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SOME ELEMENTARY EXAMPLES OF LEAST SQUARES

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CONTENTS.

INTRODUCTION.

It is intended in this article to discuss some of the fundamental processes of the method of least squares. No stress will be laid upon the theory on which the method rests, but it is thought that some profit may result from a consideration of what is really done when the method is applied to sets of observations. It is always advisable to get some insight into the elementary processes of any method before proceeding to apply them to more complicated problems in which it may be difficult to visualize just what is being done to bring about the desired end.

The problem, in general, is to determine a set of corrections to observations, such that the sum of the squares of these corrections is a minimum, and, at the same time, such that the observations may be so changed as to eliminate all inconsistencies. This, of course, is when all the observations have equal weight. On the other hand, when the observations have unequal weights, the problem is to make the sum of the pv^2 a minimum, p being the weight of the observation. In any case we are brought back to the subject of maxima and minima as treated in an ordinary course in calculus. However, since all the work in this bureau is carried out along the lines of standardized forms, we often lose sight of the connection between what we are doing and what we know about ordinary maxima and minima. In other words, we are like the man who could not see the forest for the trees, or who could not see the city for the houses.

The writer has attempted to emphasize this connection between ordinary maxima and minima and the process of least squares by means of a few illustrative examples that are characterized by their brevity and by their ease of presentation. No proof of whether a maximum or a minimum is obtained is given, as it is evident which of the two is obtained in every illustration cited.

EXAMPLE OF THE DETERMINATION OF A MAXIMUM.

Let us first determine the rectangle of the greatest area that can be found with one corner at the origin, with the sides along the axes, and with the opposite corner on the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

We now want to make xy a maximum subject to the condition expressed by the equation above. We can proceed in three different ways, and we shall now illustrate these methods.

Method I. COMPLETE DIFFERENTIATION.

We first differentiate completely the function to be made a maximum and set it equal to zero, and then differentiate the equation of condition and eliminate either dx or dy from these equations.

$$xdy+ydx=0,$$

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy=0.$$

From the first equation we get

$$xdy=-ydx,$$

or

$$dy=-\frac{y}{x}dx.$$

By substituting this value in the second equation we have

$$\frac{x}{a^2}dx-\frac{y^2}{b^2x}dx=0,$$

or

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}.$$

Substituting this in the equation of condition we get

$$\frac{2y^2}{b^2}=1,$$

$$y=\frac{b}{\sqrt{2}},$$

$$x=\frac{a}{\sqrt{2}},$$

$$xy=\frac{ab}{2}.$$

Method II. INDEPENDENT UNKNOWNS.

As a second method we can eliminate either x or y from the function xy by substituting its value from the equation of condition

$$y=\frac{b}{a}\sqrt{a^2-x^2}.$$

This gives us the function

$$U=\frac{b}{a}x\sqrt{a^2-x^2}.$$

Differentiate this function with respect to x and set the result equal to zero.

$$\frac{b}{a}\sqrt{a^2-x^2}-\frac{b}{a}\frac{x^2}{\sqrt{a^2-x^2}}=0.$$

Dividing by $\frac{b}{a}$ and clearing of fractions we get

$$a^2-x^2-x^2=0,$$

or

$$2x^2=a^2.$$

Therefore,

$$x=\frac{a}{\sqrt{2}}$$

and

$$y=\frac{b}{\sqrt{2}}.$$

Method III. LAGRANGIAN MULTIPLIERS.

The third method is probably of most interest to us because it introduces the Lagrangian multiplier, which is the basis of our method of correlates in our least-squares work. Let us take the function

$$U=xy+C\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).$$

We may now equate $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$ each separately equal to zero and then solve these resulting equations for C. This gives us the two equations from which C results as below:

$$y+\frac{2Cx}{a^2}=0,$$

$$x+\frac{2Cy}{b^2}=0,$$

$$C=-\frac{a^2y}{2x}=-\frac{b^2x}{2y}.$$

Therefore,

$$\frac{y^2}{b^2}=\frac{x^2}{a^2},$$

and as a result

$$x=\frac{a}{\sqrt{2}}, \text{ and } y=\frac{b}{\sqrt{2}}.$$

$$xy=\frac{ab}{2} \text{ as before.}$$

EXAMPLE OF THE DETERMINATION OF A MINIMUM WITH ONE CONDITION.

Now let us make x^2+y^2 a minimum subject to the condition $x+y=a$. This amounts to determining the square of the length of the perpendicular from the origin upon the line $x+y=a$; and if x and y are allowed to vary after the determination, we shall have a circle with center at the origin and tangent to the given line. In this problem also we can proceed in any one of the three ways.

Method I. COMPLETE DIFFERENTIATION.

$$xdx+ydy=0,$$

$$dx+dy=0,$$

$$-xdy+ydy=0,$$

$$x=y,$$

$$x=\frac{a}{2}=y,$$

$$x^2+y^2=\frac{a^2}{2}.$$

Method II. INDEPENDENT UNKNOWNS.

$$y=a-x,$$

$$U=x^2+(a-x)^2,$$

$$\frac{1}{2}\frac{dU}{dx}=x-(a-x)=0,$$

$$2x=a,$$

$$x=\frac{a}{2},$$

$$y=\frac{a}{2},$$

$$x^2+y^2=\frac{a^2}{2}.$$

Method III. LAGRANGIAN MULTIPLIERS.

$$U = x_2 + y_2 - 2C(x+y-a).$$

$-2C$ is used for the Lagrangian multiplier for convenience.

$$\frac{1}{2}\frac{\partial U}{\partial x} = x - C = 0,$$

$$\frac{1}{2}\frac{\partial U}{\partial y} = y - C = 0.$$

$$x = C,$$

$$y = C,$$

$$x + y = a,$$

$$2C = a,$$

$$C = \frac{a}{2},$$

$$x^2 + y^2 = \frac{a^2}{2}.$$

EXAMPLE OF THE DETERMINATION OF A MINIMUM WITH TWO CONDITIONS.

Now let us make x^2+y^2 a minimum subject to the two conditions $x+y=a$ and $x=b$. This is equivalent to determining the square of the line joining the origin and the intersection of the two lines, or with x and y variable it becomes the circle with center at the origin which passes through the intersection of the lines $x+y=a$ and $x=b$. As a matter of fact, we have enough elements to determine the problem at once; that is,

$$x^2 + y^2 = (a-b)^2 + b^2.$$

Method III. LAGRANGIAN MULTIPLIERS.

We shall apply the third method, that of Lagrangian multipliers.

$$U = x^2 + y^2 - 2C_1(x+y-a) - 2C_2(x-b),$$

$$\frac{1}{2}\frac{\partial U}{\partial x} = x - C_1 - C_2 = 0,$$

$$x = C_1 + C_2$$

$$\frac{1}{2}\frac{\partial U}{\partial y} = y - C_1 = 0,$$

$$y = C_1.$$

From $x+y-a=0$ we get

$$2C_1 + C_2 - a = 0,$$

and from $x-b=0$ we get

$$C_1 + C_2 - b = 0.$$

These are really normal equations that we can solve for C_1 and C_2.

$$C_1 = a - b,$$

$$C_2 = 2b - a,$$

Therefore
$$x = C_1 + C_2 = b,$$

$$y = C_1 = a - b,$$

$$x^2 + y^2 = (a-b)^2 + b^2.$$

RELATION BETWEEN LEAST SQUARES AND ARITHMETIC MEAN.

Let us now apply the method of least squares to the measurement of a single quantity and show that the result gives us the arithmetic mean. Let us take the following five values of a measurement:

18.21,

18.19,

18.30,

18.25,

18.20.

Let M be the value after the adjustment.
Then,

$$M - 18.21 = v_1,$$

$$M - 18.19 = v_2,$$

$$M - 18.30 = v_3,$$

$$M - 18.25 = v_4,$$

$$M - 18.20 = v_5,$$

$$U = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2$$
$$= (M - 18.21)^2 + (M - 18.19)^2 + (M - 18.30)^2 + (M - 18.25)^2 + (M - 18.20)^2.$$

$$\frac{1}{2}\frac{dU}{dM} = M - 18.21 + M - 18.19 + M - 18.30 + M - 18.25 + M - 18.20 = 0.$$

$$5M = 91.15,$$

$$M = 18.23.$$

It will be seen that this is the arithmetic mean of the five observations.

LEAST SQUARES APPLIED TO OBSERVATIONS WITH TWO CONDITIONS.

In most problems that we have to solve we have the choice of using either observation equations or condition equations. Sometimes the one method is the more direct and sometimes the other. To illustrate the matter, let us take a short problem. Let us suppose that a length is measured as a whole, and then the same length is measured in two parts and again in two different parts.

Full length, $201.71 + v_1$,

$$\text{First set of part measurements}\begin{cases} 75.81 + v_2 \\ 125.22 + v_3, \end{cases}$$

$$\text{Second set of part measurements}\begin{cases} 100.03 + v_4 \\ 100.76 + v_5. \end{cases}$$

METHOD OF OBSERVATION EQUATIONS.

Assume the approximate values

$$201 + x_1,$$

$$76 + x_2,$$

$$100 + x_3.$$

$$\left.\begin{aligned} v_1 &= x_1 - 0.71, \\ v_2 &= x_2 + 0.19, \\ v_3 &= x_1 - x_2 - 0.22, \\ v_4 &= x_3 - 0.03, \\ v_5 &= x_1 - x_3 + 0.24. \end{aligned}\right\}\text{Observation equations.}$$

$$U = (x_1 - 0.71)^2 + (x_2 + 0.19)^2 + (x_1 - x_2 - 0.22)^2 + (x_3 - 0.03)^2 + (x_1 - x_3 + 0.24)^2.$$

$$\frac{1}{2}\frac{\partial U}{\partial x_1} = x_1 - 0.71 + x_1 - x_2 - 0.22 + x_1 - x_3 + 0.24 = 0$$

or

$$3\,x_1 - x_2 - x_3 - 0.69 = 0.$$

$$\frac{1}{2}\frac{\partial U}{\partial x_2} = x_2 + 0.19 - x_1 + x_2 + 0.22 = 0,$$

or

$$-x_1 + 2x_2 + 0.41 = 0.$$

$$\frac{1}{2}\frac{\partial U}{\partial x_3} = x_3 - 0.03 - x_1 + x_3 - 0.24 = 0,$$

or

$$-x_1 + 2x_3 - 0.27 = 0.$$

We thus have the three normal equations,

$$3x_1-x_2-x_3= +0.69,$$

$$-x_1+2x_2= -0.41,$$

$$-x_1+2x_3= +0.27$$

From the second of these equations we get

$$x_2=\frac{1}{2}(x_1-0.41),$$

and from the third

$$x_3=\frac{1}{2}(x_1+0.27).$$

By substituting these values in the first normal and solving for x_1 we can get the values of the three x's,

$$3x_1-\frac{1}{2}(x_1-0.41)-\frac{1}{2}(x_1+0.27)=0.69,$$

$$6x_1-x_1+0.41-x_1-0.27=1.38.$$

$$4x_1=1.24,$$

$$x_1=+0.31,$$

$$x_2=-0.05,$$

$$x_3=+0.29.$$

With these values the v's result at once from the observation equations.

$$v_1=-0.40,$$

$$v_2=+0.14,$$

$$v_3=+0.14,$$

$$v_4=+0.26,$$

$$v_5=+0.26.$$

METHOD OF CONDITION EQUATIONS WITH USE OF CORRELATES.

This same problem could be solved by equations of condition. It is evident that the following two equations must be true:

$$201.71+v_1-(75.81+v_2+125.22+v_3)=0,$$

$$201.71+v_1-(100.03+v_4+100.76+v_5)=0.$$

or

$$v_1-v_2-v_3+0.68=0,$$

and

$$v_1-v_4-v_5+0.92=0.$$

By correlates we have

$$U = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2 - 2C_1(v_1 - v_2 - v_3 + 0.68)$$
$$- 2C_2(v_1 - v_4 - v_5 + 0.92).$$

$$\frac{1}{2}\frac{\partial U}{\partial v_1} = v_1 - C_1 - C_2 = 0, \qquad v_1 = C_1 + C_2.$$

$$\frac{1}{2}\frac{\partial U}{\partial v_2} = v_2 + C_1 = 0, \qquad v_2 = -C_1.$$

$$\frac{1}{2}\frac{\partial U}{\partial v_3} = v_3 + C_1 = 0, \qquad v_3 = -C_1.$$

$$\frac{1}{2}\frac{\partial U}{\partial v_4} = v_4 + C_2 = 0, \qquad v_4 = -C_2.$$

$$\frac{1}{2}\frac{\partial U}{\partial v_5} = v_5 + C_2 = 0, \qquad v_5 = -C_2.$$

By substituting these values in the condition equations we get the two normal equations

$$3C_1 + C_2 = -0.68,$$
$$C_1 + 3C_2 = -0.92.$$

Solving these we obtain C_1 and C_2.

$$9C_1 + 3C_2 = -2.04,$$
$$C_1 + 3C_2 = -0.92,$$
$$8C_1 = -1.12,$$
$$C_1 = -0.14,$$
$$C_2 = -0.26.$$

Therefore, as before

$$v_1 = -0.40,$$
$$v_2 = +0.14,$$
$$v_3 = +0.14,$$
$$v_4 = +0.26,$$
$$v_5 = +0.26.$$

METHOD OF CONDITION EQUATIONS WITH INDEPENDENT UNKNOWNS.

We can take the same two equations and apply least squares by the method of independent unknowns,

$$v_3 = +v_1 - v_2 + 0.68,$$

$$v_5 = +v_1 - v_4 + 0.92.$$

$$U = v_1^2 + v_2^2 + (v_1 - v_2 + 0.68)^2 + v_4^2 + (v_1 - v_4 + 0.92)^2.$$

$$\frac{1}{2}\frac{\partial U}{\partial v_1} = v_1 + v_1 - v_2 + 0.68 + v_1 - v_4 + 0.92 = 0,$$

or

$$3v_1 - v_2 - v_4 + 1.60 = 0.$$

$$\frac{1}{2}\frac{\partial U}{\partial v_2} = v_2 - v_1 + v_2 - 0.68 = 0,$$

or

$$-v_1 + 2v_2 - 0.68 = 0.$$

$$\frac{1}{2}\frac{\partial U}{\partial v_4} = v_4 - v_1 + v_4 - 0.92 = 0,$$

or

$$-v_1 + 2v_4 - 0.92 = 0.$$

From the second equation we have

$$v_2 = \frac{1}{2}(v_1 + 0.68),$$

and from the third

$$v_4 = \frac{1}{2}(v_1 + 0.92).$$

By substituting these values in the first equation and solving for v_1 we get

$$3v_1 - \frac{1}{2}(v_1 + 0.68) - \frac{1}{2}(v_1 + 0.92) + 1.60 = 0,$$

$$6v_1 - v_1 - 0.68 - v_1 - 0.92 + 3.20 = 0,$$

$$4v_1 = -1.60,$$

$$v_1 = -0.40.$$

The other v's follow from their equations.

$$v_2 = +0.14,$$

$$v_3 = +0.14,$$

$$v_4 = +0.26,$$

$$v_5 = +0.26.$$

METHOD OF CONDITION EQUATIONS WITH COMPLETE DIFFERENTIATION.

We have the third possibility in that we may take the function

$$U = v_1^2 + v_2^2 + v_3^2 + v_4^2 + v_5^2,$$

and the equations of condition

$$v_1 - v_2 - v_3 = -0.68,$$

and

$$v_1 - v_4 - v_5 = -0.92,$$

and differentiate them completely as they stand and equate the complete derivative of the sum of the squares to zero.

$$\frac{1}{2} dU = v_1 dv_1 + v_2 dv_2 + v_3 dv_3 + v_4 dv_4 + v_5 dv_5 = 0,$$

$$dv_1 - dv_2 - dv_3 = 0,$$

$$dv_1 - dv_4 - dv_5 = 0.$$

From the last two equations we get

$$dv_3 = +dv_1 - dv_2,$$

and

$$dv_5 = dv_1 - dv_4.$$

By substituting these values in the first equation we get

$$v_1 dv_1 + v_2 dv_2 + v_3(+dv_1 - dv_2) + v_4 dv_4 + v_5(+dv_1 - dv_4) = 0.$$

After rearrangement this becomes

$$(v_1 + v_3 + v_5) dv_1 + (v_2 - v_3) dv_2 + (v_4 - v_5) dv_4 = 0.$$

Since dv_1, dv_2, and dv_4 are arbitrary, the coefficients must be zero to satisfy the equation if the equation is to be identically satisfied for any values of the dv's.

Therefore,

$$v_1 + v_3 + v_5 = 0,$$

$$v_2 - v_3 = 0,$$

and

$$v_4 - v_5 = 0.$$

These equations, taken with the equations of condition,

$$v_1 - v_2 - v_3 = -0.68,$$

$$v_1 - v_4 - v_5 = -0.92,$$

give the five equations necessary for the solution.

By solving as follows the values of the v's are obtained:

$$v_3 = v_2,$$

$$v_5 = v_4,$$

$$v_1 + v_2 + v_4 = 0,$$

$$v_1 - 2v_2 = -0.68,$$

$$v_1 - 2v_4 = -0.92,$$

$$v_2 = \frac{1}{2}(v_1 + 0.68),$$

$$v_4 = \frac{1}{2}(v_1 + 0.92),$$

$$2v_1 + v_1 + 0.68 + v_1 + 0.92 = 0,$$

$$4v_1 = -1.60,$$

$$v_1 = -0.40,$$

$$v_2 = +0.14,$$

$$v_3 = +0.14,$$

$$v_4 = +0.26,$$

$$v_5 = +0.26.$$

LEAST SQUARES APPLIED TO OBSERVATIONS OF DIFFERENT WEIGHT WITH TWO CONDITIONS.

As an illustration of weighting, let us take the same problem and assign the weights 2, 1, 3, 2, 1 to the observations.

METHOD OF OBSERVATION EQUATIONS.

The function then becomes,

$$U = 2v_1^2 + v_2^2 + 3v_3^2 + 2v_4^2 + v_5^2$$

$$= 2(x_1 - 0.71)^2 + (x_2 + 0.19)^2 + 3(x_1 - x_2 - 0.22)^2 + 2(x_3 - 0.03)^2 + (x_1 - x_3 + 0.24)^2.$$

$$\frac{1}{2}\frac{\partial U}{\partial x_1} = 2x_1 - 1.42 + 3x_1 - 3x_2 - 0.66 + x_1 - x_3 + 0.24,$$

or

$$6x_1 - 3x_2 - x_3 = +1.84.$$

$$\frac{1}{2}\frac{\partial U}{\partial x_2} = x_2 + 0.19 - 3x_1 + 3x_2 + 0.66 = 0.$$

or

$$-3x_1 + 4x_2 = -0.85.$$

$$\frac{1}{2}\frac{\partial U}{\partial x_3} = 2x_3 - 0.06 - x_1 + x_3 - 0.24 = 0,$$

or

$$-x_1 + 3x_3 = +0.30.$$

The three normal equations are now as follows:

$$6x_1-3x_2-x_3=+1.84,$$

$$-3x_1+4x_2=-0.85,$$

$$-x_1+3x_3=+0.30.$$

By solving these equations as given below we can get the values of the x's:

$$x_2=\frac{1}{4}(3x_1-0.85),$$

$$x_3=\frac{1}{3}(x_1+0.30),$$

$$6x_1-\frac{3}{4}(3x_1-0.85)-\frac{1}{3}(x_1+0.30)=+1.84,$$

$$72x_1-27x_1+7.65-4x_1-1.20=+22.08,$$

$$41x_1=+15.63,$$

$$x_1=+0.38,$$

$$x_2=+0.07,$$

$$x_3=+0.23.$$

With these values the v's may be computed from the observation equations on page 7.

$$v_1=-0.33,$$

$$v_2=+0.26,$$

$$v_3=+0.09,$$

$$v_4=+0.20,$$

$$v_5=+0.39.$$

METHOD OF CONDITION EQUATIONS.

With the condition equations we proceed as follows:

$$U=2v_1^2+v_2^2+3v_3^2+2v_4^2+v_5^2-2C_1(v_1-v_2-v_3+0.68)-2C_2(v_1-v_4-v_5+0.92),$$

$$2v_1=C_1+C_2,$$

$$v_2=-C_1,$$

$$3v_3=-C_1,$$

$$2v_4=-C_2,$$

$$v_5=-C_2.$$

By substituting these values in the equations of condition we get

$$\frac{1}{2}(C_1+C_2)+C_1+\frac{1}{3}C_1+0.68=0,$$

or

$$\frac{11}{6}C_1+\frac{1}{2}C_2=-0.68,$$

and

$$\frac{1}{2}(C_1+C_2)+\frac{1}{2}C_2+C_2+0.92=0,$$

or

$$\frac{1}{2}C_1+2C_2=-0.92.$$

When these equations are cleared of fractions, we have

$$11C_1+3C_2=-4.08,$$

and

$$C_1+4C_2=-1.84.$$

We solve these two equations for C_1 and C_2, as follows:

$$C_1=-4C_2-1.84,$$

$$-44C_2-20.24+3C_2=-4.08,$$

$$-41C_2=+16.16,$$

$$C_2=-0.394,$$

$$C_1=-0.264.$$

The v's may now be computed from their values expressed in terms of the C's.

$$v_1=-0.33,$$

$$v_2=+0.26,$$

$$v_3=+0.09,$$

$$v_4=+0.20,$$

$$v_5=+0.39.$$

SOLUTION OF SIMULTANEOUS EQUATIONS BY LEAST SQUARES.

It is interesting to note that a set of any number of linear simultaneous equations can be solved by the method of least squares either by means of correlates or by setting each equation equal to a v and then by treating them as observation equations. We have already illustrated the method of correlates when we made x^2+y^2 a minimum subject to the conditions $x+y=a$ and $x=b$. Of course, we get the same number of linear equations that we had at first, but they are symmetrical and can be solved as ordinary normal equations, which is an advantage when the number of equations is large.

We shall now illustrate with two equations by means of observation equations.

METHOD OF OBSERVATION EQUATIONS.

$$x+y=3,$$

$$2x+y=4.$$

Equate each expression to a v instead of zero.

$$x+y-3=v_1,$$

$$2x+y-4=v_2,$$

$$U=v_1^2+v_2^2$$

$$=(x+y-3)^2+(2x+y-4)^2.$$

$$\frac{1}{2}\frac{\partial U}{\partial x}=x+y-3+4x+2y-8=0,$$

or

$$5x+3y=11.$$

$$\frac{1}{2}\frac{\partial U}{\partial y}=x+y-3+2x+y-4=0,$$

or

$$3x+2y=7.$$

We now solve these two equations for x and y.

$$9x+6y=21,$$

$$10x+6y=22,$$

$$x=1,$$

$$2y=4,$$

$$y=2.$$

METHOD OF CONDITION EQUATIONS WITH USE OF CORRELATES.

As a final example we shall solve three simultaneous linear equations in x, y, and z by the method of correlates. In geometrical terms this will really determine the square upon the line joining the origin to the point of intersection of the three planes represented by the three linear equations. After solution with variable x, y, and z we shall have a sphere with center at the origin and passing through the point of intersection of the three planes.

Let us take the three equations,

$$x+y+z=6,$$

$$2x-y+z=3,$$

$$3x-2y-z=-4.$$

To show more clearly the basis of the correlate multipliers, we shall differentiate the function $x^2+y^2+z^2$ completely and equate the result to zero, and so also for each of the three equations. In this way we get the four equations:

$$xdx+ydy+zdz=0,$$
$$dx+dy+dz=0,$$
$$2dx-dy+dz=0,$$
$$3dx-2dy-dz=0.$$

Let us now multiply the second equation by C_1, the third equation by C_2, and the fourth equation by C_3 and then subtract the sum of these three products from the first equation. The result is:

$$(x-C_1-2C_2-3C_3)dx+(y-C_1+C_2+2C_3)dy+(z-C_1-C_2+C_3)dz=0.$$

We can now determine the C's by equating the coefficients of dx, dy, and dz, respectively, to zero. We then have,

$$x-C_1-2C_2-3C_3=0,$$
$$y-C_1+C_2+2C_3=0,$$
$$z-C_1-C_2+C_3=0,$$

or

$$x=C_1+2C_2+3C_3,$$
$$y=C_1-C_2-2C_3,$$
$$z=C_1+C_2-C_3.$$

When these values are substituted in the three original equations, we get the three normal equations,

$$3C_1+2C_2=6,$$
$$2C_1+6C_2+7C_3=3,$$
$$7C_2+14C_3=-4.$$

If these three equations are solved for the C's, we get

$$C_1=+\frac{10}{7},$$
$$C_2=+\frac{6}{7},$$
$$C_3=-\frac{5}{7}.$$

By substituting these values in the expressions for x, y, and z, in terms of the C's, we get

$$x=1,$$

$$y=2,$$

$$z=3.$$

Therefore,

$$x^2+y^2+z^2=14.$$

CONCLUSION.

In applying the method of least squares to extensive sets of observations it is necessary to make use of tabulations, and it is not so easy to see the various steps of the process. From these few illustrative easy examples it should be evident how the result desired is obtained. For this reason it is well to give some thought to the proper understanding of these elementary steps, and later the more complicated adjustments will cease to be mysterious. The rational process is present however much it may appear to be obscured. It is hoped that the preceding examples will be of aid to clear and careful thinking in the application of the method of least squares.

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