \in \mathcal{P}(\mathbb{Q}^+)$,

$$\tilde{d}(Y, Y') = \sup(\{0\} \cup (Y \triangle Y'))$$

where again $Y \triangle Y' = (Y \cup Y') \setminus (Y \cap Y')$. Clearly, $\tilde{d}(Y, Y') = 0$ if and only if $Y = Y'$.

Set $Z = \bigcup\{C_n | n \in \mathbb{N}\}$; notice that $Z \in \mathcal{E}$. In fact, the following works for every $Z \in \mathcal{E}$ such that $\bigcup C_n \subseteq Z$. Let $\delta > 0$. Pick $k \in \mathbb{N}$ such that

$r_k < \delta$. Let $Z', Z'' \in \mathcal{E}$ such that $C_k \subseteq Z' \cap Z''$, and let $Y' \in F(Z')$ and $Y'' \in F(Z'')$. By condition (B) we have $R_k \subseteq Y' \cap Y''$.

Moreover, one can show that

$$\forall r \in Y' \triangle Y''(r < r_k). \tag{2}$$

In view of our choice of k, this would suffice to conclude that $\tilde{d}(Y', Y'') < \delta$ for all $Y' \in F(Z')$ and $Y'' \in F(Z'')$. In turn, this makes condition $\tilde{\psi}(Z, F, \tilde{d})$ satisfied.

To prove (2) we show that if $r \in Y'$ and $r \geq r_k$, then $r \in Y''$ (the other case can be treated symmetrically). Let $r \in Y'$. Then, by condition (A), there exists $c \in Z'$ such that $B_r(c) \subseteq A$. In particular, if $r \geq r_k$, then $c \in A_k$. Hence,

$$c \in A_k \subseteq \bigcup_{e \in C_k} B_{r(e)}(e) \subseteq A,$$

where, recalling the notation we have introduced at the beginning of this proof, $r(e) \in R_k$, for every $e \in C_k$. This means that, for some $e \in C_k$, we have $c \in B_{r(e)}(e)$. Since, as we have noticed, $R_k \subseteq Y' \cap Y''$ by (B), condition (C) applied to $r' = r(e)$ and $c' = e$ yields in particular that $r = r(e) \in Y''$. Hence (2).

As we have noticed, this entails $\tilde{\psi}(Z, F, \tilde{d})$ for all $Z \in \mathcal{E}$ such that $\bigcup\{C_n | n \in \mathbb{N}\} \subseteq Z$. Let $\mathcal{D}$ be the collection of those Z's (that is, $Z \in \mathcal{D}$ iff $Z \in \mathcal{E}$ and $\bigcup\{C_n | n \in \mathbb{N}\} \subseteq Z$). Then, by $\mathrm{AP}^C_P(\mathbb{N})$, there exists a function f^* such that $f^*(Z) \in F(Z)$ for every $Z \in \mathcal{D}$. Now, for any such f^*, one has $\bigcup\{R_n | n \in \mathbb{N}\} \subseteq f^*(Z)$ in view of (B). In fact, the argument we have used in order to prove (2) shows that $f^*(Z) = \bigcup\{R_n | n \in \mathbb{N}\}$.

This allows to conclude, following Borel's lines of thought:

$$A = \bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} \bigcup_{c \in C_n} B_{r(c)}(c)\,.$$

Note that $r(c) \in f^*(Z)$ for every $c \in C_n \subseteq Z$. ⊣

It should be clear that AP helps to make the proof of Borel "more deterministic" inasmuch as it now contains a law, the function f^*, which assigns the elements of the given open set to the desired balls. In turn, this reduces the impact of the implicit reference to AC in the definition of openness Levi and Borel started from.

For the sake of the preceding argument, it should be observed that $\tilde{d}$ is the pointwise distance function on the codomain of the MV-function F (which, incidentally, is the power set of the set of the positive rationals). This should be compared with the definition of a non-pointwise metric d, as we will do it for the sake of proving Proposition 3.10 below.

In contemporary mathematics one would presumably proceed in a different manner. One would rather accept the very definition of openness Levi seems to be troubled with, as well as the definition of "A is countably coverable", for any open subset A of a given metric space (X, d_X), by diminishing the appeal to AC as we indicated after Definition 3.1. Then, one would observe that the property that Theorem 3.3 deals with can be generalized as follows. Recall first the following notions:

DEFINITION 3.5.

(1) A subset S of a metric space (X, d_X) is *dense* if for every $a \in X$, and for every $\varepsilon > 0$, there exists $s \in S$ such that $d_X(a, s) < \varepsilon$.
(2) A metric space (X, d_X) is *separable* if there exists a countable $S \subseteq X$ which is dense in (X, d_X).

For instance, $\mathbb{R}^n$ is separable because $\mathbb{Q}^n$ is dense in it. The general result we have in mind reads as follows:

THEOREM 3.6. *In a separable metric space (X, d_X), every open subset A of X is countably coverable.*

Although this result is somewhat trivial, it is hard if not impossible to find it in textbooks. In order to detect where one needs AC, we next prove it in all details.[14]

A classical proof of Theorem 3.6 would indeed make use of $\mathrm{AC}_{\mathbb{N}}$, albeit only for the sake of proving Lemma 3.9 below, which states that openness alone suffices to obtain the result, whereas $\mathrm{AC}_{\mathbb{N}}$ remains unused for Proposition 3.8 below, which generalizes Borel's Theorem 3.3 to separable metric spaces. Moreover, the argument for the latter is based on assigning elements of an open set to radii—according to a rule, as desired by Levi, but without any appeal to AP.

We first prove the following little lemma from metric topology:

LEMMA 3.7. *In a metric space (X, d_X), and for every open subset A of X:*

(i) *if $S \subseteq X$ is dense in (X, d_X), then $S_A = S \cap A$ is dense in A;*
(ii) *if A is bounded, $a \in A$, and*

$$r(a) = \sup\{r > 0 \mid B_r(a) \subseteq A\},$$

then $B_{r(a)}(a) \subseteq A$.

PROOF.

(i) Let (X, d_X) be a metric space with $S \subseteq X$ dense in X. Now, by the openness of A, for all $a \in A$ there exists $r > 0$ such that $B_r(a) \subseteq A$. By S being dense in (X, d_X), for every $\varepsilon \leq r$ there exists $s \in S$ such that $d_X(a, s) < \varepsilon$. In particular, $s \in A$.
(ii) Let A and $r(a)$ be as in the hypothesis. The definition of $r(a)$ is legitimate since, by A being bounded, the set $R_a = \{r > 0 \mid B_r(a) \subseteq A\}$ has an upper bound, and by A being open, R_a is nonempty. Now let $b \in B_{r(a)}(a)$. This means that $\varepsilon = d_X(a, b) < r(a)$. By definition of the $r(\cdot)$ function, there exists $r < r(a)$ such that $\varepsilon < r$ and $B_r(a) \subseteq A$. This implies $b \in B_r(a) \subseteq A$ as desired. $\dashv$

The preceding lemma yields the following main step towards Theorem 3.6:

PROPOSITION 3.8. *Let (X, d_X) be a separable metric space. Then, if $A \subseteq X$ is open* and bounded, *A is countably coverable.*

[14] Note that Theorem 3.6 is equivalent, over a fragment of CZF, to the Kripke Schema in the form "Every subset of $\mathbb{N}$ is countable". See [29] and [40] for details.

PROOF. If A is as in the hypothesis, S is a countable and dense subset of X, and S_A and $r(a)$ for $a \in A$ are as in Lemma 3.7, then it suffices to observe that $A \subseteq \bigcup_{s \in S_A} B_{r(s)}(s)$, for then actually $A = \bigcup_{s \in S_A} B_{r(s)}(s)$ in view of Lemma 3.7, part (ii).

To see this, take $a \in A$, and $r > 0$ such that $B_{2r}(a) \subseteq A$. For example, one can set $2r = r(a)$. Then, take $s \in S_A$ such that $s \in B_r(a)$, which entails $B_r(s) \subseteq B_{2r}(a) \subseteq A$. By definition, it follows that $r \leq r(s)$ and $a \in B_r(s) \subseteq B_{r(s)}(s)$. ⊣

Note that $r(a)$ is *defined* rather than chosen; whence no use at all of AC is required.[15]

Then, as it was said, one further needs the following result in order to obtain Theorem 3.6 above:

LEMMA 3.9. *$AC_{\mathbb{N}}$ proves that in a metric space (X, d_x), if for every $A \subseteq X$ "A open and bounded" implies that A is countably coverable, then for every $A \subseteq X$ "A open" implies that A is countably coverable.*

PROOF. Let $A \subseteq X$ be open. Take $a \in X$ to be arbitrary but fixed. Define, for every $n \in \mathbb{N}$,

$$A_n = A \cap B_n(a).$$

Then, each of these sets A_n is an open and bounded subset of X. By hypothesis, for every $n \in \mathbb{N}$ there exists a countable family C_n of open balls whose union equals A_n. Note that this move requires $AC_{\mathbb{N}}$. Set $\mathcal{F} := \bigcup_{n \in \mathbb{N}} C_n$. By $AC_{\mathbb{N}}$ this $\mathcal{F}$ is countable and we have:

$$A = \bigcup_n A_n = \bigcup_n \bigcup C_n = \bigcup \mathcal{F},$$

which ends the proof. ⊣

With Proposition 3.8 and Lemma 3.9 we have proved Theorem 3.6 too. We finally observe that one also can—by precisely making use of the trick we have used to prove Proposition 2.9—prove Proposition 3.8 by means of AP. So, the following proposition is the same as Proposition 3.8, except for the additional hypothesis of AP.

PROPOSITION 3.10. *AP^C proves that every open and bounded subset of a separable metric space (X, d_X) is countably coverable.*

PROOF. Let $A \subseteq X$ be open and bounded, $S \subseteq X$ countable and dense, and S_A as in Lemma 3.7, part (i). Since A is open and S_A is dense in A, we can define an MV-function $F : S_A \rightsquigarrow \mathbb{Q}^+$ such that, for every $s \in S_A$, $t \in F(s)$ if and only if $B_t(s) \subseteq A$ and $P(s, t)$ holds, where the latter is defined by

$$P(s, t) \equiv \forall a \in A (s \in B_{\frac{r(a)}{2}}(a) \to \frac{r(a)}{2} \leq t) .$$

Here $r(a)$ is as in Lemma 3.7, part (ii). Now, F is well defined by choosing, e.g., $t = r(s)$: in fact, $B_{r(s)}(s) \subseteq A$ by Lemma 3.7, part (i) since $s \in S_A \subseteq A$;

[15]To have that S_A is countable as a subset of the countable set S, one needs Kripke's schema – see footnote 14.

and $P(s, r(s))$ holds because if $a \in A$, then $B_{r(a)}(a) \subseteq A$ again by Lemma 3.7, part (i). Hence if $s \in B_{r(a)/2}(a)$, then $B_{r(a)/2}(s) \subseteq B_{r(a)}(a) \subseteq A$: that is, $r(a)/2 \leq r(s)$ as required.

Note that $F(s)$ is a bounded subset of $\mathbb{Q}^+$ for every $s \in S_A$, because A is bounded.

Now we can use the "singleton trick" we have used in the proof of Proposition 2.9. So, set $\mathcal{E} = \mathcal{P}_1(S_A)$, which, we recall, is the set of singletons of elements of S_A. Let $F_1 : \mathcal{E} \rightsquigarrow \mathbb{Q}^+$ be defined by $F_1(\{s\}) = F(s)$ for every $s \in S_A$, and let $d : \mathcal{P}(\mathbb{Q}^+) \times \mathcal{P}(\mathbb{Q}^+) \to \mathbb{Q}_0^+$ be given by, for every $R, R' \subseteq \mathbb{Q}^+$,

$$d(R, R') = \sup(\{0\} \cup (R \triangle R')),$$

where again $R \triangle R' = (R \cup R') \setminus (R \cap R')$. Now, as noticed before, $d(R, R') = 0$ if and only if $R = R'$. Furthermore, by Lemma 2.8, part (a) the precondition $\psi(Z, F_1, d)$ of AP^C holds for every $Z \in \mathcal{E}$. Hence by AP^C there exists a function $f_1 : \mathcal{E} \to \mathbb{Q}^+$ such that $f_1(\{s\}) \in F_1(\{s\}) = F(s)$, for every $s \in S_A$. Now define $f^* : S_A \to \mathbb{Q}^+$ by setting $f^*(s) = f_1(\{s\})$ for every $s \in S_A$, and let $C = \{B_{f^*(s)}(s) \mid s \in S_A\}$. Since S_A is countable, C is countable as well. We claim that $A = \bigcup C$.

In fact, $\bigcup C \subseteq A$, for $f^*(s) \in F(s)$ for every $s \in S_A$. As for the converse, $A \subseteq \bigcup C$, let $a \in A$. Let $2r = r(a)$. Since S_A is dense in A, there exists $s \in S_A$ such that $s \in B_r(a)$, i.e., $a \in B_r(s)$. Also, $r \leq f^*(s)$ for $f^*(s) \in F(s)$ and $P(s, f^*(s))$ holds. In all, $a \in B_r(s) \subseteq B_{f^*(s)}(s)$. ⊣

Now, besides involving AP in its non-pointwise form, the foundational interest of this result is limited. For, we have here made use of a choice principle (AP) while we have seen before that no principle of this sort is needed (Proposition 3.8). By the way, the form AP^C of AP we have assumed in Proposition 3.10 implies $\mathrm{AC}_{\mathbb{N}}$ (in fact it applies AC, see Proposition 2.9, part 1), which in turn is needed to obtain the more refined result Theorem 3.6. All this naturally prompts the remarks that we make next, which we hold for important for any further study of AP.

§4. Final comments, and further work. It turns out quite clearly that the issue we have been concerned with has a philosophical as well as a more technical facet. Let us focus first on the philosophical aspect. According to Levi's view of mathematics, the act of picking elements from given collections of objects is to be regarded as a primitive operation. However, Levi's critique of Zermelo's axiom requires that this very operation must be restricted by certain conditions.[16] Now AP is intended to balance between these two requests of Levi's. Clearly, it would be important to classify the strength of AP in order to be in a position to evaluate such an equilibrium properly. The problem of finding connections of AP with the usual principles of choice is in fact most pressing.

[16] It should be observed that Levi's criticism of Zermelo's principle is motivated somewhat indirectly, as it depends upon the well-known equivalence between AC and the well-ordering principle. It is this latter result which Levi thought to be totally unjustified, as it turns out to be based on an *ad hoc* enlargement of an originally chosen deductive domain (see [21, pp. 309–312], [22, pp. 168–169], and [23, pp. 69–70]).

However, this is not the only issue. Something else turns out from reflecting on the specific features of Levi's principle. First of all, it should be noticed that AP requires that the choice of a deductive domain is made *first*. As Proposition 2.9 shows, this in turn has an impact on the strength of the principle itself. Hence, it must be made clear how much commitment one is inclined to make in this sense. Levi has only made fairly general remarks on the latter point. Apparently, he has considered the assumption that the set $\mathbb{R}$ of real numbers be part of any chosen deductive domain to be a minimum requirement for relevant parts of mathematics to make sense.

But he did not want to go too far beyond vague indications of this sort either. In a review of one of Levi's paper in Spanish [24],[17] Quine [32] criticizes Levi for having left obscure, among others, the notion of real number, and in particular the one of deductive domain; and clearly this is how one would react to a first reading of Levi's description of these notions as "primitive ideas" and "simple intuitions". However, yet another interpretation is possible, to which Quine perhaps subconsciously alleges when lucidly stating at the end of his review [32]:

> Constructivity, in general, is relative: a proof is constructive with respect to a given domain when it can be carried out wholly therein.

In fact, Levi's "primitive ideas" and "simple intuitions" can — and from a contemporary perspective presumably should — be seen as "primitive" and "simple" relative to the given context: as "black boxes" one takes for granted and one does not want to specify any further, at least not for the moment.

We are well aware that our view of Levi's wording is somehow in conflict with many a logician's preference for *absolute* concepts, entirely based on safe grounds, but our study of AP at work has brought us to believe that this position would not do full justice to Levi's work right at the forefront of early modern mathematics. By the way, pursuing the suggested interpretation helps to make sense of some remarks with which Levi relates AP to the debate on the foundations of mathematics. He did not want to commit himself to one or another of the parties involved therein, insofar as they may influence the choice of deductive domains. Quite on the contrary, he wished to present AP as an useful tool in order to clarify what disagreements from this debate amounted to in the end.

All this seems to be the meaning of what he wrote in [22, pp. 166–167] to Hilbert (emphasis is ours):

> For the school of mathematicians that we may call finitist, the only admissible deductive domain is the one which contains just one set: the set of integers: certainly it would not be possible to speak of "any real number whatsoever", as this would amount to conceive an arbitrarily given collection of fractions approximating a real number (a necessarily infinite collection). In this deductive domain one will

[17] For the time being we have not got hold of this paper, which Levi has written during his exile in Argentina. The title of it suggests that it may consist of a Spanish translation of [23].

> not be able to speak of anything but particular real numbers, each of which is defined by assigning to it an approximation procedure by rationals. And, it is neither this nor any other, similar conclusion that keeps me off from mathematicians of this sort, but *their exclusive consideration of the deductive domain to which they would like to restrict mathematical analysis*.

This somewhat pragmatic approach to deductive domains was maybe the one that Levi favored, and more traces can be found elsewhere in his writings. For instance, Levi mentions in [23, pp. 73–74] two problems which he thought to be at the basis of an axiom-oriented research on deductive domains. The two problems in question are:

1. Given a set of mathematical statements, to determine the most restricted deductive domain which makes them provable.
2. Given a deductive domain, to determine the set of statements which are made provable by it.

Apart from resemblances to the Friedman–Simpson programme of reverse mathematics [43], this may be seen as a reformulation of the soundness and completeness problem in formal logic, with deductive domains playing the role of model-like structures. Our way of understanding deductive domains, however, equally justifies an approach which looks at them as inner-mathematical objects, as has been done with related constructions in metamathematics, e.g., with toposes as universes in the context of constructive mathematics. This suggests to try to develop the present contribution in the direction of formal axiomatics.

Moreover this quite naturally prompts us to consider an aspect which is more relevant for the technical side of Levi's proposal. In the application of AP we have considered here, it is hard to avoid a certain feeling of *artificiality*, or *ad-hocness*, in the strategy that is used. Part of this feeling goes back to what we have just said: one has to choose a certain deductive domain first. Of course, this must be done *conveniently*, namely it must be such that the desired result follows. Now, should this choice not be considered as part of the foundational purpose, that is, as part of the attempt of making more acceptable a result for which a disputable choice principle is used? Note that, by definition, the choice of a deductive domain encapsulates an assumption concerning what collections of mathematical objects one is free to pick elements from. Hence it can well be the case that this choice of a deductive domain turns out to be as disputable as the choice principle one is struggling with. The "singleton trick" we have used in the proof of Proposition 2.9 shows that one should also carefully consider the specific features of Levi's principle in this respect.

There is another possible criticism going in the same direction. As a matter of fact, the choice of the deductive domain is not the only one upon which the final result may depend: in addition one has to set up the distance function d. Here too it is required to find a reasonable balance between defining d *conveniently*, in such a way that the desired result obtains, and by doing it *naturally* with respect to the mathematical setting one is working within.

Levi was aware of this issue (see [21, p. 320], for instance), and admitted that it would point to a certain "arbitrariness" in the application of AP. He nonetheless defended his principle by stressing that the "natural rule" is that the function d "should be chosen in such a way to be *convenient* for the problem to be treated" (emphasis is our).

Now, the application of AP to problems in the theory of metric spaces, as those we have considered in Section 3, allows to put this criticism in a stronger form, for there a metric is already given. As we said, the feature of a distance function exceeds the properties of metrics as these are usually defined in some respects. However, one cannot resist the temptation that, at least in these cases, the choice of d should be somehow connected to the mathematical *object* under scrutiny. Again, Levi could reply that the function d required for the purpose of applying AP must be chosen according to the choice of the deductive domain, which in turn must rather suit for the mathematical *problem* one is dealing with.

If not as a criticism of Levi's proposal, this at least poses a problem to ponder for any future investigations about AP: *Is there a significant application of AP, possibly in the realm of metric spaces, which is* natural *in the sense that it holds true with respect to nonartificial, or* ad hoc *choices of the deductive domain and the distance function* d?

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