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ELLIPTIC FUNCTIONS APPLIED TO CONFORMAL WORLD MAPS

BY

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PREFACE

In the twenty-fourth volume of the Quarterly Journal of Mathematics, 1890, A. C. Dixon, of Trinity College, Cambridge, published a paper, "On the doubly periodic functions arising out of the curve $x^3+y^3-3\alpha xy=1$." On reading this excellent paper the thought was suggested that it might be of interest to consider in more detail the case arising when α is taken as zero. The author at once started upon an investigation, and some results were obtained that are considered to be of rather general interest. The formulas developed for the functions connected with the original curve are, in general, valid for the curve $x^3+y^3=1$, if the α in them is set equal to zero. These results formed the starting data for the further investigations.

In 1864 H. A. Schwarz proved that a circle could be mapped conformally upon a regular polygon of n sides by means of the integral

$$w=\int_0^x \frac{dx}{(1-x^n)^{2/n}}$$

The same thing was shown by Weierstrass in 1866. Consideration of this fact gave an added reason for investigating the properties of this particular class of Abelian functions which are, in fact, elliptic functions with rather unusual characteristics.

In the second volume (1879) of the American Journal of Mathematics, C. S. Peirce, at that time an assistant in the United States Coast and Geodetic Survey, published an account of a conformal projection of the sphere within a square. He called this projection the quincuncial projection of the sphere. In Annales Hydrographiques, second series, volume 9, 1887, Lieut. E. Guyou published the theory of a related projection derived in an entirely different manner. These were the first examples of the application of elliptic functions to the construction of maps for geographic purposes. The purpose of this publication is to illustrate a number of projections most of which depend upon elliptic functions or elliptic integrals, although some are defined by Abelian integrals that have been developed in series.

These projections are interesting applications of the theory of functions of a complex variable to cartography. It is hoped that this general theory, both of the functions and of their application to the construction of maps, may be found of interest to those who are working along this line of investigation.

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ELLIPTIC FUNCTIONS APPLIED TO CONFORMAL WORLD MAPS

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ELLIPTIC FUNCTIONS CONNECTED WITH THE CURVE $x^3+y^3=1$

The theory of the elliptic functions is generally developed by use of integrals connected with the tacnodal quartic

$$y^2=(1-x^2)\ (1-k^2x^2)\ \text{(Jacobian theory)},$$

or with the cubic

$$y^2=4x^3-g_2x-g_3\ \text{(Weierstrassian theory)}.$$

There are other cubic and quartic curves that could be used as a basis for the theory; in fact the lemniscate functions are developed from the curve $y^2=1-x^4$. It is interesting and instructive to make use of some simple curve to serve as a basis for the theory. We shall attempt a short development based upon the curve $x^3+y^3=1$.

Let us start with the Abelian integral of the first kind

$$w=\int_0^x\frac{dx}{y^2}=\int_0^x\frac{dx}{(1-x^3)^{2/3}}.$$

We shall now invert this integral by setting

$$x=\text{sm}\ w;$$

sm w being a function the properties of which we aim to investigate. We shall also let $y=\text{cm}\ w$, cm w being an auxiliary function that is to be investigated at the same time. The fundamental algebraic relation between these functions is given at once as

$$\text{sm}^3\ w+\text{cm}^3\ w=1.$$

From consideration of the integral we have at once

$$\text{sm}\ (0)=0,$$

hence

$$\text{cm}\ (0)=1.$$

By differentiating the equation of the curve, we get

$$\frac{dx}{y^2}=-\frac{dy}{x^2},$$

therefore w, which equals the integral $\int_0^x \frac{dx}{y^2}$, also equals $\int_y^1 \frac{dy}{x^2}$,

or $w = \int^x \frac{dx}{(1-x^3)^{2/3}} = \int_y^1 \frac{dy}{(1-y^3)^{2/3}}$.

The definite integral $\int_0^1 \frac{dx}{y^2}$ is a constant that we shall denote by K;

sm $K=1$ and cm $K=0$.

Now let $$w' = \int_y^1 \frac{dy}{x^2} = \int_0^1 \frac{dy}{(1-y^3)^{2/3}} - \int^y \frac{dy}{(1-y^3)^{2/3}}.$$

If, now, y equals the x in the integral $w = \int_0^x \frac{dx}{(1-x^3)^{2/3}}$, we shall

have $w' = K - w$ and sm $w =$ cm $w' =$ cm $(K-w)$.
By setting w for $K-w$, we get cm $w =$ sm $(K-w)$.
By differentiating the expression

$$w = \int_0^x \frac{dx}{y^2} = \int_y^1 \frac{dy}{x^2},$$

we get

$$\frac{dw}{dx} = \frac{1}{y^2},$$

and

$$\frac{dw}{dy} = -\frac{1}{x^2},$$

or

$$\frac{d}{dw} \text{ sm } w = \text{cm}^2\, w,$$

and

$$\frac{d}{dw} \text{ cm } w = -\text{sm}^2\, w.$$

This last differential equation shows that cm $w =$ sm $(K-w)$ in which K is the constant of integration. We have

$$\frac{\frac{d}{dw} \text{ cm } w}{\text{sm}^2\, w} = -1,$$

$$\frac{\frac{d}{dw} \text{ cm } w}{(1-\text{cm}^3\, w)^{2/3}} = -1.$$

By integration $\text{sm}^{-1}(\text{cm } w) = A - w$, or $\text{cm } w = \text{sm }(A-w)$. Now let $w=0$, then we have $\text{sm } A = 1$, or A can have the value K, therefore $\text{cm } w = \text{sm }(K-w)$.

Considered as functions of w, sm w and cm w are uniform functions of w. It is evident that sm w and $\frac{d}{dw}\text{sm } w$ can not both be equal to zero for the same value of w. Hence there are no branch points at which this condition is fulfilled. The same is true of cm w. The only other condition for branch points is that some one of the successive differential coefficients should become infinite.

But

$$\frac{d}{dw}\text{sm } w = \text{cm}^2 w,$$

and

$$\frac{d}{dw}\text{cm } w = -\text{sm}^2 w.$$

Hence no differential coefficient of either can become infinite unless either sm w or cm w becomes infinite. Also from the equation

$$\text{sm}^3 w + \text{cm}^3 w = 1,$$

it follows that, if one of these is infinite, the other is infinite also and that in such case $\frac{\text{cm } w}{\text{sm } w}$ is finite.

Also

$$\frac{d}{dw}\frac{1}{\text{sm } w} = -\left(\frac{\text{cm } w}{\text{sm } w}\right)^2,$$

and

$$\frac{d}{dw}\frac{\text{cm } w}{\text{sm } w} = -\frac{1}{\text{sm}^2 w}.$$

Hence, the differential coefficients of $\frac{1}{\text{sm } w}$ and $\frac{\text{cm } w}{\text{sm } w}$ are rational and integral functions of $\frac{1}{\text{sm } w}$ and $\frac{\text{cm } w}{\text{sm } w}$.

Thus when sm w is infinite, it is still a uniform function. Hence, sm w is everywhere uniform and by a similar process of reasoning the same can be proved of cm w.

In the vicinity of $w=0$, sm w and cm w can be developed in series of ascending powers of w. The series for sm w can be derived either by reversion of the series for the integral or by differentiation and the use of Maclaurin's development. The series becomes:

$$\text{sm } w = w - \frac{1}{6}w^4 + \frac{2}{63}w^7 - \frac{13}{2268}w^{10} + \frac{23}{22113}w^{13} - \cdots\cdots$$

For cm w, we get

$$\text{cm } w = 1 - \frac{1}{3}w^3 + \frac{1}{18}w^6 - \frac{23}{2268}w^9 + \frac{25}{13608}w^{12} - \cdots\cdots$$

Also

$$\frac{1}{\text{sm } w}=\frac{1}{w}+\frac{1}{6}w^2-\frac{1}{252}w^5+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ ,$$

and

$$\frac{\text{cm } w}{\text{sm } w}=\frac{1}{w}-\frac{1}{6}w^2-\frac{1}{252}w^5-\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$$

These series can be considered as definitions of the functions in the vicinity of $w=0$.

We have

$$\frac{d}{dw}\frac{1}{\text{sm } w}=-\left(\frac{\text{cm } w}{\text{sm } w}\right)^2,$$

or

$$-\frac{d}{dw}\frac{1}{\text{sm } w}=\left(-\frac{\text{cm } w}{\text{sm } w}\right)^2,$$

while

$$\left(-\frac{\text{cm } w}{\text{sm } w}\right)^3+\left(\frac{1}{\text{sm } w}\right)^3=1.$$

Therefore

$$\left[1-\left(\frac{1}{\text{sm } w}\right)^3\right]^{2/3}=-\frac{d}{dw}\frac{1}{\text{sm } w},$$

or

$$-dw=\frac{d\left(\frac{1}{\text{sm } w}\right)}{\left[1-\left(\frac{1}{\text{sm } w}\right)^3\right]^{2/3}},$$

and by integration

$$C-w=\text{sm}^{-1}\left(\frac{1}{\text{sm } w}\right),$$

or

$$\frac{1}{\text{sm } w}=\text{sm } (C-w),$$

C being the constant of integration.

Let $w=K$, then

$$\frac{1}{\text{sm } K}=\text{sm } (C-K),$$

or

$$\text{sm } (C-K)=1.$$

Therefore C may have the value $2K$ and

$$\text{sm } (2K-w)=\frac{1}{\text{sm } w}.$$

It is evident then that

$$\text{cm } (2K-w)=-\frac{\text{cm } w}{\text{sm } w}$$

In these equations by putting $K-w$ in place of w we obtain the relations

$$\text{sm}\ (K+w)=\frac{1}{\text{cm}\ w},$$

and

$$\text{cm}\ (K+w)=-\frac{\text{sm}\ w}{\text{cm}\ w}.$$

Hence

$$\text{sm}\ (2K+w)=-\frac{\text{cm}\ w}{\text{sm}\ w},$$

and

$$\text{cm}\ (2K+w)=\frac{1}{\text{sm}\ w}.$$

Finally

$$\text{sm}\ (3K+w)=\text{sm}\ w,$$

and

$$\text{cm}\ (3K+w)=\text{cm}\ w.$$

The functions are therefore periodic functions of w, the period being $3K$.

In the expression

$$\text{sm}\ (K-w)=\text{cm}\ w,$$

put $K+w$ in place of w and we get

$$\text{sm}\ (-w)=\text{cm}\ (K+w)=-\frac{\text{sm}\ w}{\text{cm}\ w},$$

and from

$$\text{cm}\ (K-w)=\text{sm}\ w,$$

we get

$$\text{cm}\ (-w)=\text{sm}\ (K+w)=\frac{1}{\text{cm}\ w}.$$

The integral $\int_0^1 \frac{dx}{y^2}$ is many valued, since it can have any one of the values $K+3mK$, in which m is an integer, either positive or negative. By K we shall understand the result of integrating along the axis of reals from 0 to 1, just as we do in the case of the integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$.

It is thus seen that we have the following values:

$$\text{sm}\ K=1,$$

$$\text{cm}\ K=0,$$

$$\text{sm}\ 2K=\infty,$$

$$\text{cm}\ 2K=-\infty,$$

$$\text{sm}\ 3K=0,$$

$$\text{cm}\ 3K=1.$$

We therefore have the following results:

$$K=\int_0^1 \frac{dx}{y^2},$$

$$2K=\int_0^\infty \frac{dx}{y^2},$$

or

$$K=\int_1^\infty \frac{dx}{y^2},$$

also

$$2K=\int_{-\infty}^1 \frac{dy}{x^2},$$

or

$$K=\int_{-\infty}^0 \frac{dy}{x^2},$$

the integrals being taken over the real values of the variable in each case.

If we denote the roots of the equation $x^3=1$ by t, t^2, and $t^3=1$, t will have the value $-\frac{1}{2}+\frac{\sqrt{3}}{2}\,i$, in which i denotes as usual $\sqrt{-1}$. The equation $x^3+y^3=1$ is satisfied not only by sm w and cm w, but also by t sm w or t^2 sm w together with cm w. The series for sm w given on page 3 shows that t sm $w=$ sm tw and t^2 sm $w=$ sm t^2w, since sm $w=wP(w^3)$ in which $P(w^3)$ denotes an integral power series in ascending powers of w^3. On the other hand, cm w is an integral power series in ascending powers of w^3 (see p. 3), so that cm $tw=$ cm w and cm $t^2w=$ cm w.

It can now be proved that $3tK$ is a period and that $3t^2K$ is also a period of the functions. It should be noted that these two complex periods and the real period are not independent; in fact, a uniform function of a single variable can not have more than two independent periods. It is obvious that, in this case, we have $K+tK+t^2K=0$, since

$$1+t+t^2=0.$$

$$\text{sm}\ tK=t\ \text{sm}\ K=t,$$

$$\text{cm}\ tK=\text{cm}\ K=0,$$

$$\text{sm}\ t^2K=t^2\ \text{sm}\ K=t^2,$$

$$\text{cm}\ t^2K=\text{cm}\ K=0.$$

$$\operatorname{sm}(w+tK)=t\operatorname{sm}(t^2w+K)=\frac{t}{\operatorname{cm}t^2w}=\frac{t}{\operatorname{cm}w},$$

$$\operatorname{cm}(w+tK)=\operatorname{cm}(t^2w+K)=-\frac{\operatorname{sm}t^2w}{\operatorname{cm}t^2w}=-t^2\frac{\operatorname{sm}w}{\operatorname{cm}w},$$

$$\operatorname{sm}(w+2tK)=t\operatorname{sm}(t^2w+2K)=-t\frac{\operatorname{cm}t^2w}{\operatorname{sm}t^2w}=-t^2\frac{\operatorname{cm}w}{\operatorname{sm}w},$$

$$\operatorname{cm}(w+2tK)=\operatorname{cm}(t^2w+2K)=\frac{1}{\operatorname{sm}t^2w}=\frac{1}{t^2\operatorname{sm}w}=\frac{t}{\operatorname{sm}w},$$

$$\operatorname{sm}(w+3tK)=t\operatorname{sm}(t^2w+3K)=t\operatorname{sm}t^2w=\operatorname{sm}w,$$

$$\operatorname{cm}(w+3tK)=\operatorname{cm}(t^2w+3K)=\operatorname{cm}t^2w=\operatorname{cm}w.$$

In a similar way we may show that for the values with t^2 we have

$$\operatorname{sm}(w+t^2K)=\frac{t^2}{\operatorname{cm}w},$$

$$\operatorname{cm}(w+t^2K)=-t\frac{\operatorname{sm}w}{\operatorname{cm}w},$$

$$\operatorname{sm}(w+2t^2K)=-t\frac{\operatorname{cm}w}{\operatorname{sm}w},$$

$$\operatorname{cm}(w+2t^2K)=\frac{t^2}{\operatorname{sm}w},$$

and

$$\operatorname{sm}(w+3t^2K)=\operatorname{sm}w,$$

$$\operatorname{cm}(w+3t^2K)=\operatorname{cm}w.$$

If we denote $\frac{1}{2}+\frac{\sqrt{3}}{2}i$ by s, s will be a root of the equation $x^6=1$; also $s^2=t$, and $s=1+t=-t^2$. We see then that $3sK$ is a period of the functions, since it is equal to $3K+3tK$ or to $-3t^2K$. We have then

$$\operatorname{sm}(w+sK)=\operatorname{sm}(w-t^2K)=t^2\operatorname{sm}(tw-K)=-t^2\frac{\operatorname{sm}(K-tw)}{\operatorname{cm}(K-tw)}$$

$$=-t^2\frac{\operatorname{cm}tw}{\operatorname{sm}tw}=-t\frac{\operatorname{cm}w}{\operatorname{sm}w},$$

$$\operatorname{cm}(w+sK)=\operatorname{cm}(w-t^2K)=\operatorname{cm}(tw-K)=\frac{1}{\operatorname{cm}(K-tw)}$$

$$=\frac{1}{\operatorname{sm}tw}=\frac{t^2}{\operatorname{sm}w},$$

$$\operatorname{sm}(w+2sK)=\operatorname{sm}(w-2t^2K)=\operatorname{sm}(w+t^2K)=\frac{t^2}{\operatorname{cm}w},$$

$$\operatorname{cm}(w+2sK)=\operatorname{cm}(w-2t^2K)=\operatorname{cm}(w+t^2K)=-t\frac{\operatorname{sm}w}{\operatorname{cm}w},$$

$$\operatorname{sm}(w+3sK)=\operatorname{sm}(w-3t^2K)=\operatorname{sm}w,$$

$$\operatorname{cm}(w+3sK)=\operatorname{cm}(w-3t^2K)=\operatorname{cm}w.$$

These functions, being uniform functions of w with two independent periods and having no singularities except poles in the whole complex plane, are elliptic functions of w. We note that the periods do not divide naturally into halves and fourths, as is the case with the trigonometric functions and with the Jacobi and Weierstrassian elliptic functions, but into thirds corresponding to the three parts into which the curve is divided by the three collinear inflections. The curve cuts the axis of x at $x=1$ and is perpendicular to the axis with a point of inflection at the point of intersection. The same thing is true in regard to the axis of y. The line $x+y=0$ is an asymptote of the curve. The third real point of inflection is at infinity. The form of the curve is shown in Figure 1.

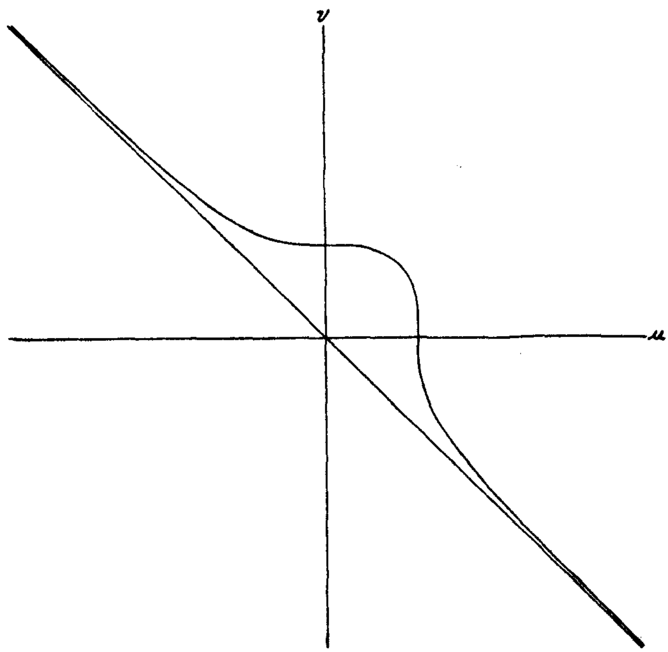

FIG. 1.—The curve of $u^3+v^3=1$

We also have the following results:

$$\operatorname{sm}(tK-w)=\frac{t}{\operatorname{cm}(-w)}=t\operatorname{cm}w,$$

$$\operatorname{cm}(tK-w)=-t^2\frac{\operatorname{sm}(-w)}{\operatorname{cm}(-w)}=t^2\operatorname{sm}w,$$

$$\operatorname{sm}(2tK-w)=-t^2\frac{\operatorname{cm}(-w)}{\operatorname{sm}(-w)}=\frac{t^2}{\operatorname{sm}w},$$

$$\operatorname{cm}(2tK-w)=\frac{t}{\operatorname{sm}(-w)}=-t\frac{\operatorname{cm}w}{\operatorname{sm}w},$$

$$\mathrm{sm}\,(t^2K-w)=t^2\,\mathrm{cm}\,w,$$

$$\mathrm{cm}\,(t^2K-w)=t\,\mathrm{sm}\,w,$$

$$\mathrm{sm}\,(2t^2K-w)=\frac{t}{\mathrm{sm}\,w},$$

$$\mathrm{cm}\,(2t^2K-w)=-t^2\,\frac{\mathrm{cm}\,w}{\mathrm{sm}\,w}.$$

Therefore we find the following relations:

$$\mathrm{sm}\,(tK-K)=\mathrm{sm}\,(tK-t^2K)=\mathrm{sm}\,(t^2K-K)=\mathrm{sm}\,(t^2K-tK)$$
$$=\mathrm{sm}\,(K-tK)=\mathrm{sm}\,(K-t^2K)=0,$$
$$\mathrm{cm}\,(tK-K)=\mathrm{cm}\,(K-t^2K)=\mathrm{cm}\,(t^2K-tK)=t^2,$$
$$\mathrm{cm}\,(K-tK)=\mathrm{cm}\,(t^2K-K)=\mathrm{cm}\,(tK-t^2K)=t,$$
$$\mathrm{sm}\,(tK-K+w)=\mathrm{sm}\,(K-t^2K+w)=\mathrm{sm}\,(t^2K-tK+w)=t\,\mathrm{sm}\,w,$$
$$\mathrm{cm}\,(tK-K+w)=\mathrm{cm}\,(K-t^2K+w)=\mathrm{cm}\,(t^2K-tK+w)=t^2\,\mathrm{cm}\,w.$$

Also

$$\mathrm{sm}\,(K+tK+t^2K)=0,$$

and

$$\mathrm{cm}\,(K+tK+t^2K)=1.$$

NUMERICAL VALUE OF K

The numerical value of K can be computed by means of the beta function. In the integral

$$K=\int_0^1\frac{dx}{(1-x^3)^{\frac{2}{3}}},$$

let

$$x=z^{\frac{1}{3}},$$

then

$$dx=\frac{1}{3}z^{-\frac{2}{3}}dz,$$

and we get

$$K=\frac{1}{3}\int_0^1 z^{-\frac{2}{3}}(1-z)^{-\frac{2}{3}}\,dz=\frac{1}{3}B\left(\frac{1}{3},\frac{1}{3}\right)=\frac{1}{3}\frac{\left[\Gamma\left(\frac{1}{3}\right)\right]^2}{\Gamma\left(\frac{2}{3}\right)}.$$

By use of Legendre's table of the gamma functions we can compute the following values:

$$\log K=0.24714775222484,$$

$$K=1.76663875,$$

$$K=\frac{\left[\Gamma\left(\frac{1}{3}\right)\right]^3}{2\sqrt{3}\pi}.$$

$$\log\,\Gamma\left(\frac{1}{3}\right)=0.42796274931426.$$

GEOMETRICAL INTERPRETATION OF w

The quantity w has a geometrical intrepretation when x and y are confined to real values. In the curve $x^3+y^3=1$ we have $x=\text{sm } w$ and $y=\text{cm } w$. The polar element of area is $\frac{1}{2}(xdy-ydx)$; that is, $\frac{1}{2}(-\text{sm}^3 w-\text{cm}^3 w)\,dw$ or $-\frac{1}{2}dw$. If the angle θ of the radius vector is to turn counterclockwise or in the positive direction, we should let $y=\text{sm } w$ and $x=\text{cm } w$; the element of area which is now traced out in the positive direction by the radius vector is equal to $+\frac{1}{2}dw$. The quantity w is therefore twice the polar area from $w=0$ to the given value of w. The quantity K is twice the area of the part included in the first quadrant, or twice the area between the curve and its asymptote in either the second or the fourth quadrant.

ZEROS AND INFINITIES OF THE FUNCTIONS

Let us take as fundamental periods for the functions sm w and cm w the values $3K$ and $3tK$. The period parallelogram will then be such as is illustrated in Figure 2. This period parallelogram is obviously

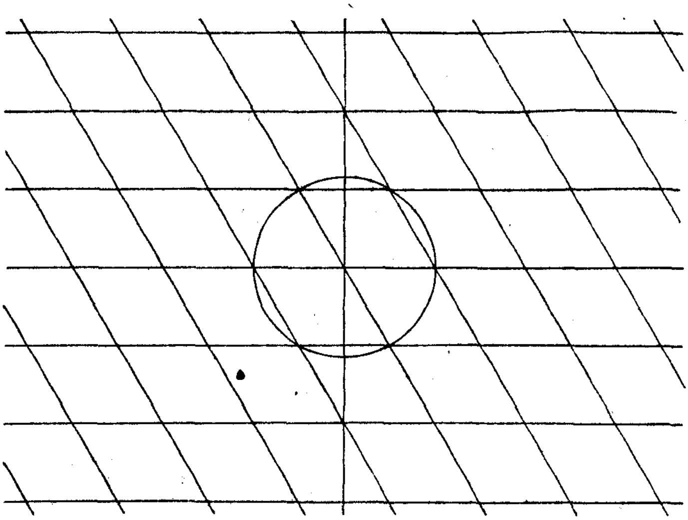

FIG. 2—The complex plane divided into period parallelograms

a rhombus, since the absolute values of the two periods are equal; that is, tK is just K turned about the origin through an angle of 120°.

We shall now investigate the zeros and infinities of the functions in this fundamental parallelogram. Since $\frac{d}{dw}\text{sm } w=\text{cm}^2 w$ and cm w is given in terms of sm w by a cubic equation, the number of really distinct arguments for which sm w has a given value is three, or sm w is an elliptic function of w of the third order, and hence there must be three zeros and three infinities in the period parallelogram. The same reasoning is applicable to cm w. If $x=\text{sm } w$, $\frac{dw}{dx}$ has three values and therefore w has three noncongruent values for each value of x. We know, therefore, that both sm w and cm w have three zeros and three infinities that are noncongruent in the fundamental parallelogram. For sm w we have the three zeros $w=0$, $2K+tK$, and $K+2tK$; moreover, these three values of w are the only values in the period parallelogram for which sm $w=0$. According to the theory of elliptic functions the sum of the w values for which the function sm w becomes zero can differ from the sum of the w values for which it becomes infinite only by integral multiples of the periods. That is, $\Sigma w_0 - \Sigma w_\infty$ must be congruent to zero modulus $3K$, $3tK$. For the zeros of sm w, we have $\Sigma w_0 = 3K+3tK$; that is, this sum is itself congruent to zero modulus $3K$, $3tK$; therefore, for sm w, we must have Σw_∞ congruent to zero modulus $3K$, $3tK$. We find that the infinities of sm w are given for $w=2K$, $2tK$, and $K+tK$; giving as it should the sum $3K+3tK$, congruent to zero, modulus $3K$, $3tK$. The zeros of cm w are given by $w=K$, tK, and $2K+2tK$. The infinities of cm w are the same as those of sm w. Therefore, with cm w, we have each of the sums of the w's congruent to zero, modulus $3K$, $3tK$.

The theory of elliptic functions also requires the sum of the residues of the infinities to be equal to zero. A residue is the coefficient of the term that has the first power of w in the denominator of the development in series in the neighborhood of the point. For sm w we have

$$\text{sm}\ (w+2K) = -\frac{\text{cm}\ w}{\text{sm}\ w} = -\frac{1}{w} + P\ (w),$$

$$\text{sm}\ (w+2tK) = -t^2\frac{\text{cm}\ w}{\text{sm}\ w} = -\frac{t^2}{w} + t^2\ P\ (w),$$

$$\text{sm}\ (w+K+tK) = -t\frac{\text{cm}\ w}{\text{sm}\ w} = -\frac{t}{w} + tP\ (w),$$

in which $P\ (w)$ is an integral power series in the variable w. We note that each of these infinities is of the first order and that the residues are -1, $-t^2$, and $-t$ with the sum $-1-t^2-t=0$ as it should be. In the same way it can be seen that the zeros of cm w are of the first order, and that the residues of cm w are, respectively, 1, t, and t^2 with their sum equal to zero. Of course, a priori consideration would show that each of these infinities must be of the first order, since an infinity of the second order must be counted as two infinities. Since sm w and cm w each have three zeros of the first order, they can not have more than three infinities in the fundamental period parallelogram. We have found that there are three values of w for which each of the functions becomes infinite; therefore each of these infinities must be of the first order, since the number of zeros must equal the number of infinities in the fundamental parallelogram.

ADDITION THEOREM

Since these functions are uniform elliptic functions, they must have addition theorems. The curve $x^3+y^3=1$ has no node and no cusp and its deficiency is equal to unity. (See fig. 1.) Hence, the integral $\int \frac{dx}{y^2}$ is an Abelian integral of the first kind which has no points of discontinuity. Thus

$$\int_0^{x_1} \frac{dx}{y^2} + \int_0^{x_2} \frac{dx}{y^2} + \int_0^{x_3} \frac{dx}{y^2} = \text{constant},$$

if x_1, x_2, and x_3 be values of x that satisfy the equations $Ax+By+C=0$ and $x^3+y^3=1$. Now, in the equation

$$A \text{ sm } w+B \text{ cm } w+C=0,$$

the left-hand member is an elliptic function of w of the third order. It has, therefore, three zeros and three infinities in the fundamental period parallelogram. We know that the sum of the w's for which it becomes infinite is congruent to zero modulus $3K$, $3tK$; consequently, the sum of the w's for which it becomes zero must also be congruent to zero modulus $3K$, $3tK$.

Let w_1, w_2, and w_3 be the three arguments. Then

$$A \text{ sm } w_1+B \text{ cm } w_1+C=0,$$

$$A \text{ sm } w_2+B \text{ cm } w_2+C=0,$$

$$A \text{ sm } w_3+B \text{ cm } w_3+C=0,$$

$$\begin{vmatrix} \text{sm } w_1, & \text{cm } w_1, & 1 \\ \text{sm } w_2, & \text{cm } w_2, & 1 \\ \text{sm } w_3, & \text{cm } w_3, & 1 \end{vmatrix} =0$$

with $w_1+w_2+w_3 \equiv 0$, mod $3K$, $3tK$.

We must now determine sm w_3 and cm w_3 so that they satisfy the determinant and also the equation $\text{sm}^3\, w_3+\text{cm}^3\, w_3=1$. To shorten the work, let us denote sm w_1 by s_1, cm w_1 by c_1, sm w_2 by s_2, cm w_2 by c_2, sm w_3 by s_3, and cm w_3 by c_3.

If

$$s_3=\frac{Ps_1+Qs_2}{P+Q} \text{ and } c_3=\frac{Pc_1+Qc_2}{P+Q},$$

the determinant will be satisfied. The value of the ratio of P and Q can be determined by substitution in the equation $s_3{}^3+c_3{}^3=1$.

$$\left(\frac{Ps_1+Qs_2}{P+Q}\right)^3+\left(\frac{Pc_1+Qc_2}{P+Q}\right)^3=1,$$

or

$$P^3s_1{}^3+3P^2Qs_1{}^2s_2+3PQ^2s_1s_2{}^2+Q^3s_2{}^3+P^3c_1{}^3+3P^2Qc_1{}^2c_2$$

$$+3PQ^2c_1c_2{}^2+Q^3c_2{}^3=P^3+3P^2Q+3PQ^2+Q^3.$$

On canceling the equal terms and rearranging we get

$$P\ (s_1{}^2s_2+c_1{}^2c_2-1)=-Q\ (s_1s_2{}^2+c_1c_2{}^2-1),$$

or

$$-\frac{P}{Q}=\frac{s_1s_2{}^2+c_1c_2{}^2-1}{s_1{}^2s_2+c_1{}^2c_2-1}=\frac{s_2c_2+s_1c_2+s_2c_1}{s_1c_1+s_2c_1+s_1c_2}.$$

The second expression for $-\frac{P}{Q}$ can be shown to be equal to the first by cross multiplication; that is,

$$(s_1s_2{}^2+c_1c_2{}^2-1)\ (s_1c_1+s_2c_1+s_1c_2)=(s_1{}^2s_2+c_1{}^2c_2-1)\ (s_2c_2+s_1c_2+s_2c_1).$$

By substituting the second value of $\frac{P}{Q}$ and reducing, we get

$$s_3 = -\frac{s_1^2 c_2 - s_2^2 c_1}{s_1 c_1 - s_2 c_2},$$

and

$$c_3 = \frac{s_1 c_2^2 - s_2 c_1^2}{s_1 c_1 - s_2 c_2}.$$

But

$$\text{sm } (w_1 + w_2) = \text{sm } (-w_3) = -\frac{s_3}{c_3},$$

and

$$\text{cm } (w_1 + w_2) = \text{cm } (-w_3) = \frac{1}{c_3},$$

therefore

$$\text{sm } (u+v) = \frac{\text{sm}^2 u \text{ cm } v - \text{sm}^2 v \text{ cm } u}{\text{sm } u \text{ cm}^2 v - \text{sm } v \text{ cm}^2 u},$$

and

$$\text{cm } (u+v) = \frac{\text{sm } u \text{ cm } u - \text{sm } v \text{ cm } v}{\text{sm } u \text{ cm}^2 v - \text{sm } v \text{ cm}^2 u}.$$

These addition formulas can also be verified by partial differentiation. If $\frac{\partial}{\partial u} f(u, v) = \frac{\partial}{\partial v} f(u, v)$, the function $f(u, v)$ must be a function of $u+v$. Denoting sm u by s_1 and sm v by s_2, etc., we find

$$\frac{\partial}{\partial u} \frac{s_1^2 c_2 - s_2^2 c_1}{s_1 c_2^2 - s_2 c_1^2} = \left(\frac{s_1 c_1 - s_2 c_2}{s_1 c_2^2 - s_2 c_1^2} \right)^2 = \frac{\partial}{\partial v} \frac{s_1^2 c_2 - s_2^2 c_1}{s_1 c_2^2 - s_2 c_1^2}.$$

Hence, the given function is a function of $u+v$. Moreover, it becomes sm u when $v=0$; therefore it represents the function sm $(u+v)$. By the same process of reasoning the formula for cm $(u+v)$ can be shown to be correct.

Letting v become negative and substituting the value of sm $(-v)$ and cm $(-v)$ (see p. 5), we obtain

$$\text{sm } (u-v) = \frac{\text{sm}^2 u \text{ cm } v - \text{sm}^2 v \text{ cm } u}{\text{sm } u + \text{sm } v \text{ cm } v \text{ cm}^2 u},$$

and

$$\text{cm } (u-v) = \frac{\text{sm } v + \text{sm } u \text{ cm } u \text{ cm}^2 v}{\text{sm } u + \text{sm } v \text{ cm } v \text{ cm}^2 u}.$$

By multiplication and use of the fundamental algebraic equation $\text{sm}^3 u + \text{cm}^3 u = 1$, we can prove the identity

$$(s_1 + s_2 c_2 c_1^2)\ (s_1 c_2^2 - s_2 c_1^2) = (s_1^2 c_2 - s_2^2 c_1)\ (c_2 + s_1 c_1 s_2^2).$$

This gives a second form for the addition formula for sm $(u+v)$ from which a third form can be obtained by the interchange of u and v. Hence we have

$$\text{sm } (u+v) = \frac{\text{sm}^2 u \text{ cm } v - \text{sm}^2 v \text{ cm } u}{\text{sm } u \text{ cm}^2 v - \text{sm } v \text{ cm}^2 u}$$

$$= \frac{\text{sm } u + \text{sm } v \text{ cm } v \text{ cm}^2 u}{\text{cm } v + \text{sm } u \text{ cm } u \text{ sm}^2 v} = \frac{\text{sm } v + \text{sm } u \text{ cm } u \text{ cm}^2 v}{\text{cm } u + \text{sm } v \text{ cm } v \text{ sm}^2 u}.$$

These formulas also give

$$\operatorname{sm}(u-v)=\frac{\operatorname{sm}^2 u \operatorname{cm} v-\operatorname{sm}^2 v \operatorname{cm} u}{\operatorname{sm} u+\operatorname{sm} v \operatorname{cm} v \operatorname{cm}^2 u}$$

$$=\frac{\operatorname{sm} u \operatorname{cm}^2 v-\operatorname{sm} v \operatorname{cm}^2 u}{\operatorname{cm} v+\operatorname{sm} u \operatorname{cm} u \operatorname{sm}^2 v}=\frac{\operatorname{sm} u \operatorname{cm} u-\operatorname{sm} v \operatorname{cm} v}{\operatorname{cm} u \operatorname{cm}^2 v-\operatorname{sm}^2 u \operatorname{sm} v}.$$

By using the relation $\operatorname{sm}(K-u-v)=\operatorname{cm}(u+v)$ we can derive three similar formulas for $\operatorname{cm}(u+v)$ by replacing $\operatorname{sm} u$ by $\operatorname{sm}(K-u)=\operatorname{cm} u$ and $\operatorname{cm} u$ by $\operatorname{cm}(K-u)=\operatorname{sm} u$ and by letting v become negative; or we can merely make the substitution for u in the addition formula for $\operatorname{sm}(u-v)$. By this means we get the following three forms:

$$\operatorname{cm}(u+v)=\frac{\operatorname{sm} u \operatorname{cm} u-\operatorname{sm} v \operatorname{cm} v}{\operatorname{sm} u \operatorname{cm}^2 v-\operatorname{sm} v \operatorname{cm}^2 u}$$

$$=\frac{\operatorname{cm} u \operatorname{cm}^2 v-\operatorname{sm}^2 u \operatorname{sm} v}{\operatorname{cm} v+\operatorname{sm} u \operatorname{cm} u \operatorname{sm}^2 v}=\frac{\operatorname{cm} v \operatorname{cm}^2 u-\operatorname{sm}^2 v \operatorname{sm} u}{\operatorname{cm} u+\operatorname{sm} v \operatorname{cm} v \operatorname{sm}^2 u}.$$

Also

$$\operatorname{cm}(u-v)=\frac{\operatorname{sm} v+\operatorname{sm} u \operatorname{cm} u \operatorname{cm}^2 v}{\operatorname{sm} u+\operatorname{sm} v \operatorname{cm} v \operatorname{cm}^2 u}$$

$$=\frac{\operatorname{cm} u+\operatorname{sm} v \operatorname{cm} v \operatorname{sm}^2 u}{\operatorname{cm} v+\operatorname{sm} u \operatorname{cm} u \operatorname{sm}^2 v}=\frac{\operatorname{cm} v \operatorname{cm}^2 u-\operatorname{sm}^2 v \operatorname{sm} u}{\operatorname{cm} u \operatorname{cm}^2 v-\operatorname{sm}^2 u \operatorname{sm} v}.$$

FORMULAS RESULTING FROM THE ADDITION THEOREM

Let $\operatorname{sm} u=s_1$, $\operatorname{cm} u=c_1$, $\operatorname{sm} v=s_2$, and $\operatorname{cm} v=c_2$ and we get

$$\operatorname{sm}(u+v)\operatorname{sm}(u-v)=\frac{s_1^2c_2-s_2^2c_1}{c_2+s_1c_1s_2^2},$$

$$\operatorname{cm}(u+v)\operatorname{cm}(u-v)=\frac{c_2c_1^2-s_2^2s_1}{c_2+s_1c_1s_2^2},$$

$$\operatorname{sm}(u+v)\operatorname{cm}(u-v)=\frac{s_2+s_1c_1c_2^2}{c_2+s_1c_1s_2^2},$$

$$\operatorname{cm}(u+v)\operatorname{sm}(u-v)=\frac{s_1c_1-s_2c_2}{c_2+s_1c_1s_2^2},$$

$$\frac{\operatorname{sm}(u+v)}{\operatorname{sm}(u-v)}=\frac{s_1+s_2c_2c_1^2}{s_1c_2^2-s_2c_1^2},$$

$$\frac{\operatorname{cm}(u+v)}{\operatorname{sm}(u-v)}=\frac{c_1c_2^2-s_1^2s_2}{s_1c_2^2-s_2c_1^2},$$

$$\operatorname{sm}(u+v)-\operatorname{sm}(u-v)=\frac{(1+c_2)\,(s_1-s_1c_2+s_2c_1^2)}{c_2+s_1c_1s_2^2},$$

$$\operatorname{cm}(u-v)-\operatorname{cm}(u+v)=\frac{(1+c_2)\,(c_1-c_1c_2+s_2s_1^2)}{c_2+s_1c_1s_2^2},$$

$$\operatorname{sm}(u+v)-\operatorname{cm}(u-v)=\frac{(s_1-c_1)\,(1-s_2c_2c_1-s_1s_2c_2)}{c_2+s_1c_1s_2^2}$$

$$\operatorname{cm}(u+v)-\operatorname{sm}(u-v)=\frac{(c_1-s_1)\,(c_2^2+s_2c_1+s_2s_1)}{c_2+s_1c_1s_2^2}.$$

We also obtain the following formulas in which sm $a=s$, cm $a=c$, sm $u=s_1$, cm $u=c_1$, sm $v=s_2$, and cm $v=c_2$.

$$\text{cm }(a-u)+\text{sm }(a-u)\text{ cm }(a+u)\text{ sm }(a+v)\text{ sm }(a-v)$$
$$=\frac{c\,(1+s^3)\,(c_2+s_1c_1s_2^2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)},$$

$$\text{sm }(a+u)+\text{cm }(a+u)\text{ sm }(a-u)\text{ cm }(a+v)\text{ sm }(a-v)$$
$$=\frac{s\,(1+c^3)\,(c_2+s_1c_1s_2^2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)},$$

$$\text{cm }(a-u)\text{ cm }(a-v)\text{ cm }(a+v)-\text{sm }(a+u)\text{ sm }(a+v)\text{ sm }(a-v)$$
$$=\frac{(c^3-s^3)\,(c_2+s_1\,c_1\,s_2^2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)},$$

$$\text{sm }(a+u)\text{ cm }(a-v)-\text{sm }(a+v)\text{ cm }(a-u)$$
$$=\frac{(c^3-s^3)\,(s_1c_1-s_2c_2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)},$$

$$\text{sm }(a-v)\text{ cm }(a-u)\text{ cm }(a+v)-\text{sm }(a-u)\text{ cm }(a-v)\text{ cm }(a+u)$$
$$=\frac{c\,(1+s^3)\,(s_1c_1-s_2c_2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)},$$

$$\text{sm }(a+u)\text{ sm }(a-v)\text{ cm }(a+v)-\text{sm }(a+v)\text{ sm }(a-u)\text{ cm }(a+u)$$
$$=\frac{s\,(1+c^3)\,(s_1c_1-s_2c_2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)},$$

If we divide any one of these equations by any other with or without changing the sign of either u or v, the right-hand side will be expressible in terms of $2a$, $u+v$, and $u-v$; for we have

$$\text{sm }2a=\frac{s\,(1+c^3)}{c\,(1+s^3)}\text{ and cm }2a=\frac{c^3-s^3}{c\,(1+s^3)}.$$

This process gives relations among the sm's and cm's of u_1, u_2, u_3, u_4, u_1+u_2, u_1+u_3, and u_1+u_4, if we take the sum of the four u's equal to zero; for we may take $u_1=a+u$, $u_2=a-u$, $u_3=-a+v$, and $u_4=-a-v$ as the four arguments the sum of which is identically equal to zero.

As an example we have

$$\text{cm }(a+u)-\text{sm }(a-u)\text{ sm }(-a+v)\text{ sm }(-a-v)$$
$$=\frac{\text{cm }(a+u)\text{ cm }(a+v)\text{ cm }(a-v)-\text{sm }(a-u)\text{ sm }(a+v)\text{ sm }(a-v)}{\text{cm }(a+v)\text{ cm }(a-v)}$$
$$=\frac{(c^3-s^3)\,(c_1^2c_2-s_1s_2^2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)}\cdot\frac{c_2+scs_2^2}{c^2c_2-ss_2^2}=\frac{(c^3-s^3)\,(c_1^2c_2-s_1s_2^2)}{(c_1+scs_1^2)\,(c^2c_2-ss_2^2)},$$

$$\text{cm }(a-u)-\text{sm }(a+u)\text{ sm }(-a+v)\text{ sm }(-a-v)$$
$$=\frac{\text{cm }(a-u)\text{ cm }(a+v)\text{ cm }(a-v)-\text{sm }(a+u)\text{ sm }(a+v)\text{ sm }(a-v)}{\text{cm }(a+v)\text{ cm }(a-v)}$$
$$=\frac{(c^3-s^3)\,(c_2+s_1c_1s_2^2)}{(c_1+scs_1^2)\,(c_2+scs_2^2)}\cdot\frac{c_2+scs_2^2}{c^2c_2-ss_2^2}=\frac{(c^3-s^3)\,(c_2+s_1c_1s_2^2)}{(c_1+scs_1^2)\,(c^2c_2-ss_2^2)}.$$

By division we get

$$\frac{(c^3-s^3)\ (c_1^2c_2-s_1s_2^2)}{(c_1+scs_1^2)\ (c^2c_2-ss_2^2)}\cdot\frac{(c_1+scs_1^2)\ (c^2c_2-ss_2^2)}{(c^3-s^3)\ (c_2+s_1c_1s_2^2)}$$

$$=\frac{\text{cm }(a+u)-\text{sm }(a-u)\text{ sm }(-a+v)\text{ sm }(-a-v)}{\text{cm }(a-u)-\text{sm }(a+u)\text{ sm }(-a+v)\text{ sm }(-a-v)},$$

or

$$\frac{c_1^2c_2-s_1s_2^2}{c_2+s_1c_1s_2^2}=\frac{\text{cm }u_1-\text{sm }u_2\text{ sm }u_3\text{ sm }u_4}{\text{cm }u_2-\text{sm }u_1\text{ sm }u_3\text{ sm }u_4}.$$

But

$$\frac{c_1^2c_2-s_1s_2^2}{c_2+s_1c_1s_2^2}=\text{cm }(u+v)\text{ cm }(u-v),$$

also

$$u+v=u_1+u_3,$$

$$u-v=u_1+u_4.$$

Therefore

$$\text{cm }(u_1+u_3)\text{ cm }(u_1+u_4)=\frac{\text{cm }u_1-\text{sm }u_2\text{ sm }u_3\text{ sm }u_4}{\text{cm }u_2-\text{sm }u_1\text{ sm }u_3\text{ sm }u_4}.$$

A great variety of such formulas could be derived.

If $u_1+u_2+u_3=0$, we have

$$s_3=-\frac{s_1^2c_2-s_2^2c_1}{s_1c_1-s_2c_2},$$

$$c_3=\frac{s_1c_2^2-s_2c_1^2}{s_1c_1-s_2c_2}.$$

Hence, we have

$$c_1c_2c_3+s_1s_2s_3=1, \qquad \text{(a)}$$

$$s_1c_2+s_2c_3+s_3c_1=0, \qquad \text{(b)}$$

$$s_2c_1+s_3c_2+s_1c_3=0. \qquad \text{(c)}$$

Of these relations (c) comes from (a) by putting $-K-u_1$, $K-u_2$, and $-u_3$ for u_1, u_2, and u_3, respectively, of which the sum of the arguments is still equal to zero.

From (b) and (c) we get

$$\frac{s_1}{c_1^2-c_2c_3}=\frac{s_2}{c_2^2-c_3c_1}=\frac{s_3}{c_3^2-c_1c_2}=\left\{\frac{s_1^2-s_2s_3}{c_1(c_1^3+c_2^3+c_3^3-3c_1c_2c_3)}\right\}^{\frac{1}{2}}$$

$$=\frac{s_1c_1+s_2c_2+s_3c_3}{c_1^3+c_2^3+c_3^3-3c_1c_2c_3}=\left\{\frac{s_1^3+s_2^3+s_3^3-3s_1s_2s_3}{(s_1c_1+s_2c_2+s_3c_3)\ (c_1^3+c_2^3+c_3^3-3c_1c_2c_3)}\right\}^{\frac{1}{2}}$$

$$=\frac{s_1^3+s_2^3+s_3^3-3s_1s_2s_3}{(s_1c_1+s_2c_2+s_3c_3)^2}.$$

Also we have as proof of the last value the following analysis:

$$(s_1+ts_2+t^2s_3)\ (c_1+t^2c_2+tc_3)=s_1c_1+s_2c_2+s_3c_3,$$

$$(s_1+t^2s_2+ts_3)\ (c_1+tc_2+t^2c_3)=s_1c_1+s_2c_2+s_3c_3,$$

$$(s_1+s_2+s_3)\ (c_1+c_2+c_3)=s_1c_1+s_2c_2+s_3c_3,$$

therefore we get

$$(s_1^3+s_2^3+s_3^3-3s_1s_2s_3)\ (c_1^3+c_2^3+c_3^3-3c_1c_2c_3)=(s_1+ts_2+t^2s_3)\ (s_1+t^2s_2+ts_3)\ (s_1+s_2+s_3)\ (c_1+t^2c_2+tc_3)\ (c_1+tc_2+t^2c_3)\ (c_1+c_2+c_3)=(s_1c_1+s_2c_2+s_3c_3)^3.$$

FORMULAS FOR cm u EQUAL TO A CONSTANT

If we let cm u assume a certain value, we shall have cm $u_1=$ cm u_2 $=$ cm u_3 and sm u will be determined by the cubic equation

$$\text{sm}^3\,u=1-\text{cm}^3\,u.$$

Therefore,

$$\begin{aligned} &s_1+s_2+s_3=0,\\ &s_1s_2+s_2s_3+s_3s_1=0,\\ &s_1s_2s_3=1-c^3=s_1^3,\\ &s_2s_3=s_1^2,\\ &s_2+s_3=-s_1,\\ &s_2-s_3=\pm\sqrt{3}is_1,\\ &s_2=ts_1 \text{ or } t^2s_1,\\ &s_3=t^2s_1 \text{ or } ts_1. \end{aligned}$$

Also,

$$\begin{vmatrix} s_1, & c_1, & 1\\ s_2, & c_2, & 1\\ s_3, & c_3, & 1 \end{vmatrix}=c\begin{vmatrix} s_1, & 1, & 1\\ s_2, & 1, & 1\\ s_3, & 1, & 1 \end{vmatrix}=0.$$

Therefore, $u_1+u_2+u_3$ is congruent to zero modulus $3K$, $3tK$, since the three points lie on a straight line. If we take the sum as equal to zero and not merely congruent to zero, we shall have

$$\begin{aligned} &u_2=tu_1 \text{ or } t^2u_1,\\ &u_3=t^2u_1 \text{ or } tu_1. \end{aligned}$$

Let us use the first values since the second pair merely interchange u_2 and u_3. We have then

$$u_1+u_2+u_3=u_1+tu_1+t^2u_1=u_1(1+t+t^2)=0.$$

SIMILAR FORMULAS FOR sm u EQUAL TO A CONSTANT

On the other hand if we wish to let sm u equal a constant, we may take the relation

$$\text{cm}\ (K-u)=\text{sm}\ u,$$

and

$$\text{sm}\ (K-u)=\text{cm}\ u.$$

We have now

$$s_1=s_2=s_3,$$

and

$$\begin{aligned} &c_1+c_2+c_3=0,\\ &c_1c_2+c_2c_3+c_1c_3=0,\\ &c_1c_2c_3=1-s^3=c_1^3, \end{aligned}$$

hence,

$$\begin{aligned} &c_2=tc_1,\\ &c_3=t^2c_1. \end{aligned}$$

We shall then have

$$u'_1 = K - u_1,$$
$$u'_2 = K - tu_1,$$
$$u'_3 = K - t^2u_1,$$

and the sum is congruent to zero modulus $3K$.
To make the sum equal to zero, we may take

$$u'_1 = K - u_1,$$
$$u'_2 = K - tu_1,$$
$$u'_3 = -2K - t^2u_1,$$

the sum of which is evidently equal to zero. The equation $K + tK + t^2K$ is a particular case of cm u equaling a constant, namely, that in which cm u is equal to zero.

FURTHER ADDITION FORMULAS

We can now derive some new forms of the addition formulas. Let us take the case of $c_2 = \text{cm } v = \text{constant}$; then sm $v_1 = s_2$, sm $v_2 = ts_2$, sm $v_3 = t^2s_2$. Let us denote sm u by s_1 and cm u by c_1. We have then

$$(s_1c_2^2 - ts_2c_1^2)\ (s_1c_2^2 - t^2s_2c_1^2) = s_1^2\ c_2^4 + s_1s_2\ c_1^2c_2^2 + c_1^4s_2^2,$$

$$\text{sm}(u+v) = \frac{s_1^2c_2 - s_2^2c_1}{s_1c_2^2 - s_2c_1^2} \cdot \frac{s_1^2c_2^4 + s_1s_2c_1^2c_2^2 + c_1^4s_2^2}{s_1^2c_2^4 + s_1s_2c_1^2c_2^2 + c_1^4s_2^2}$$

$$= \frac{(c_2^3 - c_1^3)\ (s_1c_2^2 + s_2c_1^2 - s_1^2s_2^2\ c_1c_2)}{(c_2^3 - c_1^3)\ (c_1^3 + s_1^3c_2^3)}$$

$$= \frac{s_1c_2^2 + s_2c_1^2 - s_1^2s_2^2c_1c_2}{c_1^3 + s_1^3c_2^3},$$

$$\text{sm}(2K - u - v) = \text{sm}\ (K - u + K - v) = \frac{1}{\text{sm}\ (u+v)},$$

$$\text{sm}\ (u+v) = \frac{s_1^3 + c_1^3s_2^3}{c_1s_2^2 + c_2s_1^2 - s_1s_2c_1^2c_2^2}.$$

By other similar transformations we get finally the six forms, as follows:

$$\text{sm}\ (u+v) = \frac{s_1c_2^2 + s_2c_1^2 - s_1^2s_2^2c_1c_2}{c_1^3 + s_1^3c_2^3} = \frac{s_1^3 + c_1^3s_2^3}{c_1s_2^2 + c_2s_1^2 - s_1s_2c_1^2c_2^2}$$

$$= \frac{1 - c_1^3\ c_2^3}{s_1^2c_2 + s_2^2c_1 - s_1s_2\ c_1^2c_2^2} = \frac{s_1c_2^2 + s_2c_1^2 - s_1^2s_2^2c_1c_2}{1 - s_1^3s_2^3}$$

$$= \frac{s_1c_1\ (1 + c_2^3) + s_2c_2\ (1 + c_1^3)}{2c_1c_2 + s_1s_2\ (s_1c_2^2 + s_2c_1^2)} = \frac{(s_1c_1 + s_2c_2)\ c_1c_2 + s_1^2s_2^2}{(s_1c_1 + s_2c_2)\ s_1s_2 + c_1^2c_2^2}.$$

Also we have the corresponding formulas for cm $(u+v)$, as follows:

$$\text{cm}\ (u+v) = \frac{c_1c_2 - s_2s_1^2c_2^2 - c_1^2s_2^2s_1}{c_1^3 + s_1^3c_2^3} = \frac{c_2^3 - s_1^3}{c_1^2c_2^2 + s_1s_2^2c_2 + s_1^2s_2c_1}$$

$$= \frac{s_2^2c_2c_1^2 + s_1^2c_1c_2^2 - s_1s_2}{s_1^2c_2 + s_2^2c_1 - s_1s_2c_1^2c_2^2} = \frac{c_1c_2 - s_1^2s_2c_2^2 - s_1s_2^2c_1^2}{1 - s_1^3s_2^3}$$

$$= \frac{2c_1^2c_2^2 - s_1s_2\ (s_1c_1 + s_2c_2)}{2c_1c_2 + s_1s_2\ (s_1c_2^2 + s_2c_1^2)} = \frac{(s_1^2c_2 + s_2^2c_1)\ c_1c_2 - s_1s_2}{s_1^2c_2 + s_2^2c_1 - s_1s_2c_1^2c_2^2}.$$

In all of these formulas we may interchange u and v which would give 12 formulas from the 6. Also by substituting $-v$ for v we can get formulas for $u-v$ or by substituting $K-u$ in a formula for sm $(u+v)$ we can get the corresponding formula for cm $(u-v)$, since sm $(K-u+v)=$ cm $(u-v)$; also cm $(K-u+v)=$ sm $(u-v)$ and we can make a similar substitution in the formulas for cm $(u+v)$ and get the formulas for sm $(u-v)$.

FURTHER RESULTS OF THE ADDITION FORMULAS

By use of these formulas we derive the following results:

$$\text{sm}\,(u+v)+\text{sm}\,(u+tv)+\text{sm}\,(u+t^2v)=\frac{3s_1c_2^2}{c_1^3+s_1^3c_2^3},$$

$$\text{sm}\,(u+tv)\,\text{sm}\,(u+t^2v)+\text{sm}\,(u+t^2v)\,\text{sm}\,(u+v)+\text{sm}\,(u+v)\,\text{sm}\,(u+tv)=\frac{3c_2s_1^2}{c_1^3+s_1^3c_2^3},$$

$$\text{sm}\,(u+v)\,\text{sm}\,(u+tv)\,\text{sm}\,(u+t^2v)=\frac{1-c_1^3c_2^3}{c_1^3+s_1^3c_2^3},$$

$$\text{cm}\,(u+v)+\text{cm}\,(u+tv)+\text{cm}\,(u+t^2v)=\frac{3c_1c_2}{c_1^3+s_1^3c_2^3},$$

$$\text{cm}\,(u+tv)\,\text{cm}\,(u+t^2v)+\text{cm}\,(u+t^2v)\,\text{cm}\,(u+v)+\text{cm}\,(u+v)\,\text{cm}\,(u+tv)=\frac{3c_1^2c_2^2}{c_1^3+s_1^3c_2^3},$$

$$\text{cm}\,(u+v)\,\text{cm}\,(u+tv)\,\text{cm}\,(u+t^2v)=\frac{c_2^3-s_1^3}{c_1^3+s_1^3c_2^3},$$

$$\text{sm}\,(u-v)+\text{sm}\,(u-tv)+\text{sm}\,(u-t^2v)=\frac{3s_1c_2}{s_1^3+c_1^3c_2^3},$$

$$\text{sm}\,(u-tv)\,\text{sm}\,(u-t^2v)+\text{sm}\,(u-t^2v)\,\text{sm}\,(u-v)+\text{sm}\,(u-v)\,\text{sm}\,(u-tv)=\frac{3c_2^2s_1^2}{s_1^3+c_1^3c_2^3},$$

$$\text{sm}\,(u-v)\,\text{sm}\,(u-tv)\,\text{sm}\,(u-t^2v)=\frac{c_2^3-c_1^3}{s_1^3+c_1^3c_2^3},$$

$$\text{cm}\,(u-v)+\text{cm}\,(u-tv)+\text{cm}\,(u-t^2v)=\frac{3c_1c_2^2}{s_1^3+c_1^3c_2^3},$$

$$\text{cm}\,(u-tv)\,\text{cm}\,(u-t^2v)+\text{cm}\,(u-t^2v)\,\text{cm}\,(u-v)+\text{cm}\,(u-v)\,\text{cm}\,(u-tv)=\frac{3c_2c_1^2}{s_1^3+c_1^3c_2^3},$$

$$\text{cm}\,(u-v)\,\text{cm}\,(u-tv)\,\text{cm}\,(u-t^2v)=\frac{1-s_1^3c_2^3}{s_1^3+c_1^3c_2^3},$$

$$\text{sm}\,(u+v)\,\text{cm}\,(u+tv)\,\text{cm}\,(u+t^2v)+\text{cm}\,(u+v)\,\text{sm}\,(u+tv)\,\text{cm}\,(u+t^2v)+\text{cm}\,(u+v)\,\text{cm}\,(u+tv)\,\text{sm}\,(u+t^2v)=\frac{3s_1c_1^2c_2}{c_1^3+s_1^3c_2^3},$$

$$\text{cm}(u+v)\,\text{sm}(u+tv)\,\text{sm}(u+t^2v)+\text{sm}(u+v)\,\text{cm}(u+tv)\,\text{sm}(u+t^2v)$$
$$+\text{sm}(u+v)\,\text{sm}(u+tv)\,\text{cm}(u+t^2v)=\frac{3s_1^2c_1c_2^2}{c_1^3+s_1^3c_2^3},$$

$$\text{sm}(u+tv)\,\text{cm}(u+t^2v)+\text{sm}(u+t^2v)\,\text{cm}(u+tv)$$
$$+\text{sm}(u+v)\,\text{cm}(u+tv)+\text{sm}(u+v)\,\text{cm}(u+t^2v)$$
$$+\text{cm}(u+v)\,\text{sm}(u+tv)+\text{cm}(u+v)\,\text{sm}(u+t^2v)$$
$$=\frac{3s_1c_1(1+c_2^3)}{c_1^3+s_1^3c_2^3}.$$

In the last three formulas by changing v to $-v$ we can derive the corresponding formulas for $u-v$, $u-tv$, and $u-t^2v$.

To derive formulas for which sm v is a constant we can substitute $K-v$, $K-tv$, and $K-t^2v$ for v, tv, and t^2v since it is not necessary that the arguments should sum up to zero. These formulas give nothing new, since they are merely special relations which can be derived from the arguments for which cm v equals a constant. We shall give one example to illustrate the procedure.

$$\text{sm}(u+K-v)+\text{sm}(u+K-tv)+\text{sm}(u+K-t^2v)$$

$$=\frac{1}{\text{cm}(u-v)}+\frac{1}{\text{cm}(u-tv)}+\frac{1}{\text{cm}(u-t^2v)}$$

$$=\frac{\text{cm}(u-tv)\,\text{cm}(u-t^2v)+\text{cm}(u-t^2v)\,\text{cm}(u-v)+\text{cm}(u-tv)\,\text{cm}(u-v)}{\text{cm}(u-v)\,\text{cm}(u-tv)\,\text{cm}(u-t^2v)}$$

$$=\frac{3c_2c_1^2}{1-s_1^3c_2^3}.$$

Now, denoting $K-v$ by v_1, $K-tv$ by v_2, and $K-t^2v$ by v_3, we shall have to replace v by $K-v_1$ in the right-hand member which merely interchanges the functions with subscript 2.

Therefore

$$\text{sm}(u+v_1)+\text{sm}(u+v_2)+\text{sm}(u+v_3)=\frac{3s_2c_1^2}{1-s_1^3s_2^3}.$$

In a similar way a great number of other formulas can be developed.

A number of interesting products can be derived such as the following:

$$[1-\text{sm}(u+v)]\,[1-\text{sm}(u+tv)]\,[1-\text{sm}(u+t^2v)]=\frac{(c_2-s_1)^3}{s_1^3c_2^3+c_1^3},$$

$$[1-t\,\text{sm}(u+v)]\,[1-t\,\text{sm}(u+tv)]\,[1-t\,\text{sm}(u+t^2v)]=\frac{(c_2-ts_1)^3}{s_1^3c_2^3+c_1^3},$$

$$[1-\text{cm}(u+v)]\,[1-\text{cm}(u+tv)]\,[1-\text{cm}(u+t^2v)]=\frac{(1-c_1c_2)^3}{s_1^3c_2^3+c_1^3},$$

$$[1-t\,\text{cm}(u+v)]\,[1-t\,\text{cm}(u+tv)]\,[1-t\,\text{cm}(u+t^2v)]=\frac{(1-tc_1c_2)^3}{s_1^3c_2^3+c_1^3},$$

$$[\text{cm}\ (u+v)+\text{sm}\ (u+v)]\ [\text{cm}\ (u+tv)+\text{sm}\ (u+tv)]\ [\text{cm}\ (u+t^2v) + \text{sm}\ (u+t^2v)] = \frac{(c_1+s_1c_2)^3}{s_1^3c_2^3+c_1^3} = \frac{c_1^2+2s_1c_1c_2+s_1^2c_2^2}{c_1^2-s_1c_1c_2+s_1^2c_2^2},$$

$$[\text{cm}\ (u+v)+t^2\ \text{sm}\ (u+v)]\ [\text{cm}\ (u+tv)+t^2\ \text{sm}\ (u+tv)]\ [\text{cm}\ (u+t^2v) + t^2\ \text{sm}\ (u+t^2v)] = \frac{(c_1+t^2s_1c_2)^3}{c_1^3+s_1^3c_2^3} = \frac{c_1^2+2t^2s_1c_1c_2+ts_1^2c_2^2}{c_1^2-t^2s_1c_1c_2+ts_1^2c_2^2},$$

$$[1-\text{sm}\ (u-v)]\ [1-\text{sm}\ (u-tv)]\ [1-\text{sm}\ (u-t^2v)] = \frac{(1-s_1c_2)^3}{c_1^3c_2^3+s_1^3},$$

$$[1-t\ \text{sm}\ (u-v)]\ [1-t\ \text{sm}\ (u-tv)]\ [1-t\ \text{sm}\ (u-t^2v)] = \frac{(1-ts_1c_2)^3}{c_1^3c_2^3+s_1^3},$$

$$[1-\text{cm}\ (u-v)]\ [1-\text{cm}\ (u-tv)]\ [1-\text{cm}\ (u-t^2v)] = \frac{(c_2-c_1)^3}{c_1^3c_2^3+s_1^3},$$

$$[1-t\ \text{cm}\ (u-v)]\ [1-t\ \text{cm}\ (u-tv)]\ [1-t\ \text{cm}\ (u-t^2v)] = \frac{(c_2-tc_1)^3}{c_1^3c_2^3+s_1^3},$$

$$[\text{cm}\ (u-v)+\text{sm}\ (u-v)]\ [\text{cm}\ (u-tv)+\ \text{sm}\ (u-tv)]\ [\text{cm}\ (u-t^2v) + \text{sm}\ (u-t^2v)] = \frac{(s_1+c_1c_2)^3}{s_1^3+c_1^3c_2^3} = \frac{s_1^2+2s_1c_1c_2+c_1^2c_2^2}{s_1^2-s_1c_1c_2+c_1^2c_2^2},$$

$$[\text{cm}\ (u-v)+t\ \text{sm}\ (u-v)]\ [\text{cm}\ (u-tv)+t\ \text{sm}\ (u-tv)]\ [\text{cm}\ (u-t^2v) + t\ \text{sm}\ (u-t^2v)] = \frac{(s_1+t^2c_1c_2)^3}{s_1^3+c_1^3c_2^3} = \frac{s_1^2+2t^2s_1c_1c_2+tc_1^2c_2^2}{s_1^2-t^2s_1c_1c_2+tc_1^2c_2^2}.$$

By a substitution similar to that above we can derive such formulas for v_1, v_2, and v_3 for which sm v is equal to a constant.

FORMULAS FOR MULTIPLE ARGUMENTS

If we let $u=v$ in the ordinary addition formulas

$$\text{sm}\ (u+v) = \frac{s_1+s_2c_2c_1^2}{c_2+s_1c_1s_2^2},$$

and

$$\text{cm}\ (u+v) = \frac{c_2c_1^2-s_1s_2^2}{c_1+s_2c_2s_1^2},$$

we get the values for sm $2u$ and cm $2u$

$$\text{sm}\ 2u = \frac{s\ (1+c^3)}{c\ (1+s^3)},$$

$$\text{cm}\ 2u = \frac{c^3-s^3}{c\ (1+s^3)}.$$

Again, if we let $v=2u$ and then substitute the values of sm $2u$ and cm $2u$, we get

$$\text{sm}\ 3u = \frac{sc\ (1+c^3+s^3+c^6-c^3s^3+s^6)}{c^3+3c^3s^3-s^6+c^6s^3},$$

and

$$\text{cm}\ 3u = \frac{c^6-s^3-3s^3c^3-c^3s^6}{c^3+3c^3s^3-s^6+c^6s^3}.$$

We can show by induction that sm nu and cm nu are rational functions of s^3 and c^3 multiplied, respectively, by

$$sc \text{ and } 1 \text{ if } n \equiv 0 \pmod{3},$$

$$s \text{ and } c \text{ if } n \equiv 1 \pmod{3},$$

$$\frac{s}{c} \text{ and } \frac{1}{c} \text{ if } n \equiv -1 \pmod{3},$$

n being any whole number either positive or negative.

DIFFERENTATION AND INTEGRATION

A complete algorithm for the functions must include the consideration of differentiation and integration. We have

$$\frac{d}{du}\,\text{sm}\; u = \text{cm}^2 u,$$

$$\frac{d}{du}\,\text{cm}\; u = -\text{sm}^2 u,$$

then

$$\frac{d}{du}\,(\text{sm}\; u\; \text{cm}\; u) = \text{cm}^3 u - \text{sm}^3 u,$$

$$\frac{d}{du}\,\frac{s^2}{c} = \frac{s}{c^2} + sc = \text{sm}\; u\; \text{cm}\; u - \text{sm}\;(-u)\; \text{cm}\;(-u),$$

$$\frac{d}{du}\,\frac{c^2}{s} = -\frac{c}{s^2} - sc,$$

$$\frac{d}{du}\,(c+s) = (c+s)\;(c-s),$$

$$\frac{d}{du}\,\log\,(c+s) = c - s,$$

$$\frac{d}{du}\,\log\,(c+ts) = tc - t^2 s,$$

$$\frac{d}{du}\,\log\,(c+t^2 s) = t^2 c - ts.$$

Therefore

$$(t-t^2)\;c = \frac{d}{du}\,[t^2 \log\,(c+t^2 s) - t \log\,(c+ts)],$$

and

$$(t-t^2)\;s = \frac{d}{du}\,[t \log\,(c+t^2 s) - t^2 \log\,(c+ts)].$$

Hence, the integral of cm $u\,du$ and sm $u\,du$, as also $\text{cm}^2 u\,du$ and $\text{sm}^2 u\,du$, may be expressed in terms of elementary functions. Thus also we can integrate $\frac{du}{\text{cm}\,u}$, $\frac{du}{\text{sm}\,u}$, $\frac{\text{cm}\,u}{\text{sm}\,u}\,du$, $\frac{\text{sm}\,u}{\text{cm}\,u}\,du$, as also $\frac{du}{\text{cm}^2 u}$, $\frac{du}{\text{sm}^2 u}$, $\frac{\text{cm}^2 u}{\text{sm}^2 u}\,du$, and $\frac{\text{sm}^2 u}{\text{cm}^2 u}\,du$.

If we wish to integrate any rational integral function of sm u and cm u, it may be brought down to the second degree in cm u by means of the fundamental cubic equation. Hence, we need only to consider the integrals of $\text{sm}^n u\,\text{cm}^2 u\,du$, $\text{sm}^n u\,\text{cm}\,u\,du$, and $\text{sm}^n u\,du$. The first of these is integrable at once as follows:

$$\int_0^u \text{sm}^n u\,\text{cm}^2 u\,du = \frac{1}{n+1}\,\text{sm}^{n+1} u.$$

The other two integrals have reduction formulas which we shall now give.

$$\int_0^u \text{sm}^n u\,\text{cm}\,u\,du = -\frac{1}{n}\,\text{sm}^{n-2} u\,\text{cm}^2 u + \frac{n-2}{n}\int_0^u \text{sm}^{n-3} u\,\text{cm}\,u\,du,$$

and

$$\int_0^u \text{sm}^n u\,du = -\frac{1}{n-1}\,\text{sm}^{n-2} u\,\text{cm}\,u + \frac{n-2}{n-1}\int_0^u \text{sm}^{n-3} u\,du.$$

By means of these formulas any one of the forms can be integrated with no more complicated functions than logarithms, except those that result in sm u cm $u\,du$. This form can result only from the reduction of the integral of $\text{sm}^n u\,\text{cm}\,u\,du$. Let us denote this irreducible integral by $f(u)$. We have then

$$f(u) = \int_0^u \text{sm}\,u\,\text{cm}\,u\,du.$$

From inspection we note that the integral of $\text{sm}^n u\,\text{cm}^m u\,du$ will result in this form if $m \equiv 1$ (mod. 3) and if $m \equiv n$ (mod. 3).

If we wish to integrate any rational fractional function of sm u and cm u, we can first reduce the denominator to the form $P + Q\,\text{cm}\,u + R\,\text{cm}^2 u$ by means of the fundamental cubic, P, Q, and R being rational integral functions of sm u. If we now multiply both numerator and denominator by $(P + Qt\,\text{cm}\,u + Rt^2\,\text{cm}^2 u) \times (P + Qt^2\,\text{cm}\,u + Rt\,\text{cm}^2 u)$, the denominator will take the form $P^3 + Q^3\,\text{cm}^3 u + R^3\,\text{cm}^6 u - 3\,PQR\,\text{cm}^3 u$.

If we now substitute for $\text{cm}^3 u$ its value $1 - \text{sm}^3 u$, the denominator becomes a rational integral function of sm u. We can then reduce the numerator so that it will contain only cm u and $\text{cm}^2 u$. The integrand can next be broken up into partial fractions. If an integral part occurs, it can be integrated as already indicated. Thus, the only new forms that we need to consider are the following three:

$$\int \frac{\text{cm}^2 u\,du}{(\text{sm}\,u + a)^n},\quad \int \frac{\text{cm}\,u\,du}{(\text{sm}\,u + a)^n},\quad \text{and} \int \frac{du}{(\text{sm}\,u + a)^n}.$$

Let us denote these by the symbols P_n, Q_n, and R_n, respectively. We have then the formulas

$$P_n = \text{constant} - \frac{1}{(n-1)(\text{sm}\,u+a)^{n-1}}$$

$$P_1 = \text{constant} + \log\,(\text{sm}\,u+a)$$

$$(n-1)(1+a^3)\,R_n = \text{constant} - \frac{\text{cm}\,u}{(\text{sm}\,u+a)^{n-1}}$$

$$+(3n-4)\,a^2R_{n-1}-(3n-5)\,aR_{n-2}+(n-2)R_{n-3},$$

$$(n-1)(1+a^3)\,Q_n = \text{constant} - \frac{\text{cm}^2\,u}{(\text{sm}\,u+a)^{n-1}}$$

$$+(3n-5)\,a^2Q_{n-1}-(3n-7)\,aQ_{n-2}+(n-3)\,Q_{n-3}.$$

By means of these reduction formulas we can express Q_n and R_n in terms of Q_{-1}, R_{-1}, Q_0, R_0, Q_1, and R_1, of which the last two are the only new irreducible forms. Thus, we have only to add to the former irreducibles the integral $\int \frac{\text{cm}\,u+B}{\text{sm}\,u+A}\,du$, in which A and B are constants.

PROPERTIES OF $f(u)$

The functions sm u and cm u are uniform continuous functions having no other singularities except poles; hence the function sm u cm u is also uniform, all the infinities of which are of the second order. Hence, the integral of this product is a uniform function of u with poles of the first order. We have then

$$f(u) = \int_0^u \text{sm}\,u\ \text{cm}\,u\,du.$$

To derive the addition theorem, let us suppose that $u+v$ equals a constant.

Then $du = -dv$,

$$\text{and } \frac{d}{du}\Big[f(u)+f(v)\Big] = \text{sm}\,u\ \text{cm}\,u - \text{sm}\,v\ \text{cm}\,v$$

$$= \text{cm}\,(u+v)\,(\text{sm}\,u\ \text{cm}^2\,v - \text{sm}\,v\ \text{cm}^2\,u)$$

$$= -\text{cm}\,(u+v)\,\frac{d}{du}\,(\text{sm}\,u\ \text{sm}\,v).$$

By integration we get

$$f(u)+f(v)-f(u+v) = -\text{cm}\,(u+v)\ \text{sm}\,u\ \text{sm}\,v,$$

since $u+v$ is a constant and $f(u)$ vanishes with u.

We have the following examples of the addition theorem:

$$f(u)+f(-u)=\frac{\mathrm{sm}^2 u}{\mathrm{cm}\, u},$$

$$f(u)+f(K-u)=f(K)=k, \text{ let us say,}$$

$$f(K+u)=k+f(u)-\frac{\mathrm{sm}^2 u}{\mathrm{cm}\, u},$$

$$f(2K+u)=2k+f(u)+\frac{1}{\mathrm{sm}\, u\ \mathrm{cm}\, u}-\frac{\mathrm{sm}^2 u}{\mathrm{cm}\, u}=2k+f(u)+\frac{\mathrm{cm}^2 u,}{\mathrm{sm}\, u}$$

$$f(3K+u)-f(u)=3k, \text{ a constant.}$$

$$f(u)+f(tK-u)=f(tK)=k', \text{ we shall say,}$$

$$f(tK+u)=k'+f(u)-\frac{\mathrm{sm}^2 u}{\mathrm{cm}\, u},$$

$$f(2tK+u)=2k'+f(u)+\frac{\mathrm{cm}^2 u}{\mathrm{sm}\, u},$$

$$f(3tK+u)-f(u)=3k',$$

$$f(u)+f(t^2K-u)=f(t^2K)=k'', \text{ let us say,}$$

$$f(t^2K+u)=k''+f(u)-\frac{\mathrm{sm}^2 u}{\mathrm{cm}\, u},$$

$$f(2t^2K+u)=2k''+f(u)+\frac{\mathrm{cm}^2 u}{\mathrm{sm}\, u},$$

$$f(3t^2K+u)-f(u)=3k''.$$

We also have

$$f(tK-K+u)=k'-k+f(u),$$

and so on.

The function $f(u)$ becomes infinite of the first order for the values $2K$, $2tK$, and $2t^2K$ of u.

The addition theorem shows that we have

$$f(2K-u)+f(2K+u)=f(4K)=4k,$$

with similar formulas for $2tK$ and $2t^2K$.

If $u_1+u_2+u_3=0$, we have

$$f(u_1)+f(u_2)+f(u_3)=\frac{\mathrm{sm}^2 u_3-\mathrm{sm}\, u_1\ \mathrm{sm}\, u_2}{\mathrm{cm}\, u_3}$$

$$=\frac{\mathrm{sm}^2 u_1-\mathrm{sm}\, u_2\ \mathrm{sm}\, u_3}{\mathrm{cm}\, u_1}$$

$$=\frac{\mathrm{sm}^2 u_2-\mathrm{sm}\, u_3\ \mathrm{sm}\, u_1}{\mathrm{cm}\, u_2}$$

Then

$$f(u)+f(tu)+f(t^2u)=0,$$

and consequently

$$k+k'+k''=0.$$

INTEGRALS OF THE THIRD KIND

The other irreducible integral was the form $\int \frac{\text{cm } u+B}{\text{sm } u+A} du$. Let us consider the form $\int \frac{A_1 \text{ cm } u+B_1 \text{ sm } u+C_1}{A_2 \text{ cm } u+B_2 \text{ sm } u+C_2} du$, which includes the former integral as a special form.

The integrand generally becomes infinite for three values of u the sum of which is congruent to zero, modulus $3K$, $3tK$. But we know from the theory of Abelian functions that it can be expressed in terms of u and of two integrals, each of which has but two points of discontinuity. The most general form of such an integral is

$$\int \frac{\begin{vmatrix} \text{cm } u, & \text{cm } a_1, & \text{cm } a_2 \\ \text{sm } u, & \text{sm } a_1, & \text{sm } a_2 \\ 1, & 1, & 1 \end{vmatrix}}{\begin{vmatrix} \text{cm } u, & \text{cm } b_1, & \text{cm } b_2 \\ \text{sm } u, & \text{sm } b_1, & \text{sm } b_2 \\ 1, & 1, & 1 \end{vmatrix}} du,$$

in which $a_1+a_2 \equiv b_1+b_2$ (mod. $3K$, $3tK$), the discontinuities being for the values b_1 and b_2 of u. We need to take only one particular pair of values of a_1 and a_2, for if U be the integral for any one pair a_1 and a_2, and U' that for a'_1 and a'_2, there is a relation $U'=A\,U+Bu$.

Let us then write $b_1=a+b$, $b_2=a-b$, and take $a_1=a+K-tK$, $a_2=a-K+tK$.

The numerator then becomes

$$\begin{vmatrix} \text{cm } u, & t \text{ cm } a, & t^2 \text{ cm } a \\ \text{sm } u, & t^2 \text{ sm } a, & t \text{ sm } a \\ 1, & 1, & 1 \end{vmatrix}$$

which is a constant multiple of

$$\text{sm } a \text{ cm } u+\text{cm } a \text{ sm } u+\text{sm } a \text{ cm } a.$$

The denominator becomes the determinant

$$\begin{vmatrix} \text{cm } u, & \text{cm } (a+b), & \text{cm } (a-b) \\ \text{sm } u, & \text{sm } (a+b), & \text{sm } (a-b) \\ 1, & 1, & 1 \end{vmatrix}$$

which on development, after multiplication by a constant, becomes

$$(1-\text{cm}^2 b)\ (\text{sm } a \text{ cm } u+\text{cm } a \text{ sm } u+\text{sm } a \text{ cm } a)+\text{sm } b\ (1+\text{cm } b)\ (\text{cm}^2 a \text{ cm } u+\text{sm}^2 a \text{ sm } u-1).$$

Neglecting constant factors, we can adopt the form for the integral

$$\int \frac{du}{\dfrac{\text{cm}^2 a \text{ cm } u+\text{sm}^2 a \text{ sm } u-1}{\text{sm } a \text{ cm } u+\text{cm } a \text{ sm } u+\text{sm } a \text{ cm } a}+\dfrac{1-\text{cm } b}{\text{sm } b}}$$

But we have

$$\frac{\text{cm}^2 a \text{ cm } u+\text{sm}^2 a \text{ sm } u-1}{\text{sm } a \text{ cm } u+\text{cm } a \text{ sm } u+\text{sm } a \text{ cm } a}=\frac{\text{cm}^2 u \text{ cm } a+\text{sm}^2 u \text{ sm } a-1}{\text{sm } u \text{ cm } a+\text{cm } u \text{ sm } a+\text{sm } u \text{ cm } u}$$

$$=\frac{\text{cm}^2 a \text{ cm } u+\text{sm}^2 a \text{ sm } u-\text{cm}^2 u \text{ cm } a-\text{sm}^2 u \text{ sm } a}{\text{sm } a \text{ cm } a-\text{sm } u \text{ cm } u}$$

$$=-\frac{\text{cm } a \text{ cm}^2 u-\text{sm}^2 a \text{ sm } u}{\text{sm } a \text{ cm } a-\text{sm } u \text{ cm } u}-\frac{\text{cm } u \text{ cm}^2 a-\text{sm}^2 u \text{ sm } a}{\text{sm } u \text{ cm } u-\text{sm } a \text{ cm } a}$$

$$=-\frac{1}{\text{sm }(a-u)}-\frac{1}{\text{sm }(u-a)}=-\text{sm }(u-a-K)-\text{cm }(u-a-K).$$

Also

$$\frac{1}{\text{sm } b}-\frac{\text{cm } b}{\text{sm } b}=\text{cm }(b-K)+\text{sm }(b-K).$$

Hence, if we put $K+b$ for b and $u+a+K$ for u, we have brought the third kind of Abelian integral to the form

$$\int\frac{du}{\text{cm } u+\text{sm } u-\text{cm } b-\text{sm } b},$$

which is discontinuous for the values b and $K-b$ of u.

The integral reduces to the second kind if $b=2K$, since the discontinuities coalesce. Since $\text{cm } 2K+\text{sm } 2K=0$, the integral of the second kind becomes

$$\int_0^u\frac{du}{\text{cm u}+\text{sm u}}=\int_0^u(\text{cm}^2 u+\text{sm}^2 u-\text{sm } u \text{ cm } u)\,du=\text{sm } u-\text{cm } u-f(u).$$

By multiplying both numerator and denominator by the same factor we get

$$\int\frac{du}{\text{cm} u+\text{sm} u-\text{cm} b-\text{sm} b}=\int\frac{\text{cm}^2 u-\text{sm } u \text{ cm } u+\text{sm}^2 u+(\text{cm } u+\text{sm } u+\text{cm } b+\text{sm } b)(\text{cm } b+\text{sm } b)}{\text{cm}^3 u+\text{sm}^3 u+3\text{ cm } u \text{ sm } u\,(\text{cm } b+\text{sm } b)-(\text{cm } b+\text{sm } b)^3}\,du$$

$$=\frac{1}{\text{cm } b-\text{sm } b}\int\frac{(\text{cm } b-\text{sm } b)(\text{cm}^2 u-\text{sm } u \text{ cm } u+\text{sm}^2 u)+(\text{cm } u+\text{sm } u+\text{cm } b+\text{sm } b)(\text{cm}^2 b-\text{sm}^2 b)}{3(\text{cm } b+\text{sm } b)(\text{sm } u \text{ cm } u-\text{sm } b \text{ cm } b)}\,du$$

$$=\frac{1}{3(\text{cm } b+\text{sm } b)}\left\{\int\frac{\text{cm}^2 b+\text{sm } b \text{ cm } b+\text{sm}^2 b}{\text{sm } u \text{ cm } u-\text{sm } b \text{ cm } b}\,du-u\right\}$$
$$+\frac{1}{3(\text{cm}^2 b-\text{sm}^2 b)}\left\{\int\frac{du}{\text{sm }(u-b)}+\int\frac{\text{cm }(u-b)}{\text{sm }(u-b)}\,du+\int\frac{du}{\text{cm}(u+b)}\right.$$
$$\left.+\int\frac{\text{sm }(u+b)}{\text{cm }(u+b)}\,du\right\}$$

$$=\frac{1}{3(\text{cm } b+\text{sm } b)}\left\{\int\frac{\text{cm}^2 b+\text{sm } b \text{ cm } b+\text{sm}^2 b}{\text{sm } u \text{ cm } u-\text{sm } b \text{ cm } b}\,du-u\right\}$$
$$+\frac{1}{3(\text{cm}^2 b-\text{sm}^2 b)}\left\{\int[\text{cm }(2K+u-b)-\text{sm }(2K+u-b)]\,du\right.$$
$$\left.+\int[\text{cm }(-u-b)-\text{sm }(-u-b)]\,du\right\}$$

$$=\frac{1}{3(\text{cm } b+\text{sm } b)}\left\{\int\frac{\text{cm}^2 b+\text{sm } b \text{ cm } b+\text{sm}^2 b}{\text{sm } u \text{ cm } u-\text{sm } b \text{ cm } b}du-u\right\}$$
$$+\frac{1}{3(\text{cm}^2 b-\text{sm}^2 b)}\log\frac{\text{cm }(2K+u-b)+\text{sm }(2K+u-b)}{\text{cm }(-u-b)+\text{sm }(-u-b)}.$$

Putting $2K+b$ for b in the integral in the above expression, it becomes a constant multiple of the integral $\int\frac{du}{\text{cm } b+\text{sm}^2 b \text{ sm } u \text{ cm } u}$. Hence, the integral of a rational function of sm u and cm u can be expressed in terms of sm u, cm u, $f(u)$ and $\int\frac{du}{\text{cm } b+\text{sm}^2 b \text{ sm } u \text{ cm } u}$.

THE FUNCTION $g(u)$

Let us assume a function of u such that

$$\frac{d}{du}\log g(u)=f(u),$$

and therefore

$$\frac{g(u)}{g(0)}=e^{\int_0^u f(u)\,du}.$$

If u be very small, we have

sm $(2K+u)$ cm $(2K+u)=$ sm $(2tK+u)$ cm $(2tK+u)=$ sm $(2t^2K+u)$ cm $(2t^2K+u)=-\frac{\text{cm } u}{\text{sm}^2 u}=-\frac{1}{u^2}+$positive powers of u.

Hence, if $f(u)$ becomes infinite for any value a of u, $f(u)$ behaves in the neighborhood of a like $\frac{1}{u-a}$, and therefore $g(u)$ behaves in the same neighborhood like a constant multiple of $u-a$. Therefore $\frac{g(u)}{g(0)}$ does not become infinite for any finite value of u, and it is, moreover, a uniform continuous function of u, since $f(u)$ is such a function. It can therefore be expanded in a series of ascending powers of u, which is unconditionally convergent for all values of u. This development will be given later in this publication.

By integrating the expressions containing $f(u)$ on page 25, we get the following formulas:

$$\frac{g(-u)}{g(u)}=\text{cm } u,$$

$$g(u)\, g(K)=g\,(0)\, g(K-u)\, e^{ku},$$

$$\frac{g(K+u)}{g(u)}=e^{ku}\text{ cm } u\,\frac{g(K)}{g(0)},$$

$$\frac{g(2K+u)}{g(u)}=e^{2(u-K)k}\text{ sm } u\,\frac{g(3K)}{g(K)},$$

$$\frac{g(3K+u)}{g(u)}=e^{3ku}\,\frac{g(3K)}{g(0)}.$$

In these formulas we may throughout put tK and k' or t^2K and k'' for K and k, respectively, provided that in the fourth formula we multiply the left-hand side by t or t^2 as the case may be.

We have also

$$\frac{g(u+K_1-K_2)}{g(u)}=e^{(k_1-k_2)u}\frac{g(K_1-K_2)}{g(0)},$$

in which K_1 and K_2 denote any two of the periods K, tK, and t^2K, and similarly k_1 and k_2 denote any two of the quantities k, k', and k''. In

$$\frac{g(K+u)}{g(u)}=e^{ku}\operatorname{cm}u\frac{g(K)}{g(0)},$$

substitute the value of $\operatorname{cm}u=\frac{g(-u)}{g(u)}$,

and we get

$$g(K+u)=e^{ku}g(-u)\frac{g(K)}{g(0)}.$$

Now let $u=-\frac{1}{2}K$ and we have

$$g\left(\frac{1}{2}K\right)=e^{-\frac{1}{2}kK}g\left(\frac{1}{2}K\right)\frac{g(K)}{g(0)}$$

Therefore

$$g(K)=e^{\frac{1}{2}kK}g(0).$$

In the expression $\operatorname{cm}u=\frac{g(-u)}{g(u)}$, let $u=-\frac{3}{2}K$,

$$\operatorname{cm}\left(-\frac{3}{2}K\right)=\frac{1}{\operatorname{cm}\left(\frac{3}{2}K\right)}=\frac{1}{\operatorname{cm}\left(K+\frac{1}{2}K\right)}=-\frac{\operatorname{cm}\left(\frac{1}{2}K\right)}{\operatorname{sm}\left(\frac{1}{2}K\right)},$$

but

$$\operatorname{cm}\left(\frac{1}{2}K\right)=\operatorname{sm}\left(\frac{1}{2}K\right),$$

therefore

$$g\left(\frac{3}{2}K\right)=-g\left(-\frac{3}{2}K\right).$$

Now, in the formula

$$\frac{g(3K+u)}{g(u)}=e^{3ku}\frac{g(3K)}{g(0)},$$

let $u=-\frac{3}{2}K$ and we get

$$\frac{g\left(\frac{3}{2}K\right)}{g\left(-\frac{3}{2}K\right)}=e^{-\frac{9}{2}kK}\frac{g(3K)}{g(0)}=-1,$$

therefore

$$g(3K) = -e^{\frac{9}{2}kK} g(0).$$

We have defined $f(u)$ by the integral

$$f(u) = \int_0^u \text{sm } u \text{ cm } u \, du,$$

then

$$f(tu) = \int_0^u t \text{ sm } tu \text{ cm } tu \, du = t^2 \int_0^u \text{sm } u \text{ cm } u \, du,$$

therefore

$$f(tu) = t^2 f(u);$$

in like manner

$$f(t^2 u) = t f(u).$$

In particular

$$k' = t^2 k \text{ and } k'' = tk.$$

Hence

$$k + t^2 k + tk = 0.$$

We can now find the values

$$g(tK) = e^{\frac{1}{2}kK} g(0),$$
$$g(t^2 K) = e^{\frac{1}{2}kK} g(0),$$
$$g(3tK) = g(3t^2 K) = g(3K) = -e^{\frac{9}{2}kK} g(0),$$
$$g(K_1 - K_2) = e^{kK + k_1 K_2} g(0).$$

We note in particular that

$$g(tu) = g(t^2 u) = g(u).$$

The Abelian function $g(u)$ must therefore be a series in u^3.

We have already shown that

$$\text{cm } u = \frac{g(-u)}{g(u)},$$

but

$$g(u) = e^{ku} \frac{g(0)}{g(K)} g(K - u),$$

hence

$$g(-u) = e^{-ku} \frac{g(0)}{g(K)} g(K + u).$$

But

$$\frac{g(0)}{g(K)} = e^{-\frac{1}{2}kK},$$

so that

$$g(-u) = e^{-ku - \frac{1}{2}kK} g(K + u),$$

hence

$$\text{cm } u = e^{-ku - \frac{1}{2}kK} \frac{g(K + u)}{g(u)},$$
$$= e^{-t^2 ku - \frac{1}{2}kK} \frac{g(tK + u)}{g(u)},$$
$$= e^{-tku - \frac{1}{2}kK} \frac{g(t^2 K + u)}{g(u)}.$$

Also we have

$$\text{sm } u = \text{cm } (K-u) = \frac{g(u-K)}{g(K-u)},$$

$$= e^{ku-\frac{1}{2}kK} \frac{g(u-K)}{g(u)},$$

$$= te^{t^2ku-\frac{1}{2}kK} \frac{g(u-tK)}{g(u)},$$

$$= t^2 e^{tku-\frac{1}{2}kK} \frac{g(u-t^2K)}{g(u)}.$$

We have

$$g(u) = g(K_1-u)e^{k_1u-\frac{1}{2}kK},$$

hence

$$g\left(\frac{1}{2}K_1+u\right) = g\left(\frac{1}{2}K_1-u\right)e^{k_1u},$$

or

$$e^{-\frac{1}{2}k_1u} g\left(\frac{1}{2}K_1+u\right) = e^{\frac{1}{2}k_1u} g\left(\frac{1}{2}K_1-u\right).$$

therefore

$$e^{-\frac{1}{2}k_1u}\, g\left(\frac{1}{2}K_1+u\right) \text{ is an even function of } u.$$

ABELIAN FUNCTIONS CONNECTED WITH sm u AND cm u

Let us denote the function

$$e^{ku-\frac{1}{2}kK} g(u-K) \text{ by } h(u),$$

and we have

$$\text{sm } u = \frac{h(u)}{g(u)}.$$

and

$$\text{cm } u = \frac{g(-u)}{g(u)}.$$

These functions are Abelian functions which can be expanded into series that are convergent for all values of u. They can also be expressed as Fourier series, or as infinite products. From the fundamental algebraic equation between sm u and cm u we get

$$h^3(u) + g^3(-u) = g^3(u).$$

Also from

$$\frac{d}{du} \text{sm } u = \text{cm}^2 u,$$

we get

$$g(u)h'(u) - h(u)g'(u) = g^2(-u),$$

and from

$$\frac{d}{du} \text{cm } u = -\text{sm}^2 u,$$

$$g(u)\, g'(-u) + g(-u)\, g'(u) = h^2(u).$$

From the addition formula of $f(u)$ (see p. 24), we have

$$f(u)+f(a)-f(u+a)=-\text{cm}\,(u+a)\ \text{sm}\,u\ \text{sm}\,a,$$

$$f(u)+f(-a)-f(u-a)=+\frac{\text{cm}\,(u-a)\ \text{sm}\,u\ \text{sm}\,a}{\text{cm}\,a}.$$

By subtraction we get

$$\begin{aligned}
&f(a)-f(-a)-f(u+a)+f(u-a)\\
&=-\frac{\text{sm}\,u\ \text{sm}\,a}{\text{cm}\,a}\left[\text{cm}\,(u+a)\ \text{cm}\,a+\text{cm}\,(u-a)\right]\\
&=-\frac{\text{sm}\,u\ \text{cm}\,u\ \text{sm}\,a}{\text{cm}\,a}\left(\frac{1+\text{cm}^3\,a}{\text{cm}\,a+\text{sm}\,u\ \text{cm}\,u\ \text{sm}^2\,a}\right),
\end{aligned}$$

but

$$f(a)+f(-a)=\frac{\text{sm}^2\,a}{\text{cm}\,a},$$

hence, by addition, we get

$$\begin{aligned}
&2f(a)-f(u+a)+f(u-a)\\
&=-\frac{\text{sm}\,a}{\text{cm}\,a}\left(\frac{\text{sm}\,u\ \text{cm}\,u+\text{sm}\,u\ \text{cm}\,u\ \text{cm}^3\,a}{\text{cm}\,a+\text{sm}\,u\ \text{cm}\,u\ \text{sm}^2\,a}-\text{sm}\,a\right)\\
&=-\frac{2\ \text{sm}\,u\ \text{cm}\,u\ \text{sm}\,a\ \text{cm}^2\,a-\text{sm}^2\,a}{\text{cm}\,a+\text{sm}\,u\ \text{cm}\,u\ \text{sm}^2\,a}\\
&=-\frac{d}{da}\log\,(\text{cm}\,a+\text{sm}\,u\ \text{cm}\,u\ \text{sm}^2\,a).
\end{aligned}$$

By integrating this equation with respect to a from 0 to a we get

$$\begin{aligned}
\frac{g(u+a)\ g(u-a)\ g^2(0)}{g^2(a)\ g^2(u)}&=\text{cm}\,a+\text{sm}\,u\ \text{cm}\,u\ \text{sm}^2\,a\\
&=\frac{g(-a)}{g(a)}+\frac{h(u)}{g(u)}\,\frac{g(-u)}{g(u)}\,\frac{h^2(a)}{g^2(a)},
\end{aligned}$$

hence

$$g(u+a)\ g(u-a)\ g^2(0)=g(-a)\ g(a)\ g^2(u)+h(u)\ g(-u)\ h^2(a).$$

If we denote the integral

$$\int_0^u\frac{2\ \text{sm}\,u\ \text{cm}\,u\ \text{sm}\,a\ \text{cm}^2\,a-\text{sm}^2\,a}{\text{sm}\,u\ \text{cm}\,u\ \text{sm}^2\,a+\text{cm}\,a}\,du$$

by $Q(u, a)$, we have

$$\begin{aligned}
Q(u,a)&=\int_0^u[f(u+a)-f(u-a)-2f(a)]du\\
&=\log\frac{g(u+a)}{g(u-a)}-2uf(a)+\log\,\text{cm}\,a.
\end{aligned}$$

Hence, any rational function of cm u and sm u may be integrated by means of the function $g(u)$ and its derivative $g'(u)$.

RELATIONS BETWEEN THE ABELIAN FUNCTIONS

We have defined $h(u)$ as

$$h(u)=e^{\mathrm{ku}-\frac{1}{2}\mathrm{kK}}g(u-K),$$

hence

$$h(u+K)=e^{\mathrm{ku}+\frac{1}{2}\mathrm{kK}}g(u).$$

Also

$$g(u+K)=e^{\mathrm{ku}+\frac{1}{2}\mathrm{kK}}g(-u),$$

$$g(-u-K)=-e^{\mathrm{ku}+\frac{1}{2}\mathrm{kK}}h(u).$$

In the formula

$$\mathrm{cm}\,a+\mathrm{sm}\,u\,\mathrm{cm}\,u\,\mathrm{sm}^2 a=\frac{g(u+a)\;g(u-a)\;g^2(0)}{g^2(a)\;g^2(u)},$$

if we multiply both sides by

$$\mathrm{sm}\;(u+a)=\frac{h(u+a)}{g(u+a)}$$

we get

$$\mathrm{sm}\,u+\mathrm{sm}\,a\,\mathrm{cm}\,a\,\mathrm{cm}^2 u=\frac{h(u+a)\;g(u-a)\;g^2(0)}{g^2(a)\;g^2(u)},$$

or

$$h(u)\;g(u)\;g^2(a)+h(a)\;g(-a)\;g^2(-u)=h(u+a)\;g(u-a)\;g^2(0),$$

$$\mathrm{sm}\,u\,\mathrm{cm}^2 a-\mathrm{sm}\,a\,\mathrm{cm}^2 u=\frac{g(u+a)\;h(u-a)\;g^2(0)}{g^2(a)\;g^2(u)},$$

or

$$h(u)\;g(u)\;g^2(-a)-h(a)\;g(a)\;g^2(-u)=g(u+a)\;h(u-a)\;g^2(0),$$

$$\mathrm{cm}\;u\;\mathrm{cm}^2\,a-\mathrm{sm}^2\,u\;\mathrm{sm}\;a=\frac{g(-u-a)\;g(u-a)\;g^2(0)}{g^2(a)\;g^2(u)},$$

or

$$g(-u)\;g(u)\;g^2(-a)-h^2(u)\;h(a)\;g(a)=g(-u-a)\;g(u-a)\;g^2(0),$$

$$\mathrm{sm}\;u\;\mathrm{cm}\;u-\mathrm{sm}\;a\;\mathrm{cm}\;a=\frac{g(-u-a)\;h(u-a)\;g^2(0)}{g^2(a)\;g^2(u)},$$

or

$$h(u)\;g(-u)\;g^2(a)-h(a)\;g(-a)\;g^2(u)=g(-u-a)\;h(u-a)\;g^2(0),$$

$$\mathrm{sm}^2\,u\;\mathrm{cm}\;a-\mathrm{sm}^2\,a\;\mathrm{cm}\;u=\frac{h(u+a)\;h(u-a)\;g^2(0)}{g^2(a)\;g^2(u)},$$

or

$$h^2(u)\;g(-a)\;g(a)-h^2(a)\;g(-u)\;g(u)=h(u+a)\;h(u-a)\;g^2(0).$$

We have

$$f(u)+f(a)-f(u+a)=-\mathrm{cm}\;(u+a)\;\mathrm{sm}\;u\;\mathrm{sm}\;a,$$

$$f(u)+t^2 f(a)-f(u+ta)=-t\;\mathrm{cm}\;(u+ta)\;\mathrm{sm}\;u\;\mathrm{sm}\;a,$$

$$f(u)+tf(a)-f(u+t^2 a)=-t^2\;\mathrm{cm}\;(u+t^2 a)\;\mathrm{sm}\;u\;\mathrm{sm}\;a,$$

hence, by addition, we get

$$3f(u)-f(u+a)-f(u+ta)-f(u+t^2a)=+\frac{3c_1^2s_1^2s_2^3}{c_1^3+s_1^3c_2^3}$$

$$=-\frac{-3c_1^2s_1^2+3s_1^2c_1^2c_2^3}{c_1^3+s_1^3c_2^3}=-\frac{d}{du}\log\left(\text{cm}^3 u+\text{sm}^3 u \text{ cm}^3 a\right).$$

Integrating with respect to u from 0 to u and remembering that $g(ta)=g(a)$ and $g(t^2a)=g(a)$, we get

$$\text{cm}^3 u+\text{sm}^3 u \text{ cm}^3 a=\frac{g(u+a)\,g(u+ta)\,g(u+t^2a)\,g^3(0)}{g^3(u)\,g^3(a)}.$$

Also

$$\text{sm } u \text{ cm}^2 a+\text{sm } a \text{ cm}^2 u-\text{sm}^2 u \text{ sm}^2 a \text{ cm } u \text{ cm } a$$

$$=\frac{h(u+a)\,g(u+ta)\,g(u+t^2a)\,g^3(0)}{g^3(u)\,g^3(a)}.$$

In a similar way we may continue to develop relations between these Abelian functions.

THE FUNCTION $Q(u, a)$, ADDITION OF ARGUMENTS

We have derived the formula

$$Q(u, a)=\log\frac{g(u+a)}{g(u-a)}-2uf(a)+\log \text{cm } a;$$

hence

$$Q(v, a)=\log\frac{g(v+a)}{g(v-a)}-2vf(a)+\log \text{cm } a,$$

and

$$Q(u+v, a)=\log\frac{g(u+v+a)}{g(u+v-a)}-2(u+v)f(a)+\log \text{cm } a.$$

Thus we get

$$Q(u, a)+Q(v, a)-Q(u+v, a)=\log\frac{g(u+a)\,g(v+a)\,g(u+v-a)\,g(-a)}{g(u-a)\,g(v-a)\,g(u+v+a)\,g(a)}.$$

In the formula

$$\text{cm } a+\text{sm } u \text{ cm } u \text{ sm}^2 a=\frac{g(u+a)\,g(u-a)\,g^2(0)}{g^2(a)\,g^2(u)},$$

set

$$a=\frac{1}{2}(u-v),$$

$$u=\frac{1}{2}(u+v)+a,$$

and we get

$$\text{cm }\frac{1}{2}(u-v)+\text{sm}^2\frac{1}{2}(u-v)\text{ sm}\left\{\frac{1}{2}(u+v)+a\right\}\text{cm}\left\{\frac{1}{2}(u+v)+a\right\}$$

$$=\frac{g(u+a)\,g(v+a)\,g^2(0)}{g^2\frac{1}{2}(u-v)\,g^2\left\{\frac{1}{2}(u+v)+a\right\}}.$$

Again set

$$a=\frac{1}{2}(u+v),$$

$$u=\frac{1}{2}(u+v)+a,$$

and we have

$$\operatorname{cm}\frac{1}{2}(u+v)+\operatorname{sm}^2\frac{1}{2}(u+v)\operatorname{sm}\left\{\frac{1}{2}(u+v)+a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)+a\right\}$$

$$=\frac{g(u+v+a)\,g(a)\,g^2(0)}{g^2\frac{1}{2}(u+v)\;g^2\left\{\frac{1}{2}(u+v)+a\right\}}.$$

Also

$$\operatorname{cm}\frac{1}{2}(u+v)+\operatorname{sm}^2\frac{1}{2}(u+v)\operatorname{sm}\left\{\frac{1}{2}(u+v)-a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)-a\right\}$$

$$=\frac{g(u+v-a)\,g(-a)\,g^2(0)}{g^2\frac{1}{2}(u+v)\;g^2\left\{\frac{1}{2}(u+v)-a\right\}},$$

and

$$\operatorname{cm}\frac{1}{2}(u-v)+\operatorname{sm}^2\frac{1}{2}(u-v)\operatorname{sm}\left\{\frac{1}{2}(u+v)-a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)-a\right\}$$

$$=\frac{g(u-a)\,g(v-a)\,g^2(0)}{g^2\frac{1}{2}(u-v)\;g^2\left\{\frac{1}{2}(u+v)-a\right\}}.$$

Hence

$$\frac{\operatorname{cm}\frac{1}{2}(u-v)+\operatorname{sm}^2\frac{1}{2}(u-v)\operatorname{sm}\left\{\frac{1}{2}(u+v)+a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)+a\right\}}{\operatorname{cm}\frac{1}{2}(u-v)+\operatorname{sm}^2\frac{1}{2}(u-v)\operatorname{sm}\left\{\frac{1}{2}(u+v)-a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)-a\right\}}$$

$$\times\frac{\operatorname{cm}\frac{1}{2}(u+v)+\operatorname{sm}^2\frac{1}{2}(u+v)\operatorname{sm}\left\{\frac{1}{2}(u+v)-a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)-a\right\}}{\operatorname{cm}\frac{1}{2}(u+v)+\operatorname{sm}^2\frac{1}{2}(u+v)\operatorname{sm}\left\{\frac{1}{2}(u+v)+a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)+a\right\}}$$

$$=\frac{g(u+a)\,g(v+a)\,g(u+v-a)\,g(-a)}{g(u-a)\,g(v-a)\,g(u+v+a)\,g(a)}.$$

Hence

$$Q(u,a)+Q(v,a)-Q(u+v,a)=$$

$$\log\left[\frac{\operatorname{cm}\frac{1}{2}(u-v)+\operatorname{sm}^2\frac{1}{2}(u-v)\operatorname{sm}\left\{\frac{1}{2}(u+v)+a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)+a\right\}}{\operatorname{cm}\frac{1}{2}(u-v)+\operatorname{sm}^2\frac{1}{2}(u-v)\operatorname{sm}\left\{\frac{1}{2}(u+v)-a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)-a\right\}}\right.$$

$$\left.\times\frac{\operatorname{cm}\frac{1}{2}(u+v)+\operatorname{sm}^2\frac{1}{2}(u+v)\operatorname{sm}\left\{\frac{1}{2}(u+v)-a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)-a\right\}}{\operatorname{cm}\frac{1}{2}(u+v)+\operatorname{sm}^2\frac{1}{2}(u+v)\operatorname{sm}\left\{\frac{1}{2}(u+v)+a\right\}\operatorname{cm}\left\{\frac{1}{2}(u+v)+a\right\}}\right]$$

$$=\frac{1}{2}\log\left[\frac{\text{cm }(v-a)+\text{sm}^2\,(v-a)\text{ sm }(u-a)\text{ cm }(u-a)}{\text{cm }(v+a)+\text{sm}^2\,(v+a)\text{ sm }(u+a)\text{ cm }(u+a)}\right.$$

$$\left.\times\frac{\text{cm }a+\text{sm}^2\,a\text{ sm }(u+v+a)\text{ cm }(u+v+a)}{\text{cm }a+\text{sm}^2\,a\text{ sm }(u+v-a)\text{ cm }(u+v-a)}\text{ cm}^2\,a\right].$$

This last form can be verified in the same way as the preceding one.

Now we have

$$\frac{1-\text{sm }(u_2-u_1)\text{ cm }(u_2+u_1)\text{ sm }(u_1+u_3)\text{ sm }(u_1-u_3)}{1-\text{sm }(u_4-u_1)\text{ cm }(u_4+u_1)\text{ sm }(u_1+u_3)\text{ sm }(u_1-u_3)}$$

$$=\frac{c_3+s_2c_2s_3^2}{c_1+s_2c_2s_1^2}\times\frac{c_1+s_4c_4s_1^2}{c_3+s_4c_4s_3^2};$$

in this expression let $u_1=\frac{1}{2}(u+v)$, $u_2=\frac{1}{2}(u+v)+a$,

$$u_3=\frac{1}{2}(u-v),\ u_4=\frac{1}{2}(u+v)-a.$$

With these values we see that

$$\frac{c_3+s_3^2s_2c_2}{c_3+s_3^2s_4c_4}\times\frac{c_1+s_1^2s_4c_4}{c_1+s_1^2s_2c_2}=\frac{1-\text{sm }a\text{ cm }(u+v+a)\text{ sm }u\text{ sm }v}{1-\text{sm }(-a)\text{ cm }(u+v-a)\text{ sm }u\text{ sm }v}.$$

Hence

$$Q(u,a)+Q(v,a)-Q(u+v,a)$$

$$=\log\left[\frac{1-\text{sm }a\text{ cm }(u+v+a)\text{ sm }u\text{ sm }v}{1-\text{sm }(-a)\text{ cm }(u+v-a)\text{ sm }u\text{ sm }v}\right]$$

since $\log\left(\frac{c_3+s_3^2s_2c_2}{c_3+s_3^2s_4c_4}\times\frac{c_1+s_1^2s_4c_4}{c_1+s_1^2s_2c_2}\right)$ is the complicated expression given above for $Q(u,a)+Q(v,a)-Q(u+v,a)$.

INTERCHANGE OF ARGUMENT AND PARAMETER

$$Q(u,a)-Q(a,u)=\log\frac{g(a-u)}{g(u-a)}-2uf(a)+2af(u)+\log\frac{\text{cm }a}{\text{cm }u}$$

$$=\log\frac{\text{cm}(u-a)\text{ cm }a}{\text{cm }u}-2uf(a)+2af(u).$$

Hence

$$Q(u,a)+Q(u,b)-Q(u,a+b)=Q(a,u)+Q(b,u)-Q(a+b,u)$$

$$+\log\frac{\text{cm}(u-a)\text{ cm}(u-b)\text{ cm }a\text{ cm }b}{\text{cm }u\text{ cm }(a+b)\text{ cm }(u-a-b)}+2u\text{ cm}(a+b)\text{ sm }a\text{ sm }b,$$

or

$$Q(u, a) + Q(u, b) - Q(u, a+b) = \log\left[\frac{\text{cm}(u-a)\ \text{cm}\ (u-b)\ \text{cm}\ a\ \text{cm}\ b}{\text{cm}\ u\ \text{cm}\ (a+b)\ \text{cm}\ (u-a-b)}\right.$$

$$\left.\times\frac{1-\text{sm}\ u\ \text{cm}(u+a+b)\ \text{sm}\ a\ \text{sm}\ b}{1-\text{sm}(-u)\ \text{cm}(a+b-u)\ \text{sm}\ a\ \text{sm}\ b}\right] + 2u\ \text{cm}\ (a+b)\ \text{sm}\ a\ \text{sm}\ b.$$

This gives the theorem for the addition of parameters.

NUMERICAL VALUES

From the expression

$$\text{sm}\ (K-u) = \text{cm}\ u,$$

we get

$$\text{sm}\,\frac{1}{2}\,K = \text{cm}\,\frac{1}{2}\,K,$$

hence

$$2\ \text{sm}^3\,\frac{1}{2}\,K = 1,$$

and

$$\text{sm}\,\frac{1}{2}\,K = \text{cm}\,\frac{1}{2}\,K = 2^{-\frac{1}{3}}.$$

Also

$$\text{sm}\left(\frac{3}{2}K\right) = \text{sm}\left(K + \frac{1}{2}K\right) = \frac{1}{\text{cm}\,\frac{1}{2}K} = 2^{\frac{1}{3}},$$

$$\text{cm}^3\left(\frac{3}{2}K\right) = 1 - \text{sm}^3\,\frac{3}{2}\,K = 1-2 = -1,$$

$$\text{cm}\left(\frac{3}{2}K\right) = -1.$$

$$k = \int_0^K \text{sm}\ u\ \text{cm}\ u\ du = \int_0^K \frac{\text{sm}\ u\ \text{cm}^2\ u\ du}{\text{cm}\ u} = \int_0^1 \frac{x\ dx}{(1-x^3)^{\frac{1}{3}}}$$

$$= \frac{1}{3}\,B\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{1}{3}\,\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right)} = \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}.$$

From Legendre's table we have

$$\log \Gamma\left(\frac{2}{3}\right) = 0.13165649168402,$$

and as already given

$$\log \Gamma\left(\frac{1}{3}\right) = 0.42796274931426.$$

Hence,

$$\log k = 9.83535023405378 - 10$$

$$k = 0.6840634.$$

We have

$$kK=\frac{1}{3}\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{2}{3}\right)}\cdot\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}=\frac{1}{3}\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}.$$

TRANSFORMATIONS

We have already listed several transformations which give the values of cm u and sm u for various combinations of one-third of the periods. If we denote $e^{\frac{\pi i}{3}}$ by s, then $3sK$ is a period for the functions, since $3sK=3K+3tK=-3t^2K$. To investigate the behavior of the functions for an argument su, we have

$$\text{sm}\ su=\text{sm}(-t^2u)=-\frac{\text{sm}\ t^2u}{\text{cm}\ t^2u}=-t^2\frac{\text{sm}\ u}{\text{cm}\ u}=s\frac{\text{sm}\ u}{\text{cm}\ u}$$

$$\text{cm}\ su=\text{cm}(-t^2u)=\frac{1}{\text{cm}\ t^2u}=\frac{1}{\text{cm}\ u}.$$

This transformation tells us some facts regarding the Abelian functions $h(u)$, $g(u)$, and $g(-u)$. In the first place since $g(u)$ is not equal to $g(-u)$, we know that in the series of powers of u^3 that expresses $g(u)$ the odd powers are present as well as the even powers. Since $s^3=-1$, we have

$$g(su)=g(-u),$$

$$g(-su)=g(u),$$

hence

$$\text{cm}\ su=\frac{g(-su)}{g(su)}=\frac{g(u)}{g(-u)}=\frac{1}{\text{cm}\ u}.$$

The function $h(u)$, which is zero for $u=0$, must be of the form $uP(u^6)$, in which $P(u^6)$ denotes a power series in u^6. It results that $h(su)=sh(u)$ and we have

$$\text{sm}\ su=\frac{h(su)}{g(su)}=s\frac{h(u)}{g(-u)}=s\frac{\frac{h(u)}{g(u)}}{\frac{g(-u)}{g(u)}}=s\frac{\text{sm}\ u}{\text{cm}\ u}.$$

We know then that $h(u)$ is an odd function of u. The product $g(u)\,g(-u)$ must be an even function of u. The fundamental periods could be taken as $3K$ and $3sK$, although in some respects $3K$ and $3tK$ are preferable as sides of the fundamental parallelogram. It should be noted that the area of the rhombus would be the same in either case.

TRANSFORMATION FOR THE ADDITION OF A HALF PERIOD TO THE ARGUMENT

Another transformation that might be noted is that for the addition of the half period.

$$\operatorname{sm}\left(u+\frac{3K}{2}\right)=\frac{\operatorname{sm}^2 u+2^{\frac{2}{3}} \operatorname{cm} u}{2^{\frac{1}{3}} \operatorname{cm}^2 u-\operatorname{sm} u}$$

$$=\frac{\operatorname{sm} u-2^{\frac{1}{3}} \operatorname{cm}^2 u}{2^{\frac{2}{3}} \operatorname{sm} u \operatorname{cm} u-1}$$

$$=\frac{2^{\frac{1}{3}}+\operatorname{sm} u \operatorname{cm} u}{\operatorname{cm} u-2^{\frac{1}{3}} \operatorname{sm}^2 u},$$

$$\operatorname{cm}\left(u+\frac{3K}{2}\right)=\frac{\operatorname{sm} u \operatorname{cm} u+2^{\frac{1}{3}}}{\operatorname{sm} u-2^{\frac{1}{3}} \operatorname{cm}^2 u}$$

$$=\frac{\operatorname{cm} u-2^{\frac{1}{3}} \operatorname{sm}^2 u}{2^{\frac{2}{3}} \operatorname{sm} u \operatorname{cm} u-1}$$

$$=\frac{\operatorname{cm}^2 u+2^{\frac{2}{3}} \operatorname{sm} u}{2^{\frac{1}{3}} \operatorname{sm}^2 u-\operatorname{cm} u}.$$

Evidently the argument $u-\frac{3K}{2}$ should give the same transformation or a second application of the transformation should restore the original functions.

TRANSFORMATION FOR A PURELY IMAGINARY ARGUMENT

An imaginary transformation can be applied by noting the identity

$$i=\frac{t-t^2}{\sqrt{3}}.$$

Hence we have

$$\operatorname{sm} iu=\operatorname{sm}\left(t\frac{u}{\sqrt{3}}-t^2\frac{u}{\sqrt{3}}\right)=\frac{(t-1) \operatorname{sm}\frac{u}{\sqrt{3}} \operatorname{cm}\frac{u}{\sqrt{3}}}{1+t \operatorname{cm}^3\frac{u}{\sqrt{3}}}$$

$$=\frac{\sqrt{3}\, i \operatorname{sm}\frac{u}{\sqrt{3}} \operatorname{cm}\frac{u}{\sqrt{3}}}{1+t^2 \operatorname{sm}^3\frac{u}{\sqrt{3}}}$$

$$=\frac{\sqrt{3}\, i \operatorname{sm}\frac{u}{\sqrt{3}} \operatorname{cm}\frac{u}{\sqrt{3}}}{\operatorname{cm}^3\frac{u}{\sqrt{3}}-t \operatorname{sm}^3\frac{u}{\sqrt{3}}},$$

$$\text{cm}\, iu = \text{sm}\left(t\frac{u}{\sqrt{3}} - t^2\frac{u}{\sqrt{3}}\right) = \frac{t + \text{cm}^3\, \frac{u}{\sqrt{3}}}{1 + t\,\text{cm}^3\, \frac{u}{\sqrt{3}}}$$

$$= \frac{1 + t\,\text{sm}^3\frac{u}{\sqrt{3}}}{1 + t^2\,\text{sm}^3\frac{u}{\sqrt{3}}}$$

$$= \frac{\text{cm}^3\frac{u}{\sqrt{3}} - t^2\,\text{sm}^3\,\frac{u}{\sqrt{3}}}{\text{cm}^3\frac{u}{\sqrt{3}} - t\,\text{sm}^3\,\frac{u}{\sqrt{3}}}.$$

A second application of this transformation will give sm $(-u)$ and cm $(-u)$ expressed in terms of the functions with the argument $\frac{u}{3}$, or by multiplying the argument by 3 we shall have sm $(-3u)$ and cm $(-3u)$ expressed in terms of sm u and cm u. This imaginary transformation will be found very useful when we come to discuss the use of the function for mapping purposes.

TRANSFORMATION FOR THE ADDITION OF ½K TO THE ARGUMENT

As another transformation we have

$$\text{sm}\left(u + \frac{K}{2}\right) = \frac{2^{\frac{1}{3}}\,\text{sm}^2\, u - \text{cm}\, u}{\text{sm}\, u - 2^{\frac{1}{3}}\,\text{cm}^2\, u}$$

$$= \frac{2^{\frac{1}{3}}\,\text{sm}\, u + \text{cm}^2\, u}{2^{\frac{1}{3}} + \text{sm}\, u\, \text{cm}\, u}$$

$$= \frac{2^{\frac{1}{3}} + \text{sm}\, u\,\text{cm}\, u}{2^{\frac{1}{3}}\,\text{cm}\, u + \text{sm}^2\, u},$$

$$\text{cm}\left(u + \frac{K}{2}\right) = \frac{2^{\frac{1}{3}}\,\text{sm}\, u\,\text{cm}\, u - 1}{\text{sm}\, u - 2^{\frac{1}{3}}\,\text{cm}^2\, u}$$

$$= \frac{\text{cm}\, u - 2^{\frac{1}{3}}\,\text{sm}^2\, u}{2^{\frac{1}{3}} + \text{sm}\, u\,\text{cm}\, u}$$

$$= \frac{2^{\frac{1}{3}}\,\text{cm}^2\, u - \text{sm}\, u}{2^{\frac{1}{3}}\,\text{cm}\, u + \text{sm}^2\, u}.$$

SECOND TRANSFORMATION FOR A PURELY IMAGINARY ARGUMENT

We shall now note some transformations that change the curve $x^3 + y^3 = 1$ into the more general curve $x^3 + y^3 - 3axy = 1$.

Let us start with the expressions

$$\text{sm}\left(\frac{iu}{\sqrt{3}}\right) = -t\,\frac{\text{cm}\, u + \text{sm}\, u - 1}{t^2\,\text{cm}\, u + t\,\text{sm}\, u - 1},$$

$$\text{cm}\left(\frac{iu}{\sqrt{3}}\right) = -t^2\,\frac{t\,\text{cm}\, u + t^2\,\text{sm}\, u - 1}{t^2\,\text{cm}\, u + t\,\text{sm}\, u - 1}.$$

It will be found that these expressions satisfy the equation

$$\mathrm{sm}^3\left(\frac{iu}{\sqrt{3}}\right)+\mathrm{cm}^3\left(\frac{iu}{\sqrt{3}}\right)-6\ \mathrm{sm}\left(\frac{iu}{\sqrt{3}}\right)\mathrm{cm}\left(\frac{iu}{\sqrt{3}}\right)=1,$$

so that $\alpha=2$.

We should have then

$$\frac{d}{du}\,\mathrm{sm}\left(\frac{iu}{\sqrt{3}}\right)=\mathrm{cm}^2\left(\frac{iu}{\sqrt{3}}\right)-2\ \mathrm{sm}\left(\frac{iu}{\sqrt{3}}\right),$$

and

$$\frac{d}{du}\,\mathrm{cm}\left(\frac{iu}{\sqrt{3}}\right)=-\mathrm{sm}^2\left(\frac{iu}{\sqrt{3}}\right)+2\ \mathrm{cm}\left(\frac{iu}{\sqrt{3}}\right).$$

It can easily be verified that these relations are valid. It should be noted that if α is present in the original functions this transformation is expressed as

$$\mathrm{sm}\left\{\frac{iu\ (1+\alpha)}{\sqrt{3}},\ \frac{2-\alpha}{1+\alpha}\right\},$$

and

$$\mathrm{cm}\left\{\frac{iu\ (1+\alpha)}{\sqrt{3}},\ \frac{2-\alpha}{1+\alpha}\right\},$$

in which $\frac{2-\alpha}{1+\alpha}$ denotes the new alpha. Since the alpha to start with in this case was zero, we have as above

$$\mathrm{sm}\left(\frac{iu}{\sqrt{3}},\ 2\right),$$

and

$$\mathrm{cm}\left(\frac{iu}{\sqrt{3}},\ 2\right).$$

If we apply the same transformation a second time we shall get

$$\mathrm{sm}\ (-u,\ 0)=-\frac{\mathrm{sm}\ u}{\mathrm{cm}\ u},$$

and

$$\mathrm{cm}\ (-u,\ 0)=\frac{1}{\mathrm{cm}\ u}.$$

It is easy to verify this fact and it gives a good check on the transformation.

It is interesting to note that we have already deduced an imaginary transformation that keeps the functions in the class of $\alpha=0$. This particular one could not be derived in a similar manner.

We can derive the relation between the new f and g functions and those of the original functions,

$$f(U,\ 2)=\int_0^U CS\ dU,$$

$$=\frac{i}{\sqrt{3}}\int_0^u\frac{(c+s-1)\ (tc+t^2s-1)}{(t^2c+ts-1)^2}\,du.$$

It can easily be verified that

$$-\frac{1}{3}\,\frac{(c+s-1)\,(tc+t^2s-1)}{(t^2c+ts-1)^2}=sc-\frac{1}{3}+\frac{d}{du}\,\frac{tc^2-t^2s^2}{t^2c+ts-1},$$

and therefore

$$\frac{i}{\sqrt{3}}f(U,\,2)=f(u)-\frac{1}{3}\,u+\frac{tc^2-t^2s^2}{t^2c+ts-1}+\frac{i}{\sqrt{3}}.$$

A second integration gives

$$\frac{g(U,\,2)}{g(0,\,2)}=\frac{g(u)}{g(0)}\;e^{-\frac{1}{6}u^2+\frac{i}{\sqrt{3}}u}\left[t^2\ \mathrm{cm}\ u+t\ \mathrm{sm}\ u-1\right].$$

Two other transformations may be derived from these by putting u equal either to tu or to t^2u. Such formulas could be written down at once.

TRANSFORMATION OF THE SECOND ORDER

We have seen that $\mathrm{cm}\,\frac{3}{2}K=-1$ and that $\mathrm{sm}^3\,\frac{3}{2}K=2$. Let b represent the real root of this equation. Then from the transformation formulas on page 39, putting s, c, s', c' for $\mathrm{sm}\ u$, $\mathrm{cm}\ u$, $\mathrm{sm}\left(u+\frac{3}{2}K\right)$ and $\mathrm{cm}\left(u+\frac{3}{2}K\right)$ respectively, we have

$$s+s'+bcc'=0,$$

$$c+c'+bss'=0,$$

$$sc'+s'c=b.$$

From these results it is seen that

$$(s+s')^3+(c+c')^3=2+3cc'(c+c')+3ss'(s+s')=b^3-\frac{6}{b}\,(s+s')\,(c+c').$$

Therefore

$$(ss')^3+(cc')^3-3\,bss'cc'+1=0,$$

while we also find

$$\frac{d}{du}\,(-ss')=-b\,(cc')^2+b^2ss'.$$

Furthermore, when

$$u=0,\ \ -ss'=0,\ \ -cc'=1.$$

Hence

$$\mathrm{sm}\,(-bu,-b)=-\mathrm{sm}\ u\ \mathrm{sm}\left(u+\frac{3K}{2}\right)=\frac{s(s^3+b^2c)}{s-bc^2},$$

$$\mathrm{cm}\,(-bu,-b)=-\mathrm{cm}\ u\ \mathrm{cm}\left(u+\frac{3K}{2}\right)=-\frac{c(sc+b)}{s-bc^2}.$$

There are three such transformations, one for each of the roots of the equation $b^3=2$.

We find that

$$\text{sm}\left(-\frac{3bt^2K}{2}\right)=-\text{sm}\,\frac{3t^2K}{2}\,\text{sm}\left(-\frac{3tK}{2}\right)=-\text{sm}^2\,\frac{3K}{2}=-2^{\frac{2}{3}}=-b^2.$$

When an α is present, this transformation is expressed as

$$\text{sm}\left(-bu,\frac{b\alpha-2}{b^2}\right),$$

that is, the new α is $\frac{b\alpha-2}{b^2}$.

Now we shall transform sm $(-bu,\ -b)$ or sm $(-2^{\frac{1}{3}}u,\ -2^{\frac{1}{3}})$ with the new $b=-2^{\frac{1}{3}}$, then

$$\text{sm}\ (-2u)=-\text{sm}(-2^{\frac{1}{3}}u,\ -2^{\frac{1}{3}})\ \text{sm}\left(-2^{\frac{1}{3}}u-\frac{3}{2}\,2^{\frac{1}{3}}t^2K,\ -2^{\frac{1}{3}}\right)$$

$$=-\text{sm}\ u\ \text{sm}\left(u+\frac{3K}{2}\right)\text{sm}\left(u+\frac{3tK}{2}\right)\text{sm}\left(u+\frac{3t^2K}{2}\right),$$

since the new α is zero. We can thus duplicate the argument by two second-degree transformations followed by a negative transformation.

We have, further,

$$f(-bu,\ -b)=-b\int_0^u ss'cc'du=-\frac{1}{b}\int_0^u(s+s')\ (c+c')\ du$$

$$=-\frac{1}{b}\int_0^u[sc+s'c'+sc'+s'c]\ du=-\frac{1}{b}\int_0^u(sc+s'c'+b)\ du$$

$$=-\frac{1}{b}\left[f(u)+f\left(u+\frac{3K}{2}\right)-f\left(\frac{3K}{2}\right)+bu\right].$$

From the formulas on page 25 it may be seen that

$$f\left(\frac{3K}{2}\right)=\frac{3k}{2}-\frac{1}{b}=\frac{3k}{2}-\frac{1}{2^{\frac{1}{3}}}.$$

A second integration gives the formula

$$\frac{g(-bu,-b)}{g(0,-b)}=\frac{g(u)\,g\left(u+\frac{3}{2}K\right)}{g(0)g\left(\frac{3}{2}K\right)}\,e^{\frac{1}{2}bu^2-\left(\frac{3}{2}k-\frac{1}{b}\right)u}.$$

TRANSFORMATION OF THE THIRD ORDER

Let us suppose that we wish to construct functions with the periods K and $3tK$. The functions sm u sm $(u+K)$ sm $(u+2K)$ and cm u cm $(u+K)$ cm $(u+2K)$ are each equal to -1, and hence are not available for our purpose. If we set

$$x=\text{sm}\ u+\text{sm}\ (u+K)+\text{sm}(u+2K),$$

and

$$y=\text{cm}\ u+\text{cm}(u+K)+\text{cm}(u+2K),$$

then x and y are such functions as we want.

$$x=s+\frac{1}{c}-\frac{c}{s},$$

$$y=c+\frac{1}{s}-\frac{s}{c}.$$

so that we get y from x by the interchange of s and c.

$$\frac{dx}{du}=c^2+\frac{s^2}{c^2}+\frac{1}{s^2}=y^2+2x,$$

$$\frac{dy}{du}=-s^2-\frac{c^2}{s^2}-\frac{1}{c^2}=-x^2-2y.$$

Hence

$$x^2\frac{dx}{du}+y^2\frac{dy}{du}+2\left(x\frac{dy}{du}+y\frac{dx}{du}\right)=0,$$

or

$$x^3+y^3+6xy=\text{a constant}.$$

To find the value of this constant, let

$$u=\frac{1}{2}K,\text{ then }x=y=\frac{1}{\sqrt[3]{2}}-1+\sqrt[3]{2},\ x^3+y^3+6xy=2x^2(x+3)$$

$$=2\left(\frac{1}{\sqrt[3]{2}}-1+\sqrt[3]{2}\right)^2\left(\frac{1}{\sqrt[3]{2}}+2+\sqrt[3]{2}\right)=9.$$

Thus

$$\frac{9}{y^3}+\left(-\frac{x}{y}\right)^3-\frac{6x}{y^2}=1,\quad \frac{d}{du}\frac{1}{y}=\frac{x^2}{y^2}+2\frac{1}{y},$$

while, when

$$u=0,\ \frac{1}{y}=0\text{ and }-\frac{x}{y}=1.$$

Then

$$\text{sm}\left(3^{\frac{1}{2}}u,-\frac{2}{3^{\frac{1}{2}}}\right)=\frac{3^{\frac{1}{2}}sc}{c-s^2+sc^2},$$

$$\text{cm}\left(3^{\frac{1}{2}}u,-\frac{2}{3^{\frac{1}{2}}}\right)=\frac{-s+c-s^2c}{c-s^2+sc^2}.$$

Two other similar transformations could be made starting with the expressions

$$\text{sm } u+\text{sm } (u+tK)+\text{sm } (u+2tK),$$

and

$$\text{sm } u+\text{sm } (u+t^2K)+\text{sm } (u+2t^2K),$$

which, however, have no added interest.

Transformations of higher orders could be obtained, but since they all lead to functions other than those with $\alpha=0$ they are rather beside our present purpose. It is more directly in line with our work in hand to derive the transformations that reduce the functions sm u and cm u to the Weierstrassian functions and to those of Jacobi. Such transformations will now be given.

TRANSFORMATIONS REDUCING sm u AND cm u TO THE CONGRUENT WEIERSTRASSIAN FUNCTIONS

We note that sm u cm u has a double infinity for the value $2K$ as also for $2tK$ and $2t^2K$. At $2K$ the infinity behaves like $-\frac{1}{(2K-u)^2}$. Let us apply the transformation $u=2K+u$, so that the double infinity may be transferred to the origin,

$$\text{sm } (2K+u) \text{ cm } (2K+u)=-\frac{\text{cm } u}{\text{sm}^2 u}=-\frac{1}{u^2}+P(u),$$

$P(u)$ denoting an integral power series in u. The Weierstrassian function $p(u)$ has a double infinity at the origin, and in this neighborhood it is of the form $p(u)=\frac{1}{u^2}+Q(u)$, $Q(u)$, denoting an integral power series in u. This would lead us to suspect that $p(u)$ might be equal to $\frac{\text{cm } u}{\text{sm}^2 u}$. To verify this supposition, let

$$x=\frac{\text{cm } u}{\text{sm}^2 u},$$

$$\frac{dx}{du}=-\frac{1+\text{cm}^3 u}{\text{sm}^3 u},$$

$$4x^3+1=\frac{4\text{ cm}^3 u+\text{sm}^6 u}{\text{sm}^6 u}=\frac{4\text{ cm}^3 u+(1-\text{cm}^3 u)^2}{\text{sm}^6 u}=\frac{(1+\text{cm}^3 u)^2}{\text{sm}^6 u}.$$

Therefore

$$\frac{dx}{du}=-\sqrt{4x^3+1}.$$

We have then

$$u=\int_x^\infty \frac{dx}{\sqrt{4x^3+1}},$$

and

$$x=p(u),\ (g_2=0,\ g_3=-1)$$

$$e_1=-2^{-\frac{2}{3}},\ e_2=-2^{-\frac{2}{3}}t,\ e_3=-2^{-\frac{2}{3}}t^2.$$

Hence we find that $p(u)$ does equal the given function $\frac{\text{cm}\,u}{\text{sm}^2\,u}$,

or

$$p(u)=\frac{\text{cm}\,u}{\text{sm}^2\,u}.$$

The right-hand member has the real period $3K$; hence $\omega_1=\frac{3}{2}K$,

$$\omega_1=\int_{-\frac{1}{2^{\frac{2}{3}}}}^{\infty}\frac{dx}{\sqrt{4x^3+1}}=\int_0^{\infty}\frac{dx}{\sqrt{4x^3+1}}+\int_{-\frac{1}{2^{\frac{2}{3}}}}^{0}\frac{dx}{\sqrt{4x^3+1}}.$$

In the first integral, in the right-hand member, let $4x^3=u$,

$$x=\frac{1}{2^{\frac{2}{3}}}u^{\frac{1}{3}},$$

$$dx=\frac{1}{3\cdot 2^{\frac{2}{3}}}u^{-\frac{2}{3}}du,$$

and we have

$$\int_0^{\infty}\frac{dx}{\sqrt{4x^3+1}}=\frac{1}{3\cdot 2^{\frac{2}{3}}}\int_0^{\infty}\frac{u^{-\frac{2}{3}}du}{(1+u)^{\frac{1}{2}}}=\frac{1}{3\cdot 2^{\frac{2}{3}}}B\left(\frac{1}{3},\frac{1}{6}\right).$$

From the definition of $p(u)$, we see that

$p(K)=0$; hence

$$K=\frac{1}{3\cdot 2^{\frac{2}{3}}}B\left(\frac{1}{3},\frac{1}{6}\right)=\frac{1}{3\cdot 2^{\frac{2}{3}}}\,\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{2}\right)}.$$

In the second integral, in the right-hand member, let

$$4x^3=-u,$$

$$x=-\frac{1}{2^{\frac{2}{3}}}u^{\frac{1}{3}},$$

$$dx=-\frac{1}{3\cdot 2^{\frac{2}{3}}}u^{-\frac{2}{3}}du,$$

then

$$\int_{-\frac{1}{2^{2/3}}}^{0} \frac{dx}{\sqrt{4x^3+1}} = \frac{1}{3\cdot 2^{2/3}} \int_0^1 u^{-\frac{2}{3}} (1-u)^{-\frac{1}{2}} du = \frac{1}{3\cdot 2^{2/3}} B\left(\frac{1}{3}, \frac{1}{2}\right)$$

$$= \frac{1}{3\cdot 2^{2/3}} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)}.$$

If we call this value z, we have

$$\frac{z}{K} = \frac{\frac{1}{3\cdot 2^{2/3}} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)}}{\frac{1}{3\cdot 2^{2/3}} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{2}\right)}} = \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{5}{6}\right)} = \frac{\pi}{\frac{\pi}{\sin\frac{\pi}{6}}} = \sin\frac{\pi}{6} = \frac{1}{2}.$$

Therefore, as already stated

$$\omega_1 = K + \frac{1}{2} K = \frac{3}{2} K.$$

For the complex period, we shall take the definite integral

$$\omega_2 = \int_{-\frac{t}{2^{2/3}}}^{\infty} \frac{dx}{\sqrt{4x^3+1}} = \int_{-\frac{t}{2^{2/3}}}^{0} \frac{dx}{\sqrt{4x^3+1}} + \int_0^{\infty} \frac{dx}{\sqrt{4x^3+1}} = \int_{-\frac{t}{2^{2/3}}}^{0} \frac{dx}{\sqrt{4x^3+1}} + K.$$

In the first part of the right-hand member set $x = tx$, and we have

$$\int_{-\frac{t}{2^{2/3}}}^{0} \frac{dx}{\sqrt{4x^3+1}} = t \int_{-\frac{1}{2^{2/3}}}^{0} \frac{dx}{\sqrt{4x^3+1}} = \frac{1}{2} tK.$$

Therefore

$$\omega_2 = K + \frac{1}{2} tK,$$

and the full period is

$$2\omega_2 = 2K + tK.$$

A Weierstrass p function constructed with the periods $3K$ and $2K + tK$ will be congruent with the functions sm u and cm u. The relation of the period parallelograms is shown in Figure 3, in which the fundamental periods for sm u and cm u are taken as $3K$ and $3sK$. It should be noted that the area of the parallelogram for $p(u)$ has one-third the area of that for sm u and cm u. This is evidently theoretically required, for each zero of sm u is a double infinity for $p(u)$;

also $p(u)$ has but the single double infinity in the fundamental parallelogram. The six infinities of $\frac{\text{cm } u}{\text{sm}^2 u}$ due to the duplicate infinity for each zero of sm u are divided into the three double infinities in the three parallelograms of $p(u)$.

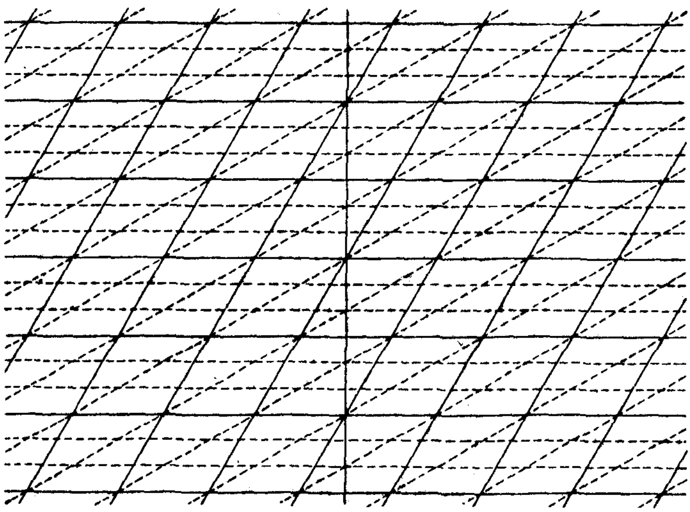

FIG. 3—Relation of the period parallelograms of the Dixon elliptic functions to those of the congruent Weierstrassian functions

IDENTIFICATION OF THE ABELIAN FUNCTIONS WITH KNOWN WEIERSTRASSIAN FUNCTIONS

This relation that we have established gives us the means of identifying many of the functions already employed. We have used the relation

$$\text{sm } u = \frac{h(u)}{g(u)},$$

$$\log \text{sm } u = \log h(u) - \log g(u),$$

$$\frac{d}{du} \log h(u) - \frac{d}{du} \log g(u) = \frac{\text{cm}^2 u}{\text{sm } u},$$

$$\frac{d^2}{du^2} \log h(u) - \frac{d^2}{du^2} \log g(u) = -\frac{\text{cm } u}{\text{sm}^2 u} - \text{sm } u \text{ cm } u,$$

but by definition

$$\frac{d^2}{du^2} \log g(u) = \text{sm } u \text{ cm } u,$$

hence

$$\frac{d^2}{du^2} \log h(u) = -\frac{\text{cm } u}{\text{sm}^2 u}.$$

In the Weierstrassian theory we have

$$\frac{d^2}{du^2}\log\sigma(u)=-p(u).$$

Therefore

$$\frac{d^2}{du^2}\log h(u)=\frac{d^2}{du^2}\log\sigma(u);$$

by one integration this gives

$$\frac{d}{du}\log h(u)=\frac{d}{du}\log\sigma(u)+A,$$

and a second integration gives

$$\log h(u)=\log\sigma(u)+Au+\log B,$$

or

$$h(u)=Be^{Au}\sigma(u),$$

in which A and B are constants to be determined.

Whatever expression we adopt for $g(u)$, we can make $g(0)$ equal to unity, for it would only be necessary to divide the adopted expression by the constant $g(0)$, and then the value of the resulting expression would be unity for $u=0$. Hereafter we shall consider $g(0)$ to be unity unless a special note is made to the contrary. Now $\frac{\operatorname{sm} u}{u}$ converges to unity as u converges to zero. But

$$\operatorname{sm} u=\frac{h(u)}{g(u)},$$

$$\frac{\operatorname{sm} u}{u}=\frac{\frac{h(u)}{u}}{g(u)},$$

hence $\frac{\frac{h(u)}{u}}{g(u)}$ must converge to unity for $u=0$, and since $g(0)=1$, $\left[\frac{h(u)}{u}\right]_{u=0}$ must equal 1. We know that $\frac{\sigma(u)}{u}$ equals unity when u equals zero. Therefore

$$\frac{h(u)}{u}=Be^{Au}\frac{\sigma(u)}{u}$$

becomes $B=1$ for $u=0$.

Hence

$$h(u)=e^{Au}\sigma(u).$$

From the Weierstrassian theory we have

$$\frac{\sigma(u+\omega)}{\sigma(u-\omega)}=-e^{2\eta u}.$$

Now $\omega=\frac{3}{2}K$, but we must determine the relation between η and k. We have

$$p(2K-u)=-\text{sm }u\text{ cm }u,$$

and hence, by multiplying by du and integrating from 0 to u, we obtain

$$-\zeta(2K-u)+\zeta(2K)=f(u).$$

Let $u=K$ and then $\frac{1}{2}K$ and we obtain the results

$$-\zeta(K)+\zeta(2K)=k,$$

$$-\zeta\left(\frac{3}{2}K\right)+\zeta(2K)=f\left(\frac{1}{2}K\right).$$

On page 25 we have the formula

$$f(u)+f(K-u)=k,$$

or

$$2f\left(\frac{1}{2}K\right)=k,$$

$$f\left(\frac{1}{2}K\right)=\frac{k}{2}.$$

Also from above

$$2\zeta(2K)=f(4K),$$

but

$$f(3K+u)-f(u)=3k,$$

$$f(4K)-k=3k,$$

or

$$f(4K)=4k,$$

hence

$$2\zeta(2K)=4k,$$

$$\zeta(2K)=2k.$$

Since $\frac{3}{2}K$ is a half period

$$\zeta\left(\frac{3}{2}K\right)=\eta,$$

hence

$$-\eta+2k=\frac{1}{2}k,$$

or

$$\eta=\frac{3}{2}k.$$

Returning to the Weierstrassian formula

$$\frac{\sigma(u+\omega)}{\sigma(u-\omega)} = -e^{2\eta u},$$

we see that it becomes

$$\frac{\sigma\left(u+\frac{3}{2}K\right)}{\sigma\left(u-\frac{3}{2}K\right)} = -e^{3ku}.$$

Now let $u=\frac{1}{2}K$ and we have

$$\frac{\sigma(2K)}{\sigma(K)} = e^{\frac{3}{2}kK}.$$

But

$$p'(u) = -\frac{\sigma(2u)}{\sigma^4(u)},$$

$$p'(K) = -\frac{\sigma(2K)}{\sigma^4(K)} = -1,$$

since $p'(K)$ is equal to -1.
Hence

$$\sigma(2K) = \sigma^4(K),$$

which combined with the expression above gives

$$\sigma^3(K) = e^{\frac{3}{2}kK},$$

or

$$\sigma(K) = e^{\frac{1}{2}kK}.$$

Returning now to the formula

$$h(u) = e^{Au}\,\sigma(u),$$

and remembering that

$$h(u+K) = e^{ku+\frac{1}{2}kK}\,g(u),$$

we get

$$h(K) = e^{\frac{1}{2}kK} = \sigma(K),$$

hence

$$h(K) = e^{AK}\,\sigma(K),$$

or

$$e^{AK} = 1,$$

and

$$A = 0.$$

We have finally established the fact that $h(u)$ and $\sigma(u)$ are identical. We have

$$p(u) = \frac{\sigma(K+u)\sigma(K-u)}{\sigma^2(u)\sigma^2(K)} = \frac{\mathrm{cm}\,u}{\mathrm{sm}^2 u} = \frac{g(u)g(-u)}{h^2(u)}.$$

Since $h(u)$ and $\sigma(u)$ are identical, we have

$$g(u)\,g(-u) = \frac{\sigma(K+u)\sigma(K-u)}{\sigma^2(K)} = e^{-kK}\,\sigma(K+u)\,\sigma(K-u).$$

We have defined $g(u)$, such that

$$\frac{d^2}{du^2}\log g(u) = \text{sm } u \text{ cm } u.$$

Hence

$$g(u)g''(u) - [g'(u)]^2 = h(u)\, g(-u).$$

Also

$$\frac{d^2}{du^2}\log h(u) = -\frac{\text{cm } u}{\text{sm}^2 u},$$

and so

$$h(u)h''(u) - [h'(u)]^2 = -g(u)\, g(-u).$$

From

$$\frac{g(-u)}{g(u)} = \text{cm } u,$$

we have

$$\frac{d^2}{du^2}\log g(-u) - \frac{d^2}{du^2}\log g(u) = -\frac{\text{sm } u}{\text{cm}^2 u} - \text{sm } u \text{ cm}$$

Therefore

$$\frac{d^2}{du^2}\log g(-u) = -\frac{\text{sm } u}{\text{cm}^2 u},$$

or

$$-g(-u)\, g''(-u) + [g'(-u)]^2 = h(u)\, g(u)$$

From the formula

$$p(u) + \frac{1}{2^{\frac{1}{3}}} = \frac{\sigma_1{}^2(u)}{\sigma^2(u)},$$

we have

$$g(u)\, g(-u) + \frac{1}{2^{\frac{1}{3}}}\, \sigma^2(u) = \sigma_1{}^2(u).$$

Also

$$p(u) + \frac{t}{2^{\frac{1}{3}}} = \frac{\sigma_2{}^2(u)}{\sigma^2(u)},$$

or

$$g(u)\, g(-u) + \frac{t}{2^{\frac{1}{3}}}\, \sigma^2(u) = \sigma_2{}^2(u),$$

and

$$p(u) + \frac{t^2}{2^{\frac{1}{3}}} = \frac{\sigma_3{}^2(u)}{\sigma^2(u)},$$

or

$$g(u)\, g(-u) + \frac{t^2}{2^{\frac{1}{3}}}\, \sigma^2(u) = \sigma_3{}^2(u).$$

From these we see that

$$\sigma_1(t^2 u) = \sigma_2(u),$$

$$\sigma_1(tu) = \sigma_3(u),$$

since $\sigma(t^2 u) = h(t^2 u) = t^2 h(u) = t^2 \sigma(u)$,
and so also for tu.

SOME NUMERICAL VALUES OF THE ABELIAN FUNCTIONS

We have already seen that with $g(0)=1$

$$g(K)=e^{\frac{1}{2}kK},$$

$$h(K)=\sigma\ (K)=e^{\frac{1}{2}kK},$$

$$g(-K)=0,$$

$$g(2K)=0,$$

$$h(2K)=e^{2kK},$$

$$g(-2K)=-e^{2kK},$$

$$g(3K)=-e^{\frac{9kK}{2}},$$

$$h(3K)=0,$$

$$g(-3K)=-e^{\frac{9kK}{2}},$$

$$\sigma\left(\frac{3}{2}K\right)=h\left(\frac{3}{2}K\right)=\frac{\sqrt[3]{2}\,e^{\frac{9kK}{8}}}{\sqrt[4]{3}},$$

$$g\left(\frac{3}{2}K\right)=\frac{1}{\sqrt[3]{2}}h\left(\frac{3}{2}K\right)=\frac{1}{\sqrt[4]{3}}e^{\frac{9kK}{8}},$$

$$g\left(-\frac{3}{2}K\right)=-g\left(\frac{3}{2}K\right)=-\frac{1}{\sqrt[4]{3}}e^{\frac{9kK}{8}},$$

$$\sigma\left(\frac{1}{2}K\right)=h\left(\frac{1}{2}K\right)=\frac{1}{\sqrt[4]{3}}e^{\frac{1}{8}kK},$$

$$g\left(\frac{1}{2}K\right)=\sqrt[3]{2}\,h\left(\frac{1}{2}K\right)=\frac{\sqrt[3]{2}}{\sqrt[4]{3}}e^{\frac{1}{8}kK},$$

$$g\left(-\frac{1}{2}K\right)=\frac{1}{\sqrt[3]{2}}\,g\left(\frac{1}{2}K\right)=\frac{1}{\sqrt[4]{3}}\,e^{\frac{1}{8}kK}.$$

DEVELOPMENT IN SERIES

Let us assume the series

$$\operatorname{sm} u=a_0u+a_1u^4+a_2u^7+a_3u^{10}+a_4u^{13}+\ \ldots\ldots\ldots,$$

$$\operatorname{cm} u=b_0+b_1u^3+b_2u^6+b_3u^9+b_4u^{12}+\ \ldots\ldots\ldots,$$

then

$$(3n+1)\ a_n=\sum_{s=0}^{s=n}b_sb_{n-s},$$

$$3nb_n=-\sum_{s=0}^{s=n-1}a_sa_{n-s-1},$$

n being a whole number greater than zero and $a_0=1$, $b_0=1$. This gives recurring formulas from which the successive a's and b's can be computed.

We have

$$\frac{d^2}{du^2}\log g(u)=\text{sm } u \text{ cm } u=u-\frac{1}{2}u^4+\frac{1}{7}u^7-\frac{1}{28}u^{10}+\frac{3}{364}u^{13}-\ \ldots,$$

which by one integration from 0 to u becomes

$$\frac{d}{du}\log g(u)=\frac{1}{2}u^2-\frac{1}{10}u^5+\frac{1}{56}u^8-\frac{1}{308}u^{11}+\frac{3}{5096}u^{14}-\ \ldots\ldots,$$

and by a second integration

$$\log g(u)=\frac{1}{6}u^3-\frac{1}{60}u^6+\frac{1}{504}u^9-\frac{1}{3696}u^{12}+\frac{1}{25480}u^{15}-\ \ldots\ldots,$$

and

$$g(u)=1+\frac{1}{6}u^3-\frac{1}{360}u^6-\frac{1}{45360}u^9+\ \ldots\ldots\ldots,$$

$$g(-u)=1-\frac{1}{6}u^3-\frac{1}{360}u^6+\frac{1}{45360}u^9+\ \ldots\ldots,$$

In a similar way, we find that

$$h(u)=u+\frac{1}{840}u^7-\ \ldots\ldots,$$

or $h(u)$ can be written down from the known development of $\sigma(u)$ for $g_2=0$ and $g_3=-1$.

These series confirm our former conclusion that $g(u)$ was a power series in u^3 and that $h(u)$ was u times a power series in u^6. These developments are not particularly important, since for computation it is more convenient to use trigonometric series.

TRIGONOMETRIC DEVELOPMENTS

Since $h(u)$ is identical with $\sigma(u)$ we can adapt the expressions given for $\sigma(u)$ to serve for $h(u)$

$$q=e^{\pi i\frac{\omega_2}{\omega_1}}=e^{\pi i\frac{2K+iK}{3K}}=e^{\pi i\frac{\frac{3}{2}K+\frac{\sqrt{3}}{2}iK}{3K}}$$

$$=e^{\frac{\pi i}{2}-\frac{\pi}{2\sqrt{3}}}=e^{\frac{\pi i}{2}}e^{-\frac{\pi}{2\sqrt{3}}}=ie^{-\frac{\pi}{2\sqrt{3}}}.$$

If we take $q=e^{-\frac{\pi}{2\sqrt{3}}}$ and then substitute iq for q in the expressions for $\sigma(u)$, we shall have the correct expression for $h(u)$.

$$h(u)=\frac{3K}{\pi}e^{\frac{ku^2}{2K}}\sin\frac{\pi u}{3K}\prod_1^\infty\frac{1-(-1)^m2q^{2m}\cos\frac{2\pi u}{3K}+q^{4m}}{[1-(-1)^m q^{2m}]^2},$$

$$g(u) = e^{-ku-\frac{1}{2}kK}\, h(u+K),$$

$$= \frac{3K}{\pi} e^{\frac{ku^2}{2K}} \sin\left(\frac{\pi u}{3K}+\frac{\pi}{3}\right) \prod_1^\infty \frac{1-(-1)^m 2q^{2m}\cos\left(\frac{2\pi u}{3K}+\frac{2\pi}{3}\right)+q^{4m}}{[1-(-1)^m q^{2m}]^2},$$

$$g(-u) = \frac{3K}{\pi} e^{\frac{ku^2}{2K}} \sin\left(\frac{\pi}{3}-\frac{\pi u}{3K}\right) \prod_1^\infty \frac{1-(-1)^m 2q^{2m}\cos\left(\frac{2\pi}{3}-\frac{2\pi u}{3K}\right)+q^{4m}}{[1-(-1)^m q^{2m}]^2},$$

$$h(u) = \frac{3K}{2\pi} e^{\frac{ku^2}{2K}} \frac{2q^{\frac{1}{4}}\sin\frac{\pi u}{3K} - 2q^{\frac{9}{4}}\sin\frac{3\pi u}{3K} + 2q^{\frac{25}{4}}\sin\frac{5\pi u}{3K} - \dots}{q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - \dots}.$$

Now substitute iq for q and drop the factor $(iq)^{\frac{1}{4}}$ from numerator and denominator and we get

$$h(u) = \frac{3K}{\pi} e^{\frac{ku^2}{2K}} \frac{\sin\frac{\pi u}{3K} + q^2\sin\frac{3\pi u}{3K} - q^6 \sin\frac{5\pi u}{3K} - \dots}{1+3q^2-5q^6 - \dots},$$

$$g(u) = \frac{3K}{\pi} e^{\frac{ku^2}{2K}} \frac{\sin\left(\frac{\pi u}{3K}+\frac{\pi}{3}\right) + q^2\sin\left(\frac{3\pi u}{3K}+\frac{3\pi}{3}\right) - q^6\sin\left(\frac{5\pi u}{3K}+\frac{5\pi}{3}\right) - \dots}{1+3q^2-5q^6 - \dots}$$

These expressions give

$$\operatorname{sm} u = \frac{\begin{aligned}&\sin\frac{\pi u}{3K} + q^2\sin\frac{3\pi u}{3K} - q^6\sin\frac{5\pi u}{3K} - q^{12}\sin\frac{7\pi u}{3K} + q^{20}\sin\frac{9\pi u}{3K}\\ &+ q^{30}\sin\frac{11\pi u}{3K} - q^{42}\sin\frac{13\pi u}{3K} - q^{56}\sin\frac{15\pi u}{3K} + \dots\end{aligned}}{\begin{aligned}&\sin\left(\frac{\pi u}{3K}+\frac{\pi}{3}\right) + q^2\sin\left(\frac{3\pi u}{3K}+\frac{3\pi}{3}\right) - q^6\sin\left(\frac{5\pi u}{3K}+\frac{5\pi}{3}\right) - q^{12}\sin\left(\frac{7\pi u}{3K}\right.\\ &\left.+\frac{7\pi}{3}\right) + q^{20}\sin\left(\frac{9\pi u}{3K}+\frac{9\pi}{3}\right) + q^{30}\sin\left(\frac{11\pi u}{3K}+\frac{11\pi}{3}\right)\\ &- q^{42}\sin\left(\frac{13\pi u}{3K}+\frac{13\pi}{3}\right) - q^{56}\sin\left(\frac{15\pi u}{3K}+\frac{15\pi}{3}\right) + \dots\end{aligned}}$$

$$\operatorname{cm} u = \frac{\begin{aligned}&\sin\left(\frac{\pi}{3}-\frac{\pi u}{3K}\right) + q^2\sin\left(\frac{3\pi}{3}-\frac{3\pi u}{3K}\right) - q^6\sin\left(\frac{5\pi}{3}-\frac{5\pi u}{3K}\right) - q^{15}\sin\left(\frac{7\pi}{3}\right.\\ &\left.-\frac{7\pi u}{3K}\right) + q^{20}\sin\left(\frac{9\pi}{3}-\frac{9\pi u}{3K}\right) + q^{30}\sin\left(\frac{11\pi}{3}-\frac{11\pi u}{3K}\right)\\ &- q^{42}\sin\left(\frac{13\pi}{3}-\frac{13\pi u}{3K}\right) - q^{56}\sin\left(\frac{15\pi}{3}-\frac{15\pi u}{3K}\right) + \dots\end{aligned}}{\begin{aligned}&\sin\left(\frac{\pi}{3}+\frac{\pi u}{3K}\right) + q^2\sin\left(\frac{3\pi}{3}+\frac{3\pi u}{3K}\right) - q^6\sin\left(\frac{5\pi}{3}+\frac{5\pi u}{3K}\right) - q^{15}\sin\left(\frac{7\pi}{3}\right.\\ &\left.+\frac{7\pi u}{3K}\right) + q^{20}\sin\left(\frac{9\pi}{3}+\frac{9\pi u}{3K}\right) + q^{3}\sin\left(\frac{11\pi}{3}+\frac{11\pi u}{3K}\right)\\ &- q^{42}\sin\left(\frac{13\pi}{3}+\frac{13\pi u}{3K}\right) - q^{56}\sin\left(\frac{15\pi}{3}+\frac{15\pi u}{3K}\right) + \dots\end{aligned}}$$

in which $q = e^{-\frac{\pi}{2\sqrt{3}}}$.

SOME CONSTANT RESULTS

$$\frac{3K}{\pi}=\frac{1+3q^2-5q^6-7q^{12}+9q^{20}+11q^{30}-13q^{42}-15q^{56}+\ \ldots\ldots}{\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}q^6-\frac{\sqrt{3}}{2}q^{12}-\frac{\sqrt{3}}{2}q^{30}-\frac{\sqrt{3}}{2}q^{42}-\ \ldots\ldots},$$

or

$$\frac{3\sqrt{3}K}{2\pi}=\frac{1+3q^2-5q^6-7q^{12}+9q^{20}+11q^{30}-13q^{42}-15q^{56}+\ \ldots\ldots}{1+q^6-q^{12}-q^{30}-q^{42}-\ \ldots\ldots}.$$

$$\sqrt[3]{2}=\frac{1-q^2-q^6+q^{12}+q^{20}-q^{30}-q^{42}+q^{56}+\ \ldots\ldots}{\frac{1}{2}+q^2-\frac{1}{2}q^6+\frac{1}{2}q^{12}-q^{20}-\frac{1}{2}q^{30}-\frac{1}{2}q^{42}-q^{56}+\ldots\ldots}.$$

$$\frac{2\pi}{3\sqrt[4]{3}K}=\frac{1+2q^2-q^6+q^{12}-2q^{20}-q^{30}-q^{42}-2q^{56}+\ \ldots\ldots}{1+3q^2-5q^6-7q^{12}+9q^{20}+11q^{30}-13q^{42}-15q^{56}+\ \ldots\ldots},$$

$$\frac{2\pi}{3\sqrt[4]{3}K}=\prod_{1}^{\infty}\frac{1-(-1)^m q^{2m}+q^{4m}}{[1-(-1)^m q^{2m}]^2}.$$

A SECOND TRANSFORMATION TO WEIERSTRASSIAN FUNCTIONS

The function $\frac{\text{sm}\ u}{1-\text{cm}\ u}$ has a double infinity at the origin, and hence we shall examine its behavior. Let us set it equal to y,

$$y=\frac{\text{sm}\ u}{1-\text{cm}\ u},$$

then

$$\frac{dy}{du}=\frac{(1-\text{cm}\ u)\ \text{cm}^2\ u-\text{sm}^3\ u}{(1-\text{cm}\ u)^2}$$

$$=\frac{\text{cm}^2\ u-1}{(1-\text{cm}\ u)^2}=-\frac{1+\text{cm}\ u}{1-\text{cm}\ u},$$

and

$$4y^3-1=\frac{4\ \text{sm}^3\ u-(1-\text{cm}\ u)^3}{(1-\text{cm}\ u)^3}$$

$$=\frac{4\ \text{sm}^3\ u-1+3\ \text{cm}\ u-3\ \text{cm}^2\ u+\text{cm}^3\ u}{(1-\text{cm}\ u)^3}$$

$$=\frac{3\ \text{sm}^3\ u+3\ \text{cm}\ u-3\ \text{cm}^2\ u}{(1-\text{cm}\ u)^3}$$

$$=\frac{3\ (1+\text{cm}\ u+\text{cm}^2\ u)+3\ \text{cm}\ u}{(1-\text{cm}\ u)^2}$$

$$=\frac{3\ (1+\text{cm}\ u)^2}{(1-\text{cm}\ u)^2}.$$

Therefore

$$\frac{dy}{du} = -\sqrt{3}\sqrt{4y^3-1},$$

and

$$\frac{u}{\sqrt{3}} = \int_y^\infty \frac{dy}{\sqrt{4y^3-1}},$$

so that

$$y = p\left(\frac{u}{\sqrt{3}}\right),$$

and we have

$$p\left(\frac{u}{\sqrt{3}}\right) = \frac{\text{sm } u}{1-\text{cm } u}.$$

This function gives us the means of expressing sm u and cm u in terms of the p function.

We have

$$\frac{1}{\sqrt{3}} p'\left(\frac{u}{\sqrt{3}}\right) = -\frac{1+\text{cm } u}{1-\text{cm } u},$$

$$1+\frac{1}{\sqrt{3}} p'\left(\frac{u}{\sqrt{3}}\right) = -\frac{2 \text{ cm } u}{1-\text{cm } u},$$

$$1-\frac{1}{\sqrt{3}} p'\left(\frac{u}{\sqrt{3}}\right) = \frac{2}{1-\text{cm } u},$$

so that by division we get

$$\text{cm } u = \frac{p'\left(\frac{u}{\sqrt{3}}\right)+\sqrt{3}}{p'\left(\frac{u}{\sqrt{3}}\right)-\sqrt{3}}.$$

Moreover

$$\text{sm } u = (1-\text{cm } u)\, p\left(\frac{u}{\sqrt{3}}\right)$$

$$= \frac{2\sqrt{3}\, p\left(\frac{u}{\sqrt{3}}\right)}{\sqrt{3}-p'\left(\frac{u}{\sqrt{3}}\right)}.$$

This is the solution in Weierstrassian form of the problem considered by Cayley "On the elliptic function solution of the equation $x^3+y^3=1$" (Collected Mathematical Papers, vol. 12, p. 35). We shall give further consideration to this problem later in this publication.

The roots of the equation $4x^3-1=0$, are $x=\frac{1}{2^{\frac{2}{3}}}, \frac{t}{2^{\frac{2}{3}}}$, and $\frac{t^2}{2^{\frac{2}{3}}}$. When $p\left(\frac{u}{\sqrt{3}}\right)$ has any one of these values, then $p'\left(\frac{u}{\sqrt{3}}\right)=0$; but when $p'\left(\frac{u}{\sqrt{3}}\right)$ equals zero cm $u=-1$. The solution of this equation will therefore give the half periods for the function $p\left(\frac{u}{\sqrt{3}}\right)$. The values of u for which cm $u=-1$ in the fundamental parallellogram are $\frac{3}{2}K$, $\frac{3}{2}tK$, and $\frac{3}{2}t^2K$. The periods for this function are therefore the same as those for the functions sm u and cm u. The fundamental periods can be taken as $3K$ and $3sK$ if it is desired. Of course, if the function is expressed as $p(v)$, then v must be increased by $\sqrt{3}K$ and $\sqrt{3}sK$ to make the period. We have then

$$p\left(\frac{u+3K}{\sqrt{3}}\right)=p\left(\frac{u}{\sqrt{3}}+\sqrt{3}K\right)=p\left(\frac{u}{\sqrt{3}}\right),$$

and

$$p\left(\frac{u+3sK}{\sqrt{3}}\right)=p\left(\frac{u}{\sqrt{3}}+\sqrt{3}sK\right)=p\left(\frac{u}{\sqrt{3}}\right).$$

This transformation is not as interesting in itself as the former one, but it is important as furnishing a means for computing a table of the functions sm u and cm u. A table of the p and p' functions for this case of the Weierstrassian functions was computed by A. G. Haddock and published by A. G. Greenhill in connection with an article entitled "On the trajectory of a particle for the cubic law of resistance" (Proc. Royal Artill. Instit., vol. 17, pp. 1–36, 1889). The table is also given by Jahnke und Emde in "Funktionentafeln mit Formeln und Kurven," pages 73 to 75. By using the tabulated values for p and p' a table was computed for sm u and cm u by employing the relation just established. Two serious errors were noted in the Weierstrass table; the value of $p(u)$ for $r=3$ should be 1537.9625 instead of 1468.820 and the value of $p'(u)$ for $r=35$ should be -75.9603 instead of -73.4302, as given in the table both in the Artillery Journal and in the Jahnke und Emde publication. It should be borne in mind that $r=120$ is the value corresponding to K for which value $p(u)=0$.

TRANSFORMATIONS OF THE FUNCTIONS sm *u* AND cm *u* TO THE JACOBI FORMS

Let us consider the form

$$\sqrt[3]{4}x=-1+\sqrt{3}\cot^2\frac{\phi}{2},$$

$$\sqrt[3]{4}\frac{dx}{d\phi}=-\sqrt{3}\cot\frac{\phi}{2}\operatorname{cosec}^2\frac{\phi}{2}.$$

$$4x^3+1=3\sqrt{3}\cot^2\frac{\phi}{2}-9\cot^4\frac{\phi}{2}+3\sqrt{3}\cot^6\frac{\phi}{2},$$

$$\sqrt{4x^3+1}=3^{\frac{3}{4}}\cot\frac{\phi}{2}\operatorname{cosec}^2\frac{\phi}{2}\sqrt{\sin^4\frac{\phi}{2}-\sqrt{3}\sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2}+\cos^4\frac{\phi}{2}},$$

$$-\frac{\sqrt[3]{4}dx}{\sqrt{4x^3+1}}=\frac{d\phi}{\sqrt[4]{3}\sqrt{\frac{1}{4}(1-\cos\phi)^2-\frac{\sqrt{3}}{4}(1-\cos^2\phi)+\frac{1}{4}(1+\cos\phi)^2}}$$

$$=\frac{2d\phi}{\sqrt[4]{3}\sqrt{2+2\cos^2\phi-\sqrt{3}+\sqrt{3}\cos^2\phi}}$$

$$=\frac{2d\phi}{\sqrt[4]{3}\sqrt{2-\sqrt{3}+(2+\sqrt{3})\cos^2\phi}}$$

$$=\frac{2d\phi}{\sqrt[4]{3}\sqrt{4-(2+\sqrt{3})\sin^2\phi}}$$

$$=\frac{d\phi}{\sqrt[4]{3}\sqrt{1-\frac{2+\sqrt{3}}{4}\sin^2\phi}},$$

hence

$$\int_x^\infty \frac{dx}{\sqrt{4x^3+1}}=\frac{1}{\sqrt[3]{4}\sqrt[4]{3}}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{2+\sqrt{3}}{4}\sin^2\phi}}.$$

If we start with

$$u=\int_0^x \frac{dx}{(1-x^3)^{\frac{2}{3}}},$$

we have shown (p. 46) that

$$y=\frac{(1-x^3)^{\frac{1}{3}}}{x^2}$$

reduces the integral to

$$u=\int_y^\infty \frac{dy}{\sqrt{4y^3+1}},$$

and the further transformation

$$y=-\frac{1}{\sqrt[3]{4}}+\frac{\sqrt{3}}{\sqrt[3]{4}}\frac{1+\cos\phi}{1-\cos\phi}$$

gives the result

$$\sqrt[3]{4}\sqrt[4]{3}u=\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{2+\sqrt{3}}{4}\sin^2\phi}},$$

so that

$$\phi=\operatorname{am}\sqrt[3]{4}\sqrt[4]{3}u,$$

and

$$\cos\phi=\operatorname{cn}\sqrt[3]{4}\sqrt[4]{3}\,u.$$

The transformation therefore becomes

$$p(u)=-\frac{1}{\sqrt[3]{4}}+\frac{\sqrt{3}}{\sqrt[3]{4}}\frac{1+\operatorname{cn}\sqrt[3]{4}\sqrt[4]{3}\,u}{1-\operatorname{cn}\sqrt[3]{4}\sqrt[4]{3}\,u},$$

with

$$\kappa^2=\frac{2+\sqrt{3}}{4},$$

or

$$\kappa=\frac{\sqrt{3}+1}{2\sqrt{2}}=\sin 75^\circ.$$

If L denotes the half period of the Jacobi functions, the fraction is equal to zero when $\sqrt[3]{4}\ \sqrt[4]{3}\ u=2L$; however, when the fraction is zero $u=\frac{3K}{2}$. Therefore we have

$$2L=\sqrt[3]{4}\sqrt[4]{3}\frac{3K}{2},$$

$$L=\frac{3^{\frac{5}{4}}}{4^{\frac{2}{3}}}K.$$

Legendre's table shows that this result is correct.

We can express $\mathrm{sm}^3\ u$ and $\mathrm{cm}^3\ u$ in terms of $p'(u)$. We find

$$\mathrm{sm}^3\ u=\frac{2}{1-p'(u)},$$

$$\mathrm{cm}^3\ u=\frac{p'(u)+1}{p'(u)-1}.$$

However, we can not express sm u and cm u in rational real values in terms of this p function.

On the other hand, we can express cn $\sqrt[3]{4}\ \sqrt[4]{3}\ u$ rationally in terms of $p(u)$ as also of sm u and cm u.

$$\mathrm{cn}\ \sqrt[3]{4}\sqrt[4]{3}u=\frac{\sqrt[3]{4}\ p(u)+1-\sqrt{3}}{\sqrt[3]{4}\ p(u)+1+\sqrt{3}},$$

and

$$\mathrm{cn}\ \sqrt[3]{4}\ \sqrt[4]{3}\,u=\frac{\sqrt[3]{4}\ \mathrm{cm}\ u+(1-\sqrt{3})\ \mathrm{sm}^2\ u}{\sqrt[3]{4}\ \mathrm{cm}\ u+(1+\sqrt{3})\ \mathrm{sm}^2\ u},$$

and finally, after substituting the value of $p'(u)$ in the above expressions, we get

$$\mathrm{sm}^3\ u=\frac{2(1-\mathrm{cn}\ \sqrt[3]{4}\ \sqrt[4]{3}\ u)^{3/2}}{2\cdot 3^{3/4}\ \mathrm{dn}\ \sqrt[3]{4}\ \sqrt[4]{3}\,u\ (1+\mathrm{cn}\ \sqrt[3]{4}\ \sqrt[4]{3}\ u)^{1/2}+(1-\mathrm{cn}\ \sqrt[3]{4}\ \sqrt[4]{3}\,u)^{3/2}}$$

$$\mathrm{cm}^3\ u=\frac{2\cdot 3^{3/4}\ \mathrm{dn}\ \sqrt[3]{4}\ \sqrt[4]{3}\ u(1+\mathrm{cn}\ \sqrt[3]{4}\ \sqrt[4]{3}\ u)^{1/2}-(1-\mathrm{cn}\ \sqrt[3]{4}\ \sqrt[4]{3}\ u)^{3/2}}{2\cdot 3^{3/4}\ \mathrm{dn}\ \sqrt[3]{4}\ \sqrt[4]{3}\ \ u(1+\mathrm{cn}\ \sqrt[3]{4}\ \sqrt[4]{3}\,u)^{1/2}+(1-\mathrm{cn}\sqrt[3]{4}\ \sqrt[4]{3}\,u)^{3/2}}$$

TRANSFORMATION OF THE SECOND p FUNCTION TO THE JACOBI FORM

We shall now transform the other p function to the Jacobi form.

$$\sqrt[3]{4}\,z=1+\sqrt{3}\ \cot^2\frac{\phi}{2}$$

$$\sqrt[3]{4}\,dz=-\sqrt{3}\ \cot\frac{\phi}{2}\ \mathrm{cosec}^2\frac{\phi}{2}\ d\phi,$$

$$4z^3-1=3\sqrt{3}\ \cot^2\frac{\phi}{2}+9\ \cot^4\frac{\phi}{2}+3\sqrt{3}\ \cot^6\frac{\phi}{2},$$

$$\sqrt{4z^3-1}=3^{\frac{3}{4}}\cot\frac{\phi}{2}\operatorname{cosec}^2\frac{\phi}{2}\sqrt{\sin^4\frac{\phi}{2}+\sqrt{3}\sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2}+\cos^4\frac{\phi}{2}},$$

$$-\frac{dz}{\sqrt{4z^3-1}}=\frac{1}{\sqrt[4]{3}\sqrt[3]{4}}\frac{d\phi}{\sqrt{\sin^4\frac{\phi}{2}+\sqrt{3}\sin^2\frac{\phi}{2}\cos^2\frac{\phi}{2}+\cos^4\frac{\phi}{2}}}$$

$$=\frac{1}{\sqrt[4]{3}\sqrt[3]{4}}\frac{d\phi}{\sqrt{\frac{1}{4}(1-\cos\phi)^2+\frac{\sqrt{3}}{4}(1-\cos^2\phi)+\frac{1}{4}(1+\cos\phi)^2}}$$

$$=\frac{\sqrt[6]{4}}{\sqrt[4]{3}}\frac{d\phi}{\sqrt{2+2\cos^2\phi+\sqrt{3}-\sqrt{3}\cos^2\phi}}$$

$$=\frac{\sqrt[6]{4}\,d\phi}{\sqrt[4]{3}\sqrt{2+\sqrt{3}+(2-\sqrt{3})\cos^2\phi}}$$

$$=\frac{\sqrt[6]{4}\,d\phi}{\sqrt[4]{3}\sqrt{4-(2-\sqrt{3})\sin^2\phi}}$$

$$=\frac{1}{\sqrt[4]{3}\sqrt[3]{4}}\frac{d\phi}{\sqrt{1-\frac{2-\sqrt{3}}{4}\sin^2\phi}},$$

hence

$$\int_z^{\infty}\frac{dz}{\sqrt{4z^3-1}}=\frac{1}{\sqrt[4]{3}\sqrt[3]{4}}\int_0^{\phi}\frac{d\phi}{\sqrt{1-\frac{2-\sqrt{3}}{4}\sin^2\phi}}.$$

We have already seen (p. 57) that starting with

$$u=\int_0^x\frac{dx}{(1-x^3)^{\frac{2}{3}}},$$

$$z=\frac{x}{1-(1-x^3)^{\frac{1}{3}}},$$

reduces the integral to

$$\frac{u}{\sqrt{3}}=\int_z^{\infty}\frac{dz}{\sqrt{4z^3-1}}$$

and the final relation

$$\sqrt[3]{4}\,p\left(\frac{u}{\sqrt{3}}\right)=1+\sqrt{3}\cot^2\frac{\phi}{2}$$

reduces the integral to

$$\frac{u}{\sqrt{3}}=\frac{1}{\sqrt[4]{3}\sqrt[3]{4}}\int_0^{\phi}\frac{d\phi}{\sqrt{1-\frac{2-\sqrt{3}}{4}\sin^2\phi}}$$

or

$$\frac{\sqrt[3]{4}}{\sqrt[4]{3}}\,u=\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{2-\sqrt{3}}{4}\sin^2\phi}}.$$

Hence

$$\phi=\operatorname{am}\frac{\sqrt[3]{4}}{\sqrt[4]{3}}u$$

and

$$\cos\phi=\operatorname{cn}\frac{\sqrt[3]{4}}{\sqrt[4]{3}}u.$$

Hence

$$\sqrt[3]{4}\,p\left(\frac{u}{\sqrt{3}}\right)=1+\sqrt{3}\,\frac{1+\operatorname{cn}\frac{\sqrt[3]{4}}{\sqrt[4]{3}}u}{1-\operatorname{cn}\frac{\sqrt[3]{4}}{\sqrt[4]{3}}u},$$

with

$$\kappa^2=\frac{2-\sqrt{3}}{4},$$

or

$$\kappa=\frac{\sqrt{3}-1}{2\sqrt{2}}=\sin 15^\circ.$$

In this case

$$2L=\frac{\sqrt[3]{4}}{\sqrt[4]{3}}\,\frac{3}{2}\,K,$$

or

$$L=\frac{3^{\frac{3}{4}}}{4^{\frac{2}{3}}}\,K,$$

which can also be tested by means of Legendre's tables of the first elliptic integrals.

CAYLEY'S PROBLEM

This is the function which Cayley used in the memoir referred to on page 57. The same transformation is given as a problem in Whittaker and Watson's Modern Analysis, second edition, page 526. There are two mistakes in the statement as there given, so that the student would have difficulty in proving the exercise. These errors will be pointed out in the course of the analysis that will now be given.

We have already derived the formulas

$$\operatorname{sm} u=\frac{2\sqrt{3}\,p\left(\frac{u}{\sqrt{3}}\right)}{\sqrt{3}-p'\left(\frac{u}{\sqrt{3}}\right)},$$

and

$$\operatorname{cm} u = \frac{p'\left(\frac{u}{\sqrt{3}}\right) + \sqrt{3}}{p'\left(\frac{u}{\sqrt{3}}\right) - \sqrt{3}}.$$

But

$$\sqrt[3]{4}\, p\left(\frac{u}{\sqrt{3}}\right) = 1 + \sqrt{3}\, \frac{1 + \operatorname{cn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u}{1 - \operatorname{cn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u},$$

so that

$$p'\left(\frac{u}{\sqrt{3}}\right) = -\frac{2 \cdot 3^{\frac{1}{4}} \operatorname{sn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u \operatorname{dn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u}{\left(1 - \operatorname{cn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u\right)^2}.$$

By substituting these values we get

$$\operatorname{sm} u = \frac{\sqrt[3]{2}\left(1 - \operatorname{cn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u\right)^2 + \sqrt[3]{2}\sqrt{3}\left(1 - \operatorname{cn}^2 \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u\right)}{\left(1 - \operatorname{cn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u\right)^2 + 2\sqrt[4]{3} \operatorname{sn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u \operatorname{dn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u},$$

$$\operatorname{cm} u = \frac{2\sqrt[4]{3} \operatorname{sn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u \operatorname{dn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u - \left(1 - \operatorname{cn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u\right)^2}{2\sqrt[4]{3} \operatorname{sn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u \operatorname{dn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u + \left(1 - \operatorname{cn} \frac{\sqrt[3]{4}}{\sqrt[4]{3}} u\right)^2}.$$

To make the expression simpler, we shall replace $\frac{\sqrt[3]{4}}{\sqrt[4]{3}} u$ by u and we get

$$\operatorname{sm} \frac{\sqrt[4]{3}}{\sqrt[3]{4}} u = \frac{\sqrt[3]{2}\,(1 - \operatorname{cn} u)[1 + \sqrt{3} + (\sqrt{3} - 1) \operatorname{cn} u]}{2\sqrt[4]{3} \operatorname{sn} u \operatorname{dn} u + (1 - \operatorname{cn} u)^2},$$

and

$$\operatorname{cm} \frac{\sqrt[4]{3}}{\sqrt[3]{4}} u = \frac{2\sqrt[4]{3} \operatorname{sn} u \operatorname{dn} u - (1 - \operatorname{cn} u)^2}{2\sqrt[4]{3} \operatorname{sn} u \operatorname{dn} u + (1 - \operatorname{cn} u)^2}.$$

The first expression can also be transformed as follows:

$$\operatorname{sm} \frac{\sqrt[4]{3}}{\sqrt[3]{4}} u = \frac{\sqrt[3]{2}\,(\sqrt{3} + 1)(1 - \operatorname{cn} u)\left(1 + \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \operatorname{cn} u\right)}{2\sqrt[4]{3} \operatorname{sn} u \operatorname{dn} u + (1 - \operatorname{cn} u)^2},$$

but

$$\frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \tan \frac{\pi}{12},$$

and

$$\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}},$$

or

$$\sqrt{3}+1=2^{\frac{3}{2}}\cos\frac{\pi}{12},$$

and

$$\sqrt[3]{2}\,(\sqrt{3}+1)=2^{\frac{11}{6}}\cos\frac{\pi}{12}.$$

Now the expression becomes

$$\operatorname{sm}\frac{\sqrt[4]{3}}{\sqrt[3]{4}}u=\frac{2^{\frac{11}{6}}\cos\frac{\pi}{12}(1-\operatorname{cn}u)\left(1+\tan\frac{\pi}{12}\operatorname{cn}u\right)}{2\sqrt[4]{3}\operatorname{sn}u\operatorname{dn}u+(1-\operatorname{cn}u)^2}.$$

This is the form in which the solution is given in Whittaker and Watson previously referred to. It should be noted that y in their expression corresponds to the sm function as given here. It may be seen that in their expression for y the factor $2^{\frac{3}{2}}$ should be $2^{\frac{11}{6}}$ and the u should be $\frac{\sqrt[4]{3}}{\sqrt[3]{4}}$ times the integral instead of $\sqrt[3]{2}\sqrt[4]{3}$ as given by them.

In Cayley's memoir already referred to the coordinates are given in the form

$$x=\frac{2\sqrt[4]{3}\operatorname{sn}u\operatorname{dn}u-(1+\operatorname{cn}u)^2}{2\sqrt[4]{3}\operatorname{sn}u\operatorname{dn}u+(1+\operatorname{cn}u)^2},$$

$$y=\frac{\sqrt[3]{2}\,(1+\operatorname{cn}u)[\sqrt{3}+1-(\sqrt{3}-1)\operatorname{cn}u]}{2\sqrt[4]{3}\operatorname{sn}u\operatorname{dn}u+(1+\operatorname{cn}u)^2}.$$

This result is obtained from the former result by increasing u by $2L$, which changes the sign of cn u and sn u. We must then set x equal to $\frac{1}{\operatorname{cm}}$ and y to $-\frac{\operatorname{sm}}{\operatorname{cm}}$, which, of course, is a transformation of the x and y. In Cayley's collected papers there is a mistake of sign in the formula for y at the top of page 36 of volume 12; it can be checked up by the correct form for y^3 given near the bottom of the page. There are several other typographical errors in this memoir in the collected papers.

Various other forms could be given for x and y by transformations of the elliptic functions. We shall list a few.

$$x=\frac{2\sqrt[4]{3}\operatorname{sn}u\operatorname{dn}u+(1+\operatorname{cn}u)^2}{2\sqrt[4]{3}\operatorname{sn}u\operatorname{dn}u-(1+\operatorname{cn}u)^2},$$

$$y=\frac{2^{\frac{11}{6}}\cos\frac{\pi}{12}(1+\operatorname{cn}u)\left(1-\tan\frac{\pi}{12}\operatorname{cn}u\right)}{(1+\operatorname{cn}u)^2-2\sqrt[4]{3}\operatorname{sn}u\operatorname{dn}u},$$

$$x=\frac{2\sqrt[4]{3}\cos\frac{\pi}{12}\operatorname{cn}u-\left(\operatorname{dn}u+\cos\frac{\pi}{12}\operatorname{sn}u\right)^2}{2\sqrt[4]{3}\cos\frac{\pi}{12}\operatorname{cn}u+\left(\operatorname{dn}u+\cos\frac{\pi}{12}\operatorname{sn}u\right)^2},$$

$$y=\frac{2^{\frac{11}{8}}\cos\frac{\pi}{12}\left(\operatorname{dn}u+\cos\frac{\pi}{12}\operatorname{sn}u\right)\left(\operatorname{dn}u-\sin\frac{\pi}{12}\operatorname{sn}u\right)}{2\sqrt[4]{3}\cos\frac{\pi}{12}\operatorname{cn}u+\left(\operatorname{dn}u+\cos\frac{\pi}{12}\operatorname{sn}u\right)^2},$$

$$x=\frac{2\sqrt[4]{3}\,i\operatorname{sn}u\operatorname{dn}u-(\operatorname{cn}u-1)^2}{2\sqrt[4]{3}\,i\operatorname{sn}u\operatorname{dn}u+(\operatorname{cn}u-1)^2},$$

$$y=\frac{2^{\frac{11}{8}}\cos\frac{\pi}{12}(\operatorname{cn}u-1)\left(\operatorname{cn}u+\tan\frac{\pi}{12}\right)}{2\sqrt[4]{3}\,i\operatorname{sn}u\operatorname{dn}u+(\operatorname{cn}u-1)^2}.$$

In the last set of values, we have

$$\kappa=\sin 75^\circ$$

in place of sin 15° as in the other sets. This last set of values could be derived from the expressions for $\operatorname{sm}^3 u$ and $\operatorname{cm}^3 u$ given on page 60. Many more values could be derived, but enough have been listed to illustrate the various ways of possible expression.

IMAGINARY TRANSFORMATION OF $p(u)$ INTO $p\left(\frac{u}{\sqrt{3}}\right)$

The function

$$p(u)=\frac{\operatorname{cm}u}{\operatorname{sm}^2 u}$$

is the imaginary transformation of the function

$$p\left(\frac{u}{\sqrt{3}}\right)=\frac{\operatorname{sm}u}{1-\operatorname{cm}u}.$$

We can show this directly; it is known that $p\left(\frac{iu}{\sqrt{3}}\right)=-p\left(\frac{u}{\sqrt{3}}\right)$ with g_3 changed into $-g_3$ since $g_2=0$. Now, substituting for sm iu and cm iu their values given on pages 39 and 40, we get

$$-p\left(\frac{u}{\sqrt{3}}\right)=\frac{\dfrac{(t-t^2)\operatorname{sm}\frac{u}{\sqrt{3}}\operatorname{cm}\frac{u}{\sqrt{3}}}{1+t^2\operatorname{sm}^3\frac{u}{\sqrt{3}}}}{1-\dfrac{1+t\operatorname{sm}^3\frac{u}{\sqrt{3}}}{1+t^2\operatorname{sm}^3\frac{u}{\sqrt{3}}}}$$

$$=\frac{(t-t^2)\operatorname{sm}\frac{u}{\sqrt{3}}\operatorname{cm}\frac{u}{\sqrt{3}}}{-(t-t^2)\operatorname{sm}^3\frac{u}{\sqrt{3}}},$$

or

$$p\left(\frac{u}{\sqrt{3}}\right)=\frac{\text{cm}\left(\frac{u}{\sqrt{3}}\right)}{\text{sm}^2\left(\frac{u}{\sqrt{3}}\right)}.$$

We can now replace $\frac{u}{\sqrt{3}}$ by u and we have

$$p(u)=\frac{\text{cm } u}{\text{sm}^2 u}.$$

TRANSFORMATION OF $p\left(\frac{u}{\sqrt{3}}\right)$ TO $p(u)$

In the formula for the imaginary transformation of cm u let us replace u by $\sqrt{3}u$ and we get

$$\text{cm } \sqrt{3}\, iu=\frac{1+t \text{ sm}^3 u}{1+t^2 \text{ sm}^3 u},$$

hence

$$\text{sm}^3 u=\frac{1-\text{cm } \sqrt{3}\, iu}{t^2 \text{ cm } \sqrt{3}\, iu-t}=\frac{1-\text{cm } \sqrt{3}\, iu}{t(t \text{ cm } \sqrt{3}\, iu-1)}$$

and

$$\text{cm}^3 u=\frac{-t \text{ cm } \sqrt{3}\, iu+t^2}{t^2 \text{ cm } \sqrt{3}\, iu-t}=\frac{t-\text{cm } \sqrt{3}\, iu}{t \text{ cm } \sqrt{3}\, iu-1}.$$

With these values let us make the substitution in

$$p^3(u)=\frac{\text{cm}^3 u}{\text{sm}^6 u}$$

$$=\frac{-(\text{cm } \sqrt{3}\, iu-t)\, t^2(t \text{ cm } \sqrt{3}\, iu-1)}{(1-\text{cm } \sqrt{3}\, iu)^2}$$

$$=-\frac{1+\text{cm } \sqrt{3}\, iu+\text{cm}^2 \sqrt{3}\, iu}{(1-\text{cm } \sqrt{3}\, iu)^2}$$

$$=-\frac{\text{sm}^3 \sqrt{3}\, iu}{(1-\text{cm } \sqrt{3}\, iu)^3},$$

or

$$p(u)=-\frac{\text{sm } \sqrt{3}\, iu}{1-\text{cm } \sqrt{3}\, iu}.$$

Now, replace $\sqrt{3}\, iu$ by u and put in place of $p\left(-\frac{iu}{\sqrt{3}}\right)$ its value $-p\left(\frac{u}{\sqrt{3}}\right)$, in which g_3 is now equal to -1 instead of $+1$ as in the former $p(u)$, and we get

$$p\left(\frac{u}{\sqrt{3}}\right)=\frac{\operatorname{sm} u}{1-\operatorname{cm} u}.$$

The imaginary transformation can be applied directly to the formula

$$p\left(\frac{u}{\sqrt{3}}\right)=\frac{1}{2^{\frac{2}{3}}}+\frac{\sqrt{3}}{2^{\frac{2}{3}}}\,\frac{1+\cos\phi}{1-\cos\phi},$$

but we must remember that $\frac{iu}{\sqrt{3}}$ changes to u just the same as L changes into L' in the Jacobi or Legendre form,

$$p\left(\frac{iu}{\sqrt{3}}\right)=\frac{1}{2^{\frac{2}{3}}}+\frac{\sqrt{3}}{2^{\frac{2}{3}}}\,\frac{1+\cos i\phi}{1-\cos i\phi}$$

$$=\frac{1}{2^{\frac{2}{3}}}+\frac{\sqrt{3}}{2^{\frac{2}{3}}}\,\frac{1+\cosh\phi}{1-\cosh\phi}.$$

Now, by the Mercator transformation we can write $\cosh\phi=\sec\theta$

$$-p(u)=\frac{1}{\sqrt[3]{4}}+\frac{\sqrt{3}}{\sqrt[3]{4}}\,\frac{\cos\theta+1}{\cos\theta-1}$$

or

$$p(u)=-\frac{1}{\sqrt[3]{4}}+\frac{\sqrt{3}}{\sqrt[3]{4}}\,\frac{1+\cos\theta}{1-\cos\theta}.$$

This is the formula for transformation of the Weierstrassian form for $g_2=0,\ g_3=1$ into $g_2=0,\ g_3=-1$.

Hereafter we shall call the functions sm u and cm u the Dixon elliptic functions of u, since Dixon has written such an excellent memoir upon the whole series of functions connected with the curve $x^3+y^3-3axy=1$. Much use has been made of his work in deriving the theory of the special class of such functions treated in this publication.

APPLICATION OF THE DIXON ELLIPTIC FUNCTIONS TO MAP PROJECTIONS

As has already been stated, H. A. Schwarz, of Halle, in 1864, called attention to the fact that a circle could be mapped conformally upon a regular polygon of n sides by means of the function

$$w=\int_0^x\frac{dx}{(1-x^n)^{\frac{2}{n}}}.$$

In 1866 Weierstrass gave the same function in a memoir to the Berlin Academy. In 1879 the first geographic map depending upon a function of this kind was made by C. S. Peirce of this survey. We shall now show in what manner the general function can be adapted for a particular series of projections.

We start with the function sm w, but for computation purposes the real and the imaginary parts of the function must be separated. To attain this end, let $w=u+iv$ and $\bar{w}=u-iv$. Then w and $\bar{w}$ are conjugate complex quantities. Let us suppose that sm $w=Re^{ic}$; then sm $\bar{w}=Re^{-ic}$. Let us further assume that cm $w=re^{-id}$ and then cm $\bar{w}=re^{id}$.

The values of r and d can be computed from the relation

$$\text{cm}^3 w=1-\text{sm}^3 w,$$

or

$$r^3e^{-3id}=1-R^3e^{3ic}.$$

Taking the Naperian logarithms of both members we get

$$3\log r-3id=\frac{1}{2}\log\left[(1-R^3\cos 3c)^2+R^6\sin^2 3c\right]$$
$$+i\tan^{-1}\left(\frac{-R^3\sin 3c}{1-R^3\cos 3c}\right).$$

By equating the real parts and then the imaginary parts we obtain after reduction

$$r^6=1-2R^3\cos 3c+R^6,$$

$$\tan 3d=\frac{R^3\sin 3c}{1-R^3\cos 3c}.$$

$$1+\tan^2 3d=\sec^2 3d=\frac{1-2R^3\cos 3c+R^6}{(1-R^3\cos 3c)^2},$$

or

$$\frac{r^6}{(1-R^3\cos 3c)^2}=\frac{1}{\cos^2 3d},$$

hence

$$r^3=\frac{1-R^3\cos 3c}{\cos 3d}.$$

After d is determined from its tangent it is more convenient to compute r from the form last given. When r and d are determined numerically, we know the values of cm $w=re^{-id}$ and cm $\bar{w}=re^{id}$

Now we have

$$\text{sm}\,(w+\bar{w})=\text{sm}\,2u=\frac{\text{sm}\,w+\text{cm}^2 w\,\text{sm}\,\bar{w}\,\text{cm}\,\bar{w}}{\text{cm}\,\bar{w}+\text{sm}\,w\,\text{cm}\,w\,\text{sm}^2\,\bar{w}},$$

or

$$\text{sm}\,2u=\frac{Re^{ic}+Rr^3e^{-i(c+d)}}{re^{id}+R^3re^{-i(c+d)}}$$

$$=\frac{R\ \left(e^{i(c-d)}+r^3e^{-i(c+2d)}+R^3e^{i(2c+d)}+R^3r^3\right)}{r\left[1+2R^3\cos\,(c+2d)+R^6\right]}.$$

Since sm $2u$ is entirely real the imaginary part of the right-hand member must be identically equal to zero. We have, therefore,

$$\text{sm}\,2u=\frac{R\left[\cos\,(c-d)+r^3\cos\,(c+2d)+R^3\cos\,(2c+d)+R^3r^3\right]}{r\left[1+2R^3\cos\,(c+2d)+R^6\right]},$$

as also the identity

$$\sin\,(c-d)-r^3\sin\,(c+2d)+R^3\sin\,(2c+d)=0.$$

This identity is a consequence of the relations already established between the various functions. Again we have

$$\text{sm}\,(w-\bar{w})=\text{sm}\,2iv=\frac{\text{sm}\,w\,\text{cm}^2\,\bar{w}-\text{cm}^2\,w\,\text{sm}\,\bar{w}}{\text{cm}\,\bar{w}+\text{sm}\,w\,\text{cm}\,w\,\text{sm}^2\,\bar{w}}$$

$$=\frac{Rr^2e^{i(c+2d)}-Rr^2e^{-i(c+2d)}}{re^{id}+R^3re^{-i(c+d)}},$$

$$\text{sm}\,2iv=\frac{2iRr\sin\,(c+2d)}{e^{id}+R^3e^{-i(c+d)}}.$$

But from the imaginary transformation given on page 39 we have

$$\text{sm}\,2iv=\frac{\sqrt{3}\,i\,\text{sm}\,\frac{2v}{\sqrt{3}}\,\text{cm}\,\frac{2v}{\sqrt{3}}}{1+t^2\,\text{sm}^3\,\frac{2v}{\sqrt{3}}},$$

therefore

$$\frac{\sqrt{3}\,i\,\text{sm}\,\frac{2v}{\sqrt{3}}\,\text{cm}\,\frac{2v}{\sqrt{3}}}{1+t^2\,\text{sm}^3\,\frac{2v}{\sqrt{3}}}=\frac{2iRr\sin\,(c+2d)}{e^{id}+R^3e^{-i(c+d)}},$$

or by dividing by i and equating the reciprocals, we get

$$\frac{1-\frac{1}{2}\,\text{sm}^3\,\frac{2v}{\sqrt{3}}-\frac{\sqrt{3}}{2}\,i\,\text{sm}^3\,\frac{2v}{\sqrt{3}}}{\sqrt{3}\,\text{sm}\,\frac{2v}{\sqrt{3}}\,\text{cm}\,\frac{2v}{\sqrt{3}}}$$

$$=\frac{\cos d+i\sin d+R^3\cos\,(c+d)-R^3i\sin\,(c+d)}{2Rr\sin\,(c+2d)}.$$

Now, by equating the real parts and then the imaginary parts we obtain

$$\frac{1-\frac{1}{2}\,\text{sm}^3\,\frac{2v}{\sqrt{3}}}{\sqrt{3}\,\text{sm}\,\frac{2v}{\sqrt{3}}\,\text{cm}\,\frac{2v}{\sqrt{3}}}=\frac{\cos d+R^3\cos\,(c+d)}{2Rr\sin\,(c+2d)},$$

and

$$\frac{\text{sm}^2\,\frac{2v}{\sqrt{3}}}{\text{cm}\,\frac{2v}{\sqrt{3}}}=-\frac{\sin d-R^3\sin\,(c+d)}{Rr\sin\,(c+2d)}.$$

Since both of these equations are valid, we can divide the first by the second and thus obtain

$$\frac{1}{\sqrt{3}\ \mathrm{sm}^3 \frac{2v}{\sqrt{3}}} - \frac{1}{2\sqrt{3}} = \frac{\cos d + R^3 \cos(c+d)}{2[R^3 \sin(c+d) - \sin d]},$$

or

$$\frac{1}{\mathrm{sm}^3 \frac{2v}{\sqrt{3}}} = \frac{\frac{\sqrt{3}}{2}\cos d - \frac{1}{2}\sin d + R^3\left[\frac{\sqrt{3}}{2}\cos(c+d) + \frac{1}{2}\sin(c+d)\right]}{R^3 \sin(c+d) - \sin d}$$

$$= \frac{\sin\left(\frac{\pi}{3} - d\right) + R^3 \sin\left(\frac{\pi}{3} + c + d\right)}{R^3 \sin(c+d) - \sin d};$$

hence finally this becomes

$$\mathrm{sm}^3 \frac{2v}{\sqrt{3}} = \frac{R^3 \sin(c+d) - \sin d}{R^3 \sin\left(\frac{\pi}{3} + c + d\right) + \sin\left(\frac{\pi}{3} - d\right)}.$$

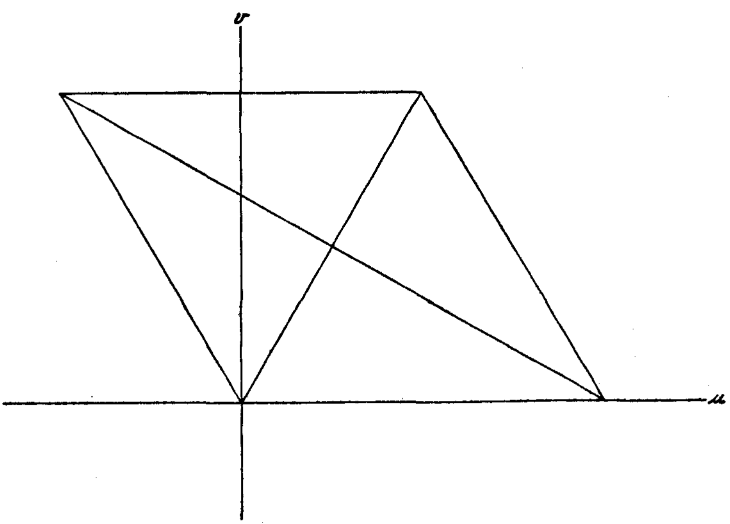

FIG. 4.—Relation of the Dixon rhombus to the axes of coordinates

If the numerical value of sm $w = $ sm $(u + iv)$ is given, we can compute cm w, and hence r and d; with these values by means of the above formulas we can compute the value of sm $2u$ and that of sm $\frac{2v}{\sqrt{3}}$. Then, by means of a table of the functions we can determine u and v which are the coordinates of the projection. These rectangular coordinates, u and v, are laid off along the axes as illustrated in Figure 4. The relation of the rhombus to the axes of coordinates is illustrated in the same diagram. If sm w is equated to any analytic function of the isometric coordinates of the sphere, the resulting projection will be conformal, as was proved by Gauss and the proof of which is given in any treatise on the general theory of conformal projections.[1] Any analytic function of the complex variable in the stereographic projection plane will serve the purpose in this case and will of necessity give a conformal projection of the original sphere or of a part of it, as the case may be.

The formulas for the computation of sm $2u$ and sm $\frac{2v}{\sqrt{3}}$ are somewhat involved, and the amount of calculation required is considerable, although it is all easy logarithmic work including the trigonometric functions. For all coordinates lying moderately near the origin it is more convenient to use a series development of the integral.

$$w=\int_0^x \frac{dx}{(1-x^3)^{\frac{2}{3}}}=x+\frac{2}{3}\cdot\frac{1}{4}x^4+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{7}x^7+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{8}{9}\cdot\frac{1}{10}x^{10}+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{8}{9}\cdot\frac{11}{12}\cdot\frac{1}{13}x^{13}+$$

$$\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{8}{9}\cdot\frac{11}{12}\cdot\frac{14}{15}\cdot\frac{1}{16}x^{16}+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot,$$

or

$$w=u+iv=Re^{ic}+\frac{2}{3}\cdot\frac{1}{4}R^4e^{4ic}+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{7}R^7e^{7ic}+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot$$

Therefore

$$u=R\cos c+\frac{2}{3}\cdot\frac{1}{4}R^4\cos 4c+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{7}R^7\cos 7c+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot,$$

$$v=R\sin c+\frac{2}{3}\cdot\frac{1}{4}R^4\sin 4c+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{7}R^7\sin 7c+\cdot\cdot\cdot\cdot\cdot\cdot\cdot\cdot$$

The logarithms of the coefficients of the above series are as follows, beginning with the second:

$\log a_2=9.2218487-10$	$\log a_{15}=7.849400-10$
$\log a_3=8.8996294-10$	$\log a_{16}=7.81035-10$
$\log a_4=8.6935749-10$	$\log a_{17}=7.77377-10$
$\log a_5=8.5418430-10$	$\log a_{18}=7.73936-10$
$\log a_6=8.4217031-10$	$\log a_{19}=7.70688-10$
$\log a_7=8.3222459-10$	$\log a_{20}=7.67613-10$
$\log a_8=8.2373875-10$	$\log a_{21}=7.64693-10$
$\log a_9=8.1633868-10$	$\log a_{22}=7.61913-10$
$\log a_{10}=8.097778-10$	$\log a_{23}=7.59261-10$
$\log a_{11}=8.038851-10$	$\log a_{24}=7.56724-10$
$\log a_{12}=7.985370-10$	$\log a_{25}=7.54294-10$
$\log a_{13}=7.936413-10$	$\log a_{26}=7.51962-10$
$\log a_{14}=7.891274-10$	$\log a_{27}=7.49721-10$

In all of the projections depending upon the Dixon elliptic functions the period parallelogram is a rhombus with one pair of the angles equal to 120° and the other pair equal to 60°. In fact, any

[1] See General Theory of the Lambert Conformal Conic Projection, United States Coast and Geodetic Survey Spec. Pub. No. 53.

projection defined by the Schwarz integral, $\int_0^x \frac{dx}{(1-x^n)^{\frac{2}{n}}}$, will be connected with a rhombus which becomes a square in case $n=4$. This class of projections has accordingly been given the generic name of rhombic projections. An indefinite number of individual conformal projections belong to the rhombic class. A few of the most interesting members of this class have been computed for this publication, the analytical development of which will now be given.

PROJECTION OF THE SPHERE IN A REGULAR HEXAGON

If sm w is placed equal to the complex variable in the polar stereographic plane, we shall have the Northern Hemisphere mapped within an equilateral triangle, and the Southern Hemisphere will fill the remainder of the regular hexagon separated into three distinct parts of 120° of longitude each. We have then

$$\text{sm } w = \tan \frac{p}{2} e^{i\lambda},$$

in which p is the complement of the latitude and λ is the longitude reckoned from some chosen point. In the general formulas R becomes equal to $\tan \frac{p}{2}$ and c becomes equal to λ.

We have, therefore,

$$\tan 3d = \frac{\tan^3 \frac{p}{2} \sin 3\lambda}{1 - \tan^3 \frac{p}{2} \cos 3\lambda} \qquad \text{(see p. 68)},$$

$$r^3 = \frac{1 - \tan^3 \frac{p}{2} \cos 3\lambda}{\cos 3d} \qquad \text{(see p. 68)},$$

$$\text{sm } 2u = \frac{\tan \frac{p}{2} \left[\cos (\lambda - d) + r^3 \cos (\lambda + 2d) + \tan^3 \frac{p}{2} \cos (2\lambda + d) + r^3 \tan^3 \frac{p}{2} \right]}{r \left[1 + 2 \tan^3 \frac{p}{2} \cos (\lambda + 2d) + \tan^6 \frac{p}{2} \right]} \qquad \text{(see p. 68)},$$

$$\text{sm}^3 \frac{2v}{\sqrt{3}} = \frac{\tan^3 \frac{p}{2} \sin (\lambda + d) - \sin d}{\tan^3 \frac{p}{2} \sin \left(\frac{\pi}{3} + \lambda + d \right) + \sin \left(\frac{\pi}{3} - d \right)} \qquad \text{(see p. 70)},$$

and the series development becomes

$$u = \tan \frac{p}{2} \cos \lambda + \frac{2}{3} \cdot \frac{1}{4} \tan^4 \frac{p}{2} \cos 4\lambda + \ldots \ldots \ldots,$$

$$v = \tan \frac{p}{2} \sin \lambda + \frac{2}{3} \cdot \frac{1}{4} \tan^4 \frac{p}{2} \sin 4\lambda + \ldots \ldots \ldots \text{(see p. 71)}.$$

The meridian for $\lambda = 60°$ projects into the straight line drawn to the second vertex of the regular hexagon, the first vertex lying upon the axis of u. When $\lambda = 60°$

$$\text{sm}\, w' = s \tan \frac{p}{2}.$$

Now, place sw in place of w' in which case w lies along the straight line drawn from the origin to the second vertex; then

$$\text{sm}\, w' = \text{sm}\, sw = s \tan \frac{p}{2},$$

but

$$\text{sm}\, sw = s\, \frac{\text{sm}\, w}{\text{cm}\, w},$$

hence

$$\frac{\text{sm}\, w}{\text{cm}\, w} = \tan \frac{p}{2}.$$

This gives us

$$\frac{1}{\text{cm}\, w} = \left(1 + \tan^3 \frac{p}{2}\right)^{\frac{1}{3}},$$

or

$$\text{sm}\, w = \frac{\tan \frac{p}{2}}{\left(1 + \tan^3 \frac{p}{2}\right)^{\frac{1}{3}}}.$$

This is the most convenient form for computing the values along this meridian, and the analysis proves that the meridian is represented by the straight line. After w is determined by this formula we have $u = \frac{1}{2}\, w$ and $v = \frac{\sqrt{3}}{2}\, w$.

The Equator in the first of the three rhombuses is represented by the long diagonal; that is, the straight line joining the first vertex of the hexagon with the third vertex. Along the Equator the complex variable becomes $e^{i\lambda}$; that is, the Equator is represented in the stereographic projection by the circle of unit radius. In this case

$$\text{sm}\, w' = e^{i\lambda}.$$

Now let

$$w' = K + siw,$$

in which case w is reckoned along this long diagonal, and we have

$$\text{sm}\, w' = \text{sm}\, (K + siw) = e^{i\lambda}.$$

But

$$\text{sm}\, (K + siw) = \frac{1}{\text{cm}\, siw} = \text{cm}\, iw = \frac{1 + t\, \text{sm}^3 \frac{w}{\sqrt{3}}}{1 + t^2\, \text{sm}^3 \frac{w}{\sqrt{3}}}.$$

To investigate whether we were justified in assuming this equality, we must see whether the absolute value of this expression is equal to unity. We see that the numerator is the conjugate of the denominator; hence

$$(\text{absolute value})^2 = \frac{1+t\,\mathrm{sm}^3\frac{w}{\sqrt{3}}}{1+t^2\,\mathrm{sm}^3\frac{w}{\sqrt{3}}} \cdot \frac{1+t^2\,\mathrm{sm}^3\frac{w}{\sqrt{3}}}{1+t\,\mathrm{sm}^3\frac{w}{\sqrt{3}}} = 1.$$

We are therefore justified in assuming the equality

$$\frac{1+t\,\mathrm{sm}^3\frac{w}{\sqrt{3}}}{1+t^2\,\mathrm{sm}^3\frac{w}{\sqrt{3}}} = e^{i\lambda}.$$

It follows that

$$\frac{1+t^2\,\mathrm{sm}^3\frac{w}{\sqrt{3}}}{1+t\,\mathrm{sm}^3\frac{w}{\sqrt{3}}} = e^{-i\lambda}.$$

By addition we have

$$\frac{1-\mathrm{sm}^3\frac{w}{\sqrt{3}}-\frac{1}{2}\mathrm{sm}^6\frac{w}{\sqrt{3}}}{1-\mathrm{sm}^3\frac{w}{\sqrt{3}}+\mathrm{sm}^6\frac{w}{\sqrt{3}}} = \cos\lambda,$$

and by subtraction

$$\frac{\sqrt{3}\,\mathrm{sm}^3\frac{w}{\sqrt{3}}-\frac{\sqrt{3}}{2}\mathrm{sm}^6\frac{w}{\sqrt{3}}}{1-\mathrm{sm}^3\frac{w}{\sqrt{3}}+\mathrm{sm}^6\frac{w}{\sqrt{3}}} = \sin\lambda.$$

Furthermore, we get

$$\frac{\frac{3}{2}\mathrm{sm}^6\frac{w}{\sqrt{3}}}{1-\mathrm{sm}^3\frac{w}{\sqrt{3}}+\mathrm{sm}^6\frac{w}{\sqrt{3}}} = 1-\cos\lambda,$$

and

$$\frac{2}{\sqrt{3}\,\mathrm{sm}^3\frac{w}{\sqrt{3}}}-\frac{1}{\sqrt{3}} = \frac{\sin\lambda}{1-\cos\lambda} = \cot\frac{\lambda}{2},$$

or

$$\frac{1}{\mathrm{sm}^3\frac{w}{\sqrt{3}}} = \frac{1}{2}+\frac{\sqrt{3}}{2}\cot\frac{\lambda}{2} = \frac{\frac{1}{2}\sin\frac{\lambda}{2}+\frac{\sqrt{3}}{2}\cos\frac{\lambda}{2}}{\sin\frac{\lambda}{2}} = \frac{\sin\left(\frac{\pi}{3}+\frac{\lambda}{2}\right)}{\sin\frac{\lambda}{2}},$$

so that finally we have

$$\mathrm{sm}^3 \frac{w}{\sqrt{3}} = \frac{\sin \frac{\lambda}{2}}{\sin\left(\frac{\pi}{3}+\frac{\lambda}{2}\right)}.$$

After w is determined by this formula we have

$$u = K - \frac{\sqrt{3}}{2} w$$

$$v = \frac{1}{2} w.$$

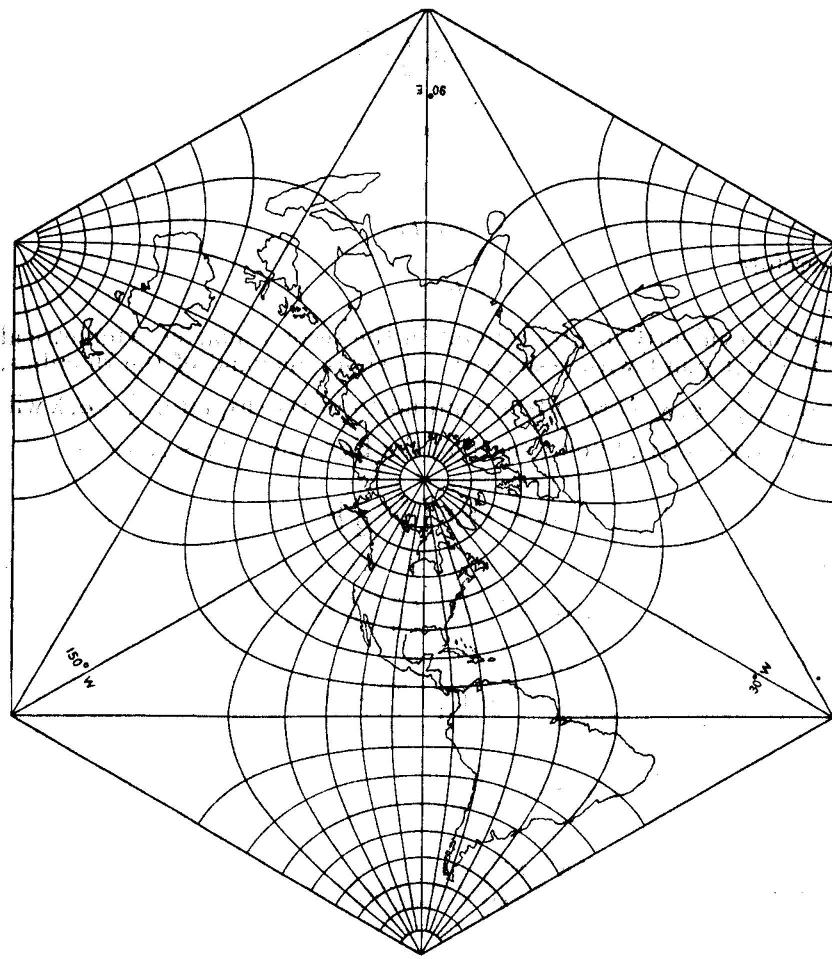

Fig. 5.—Rhombic projection of the world in a regular hexagon

The projection is shown in Figure 5. It will be seen that in each of the three rhombuses the four sections formed by the diagonals are symmetrical; each of the diagonals is a line of symmetry in the rhombus and the intersection of the diagonals is a center of symmetry for the whole rhombus. It is therefore necessary to compute the coordinates for only one of these four sections and then the others can be constructed from consideration of symmetry. This similarity of the sections of the rhombus is found in all of the projections computed with it as basis, and hence we shall prove the fact for this projection only.

If p, λ are the spherical coordinates of a point in the first quarter of the rhombus, p, $120° - \lambda$ will be the point that should be symmetrical with respect to the first point. Then we shall have

$$\text{sm } w = \tan\frac{p}{2}.e^{i\lambda},$$

and

$$\text{sm } w' = \tan\frac{p}{2}\, e^{\frac{2\pi}{3}i - i\lambda},$$

$$= t \tan\frac{p}{2} e^{-i\lambda},$$

$$= t \text{ sm } \bar{w},$$

$$= \text{sm } t\bar{w}.$$

Therefore

$$w' = t\bar{w},$$

and this evidently locates a point in the second quarter symmetrical to the first point with respect to the short diagonal of the rhombus. Again, if $\frac{\pi}{2} - \phi$, λ represents a point north of the Equator, $\frac{\pi}{2} + \phi$, λ will be the symmetrical point south of the Equator. This gives us

$$\text{sm } w = \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)e^{i\lambda}.$$

and

$$\text{sm } w' = \tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)e^{i\lambda}.$$

Then

$$\frac{1}{\text{sm } w'} = \tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right) e^{-i\lambda} = \text{sm } \bar{w},$$

or

$$\text{sm } w' = \frac{1}{\text{sm } \bar{w}}.$$

But

$$\text{sm } (sK + st^2\bar{w}) = s\,\frac{\text{sm } (K + t^2\bar{w})}{\text{cm } (K + t^2\bar{w})} = -\frac{s}{\text{sm } t^2\bar{w}} = -\frac{s}{t^2 \text{ sm } \bar{w}} = -\frac{st}{\text{sm } \bar{w}} = \frac{1}{\text{sm } \bar{w}},$$

and consequently

$$\text{sm } w' = \text{sm } (sK + st^2\bar{w}),$$

or

$$w' = sK + st^2\bar{w} = sK - t\bar{w}.$$

This point in the rhombus is the reflection of the point w on the long diagonal, and hence the two points are symmetrical with respect to the long diagonal. In a similar way we can prove that the fourth quarter is symmetrical to the third with respect to the long diagonal or to the second quarter with respect to the short diagonal, or finally to the first quarter with respect to the intersection of the diagonals. If w is any point in the first rhombus that has the spherical coordinates p, λ then p, $120° + \lambda$ should be a corresponding point in the second rhombus.

Now we have

$$\text{sm}\, w = \tan\frac{p}{2} e^{i\lambda},$$

$$\text{sm}\, w' = \tan\frac{p}{2} e^{\frac{2\pi i}{3} + i\lambda},$$

$$= t \tan\frac{p}{2} e^{i\lambda},$$

$$= t\, \text{sm}\, w$$

$$= \text{sm}\, tw,$$

or

$$w' = tw.$$

Therefore, the first rhombus turned about the origin through an angle of 120° gives the second rhombus, just as it should do. The third rhombus is again the first rhombus turned about the origin through an angle 240°. This is the projection referred to by H. A. Schwarz, and we see that the Northern Hemisphere is mapped within the equilateral triangle, just as his theorem stated that it would be. In reality the circle representing the Northern Hemisphere has been mapped from the stereographic plane within the triangle. We can also map the whole sphere in the triangle, but such a projection would not be especially important for geographic purposes, so that no computations were made for it.

To find an expression for the ratio of scale, we may start from the expression of definition of the projection. The scale is the same in all directions at any point, since the projection is conformal. If, therefore, we get the ratio of w to p at any point, this value can then be taken for the general scale ratio at that point. We have then

$$\text{sm}\, w = \tan\frac{p}{2} e^{i\lambda},$$

hence

$$\text{cm}^2 w \frac{dw}{dp} = \frac{1}{2} \sec^2\frac{p}{2} e^{i\lambda},$$

or

$$\left|\frac{dw}{dp}\right|^2 = \frac{1}{4} \frac{\sec^4\frac{p}{2}}{\text{cm}^2 w\, \text{cm}^2 \bar{w}}.$$

The symbol $\left|\frac{dw}{dp}\right|$ is used to denote the absolute value of the ratio of scale, and will so be used in the other projections.
But we have

$$\text{cm}\,(u+iv)\,\text{cm}\,(u-iv)=\frac{\text{cm}\,iv\,\text{cm}^2\,u-\text{sm}^2\,iv\,\text{sm}\,u}{\text{cm}\,iv+\text{sm}\,u\,\text{cm}\,u\,\text{sm}^2\,iv}$$

$$=\frac{\left(1-\text{sm}^3\frac{v}{\sqrt{3}}+\text{sm}^6\frac{v}{\sqrt{3}}\right)\text{cm}^2\,u+3\,\text{sm}\,u\,\text{sm}^2\frac{v}{\sqrt{3}}\,\text{cm}^2\frac{v}{\sqrt{3}}}{1-\text{sm}^3\frac{v}{\sqrt{3}}+\text{sm}^6\frac{v}{\sqrt{3}}-3\,\text{sm}\,u\,\text{cm}\,u\,\text{sm}^2\frac{v}{\sqrt{3}}\,\text{cm}^2\frac{v}{\sqrt{3}}}.$$

Hence we have finally

$$\left|\frac{dw}{dp}\right|=\frac{1}{2}\sec^2\frac{p}{2}\,\frac{1-\text{sm}^3\frac{v}{\sqrt{3}}+\text{sm}^6\frac{v}{\sqrt{3}}-3\,\text{sm}\,u\,\text{cm}\,u\,\text{sm}^2\frac{v}{\sqrt{3}}\,\text{cm}^2\frac{v}{\sqrt{3}}}{\left(1-\text{sm}^3\frac{v}{\sqrt{3}}+\text{sm}^6\frac{v}{\sqrt{3}}\right)\text{cm}^2\,u+3\,\text{sm}\,u\,\text{sm}^2\frac{v}{\sqrt{3}}\,\text{cm}^2\frac{v}{\sqrt{3}}}.$$

We know, however, that

$$\text{cm}\,w=re^{-id},$$

and

$$\text{cm}\,\overline{w}=re^{id},$$

hence

$$\text{cm}\,w\,\text{cm}\,\overline{w}=r^2;$$

this gives the above expression in the form

$$\left|\frac{dw}{dp}\right|=\frac{1}{2r^2}\sec^2\frac{p}{2}.$$

In any event it would be more convenient to compute r rather than use the complicated expression given above.

The points of discontinuity are found on the perimeter of the hexagon starting with the vertex lying on the positive section of the axis of u and including every other one; that is, the first, third, and fifth vertices. At these points r becomes equal to zero, so that $\left|\frac{dw}{dp}\right|$ becomes infinite. The factor $\sec\frac{p}{2}$ becomes infinite in the second, fourth, and sixth vertices, but r is also infinite at these points and such that $\frac{1}{r}\sec\frac{p}{2}$ is finite. These points are, therefore, ordinary points in which the conformality is preserved.

This projection is shown in Figure 5, from which it will be seen that it is not particularly well fitted for mapping the whole sphere. The projection is, however, an interesting case and even for geographic purposes the Northern Hemisphere is pretty well represented.

PROJECTION OF A HEMISPHERE IN THE RHOMBUS WITH THE POLES IN THE 120° ANGLES

If the projection is defined by the expression

$$\operatorname{sm} w = \tan^{\frac{2}{3}} \frac{p}{2} e^{\frac{2i\lambda}{3}},$$

the hemisphere will be mapped in the rhombus. When $\lambda = 180°$, $e^{\frac{2i\lambda}{3}} = t$ and

$$\operatorname{sm} w' = t \tan^{\frac{2}{3}} \frac{p}{2} = t \operatorname{sm} w = \operatorname{sm} tw$$

in which w is the value for $\lambda = 0$. Hence, $w' = tw$, or if the axis of u is turned about the origin through an angle of 120° we shall have the second side of the rhombus adjacent to the origin and w' is mapped along it.

In the general formulas R becomes equal to $\tan^{\frac{2}{3}} \frac{p}{2}$ and c becomes $\frac{2}{3}\lambda$, so that we have

$$\tan 3d = \frac{\tan^2 \frac{p}{2} \sin 2\lambda}{1 - \tan^2 \frac{p}{2} \cos 2\lambda} \qquad \text{(see p. 68)},$$

$$r^3 = \frac{1 - \tan^2 \frac{p}{2} \cos 2\lambda}{\cos 3d} \qquad \text{(see p. 68)},$$

$$\operatorname{sm} 2u = \frac{\tan^{\frac{2}{3}} \frac{p}{2} \left[\cos\left(\frac{2}{3}\lambda - d\right) + r^3 \cos\left(\frac{2}{3}\lambda + 2d\right) + \tan^2 \frac{p}{2} \cos\left(\frac{4}{3}\lambda + d\right) + r^3 \tan^2 \frac{p}{2}\right]}{r\left[1 + 2\tan^2 \frac{p}{2} \cos\left(\frac{2}{3}\lambda + 2d\right) + \tan^4 \frac{p}{2}\right]} \qquad \text{(see p. 68)},$$

$$\operatorname{sm}^3 \frac{2v}{\sqrt{3}} = \frac{\tan^2 \frac{p}{2} \sin\left(\frac{2}{3}\lambda + d\right) - \sin d}{\tan^2 \frac{p}{2} \sin\left(\frac{\pi}{3} + \frac{2}{3}\lambda + d\right) + \sin\left(\frac{\pi}{3} - d\right)} \qquad \text{(see p. 70)}.$$

The series expression follows at once from $R = \tan^{\frac{2}{3}} \frac{p}{2}$ and $c = \frac{2}{3}\lambda$. By analysis similar to that on page 73, we find that along the short diagonal the relation becomes

$$\operatorname{sm} w = \sin^{\frac{2}{3}} \frac{p}{2},$$

with

$$u = \frac{1}{2} w,$$

$$v = \frac{\sqrt{3}}{2} w.$$

Along the long diagonal we have

$$\mathrm{sm}^3 \frac{w}{\sqrt{3}} = \frac{\sin \frac{1}{3}\lambda}{\sin\left(\frac{\pi}{3} + \frac{1}{3}\lambda\right)},$$

with

$$u = K - \frac{\sqrt{3}}{2} w,$$

$$v = \frac{1}{2} w.$$

The symmetry of the four sections of the rhombus is the same as in the case of the first projection.

We can find the expression for the ratio of scale at a point by differentiating the expression of definition

$$\mathrm{sm}\, w = \tan^{\frac{2}{3}} \frac{p}{2} e^{\frac{2i}{3}\lambda},$$

$$\mathrm{cm}^2 w \frac{dw}{dp} = \frac{1}{3} \tan^{-\frac{1}{3}} \frac{p}{2} \sec^2 \frac{p}{2} e^{\frac{2i}{3}\lambda},$$

$$\left|\frac{dw}{dp}\right|^2 = \frac{1}{9r^4} \tan^{-\frac{2}{3}} \frac{p}{2} \sec^4 \frac{p}{2},$$

or

$$\left|\frac{dw}{dp}\right| = \frac{1}{3r^2} \cot^{\frac{1}{3}} \frac{p}{2} \sec^2 \frac{p}{2}.$$

This expression becomes either infinite or zero for each one of the vertices of the rhombus; hence these points are critical points of discontinuity for the functional relation. This fact is evident since 180° is represented by each one of the angles.

This projection is shown in Figure 6. The representation of the Western Hemisphere is shown in a fairly exact manner by the projection. The exceedingly bad distortions in the 60° angles are thrown in the ocean areas and hence do little harm to the land areas.

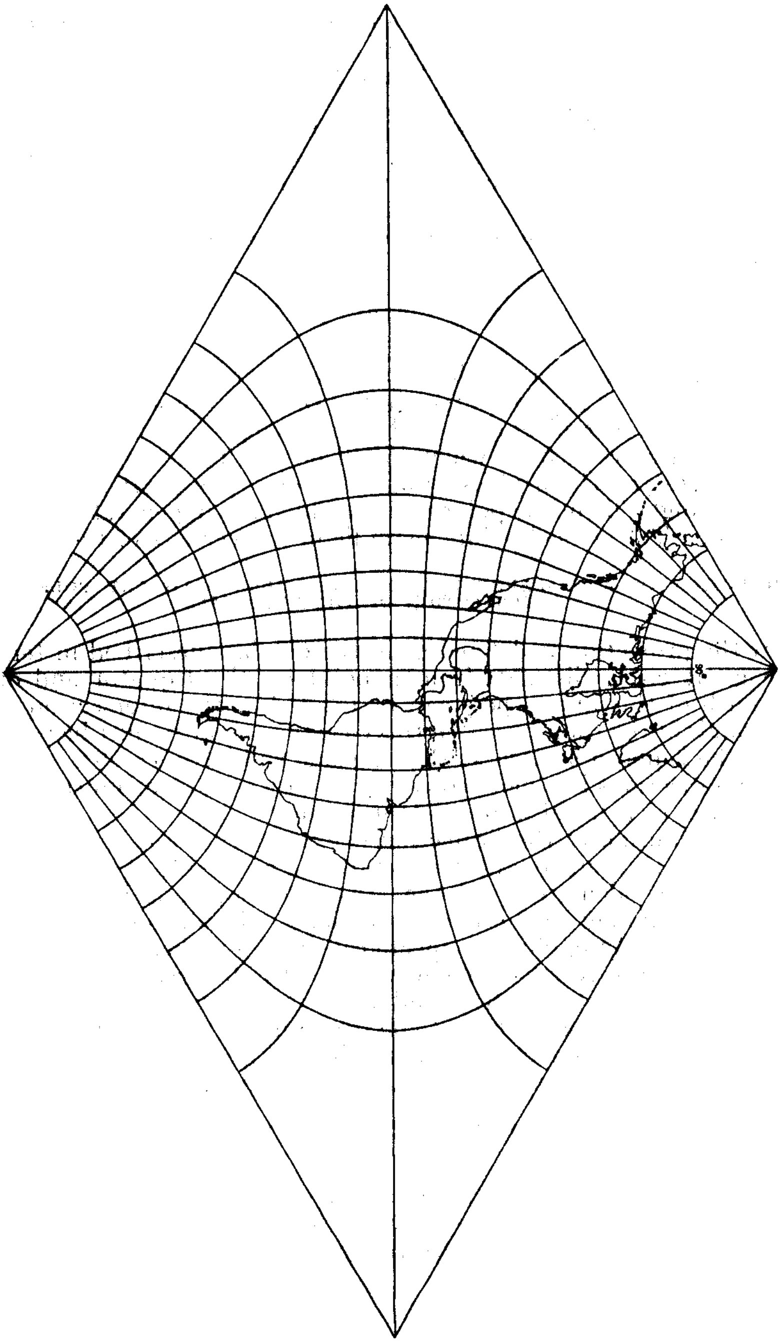

Fig. 6.—Rhombic projection of the Western Hemisphere, poles in the 120° angles

PROJECTION OF A HEMISPHERE IN THE RHOMBUS WITH THE POLES IN THE 60° ANGLES

If in place of using latitude and longitude as the spherical coordinates we employ the great circle distance and azimuth computed from a chosen point, we can locate the pole wherever we may wish within the rhombus. In the United States Coast and Geodetic Survey Special Publication No. 67, Latitude Developments Connected with Geodesy and Cartography, there is published a table of such coordinates computed from a point on the Equator. By using these coordinates and by reckoning the azimuth from the pole we can locate the poles in the 60° angles. The definition of the projection becomes

$$\operatorname{sm} w = \tan^{\frac{2}{3}} \frac{\zeta}{2} \, e^{\frac{2}{3} i \alpha}$$

in which ζ is the great circle distance and α is the azimuth reckoned from the pole. The formulas are the same as those for the last projection, with ζ replacing p and α replacing λ. The projection is shown in Figure 7, from which it will be seen that the representation of the Western Hemisphere is fairly good. The great distortions in the 60° angles lie in the polar regions which are in general of but little interest. The critical points again lie in the vertices of the rhombus.

PROJECTION OF A HEMISPHERE IN THE RHOMBUS WITH ONE POLE AT THE INTERSECTION OF THE DIAGONALS

If the projection is defined as in the one last described in terms of the great circle distance from a point on the Equator and of the azimuth of this line, a pole can be located at the intersection of the diagonals of the rhombus. In this case the azimuth must be reckoned from the Equator as zero, so that the pole will have $\alpha = 90°$. In all of these projections it is more convenient to reckon the azimuth counterclockwise, as is usual in plane coordinates.

The definition of the projection is exactly the same as that of the last projection, but for the computation a new table of r and d would have to be calculated. If the series development is used, the α would be different in the two projections. This projection is shown in Figure 8. As might be expected it gives a rather distorted representation of the Northern Hemisphere. In all three of these projections of a hemisphere in the rhombus the intersections on the diagonals are the same and do not have to be recomputed. It is only a question of rearranging them in the new table to fit the new condition of projection. This fact saves a considerable amount of recomputation.

PROJECTION OF THE WHOLE SPHERE IN THE RHOMBUS WITH THE POLES IN THE 120° ANGLES

Just as we succeeded in mapping a hemisphere within the rhombus, so we can arrange to map the whole sphere within the same area. Let us define the projection by the expression

$$\operatorname{sm} w = \tan^{\frac{1}{3}} \frac{p}{2} \, e^{\frac{1}{3} i \lambda},$$

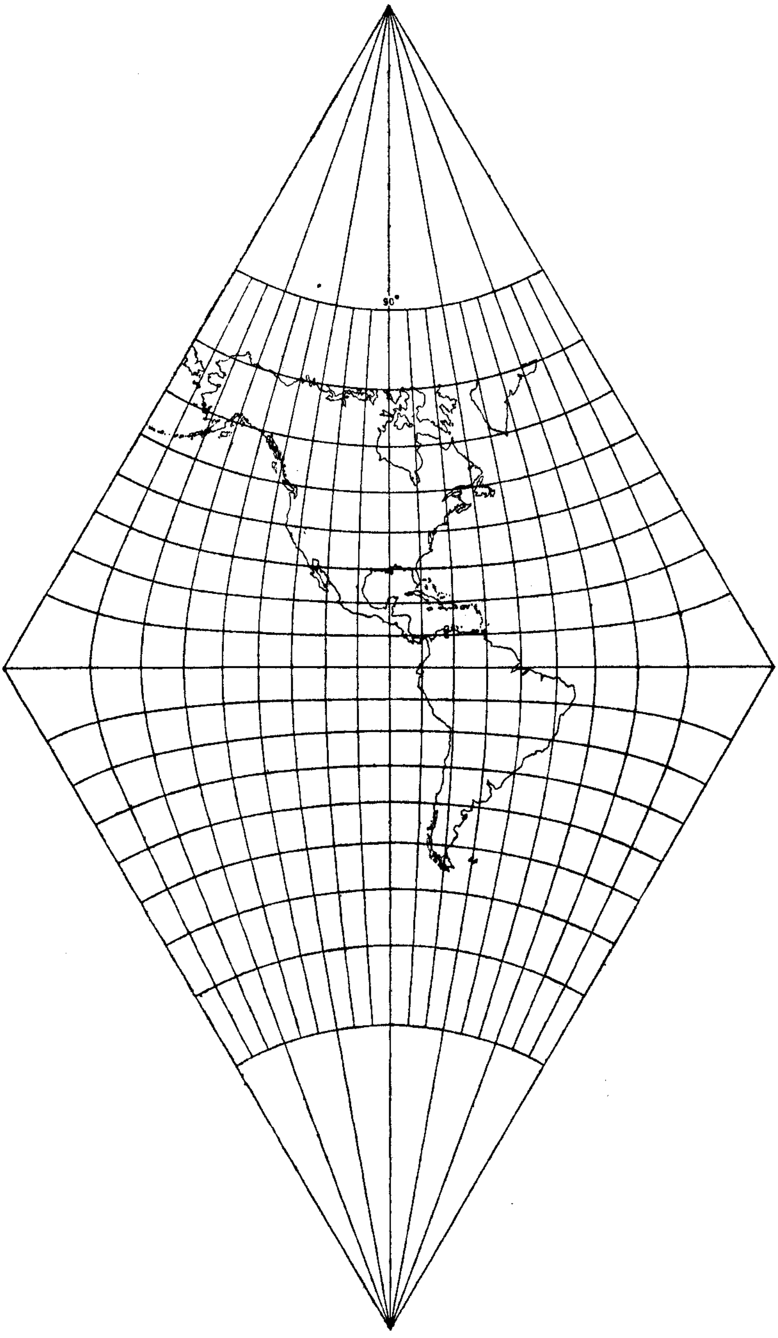

FIG. 7.—Rhombic projection of the Western Hemisphere, poles in the 60° angles

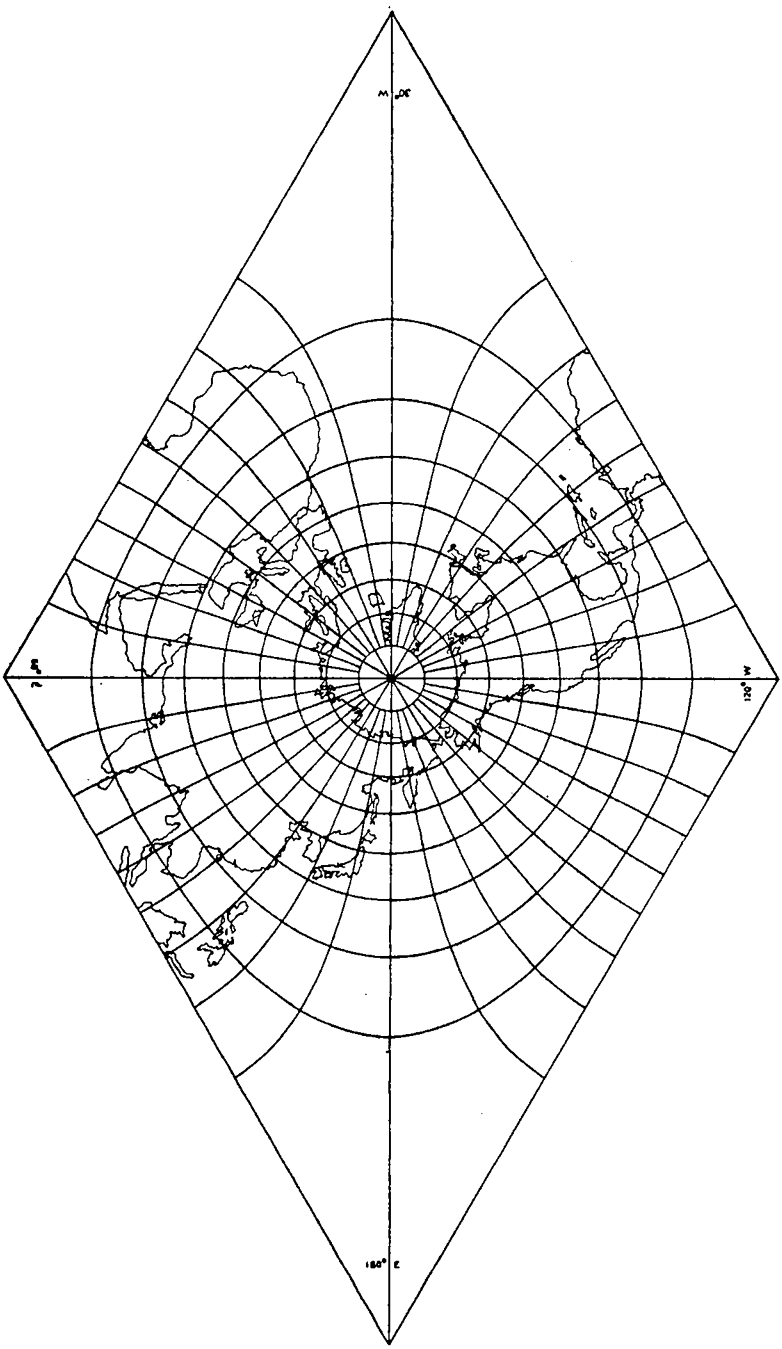

Fig. 8.—Rhombic projection of the Northern Hemisphere, pole at the center of the rhombus

Now when $\lambda = 360°$, $e^{\frac{1}{3}i\lambda} = t$ and we shall have reached the second side of the rhombus adjacent to the origin. The Equator will be represented by the long diagonal, but the 360° of longitude will be mapped upon it. With this definition R becomes $\tan^{\frac{1}{3}} \frac{p}{2}$ and c becomes $\frac{1}{3}\lambda$.

The general formulas now become as follows:

$$\tan 3d = \frac{\tan \frac{p}{2} \sin \lambda}{1 - \tan \frac{p}{2} \cos \lambda} \qquad \text{(see p. 68)}.$$

$$r^3 = \frac{1 - \tan \frac{p}{2} \cos \lambda}{\cos 3d} \qquad \text{(see p. 68)},$$

$$\operatorname{sm} 2u = \frac{\tan^{\frac{1}{3}} \frac{p}{2}\left[\cos\left(\frac{1}{3}\lambda - d\right) + r^3 \cos\left(\frac{1}{3}\lambda + 2d\right) + \tan \frac{p}{2} \cos\left(\frac{2}{3}\lambda + d\right) + r^3 \tan \frac{p}{2}\right]}{r\left[1 + 2 \tan \frac{p}{2} \cos\left(\frac{1}{3}\lambda + 2d\right) + \tan^2 \frac{p}{2}\right]} \qquad \text{(see p. 68)},$$

$$\operatorname{sm}^3 \frac{2v}{\sqrt{3}} = \frac{\tan \frac{p}{2} \sin\left(\frac{1}{3}\lambda + d\right) - \sin d}{\tan \frac{p}{2} \sin\left(\frac{\pi}{3} + \frac{1}{3}\lambda + d\right) + \sin\left(\frac{\pi}{3} - d\right)} \qquad \text{(see p. 70)},$$

$$u = \tan^{\frac{1}{3}} \frac{p}{2} \cos \frac{1}{3}\lambda + \frac{2}{3} \cdot \frac{1}{4} \tan^{\frac{4}{3}} \frac{p}{2} \cos \frac{4}{3}\lambda + \ldots\ldots\ldots,$$

$$v = \tan^{\frac{1}{3}} \frac{p}{2} \sin \frac{1}{3}\lambda + \frac{2}{3} \cdot \frac{1}{4} \tan^{\frac{4}{3}} \frac{p}{2} \sin \frac{4}{3}\lambda + \ldots\ldots\ldots \qquad \text{(see p. 71)}.$$

By analysis similar to that given on page 73 we find that along the short diagonal we have

$$\operatorname{sm} w = \frac{\tan^{\frac{1}{3}} \frac{p}{2}}{\left(1 + \tan \frac{p}{2}\right)^{\frac{2}{3}}} = \left[\frac{\sin \frac{p}{2}}{\sqrt{2} \sin\left(\frac{\pi}{4} + \frac{p}{2}\right)}\right]^{\frac{1}{3}},$$

with

$$u = \frac{1}{2} w,$$

$$v = \frac{\sqrt{3}}{2} w.$$

Along the long diagonal the expression becomes

$$\mathrm{sm}^3 \frac{w}{\sqrt{3}} = \frac{\sin \frac{1}{6}\lambda}{\sin\left(\frac{\pi}{3} + \frac{1}{6}\lambda\right)},$$

with

$$u = K - \frac{\sqrt{3}}{2} w,$$

$$v = \frac{1}{2} w.$$

Again, we can find the ratio of scale by differentiating the expression of definition

$$\mathrm{sm}\ w = \tan^{\frac{1}{3}} \frac{p}{2}\, e^{\frac{1}{3} i\lambda},$$

$$\mathrm{cm}^2 w \frac{dw}{dp} = \frac{1}{6} \tan^{-\frac{2}{3}} \frac{p}{2} \sec^2 \frac{p}{2}\, e^{\frac{1}{3} i\lambda}$$

$$\left|\frac{dw}{dp}\right|^2 = \frac{1}{36 r^4} \cot^{\frac{4}{3}} \frac{p}{2} \sec^4 \frac{p}{2},$$

or

$$\left|\frac{dw}{dp}\right| = \frac{1}{6 r^2} \cot^{\frac{2}{3}} \frac{p}{2} \sec^2 \frac{p}{2}.$$

This expression becomes either zero or infinite in each one of the vertices of the rhombus, but it is finite and different from zero at all other points. These four points are therefore the critical points for the projection and are the points of discontinuity for the representation of the functional relation. This projection is shown in Figure 9.

CONFORMAL MAP OF THE EARTH IN A SIX-POINTED STAR

The function defined by the integral

$$w = \int_0^x \frac{dx}{(1 - x^6)^{\frac{1}{3}}},$$

belongs to the class mentioned by Schwarz in his memoir. If we assume the relation

$$x = \tan \frac{p}{2}\, e^{i\lambda},$$

one hemisphere will be conformally mapped in a regular hexagon, as Schwarz stated; the other hemisphere is conformally mapped on the six triangles that complete the star.

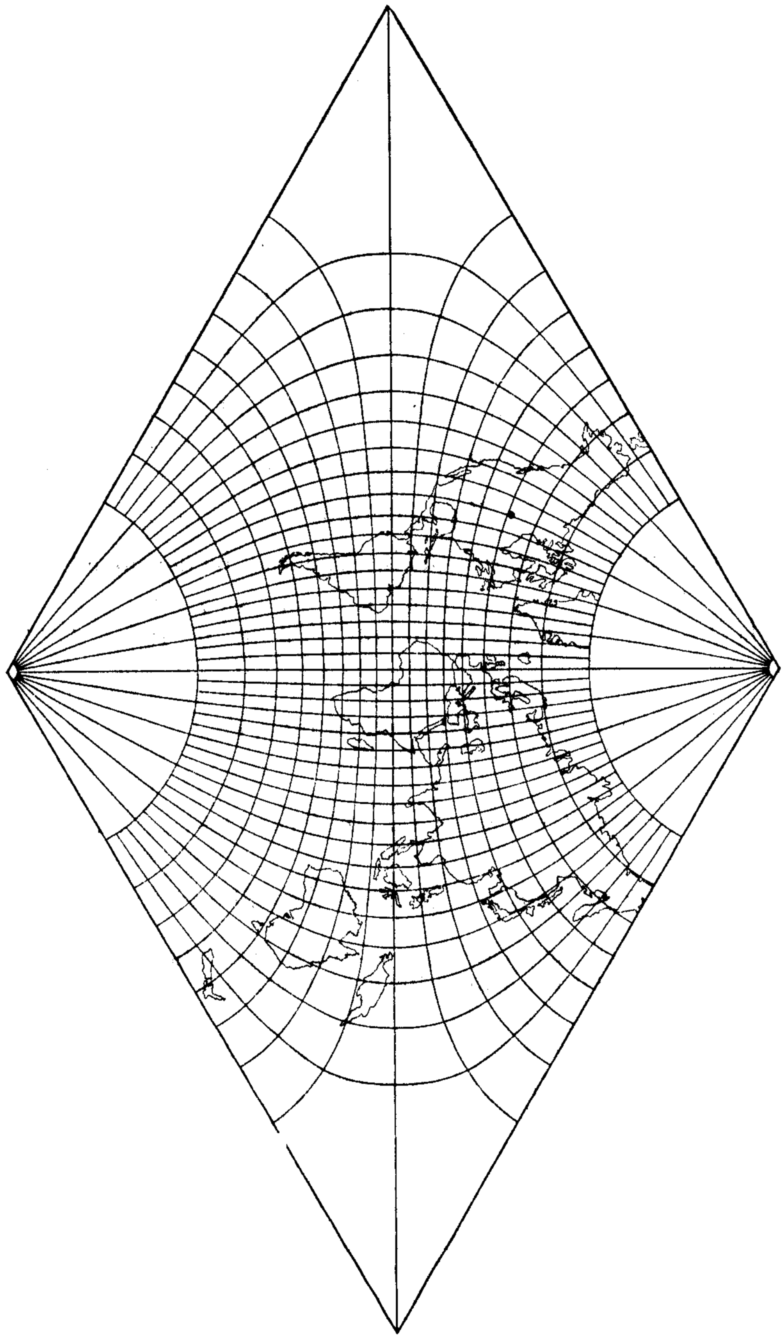

Fig. 9.—Rhombic projection of the world; poles in the 120° angles

On the other hand, if we assume the relation

$$x = \tan^{\frac{1}{3}} \frac{p}{2} e^{\frac{i\lambda}{3}}$$

the hemisphere will be mapped within the rhombus, with the poles in the 60° angles. This would give, except for the matter of scale, the same projection that we have given on page 82 and illustrated in Figure 7.

The value of the integral from 0 to 1 is easily seen to be equal to the following result:

$$\int_0^1 \frac{dx}{(1-x^6)^{\frac{1}{3}}} = \frac{1}{6} B\left(\frac{1}{6}, \frac{2}{3}\right) = \frac{1}{6} \frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{5}{6}\right)},$$

If we denote this constant by M, we find from Legendre's tables the values

$$\log M = 0.04646108844886$$

$$M = 1.11291268.$$

For computation purposes the integral may be developed in series.

$$w = \int_0^x \frac{dx}{(1-x^6)^{\frac{1}{3}}} = x + \frac{1}{3} \cdot \frac{1}{7} x^7 + \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{1}{13} x^{13} + \frac{1}{3} \cdot \frac{4}{6} \cdot \frac{7}{9} \cdot \frac{1}{19} x^{19} + \dots\dots,$$

but

$$w = u + iv, \quad \text{and } x = \tan \frac{p}{2} e^{i\lambda},$$

hence

$$u = \tan \frac{p}{2} \cos \lambda + \frac{1}{3} \cdot \frac{1}{7} \tan^7 \frac{p}{2} \cos 7\lambda + \dots\dots,$$

$$v = \tan \frac{p}{2} \sin \lambda + \frac{1}{3} \cdot \frac{1}{7} \tan^7 \frac{p}{2} \sin 7\lambda + \dots\dots$$

The logarithms of the coefficients of the series as far as it was found necessary to extend the values for computation are as follows:

$\log a_2 = 8.6777807 - 10$	$\log a_{11} = 7.11519 - 10$
$\log a_3 = 8.232844 - 10$	$\log a_{12} = 7.04730 - 10$
$\log a_4 = 7.958889 - 10$	$\log a_{13} = 6.9852 - 10$
$\log a_5 = 7.76052 - 10$	$\log a_{14} = 6.9281 - 10$
$\log a_6 = 7.60495 - 10$	$\log a_{15} = 6.8751 - 10$
$\log a_7 = 7.47696 - 10$	$\log a_{16} = 6.8258 - 10$
$\log a_8 = 7.36822 - 10$	$\log a_{17} = 6.7795 - 10$
$\log a_9 = 7.27371 - 10$	$\log a_{18} = 6.7361 - 10$
$\log a_{10} = 7.19012 - 10$	

The projection is shown in Figure 10. It will be seen that the Northern Hemisphere is fairly well represented within the regular hexagon.

By starting with a similar function of the fifth degree we can map the sphere in a five-pointed star; with the eighth degree we get an eight-pointed star, and so on.

From the symmetry of the Figure 10 it will be seen that we need to compute only coordinates for 30° of longitude north of the Equator. These values are given in the table on page 116; the other sections are replica of this quarter of one of the six rhombuses.

This integral can be inverted in terms of elliptic functions, but the expressions are so complicated that it is more convenient to use the series for computation purposes. The amount of computation required is very small because of the symmetry of the various sections, as has already been pointed out.

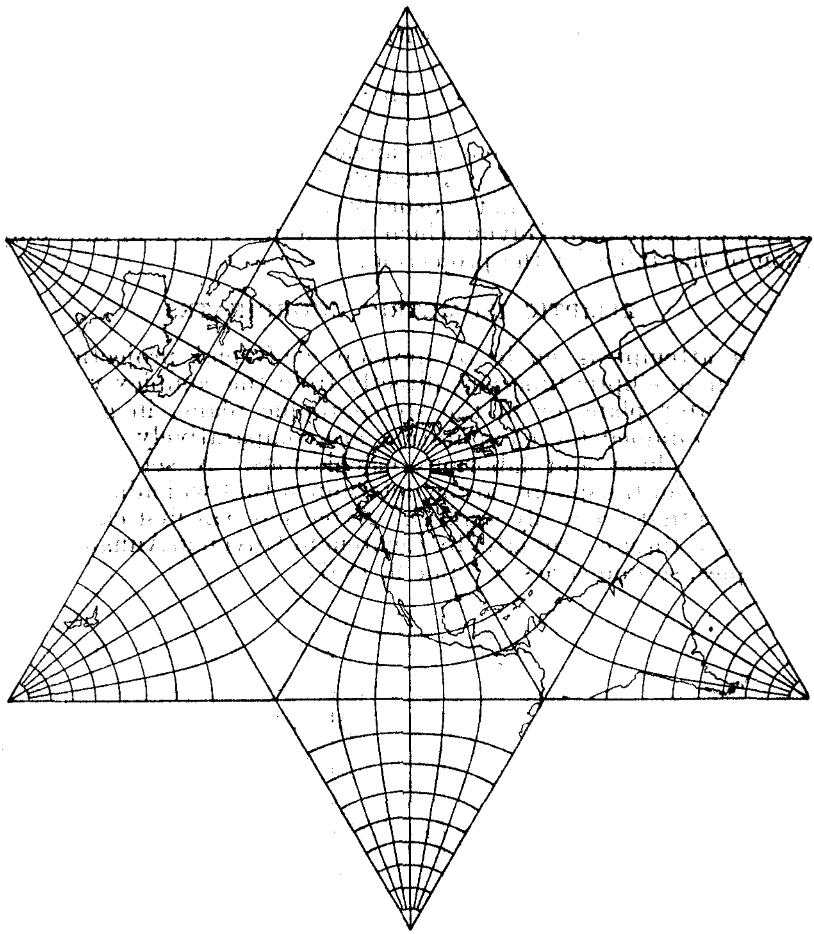

FIG. 10.—Rhombic projection of the world, in a six-pointed star

CONFORMAL PROJECTIONS IN A SQUARE

We have stated that C. S. Peirce was the first one to compute a conformal projection for geographic purposes based upon elliptic functions. This work was first published in the Report of the Superintendent of the United States Coast and Geodetic Survey for 1877 and later was published in the American Journal of Mathematics in 1879, as has already been stated.

This projection, called the quincuncial, is connected with the integral

$$w=\int_0^x \frac{dx}{(1-x^4)^{\frac{1}{2}}},$$

which is a member of the Schwarz type of integrals in which n is equal to 4. In fact we get this identical projection if we assume the relation

$$x=\tan\frac{p}{2}e^{i\lambda}.$$

We can also map a hemisphere within the square by the definition

$$x=\tan^{1/2}\frac{p}{2}e^{\frac{1}{2}i\lambda}$$

in which the poles lie in two of the angles of the square. This projection is shown in Figure 16. Although the projection could be computed with this definition in the usual way, we shall give the interesting method of Lieutenant Guyou which was actually used in the computation of the table for the projection.

ELLIPTIC ISOMETRIC COORDINATES

If an attempt is made to develop conformal projections that depend upon elliptic functions directly from the complex variable in the Mercator plane or in the stereographic plane, in general the formulas obtained for computation are comparatively complicated and require long and laborious calculations. A set of isometric coordinates for the sphere can be determined that will admit of transformations that are easily applied. Attention was first called to the existence of these coordinates by Lieutenant Guyou in Annales Hydrographiques, second series, volume 9, 1887.

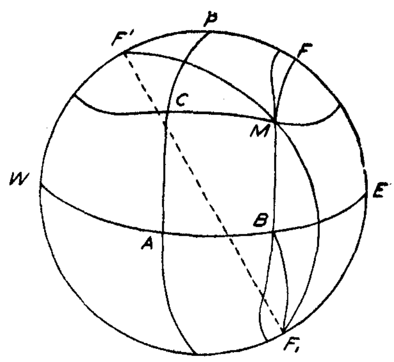

FIG. 11.—Elliptic coordinates for Guyou's projection

A spherical ellipse is the locus of points on the sphere such that the sum of the two great circle arcs which join them to two fixed points is constant. The two fixed points are called the foci of the ellipse. Let us consider two such ellipses which have one focus in common and the other foci at the opposite ends of a diameter. In Figure 11 let F be the common focus and let F' and F_1 be the other foci. Place the sphere so that P, the pole, bisects the great circle arc joining F and F'. Let $F'M=a$ and $FM=b$, then $F_1M=\pi-a$; let $PF=PF'=c$. If λ represents the longitude reckoned out from the central meridian PAP', the plane of which is perpendicular to the plane of FPF', and if ϕ represents the latitude of M, we have

$$\cos a=\cos c \sin \phi-\sin c \cos \phi \sin \lambda,$$

and

$$\cos b=\cos c \sin \phi+\sin c \cos \phi \sin \lambda.$$

If ϕ and λ are the parameters, the equations of the coordinates of the unit sphere are

$$x=\cos \phi \sin \lambda,$$

$$y=\cos \phi \cos \lambda,$$

$$z=\sin \phi.$$

Since c is taken as constant, we obtain

$$z=\frac{\cos a+\cos b}{2 \cos c}=\sec c \cos \frac{a+b}{2} \cos \frac{a-b}{2},$$

$$x=\frac{\cos b-\cos a}{2 \sin c}=\operatorname{cosec} c \sin \frac{a+b}{2} \sin \frac{a-b}{2}.$$

Now let

$$\frac{a+b}{2}=u \text{ and } \frac{a-b}{2}=v.$$

Then

$$x=\operatorname{cosec} c \sin u \sin v,$$

$$z=\sec c \cos u \cos v,$$

$$y=\sqrt{1-\operatorname{cosec}^2 c \sin^2 u \sin^2 v-\sec^2 c \cos^2 u \cos^2 v}$$

$$=\sqrt{(1-\sec^2 c \cos^2 u)\ (1-\operatorname{cosec}^2 c \sin^2 v)}.$$

The radical can of course assume either the plus or minus sign. This gives the coordinates of the sphere with u and v as curvilinear coordinates.

We can now express the element of arc upon the sphere in terms of these parameters.

$$\frac{\partial x}{\partial u}=\operatorname{cosec} c \cos u \sin v,$$

$$\frac{\partial x}{\partial v}=\operatorname{cosec} c \sin u \cos v,$$

$$\frac{\partial z}{\partial u}=-\sec c \sin u \cos v,$$

$$\frac{\partial z}{\partial v}=-\sec c \cos u \sin v,$$

$$\frac{\partial y}{\partial u}=\frac{-\operatorname{cosec}^2 c \sin u \cos u \sin^2 v+\sec^2 c \sin u \cos u \cos^2 v}{\sqrt{1-\operatorname{cosec}^2 c \sin^2 u \sin^2 v-\sec^2 c \cos^2 u \cos^2 v}},$$

$$\frac{\partial y}{\partial v}=\frac{-\operatorname{cosec}^2 c \sin^2 u \sin v \cos v+\sec^2 c \cos^2 u \sin v \cos v}{\sqrt{1-\operatorname{cosec}^2 c \sin^2 u \sin^2 v-\sec^2 c \cos^2 u \cos^2 v}},$$

$$E=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2=\frac{(\sin^2 u \cos^2 v-\cos^2 u \sin^2 v)}{\sin^2 u-\sin^2 c}$$

$$=\frac{\sin (u+v) \sin (u-v)}{\sin^2 u-\sin^2 c}=\frac{\sin a \sin b}{\sin^2 u-\sin^2 c}$$

$$F=\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}=0,$$

$$G=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2=\frac{(\sin^2 u \cos^2 v-\cos^2 u \sin^2 v)}{\sin^2 c-\sin^2 v}$$

$$=\frac{\sin (u+v) \sin (u-v)}{\sin^2 c-\sin^2 v}=\frac{\sin a \sin b}{\sin^2 c-\sin^2 v}.$$

Therefore

$$ds^2=\sin a \sin b\left(\frac{du^2}{\sin^2 u-\sin^2 c}+\frac{dv^2}{\sin^2 c-\sin^2 v}\right).$$

Since F is zero, the v curves are the orthogonal trajectories of the family of u curves.

Now, in Figure 11, let AC be denoted by n and AB by m, WAE representing the Equator. Then

$$FC=\frac{a+b}{2}=u,$$

$$F_1M=\pi-a,$$

$$F_1B=\frac{1}{2}(FM+F_1M)=\frac{\pi}{2}-\left(\frac{a-b}{2}\right)=\frac{\pi}{2}-v$$

Let A be the intersection of the Equator and the meridian, the plane of which is perpendicular to the plane of the meridian $WPEP_1$ Then, from the right spherical triangle FCP, we have

$$\sin n=\sec c \cos u,$$

and from the right spherical triangle F_1BE we get

$$\sin m=\operatorname{cosec} c \sin v.$$

By differentiating this equation, we obtain

$$\cos m\, dm=\operatorname{cosec} c \cos v\, dv,$$

but

$$\cos m=\operatorname{cosec} c\sqrt{\sin^2 c-\sin^2 v},$$

hence

$$dm = \frac{\cos v \, dv}{\sqrt{\sin^2 c - \sin^2 v}}.$$

But

$$\cos v = \sqrt{1 - \sin^2 c \sin^2 m},$$

therefore

$$\frac{dv}{\sqrt{\sin^2 c - \sin^2 v}} = \frac{dm}{\sqrt{1 - \sin^2 c \sin^2 m}}.$$

Now, by differentiating the equation for sin n, we get

$$\cos n \, dn = -\sec c \sin u \, du,$$

but

$$\cos n = \sec c \sqrt{\cos^2 c - \cos^2 u} = \sec c \sqrt{\sin^2 u - \sin^2 c},$$

hence

$$dn = -\frac{\sin u \, du}{\sqrt{\sin^2 u - \sin^2 c}}.$$

But

$$\sin u = \sqrt{1 - \cos^2 c \sin^2 n},$$

therefore

$$\frac{du}{\sqrt{\sin^2 u - \sin^2 c}} = -\frac{dn}{\sqrt{1 - \cos^2 c \sin^2 n}}.$$

The expression for the differential length of arc now becomes

$$ds^2 = \sin a \sin b \left(\frac{dm^2}{1 - \sin^2 c \sin^2 m} + \frac{dn^2}{1 - \cos^2 c \sin^2 n} \right).$$

We can now obtain a set of isometric coordinates by the following relations

$$p = \int_0^v \frac{dv}{\sqrt{\sin^2 c - \sin^2 v}} = \int_0^m \frac{dm}{\sqrt{1 - \sin^2 c \sin^2 m}},$$

and

$$q = \int_u^{\frac{\pi}{2}} \frac{du}{\sqrt{\sin^2 u - \sin^2 c}} = -\int_n^0 \frac{dn}{\sqrt{1 - \cos^2 c \sin^2 n}} = \int_0^n \frac{dn}{\sqrt{1 - \cos^2 c \sin^2 n}}.$$

The differential element of arc now becomes

$$ds^2 = \sin a \sin b \, (dp^2 + dq^2).$$

p and q are expressed as elliptic integrals of the first kind; that is,

$$p = F\,(m, \sin c),$$

$$q = F\,(n, \cos c).$$

After m and n are computed the values of p and q can be taken from Legendre's table. As a check on the computation we have the relation

$$\cos m \cos n = \sqrt{(1 - \operatorname{cosec}^2 c \sin^2 v)\,(1 - \sec^2 c \cos^2 u)} = y = \cos \phi \cos \lambda.$$

p and q form a set of isometric coordinates for the sphere and a conformal map of the sphere upon the plane can be determined by assuming the relation

$$x+iy=f(p+iq),$$

the symbol f denoting any analytic function of the complex variable $p+iq$. The simplest map of this kind is given by the equation

$$x+iy=p+iq,$$

or

$$x=p,$$

$$y=q.$$

PROJECTION OF LIEUTENANT GUYOU

The most interesting case, and the one especially treated by Lieutenant Guyou, is that in which $c=\frac{\pi}{4}$.

In this case

$$\cos a=\frac{1}{\sqrt{2}}(\sin\phi-\cos\phi\sin\lambda),$$

$$\cos b=\frac{1}{\sqrt{2}}(\sin\phi+\cos\phi\sin\lambda),$$

$$\sin m=\sqrt{2}\sin v,$$

$$\sin n=\sqrt{2}\cos u,$$

$$x=\int_0^m\frac{dm}{\sqrt{1-\frac{1}{2}\sin^2 m}},$$

and

$$y=\int_0^n\frac{dn}{\sqrt{1-\frac{1}{2}\sin^2 n}}.$$

Therefore, x and y depend upon the same integral and the hemisphere is mapped within a square, the side of which has the value $2F\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right)$ or the $2K$ value for $k=\frac{1}{\sqrt{2}}$. This gives the map as constructed by Lieutenant Guyou. (See fig. 12.) The poles are in the middle of two opposite sides of the square.

From the expression for the differential element of arc it can be seen that the ratio between the element in the plane and that on the sphere becomes infinite for the points taken as foci and their antipodal points. At these points either a or b becomes 0 or π. These points are represented by the corners of the square, and these are the critical points for the projection at which the conformality fails.

These critical points may be located anywhere upon the sphere; that is, the system of spherical ellipses can be related to the meridians and parallels in any way that we may choose. It is merely necessary to express a and b in terms of ϕ and λ for the new position of the foci.

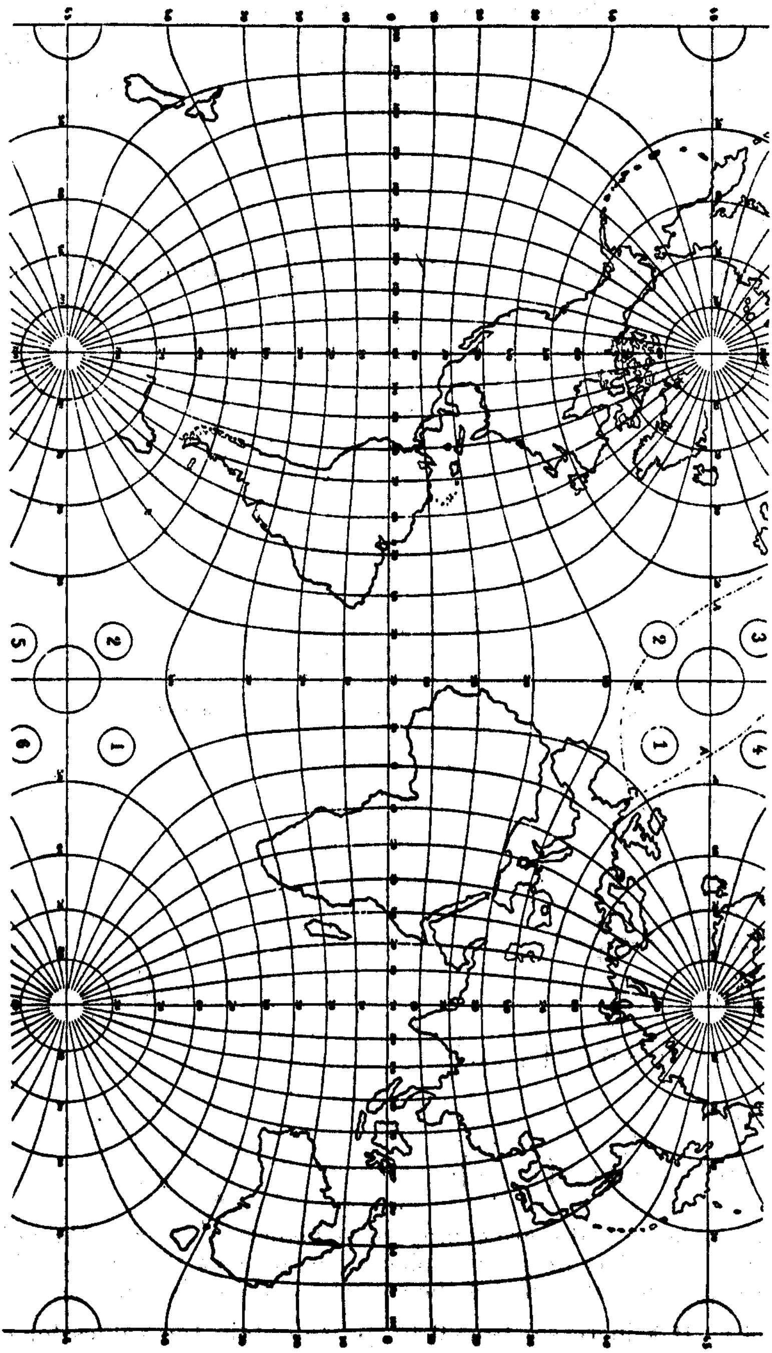

Fig. 12.—Lieutenant Guyou's projection

PEIRCE'S QUINCUNCIAL PROJECTION

Let us make the transformation that will place the foci on the Equator in place of upon a meridian. (See fig. 13.) Let P and P_1 be the poles and WAE the Equator; also suppose the planes of the great circles PEP_1W and PAP_1 perpendicular to each other. Let λ be reckoned out from PAP_1 and let F and F' be two foci so that

$$AF=AF'=c=\frac{\pi}{4}.$$

Then

$$\cos F'M=\cos F'P \cos MP+\sin F'P \sin MP \cos F'PM,$$

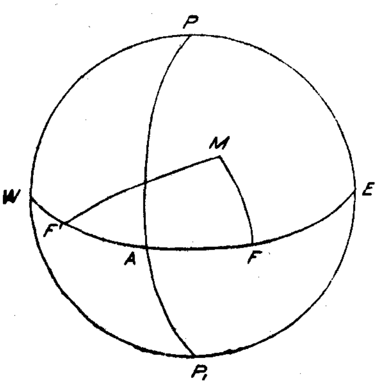

FIG. 13.—Elliptic coordinates for Peirce's quincuncial projection

but

$$F'P=\frac{\pi}{2},$$

$$MP=\frac{\pi}{2}-\phi,$$

$$\angle F'PM=\frac{\pi}{4}+\lambda,$$

therefore

$$\cos a=\cos \phi \cos\left(\frac{\pi}{4}+\lambda\right).$$

Similarly

$$\cos b=\cos \phi \cos\left(\frac{\pi}{4}-\lambda\right).$$

The angles m and n are computed as before, as are also the values of x and y. This projection places the pole at the center of the square, and the four sides of the square represent the Equator. (See fig. 14.) If λ increases from 0 to $\frac{\pi}{4}$, we obtain the values for one-eighth of a hemisphere; the symmetrical image of this with respect to the y axis completes one quarter of the hemisphere; the other three quarters are just replicas of this quarter. In this position just one-eighth of a hemisphere has actually to be computed. This is the quincuncial projection devised by C. S. Peirce and first published in the United States Coast and Geodetic Survey Superintendent's Report for 1877, as has already been stated. In this derivation the axes are differently situated from what they were in the development given by Peirce. In this case they are taken perpendicular to the

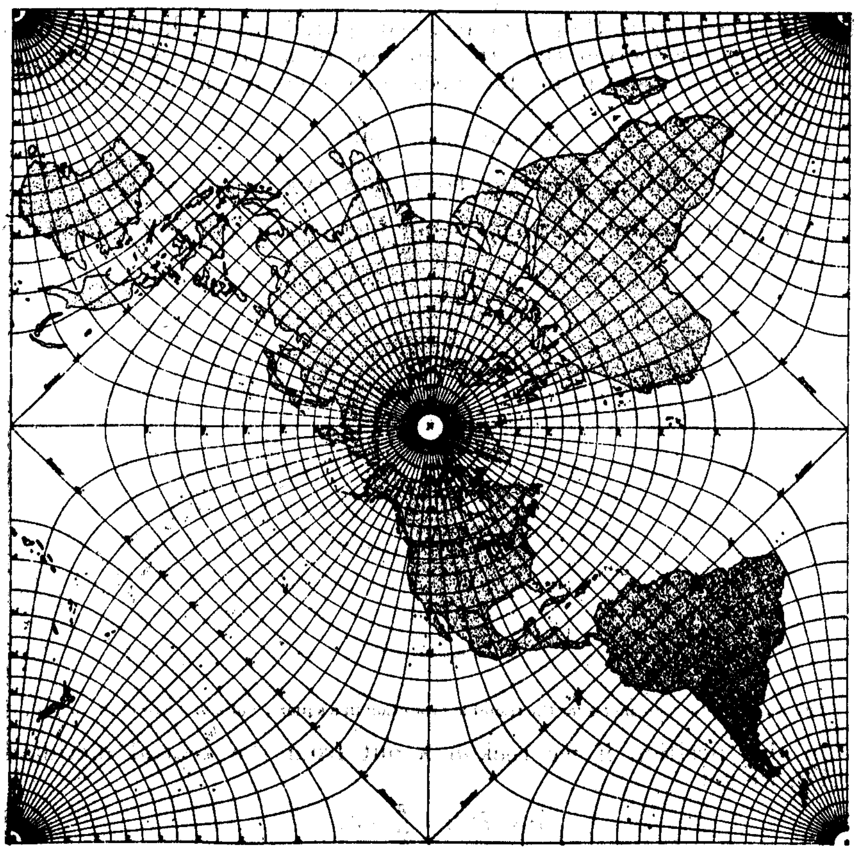

FIG. 14.—Peirce's quincuncial projection

sides of the square, but in the former development they were taken along the diagonals. The formulas for the expressions of the coordinates were complicated in the earlier development because of this fact. Peirce's table can be checked from values derived as above by the relation

$$x' = \frac{x+y}{2K},$$

$$y' = \frac{y-x}{2K},$$

x and y being the coordinates as expressed above and x' and y' being Peirce's values and K being expressed in the form,

$$K=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}.$$

In this transformation it should be noted that λ is differently reckoned in the two cases. It is reckoned out from the x axis in both cases, so that there is a difference of $\frac{\pi}{4}$ in the position of the prime meridian. It will be evident to those who have computed any of the coordinates for Peirce's projection that this method of approach is by far the simpler. In this position the check on the computation is found in the relation

$$\cos m \cos n=\sin \phi.$$

PROJECTION IN A SQUARE, POLES IN A PAIR OF THE ANGLES

As a third example let us place one focus at the pole and one on the Equator. (See fig. 15.)

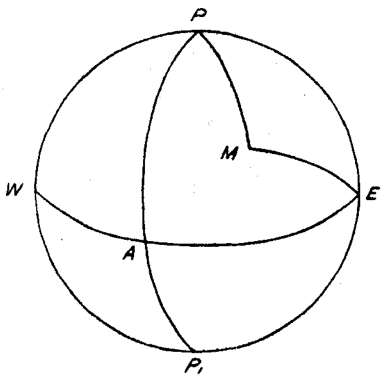

FIG. 15.—Elliptic coordinates for the rhombic projection in a square, poles in a pair of the angles

In this position if we reckon λ out from the central meridian PAP_1 we get

$$a=PM=\frac{\pi}{2}-\phi,$$

$$b=EM,$$

$$\cos b=\cos\phi \sin\lambda.$$

The angles m and n are computed in the same way as before. The check on the computation is given by the relation $\cos m \cos n=\cos\phi \cos\lambda$. The directions of the axes in this position are determined by the great circles through A bisecting the angles PAE and PAW; that is, the planes of these circles are inclined $\frac{\pi}{4}$ and $\frac{3\pi}{4}$ to the plane of the Equator. The axes in the plane will be as before parallel to the sides of the square, and the origin will be at the center of the square. One of the diagonals of the square represents the Equator and the other the central meridian, both of which are straight lines on the projection. The four quadrants of the hemisphere are symmetrical, so that it is sufficient to compute one quadrant. With the axes placed as indicated above part of the x coordinates will be positive and part of them negative. For convenience in drafting it is better to turn the axes through an angle of $\frac{\pi}{4}$. If x and y represent the values as computed and x', y' the new values, we obtain by the transformation of axes the new values

$$x' = \frac{1}{\sqrt{2}} (y + x),$$

$$y' = \frac{1}{\sqrt{2}} (y - x).$$

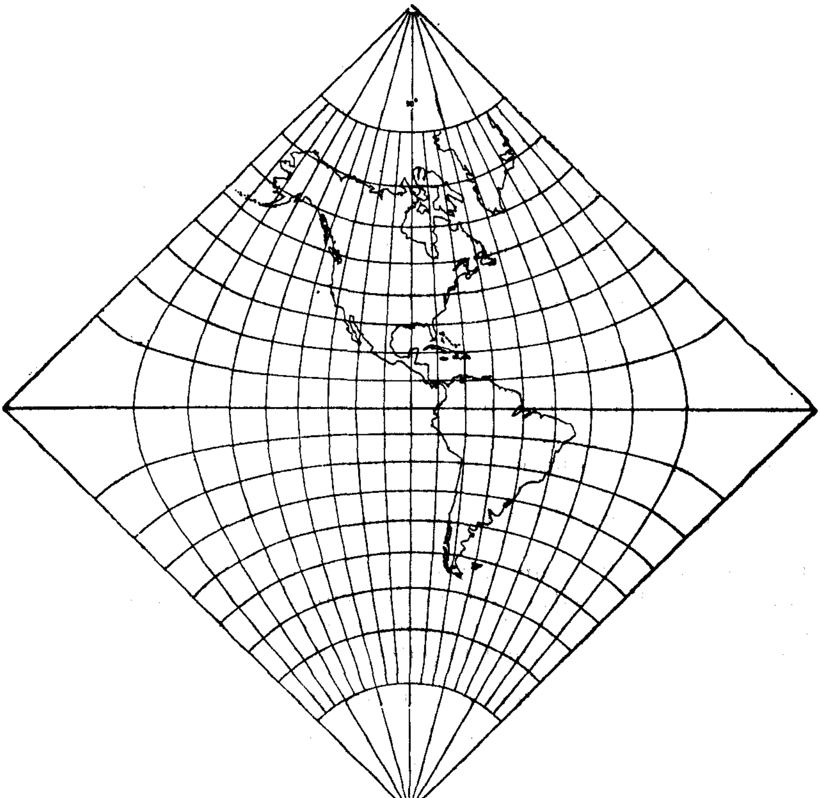

FIG. 16.—Rhombic projection of the Western Hemisphere in a square, poles in a pair of the angles

A table of these coordinates for 10° intersections of meridians and parallels computed for one quadrant of a hemisphere is given on page 116. The complete hemisphere is given by all possible combinations of signs of these coordinates. It is believed that this arrangement of the map in the square is new at least in its application to geographic maps. The projection is shown in Figure 16.

PROJECTION IN A SQUARE WHICH WOULD BE SUITABLE FOR THE UNITED STATES

The four points that are to become the critical points can be placed upon any great circle of the sphere. The best map for any particular region could be computed by adopting the central point of the region to be mapped as the pole of the great circle upon which the critical points are to be located. Assuming that the center of the United States is approximately at the intersection of the fortieth parallel with the ninety-fifth meridian, we could get a satisfactory map by assuming the foci at latitude 32° 47′ 51.″9 south and one on each side of the ninety-fifth meridian distant 57° 16′ 03.″4 in longitude from this central meridian. In this position the formula for a would be

$$\cos a = \cos (122° 47' 51.''9) \sin \phi + \sin (122° 47' 51.''9) \cos \phi \cos (\lambda - 57° 16' 03.''4),$$

and for b

$$\cos b = \cos (122° 47' 51.''9) \sin \phi + \sin (122° 47' 51.''9) \cos \phi \cos (\lambda + 57° 16' 03.''4).$$

These can be put into better shape for logarithmic computation by assuming in the first equation

$$\tan f = \tan (122° 47' 51.''9) \cos (\lambda - 57° 16' 03.''4),$$

upon which the first equation becomes

$$\cos a = \cos (122° 47' 51.''9) \sec f \sin (\phi + f).$$

Similarly, the assumption

$$\tan g = \tan (122° 47' 51.''9) \cos (\lambda + 57° 16' 03.''4),$$

reduces the second equation to the form

$$\cos b = \cos (122° 47' 51.''9) \sec g \sin (\phi + g).$$

Consideration of symmetry show that it would be sufficient to compute the half on either side of the central meridian, since this line is an axis of symmetry. The angles m and n are computed as in all of the other cases from the computed values of a and b.

FURTHER CONSIDERATION OF PEIRCE'S QUINCUNCIAL PROJECTION

The formulas that Peirce used for the quincuncial projection may be derived from the coordinates as given in this development in the following way: By turning the axes through the angle $\frac{\pi}{4}$, we get

$$x' = \frac{y + x}{\sqrt{2}},$$

$$y' = \frac{y - x}{\sqrt{2}},$$

x' and y' being coordinates for Peirce's projection. From this relation, we obtain

$$\text{cn}\,\sqrt{2}\,x' = \text{cn}\,(y+x) = \frac{\text{cn}\,x\,\text{cn}\,y - \text{sn}\,x\,\text{sn}\,y\,\text{dn}\,x\,\text{dn}\,y}{1-\frac{1}{2}\,\text{sn}^2\,x\,\text{sn}^2\,y},$$

$$\text{sn}\,x = \sin m = \sqrt{2}\,\sin\left(\frac{a-b}{2}\right),$$

$$\text{sn}\,y = \sin n = \sqrt{2}\,\cos\left(\frac{a+b}{2}\right),$$

$$\text{cn}\,x = \sqrt{1-2\sin^2\frac{a-b}{2}} = \sqrt{\cos(a-b)},$$

$$\text{dn}\,x = \sqrt{1-\sin^2\frac{a-b}{2}} = \cos\frac{a-b}{2},$$

$$\text{cn}\,y = \sqrt{1-2\cos^2\frac{a+b}{2}} = \sqrt{-\cos(a+b)},$$

$$\text{dn}\,y = \sqrt{1-\cos^2\frac{a+b}{2}} = \sin\frac{a+b}{2}.$$

Substituting these values we obtain

$$\text{cn}\,\sqrt{2}\,x' = \frac{\sqrt{-\cos(a+b)\cos(a-b)} - 2\sin\frac{a-b}{2}\cos\frac{a+b}{2}\cos\frac{a-b}{2}\sin\frac{a+b}{2}}{1-2\sin^2\frac{a-b}{2}\cos^2\frac{a+b}{2}},$$

$$= \frac{\sqrt{-\cos(a+b)\cos(a-b)} - \frac{1}{2}\sin(a+b)\sin(a-b)}{1-\frac{1}{2}(\sin a - \sin b)^2}$$

$$= \frac{\sqrt{1-\cos^2 a - \cos^2 b} + \frac{1}{2}(\cos^2 b - \cos^2 a)}{1-\frac{1}{2}\left[2-\cos^2 a - \cos^2 b - 2\sqrt{(1-\cos a)(1-\cos b)}\right]}.$$

Since λ is to be reckoned out from a meridian passing through one of the critical points, we must subtract $\frac{\pi}{4}$ from the old λ. We therefore have

$$\cos a = \cos\phi\cos\lambda,$$

$$\cos b = \cos\phi\sin\lambda.$$

When these values are substituted, we get

$$\operatorname{cn}\sqrt{2}\,x' = \frac{\sqrt{1-\cos^2\phi\cos^2\lambda-\cos^2\phi\sin^2\lambda}+\frac{1}{2}(\cos^2\phi\sin^2\lambda-\cos^2\phi\cos^2\lambda)}{1-\frac{1}{2}[2-\cos^2\phi\cos^2\lambda-\cos^2\phi\sin^2\lambda-2\sqrt{(1-\cos^2\phi\cos^2\lambda)(1-\cos^2\phi\sin^2\lambda)}]}$$

$$=\frac{\sin\phi-\frac{1}{2}\cos^2\phi\cos 2\lambda}{1-\frac{1}{2}\left[1+\sin^2\phi-2\sqrt{\sin^2\phi+\frac{1}{4}\cos^4\phi\sin^2 2\lambda}\right]}$$

$$=\frac{\sin\phi-\frac{1}{2}\cos^2\phi\cos 2\lambda}{\frac{1}{2}\cos^2\phi+\sqrt{\sin^2\phi+\frac{1}{4}\cos^4\phi\sin^2 2\lambda}}$$

$$=\frac{2\sin\phi-\cos^2\phi\cos 2\lambda}{\cos^2\phi+\sqrt{4\sin^2\phi+\cos^4\phi\sin^2 2\lambda}},$$

$$=\frac{2\tan\phi\sec\phi-\cos 2\lambda}{1+\sqrt{(2\tan\phi\sec\phi)^2+\sin^2 2\lambda}}$$

$$=\frac{(2\tan\phi\sec\phi-\cos 2\lambda)[1-\sqrt{(2\tan\phi\sec\phi)^2+\sin^2 2\lambda}]}{1-4\tan^2\phi\sec^2\phi-\sin^2 2\lambda}$$

$$=\frac{\sqrt{(2\tan\phi\sec\phi)^2+\sin^2 2\lambda}-1}{\cos 2\lambda+2\tan\phi\sec\phi}.$$

Symmetry shows us that when $\lambda'=\frac{\pi}{2}-\lambda$, x' changes into y' and y' into x'. Making this substitution and dropping the prime on λ, we get the value of y'.

$$\operatorname{cn}\sqrt{2}\,y' = \frac{\sqrt{(2\tan\phi\sec\phi)^2+\sin^2 2\lambda}-1}{2\tan\phi\sec\phi-\cos 2\lambda}.$$

This value may be checked by direct development if it is desired.

If we assume the auxiliary angles

$$\cos\alpha=\frac{\sqrt{(2\tan\phi\sec\phi)^2+\sin^2 2\lambda}-1}{\cos 2\lambda+2\tan\phi\sec\phi},$$

$$\cos\beta=\frac{\sqrt{(2\tan\phi\sec\phi)^2+\sin^2 2\lambda}-1}{2\tan\phi\sec\phi-\cos 2\lambda},$$

the coordinates become

$$x'=\frac{1}{\sqrt{2}}F\left(\alpha,\frac{1}{\sqrt{2}}\right),$$

$$y'=\frac{1}{\sqrt{2}}F\left(\beta,\frac{1}{\sqrt{2}}\right).$$

This is sufficient if we are willing to let the coordinates be expressed directly in the values of the elliptic integrals. Peirce's table is computed with the length of the semidiagonal of the square as unity. The above values must therefore be divided by $\sqrt{2}K$, since this is the length of the semidiagonal in terms of the integral. This gives, on dropping the primes,

$$x=\frac{1}{2K}F\left(\alpha, \frac{1}{\sqrt{2}}\right),$$

$$y=\frac{1}{2K}F\left(\beta, \frac{1}{\sqrt{2}}\right).$$

RATIO OF SCALE FOR THE SQUARE PROJECTION, POLES IN A PAIR OF THE ANGLES

To determine the ratio of scale in the projection illustrated in Figure 16, it is better to make use of the definition in the form

$$x=\tan^{\frac{1}{2}}\frac{p}{2}e^{\frac{1}{2}i\lambda}.$$

By differentiation with respect to p, we get

$$\frac{dx}{dw}\frac{dw}{dp}=\frac{1}{4}\cot^{\frac{1}{2}}\frac{p}{2}\sec^2\frac{p}{2}e^{\frac{1}{2}i\lambda}.$$

But

$$\frac{dx}{dw}=\sqrt{1-x^4}=\sqrt{1-\tan^2\frac{p}{2}e^{2i\lambda}},$$

hence

$$\frac{dw}{dp}=\frac{\frac{1}{4}\cot^{\frac{1}{2}}\frac{p}{2}\sec^2\frac{p}{2}e^{\frac{1}{2}i\lambda}}{\sqrt{1-\tan^2\frac{p}{2}e^{2i\lambda}}},$$

and so

$$\left|\frac{dw}{dp}\right|^2=\frac{\frac{1}{16}\cot\frac{p}{2}\sec^4\frac{p}{2}}{\sqrt{1-2\tan^2\frac{p}{2}\cos 2\lambda+\tan^4\frac{p}{2}}}$$

or

$$\left|\frac{dw}{dp}\right|=\frac{\frac{1}{4}\cot^{\frac{1}{2}}\frac{p}{2}\sec^2\frac{p}{2}}{\left(1-2\tan^2\frac{p}{2}\cos 2\lambda+\tan^4\frac{p}{2}\right)^{\frac{1}{4}}}.$$

This expression shows that the points of discontinuity lie in the angles of the square; it is evident, a priori, that these points are such since the conformality fails in each of them, 180° being mapped in a 90° angle.

CONFORMAL PROJECTION OF THE SPHERE IN AN ELLIPSE

We shall now give consideration to a projection that does not belong to the rhombic class but which bears some relation to it in that it depends upon a projection that is defined in terms of elliptic functions. In the seventieth volume of Crelle's Journal für die reine und angewandte Mathematik, page 115, H. A. Schwarz mentions the fact that an ellipse with foci at $u = \pm 1$ can be conformally mapped within a circle by the function

$$s = \sin \text{am} \left(\frac{2K}{\pi} \text{arc} \sin u \right)$$

in which $s = 0$ is the center of the circle with the radius of the circle equal to $\frac{1}{\sqrt{k}}$.

It was at once noted that conversely the circle could be mapped within the ellipse by the same relation. It seemed desirable to map the sphere within an ellipse such that the major axis would be about twice the minor axis. It was found that this end could be attained if k were taken equal to sin 65°.

Accordingly, the full definition of the projection was taken in the form

$$\sin \text{am} \left[\left(\frac{2K}{\pi} \text{arc} \sin \bar{w} \right) + \frac{1}{2} i K' \right] = \frac{1}{\sqrt{k}} \tan^{\frac{1}{2}} \frac{p}{2} e^{\frac{1}{2} i \lambda},$$

in which $\bar{w} = u - iv$.

It was found most convenient to make the computation step by step; and so, in accordance with this plan, a projection was computed with the definition

$$\sin \text{am}\, z = \frac{1}{\sqrt{k}} \tan^{\frac{1}{2}} \frac{p}{2} e^{\frac{1}{2} i \lambda}.$$

This definition maps the sphere within a rectangle with base $2K$ and altitude K'. The Equator is represented by the line $y = \frac{1}{2} K'$; the poles lie on the axis of y, one at the origin and the other at $y = K'$. This projection is illustrated in Figure 17. The projection is very much elongated and of course has no further interest than being the basis of the projection within the ellipse. As could be foreseen, the four quadrants of the rectangle are symmetrical images of any one of them. The table given on page 117 gives the coordinates for the first quarter of the projection.

From this table a new set of y values was derived by subtracting each of the y's of the table from $\frac{1}{2} K'$. The effect of this process was to move the axis of x up to the point $y = \frac{1}{2} K'$, with the coordinates

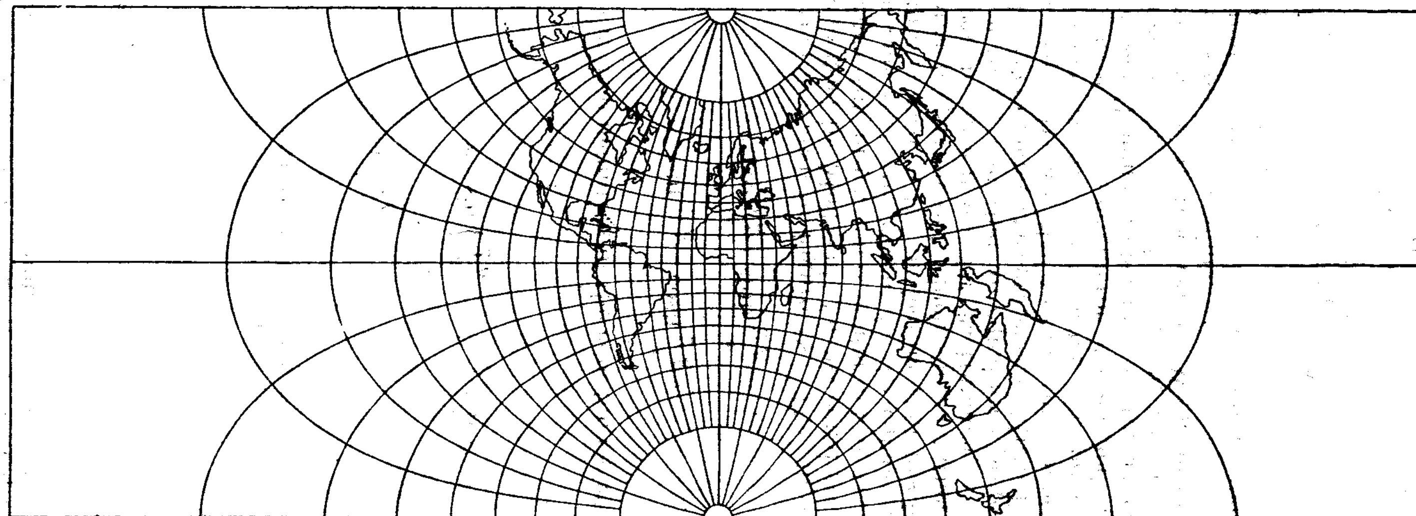

FIG. 17.—Conformal projection of the world in a rectangle

given for the section of the rectangle in the first quadrant of coordinates. If we denote this new complex variable by z', we have

$$z = \frac{2K}{\pi} \text{arc sin} \overline{w} + \frac{1}{2} iK',$$

or

$$\frac{2K}{\pi} \text{arc sin} \overline{w} = z - \frac{1}{2} iK',$$

and

$$\frac{2K}{\pi} \text{arc sin } w = \bar{z} + \frac{1}{2} iK' = z',$$

or finally

$$w = \sin \frac{\pi}{2K} z'.$$

This gives us

$$u + iv = \sin \left(\frac{\pi}{2K} x + i \frac{\pi}{2K} y\right),$$

if we let

$$z' = x + iy.$$

This gives us the final definition of the coordinates within the ellipse

$$u = \sin \frac{\pi}{2K} x \cosh \frac{\pi}{2K} y,$$

$$v = \cos \frac{\pi}{2K} x \sinh \frac{\pi}{2K} y.$$

This projection is shown in Figure 18. The ellipse is about the same as the one used by Mollweide for his equal-area projection of the sphere. Since this projection is conformal, it avoids the violent angular distortions that are present in the equal-area projections. The same ellipse is used in the projection that has been called the Aitoff equal-area projection but which should rather be called the Hammer projection, since it was Dr. E. Hammer, of Stuttgart, who called attention to the fact that an equal-area projection could be so constructed. (Petermann's Geographische Mittheilungen, Bd. 38, 1892, p. 85 et. seq.) We are glad to take this opportunity to state that we were somewhat at fault in using the name Aitoff in previous publications when the originator of the equal-area map of the world was Professor Hammer, as is shown by the article cited above.

An expression for the ratio of the linear arcs is so complicated that it is of very small practical value. We shall derive it to help in locating the discontinuous points.

If we differentiate the general expression of definition of the projection with regard to p we get

$$\text{cn} \left[\left(\frac{2K}{\pi} \text{arc sin } \overline{w}\right) + \frac{1}{2} iK'\right] \text{dn} \left[\left(\frac{2K}{\pi} \text{arc sin } \overline{w}\right) + \frac{1}{2} iK'\right]$$

$$\times \frac{2K}{\pi} \frac{dw}{dp} \frac{1}{\sqrt{1-w^2}} = \frac{1}{4} \frac{1}{\sqrt{k}} \cot^{\frac{1}{2}} \frac{p}{2} \sec^2 \frac{p}{2} e^{\frac{1}{2} i\lambda}.$$

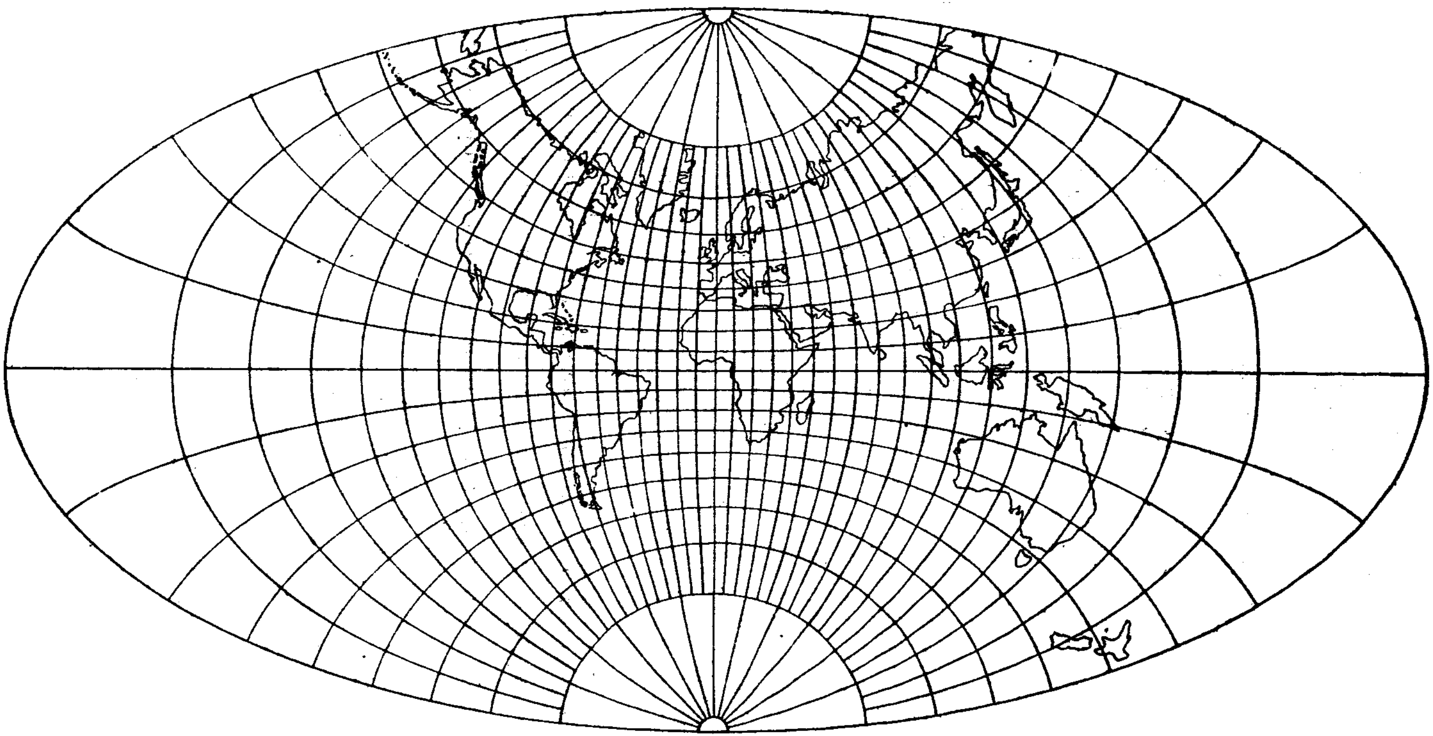

FIG. 18.—Conformal projection of the world in an ellipse

Therefore

$$\left|\frac{dw}{dp}\right|^2=\frac{\pi^2}{64K^2}\frac{1}{k}\frac{\cot\frac{p}{2}\sec^4\frac{p}{2}\sqrt{(1-u^2+v^2)^2+4u^2v^2}}{\sqrt{1-\frac{2}{k}\tan\frac{p}{2}\cos\lambda+\frac{1}{k^2}\tan^2\frac{p}{2}}\sqrt{1-2k\tan\frac{p}{2}\cos\lambda+k^2\tan^2\frac{p}{2}}}$$

or

$$\left|\frac{dw}{dp}\right|=\frac{\pi}{8\sqrt{k}K}\frac{\cot^{\frac{1}{2}}\frac{p}{2}\sec^2\frac{p}{2}[(1-u^2+v^2)^2+4u^2v^2]^{\frac{1}{4}}}{\left(1-\frac{2}{k}\tan\frac{p}{2}\cos\lambda+\frac{1}{k^2}\tan^2\frac{p}{2}\right)^{\frac{1}{4}}\left(1-2k\tan\frac{p}{2}\cos\lambda+k^2\tan^2\frac{p}{2}\right)^{\frac{1}{4}}}.$$

The critical points are thus seen to lie at $p=0°$ and $p=180°$.

ANALYSIS FOR THE PROJECTION OF THE SPHERE IN A RECTANGLE

We shall now indicate the method used in the computation of the rectangular projection. We have the definition

$$\operatorname{sn} z=\frac{1}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2}\,e^{\frac{1}{2}i\lambda},$$

and

$$\operatorname{sn} \bar{z}=\frac{1}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2}\,e^{-\frac{1}{2}i\lambda}.$$

Now assume

$$\operatorname{cn} z=r_1e^{-if},$$

and

$$\operatorname{dn} z=r_2e^{-ig},$$

then

$$\operatorname{cn} \bar{z}=r_1e^{if},$$

$$\operatorname{dn} \bar{z}=r_2e^{ig}.$$

But

$$\operatorname{cn}^2 z=1-\operatorname{sn}^2 z,$$

or

$$r_1^2e^{-2if}=1-\frac{1}{k}\tan\frac{p}{2}\,e^{i\lambda},$$

hence

$$r_1^4=1-\frac{2}{k}\tan\frac{p}{2}\cos\lambda+\frac{1}{k^2}\tan^2\frac{p}{2},$$

and

$$\tan 2f=\frac{\frac{1}{k}\tan\frac{p}{2}\sin\lambda}{1-\frac{1}{k}\tan\frac{p}{2}\cos}.$$

By an obvious reduction, we get

$$\sec^2 2f = \frac{r_1^4}{\left(1-\frac{1}{k}\tan\frac{p}{2}\cos\lambda\right)^2},$$

or

$$r_1^2 = \frac{1-\frac{1}{k}\tan\frac{p}{2}\cos\lambda}{\cos 2f}.$$

Again we have

$$\mathrm{dn}^2\, z = 1 - k^2\,\mathrm{sn}^2\, z,$$

or

$$r_2 e^{-2ig} = 1 - k\tan\frac{p}{2}\, e^{i\lambda}.$$

Hence we get as before

$$r_2^4 = 1 - 2k\tan\frac{p}{2}\cos\lambda + k^2\tan^2\frac{p}{2},$$

$$\tan 2g = \frac{k\tan\frac{p}{2}\sin\lambda}{1-k\tan\frac{p}{2}\cos\lambda},$$

or

$$r_2^2 = \frac{1-k\tan\frac{p}{2}\cos\lambda}{\cos 2g}.$$

After f and g are computed it is more convenient to use the **second** expression for r_1 and r_2.

Now we get

$$\mathrm{sn}\, 2x = \mathrm{sn}\,(z+\bar{z}) = \frac{\mathrm{sn}\, z\,\mathrm{cn}\,\bar{z}\,\mathrm{dn}\,\bar{z} + \mathrm{sn}\,\bar{z}\,\mathrm{cn}\, z\,\mathrm{dn}\, z}{1-k^2\,\mathrm{sn}^2 z\,\mathrm{sn}^2\bar{z}}$$

$$= \frac{\frac{1}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2}\, e^{\frac{1}{2}i\lambda}\, r_1 e^{if}\, r_2 e^{ig} + \frac{1}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2}\, e^{-\frac{1}{2}i\lambda}\, r_1 e^{-if}\, r_2 e^{-ig}}{1-k^2\frac{1}{k^2}\tan^2\frac{p}{2}}$$

$$= \frac{\frac{1}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2}\, r_1 r_2\left[e^{\frac{1}{2}i\lambda+if+ig} + e^{-(\frac{1}{2}i\lambda+if+ig)}\right]}{1-\tan^2\frac{p}{2}}$$

$$= \frac{\frac{2}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2}\, r_1 r_2\cos\left(\frac{1}{2}\lambda+f+g\right)}{1-\tan^2\frac{p}{2}}.$$

Also we have

$$\operatorname{sn}(2iy) = \operatorname{sn}(z-\bar{z}) = \frac{\operatorname{sn} z \operatorname{cn} \bar{z} \operatorname{dn} z - \operatorname{sn} \bar{z} \operatorname{cn} z \operatorname{dn} z}{1-k^2 \operatorname{sn}^2 z \operatorname{sn}^2 \bar{z}},$$

or

$$i\frac{\operatorname{sn}(2y, k')}{\operatorname{cn}(2y, k')} = \frac{\frac{2i}{\sqrt{k}} \tan^{\frac{1}{2}} \frac{p}{2} r_1 r_2 \sin\left(\frac{1}{2}\lambda + f + g\right)}{1-\tan^2 \frac{p}{2}},$$

hence finally

$$\tan \operatorname{am}(2y, k') = \frac{\frac{2}{\sqrt{k}} \tan^{\frac{1}{2}} \frac{p}{2} r_1 r_2 \sin\left(\frac{1}{2}\lambda + f + g\right)}{1-\tan^2 \frac{p}{2}}.$$

With these formulas we can compute am $(2x, k)$ and am $(2y, k')$; then from Legendre's table with $k = \sin 65°$ we get u and with $k' = \sin 25°$, we get v. The computations are not so formidable as they appear at first sight. A table of the values $\dfrac{\frac{2}{\sqrt{k}} \tan^{\frac{1}{2}} \frac{p}{2}}{1-\tan^2 \frac{p}{2}}$ can be made for the various latitudes and this table may be used in the computation of the coordinates.

On the Equator and the central meridian the formulas can be simplified. On the Equator, we have

$$\operatorname{sn}\left(a + \frac{1}{2}iK\right) = \frac{1}{\sqrt{k}} e^{\frac{1}{2}i\lambda},$$

or

$$\frac{\frac{1+k}{\sqrt{k}} \operatorname{sn} a + \frac{i}{\sqrt{k}} \operatorname{cn} a \operatorname{dn} a}{1+k \operatorname{sn}^2 a} = \frac{1}{\sqrt{k}} \cos \frac{1}{2}\lambda + \frac{i}{\sqrt{k}} \sin \frac{1}{2}\lambda.$$

Hence

$$\frac{(1+k) \operatorname{sn} a}{1+k \operatorname{sn}^2 a} = \cos \frac{1}{2}\lambda,$$

and

$$\frac{\operatorname{cn} a \operatorname{dn} a}{1+k \operatorname{sn}^2 a} = \sin \frac{1}{2}\lambda.$$

But

$$\frac{(1+k) \operatorname{sn} a}{1+k \operatorname{sn}^2 a} = \operatorname{sn}\left[(1+k)a, \frac{2\sqrt{k}}{1+k}\right] = \cos \frac{1}{2}\lambda.$$

Since some of the interpolations required by this formula with Legendre's table are rather laborious, we can compute sn a from the formula

$$\operatorname{sn} a = \frac{1+k-\sqrt{1-2k \cos \lambda + k^2}}{2k \cos \frac{1}{2}\lambda},$$

which is, of course, derived from

$$\frac{(1+k)\ \mathrm{sn}\ a}{1+k\ \mathrm{sn}^2\ a}=\cos\frac{1}{2}\lambda.$$

On the central meridian we have

$$\mathrm{sn}\ ia=\frac{i}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2},$$

or

$$\tan\ \mathrm{am}\ (a,\ k')=\frac{1}{\sqrt{k}}\tan^{\frac{1}{2}}\frac{p}{2}.$$

CONCLUSION

This concludes the list of projections that we wish to discuss at this time. It is believed that practically all of the varieties developed in this publication for which tables are given are new, at least as to application to geographic cartography. Many other varieties could be deduced from the principles laid down in these pages, but we have included enough to fully illustrate the methods developed. A projection of the whole world within the 60°–120° rhombus with the poles symmetrically located on the long diagonal was considered for a time, but the amount of work required for the computation seemed excessive in view of the small usefulness of such an example. Tables for the necessary coordinates of the various projections are added to this work so that they will be available for immediate use. The other points of the projections for which coordinates are not given must be plotted from consideration of symmetry. The amount of computation for the various tables was considerable. The author has had the pleasure, if such it be, of personally carrying out most of these computations. He is sure that the work will be useful in many practical applications.

TABLES

Dixon elliptic functions for $\alpha=0$

u	sm u	cm u	$\frac{\text{sm } u}{\text{cm } u}$	1−cm u	u	sm u	cm u	$\frac{\text{sm } u}{\text{cm } u}$	1−cm u
0.0000	0.0000	1.0000	0.0000	0.00000000	0.8833	0.7937	0.7937	1.0000	
.0147	.0147	1.0000	.0147	.00000106	.8980	.8029	.7843	1.0236	
.0294	.0294	1.0000	.0294	.00000851	.9128	.8118	.7747	1.0479	
.0442	.0442	1.0000	.0442	.0000287	.9275	.8205	.7649	1.0727	
.0589	.0589	.9999	.0589	.0000681	.9422	.8290	.7549	1.0982	
.0736	.0736	.9999	.0736	.0001329	.9569	.8373	.7447	1.1244	
.0883	.0883	.9998	.0883	.0002297	.9716	.8454	.7343	1.1513	
.1030	.1030	.9996	.1031	.0003648	.9864	.8532	.7237	1.1790	
.1178	.1177	.9995	.1178	.0005446	1.0011	.8608	.7128	1.2076	
.1325	.1325	.9992	.1325	.0007754	1.0158	.8681	.7018	1.2370	
.1472	.1471	.9989	.1473	.001063	1.0305	.8753	.6906	1.2673	
.1619	.1618	.9986	.1621	.001415	1.0453	.8822	.6793	1.2988	
.1767	.1765	.9982	.1768	.001836	1.0600	.8889	.6677	1.3312	
.1914	.1912	.9977	.1916	.002338	1.0747	.8953	.6560	1.3648	
.2061	.2058	.9971	.2064	.002914	1.0894	.9015	.6441	1.3996	
.2208	.2204	.9964	.2212	.00358	1.1042	.9075	.6321	1.4358	
.2356	.2350	.9957	.2361	.00435	1.1189	.9133	.6199	1.4734	
.2503	.2496	.9948	.2509	.00521	1.1336	.9189	.6075	1.5125	
.2650	.2642	.9938	.2658	.00618	1.1483	.9242	.5950	1.5532	
.2797	.2787	.9927	.2808	.00727	1.1630	.9293	.5824	1.5957	
.2944	.2932	.9915	.2957	.00847	1.1778	.9342	.5696	1.6401	
.3092	.3076	.9902	.3107	.00980	1.1925	.9388	.5567	1.6865	
.3239	.3221	.9887	.3257	.01126	1.2072	.9433	.5436	1.7351	
.3386	.3364	.9871	.3408	.01286	1.2219	.9475	.5305	1.7861	
.3533	.3507	.9854	.3559	.01460	1.2366	.9516	.5172	1.8399	
.3681	.3650	.9835	.3711	.01648	1.2514	.9554	.5038	1.8963	
.3828	.3792	.9815	.3864	.01852	1.2661	.9590	.4903	1.9559	
.3975	.3934	.9794	.4017	.02072	1.2808	.9625	.4767	2.0189	
.4122	.4075	.9763	.4171	.02308	1.2955	.9657	.4630	2.0856	
.4269	.4215	.9749	.4326	.02561	1.3103	.9688	.4493	2.1564	
.4417	.4354	.9717	.4481	.02831	1.3250	.9717	.4354	2.2316	
.4564	.4493	.9688	.4637	.03119	1.3397	.9744	.4215	2.3119	
.4711	.4630	.9657	.4795	.03426	1.3544	.9769	.4075	2.3975	
.4858	.4767	.9625	.4953	.03751	1.3692	.9793	.3934	2.4894	
.5006	.4903	.9590	.5113	.04095	1.3839	.9815	.3792	2.5873	
.5153	.5038	.9554	.5273	.04459	1.3986	.9835	.3650	2.6944	
.5300	.5172	.9516	.5435	.04842	1.4133	.9854	.3507	2.8094	
.5447	.5305	.9475	.5599	.05246	1.4280	.9871	.3364	2.9341	
.5594	.5436	.9433	.5763	.05671	1.4428	.9887	.3221	3.0701	
.5742	.5567	.9388	.5929	.06117	1.4575	.9902	.3076	3.2186	
.5889	.5696	.9342	.6097	.06584	1.4722	.9915	.2932	3.3818	
.6036	.5824	.9293	.6267		1.4869	.9927	.2787	3.5619	
.6183	.5950	.9242	.6438		1.5016	.9938	.2642	3.7620	
.6330	.6075	.9189	.6612		1.5164	.9948	.2496	3.9851	
.6478	.6199	.9133	.6787		1.5311	.9957	.2350	4.2360	
.6625	.6321	.9075	.6965		1.5458	.9964	.2204	4.5203	
.6772	.6441	.9015	.7145		1.5605	.9971	.2058	4.8448	
.6919	.6560	.8953	.7327		1.5752	.9977	.1912	5.2192	
.7067	.6677	.8889	.7512		1.5900	.9982	.1765	5.6550	
.7214	.6793	.8822	.7700		1.6047	.9986	.1618	6.1708	
.7361	.6906	.8753	.7891		1.6194	.9989	.1471	6.7889	
.7508	.7018	.8681	.8084		1.6341	.9992	.1325	7.5157	
.7655	.7128	.8608	.8281		1.6489	.9995	.1177	8.4887	
.7803	.7237	.8532	.8482		1.6636	.9996	.1030	9.7011	
.7950	.7343	.8454	.8686		1.6783	.9998	.0883	11.3187	
.8097	.7447	.8373	.8893		1.6930	.9999	.0736	13.5859	
.8244	.7549	.8290	.9106		1.7077	.9999	.0589	16.9821	
.8392	.7649	.8205	.9322		1.7225	1.0000	.0442	22.6423	
.8539	.7747	.8118	.9543		1.7372	1.0000	.0294	33.9552	
.8686	.7843	.8029	.9769		1.7519	1.0000	.0147	67.9221	
					1.7666	1.0000	.0000	∞	

Entire sphere in a regular hexagon

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°		Longitude 50°		Longitude 60°	
	x	y	x	y	x	y	x	y	x	y	x	y	x	y
0°	1.7666	0	1.0678	0.4035	0.8833	0.5100	0.7497	0.5871	0.6382	0.6515	0.5376	0.7096	0.4416	0.7650
10	.9602	0	.8913	.2317	.7782	.3764	.6714	.4747	.5730	.5504	.4793	.6138	.3868	.6700
20	.7487	0	.7219	.1552	.6586	.2823	.5816	.3805	.5007	.4583	.4179	.5222	.3328	.5764
30	.5978	0	.5833	.1140	.5443	.2167	.4894	.3037	.4252	.3759	.3552	.4357	.2802	.4853
40	.4747	0	.4654	.0864	.4392	.1675	.3996	.2398	.3499	.3021	.2927	.3545	.2294	.3973
50	.3670	0	.3606	.0651	.3425	.1274	.3137	.1845	.2761	.2349	.2313	.2778	.1806	.3128
60	.2688	0	.2647	.0471	.2519	.0925	.2316	.1347	.2045	.1725	.1714	.2050	.1335	.2313
70	.1765	0	.1738	.0307	.1657	.0605	.1526	.0883	.1349	.1134	.1132	.1350	.0881	.1526
80	.0875	0	.0862	.0152	.0822	.0299	.0758	.0438	.0670	.0562	.0562	.0670	.0437	.0758
90	.0000	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Hemisphere in the rhombus, poles in the 120° angles

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°	
	x	y	x	y	x	y	x	y	x	y
0°	1.7666	0	1.1566	0.3522	0.9966	0.4445	0.8833	0.5100	0.7910	0.5633
10	1.0617	0	1.0013	.2034	.9055	.3296	.8170	.4164	.7365	.4819
20	.8752	0	.8465	.1371	.7987	.2514	.7370	.3395	.6726	.4095
30	.7397	0	.7278	.1038	.6962	.1979	.6525	.2784	.6027	.3461
40	.6257	0	.6183	.0818	.5985	.1587	.5668	.2285	.5289	.2901
50	.5219	0	.5169	.0648	.5028	.1274	.4805	.1858	.4516	.2394
60	.4207	0	.4170	.0506	.4073	.1000	.3913	.1471	.3698	.1913
70	.3161	0	.3138	.0373	.3069	.0740	.2958	.1091	.2805	.1427
80	.1973	0	.1960	.0230	.1919	.0457	.1853	.0648	.1762	.0887
90	.0000	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Latitude	Longitude 50°		Longitude 60°		Longitude 70°		Longitude 80°		Longitude 90°	
	x	y	x	y	x	y	x	y	x	y
0°	0.7107	0.6096	0.6382	0.6515	0.5702	0.6907	0.5053	0.7282	0.4416	0.7650
10	.6635	.5360	.5956	.5830	.5308	.6253	.4676	.6644	.4049	.7014
20	.6097	.4675	.5479	.5171	.4881	.5611	.4282	.6007	.3679	.6372
30	.5504	.4039	.4968	.4537	.4422	.4975	.3868	.5364	.3300	.5716
40	.4863	.3444	.4408	.3920	.3928	.4340	.3428	.4699	.2908	.5037
50	.4178	.2877	.3800	.3309	.3392	.3691	.2951	.4029	.2496	.4324
60	.3437	.2318	.3138	.2688	.2815	.3017	.2442	.3307	.2054	.3558
70	.2616	.1726	.2394	.2027	.2145	.2285	.1865	.2513	.1564	.2710
80	.1645	.1085	.1507	.1268	.1350	.1433	.1175	.1580	.0984	.1704
90	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Hemisphere in the rhombus, poles in the 60° angles

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°	
	x	y	x	y	x	y	x	y	x	y
0°	0.0000	0	0.0984	0.1704	0.1564	0.2710	0.2054	0.3558	0.2496	0.4324
10	.1973	0	.2153	.1236	.2500	.2260	.2874	.3138	.3246	.3924
20	.3161	0	.3235	.1010	.3452	.1946	.3726	.2793	.4027	.3588
30	.4207	0	.4258	.0874	.4396	.1716	.4599	.2510	.4842	.3257
40	.5219	0	.5254	.0774	.5353	.1533	.5508	.2264	.5706	.2966
50	.6257	0	.6285	.0689	.6360	.1371	.6484	.2039	.6646	.2686
60	.7397	0	.7414	.0609	.7470	.1215	.7576	.1811	.7712	.2395
70	.8752	0	.8768	.0523	.8817	.1044	.8897	.1559	.8998	.2067
80	1.0617	0	1.0628	.0411	1.0674	.0820	1.0725	.1226	1.0810	.1628
90	1.7666	0	1.7666	.0000	1.7666	.0000	1.7666	.0000	1.7666	.0000

Latitude	Longitude 50°		Longitude 60°		Longitude 70°		Longitude 80°		Longitude 90°	
	x	y	x	y	x	y	x	y	x	y
0°	0.2908	0.5037	0.3300	0.5716	0.3679	0.6372	0.4049	0.7014	0.4416	0.7650
10	.3611	.4651	.3974	.5341	.4333	.6003	.4692	.6648	.5053	.7282
20	.4345	.4297	.4672	.4984	.5007	.5641	.5351	.6280	.5702	.6907
30	.5114	.3961	.5375	.4633	.5717	.5278	.6042	.5904	.6382	.6515
40	.5938	.3637	.6197	.4283	.6481	.4905	.6785	.5508	.7107	.6096
50	.6843	.3313	.7071	.3919	.7328	.4508	.7607	.5078	.7910	.5633
60	.7874	.2966	.8079	.3522	.8305	.4063	.8554	.4589	.8833	.5100
70	.9142	.2564	.9308	.3053	.9503	.3529	.9723	.3994	.9966	.4445
80	1.0915	.2024	1.1066	.2412	1.1197	.2792	1.1369	.3167	1.1566	.3522
90	1.7666	.0000	1.7666	.0000	1.7666	.0000	1.7666	.0000	1.7666	.0000

Hemisphere in the rhombus, pole at the intersection of the diagonals

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°	
	x	y	x	y	x	y	x	y	x	y
0°	0.0000	0.0000	0.1973	0.0000	0.3161	0.0000	0.4207	0.0000	0.5219	0.0000
10	.0984	.1704	.2140	.1258	.3221	.1977	.4224	.1010	.5201	.1010
20	.1564	.2710	.2453	.2310	.3364	.2084	.4264	.1988	.5162	.1998
30	.2054	.3558	.2776	.3221	.3536	.3012	.4311	.2914	.5095	.2929
40	.2496	.4324	.3085	.4046	.3707	.3864	.4353	.3790	.5007	.3812
50	.2908	.5037	.3373	.4817	.3869	.4676	.4383	.4618	.4904	.4645
60	.3300	.5716	.3647	.5551	.4018	.5448	.4403	.5409	.4790	.5434
70	.3679	.6372	.3910	.6263	.4158	.6195	.4412	.6169	.4669	.6192
80	.4049	.7014	.4166	.6962	.4289	.6926	.4416	.6915	.4544	.6926
90	.4416	.7650	.4416	.7650	.4416	.7650	.4416	.7650	.4416	.7650

Latitude	Longitude 50°		Longitude 60°		Longitude 70°		Longitude 80°		Longitude 90°	
	x	y	x	y	x	y	x	y	x	y
0°	0.6257	0.0000	0.7397	0.0000	0.8752	0.0000	1.0617	0.0000	1.7666	0.0000
10	.6210	.1068	.7299	.1203	.8538	.1474	1.0028	.2073	1.1566	.3522
20	.6080	.2094	.7040	.2315	.8036	.2714	.9086	.3391	.9966	.4445
30	.5888	.3053	.6668	.3305	.7486	.3710	.8229	.4309	.8833	.5100
40	.5664	.3939	.6280	.4183	.6909	.4539	.7468	.5028	.7910	.5633
50	.5420	.4759	.6012	.4892	.6378	.5258	.6783	.5640	.7107	.6096
60	.5170	.5527	.5649	.5615	.5870	.5907	.6150	.6186	.6382	.6515
70	.4905	.6255	.5154	.6363	.5369	.6510	.5555	.6693	.5702	.6907
80	.4668	.6941	.4785	.7012	.4889	.7085	.4980	.7177	.5053	.7282
90	.4416	.7650	.4416	.7650	.4416	.7650	.4416	.7650	.4416	.7650

Entire sphere in the rhombus, poles in the 120° angles

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°	
	x	*y*	*x*	*y*	*x*	*y*	*x*	*y*	*x*	*y*
0°	1.7666	0	1.2687	0.2874	1.1566	0.3522	1.0678	0.4035	0.9966	0.4445
10	1.2069	0	1.1600	.1618	1.0930	.2630	1.0152	.3312	.9543	.3840
20	1.0580	0	1.0405	.1197	.9997	.2020	.9518	.2727	.9029	.3290
30	.9487	0	.9396	.0845	.9160	.1612	.8833	.2270	.8476	.2825
40	.8551	0	.8496	.0679	.8345	.1323	.8123	.1908	.7855	.2431
50	.7671	0	.7648	.0558	.7536	.1099	.7382	.1609	.7185	.2081
60	.6845	0	.6756	.0460	.6688	.0911	.6580	.1344	.6437	.1759
70	.5789	0	.5773	.0371	.5727	.0737	.5653	.1094	.5553	.1440
80	.4507	0	.4497	.0274	.4469	.0547	.4422	.0815	.4358	.1078
90	.0000	0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Latitude	Longitude 50°		Longitude 60°		Longitude 70°		Longitude 80°		Longitude 90°	
	x	*y*	*x*	*y*	*x*	*y*	*x*	*y*	*x*	*y*
0°	0.9365	0.4793	0.8833	0.5100	0.8352	0.5377	0.7910	0.5633	0.7497	0.5871
10	.9016	.4251	.8520	.4615	.8073	.4928	.7646	.5214	.7256	.5476
20	.8582	.3755	.8144	.4148	.7796	.4497	.7370	.4807	.6983	.5089
30	.8100	.3303	.7731	.3716	.7370	.4080	.7019	.4406	.6677	.4702
40	.7561	.2893	.7255	.3304	.6845	.3666	.6634	.4005	.6326	.4308
50	.6958	.2512	.6711	.2906	.6452	.3265	.6185	.3594	.5913	.3896
60	.6267	.2147	.6074	.2509	.5865	.2846	.5643	.3159	.5411	.3450
70	.5429	.1772	.5287	.2088	.5125	.2388	.4950	.2671	.4752	.2938
80	.4277	.1335	.4181	.1584	.4070	.1823	.3946	.2054	.3810	.2274
90	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Latitude	Longitude 100°		Longitude 110°		Longitude 120°		Longitude 130°		Longitude 140°	
	x	*y*	*x*	*y*	*x*	*y*	*x*	*y*	*x*	*y*
0°	0.7107	0.6096	0.6737	0.6310	0.6382	0.6515	0.6038	0.6713	0.5702	0.6907
10	.6880	.5721	.6522	.5950	.6174	.6168	.5837	.6376	.5509	.6577
20	.6627	.5349	.6283	.5591	.5953	.5823	.5622	.6036	.5301	.6241
30	.6344	.4974	.6019	.5226	.5699	.5462	.5385	.5684	.5075	.5895
40	.6020	.4586	.5717	.4844	.5417	.5085	.5138	.5311	.4821	.5525
50	.5641	.4175	.5388	.4434	.5085	.4675	.4806	.4901	.4526	.5115
60	.5172	.3721	.4928	.3972	.4678	.4210	.4424	.4432	.4167	.4642
70	.4565	.3190	.4358	.3426	.4144	.3648	.3923	.3857	.3696	.4053
80	.3663	.2484	.3506	.2684	.3340	.2873	.3166	.3052	.2984	.3220
90	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Latitude	Longitude 150°		Longitude 160°		Longitude 170°		Longitude 180°	
	x	*y*	*x*	*y*	*x*	*y*	*x*	*y*
0°	0.5376	0.7096	0.5053	0.7282	0.4733	0.7467	0.4416	0.7650
10	.5186	.6771	.4866	.6959	.4549	.7147	.4233	.7332
20	.4984	.6441	.4670	.6634	.4357	.6821	.4044	.7004
30	.4767	.6097	.4460	.6291	.4153	.6478	.3845	.6659
40	.4525	.5727	.4228	.5921	.3929	.6106	.3628	.6284
50	.4245	.5316	.3961	.5507	.3674	.5688	.3384	.5862
60	.3905	.4837	.3640	.5022	.3370	.5196	.3096	.5362
70	.3462	.4238	.3212	.4411	.2978	.4573	.2727	.4724
80	.2795	.3378	.2599	.3526	.2397	.3663	.2189	.3791
90	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Entire sphere in a six-pointed star

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°	
	x	y	x	y	x	y	x	y
° 0	1. 1129	0	0. 9836	0. 2239	0. 9051	0. 3599	0. 8347	0. 4819
10	. 8553	0	. 8297	. 1600	. 7778	. 2943	. 7159	. 4133
20	. 7043	0	. 6908	. 1254	. 6550	. 2419	. 6031	. 3482
30	. 5784	0	. 5689	. 1012	. 5418	. 1981	. 4991	. 2882
40	. 4665	0	. 4593	. 0812	. 4380	. 1596	. 4036	. 2330
50	. 3640	0	. 3584	. 0632	. 3420	. 1245	. 3152	. 1820
60	. 2679	0	. 2639	. 0465	. 2518	. 0916	. 2321	. 1339
70	. 1763	0	. 1736	. 0306	. 1657	. 0603	. 1527	. 0882
80	. 0875	0	. 0862	. 0152	. 0822	. 0299	. 0758	. 0437
90	. 0000	0	. 0000	. 0000	. 0000	. 0000	. 0000	. 0000

Hemisphere in a square, poles in a pair of the angles

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°	
	x	y	x	y	x	y	x	y	x	y
° 0	0	0. 00000	0. 17458	0. 00000	0. 35269	0. 00000	0. 53618	0. 00000	0. 72923	0. 00000
10	0	. 17458	. 17363	. 17630	. 34983	. 18041	. 53151	. 18776	. 72196	. 19940
20	0	. 35269	. 16953	. 35524	. 34124	. 36311	. 51742	. 37713	. 70062	. 39886
30	0	. 53618	. 16267	. 53974	. 32692	. 55073	. 49425	. 57005	. 66617	. 59938
40	0	. 72923	. 15305	. 73369	. 30667	. 74672	. 46209	. 76957	. 61969	. 80349
50	0	. 93713	. 13997	. 94182	. 28190	. 95780	. 42095	. 98071	. 56145	1. 01589
60	0	1. 16817	. 12541	1. 17289	. 24672	1. 18673	. 36902	1. 21079	. 49042	1. 24475
70	0	1. 43804	. 10200	1. 44221	. 20347	1. 45479	. 30387	1. 47583	. 40304	1. 50494
80	0	1. 78611	. 07264	1. 78923	. 14479	1. 79858	. 21592	1. 81415	. 28552	1. 83581
90	0	2. 62205	. 00000	2. 62205	. 00000	2. 62205	. 00000	2. 62205	. 00000	2. 62205

Latitude	Longitude 50°		Longitude 60°		Longitude 70°		Longitude 80°		Longitude 90°	
	x	y	x	y	x	y	x	y	x	y
° 0	0. 93713	0. 00000	1. 16817	0. 00000	1. 43804	0. 00000	1. 78611	0. 00000	2. 62205	0. 00000
10	. 92593	. 21710	1. 15001	. 24474	1. 40448	. 29140	1. 70379	. 38283	2. 03027	. 59178
20	. 89382	. 43117	1. 10024	. 47923	1. 32225	. 55265	1. 55618	. 66788	1. 77959	. 84246
30	. 84315	. 64235	1. 02818	. 70081	1. 21717	. 78339	1. 40490	. 89628	1. 57912	1. 04293
40	. 77952	. 85054	. 94073	. 91349	1. 10094	. 99540	1. 25564	1. 09870	1. 39852	1. 22354
50	. 70182	1. 06355	. 84059	1. 12479	. 97579	1. 20086	1. 10454	1. 29230	1. 22354	1. 39852
60	. 60994	1. 28934	. 72647	1. 34502	. 83864	1. 41206	. 94472	1. 49031	1. 04293	1. 57912
70	. 49889	1. 54358	. 59196	1. 59030	. 68088	1. 64546	. 76467	1. 70870	. 84246	1. 77959
80	. 35303	1. 86350	. 41796	1. 89698	. 47979	1. 93612	. 53791	1. 98061	. 59178	2. 03027
90	. 00000	2. 62205	. 00000	2. 62205	. 00000	2. 62205	. 00000	2. 62205	. 00000	2. 62205

Entire sphere in a rectangle, poles at the middle points of the long sides

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°	
	x	y	x	y	x	y	x	y	x	y
0°	2.3088	0.8245	1.6063	0.8245	1.2687	0.8245	1.0598	0.8245	0.9084	0.8245
10	1.6889	0	1.4596	.4309	1.2151	.5852	1.0334	.6552	.8931	.6937
20	1.2840	0	1.2182	.2442	1.0924	.4092	.9647	.5096	.8499	.5729
30	1.0497	0	1.0241	.1616	.9564	.2957	.8730	.3896	.7885	.4675
40	.8782	0	.8642	.1161	.8261	.2209	.7709	.3084	.7114	.3786
50	.7344	0	.7264	.0864	.7037	.1679	.6686	.2405	.6309	.3032
60	.6025	0	.5952	.0646	.5833	.1267	.5611	.1847	.5325	.2370
70	.4703	0	.4673	.0466	.4588	.0921	.4451	.1356	.4267	.1764
80	.3200	0	.3188	.0298	.3142	.0589	.3067	.0874	.2965	.1149
90	.0000	0	.0000	.0000	.0000	.0000	.0000	.0000	0000	.0000

Latitude	Longitude 50°		Longitude 60°		Longitude 70°		Longitude 80°		Longitude 90°	
	x	y	x	y	x	y	x	y	x	y
0°	0.7891	0.8245	0.6902	0.8245	0.6051	0.8245	0.5299	0.8245	0.4622	0.8245
10	.7790	.7175	.6833	.7336	.6002	.7450	.5262	.7534	.4594	.7598
20	.7508	.6151	.6629	.6446	.5854	.6660	.5152	.6821	.4508	.6946
30	.7062	.5201	.6306	.5588	.5609	.5881	.4963	.6105	.4361	.6281
40	.6484	.4341	.5863	.4770	.5264	.5111	.4692	.5381	.4147	.5598
50	.5787	.3555	.5288	.3988	.4815	.4349	.4329	.4644	.3852	.4886
60	.4992	.2832	.4628	.3235	.4246	.3581	.3852	.3876	.3456	.4126
70	.4045	.2138	.3787	.2482	.3516	.2781	.3224	.3048	.2918	.3281
80	.2837	.1405	.2686	.1647	.2524	.1862	.2330	.2077	.2139	.2258
90	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Latitude	Longitude 100°		Longitude 110°		Longitude 120°		Longitude 130°		Longitude 140°	
	x	y	x	y	x	y	x	y	x	y
0°	0.4001	0.8245	0.3424	0.8245	0.2881	0.8245	0.2365	0.8245	0.1870	0.8245
10	.3979	.7647	.3407	.7685	.2868	.7715	.2355	.7738	.1861	.7756
20	.3912	.7039	.3355	.7116	.2827	.7175	.2323	.7221	.1838	.7257
30	.3797	.6419	.3264	.6529	.2755	.6615	.2267	.6683	.1796	.6736
40	.3623	.5772	.3127	.5919	.2647	.6022	.2183	.6110	.1737	.6178
50	.3392	.5087	.2935	.5246	.2498	.5382	.2062	.5481	.1639	.5563
60	.3060	.4335	.2666	.4510	.2275	.4653	.1888	.4769	.1505	.4861
70	.2603	.3483	.2284	.3656	.1960	.3799	.1635	.3917	.1308	.4012
80	.1917	.2421	.1700	.2564	.1465	.2687	.1229	.2790	.0988	.2873
90	.0000	.0000	.0000	.0000	.0900	.0000	.0000	.0000	.0000	.0000

Latitude	Longitude 150°		Longitude 160°		Longitude 170°		Longitude 180°	
	x	y	x	y	x	y	x	y
0°	0.1389	0.8245	0.0920	0.8245	0.0458	0.8245	0	0.8245
10	.1384	.7769	.0916	.7778	.0457	.7784	0	.7785
20	.1367	.7279	.0906	.7301	.0451	.7312	0	.7315
30	.1336	.6774	.0886	.6801	.0442	.6817	0	.6822
40	.1290	.6229	.0856	.6263	.0427	.6284	0	.6290
50	.1223	.5624	.0813	.5688	.0405	.5692	0	.5700
60	.1126	.4930	.0751	.4978	.0374	.5006	0	.5016
70	.0982	.4084	.0654	.4135	.0326	.4165	0	.4175
80	.0744	.2938	.0498	.2984	.0249	.3012	0	.3021
90	.0000	.0000	.0000	.0000	.0000	.0000	0	.0000

Entire sphere in an ellipse

Latitude	Longitude 0°		Longitude 10°		Longitude 20°		Longitude 30°		Longitude 40°	
	x	y	x	y	x	y	x	y	x	y
° 0	1.1615	0.0000	0.8880	0.0000	0.7599	0.0000	0.6601	0.0000	0.5794	0.0000
10	1.0598	.2419	.8679	.1480	.7455	.1108	.6509	.0881	.5732	.0732
20	.8905	.3794	.7953	.2738	.7040	.2109	.6243	.1710	.5545	.1441
30	.7608	.4465	.7082	.3577	.6454	.2965	.5844	.2488	.5263	.2108
40	.6534	.4885	.6204	.4168	.5785	.3574	.5319	.3102	.4870	.2736
50	.5566	.5186	.5354	.4609	.5075	.4098	.4745	.3664	.4426	.3293
60	.4629	.5420	.4478	.4966	.4310	.4545	.4083	.4167	.3831	.3838
70	.3654	.5609	.3574	.5265	.3460	.4941	.3316	.4639	.3145	.4363
80	.2509	.5773	.2474	.5541	.2416	.5324	.2338	.5114	.2241	.4916
90	.0000	.5909	.0000	.5909	.0000	.5909	.0000	.5909	.0000	.5909

Latitude	Longitude 50°		Longitude 60°		Longitude 70°		Longitude 80°		Longitude 90°	
	x	y	x	y	x	y	x	y	x	y
° 0	0.5115	0.0000	0.4525	0.0000	0.4002	0.0000	0.3527	0.0000	0.3092	0.0000
10	.5069	.0629	.4492	.0553	.3976	.0497	.3508	.0453	.3078	.0419
20	.4938	.1247	.4391	.1104	.3901	.0996	.3450	.0912	.3031	.0844
30	.4722	.1850	.4228	.1653	.3772	.1499	.3348	.1379	.2950	.1281
40	.4421	.2430	.3994	.2199	.3585	.2012	.3198	.1862	.2829	.1739
50	.4033	.2997	.3670	.2748	.3331	.2540	.2990	.2368	.2659	.2226
60	.3560	.3552	.3279	.3305	.2994	.3093	.2706	.2913	.2421	.2761
70	.2955	.4115	.2747	.3889	.2535	.3695	.2313	.3524	.2085	.3374
80	.2130	.4734	.2003	.4564	.1872	.4415	.1719	.4266	.1572	.4142
90	.0000	.5909	.0000	.5909	.0000	.5909	.0000	.5909	.0000	.5909

Latitude	Longitude 100°		Longitude 110°		Longitude 120°		Longitude 130°		Longitude 140°	
	x	y	x	y	x	y	x	y	x	y
° 0	0.2688	0.0000	0.2309	0.0000	0.1948	0.0000	0.1602	0.0000	0.1269	0.0000
10	.2676	.0392	.2299	.0371	.1940	.0354	.1596	.0341	.1264	.0330
20	.2640	.0793	.2270	.0749	.1916	.0716	.1577	.0689	.1250	.0667
30	.2574	.1204	.2218	.1141	.1875	.1092	.1545	.1052	.1225	.1021
40	.2475	.1640	.2139	.1546	.1812	.1493	.1495	.1442	.1191	.1401
50	.2341	.2108	.2025	.2013	.1724	.1932	.1423	.1873	.1131	.1824
60	.2141	.2633	.1863	.2526	.1588	.2439	.1317	.2367	.1049	.2310
70	.1855	.3245	.1624	.3135	.1390	.3044	.1158	.2969	.0926	.2908
80	.1405	.4032	.1242	.3935	.1067	.3853	.0893	.3783	.0717	.3728
90	.0000	.5909	.0000	.5909	.0000	.5909	.0000	.5909	.0000	.5909

Latitude	Longitude 150°		Longitude 160°		Longitude 170°		Longitude 180°	
	x	y	x	y	x	y	x	y
° 0	0.0945	0.0000	0.0626	0.0000	0.0312	0.0000	0	0.0000
10	.0941	.0328	.0623	.0317	.0311	.0314	0	.0318
20	.0931	.0656	.0617	.0641	.0308	.0635	0	.0638
30	.0912	.0999	.0606	.0982	.0302	.0973	0	.0970
40	.0885	.1371	.0587	.1350	.0294	.1337	0	.1324
50	.0844	.1786	.0561	.1746	.0280	.1745	0	.1741
60	.0785	.2267	.0523	.2238	.0260	.2221	0	.2215
70	.0694	.2863	.0462	.2830	.0231	.2811	0	.2805
80	.0539	.3686	.0361	.3654	.0180	.3635	0	.3629
90	.0000	.5909	.0000	.5909	.0000	.5909	0	.5909