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Shards for the affine symmetric group

Grant T. Barkley*[1]

[1] *Department of Mathematics, Harvard University, Cambridge, MA 02138*

Abstract. The poset of "biclosed sets" in a root system has received attention as a natural extension of the weak Bruhat order on the associated Coxeter group. We will discuss these ideas in the context of the simplest infinite Coxeter groups, the affine symmetric groups. Using the combinatorial model introduced in (Barkley–Speyer 2022), we show that many constructions for the weak order on the symmetric group have analogs for the extended weak order on the affine symmetric group. In particular, shards in the affine braid arrangement biject with completely join-irreducible elements of the extended weak order, and there is a parametrization of both objects by "type-$\widetilde{A}$ arc diagrams".

Keywords: Coxeter groups, weak order, shards

1 Introduction

The weak Bruhat order is a partial order on a Coxeter group which is studied for its connections to generalized permutahedra [6], Coxeter arrangements [4], pattern avoidance [3], preprojective algebras [5], and Catalan combinatorics [10]. Some of these connections fail to be complete or do not make sense when applied to infinite Coxeter groups. Motivated by this failure in the context of Hecke algebras [8], Matthew Dyer introduced a different but related poset associated to each Coxeter group, which is now called the **extended weak order**. In general the extended weak order strictly contains the weak Bruhat order as an order ideal, but for finite Coxeter groups the two posets coincide. There are many fascinating conjectures [7] suggesting that, often, the extended weak order is a more natural object than the usual weak Bruhat order. For example, weak Bruhat order is a lattice for any finite Coxeter group, but is never a lattice for an infinite Coxeter group (of finite rank). In contrast, the extended weak order is conjectured to always be a complete lattice. This conjecture has recently been proven for affine Coxeter groups in [2].

In this extended abstract, we will focus on the combinatorics and geometry of extended weak order in type $\widetilde{A}$, using combinatorial models introduced in [1]. Our focus will be on **shards**, certain cones in a Coxeter arrangement which, for finite Coxeter groups, govern the combinatorics of lattice quotients of the weak order. Our main theorem is the following.

*gbarkley@math.harvard.edu. Grant Barkley was supported by NSF grant DMS-1854512.

Theorem 1. *There is a canonical bijection between shards in the Coxeter arrangement of type $\widetilde{A}_n$ and completely join-irreducible elements in the extended weak order of type $\widetilde{A}_n$.*

The analogous result is known to be true for all finite Coxeter groups [9]. Importantly, the result is **false** for the weak Bruhat order of type $\widetilde{A}_n$: there is an injection from completely join-irreducible elements of weak Bruhat order to the set of shards, but it is not a bijection. The weak Bruhat order is "missing" some join-irreducibles, and the extended weak order provides them.

We prove this theorem by parametrizing both shards and complete join-irreducibles by **cyclic arc diagrams**. Arc diagrams were introduced by Nathan Reading [11] as a way of encoding the combinatorics of shards in type A. Our results can be viewed as a type-$\widetilde{A}$ analog of his. Analogs in type B and type D have also been recently introduced by Ashley Tharp in her thesis [12].

The Coxeter group of type A_n is the symmetric group S_{n+1}, and the Coxeter group of type $\widetilde{A}_n$ is the affine symmetric group $\widetilde{S}_{n+1}$. The corresponding Coxeter arrangements are the braid and affine braid arrangements, respectively. Because we are working in a context where we have explicit combinatorial models, we have attempted to avoid Coxeter-theoretic language in the body of the paper, and have included an introduction to the relationship between arc diagrams, weak order, and the geometry of these arrangements for the unfamiliar reader.

2 Weak order and the symmetric group

We begin by recalling the combinatorics of the weak Bruhat order on the symmetric group. Let S_n denote the group of permutations of the set $\{1, \ldots, n\}$. We say that the pair (a, b) is an **inversion** of π if $a < b$ and $\pi^{-1}(a) > \pi^{-1}(b)$. If we write a permutation in one-line notation, then the inversions are the pairs which are out of order. For instance, the inversions of the permutation 51423 are $(1, 5)$, $(4, 5)$, $(2, 5)$, $(3, 5)$, $(2, 4)$, $(3, 4)$. Write $N(\pi)$ for the set of inversions of π. This set determines π uniquely. The **weak order** on S_n is the partial ordering such that $u \leq v$ if and only if $N(u) \subseteq N(v)$. Figure 1 depicts the weak order on S_3.

2.1 The poset of regions

In this section, we will consider the relationship between the weak order and convex geometry. To see this, consider the **braid arrangement** $\mathcal{B}_n$. The braid arrangement consists of hyperplanes H_{ab} in $\mathbb{R}^n$, for $1 \leq a < b \leq n$, where $H_{ab} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_a = x_b\}$. In Figure 2, we've depicted (a slice through) $\mathcal{B}_3$. As illustrated in the figure, two points are in the same region (connected component of $\mathbb{R}^n \setminus \bigcup_{a<b} H_{ab}$) if and only if their coordinates are in the same order. Hence regions correspond to total orderings of

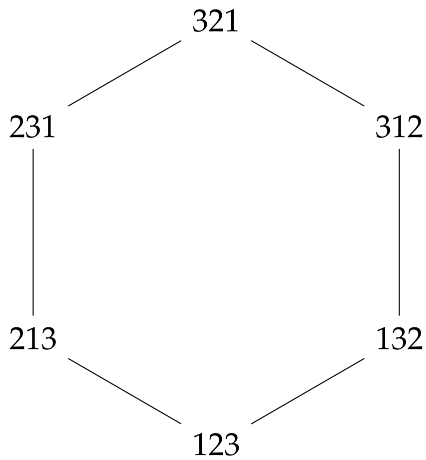

Figure 1: The Hasse diagram of weak order on S_3.

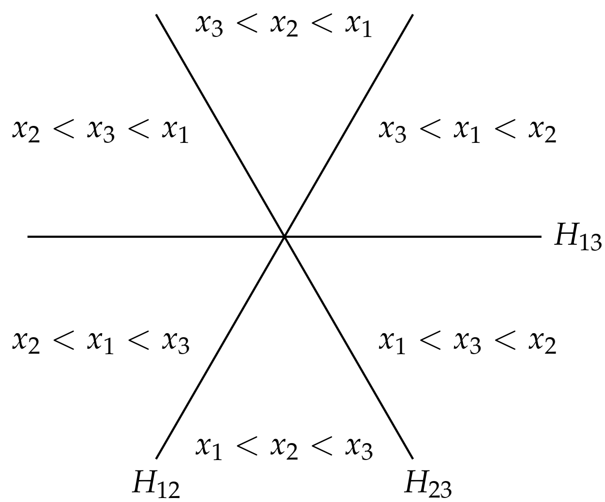

Figure 2: The intersection of $\mathcal{B}_3$ with a two-dimensional subspace of $\mathbb{R}^3$.

the coordinates $x_1, \ldots, x_n$. We can think of these total orderings as the one-line notation of a permutation, so that for instance the permutation 231 corresponds to the region whose points have coordinates satisfying $x_2 < x_3 < x_1$. This gives a bijection between elements of S_n and regions of $\mathcal{B}_n$.

There's another perspective on where this bijection comes from: there is a group action of S_n on $\mathbb{R}^n$, where π acts via $(x_1, \ldots, x_n) \mapsto (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$. This induces an action on the regions of $\mathcal{B}_n$, and this action is simply transitive. So if we fix a "base region", then the group action induces a bijection between regions and elements of S_n. If we use as a base region the region of points such that $x_1 < \cdots < x_n$, then we get the bijection outlined above.

For our purposes, the weak order is more naturally viewed as a partial order on regions of the braid arrangement than it is as a partial order on S_n. Given regions R_1 and R_2, their **separating set** is

$$\mathcal{S}(R_1, R_2) := \{H_{ab} \mid R_1 \text{ and } R_2 \text{ are in different components of } \mathbb{R}^n \setminus H_{ab}\}.$$

Now if B denotes the region with $x_1 < \cdots < x_n$, then πB is the region associated to π via the bijection above. In this case, $\mathcal{S}(B, \pi B) = \{H_{ab} \mid (a, b) \text{ is an inversion of } \pi\}$. Hence weak order can be identified with the order on regions of the braid arrangement so that $R_1 \leq R_2$ if $\mathcal{S}(B, R_1) \subseteq \mathcal{S}(B, R_2)$. This is called the **poset of regions** of $\mathcal{B}_n$.

2.2 Lattice structure

The weak order on S_n is a **complete lattice**, meaning that it admits meets (greatest lower bounds) and joins (least upper bounds) for any collection of elements. To compute the join of a list of permutations, we introduce a closure operator on sets of inversions. Write $T := \{(a,b) \mid 1 \leq a < b \leq n\}$. If $N \subseteq T$, then we define the **closure** of N to be the minimal set $\overline{N} \supseteq N$ such that if $a < b < c$ and (a,b), (b,c) are both in $\overline{N}$, then (a,c) is in $\overline{N}$. The join of two permutations π_1 and π_2, denoted $\pi_1 \vee \pi_2$, is the unique permutation with inversion set $\overline{N(\pi_1) \cup N(\pi_2)}$. More generally, the join of a family $\{\pi_i\}_{i \in I}$ has inversion set $\overline{\bigcup_{i \in I} N(\pi_i)}$. One can compute the meet of a family $\{\pi_i\}_{i \in I}$ dually: it has inversion set $(\bigcap_{i \in I} N(\pi_i))^\circ$, where $N^\circ := T \setminus \overline{T \setminus N}$ is the **interior** of N.

For an example, let's compute the join of 213 and 132. Then $N(213) = \{(1,2)\}$ and $N(132) = \{(2,3)\}$. We need to compute the closure of $N(213) \cup N(132) = \{(1,2),(2,3)\}$. The closure is forced to contain $(1,3)$, since $(1,2)$ and $(2,3)$ are both elements. Hence $\overline{N(213) \cup N(132)} = \{(1,2),(2,3),(1,3)\}$. The join $213 \vee 132$ should be the unique permutation with this inversion set, which is the permutation 321. As can be seen in Figure 1, we indeed have $213 \vee 132 = 321$.

2.3 Shards and arc diagrams

A permutation π is **join-irreducible** if it cannot be written as a join of elements strictly below π. Equivalently, π covers a unique element π_* in the weak order. The join-irreducible elements (JIs) in S_3 are 213, 132, 231, and 312. Each JI has a unique **lower wall**: an inversion (a,b) such that $\pi^{-1}(a) = 1 + \pi^{-1}(b)$. If (a,b) is the lower wall of a JI π, then $(a,b) \cdot \pi = \pi_*$. The lower walls of the JIs listed above are $(1,2)$, $(2,3)$, $(1,3)$, and $(1,3)$, respectively. Nathan Reading introduced in [11] an elegant way of parametrizing the JIs in S_n via arc diagrams.

Definition 1. A **shard arc** for S_n is the data of:

- an initial value i and a terminal value j, such that $1 \leq i < j \leq n$, and
- for each intermediate value k with $i < k < j$, a choice of "left" or "right".

We will depict shard arcs using arc diagrams, where an arc is drawn connecting the initial value to the terminal value, and where "left" or "right" at k indicates whether the arc passes over or under k, respectively. (There are many ways to draw such a diagram; we use diagrams only as an abbreviation for the data of a shard arc, so different diagrams indicating the same shard arc should be treated as equivalent.) The four shard arcs in S_4 with initial value 1 and terminal value 4 are shown in Figure 3. For space purposes, we have drawn the arcs horizontally, though the "left/right" terminology more clearly applies to arcs drawn vertically. To each shard arc for S_n, we assign a JI of the weak order on S_n. This JI is the unique permutation π with the following properties:

- If the shard arc has initial value i and terminal value j, then the unique lower wall of π is (i,j), and

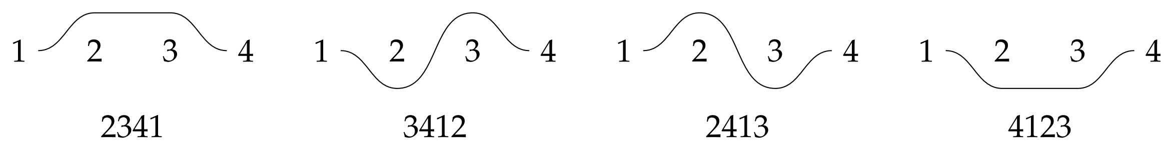

Figure 3: The arc diagrams for shard arcs in S_4 with initial value 1 and terminal value 4. Below each diagram, we have indicated the associated JI in S_4.

- For each intermediate value k, if we have chosen "left" at k, then $\pi^{-1}(k) < \pi^{-1}(j)$, and if we have chosen "right" at k, then $\pi^{-1}(k) > \pi^{-1}(i)$.

In other words, the pair j, i should appear consecutively in the one-line notation for π, and an intermediate value k should appear to the left or right of j, i according to whether we have chosen "left" or "right", respectively. The positions of non-intermediate values, and the relative ordering of intermediate values, are determined by the requirement that (i, j) is the unique lower wall of π. In Figure 3, we have indicated the JI associated to each shard arc.

Shard arcs are also related to the geometry of the braid arrangement. Given a shard arc for S_n, we will associate a convex polyhedral cone in the braid arrangement $\mathcal{B}_n$. To do so, define the half-spaces

$$H_{ab}^{+} := \{(x_1, \ldots, x_n) \mid x_a \leq x_b\} \qquad H_{ab}^{-} := \{(x_1, \ldots, x_n) \mid x_a \geq x_b\}.$$

Consider a shard arc with initial value i and terminal value j. For each intermediate value k, we pick a sign for H_{ik} and for H_{kj}, as follows:

$$\begin{array}{rcl} \text{“left” at } k & \Rightarrow & H_{ik}^{-} \text{ and } H_{kj}^{+} \\ \text{“right” at } k & \Rightarrow & H_{ik}^{+} \text{ and } H_{kj}^{-} \end{array}.$$

The polyhedral cone is defined to be the intersection of H_{ij} with the correctly signed $H_{ik}^{\pm}$ for all intermediate k. The resulting cone Σ is called a **shard** of $\mathcal{B}_n$. Shards are characterized as (the closures of) the connected components of $H_{ij} \setminus \bigcup_{i<k<j} H_{ik}$. In $\mathcal{B}_3$, there are four total shards: H_{12} and H_{23} are themselves shards, and H_{13} is the union of two shards. In Figure 2, the two shards in H_{13} are the left and right halves of H_{13}, which intersect at the origin.

We have constructed bijections shards $\Leftrightarrow$ shard arcs $\Leftrightarrow$ JIs. Let's discuss how to go directly between JIs and shards. Given any permutation π, we can consider the region of the braid arrangement πB. The lower walls (a, b) of π correspond to the hyperplanes H_{ab} in $\mathcal{S}(B, \pi B)$ which are incident to πB. (Hence the term "wall".) We can refine this further: if (a, b) is a lower wall of π, then there is a unique shard Σ contained in H_{ab} which is incident to the region πB. We say that Σ is a **lower shard** of πB. Hence there is a bijection between the lower walls of π and the lower shards of πB. If π is a JI, then there is a unique lower wall of π, and the corresponding lower shard of πB comes from the shard $\Leftrightarrow$ JI bijection. As an example, consider the JI $\pi = 231$. Then the unique lower wall of π is $(1,3)$, so the unique lower shard of πB should be contained in H_{13}. Examining Figure 2, we see that the unique lower shard of πB is the left half of H_{13}, which has shard arc 1 2 3. As expected, this is the shard arc associated to 231.

We have focused on JIs, but the notion of lower shards makes sense for any region of the braid arrangement. For a general region, there will be multiple lower shards. We can record the set of lower shards of πB in a diagram by overlaying the arc diagrams for each shard arc. For instance, the permutation 321 has diagram 1−2−3 and the permutation 123 has the empty diagram 1 2 3. Reading showed [11] that any permutation can be recovered from its arc diagram, and that the collections of shard arcs arising from this construction are exactly the **non-crossing arc diagrams**: those collections that can be drawn so no two shard arcs intersect or share an initial or terminal value.

3 Extended weak order and the affine symmetric group

Definition 2. The **affine symmetric group** $\widetilde{S}_n$ is the group of bijections $\widetilde{\pi} : \mathbb{Z} \to \mathbb{Z}$ satisfying:

(a) $\widetilde{\pi}(i+n) = \widetilde{\pi}(i) + n$ for all $i \in \mathbb{Z}$, and

(b) $\displaystyle\sum_{i=1}^{n} \widetilde{\pi}(i) = \sum_{i=1}^{n} i.$

Elements of $\widetilde{S}_n$ are **affine permutations**. The one-line notation of an affine permutation is defined similarly to a usual permutation, so that for instance the one-line notation of the identity is: $\ldots, -1, 0, 1, 2, 3, 4, 5, \ldots$. Condition (b) in Definition 2 lets us recover an affine permutation from its one-line notation. We abbreviate affine permutations via **window notation**: given a sequence of n integers $x_1, \ldots, x_n$ which have distinct residue classes mod n, we write $[x_1 x_2 \cdots x_n]$ for the unique affine permutation whose one-line notation contains $x_1, \ldots, x_n$ as a consecutive subsequence. For instance, in $\widetilde{S}_3$, the windows [123] and [012] both represent the identity permutation, whereas [102] represents the permutation $\ldots, -3, -1, 1, 0, 2, 4, 3, 5, \ldots$.

The window notations for the elements of $\widetilde{S}_2$ are shown in black in Figure 4.

3.1 Extended weak order

Let $(\prec)$ be a total ordering of the integers (a relation which is transitive, asymmetric, irreflexive, and so that for all distinct $a, b \in \mathbb{Z}$, either $a \prec b$ or $a \succ b$). The symbol $<$ will always denote the usual total ordering on the integers. If $a, b \in \mathbb{Z}$ are distinct modulo n, then we say that the pair (a, b) is an **inversion** of $(\prec)$ if $a < b$ and $b \prec a$. We write $N(\prec)$ for the set of inversions of $(\prec)$.

Definition 3 ([1])**.** The **extended weak order** for $\widetilde{S}_n$ is the poset whose elements are the total orders $(\prec)$ of $\mathbb{Z}$ satisfying the following properties:

- For all $i, j \in \mathbb{Z}$, we have $i \prec j$ if and only if $i + n \prec j + n$, and
- For all $i \in \mathbb{Z}$, if $i + n \prec i$ then there exists a k with $i + n \prec k \prec i$.

We say that $(\prec_1) \leq (\prec_2)$ in extended weak order if $N(\prec_1) \subseteq N(\prec_2)$.

We call an element of extended weak order a **translationally invariant total order** (TITO). Because we do not count the pair $(0, n)$ as an inversion, the second condition in Definition 3 is necessary to make it so any TITO is determined by its inversion set. To see the issue, consider the following two total orderings which satisfy the first condition of Definition 3 with $n = 2$:

$$\cdots \prec 0 \prec 2 \prec 4 \prec \cdots \prec \cdots \prec -1 \prec 1 \prec 3 \prec \cdots \tag{3.1}$$

$$\cdots \prec 4 \prec 2 \prec 0 \prec \cdots \prec \cdots \prec -1 \prec 1 \prec 3 \prec \cdots .$$

These two total orders have the same inversion set. We resolve this by declaring the first to be a TITO and the second to be not a TITO; alternatively, we could declare the two total orders equivalent, and the resulting theory would be the same.

Because (a, b) is an inversion of $(\prec)$ if and only if $(a + n, b + n)$ is an inversion of $(\prec)$, we will identify the pairs (a, b) and $(a + n, b + n)$. Hence

$$\cdots \prec -3 \prec -4 \prec -1 \prec -2 \prec 1 \prec 0 \prec 3 \prec 2 \prec 5 \prec 4 \prec 7 \prec 6 \prec \cdots \tag{3.2}$$

is a TITO for $\widetilde{S}_4$ with two inversions, $(0, 1)$ and $(2, 3)$. We note that using the window notation $[1, 0, 3, 2]$ is a reasonable way to encode this TITO. We will now extend window notation to allow us to encode any TITO.

Observe that any TITO $(\prec)$ splits up into **blocks**: subintervals which are order-isomorphic to the usual ordering on $\mathbb{Z}$. The blocks of (3.1) are $\cdots \prec 0 \prec 2 \prec 4 \prec \cdots$ and $\cdots \prec -1 \prec 1 \prec 3 \prec \cdots$, while there is a unique block for (3.2). The residue classes mod n of integers appearing in distinct blocks are necessarily distinct. If a block contains k residue classes, then we will use a window listing any k consecutive entries of the block. We give each block its own window and separate them by the symbol $\prec$. So, for instance, (3.1) has window notation $[2] \prec [1]$ and (3.2) has window notation $[1, 0, 3, 2]$.

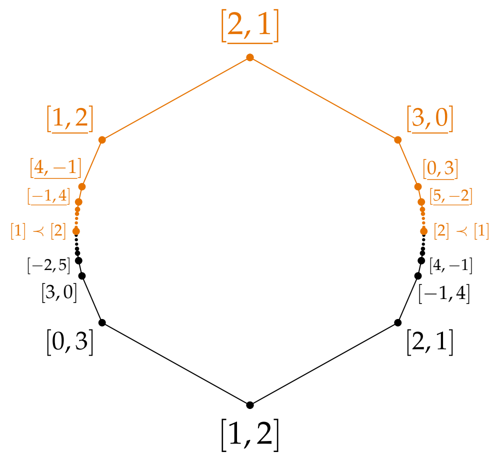

Figure 4: The Hasse diagram for extended weak order. Elements of weak Bruhat order are shown in black, and new elements from extended weak order are in orange.

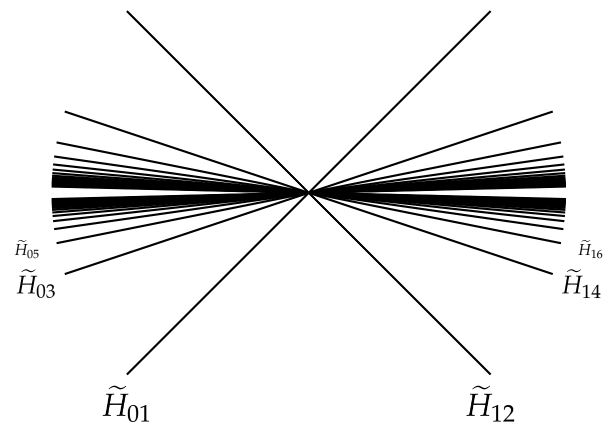

Figure 5: The intersection of the affine braid arrangement $\widetilde{\mathcal{B}}_2$ with a two-dimensional subspace of $\mathbb{R}^3$.

There is one subtlety we haven't yet addressed, which is blocks appearing "in reverse order". For example, consider the following TITO for $\widetilde{S}_4$:

$$\cdots \prec -3 \prec -1 \prec 1 \prec 3 \prec 5 \prec 7 \prec \cdots \prec \cdots \prec 6 \prec 4 \prec 2 \prec 0 \prec -2 \prec -4 \prec \cdots . \tag{3.3}$$

Based on what we have stated so far, the window notation of this TITO would be $[1,3] \prec [2,0]$. However, this does not distinguish (3.3) from the TITO

$$\cdots \prec -3 \prec -1 \prec 1 \prec 3 \prec 5 \prec 7 \prec \cdots \prec \cdots \prec -2 \prec -4 \prec 2 \prec 0 \prec 6 \prec 4 \prec \cdots .$$

To distinguish these, we will write the window notation for (3.3) as $[1,3] \prec [\underline{2,0}]$. What's going on here? It turns out there are exactly two ways to extend the consecutive sequence $2 \prec 0$ to a TITO block: either 0 is covered by $2+4$, or 0 is covered by $2-4$. Once we make that choice, the rest of the block is uniquely determined. In general, we underline a window to indicate that elements i of its block satisfy $i \prec i-n$. If a window is not underlined, then elements of its block satisfy $i \prec i+n$. The TITOs for $\widetilde{S}_2$ are shown in Figure 4.

3.2 The affine braid arrangement

The **affine braid arrangement** $\widetilde{\mathcal{B}}_n$ consists of hyperplanes $\widetilde{H}_{ab}$ in $\mathbb{R}^{n+1}$, for $a < b$ integers that are distinct modulo n. We write a general element of $\mathbb{R}^{n+1}$ as $(y, x_1, \ldots, x_n)$. Then

$$\widetilde{H}_{ab} := \{(y, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_a = x_b\},$$

where we take the convention that $x_{a+kn} = x_a + ky$ for any $k \in \mathbb{Z}$. So for instance, in $\widetilde{\mathcal{B}}_3$, we have $\widetilde{H}_{-1,9} = \{(y, x_1, x_2, x_3) \in \mathbb{R}^4 \mid x_2 - y = x_3 + 2y\}$.

When we were studying the the symmetric group, there was a bijection between elements of S_n and regions of $\mathcal{B}_n$. This is no longer true for $\widetilde{S}_n$: we can see by comparing Figure 4 and Figure 5 that there are more regions than elements of $\widetilde{S}_n$. One might wonder if elements of the extended weak order biject with regions of $\widetilde{\mathcal{B}}_n$. This fails in general. To see the problem, let's introduce the half-spaces

$$\widetilde{H}^+_{ab} := \{(y, x_1, \dots, x_n) \mid x_a \leq x_b\} \qquad \widetilde{H}^-_{ab} := \{(y, x_1, \dots, x_n) \mid x_a \geq x_b\}.$$

Given a TITO $(\prec)$, we say that $\widetilde{H}^+_{ab}$ **contains** $(\prec)$ if $a \prec b$, and similarly we say $\widetilde{H}^-_{ab}$ **contains** $(\prec)$ if $a \succ b$. We write $\mathcal{H}(\prec)$ for the collection of half-spaces containing $(\prec)$. The geometry in Figures 4 and 5 suggests that the region associated to $(\prec)$ should be the intersection of all the half-spaces in $\mathcal{H}(\prec)$. When this intersection has points in its interior, then this is a reasonable definition. But this is not always the case: for instance, the $\widetilde{S}_2$ TITO with window notation $[1] \prec [2]$ is contained in the half-spaces $\widetilde{H}^-_{01}, \widetilde{H}^-_{03}, \widetilde{H}^-_{05}, \dots$ and the half-spaces $\widetilde{H}^+_{12}, \widetilde{H}^+_{14}, \dots$. The intersection of these half-spaces is the line $y = 0$, which has empty interior. However, in this case every finite subset of $\mathcal{H}(\prec)$ has intersection with nonempty interior. (We say the TITO is **weakly separable** in this case; see [1].) A more serious problem arises for the $\widetilde{S}_4$ TITO $[0, 1] \prec [\underline{3, 2}]$. This is contained in the half-spaces $\widetilde{H}^+_{01}, \widetilde{H}^+_{14}, \widetilde{H}^-_{23}, \widetilde{H}^-_{36}$, whose intersection is contained in the hyperplane $y = 0$. This TITO is not weakly separable.

We see that not every TITO has an associated region. However, every region does have an associated TITO. Given a region R of $\widetilde{\mathcal{B}}_n$, let $\mathcal{H}(R)$ be the collection of half-spaces $\widetilde{H}^\pm_{ab}$ such that $R \subseteq \widetilde{H}^\pm_{ab}$. Then $\mathcal{H}(R)$ is equal to $\mathcal{H}(\prec)$ for a unique TITO $(\prec)$. Uniqueness follows since we can recover the inversion set of $(\prec)$ from its set of containing hyperplanes: $N(\prec) = \{(a, b) \mid \widetilde{H}^-_{ab} \in \mathcal{H}(\prec)\}$. This is the analog of the fact that we can recover the inversion set of a permutation π from the separating set $\mathcal{S}(B, \pi B)$.

3.3 Lattice structure

Like the weak order on S_n, the $\widetilde{S}_n$ extended weak order is a complete lattice [1, 2]. We can compute the join of a collection of TITOs in a similar fashion. Write $\widetilde{T} := \{(a, b) \mid a < b,\ a \not\equiv b \mod n\}$ and $\widetilde{T}_{\text{aug}} := \{(a, b) \mid a < b\}$. If $N \subseteq \widetilde{T}_{\text{aug}}$, then we define the **augmented closure** of N to be the minimal set $\overline{N}_{\text{aug}} \supseteq N$ such that if $a < b < c$ and $(a, b), (b, c)$ are both in $\overline{N}_{\text{aug}}$, then (a, c) is in $\overline{N}_{\text{aug}}$. If $N \subseteq \widetilde{T}$ is a union of inversion sets, then the **closure** of N is the set $\overline{N} := \overline{N}_{\text{aug}} \cap \widetilde{T}$. Now, the join of a family of TITOs $\{\prec_i\}_{i \in I}$ is the unique TITO with inversion set $\overline{\bigcup_{i \in I} N(\prec_i)}$. Analogously, the meet of $\{\prec_i\}_{i \in I}$ has inversion set $(\bigcap_{i \in I} N(\prec_i))^\circ$, where $N^\circ := \widetilde{T} \setminus \overline{(\widetilde{T} \setminus N)}$ is the **interior** of N.

For example, let's compute the join of $[0, 3]$ and $[2, 1]$ in the extended weak order of $\widetilde{S}_2$. We have $N([0, 3]) = \{(0, 1)\}$ and $N([2, 1]) = \{(1, 2)\}$. The augmented closure of the union $\{(0,1),(1,2)\}$ contains $(0,2)$ since $(0,1),(1,2)$ are both elements, and it contains $(1,3)$ since $(1,2),(2,3)$ are both elements. (Recall our convention that $(a,b) = (a+n, b+n)$.) Hence the augmented closure contains every pair, since it contains $(0,1),(1,3),(3,5),\ldots$ and $(1,2),(2,4),(4,6),\ldots$.

It follows that the closure $\overline{N([0,3]) \cup N([2,1])}$ is $\widetilde{T}$. Hence the join $[0,3] \vee [2,1]$ is the unique TITO with inversion set $\widetilde{T}$, which is $[\underline{2,1}]$.

3.4 Shards and arc diagrams

A TITO $(\prec)$ is **completely join-irreducible** if it cannot be written as a join of elements strictly below $(\prec)$. This implies that $(\prec)$ covers a unique element, but is a stronger condition in general. The only TITOs for $\widetilde{S}_2$ which are **not** completely join-irreducible are $[12]$, $[1] \prec [2]$, $[2] \prec [1]$, and $[\underline{2,1}]$. In this section JI will abbreviate "completely join-irreducible element".

The **lower walls** of a TITO $(\prec)$ are the inversions (a,b) so that b and a are consecutive in the total order $\prec$. There exist TITOs, like $[1] \prec [2]$, which have no lower walls. However, each JI has a unique lower wall. The goal of this section is to describe the analog of arc diagrams which parametrizes the JIs.

Definition 4. A **shard arc** for $\widetilde{S}_n$ is the data of:

- an initial value i and a terminal value j, such that $1 \leq i \leq n$ and $i < j$ and $i \not\equiv j \mod n$, and
- for each intermediate value k with $i < k < j$, a choice of "left" or "right".

These data are required to satisfy a condition which will be explained below.

We can depict these shard arcs in two ways. One is to simply draw an arc diagram for S_j, where j is the terminal value of the arc. This would fully encode the data of the shard arc. However, the conditions on the data are more well-motivated by drawing a **cyclic arc diagram**: we arrange the numbers $1,\ldots,n$ in a circle, and draw an arc starting at i, proceeding clockwise around the circle until it is of length $j-i$, then terminating (at a value congruent to j modulo n). At each intermediate value k, the arc passes k on the outside or inside of the circle depending on whether we have chosen "left" or "right", respectively. Now we can state the condition on $\widetilde{S}_n$ shard arc data: we must be able to draw the cyclic arc diagram in this way without self-crossing.

The JI associated to a shard arc is the unique weak order-minimal TITO with lower wall (i,j) and such that each intermediate value k satisfies $k \prec j$ if we chose "left" and satisfies $i \prec k$ if we chose "right". The TITOs associated to the shard arcs in Figure 6 are

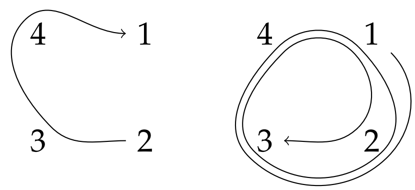

Figure 6: Two shard arcs. The arc on the left starts at 2, ends at 5, and passes 3 and 4 on the inside and outside, respectively. The arc on the right starts at 1 and ends at 11.

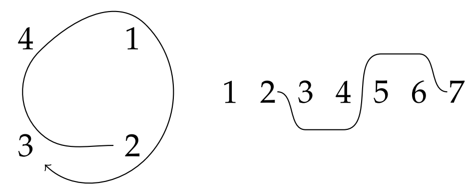

Figure 7: On the left, a cyclic arc diagram. On the right, a "straightened" version of the diagram encoding the same data. (Note that not all S_j arc diagrams give valid $\widetilde{S}_n$ shard arcs.)

$$\cdots \prec 0 \prec 1 \prec -2 \prec -1 \prec 4 \prec 5 \prec 2 \prec 3 \prec 8 \prec 9 \prec 6 \prec 7 \prec \cdots$$

$$\cdots \prec 2 \prec 7 \prec -3 \prec 0 \prec 6 \prec 11 \prec 1 \prec 4 \prec 10 \prec 15 \prec 5 \prec \cdots$$

and the TITO for the shard arc in Figure 7 is

$$\cdots - 3 \prec 1 \prec 5 \prec 9 \prec \cdots \prec 6 \prec 7 \prec 2 \prec 3 \prec -2 \prec -1 \prec \cdots \prec -4 \prec 0 \prec 4 \prec 8 \prec \cdots .$$

To construct the shard associated to a shard arc, for each intermediate value k, we pick a sign for $\widetilde{H}_{ik}$ and for $\widetilde{H}_{kj}$ as follows:

$$\begin{aligned} \text{“left” at } k &\quad\Rightarrow\quad \widetilde{H}_{ik}^{-} \text{ and } \widetilde{H}_{kj}^{+} \\ \text{“right” at } k &\quad\Rightarrow\quad \widetilde{H}_{ik}^{+} \text{ and } \widetilde{H}_{kj}^{-} \end{aligned} .$$

The shard associated to the shard arc is then defined to be the cone Σ which is the intersection of $\widetilde{H}_{ij}$ with $\widetilde{H}_{ik}^{\pm}$ for all intermediate k. Shards are characterized as (the closures of) the connected components of $\widetilde{H}_{ij} \setminus \bigcup_{i<k<j} \widetilde{H}_{ik}$.

The map sending a JI to its associated shard has a geometric description. Let $(\prec)$ be a JI with lower wall (a,b). If $\mathcal{H}(\prec) = \mathcal{H}(R)$ for some region R of $\mathcal{H}$, then the hyperplane $\widetilde{H}_{ab}$ is incident to R. The shard Σ associated to $(\prec)$ is the unique shard contained in $\widetilde{H}_{ab}$ which is incident to R: we say Σ is a **lower shard** of R. However, there exist JIs which do not come from regions, such as $[1,2] \prec [\underline{3,4}]$. Even in this case, Σ is characterized as the unique shard of $\widetilde{H}_{ab}$ which, for each intermediate k, is contained in $\widetilde{H}_{ak}^{-}$ if and only if $(\prec)$ is contained in $\widetilde{H}_{ak}^{-}$. Hence, despite the lack of a literal region to go with $(\prec)$, the geometry still behaves as if Σ is the lower shard of a "quasi-region" associated to $(\prec)$. The existence of such exotic JIs makes the following result even more surprising.

Theorem 2. *These correspondences set up bijections*

$$\textit{shards} \Leftrightarrow \textit{shard arcs} \Leftrightarrow \textit{JIs}.$$

Let i and j be the initial and terminal values of a shard arc datum. The JIs which are elements of weak Bruhat order are those with shard arc data having either $i < j < i + n$, or else $j > i + n$ and we have chosen "right" at $i + n$.

In particular, the shard arc in Figure 7, or any shard with arc data having a choice of "left" at $i + n$, does not have an associated join-irreducible element of weak Bruhat order: we truly need to go to the extended weak order to explain these shards.

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