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GENERAL THEORY OF EQUIVALENT PROJECTIONS

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GENERAL THEORY OF EQUIVALENT PROJECTIONS

INTRODUCTION

Under this general title we shall treat the general principles that underlie all projections in which the area is preserved in constant ratio throughout the map in all its parts. Such projections are also called equal-area and sometimes authalic. The latter designation was employed by M. A. Tissot in his classical work "Mémoire sur la Représéntation des Surfaces." In the course of this pamphlet we shall make much use of the principles enunciated in this publication.

The law that underlies all of the projections to be treated may be stated as follows: Every section of the resulting map must bear a constant ratio to the area of the earth that is represented by it. As a result then, the whole map will have the same ratio to the whole region represented. To accomplish this end it is necessary that the mathematical expressions that give the coordinates of the map must meet certain differential requirements. It is necessary that we first investigate this phase of the subject so that we may have at hand the guiding principles in our further treatment.

Let us think of each point of the spheroid as corresponding to a definite related point of the plane. This one to one correspondence is what we have in all kinds of maps whether equivalent or not. In practice this generally consists of expressions for x and y coordinates in terms of the latitude and longitude of the point on the earth. We shall, as usual, denote latitude and longitude by ϕ and λ, respectively. The projection will then consist of the two expressions

$$x=f(\phi,\lambda)$$
$$y=g(\phi,\lambda)$$

in which f and g denote functions which must be determined to fulfill the required conditions.

The element of area on the spheroid must first be determined. If a denotes the equatorial radius and e the eccentricity of the meridian ellipse, the radius of curvature in the meridian is

$$\rho_m=\frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}.$$

The element of length in the meridian becomes

$$ds_m=\rho_m\,d\phi=\frac{a(1-e^2)d\phi}{(1-e^2\sin^2\phi)^{3/2}}.$$

Likewise, the radius of curvature perpendicular to the meridian is

$$\rho_p=\frac{a}{(1-e^2\sin^2\phi)^{1/2}},$$

and the element of length

$$ds_p=\rho_p\cos\phi\, d\lambda=\frac{a\cos\phi\, d\lambda}{(1-e^2\sin^2\phi)^{1/2}}.$$

The element of area on the spheroid is the right triangle formed by these two elements of length as sides. This element of area therefore becomes

$$\frac{1}{2}ds_m\, ds_p=\frac{a^2(1-e^2)\cos\phi\, d\phi\, d\lambda}{2(1-e^2\sin^2\phi)^2}.$$

In the plane the three points that correspond to the ϕ, λ origin and the ends of the two elements of length are

$$x,\ x+\frac{\partial x}{\partial\lambda}d\lambda,\ x+\frac{\partial x}{\partial\phi}d\phi$$

$$y,\ y+\frac{\partial y}{\partial\lambda}d\lambda,\ y+\frac{\partial y}{\partial\phi}d\phi.$$

The element of area is the area of the triangle formed by these three points. This is given by the determinant

$$\frac{1}{2}\begin{vmatrix} x, & x+\frac{\partial x}{\partial\lambda}d\lambda, & x+\frac{\partial x}{\partial\phi}d\phi \\ y, & y+\frac{\partial y}{\partial\lambda}d\lambda, & y+\frac{\partial y}{\partial\phi}d\phi \\ 1, & 1\ , & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} x, & \frac{\partial x}{\partial\lambda}d\lambda, & \frac{\partial x}{\partial\phi}d\phi \\ y, & \frac{\partial y}{\partial\lambda}d\lambda, & \frac{\partial y}{\partial\phi}d\phi \\ 1, & 0\ , & 0 \end{vmatrix}$$

or finally

$$=\frac{1}{2}\begin{vmatrix} \frac{\partial x}{\partial\lambda}, & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\lambda}, & \frac{\partial y}{\partial\phi} \end{vmatrix} d\phi\, d\lambda=\frac{1}{2}\left(\frac{\partial x}{\partial\lambda}\frac{\partial y}{\partial\phi}-\frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\lambda}\right)d\phi\, d\lambda.$$

The condition for equivalence or equal-area therefore becomes

$$\frac{\partial x}{\partial\lambda}\frac{\partial y}{\partial\phi}-\frac{\partial x}{\partial\phi}\frac{\partial y}{\partial\lambda}=\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}.$$

It should be noted that $\frac{\partial x}{\partial\phi}$, $\frac{\partial y}{\partial\lambda}$ etc., denote partial derivatives, that is, $\frac{\partial x}{\partial\phi}$ is a derivative of x with respect to ϕ with λ considered as a constant and so with the other partial derivatives. In terms of the f and g symbols for x and y, if we denote by the subscript 1 a partial derivative with respect to λ and by the subscript 2, the same with respect to ϕ, we have

$$f_1g_2-f_2g_1=\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}.$$

We see at once that if the earth is considered as a sphere, e will become 0 and we have

$$f_1 g_2 - f_2 g_1 = a^2 \cos \phi.$$

The formula is thus reduced to a much simpler form than is the case for the spheroid.

REDUCTION OF THE SPHEROID TO THE SPHERE

In order to make use of this simplification even when we wish to consider the spheroid, we have employed a scheme of first mapping the spheroid on a sphere of equivalent area and then we can map this equivalent sphere in the plane and thus get an equal-area map without the complications due to the spheroid.

The details of the scheme are now indicated. We do not change the longitude but determine the latitude on the equivalent sphere so the map may be an equivalent one. If we denote the latitude on the equivalent sphere by β and the radius of the sphere as **c**, the element of area on this sphere would become

$$1/2\ c^2 \cos \beta\, d\beta\, d\lambda.$$

This then must equal the element of area on the spheroid.

$$1/2\ c^2 \cos \beta\, d\beta\, d\lambda = \frac{a^2(1-e^2) \cos \phi\, d\phi\, d\lambda}{2(1-e^2 \sin^2 \phi)^2}.$$

Since we are going to take the longitude the same on the equivalent sphere as on the spheroid, the $d\lambda$ will cancel out and we have left

$$c^2 \cos \beta\, d\beta = \frac{a^2(1-e^2) \cos \phi\, d\phi}{(1-e^2 \sin^2 \phi)^2}.$$

First we determine the radius c on the agreement that β and ϕ become $\pi/2$ together.

$$c^2 \int_0^{\pi/2} \cos \beta\, d\beta = a^2(1-e^2) \int_0^{\pi/2} \frac{\cos \phi\, d\phi}{(1-e^2 \sin^2 \phi)^2}\,{}^*$$

$$c^2 = a^2(1-e^2)\left[\frac{1}{2(1-e^2)} + \frac{1}{4e} \log_n\left(\frac{1+e}{1-e}\right)\right]$$

in which $\log_n$ denotes the Napierian logarithm. The radius c thus becomes the radius of a sphere equal in area to that of the spheroid. We have called β the authalic latitude and the resulting sphere the authalic sphere. The difference between ϕ and β has been developed in a series in Special Publication No. 67, "Latitude Developments Connected with Geodesy and Cartography." A table of the authalic or equivalent latitudes is also given for every half degree of ϕ. If the β latitudes are used in place of ϕ, the spheroid can be taken into account with the simplified forms for the sphere. A copy of Special Publication No. 67 should be procured by anyone who wishes to use this method of procedure.

From the equation of condition for an equivalent projection we see that the function for either x or y may be arbitrarily chosen and then the other so determined as to fulfill the given condition. If the function for x is arbitrarily chosen as

$$x = F(\lambda, \phi),$$

*See pages 7 and 8 for a method of evaluating the integral on the right.

we will find for the partial derivatives

$$\frac{\partial x}{\partial \lambda}=p$$

$$\frac{\partial x}{\partial \phi}=q.$$

In these p and q will in turn be functions (ϕ, λ). It may happen that one becomes either a constant or a function of only one of the variables. With these values of the derivatives of x the partial differential equation in y becomes,

$$p\frac{\partial y}{\partial \phi}-q\frac{\partial y}{\partial \lambda}=\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}.$$

This can be solved as total differential equations in the usual manner. The equations become

$$\frac{d\phi}{p}=-\frac{d\lambda}{q}=\frac{dy}{\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}}. \tag{1}$$

If p and q are such that these equations can be solved, we get the function for y that must be used in connection with the adopted function for x. This would give the complete solution for the given projection. Later, we shall have occasion to make use of this principle in our work.

DISTORTION IN LENGTH AND ANGLE

In order to derive expressions for the change in length we consider an infinitesimal triangle with sides ds_m, ds_p and hypotenuse ds in which ds_m is in the meridian and ds_p in the parallel and hence they are perpendicular and they form a right angled triangle with ds. The linear elements on the map become

$$dS_m=\sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2+\left(\frac{\partial y}{\partial \phi}\right)^2}\,d\phi$$

and

$$dS_p=\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2}\,d\lambda.$$

In many cases the representation of the meridians and parallels on the map will not form an orthogonal network; if the angle which they form at a point is denoted by ψ and if the angles that ds and dS form with the meridian are u, u', respectively, then we have

$$\overline{ds}^2=\overline{ds}_m^{\,2}+\overline{ds}_p^{\,2}$$

$$\tan u=\frac{ds_p}{ds_m}$$

$$\overline{dS}^2=\overline{dS_p}^2\sin^2(\pi-\psi)+[dS_m-dS_p\cos(\pi-\psi)]^2$$
$$=\overline{dS_p}^2+\overline{dS_m}^2-2dS_p\,dS_m\cos(\pi-\psi)$$
$$=\overline{dS_p}^2+\overline{dS_m}^2+2dS_p\,dS_m\cos\psi$$
$$\tan u'=\frac{dS_p\sin(\pi-\psi)}{dS_m-dS_p\cos(\pi-\psi)}=\frac{dS_p\sin\psi}{dS_m+dS_p\cos\psi}.$$

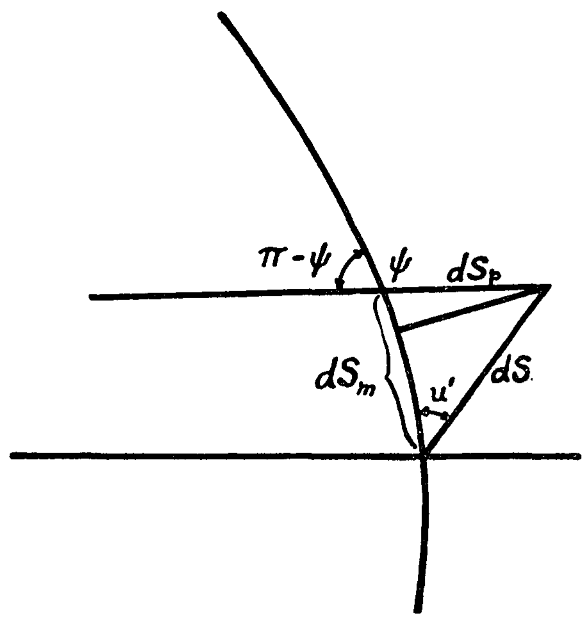

FIGURE 1.—Differential relations of azimuth and distance on the projection.

Since ϕ is reckoned positive to the north and λ to the east, dS_m is positive northward and dS_p positive eastward. In like manner ψ is the angle of intersection of the meridian and parallel between the same directions. In general, east of the central meridian ψ is greater than $\frac{\pi}{2}$ and west of the central meridian less than $\frac{\pi}{2}$. The angles u and u' are, therefore, reckoned from the north in a clockwise direction.

The area of the infinitesimal right triangle formed by ds_m, ds_p and ds on the earth is represented on the map by the area of the triangle formed by dS_m, dS_p and dS. Since the projection is equivalent we must have

$$ds_m\,ds_p=dS_m\,dS_p\sin\psi,$$

which determines the angle ψ as function of the position.

Let

$$\frac{dS_m}{ds_m}=h;\ \frac{dS_p}{ds_p}=k;\ \frac{dS}{ds}=K.$$

When we introduce the values already obtained, we get

$$h=\sqrt{\left(\frac{\partial x}{\partial \phi}\right)^2+\left(\frac{\partial y}{\partial \phi}\right)^2}\ \frac{(1-e^2 \sin^2 \phi)^{3/2}}{a\ (1-e^2)}$$

$$k=\sqrt{\left(\frac{\partial x}{\partial \lambda}\right)^2+\left(\frac{\partial y}{\partial \lambda}\right)^2}\ \frac{(1-e^2 \sin^2 \phi)^{1/2}}{a \cos \phi}$$

$$\sin \psi=\frac{1}{hk}$$

$$K^2=\frac{\left(\frac{dS_m}{ds_m}\right)^2+\left(\frac{dS_p}{ds_p}\right)^2\left(\frac{ds_p}{ds_m}\right)^2+2\left(\frac{dS_m}{ds_m}\right)\left(\frac{dS_p}{ds_p}\right)\left(\frac{ds_p}{ds_m}\right)\cos \psi}{1+\left(\frac{ds_p}{ds_m}\right)^2}$$

$$\tan u'=\frac{\left(\frac{dS_p}{ds_p}\right)\left(\frac{ds_p}{ds_m}\right)\sin \psi}{\left(\frac{dS_m}{ds_m}\right)+\left(\frac{dS_p}{ds_p}\right)\frac{ds_p}{ds_m}\cos \psi}$$

or

$$K^2=\frac{h^2+k^2 \tan^2 u+2hk \tan u \cos \psi}{1+\tan^2 u}$$

$$=h^2 \cos^2 u+k^2 \sin^2 u+2hk \sin u \cos u \cos$$

$$\tan u'=\frac{k \tan u \sin \psi}{h+k \tan u \cos \psi}$$

$$=\frac{hk \tan u \sin \psi}{h^2+hk \tan u \cos \psi}$$

but

$$hk=\frac{1}{\sin \psi}$$

$$\tan u'=\frac{\tan u}{h^2+\tan u \cot \psi}.$$

When ψ equals $\frac{\pi}{2}$ as is the case in many projections, $\cos \psi=0$ and $\sin \psi=1$. The above expressions then become,

$$K^2=h^2 \cos^2 u+k^2 \sin^2 u$$

$$h=\frac{1}{k}$$

$$\tan u'=\frac{k}{h}\tan u=k^2 \tan u=\frac{1}{h^2}\tan u.$$

$$1-\frac{\tan u'}{\tan u}=\frac{h-k}{h}$$

$$1+\frac{\tan u'}{\tan u}=\frac{h+k}{h},$$

hence

$$\frac{\tan u - \tan u'}{\tan u + \tan u'} = \frac{h-k}{h+k}$$

$$\frac{\sin (u-u')}{\sin (u+u')} = \frac{h-k}{h+k}$$

or

$$\sin (u-u') = \frac{h-k}{h+k} \sin (u+u').$$

From this we can see that $u-u'$ will be greatest when $u+u'=\pi/2$. Let this maximum value be denoted by δ, then

$$\sin \delta = \frac{h-k}{h+k} = \frac{h^2-1}{h^2+1} = \frac{1-k^2}{1+k^2}.$$

EXAMPLES OF ONE COORDINATE AS AN ARBITRARY FUNCTION

(A) Let us assume x to be a function of λ alone.

$$x = F(\lambda).$$

For the meridians on the earth λ is a constant, so in this case x on the map will be a constant for a given meridian. The representation of the meridians are therefore straight lines on the map which are parallel to the y-axis at the distance given by $x=F(\lambda)$. In this case

$$\frac{\partial x}{\partial \lambda} = F'(\lambda); \frac{\partial x}{\partial \phi} = 0.$$

Here we have $p=F'(\lambda)$; $q=0$.

Thus the equation (1) on page 4 becomes

$$\frac{dy}{\frac{a^2(1-e^2) \cos \phi}{(1-e^2 \sin^2 \phi)^2}} = \frac{d\phi}{F'(\lambda)}$$

or

$$dy = \frac{a^2(1-e^2) \cos \phi \, d\phi}{F'(\lambda)(1-e^2 \sin^2 \phi)^2}$$

$$y = \frac{a^2(1-e^2)}{F'(\lambda)} \int \frac{\cos \phi \, d\phi}{(1-e^2 \sin^2 \phi)^2} + G(\lambda),$$

in which G (λ) is a function of λ that depends upon the limits of the integral. When $\phi=0$ we have the Equator on the earth. If this is to be represented by a straight line and if it is to serve as the x-axis, we must take $G(\lambda)=0$. In this case, the above integral becomes

$$I = \int_0^\phi \frac{\cos \phi \, d\phi}{(1-e^2 \sin^2\phi)^2}$$

let

$$\sin \phi = z$$

$$I = \int_0^z \frac{dz}{(1-e^2z^2)^2}.$$

To evaluate this integral we can proceed in the following manner. The formula for integration by parts gives us

$$\int u\, dv = uv - \int v\, du.$$

In the integral

$$\int \frac{dz}{1-e^2z^2}$$

let

$$dv = dz$$

$$u = \frac{1}{1-e^2z^2}$$

then

$$v = z$$

$$v\, du = z\, du = \frac{2e^2z^2\, dz}{(1-e^2z^2)^2} = -\frac{2(1-e^2z^2)\, dz}{(1-e^2z^2)^2} + \frac{2dz}{(1-e^2z^2)^2}$$

$$= -\frac{2dz}{1-e^2z^2} + \frac{2dz}{(1-e^2z^2)^2}.$$

Hence

$$\int \frac{dz}{(1-e^2z^2)} = \frac{z}{1-e^2z^2} + 2\int \frac{dz}{1-e^2z^2} - 2\int \frac{dz}{(1-e^2z^2)^2}$$

therefore

$$\int \frac{dz}{(1-e^2z^2)^2} = \frac{z}{2(1-e^2z^2)} + \frac{1}{2}\int \frac{dz}{1-e^2z^2}$$

$$= \frac{z}{2(1-e^2z^2)} + \frac{1}{4e}\int \frac{e\, dz}{1+ez} + \frac{1}{4e}\int \frac{e\, dz}{1-ez}$$

$$= \frac{z}{2(1-e^2z^2)} + \frac{1}{4e} \log_n \frac{1+ez}{1-ez}.$$

On restoring the value of z, we get for the projection

$$y = \frac{a^2(1-e^2)}{F'(\lambda)}\left[\frac{\sin\phi}{2(1-e^2\sin^2\phi)} + \frac{1}{4e}\log_n\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)\right]$$

$$x = F(\lambda).$$

For example if $F(\lambda) = a\lambda$, $F'(\lambda) = a$, and the equations become

$$x = a\lambda$$

$$y = a(1-e^2)\left[\frac{\sin\phi}{2(1-e^2\sin^2\phi)} + \frac{1}{4e}\log_n\left(\frac{1+e\sin\phi}{1-e\sin\phi}\right)\right]$$

For the sphere $e=0$ and the equations become

$$x = a\lambda$$

$$y = a\sin\phi.$$

This is the Lambert equal-area cylindrical projection. (See fig. 2.)

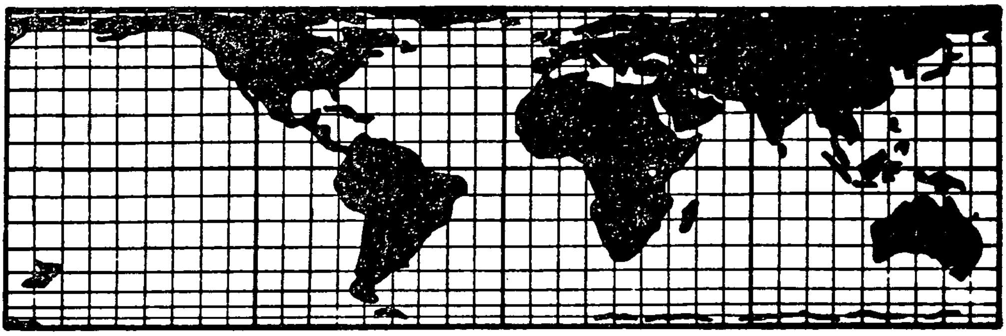

FIGURE 2.—Lambert's equivalent cylindrical projection.

Since here, for the spheroid

$$\frac{\partial x}{\partial \phi}=0, \frac{\partial x}{\partial \lambda}=a$$

$$\frac{\partial y}{\partial \phi}=\frac{a(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}; \frac{\partial y}{\partial \lambda}=0$$

$$h=\frac{\cos\phi}{\sqrt{1-e^2\sin^2\phi}}; k=\frac{\sqrt{1-e^2\sin^2\phi}}{\cos\phi},$$

hence

$$\sin\psi=1,$$

so that the meridians and parallels are perpendicular to each other as is a priori evident.

$$\tan u'=\frac{1-e^2\sin^2\phi}{\cos^2\phi}\tan u$$

$$\sin\delta=-\frac{(1-e^2)\sin^2\phi}{2-(1+e^2)\sin^2\phi}.$$

(**B**) As a second example let

$$x=f(\phi)\lambda$$

$$p=f(\phi); q=f'(\phi)\lambda.$$

Therefore, the differential equation (1) becomes

$$\frac{d\phi}{f(\phi)}=-\frac{d\lambda}{\lambda f'(\phi)}=\frac{dy}{\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}},$$

or

$$dy=\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}\frac{d\phi}{f(\phi)}$$

and

$$\frac{d\lambda}{\lambda}=-\frac{f'(\phi)}{f(\phi)}d\phi.$$

The integrals of these equations are

$$y-a^2(1-e^2)\int \frac{\cos\phi\, d\phi}{(1-e^2\sin^2\phi)^2 f(\phi)}=c_1$$

and

$$\log_n \lambda=-\log_n f(\phi)+\log_n c_2$$

or

$$\lambda f(\phi)=c_2.$$

The general integral of the above partial differential equation is given by $c_1=g\ (c_2)$ in which g denotes an arbitrary function. This gives

$$y-a^2(1-e^2)\int \frac{\cos\phi\, d\phi}{(1-e^2\sin^2\phi)^2 f(\phi)}=g[\lambda f(\phi)]=g(x).$$

If we choose for example

$$x=\rho_p(\cos\phi)\lambda=\frac{a\lambda\cos\phi}{\sqrt{1-e^2\sin^2\phi}},$$

then the lengths along the parallels will be maintained true to scale but the meridians will no longer be straight lines for, when λ is a constant, x will still depend upon ϕ. In this case

$$f(\phi)=\frac{a\cos\phi}{(1-e^2\sin^2\phi)^{1/2}};\ f'(\phi)=-\frac{a(1-e^2)\sin\phi}{(1-e^2\sin^2\phi)^{3/2}},$$

therefore

$$y=a(1-e^2)\int \frac{d\phi}{(1-e^2\sin^2\phi)^{3/2}}+g(x).$$

We can integrate this expression by the following procedure:

$$e^2\frac{d}{d\phi}\left[\frac{\sin\phi\cos\phi}{(1-e^2\sin^2\phi)^{1/2}}\right]=e^2\left[\frac{e^2\sin^2\phi\cos^2\phi}{(1-e^2\sin^2\phi)^{3/2}}+\frac{\cos^2\phi-\sin^2\phi}{(1-e^2\sin^2\phi)^{1/2}}\right]$$

$$=\frac{e^2-2e^2\sin^2\phi+e^4\sin^4\phi}{(1-e^2\sin^2\phi)^{3/2}}$$

$$=\frac{1-2e^2\sin^2\phi+e^4\sin^4\phi-(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}$$

$$=\sqrt{1-e^2\sin^2\phi}-\frac{1-e^2}{(1-e^2\sin^2\phi)^{3/2}}.$$

On integration of this equation, we get

$$(1-e^2)\int \frac{d\phi}{(1-e^2\sin^2\phi)^{3/2}}=\int \sqrt{1-e^2\sin^2\phi}\, d\phi-\frac{e^2\sin\phi\cos\phi}{(1-e^2\sin^2\phi)^{1/2}}$$

hence

$$y=aE(\phi)-\frac{ae^2\sin\phi\cos\phi}{(1-e^2\sin^2\phi)^{1/2}},$$

in which E (ϕ) is the second elliptic integral of Legendre. We have taken $g(x)=0$, so y may vanish throughout for $\phi=0$ and the equator becomes a straight line on the map. It should be noted that y is merely the length of the meridian counted from $\phi=0$. For the sphere, since e becomes zero, we have

$$x=a\lambda \cos \phi$$

$$y=a\phi.$$

With the values for the spheroid, we get

$$\frac{\partial x}{\partial \phi}=-\frac{a(1-e^2) \sin \phi}{(1-e^2 \sin^2 \phi)^{3/2}}\lambda; \ \frac{\partial x}{\partial \lambda}=\frac{a \cos \phi}{(1-e^2 \sin^2 \phi)^{1/2}}$$

$$\frac{\partial y}{\partial \phi}=\frac{a(1-e^2)}{(1-e^2 \sin^2 \phi)^{3/2}}; \ \frac{\partial y}{\partial \lambda}=0$$

Hence

$$h=\sqrt{1+\lambda^2 \sin^2 \phi}; \ k=1; \ \sin \psi=\frac{1}{\sqrt{1+\lambda^2 \sin^2 \phi}}$$

$$\cos \psi=\frac{-\lambda \sin \phi}{\sqrt{1+\lambda^2 \sin^2 \phi}}; \ \tan \psi=-\frac{1}{\lambda \sin \phi}$$

$$\tan u'=\frac{\tan u}{1-\lambda \sin \phi \tan u+\lambda^2 \sin^2 \phi}.$$

The angle ψ is the angle of the intersection of the meridian and parallel greater than $\frac{\pi}{2}$.

This formula gives us the alteration in the angle u counted from north in a clockwise direction with λ counted positive to the eastward. When west of the central meridian λ is negative and the formula is correct for the same considerations for u and also u'. When $u=\frac{\pi}{2}$, it can be seen that $\tan u'=-\frac{1}{\lambda \sin \phi}=\tan \psi$ the greater angle of intersection as it should be. When west of the central meridian λ is negative and $\tan u'$ is positive and we have the smaller angle of the intersection again as it should be. Norbert Herz in his excellent treatise on projections, "Lehrbuch der Landkartenprojektionen" makes a mistake in this formula by omitting the term $\tan u$ in the denominator. If his formula were correct, u and u' would become $\frac{\pi}{2}$ together and the meridians and parallels would be perpendicular at all points.

In actual practice we are not as much interested in the distortion of this angle as we are in the direction of the arc ds with respect to the axes of coordinates. Since the parallels are represented by straight lines parallel to the x-axis it is better to count our angle of direction from the parallel as initial. Let us take east as origin and reckon angles in a counterclockwise direction. With this convention we have the relation

$$\tan u' = \frac{\frac{\partial y}{R\,\partial\phi}\tan u + \frac{\partial y}{R'\partial\lambda}}{\frac{\partial x}{R\,\partial\phi}\tan u + \frac{\partial x}{R'\partial\lambda}},\ \tan u = \frac{R\,d\phi}{R'd\lambda}$$

in which R is the radius of curvature in the meridian and R' is the radius of curvature of the parallel.

$$\frac{\partial y}{R\,\partial\phi} = 1;\ \frac{\partial y}{R'\partial\lambda} = 0$$

$$\frac{\partial x}{R\,\partial\phi} = -\lambda\sin\phi;\ \frac{\partial x}{R'\partial\lambda} = 1$$

$$\tan u' = \frac{\tan u}{1-\lambda\sin\phi\tan u} = \frac{\tan u}{1+\cot\psi\tan u}.$$

When $u=\frac{\pi}{2}$, $\tan u' = \tan\psi$ or $u'=\psi$, the larger angle of intersection of the meridian and parallels and $\cot\psi = -\lambda\sin\phi$.

To determine the axes of Tissot's indicatrix [1] we have the two equations

$$a^2+b^2=h^2+k^2=1+1+\lambda^2\sin^2\phi=2+\lambda^2\sin^2\phi$$
$$ab=hk\sin\psi=1.$$

From these equations we find

$$a=\sqrt{1+\frac{\lambda^2}{4}\sin^2\phi}+\frac{\lambda}{2}\sin\phi$$

$$b=\sqrt{1+\frac{\lambda^2}{4}\sin^2\phi}-\frac{\lambda}{2}\sin\phi.$$

To determine the directions of the axes we must determine the maximum and minimum values of K in terms of $\tan u$. We have

$$K^2=\frac{h^2\tan^2 u+k^2+2hk\tan u\cos\psi}{1+\tan^2 u},$$

in which u and ψ are reckoned from east in a counterclockwise direction and ψ is the larger angle of intersection of the meridians and parallels and $\cot\psi = -\lambda\sin\phi$.

$$\frac{\partial K^2}{\partial(\tan u)}=\frac{2h^2\tan u+2hk\cos\psi}{1+\tan^2 u} - \frac{(h^2\tan^2 u+k^2+2hk\tan u\cos\psi)\,2\tan u}{(1+\tan^2 u)^2}=0$$

[1] For an account of Tissot's indicatrix see Special Publication No. 57, General Theory of Polyconic Projections, pp. 153 et seq.

$$(h^2 \tan u+hk \cos \psi)\ (1+\tan^2 u)-\tan u\ (h^2 \tan^2 u+k^2+2hk \tan u \cos \psi)=0$$

$$h^2 \tan u+hk \cos \psi+h^2 \tan^3 u+hk \cos \psi \tan^2 u-h^2 \tan^3 u-k^2 \tan u-2hk \tan^2 u \cos \psi=0$$

$$hk \tan^2 u \cos \psi-(h^2-k^2) \tan u-hk \cos \psi=0$$

$$\tan^2 u-\frac{h^2-k^2}{hk \cos \psi} \tan u-1=0.$$

But for this projection we have

$$h^2-k^2=\lambda^2 \sin^2 \phi=\cot^2 \psi$$

$$hk \cos \psi=\cot \psi.$$

Hence we get

$$\tan^2 u-\cot\psi \tan u=1$$

$$\tan^2 u-\cot \psi \tan u+\frac{\cot^2 \psi}{4}=1+\frac{\cot^2 \psi}{4}$$

$$\tan u=\frac{\cot \psi}{2}\pm\sqrt{1+\frac{\cot^2 \psi}{4}}$$

$$=-\frac{\lambda}{2} \sin \phi\pm\sqrt{1+\frac{\lambda^2}{4} \sin^2 \phi}.$$

Hence,

$$\tan u_1=-a$$

$$\tan u_2=b.$$

And so

$$\tan u_1'=\frac{-a}{1+a\lambda \sin \phi}$$

$$\tan u_2'=\frac{b}{1-b\lambda \sin \phi}$$

$$\tan u_1' \tan u_2'=\frac{-ab}{1+(a-b)\lambda \sin \phi-ab\lambda^2 \sin^2 \phi}$$

but

$$ab=1\ ;\ a-b=\lambda \sin \phi.$$

Therefore

$$\tan u_1' \tan u_2'=-1,$$

or the two directions on the projection are perpendicular to each other as they should be.

When the azimuths are measured from the x-axis of Tissot's indicatrix, we have the relation

$$\tan u'=\frac{b}{a} \tan u.$$

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From this relation we find

$$\frac{\tan u - \tan u'}{\tan u + \tan u'} = +\frac{a-b}{a+b},$$

or

$$\frac{\sin (u-u')}{\sin (u+u')} = +\frac{a-b}{a+b}.$$

This shows us that the maximum alteration of direction is found when $u'+u=\frac{\pi}{2}.$

If we denote this maximum value of $u-u'$ by δ, we have

$$\sin \delta = +\frac{a-b}{a+b};\ \cos \delta = \frac{2\sqrt{ab}}{a+b}$$

$$\tan \delta = +\frac{a-b}{2\sqrt{ab}};\ \tan^2\frac{\delta}{2} = \frac{1-\cos \delta}{1+\cos \delta} = \frac{(\sqrt{b}-\sqrt{a})^2}{(\sqrt{b}+\sqrt{a})^2}$$

$$\tan \frac{\delta}{2} = +\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{\frac{a}{b}}-1}{\sqrt{\frac{a}{b}}+1}$$

$$\tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right) = \sqrt{\frac{a}{b}};\ \tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right) = \sqrt{\frac{b}{a}}.$$

If

$$W = \frac{\pi}{4}+\frac{\delta}{2};\ W' = \frac{\pi}{4}-\frac{\delta}{2}$$

we have

$$W+W' = \frac{\pi}{2} \text{ and } W-W' = \delta.$$

Now, for the projection which we have been examining, we have

$$ab=1,$$

therefore

$$\tan W = a \text{ and } \tan W' = b.$$

We have found the direction on the earth of the major axis to be $\tan u_1 = -a$, therefore

$$\tan u_1 = -\tan W = \tan (-W)$$

or

$$u_1 = -W = -\frac{\pi}{4}-\frac{\delta}{2},$$

and

$$u_2 = W' = \frac{\pi}{4}-\frac{\delta}{2}$$

$$\tan u_1' = \frac{\tan u_1}{1-\lambda \sin \phi \tan u_1},$$

but

$$\tan u_1 = -a$$

$$\lambda \sin \phi = a-b,$$

therefore

$$\tan u_1' = \frac{-a}{1+a(a-b)} = \frac{-a}{1+a^2-ab} = \frac{-a}{1+a^2-1} = -\frac{1}{a} = -b$$

therefore

$$\tan u_1' = \tan\,(-W') = \tan\left(\frac{\delta}{2} - \frac{\pi}{4}\right),$$

so

$$u_1' = \frac{\delta}{2} - \frac{\pi}{4}.$$

Tissot's indicatrix thus becomes

$$\tan\left(u' + \frac{\pi}{4} - \frac{\delta}{2}\right) = \frac{b}{a}\tan\left(u + \frac{\pi}{4} + \frac{\delta}{2}\right) = b^2 \tan\left(u + \frac{\pi}{4} + \frac{\delta}{2}\right)$$

It is evident that u and u' vanish together as they should. With this projection the equation

$$\tan\delta = \frac{a-b}{2\sqrt{ab}},$$

becomes

$$\tan\delta = \frac{\lambda \sin\phi}{2} = -\frac{1}{2}\cot\psi$$

in which ψ is again the larger angle of the intersection of the meridian and parallel.

$$\text{At } \lambda = 30°,\ \phi = 30°,\ \lambda \sin\phi = \frac{\pi}{12} \text{ and } \tan\delta = \frac{\pi}{24}.$$

$$\tan\delta = 0.13089969$$

$$\delta = 7°27'27''.4$$

$$\frac{\delta}{2} = 3°43'43''.7$$

$$\frac{\pi}{4} - \frac{\delta}{2} = 41°16'16''.3$$

$$\frac{\pi}{4} + \frac{\delta}{2} = 48°43'43''.7$$

Let us apply Tissot's indicatrix for $u = 60°$.

$$\tan\left(u' + \frac{\pi}{4} - \frac{\delta}{2}\right) = b^2 \tan\,(108°43'43''.7)$$

$$b = \tan 41°16'16''.3 = 0.877631$$

$$b^2 = 0.770236$$

$$\tan\,(108°43'43''.7) = -2.94948906$$

$$\tan\left(u'+\frac{\pi}{4}-\frac{\delta}{2}\right)=-2.27180266$$

$$u'+41°16'16''.3=113°45'29''.1$$

$$u'=72°29'12''.8$$

$$\tan u'=\frac{\sqrt{3}}{1-\frac{\pi}{12}\sqrt{3}}=\frac{1.7320508}{1-\frac{\pi}{12}(1.7320508)}=\frac{1.7320508}{0.54655016}$$

$$=3.16906101$$

$$u'=72°29'12''.7$$

$$\tan\psi=-\frac{12}{\pi}=-3.81971864$$

$$\psi=104°40'14''.7$$

smaller angle=75°19'45''.3.

We see that either of the formulas for tan u' can be used since they give the same result. Tissot's indicatrix is merely another form of expression for tan u' and the one formula can be transformed into the other by a little mathematical ingenuity.

We have given a pretty full account of this projection with the purpose of illustration of the use of Tissot's indicatrix in case the parallels and meridians do not intersect at right angles.

Since the equation for y in this projection is independent of λ, this equation at once gives the parallels which consist of straight lines perpendicular to the straight line central meridian and parallel to the x-axis at the distance given by the expression of y in terms of ϕ. If we should want to get the equation for the meridian it would be necessary to eliminate ϕ from the equation for x and y. However, since the expression for y in terms of ϕ contains an elliptic integral as well as the trigonometric functions, the elimination is not readily practicable. Of course the elliptic integral could be expanded in a series to a few terms and then the elimination of ϕ could be made in the approximate equation.

Since the result in any case would not be simple and would not throw much light on the nature of the curve, it is better to dispense with the effort. However, since the form of the meridian for the spheroid does not differ much from that for the sphere, we can get a fair idea of it from the formulas for the sphere. For the sphere the equations become very simple:

$$x=a\lambda\cos\phi$$

$$y=a\phi,$$

thus the meridian has the equation

$$\frac{x}{a}=\lambda\cos\frac{y}{a}.$$

The meridian of longitude λ is therefore a cosine curve with the distance λ from the central meridian at the Equator. Since a cosine curve is exactly similar to a sine curve but differently placed with regard to the origin, this projection has been called the sinusoidal projection. (See fig. 3.) It is also sometimes called the Sanson projection or the Sanson-Flamsteed projection. We understand that this projection was employed by Mercator in his atlas and so some prefer to call it the Mercator equal-area projection. The designation sinusoidal is so well known that it would be better to designate it the Mercator sinusoidal projection to avoid confusion, if it is desired to link it with Mercator.

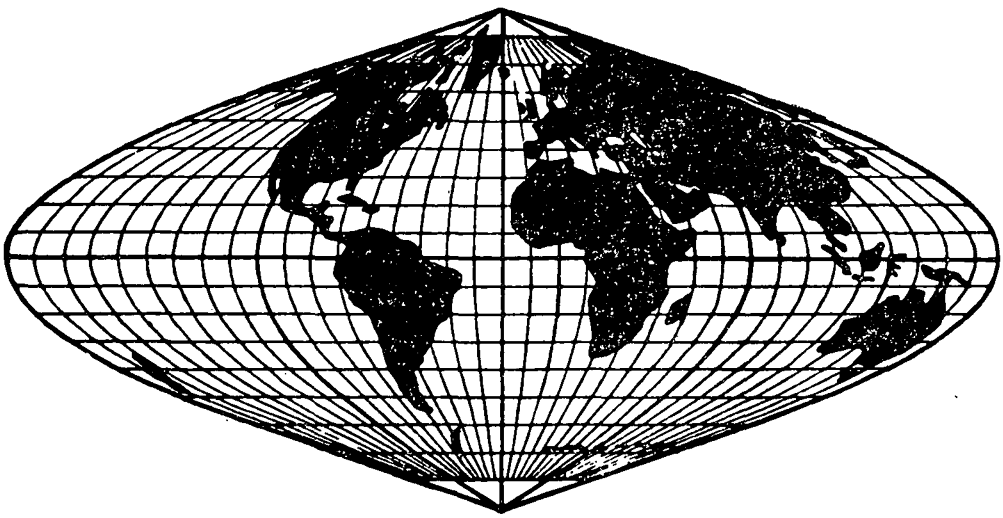


FIGURE 3.—Sinusoidal projection.

Since the equation for y for the ellipsoid is merely the length of the meridian from the Equator and the length of the straight line parallels is exactly equal to the true length of the parallels, the complete values for the construction of a map are found in the polyconic projection tables. This map is therefore one that can be constructed very easily and hence it is much in use for equal-area representation. For maps that include more than a hemisphere, it is not very well suited because of the violent distortions of the part beyond $\lambda=\frac{\pi}{2}$.

(C). If the parallels are to be represented by straight lines parallel to the x-axis, then y must be independent of λ. Therefore, y must be a function of ϕ alone.

$$y=F(\phi),$$

$$\frac{\partial y}{\partial \lambda}=0; \frac{\partial y}{\partial \phi}=F'(\phi),$$

and the differential equation to be solved becomes

$$F'(\phi)\frac{\partial x}{\partial\lambda}=\frac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}.$$

Since this is a partial differential equation in $\partial\lambda$, the expression in ϕ is considered as a constant, and so

$$x=\frac{a^2(1-e^2)\cos\phi}{F'(\phi)(1-e^2\sin^2\phi)^2}(\lambda)+f(\phi).$$

If the central meridian is to be represented by the y-axis, then x must equal zero for $\lambda=0$, and $f(\phi)$ must be zero, and the general equation of this projection becomes

$$x=\frac{a^2(1-e^2)\cos\phi}{F'(\phi)(1-e^2\sin^2\phi)^2}(\lambda)$$

$$y=F(\phi),$$

and for the sphere

$$x=\frac{a^2\lambda\cos\phi}{F'(\phi)}$$

$$y=F(\phi).$$

In these expressions the function $F(\phi)$ is still arbitrary. If we choose

$$F'(\phi)=\frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}},$$

we get the sinusoidal projection that we have just treated.

We can specify that the meridians should be curves with particular properties; but in so doing we cannot proceed arbitrarily. If the accepted curve is expressed by the function

$$G(x, y, \lambda)=0,$$

then in accordance with the assumed value of y we must have

$$G\left\{\frac{a^2(1-e^2)\lambda\cos\phi}{(1-e^2\sin^2\phi)^2\frac{dy}{d\phi}}, y, \lambda\right\}=0.$$

But from this equation λ must cancel out for from our assumption, y is a function of ϕ alone. This fact is analytically expressed by the relations

$$\frac{dG}{d\lambda}=0 \text{ or } \frac{\partial G}{\partial\lambda}+\frac{\partial G}{\partial x}\cdot\frac{\partial x}{\partial\lambda}=0,$$

and since

$$\frac{\partial x}{\partial\lambda}=\frac{x}{\lambda},$$

the relation becomes

$$\lambda\frac{\partial G}{\partial\lambda}+x\frac{\partial G}{\partial x}=0.$$

This equation expresses the fact that the function $G(x, y, \lambda)$ with respect to x and λ is homogeneous of the first degree, that is, it is merely a function of $\frac{x}{\lambda}$ and is therefore of the form

$$G\left(\frac{x}{\lambda}, y\right)=0,$$

which in fact satisfies the required condition. This can be seen at once from the fact that in the elimination of ϕ from the equations for x and y the quantities x and λ always appear in the combination $\frac{x}{\lambda}$.

(**D**) If the meridians are to be represented by ellipses, the general equation of which we write

$$\frac{x^2}{f(\lambda)^2}+\frac{y^2}{f_1(\lambda)^2}=1,$$

then, because these curves must all pass through both poles, the distance from the origin of their intersections with the central meridian must be constant and equal to half the distance between the poles. Since for $x=0$, y must equal a constant denoted by m, we must have $f_1(\lambda)=m$, and the equation of the meridian becomes when the value of x is substituted,

$$\left(\frac{a^2(1-e^2)\lambda \cos\phi}{(1-e^2\sin^2\phi)^2\frac{dy}{d\phi}f(\lambda)}\right)^2+\left(\frac{y}{m}\right)^2=1.$$

If y is to be a mere function of ϕ, then $f(\lambda)$ must be equal to $n\lambda$ in which n is a constant. The equation for the meridian now becomes

$$\frac{x^2}{(n\lambda)^2}+\frac{y^2}{m^2}=1.$$

The equation for y now becomes

$$\frac{\sqrt{m^2-y^2}}{m}=\frac{a^2(1-e^2)\cos\phi}{n(1-e^2\sin^2\phi)^2\frac{dy}{d\phi}},$$

or

$$\int_0^y\sqrt{m^2-y^2}\,dy=a^2\frac{m}{n}(1-e^2)\int_0^\phi\frac{\cos\phi\,d\phi}{(1-e^2\sin^2\phi)^2}$$

or by integration

$$\frac{1}{2}y\sqrt{m^2-y^2}+\frac{m^2}{2}\sin^{-1}\frac{y}{m}$$

$$=a^2\frac{m}{n}(1-e^2)\left[\frac{\sin\phi}{2(1-e^2\sin^2\phi)}+\frac{1}{4e}\log_n\frac{1+e\sin\phi}{1-e\sin\phi}\right]$$

in which we have taken the constant of integration as zero because y must equal zero for $\phi=0$. To this we must add the equation for x,

$$x=\frac{a^2(1-e^2)\lambda \cos \phi}{(1-e^2 \sin^2 \phi)^2 \frac{dy}{d\phi}}.$$

We can get more convenient forms for computation by assuming

$$y=m \sin \theta,$$

then the left side of the equation becomes

$$\frac{m^2}{4}(2\theta+\sin 2\theta).$$

If, for the sake of brevity, we take

$$(1-e^2)\left[\frac{\sin \phi}{2(1-e^2 \sin^2 \phi)}+\frac{1}{4e} \log_n \frac{1+e \sin \phi}{1-e \sin \phi}\right]=g(\phi),$$

then the equation becomes

$$2\theta+\sin 2\theta=\frac{4a^2 g(\phi)}{mn}.$$

The constants, m, n, that we have introduced are not independent magnitudes but are interdependent for a given ellipsoid. A zone of the ellipsoid from the Equator to latitude ϕ has the area

$$O_\phi=2\pi a^2 g(\phi),$$

hence the area of the partial zone between the meridians at $+\frac{\lambda}{2}$ and $-\frac{\lambda}{2}$ is

$$O_{\phi,\lambda}=a^2 g(\phi)\lambda.$$

The lune thus formed from pole to pole becomes

$$O_\lambda=a^2\left(1+\frac{1-e^2}{2e} \log_n \frac{1+e}{1-e}\right)\lambda,$$

since $\phi=\frac{\pi}{2}$, and we have twice the area from 0 to $\frac{\pi}{2}$. The area of the corresponding lune on the map is

$$\pi mn\frac{\lambda}{2},$$

and because of the equivalence if, for brevity, we set

$$\left(1+\frac{1-e^2}{2e} \log_n \frac{1+e}{1-e}\right)=\eta$$

we have

$$\pi mn\frac{\lambda}{2}=a^2\eta\lambda$$

or

$$\pi mn = 2a^2\eta$$

and so follows

$$2\theta + \sin 2\theta = 2\frac{\pi g(\phi)}{\eta}.$$

From this we get

$$2(1+\cos 2\theta)\frac{d\theta}{d\phi} = \frac{2\pi}{\eta}g'(\phi)$$

or

$$4\cos^2\theta\frac{d\theta}{d\phi} = \frac{2(1-e^2)}{\eta}\,\frac{\pi\cos\phi}{(1-e^2\sin^2\phi)^2},$$

therefore

$$\frac{dy}{d\phi} = m\cos\theta\frac{d\theta}{d\phi} = \frac{m}{2\cos\theta}\left(\frac{1-e^2}{\eta}\right)\frac{\pi\cos\phi}{(1-e^2\sin^2\phi)^2}$$

$$= \frac{a^2(1-e^2)}{n\cos\theta}\left[\frac{\cos\phi}{(1-e^2\sin^2\phi)^2}\right]$$

and consequently

$$x = \lambda\, n\cos\theta.$$

As a whole then we have for the projection

$$2\theta + \sin 2\theta = \frac{\pi(1-e^2)\left(\frac{\sin\phi}{1-e^2\sin^2\phi} + \frac{1}{2e}\log_n\frac{1+e\sin\phi}{1-e\sin\phi}\right)}{1+\frac{1-e^2}{2e}\log_n\frac{1+e}{1-e}}$$

$$= \pi\sin\beta$$

in which β is the authalic latitude already referred to and,

$$x = n\lambda\cos\theta$$

$$y = m\sin\theta.$$

The values of m and n are only limited to the interrelation $\pi mn = 2a^2\eta$. If we take n such that $m = \frac{\pi n}{2}$, then the 90° meridian will become a circle. After the circle to represent $\lambda = \frac{\pi}{2}$ is drawn, the parallels can be located on the circle by

$$x = r\cos\theta$$

$$y = r\sin\theta.$$

Straight lines through these points perpendicular to the y-axis give the parallels. The meridians can be computed from their equations by using the adopted values of m and n. The tables for the values of sin θ and cos θ are given in Special Publication

No. 68, so they are not reproduced here. By interpolating the values for β, the spheroid can be taken into account.

$$\frac{\partial y}{\partial \phi}=\frac{a^2(1-e^2)}{n \cos \theta}\left[\frac{\cos \phi}{(1-e^2 \sin^2 \phi)^2}\right]; \frac{\partial y}{\partial \lambda}=0$$

$$\frac{\partial x}{\partial \phi}=-n\lambda \sin \theta \frac{d\theta}{d\phi}=-\frac{n\lambda \sin \theta}{2 \cos^2 \theta}\left[\frac{\pi \cos \phi(1-e^2)}{\eta(1-e^2 \sin^2 \phi)^2}\right]$$

or

$$\frac{\partial x}{\partial \phi}=-\frac{a^2\lambda(1-e^2) \sin \theta}{m \cos^2 \theta}\left[\frac{\cos \phi}{(1-e^2 \sin^2 \phi)^2}\right]; \frac{\partial x}{\partial \lambda}=n \cos \theta,$$

from these values, we get

$$h=\frac{a \cos \phi}{n \cos \theta\sqrt{1-e^2 \sin^2 \phi}}\sqrt{1+\left(\frac{n}{m}\lambda \tan \theta\right)^2}$$

$$k=\frac{n \cos \theta\sqrt{1-e^2 \sin^2 \phi}}{a \cos \phi}$$

$$hk=\sqrt{1+\left(\frac{n}{m}\lambda \tan \theta\right)^2},$$

consequently

$$\sin \psi=\frac{1}{\sqrt{1+\left(\frac{n}{m}\lambda \tan \theta\right)^2}}; \cos \psi=\frac{-\frac{n}{m}\lambda \tan \theta}{\sqrt{1+\left(\frac{n}{m}\lambda \tan \theta\right)^2}}$$

$$\cot \psi=-\frac{n}{m}\lambda \tan \theta.$$

FIGURE 4.—Mollweide projection.

This projection was first given by Mollweide. It was further employed by Babinet who gave it the name of homalographic. (See fig. 4.)

(E) Another example of an equivalent projection with parallels represented by straight lines is the one given by Prépetit Foucaut called by him the stereographic equivalent projection because the parallels are spaced along the central meridian at the same intervals as in the stereographic projection. For the sphere, we have

$$y=a \tan \frac{\phi}{2}$$

$$\frac{dy}{d\phi}=\frac{a}{2 \cos^2 \frac{\phi}{2}},$$

therefore

$$x=\frac{a^2\lambda \cos \phi}{\frac{dy}{d\phi}}=2a\lambda \cos \phi \cos^2 \frac{\phi}{2}.$$

To determine the equation of the meridians, we have

$$\cos \frac{\phi}{2}=\frac{1}{\sqrt{1+\tan^2 \frac{\phi}{2}}}=\frac{1}{\sqrt{1+\left(\frac{y}{a}\right)^2}}=\frac{a}{\sqrt{a^2+y^2}}$$

$$\sin \frac{\phi}{2}=\frac{y}{\sqrt{a^2+y^2}}$$

$$\cos \phi=\cos^2 \frac{\phi}{2}-\sin^2 \frac{\phi}{2}=\frac{a^2-y^2}{a^2+y^2},$$

hence

$$x=2a\left(\frac{a^2-y^2}{a^2+y^2}\right)\left(\frac{a^2}{a^2+y^2}\right)\lambda.$$

The meridians on the projection are therefore curves of the fifth degree.

(F) In the Geographical Journal for November 1929, Lt. Col. J. E. E. Craster proposed three equal-area projections with straight line parallels, one with hyperbolic meridians, another with parabolic meridians and a third with elliptic meridians. The most interesting one of these is the parabolic variant. (See fig. 5.) Let us take the

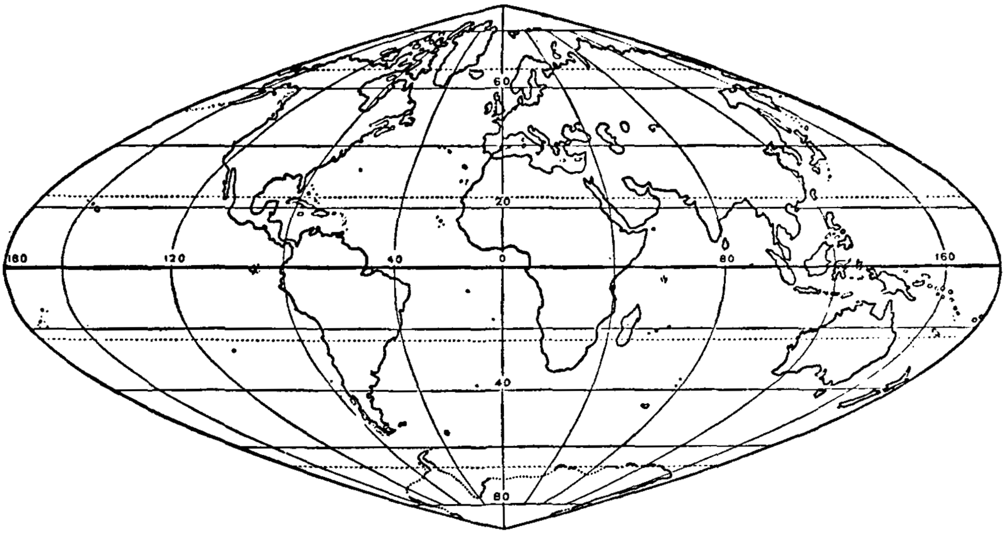


FIGURE 5.—Parabolic equivalent projection for the sphere, after Steers.

origin at the vertex of the parabola

$$y^2=\frac{1}{2}mx$$

then, if we represent the central meridian by $x=2m$, the meridian from pole to pole will equal $2m$, with the half of the whole Equator also equal to $2m$. This parabola would then bound the hemisphere on one side of the central meridian. Another inversely similar parabola cutting the x axis at the distance $2m$ on the other side of the central meridian and passing through the poles would represent the hemisphere on the other side of the central meridian. The outer meridian would not thus be a continuous curve but consist of the two parabolas meeting at the poles. We can just as well le m become equal to one and thus have the whole Equator equal 4 and the whole central meridian equal 2.

One-half of the area of a zone between the Equator and any given parallel will be represented on the map by the integral

$$\int_0^y (2-x)dy$$

or

$$\int_0^y (2-2y^2)dy.$$

The value of this integral is

$$A=2y-\frac{2}{3}y^3.$$

Since, when y is equal to 1, this map area must be equal to one-fourth of the area of the given sphere or πR^2, we must have the equation

$$\pi R^2=\frac{4}{3}.$$

Hence

$$R=0.651470.$$

Now the area of a zone on the sphere from the Equator to latitude ϕ is known to be equal to $2\pi R^2 \sin\phi$. Then the half of this zone must be represented by the expression for A in terms of y as derived above. Hence, the equation for the determination of y for the various parallels is given by the solution of the equation

$$2y-\frac{2}{3}y^3=\pi R^2 \sin\phi.$$

But we have already found πR^2 to be equal to 4/3. With this value the equation for y becomes

$$y^3-3y+2\sin\phi=0.$$

Now it happens that this equation has the root $y=2\sin\frac{\phi}{3}$. Since y is thus a function of ϕ alone, we have

$$\frac{\partial y}{\partial\lambda}=0;\ \frac{\partial y}{\partial\phi}=\frac{2}{3}\cos\frac{\phi}{3}.$$

With these values, we get

$$\frac{2}{3}\cos\frac{\phi}{3}\frac{\partial x}{\partial\lambda}=R^2\cos\phi$$

$$x=\frac{3R^2\cos\phi}{2\cos\frac{\phi}{3}}(\lambda).$$

But

$$R^2=\frac{4}{3\pi},$$

hence

$$x=\frac{2\cos\phi}{\pi\cos\frac{\phi}{3}}\lambda=\frac{2}{\pi}\lambda\left(4\cos^2\frac{\phi}{3}-3\right)$$

$$y=2\sin\frac{\phi}{3}$$

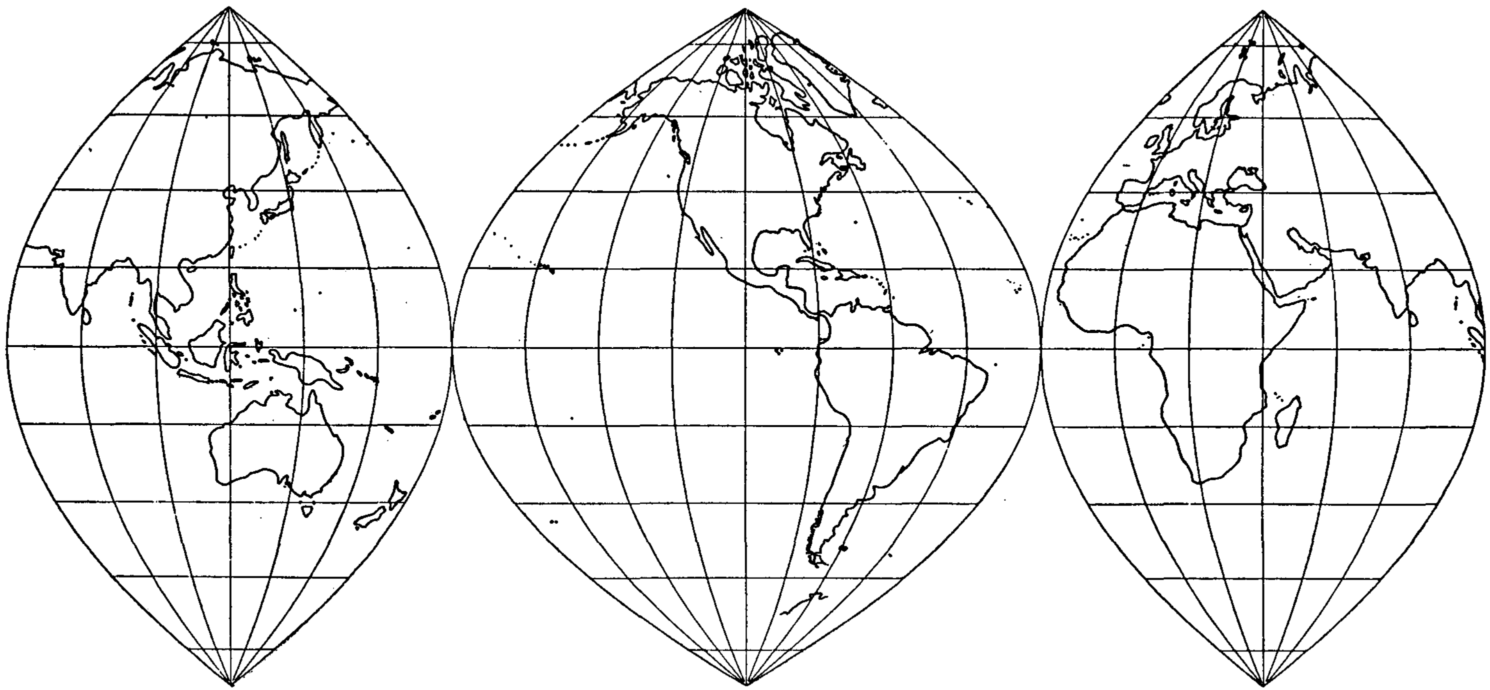
$$y^2+\frac{\pi x}{2\lambda}=1$$

$$y^2=1-\frac{\pi x}{2\lambda}.$$

This is the equation of the meridian at distance of λ from the central meridian with the line of poles for the y-axis. This is the equation of a parabola, so all of the meridians are parabolas.

For the outer meridian

$$y^2=1-\frac{x}{2}.$$



Parabolic Equal Area Projection of the World

Compiled and Arranged by

C. H. Deetz, U. S. Coast and Geodetic Survey

FIGURE 6.—Interrupted parabolic equivalent projection.

If it is desired to use this projection for the spheroid, it is only necessary to use the authalic latitudes as given in Special Publication No. 67, Latitude Developments Connected with Geodesy and Cartography, instead of ϕ in the computations. A table of the values of these latitudes denoted by β are given in this publication for every half degree of ϕ. Tables for this projection are given in Special Publication No. 68, Elements of Map Projection. We do not include hyperbolic and elliptic variants for they are similar to other projections and are therefore not of special interest.

(G) We will now give an example in which the meridians are straight lines as well as the parallels. The meridians, of course, must pass through a point which represents the pole. As a result, they must be discontinuous at the Equator, the meridians in the Southern Hemisphere being an inversely similar set of lines passing through the South Pole.

Since by use of the authalic or equal-area latitudes we can pass from the sphere to the spheroid, we will discuss the projection for the sphere. Since the parallels are to be straight lines, we must have y a function of ϕ alone.

Then
$$y=F(\phi)$$
$$x=\frac{a^2 \cos \phi}{F'(\phi)}(\lambda).$$

The equation of the straight line which is to represent the meridian must be
$$y-d=x \tan \alpha,$$
in which d is where the line intersects the y-axis or the distance of the pole from the origin or center of the map and α is the inclination of the line to the x-axis.

Since all the meridians cut the y-axis at the same point, d must be a constant; furthermore, the angle of inclination of the line depends on only the λ of the given meridian. Since this is so, we must have $\tan \alpha = G(\lambda)$ and the equation of the meridian becomes
$$y-d=xG(\lambda).$$

This equation must be satisfied by the values of x and y given above, hence
$$F(\phi)-d=\frac{a^2 \cos \phi}{F'(\phi)}(\lambda)G(\lambda).$$

This equation can only become identical, that is, independent of λ, if $G(\lambda)=\frac{1}{n\lambda}$, and then we get
$$y-d=\frac{a^2 \cos \phi}{n\frac{dy}{d\phi}},$$
or
$$(y-d)\frac{dy}{d\phi}=\frac{a^2 \cos \phi}{n}.$$

By integration of this equation, we get

$$y^2-2yd+C=\frac{2a^2 \sin \phi}{n}.$$

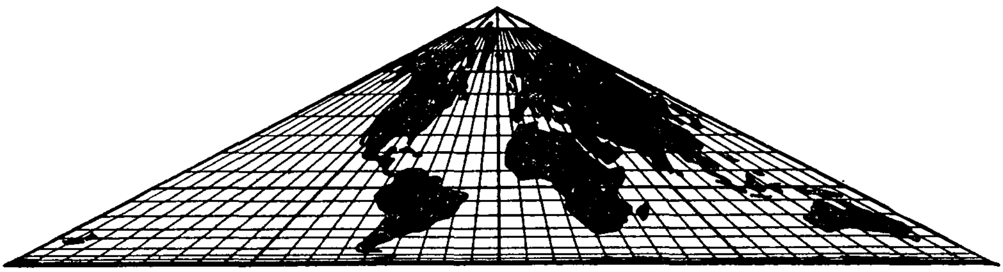


FIGURE 7.—Equivalent projection with meridians and parallels as straight lines.

Since y must be zero for $\phi=0$, C must be zero. Since further, $y=d$ for $\phi=\frac{\pi}{2}$, we get

$$-d^2=\frac{2a^2}{n},$$

and by division we have

$$\frac{2yd-y^2}{d^2}=\sin \phi.$$

The equation of the meridian becomes

$$(y-d)n\lambda=x,$$

in which d, n, and a are bound together by the relation

$$d^2n+2a^2=0.$$

If we choose d and n arbitrarily, a will be fixed by the above relation.

If $y=0$, $x=-dn\lambda$, or $-dn\lambda$ is the distance from the origin of the intersection of the meridian and the x-axis. If we wish the outer meridian to be inclined at 45° to the y-axis, then we must have $d=-dn\frac{\pi}{2}$, or $n=-\frac{2}{\pi}$. With this value of n and the relation $d^2n+2a^2=0$, we determine that $d=a\sqrt{\pi}$. The equation of the meridian thus becomes

$$(y-a\sqrt{\pi})\left(-\frac{2}{\pi}\right)\lambda=x$$

or

$$a\sqrt{\pi}-y=\frac{\pi x}{2\lambda}.$$

Returning to the equation of the parallel, we have

$$2yd-y^2=d^2 \sin \phi$$

$$y^2-2yd+d^2=d^2(1-\sin \phi)=2d^2 \sin^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

$$y-d=-\sqrt{2}d \sin \left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

or

$$y-a\sqrt{\pi}=-a\sqrt{2\pi} \sin \left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

$$y=a\sqrt{\pi}\left[1-\sqrt{2} \sin \left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right].$$

This is not a particularly important projection and we will not discuss it further.

PROJECTIONS WITH CIRCULAR PARALLELS

Let us now consider a case in which the parallels are to be represented by concentric circles. In order to treat the matter in all generality, it will be best to carry through the computation in polar coordinates. With these coordinates the element of area in the plane is expressed by $\rho\, d\rho\, d\theta$ in which ρ is the radius of the parallel and θ the angle at the center of the parallel. This element for an equivalent projection must equal the element of area on the ellipsoid.

$$\rho\, d\rho\, d\theta=-\frac{a^2(1-e^2) \cos \phi\, d\phi\, d\lambda}{(1-e^2 \sin^2 \phi)^2}.$$

The negative sign is used because as ρ increases, the latitude decreases, so the sign as used is necessary. The values ρ and θ are functions of ϕ and λ, so we must have

$$d\rho=\frac{\partial \rho}{\partial \lambda}d\lambda+\frac{\partial \rho}{\partial \phi}d\phi,$$

$$d\theta=\frac{\partial \theta}{\partial \lambda}d\lambda+\frac{\partial \theta}{\partial \phi}d\phi,$$

therefore

$$\rho\frac{\partial \rho}{\partial \phi}\frac{\partial \theta}{\partial \phi}\overline{d\phi}^2+\rho\frac{\partial \rho}{\partial \lambda}\frac{\partial \theta}{\partial \lambda}\overline{d\lambda}^2+\rho\left(\frac{\partial \rho}{\partial \phi}\frac{\partial \theta}{\partial \lambda}+\frac{\partial \rho}{\partial \lambda}\frac{\partial \theta}{\partial \phi}\right)d\phi\, d\lambda=-\frac{a^2(1-e^2) \cos \phi\, d\phi\, d\lambda}{(1-e^2 \sin^2 \phi)^2}.$$

If we adopt for ρ or θ an arbitrary value, then we can find from the above partial differential equation the value of the other unknown.

If $\rho=F(\phi, \lambda)$, a known function; then both $\frac{\partial F}{\partial \phi}$ and $\frac{\partial F}{\partial \lambda}$ are also known, and we have a partial differential equation in θ in place of which the total differential equation becomes

$$\frac{d\lambda}{F\frac{\partial F}{\partial \phi}}=\frac{d\phi}{F\frac{\partial F}{\partial \lambda}}=-\frac{d\theta}{\frac{a^2(1-e^2) \cos \phi}{(1-e^2 \sin^2 \phi)^2}}.$$

If $\theta = G(\phi, \lambda)$, a known function, then in similar fashion we have for ρ

$$\frac{d\phi}{\dfrac{\partial\theta}{\partial\lambda}} = \frac{d\lambda}{\dfrac{\partial\theta}{\partial\phi}} = -\frac{\rho\, d\rho}{\dfrac{a^2(1-e^2)\cos\phi}{(1-e^2\sin^2\phi)^2}}.$$

For a given meridian λ is a constant, hence $d\lambda = 0$; consequently on the map we have

$$d\rho_m = \frac{\partial\rho}{\partial\phi} d\phi;\ d\theta_m = \frac{\partial\theta}{\partial\phi} d\phi$$

therefore, the element of the meridian on the spheroid $ds_m = r_m\, d\phi$ is represented on the map by

$$dS_m = \sqrt{(\rho\, d\theta_m)^2 + \overline{d\rho_m}^2} = \sqrt{\left(\rho\frac{\partial\theta}{\partial\phi}\right)^2 + \left(\frac{\partial\rho}{\partial\phi}\right)}\, d\phi.$$

For a given parallel on the other hand, ϕ is constant and $d\phi = 0$; so

$$d\rho_p = \frac{\partial\rho}{\partial\lambda} d\lambda;\ d\theta_p = \frac{\partial\theta}{\partial\lambda} d\lambda$$

and the element of the parallel on the spheroid $ds_p = r_p\, d\lambda$ is represented on the map by

$$dS_p = \sqrt{\left(\rho\frac{\partial\theta}{\partial\lambda}\right)^2 + \left(\frac{\partial\rho}{\partial\lambda}\right)^2} d\lambda$$

The two ratios of scale thus become—in the meridian

$$h = \sqrt{\left(\frac{\partial\theta}{\partial\phi}\right)^2 + \left(\frac{\partial\rho}{\partial\phi}\right)^2} \frac{(1-e^2\sin^2\phi)^{3/2}}{a(1-e^2)}$$

in the parallel

$$k = \sqrt{\left(\rho\frac{\partial\theta}{\partial\lambda}\right)^2 + \left(\frac{\partial\rho}{\partial\lambda}\right)^2} \frac{\sqrt{1-e^2\sin^2\phi}}{a\cos\phi}.$$

All other relations remain exactly the same as they were in the consideration with x and y coordinates.

$$\sin\psi = \frac{1}{hk}$$

$$K^2 = h^2\cos^2 u + k^2\sin^2 u + 2hk\sin u\cos u\cos\psi$$

$$\tan u' = \frac{\tan u}{h^2 + hk\tan u\cos\psi} = \frac{\tan u}{h^2 + \tan u\cot\psi}$$

the angle ψ being as usual the larger angle of intersection.

If the representation of the parallels is to be circles, then must the value of ρ for any given parallel be a constant and hence independent of λ, or it must be a function of ϕ alone.

$$\rho = F(\phi).$$

For the determination of θ, since $\frac{\partial\rho}{\partial\lambda}=0$, we have the equation

$$\frac{\partial\theta}{\partial\lambda}=-\frac{a^2(1-e^2)\cos\phi}{F(\phi)F'(\phi)(1-e^2\sin^2\phi)^2},$$

the integral of which is

$$\theta=-\frac{a^2(1-e^2)\cos\phi}{F(\phi)F'(\phi)(1-e^2\sin^2\phi)^2}(\lambda)+g(\phi),$$

in which $g(\phi)$ is an arbitrary function of ϕ, the value of which is determined when λ is set equal to zero; then $\theta=g(\phi)$. The function $g(\phi)$ thus is the θ for the central meridian. If this meridian is to be a straight line and that from which the angle θ is reckoned, then must $g(\phi)$ be equal to zero, and the fundamental equations of this projection become

$$\rho=F(\phi)$$

$$\theta=-\frac{a^2(1-e^2)\lambda\cos\phi}{F(\phi)F'(\phi)(1-e^2\sin^2\phi)^2}.$$

(A) As the first special case, we will make the assumption that the parallel circles on the map shall be spaced at equal distances. Then will

$$F(\phi)=m-n\phi$$

and

$$F'(\phi)=-n$$

and therefore

$$\theta=+\frac{a^2(1-e^2)\lambda\cos\phi}{n(m-n\phi)(1-e^2\sin^2\phi)^2},$$

in which both m and n are arbitrary constants. If we impose the condition that the pole should be the common center for the circles that represent the parallels, then for $\phi=\frac{\pi}{2}$, ρ must be zero and

$$m-n\frac{\pi}{2}=0.$$

Therefore

$$\rho=n\left(\frac{\pi}{2}-\phi\right).$$

$$\theta=\frac{a^2(1-e^2)\lambda\cos\phi}{n^2\left(\frac{\pi}{2}-\phi\right)(1-e^2\sin^2\phi)^2}.$$

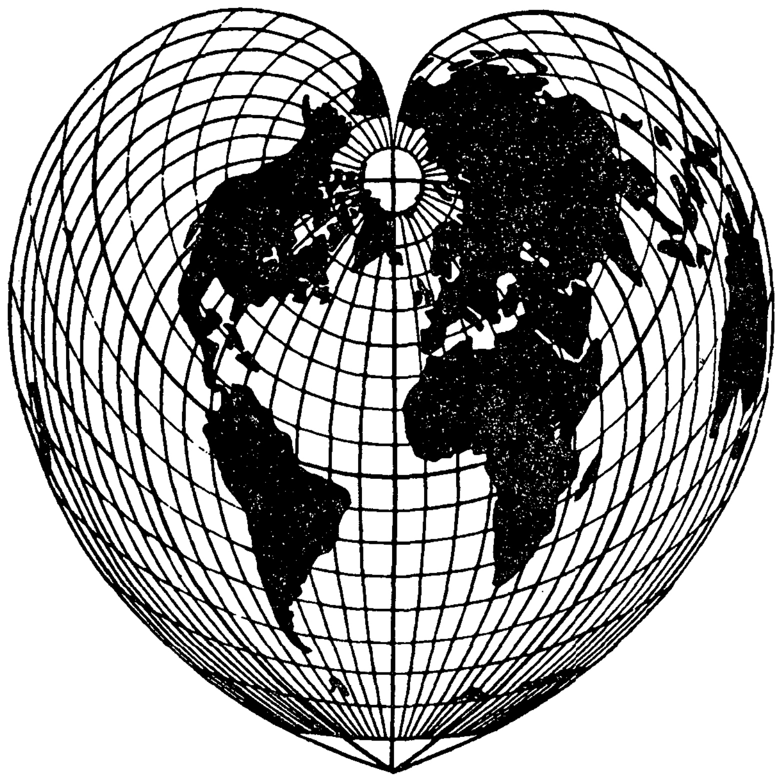


FIGURE 8.—Werner equivalent projection.

Since n is still arbitrary, we can let $n=a$ and, if instead of ϕ, we use the polar distance $\frac{\pi}{2}-\phi=p$, we get

$$\rho=ap$$

$$\theta=\frac{1-e^2}{(1-e^2\cos^2 p)^2}\left(\frac{\sin p}{p}\right)\lambda.$$

This projection was devised by Johann Werner of Nürnberg and it is called the Werner projection. (See Fig. 8.) The parallels are represented by equally spaced concentric circles with the pole as center. The spacings of the meridians on any given parallel are equal, so it is only necessary to compute one and then the others can be stepped off on the circle of the parallel on the map.

If we confine the projection to the sphere, we get

$$\rho=ap$$

$$\theta=\frac{\sin p}{p}(\lambda)$$

or

$$\rho\theta=a\lambda\sin p.$$

We thus see that the arc $\rho\theta$ of the parallel circle of latitude ϕ on the map is equal to the arc $a\lambda \sin p$ of the parallel on the sphere; because of this, the construction of the net of projection is a very simple matter. With the pole as center equidistant circles are drawn which represent the parallels on the map and on these are laid off the true lengths of the parallel arcs on the sphere. The complete map for the sphere thus becomes a heart-shaped representation. For a map that does not extend very far from the central meridian the projection is not too much distorted and could serve very well for practical use.

From the equations for the sphere we find

$$\frac{\partial\rho}{\partial\phi}=\frac{\partial\rho}{\partial p}\frac{dp}{d\phi}=-a;\ \frac{\partial\rho}{\partial\lambda}=0$$

$$\frac{\partial\theta}{\partial\phi}=\frac{\partial\theta}{\partial p}\frac{dp}{d\phi}=\left(\frac{\sin p}{p^2}-\frac{\cos p}{p}\right)\lambda;\ \frac{\partial\theta}{\partial\lambda}=\frac{\sin p}{p},$$

hence we get

$$h=\sqrt{1+\lambda^2\left(\frac{\sin p}{p}-\cos p\right)^2};\ k=1$$

$$\sin\psi=\frac{1}{h};\ \cot\psi=-\lambda\left(\frac{\sin p}{p}-\cos p\right);\ \psi>\frac{\pi}{2}.$$

If, for a given p and λ, the value of ψ has been computed, then $h = \operatorname{cosec}\psi$.

(B) We will now consider a projection in which not only the parallels are circles but also in which the meridians are straight lines that pass through a common point. Besides the projection is equivalent and consequently a conical equivalent projection. Because of the first condition we must have as before

$$\rho=F(\phi)$$

$$\theta=-\frac{a^2(1-e^2)\lambda\cos\phi}{F(\phi)F'(\phi)(1-e^2\sin^2\phi)^2}.$$

Since the meridians are to be straight lines, the radius of the circle must represent the meridian; therefore θ must be independent of ϕ, or

$$-\frac{a^2(1-e^2)\cos\phi}{F(\phi)F'(\phi)(1-e^2\sin^2\phi)^2}=m,$$

in which m is a constant. Then we have

$$\theta=m\lambda,$$

and we have for the radius

$$\rho\frac{\partial\rho}{\partial\phi}=F(\phi)F'(\phi)=-\frac{a^2(1-e^2)\cos\phi}{m(1-e^2\sin^2\phi)^2};$$

or from this by integration,

$$\rho^2+C=-\frac{a^2(1-e^2)}{m}\left[\frac{\sin\phi}{1-e^2\sin^2\phi}+\frac{1}{2e}\log_e\frac{1+e\sin\phi}{1-e\sin\phi}\right]$$

or in terms of the authalic latitude

$$\rho^2+C=-\frac{2c^2}{m}\sin\beta.$$

For the determination of C, we can make several different assumptions.

(1) If the pole is to be represented by a point, for $\phi=\frac{\pi}{2}$, ρ must equal zero. In this case $C=-\frac{2c^2}{m}$ and our equation becomes

$$\rho^2=\frac{2c^2}{m}(1-\sin\beta)=\frac{4c^2}{m}\sin^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$$

and

$$\rho=\frac{2c}{\sqrt{m}}\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$$

$$\theta=m\lambda.$$

If r_p denotes the radius of the parallel on the earth $r_p\,d\lambda$ on the earth will be represented by $\rho\,d\theta$ on the map. But $\rho\,d\theta=\rho m\,d\lambda$; therefore, the scale along the parallel becomes

$$k=\frac{m\rho}{r_p}.$$

Since the meridians and parallels intersect at right angles, we have $\sin\psi=1$ and $\psi=\frac{\pi}{2}$. Hence

$$h=\frac{r_p}{m\rho}$$

$$\tan u'=\frac{m^2\rho^2}{r_p^2}\tan u.$$

Since the meridians and parallels are orthogonal, u' and u are the correct angles for Tissot's indicatrix as appears in the last equation.

For the case of the sphere we get

$$h=\frac{\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}{\sqrt{m}};\quad k=\frac{\sqrt{m}}{\cos\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}$$

$$\tan u'=\frac{m}{\cos^2\left(\frac{\pi}{4}-\frac{\phi}{2}\right)}\tan u$$

This projection originated with J. H. Lambert and was called by Germain, Lambert's isospheric stenoteric projection. (See fig. 9.)

It is evident that m should be less than unity for, if it is greater than one, we can reduce k by reducing m and at the same time bring h nearer unity.

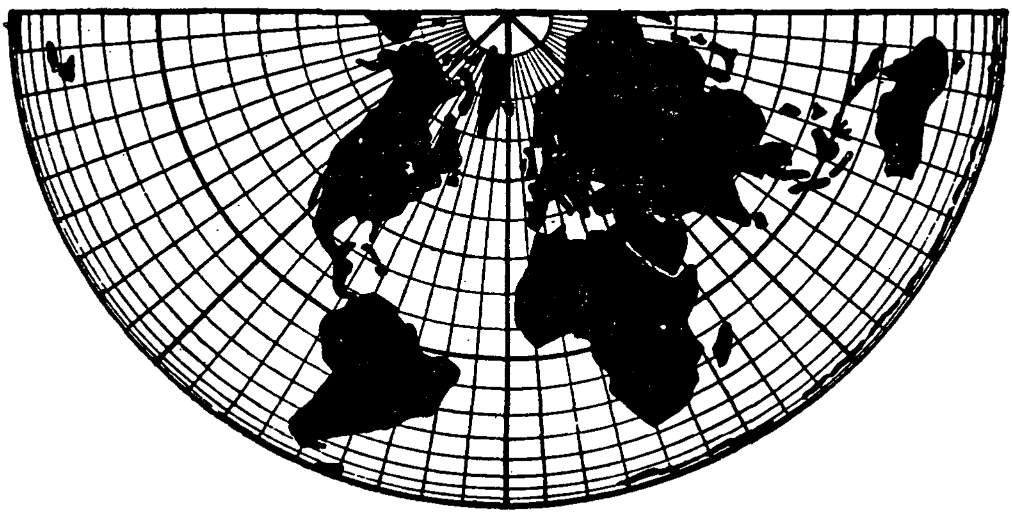


FIGURE 9.—Lambert's isospheric stenoteric projection.

(2) Since m is arbitrary, we can determine its value in such a way as to give certain advantages to the map. Let us suppose we wish to map a certain region. If ϕ_0 is the latitude of the middle parallel of the region, we can take $\sqrt{m}=\dfrac{r_0}{2c\sin\left(\frac{\pi}{4}-\frac{\beta_0}{2}\right)}$ for the parallel ϕ_0. For this parallel both h and k will be equal to unity, and the scale will be preserved along that parallel and the angles will be unaltered along the same parallel. If we wish a map that will change the angles throughout as little as possible, we can adopt the value of $m=\dfrac{r_1 r_2}{4c^2\sin\left(\frac{\pi}{4}-\frac{\beta_1}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\beta_2}{2}\right)}$. In this expression the sub one functions are for the lowest parallel and the sub two functions, the same for the upper parallel. With this value of m, the a semiaxis of Tissot's indicatrix has the same value at the upper and lower margins of the map. At ϕ_1, a lies along the parallel and at ϕ_2, along the meridian.

For a projection of this kind in which the earth is treated as a sphere, the equations become

$$\rho=\frac{2a}{\sqrt{m}}\sin\frac{p}{2}$$

$$\theta=m\lambda,$$

in which p is the polar distance or colatitude of the given parallel. The distortion equations are in terms of p

$$h=\frac{\cos\frac{p}{2}}{\sqrt{m}};\ k=\frac{\sqrt{m}}{\cos\frac{p}{2}}.$$

$$\tan u' = \frac{m}{\cos^2 \frac{p}{2}} \tan u.$$

The projection of this type with minimum distortion of angles for a hemisphere has for the value of m

$$m = \frac{1}{\sqrt{2}}; \ \sqrt{m} = \frac{1}{\sqrt[4]{2}}.$$

The alterations are null along the parallel of 24° 28′. The greatest value of a is $\sqrt[4]{2}$ or 1.189 and the smallest value of b is the inverse of this number, or 0.841 and the ratio of length of the most expanded length to the length of the most reduced is $\sqrt{2}$ or 1.414. Finally, the greatest value of $\sin \delta$ is $\tan^2 \frac{\pi}{8}$, which corresponds to $\delta = 9°53'$ or for 2δ, 19°45′.

For the representation of the whole globe by means of two maps, it is not best to adopt equatorial projections, that is, maps with the poles as centers, but what are called meridian projections, or ones with centers on the Equator. In this way North and South America will not be separated and Africa will not be separated into two parts by the Equator. For the one center it would be best to locate it on the Equator at 70° east longitude; and for the other at 110° west longitude. The maps would thus be transverse conic projections. A table of distances and azimuths from a point on the Equator are given in Special Publication No. 67 already referred to. If one wished to take into account the spheroid, it would only be necessary to compute such a table using the authalic latitudes (β) instead of the ordinary latitudes (ϕ). The map instead of forming a complete circle would be reduced to a sector of about 255°. The division line for the vacant sector could be chosen in each so that one would lie in the Indian Ocean and the other in the Pacific Ocean, thus leaving the land areas intact.

We have computed such a table for every 5 degrees of latitude and longitude and the results are given in the table on page 68. These values can be used for any equivalent projection with center on the Equator or for transverse conic projections with pole on the Equator, such as suggested above.

This projection has null deformation at a radial distance from the center of approximately 65° 30′; from this it results that in the Eastern Hemisphere, a circle through Africa, Europe, Asia and Australia has null deformation throughout its entire length. The circle in the Western Hemisphere which possesses the same property passes through the Americas. Thus, not only are the deformations small but also the number of points where they attain their maximum values are very limited in number. The projection is rather easy to construct and lends itself readily to determine the alterations in length and azimuth. The deformations are the same at points equidistant from the center and a line drawn from the center to any point gives one of the directions for Tissot's indicatrix with a perpendicular thereto at the point for the other direction. These two directions are orthogonal, both on the globe and on the map. The deviations starting from either of these axes do not amount to 10° anywhere on the map.

It is on these axes that the maximum and minimum scale distortions are found which in no case differ from true scale by more than 20 percent.

Instead of taking $m=\frac{1}{\sqrt{2}}=0.707$, one could take in round numbers $m=\frac{3}{4}=0.75$ which would be convenient for computation. The angle of the vacant sector on the map would be only 90° instead of 105°. The circumference of null deformation would be approximately 60° from the center and would pass through the continents close to their central regions. The alterations would be slightly increased but they would be diminished toward the center, that is, in the region where they change most slowly. At the center of the map, we should have

$$a=\frac{2}{\sqrt{3}}=1.155;\ a^2=\frac{4}{3}=1.333$$

$$\sin\delta=\frac{1}{7};\qquad 2\delta=16°26',$$

and on the edges

$$a=\sqrt{\frac{3}{2}}=1.225;\ a^2=\frac{3}{2}=1.500$$

$$\sin\delta=\frac{1}{5};\qquad 2\delta=23°04'.$$

Besides the representation of the two hemispheres just described, it is sometimes desired to represent the whole world on one map. If it is desired to have an equal-area map of this kind, it would be necessary to use a conic projection with minimum deformations between the north pole and 50° south latitude. The deformation beyond the parallel of 50° south would not be troublesome as no land of importance lies beyond that point, since only a tip of South America extends further south. The north pole should be taken as the center and the separation should be made at 170° west longitude which passes through Bering Strait and does not meet any land area. This projection corresponds to $m=0.342$; it does not produce any deformation along the parallel of 18°25′ south; at the north pole, a singular point of the projection, 2δ amounts to 118°26′. The greatest value of 2δ besides this point is 58°43′; of a, 1.710 and of a^2, 2.924.

LAMBERT'S AZIMUTHAL EQUIVALENT PROJECTION

(3) If, in the above-described Lambert's projection, we take $m=1$, θ will equal λ and the parallels form complete circles. With the authalic latitudes we have

$$\rho=2c\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right)=2c\sin\frac{p'}{2}.$$

If $p'=\frac{\pi}{2}-\beta$

$$h=\frac{r_1}{\rho};\ k=\frac{\rho}{r_1}.$$

(4) Returning to the cone, we would have for the sphere

$$\rho^2+C=-2a^2\frac{\sin\phi}{m}.$$

Let us suppose we wish the cone to be tangent to the middle parallel; then $m=\sin\phi_0$ and we have

$$\rho^2+C=-2a^2\frac{\sin\phi}{\sin\phi_0},$$

and when $\phi=\phi_0$,

$$\rho=a\cot\phi_0.$$

Hence

$$a^2\cot^2\phi_0+C=-2a^2$$

or

$$\rho^2-a^2\cot^2\phi_0=2a^2\left(1-\frac{\sin\phi}{\sin\phi_0}\right)$$

$$\rho^2=\frac{a^2}{\sin^2\phi_0}(1+\sin^2\phi_0-2\sin\phi_0\sin\phi)$$

$$\rho=\frac{a}{\sin\phi_0}\sqrt{1+\sin^2\phi_0-2\sin\phi_0\sin\phi}.$$

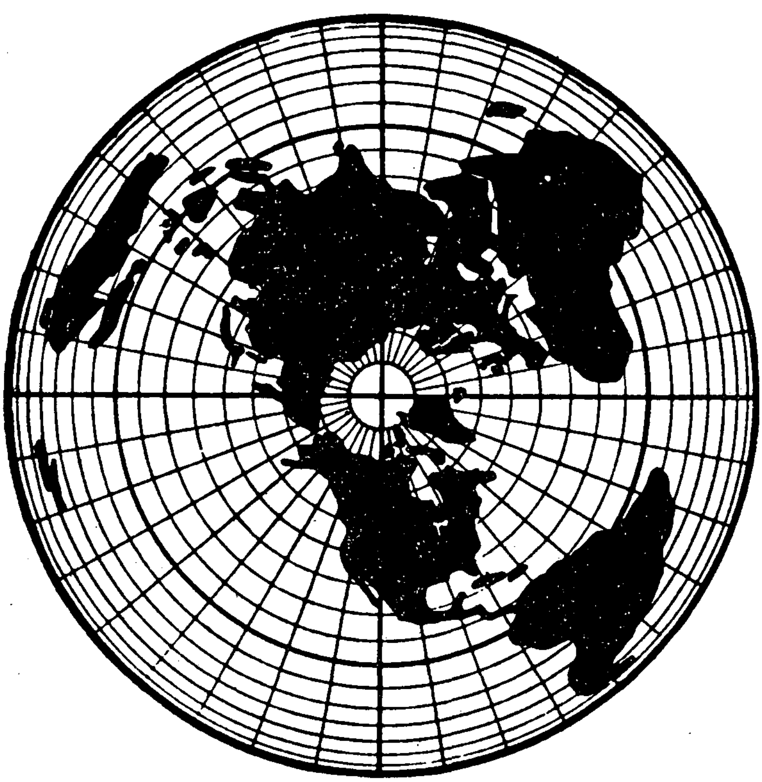


FIGURE 10.—Lambert's azimuthal equivalent projection.

The distance of any parallel from the middle parallel becomes

$$q=\rho-\rho_0=\frac{a}{\sin\phi_0}\left[\sqrt{1+\sin^2\phi_0-2\sin\phi_0\sin\phi}-\cos\phi_0\right]$$

$$=\frac{a}{\sin\phi_0}\frac{1+\sin^2\phi_0-2\sin\phi_0\sin\phi-\cos^2\phi_0}{\sqrt{1+\sin^2\phi_0-2\sin\phi_0\sin\phi}+\cos\phi_0}$$

$$=\frac{2a(\sin\phi_0-\sin\phi)}{\sqrt{1+\sin^2\phi_0-2\sin\phi_0\sin\phi}+\cos\phi_0}.$$

For $\phi_0=0$, $q=-a\sin\phi$ and ρ_0 becomes ∞ as do all of the ρ's, but so that $\rho\theta=a\lambda$. It thus passes into Lambert's equivalent cylinder projection for the sphere, with

$$x=a\lambda$$

$$y=a\sin\phi.$$

BONNE'S PROJECTION

The spacing of the concentric circles representing the parallels are to be the same as that upon the ellipsoid. We will start with the equations

$$\rho=F(\phi)$$

$$\theta=-\frac{a^2(1-e^2)\cos\phi}{F(\phi)F'(\phi)(1-e^2\sin^2\phi)^2}\lambda.$$

If C is the radius of the middle parallel of latitude ϕ_0, then we will have

$$\rho=C+s_0-s,$$

in which s_0 and s are the lengths of the meridian to ϕ_0 and ϕ, respectively, reckoned from some initial point or from the Equator as may be. But we know that we have

$$ds=-\frac{a(1-e^2)d\phi}{(1-e^2\sin^2\phi)^{3/2}},$$

hence

$$\rho=F(\phi)=C-a(1-e^2)\int\frac{d\phi}{(1-e^2\sin^2\phi)^{3/2}}$$

$$F'(\phi)=\frac{ds}{d\phi}=-\frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}}.$$

Therefore

$$\theta=\frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}\left[\frac{\lambda}{F(\phi)}\right].$$

From this it follows that

$$\theta F(\phi)=\rho\theta=\frac{a\cos\phi}{\sqrt{1-e^2\sin^2\phi}}(\lambda).$$

But since $\frac{a \cos \phi}{\sqrt{1-e^2 \sin^2 \phi}}$ is the radius of the parallel on the earth, the expression given above for $\rho\theta$ is the arc length of the parallel for the longitude difference λ, and $\rho\theta$ is the corresponding arc on the projection; it thus appears that the scale along the parallel is preserved and that $k=1$. Of course the meridians and parallels do not intersect at right angles for θ is a function of both ϕ and λ. We have the following partial differentials.

$$\frac{\partial \rho}{\partial \phi}=-\frac{a(1-e^2)}{(1-e^2 \sin^2 \phi)^{3/2}};\ \frac{\partial \rho}{\partial \lambda}=0.$$

$$\frac{\partial \theta}{\partial \phi}=-\frac{a(1-e^2) \sin \phi}{(1-e^2 \sin^2 \phi)^{3/2}}\left(\frac{\lambda}{\rho}\right)+\frac{a^2(1-e^2) \cos \phi}{(1-e^2 \sin^2 \phi)^2}\left(\frac{\lambda}{\rho^2}\right)$$

$$=-\frac{a(1-e^2)\lambda}{\rho(1-e^2 \sin^2 \phi)^{3/2}}\left[\sin \phi-\frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right];$$

$$\frac{\partial \theta}{\partial \lambda}=\frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}.$$

From these we get

$$h=\sqrt{1+\lambda^2\left(\sin \phi-\frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2}$$

$$k=1$$

$$\sin \psi=\frac{1}{\sqrt{1+\lambda^2\left(\sin \phi-\frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2}}$$

$$\cos \psi=\frac{-\lambda\left(\sin \phi-\frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)}{\sqrt{1+\lambda^2\left(\sin \phi-\frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2}} \qquad \psi>\frac{\pi}{2},$$

$$\cot \psi=-\lambda\left(\sin \phi-\frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)$$

$$h=\operatorname{cosec} \psi.$$

The expression for h shows that the scale along the meridians is not preserved on the map; only on the central meridian is the scale true. As long as λ is small, the distortions will also be small. For $\lambda=0$, $\psi=\frac{\pi}{2}$; hence the parallels are perpendicular to the central meridian. In addition, for the meridians to be perpendicular to the middle parallel, the function of ϕ in parentheses for $\sin \psi$ must vanish for ϕ_0, or

$$\sin \phi_0-\frac{a \cos \phi_0}{\rho_0\sqrt{1-e^2 \sin^2 \phi_0}}=0,$$

from which we get

$$\rho_0=\frac{a \cot \phi_0}{\sqrt{1-e^2 \sin^2 \phi_0}}=C$$

that is, the radius for the parallel of ϕ_0 is the tangent to the ellipsoid at ϕ_0 and the cone is tangent to the spheroid along this parallel.

s is the length of the meridian as already stated. The values of s in meters for every minute of ϕ from the Equator to latitude 75° are given in the Publication G–43, Tables for the Machine Computation of Geodetic Positions. A table of the values extending from the equator to the pole will soon be ready for use in such projections.

This projection is generally known as the Bonne projection, but authorities ascribe it to Ptolemy and to Mercator. Herz prefers to call it the Mercator equivalent projection. However, we retain the name that is most commonly applied to it. This projection was used for a map of France and it is sometimes known by the title, Projection du Dépôt de la Guerre or Projection de la Carte de France.

We will now investigate the distortions due to this projection.

$$\tan \delta = \frac{\lambda}{2}\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right) = -\frac{1}{2} \cot \psi.$$

Since h and k are conjugate semidiameters of Tissot's ellipse, we have

$$a^2+b^2=h^2+k^2=2+\lambda^2\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2$$

$$ab=1$$

$$(a+b)^2=4+\lambda^2\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2$$

$$(a-b)^2=\lambda^2\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2$$

(Note that the a on the left is not the same a as on the right.)

$$a+b=\sqrt{4+\lambda^2\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2}$$

$$a-b=\lambda\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)$$

$$a=\sqrt{1+\frac{\lambda^2}{4}\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2}+\frac{\lambda}{2}\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)$$

$$b=\sqrt{1+\frac{\lambda^2}{4}\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right)^2}-\frac{\lambda}{2}\left(\sin \phi - \frac{a \cos \phi}{\rho\sqrt{1-e^2 \sin^2 \phi}}\right).$$

On the projection we have

$$\tan u' = -\frac{d\rho}{\rho\, d\theta} = -\frac{\frac{\partial \rho}{\partial \phi}d\phi + \frac{\partial \rho}{\partial \lambda}d\lambda}{\rho\frac{\partial \theta}{\partial \phi}d\phi + \rho\frac{\partial \theta}{\partial \lambda}d\lambda}.$$

On the earth for the corresponding curve, we have

$$\tan u = \frac{\rho_m\, d\phi}{r_p\, d\lambda}$$

in which $r_p = \rho_p \cos \phi$.

But

$$\tan u' = -\frac{\dfrac{1}{\rho_m}\dfrac{\partial \rho}{\partial \phi}(\rho_m\, d\phi) + \dfrac{1}{r_p}\dfrac{\partial \rho}{\partial \lambda}(r_p\, d\lambda)}{\dfrac{\rho}{\rho_m}\dfrac{\partial \theta}{\partial \phi}(\rho_m\, d\phi) + \dfrac{\rho}{r_p}\dfrac{\partial \theta}{\partial \lambda}(r_p\, d\lambda)}$$

or

$$\tan u' = -\frac{\dfrac{1}{\rho_m}\dfrac{\partial \rho}{\partial \phi}\tan u + \dfrac{1}{r_p}\dfrac{\partial \rho}{\partial \lambda}}{\dfrac{\rho}{\rho_m}\dfrac{\partial \theta}{\partial \phi}\tan u + \dfrac{\rho}{r_p}\dfrac{\partial \theta}{\partial \lambda}}$$

On substituting the partial derivative values given on page 40, we get

$$\tan u' = \frac{\tan u}{1 - \lambda\left(\sin \phi - \dfrac{a \cos \phi}{\rho\sqrt{1 - e^2 \sin^2 \phi}}\right)\tan u},$$

in which u is reckoned from the east in counterclockwise direction and u' is reckoned from the tangent to the parallel at that point on the projection.

We have already proved that the direction of the major axis of the Tissot ellipse is given on the earth as $-\tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right)$ and of the minor axis, $\tan\left(\frac{\pi}{4} - \frac{\delta}{2}\right)$. These values are $-a$ and b, respectively. The formula above for $\tan u'$ can also be written

$$\tan u' = \frac{\tan u}{1 - (a - b)\tan u} = \frac{\tan u}{1 + \cot \psi \tan u}.$$

By substituting the above values for $\tan u$, we find for the corresponding values of $\tan u'$

$$\tan u_1' = \frac{-a}{1 + (a - b)a} = \frac{-a}{1 + a^2 - ab} = -\frac{1}{a} = -b$$

$$\tan u_2' = \frac{b}{1 - (a - b)b} = \frac{b}{1 - ab + b^2} = \frac{1}{b} = a$$

$$\tan u_1' \tan u_2' = -ab = -1.$$

Hence, the two directions on the projection are perpendicular to each other as well as are the corresponding directions on the earth.

We thus find

$$\tan u_1' = -\tan\left(\frac{\pi}{4} - \frac{\delta}{2}\right) = \tan\left(\frac{\delta}{2} - \frac{\pi}{4}\right)$$

$$\tan u_2' = \tan\left(\frac{\pi}{4} + \frac{\delta}{2}\right),$$

hence

$$u_1'=\frac{\delta}{2}-\frac{\pi}{4};\ u_1=-\frac{\pi}{4}-\frac{\delta}{2}$$

$$u_2'=\frac{\pi}{4}+\frac{\delta}{2};\ u_2=\frac{\pi}{4}-\frac{\delta}{2}$$

Tissot's indicatrix thus becomes

$$\tan\left(u'+\frac{\pi}{4}-\frac{\delta}{2}\right)=\frac{b}{a}\tan\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)=b^2\tan\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right).$$

We see that u and u' vanish together. When $u=\frac{\pi}{2}$

$$\tan\left(u'+\frac{\pi}{4}-\frac{\delta}{2}\right)=-\frac{b}{a}\cot\left(\frac{\pi}{4}+\frac{\delta}{2}\right)=-\frac{b}{a^2}=-b^3$$

$$\frac{\tan u'+b}{1-b\tan u'}=-b^3$$

$$\tan u'+b=-b^3+b^4\tan u'$$

$$(1-b^4)\tan u'=-b-b^3$$

$$(a^2b^2-b^4)\tan u'=-(ab^2+b^3)$$

$$\tan u'=-\frac{a+b}{a^2-b^2}=-\frac{1}{a-b}=\tan\psi$$

hence $u'=\psi$, the larger angle of intersection of the parallels and meridians.

$$\tan\delta=\tan\left[\left(\frac{\pi}{4}+\frac{\delta}{2}\right)-\left(\frac{\pi}{4}-\frac{\delta}{2}\right)\right]=\frac{a-b}{1+ab}=\frac{1}{2}(a-b)$$

$$=\frac{\lambda}{2}\left(\sin\phi-\frac{a\cos\phi}{\rho\sqrt{1-e^2\sin^2\phi}}\right)$$

$$=-\frac{1}{2}\cot\psi$$

as already given.

By means of this indicatrix, the azimuth distortions can be computed at a point and the scale in a given direction could be computed. For the ellipse, we have

$$x=a\cos\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)$$

$$y=b\sin\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)$$

$$x^2+y^2=a^2\cos^2\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)+b^2\sin^2\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)$$

$$=a^2\left[1-\frac{a^2-b^2}{a^2}\sin^2\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)\right],$$

hence, scale in the direction α reckoned from the east point is given as

$$\text{scale}=\sqrt{x^2+y^2}=a\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)}$$

ALBERS EQUAL-AREA PROJECTION

We now come to one of the most important of the equivalent projections. Let us return to the conic projection with the general equations already found on page 33 in which we will change the constant m to n to agree with previous usage.

$$\theta=n\lambda$$

$$\rho^2+C=-\frac{2c^2}{n}\sin\beta,$$

with β denoting the authalic latitude.

We now put in the condition that the scale shall be held true along two parallels called the standard parallels. We must then have the relations

$$\rho_1\theta=\rho_1 n\lambda=\frac{a\cos\phi_1}{\sqrt{1-e^2\sin^2\phi_1}}\lambda$$

$$\rho_2\theta=\rho_2 n\lambda=\frac{a\cos\phi_2}{\sqrt{1-e^2\sin^2\phi_2}}\lambda.$$

Substituting these values of the ρ's after dividing out the λ and first multiplying the general equation by n^2

$$\rho^2n^2+n^2C=-2c^2n\sin\beta,$$

we get

$$\frac{a^2\cos^2\phi_1}{1-e^2\sin^2\phi_1}+n^2C=-2c^2n\sin\beta_1$$

$$\frac{a^2\cos^2\phi_2}{1-e^2\sin^2\phi_2}+n^2C=-2c^2n\sin\beta_2$$

The first terms in the left-hand member of these equations are r_1^2 and r_2^2, respectively, in which the r's are the radii of the parallels of ϕ_1 and ϕ_2, respectively. These, in turn, are equal to $N_1^2\cos^2\phi_1$ and $N_2^2\cos^2\phi_2$ in which the N's are what are called the great normals of the ellipsoid or the radius of curvature perpendicular to the meridian. By subtracting the two equations, we get

$$N_1^2\cos^2\phi_1-N_2^2\cos^2\phi_2=2c^2n\ (\sin\beta_2-\sin\beta_1)$$

or

$$n=\frac{N_1^2\cos^2\phi_1-N_2^2\cos^2\phi_2}{2c^2\ (\sin\beta_2-\sin\beta_1)}=\frac{r_1^2-r_2^2}{2c^2(\sin\beta_2-\sin\beta_1)}.$$

This equation serves for the computation of the value of n. We get the values of ρ_1 and ρ_2 directly from their equation in terms of ϕ_1 and ϕ_2 respectively.

$$\rho_1=\frac{a\cos\phi_1}{n\sqrt{1-e^2\sin^2\phi_1}}=\frac{N_1}{n}\cos\phi_1=\frac{r_1}{n}$$

$$\rho_2=\frac{a\cos\phi_2}{n\sqrt{1-e^2\sin^2\phi_2}}=\frac{N_2}{n}\cos\phi_1=\frac{r_2}{n}.$$

By setting ρ equal to ρ_1 in the general equation, we get

$$\rho_1^2+C=-\frac{2c^2}{n}\sin\beta_1.$$

Now by subtracting this equation from the general equation, we have

$$\rho^2-\rho_1^2=\frac{2c^2}{n}(\sin\beta_1-\sin\beta),$$

or

$$\rho^2=\rho_1^2+\frac{2c^2}{n}(\sin\beta_1-\sin\beta).$$

In a similar way we can get

$$\rho^2=\rho_2^2+\frac{2c^2}{n}(\sin\beta_2-\sin\beta).$$

Either of these equations can be used for the computation of the various ρ's. With a calculating machine these ρ's can be computed rather rapidly.

For this projection we find, since it is a true conic projection and $\psi=\frac{\pi}{2}$,

$$h=\frac{r}{n\rho}=\frac{a\cos\phi}{n\rho\sqrt{1-e^2\sin^2\phi}}$$

$$k=\frac{n\rho}{r}=\frac{n\rho\sqrt{1-e^2\sin^2\phi}}{a\cos\phi}$$

$$\tan u'=\frac{n^2\rho^2(1-e^2\sin^2\phi)}{a^2\cos^2\phi}\tan u.$$

This last equation is Tissot's indicatrix as, of course, it should be. It should be noted that ρ does not equal zero for $\beta=\frac{\pi}{2}$; the result is that the pole is represented by an arc of a circle and not by a point as is usually the case in conic projections. The cone is thus a truncated one and the projection is sometimes called a truncated conic projection. For mapping regions that do not extend to the pole, this feature is in no way troublesome. For a map of the United States, for instance, no one would ever know whether the map was a true conic projection or not, since only a small section of the surface of the cone would be used in any case.

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PROPOSED NEW PROJECTION

In thinking about the projection called the stereographic equal-area, it occurred to us to try a similar projection with the central meridian spaced as in the meridian Lambert azimuthal equal-area projection. (See fig. 11.) As far as we know, no such projection has been heretofore proposed. In order to take account of the spheroid, we decided to project the authalic sphere as a further study in the theory of map projection. Let us assume

$$y=2c\sin\frac{\beta}{2},$$

with c as the radius of the authalic sphere and β the authalic latitude, as already explained. Then

$$\frac{\partial y}{\partial\beta}=c\cos\frac{\beta}{2}$$

$$\frac{\partial y}{\partial\lambda}=0.$$

Hence, from the equation of condition for equivalent projections, we have

$$\frac{\partial x}{\partial\lambda}\,c\cos\frac{\beta}{2}=c^2\cos\beta,$$

or by integration

$$x=\frac{c\cos\beta}{\cos\frac{\beta}{2}}\lambda,$$

$$x=c\lambda\left(2\cos\frac{\beta}{2}-\frac{1}{\cos\frac{\beta}{2}}\right)$$

$$\frac{\partial x}{\partial\beta}=-c\lambda\left(\sin\frac{\beta}{2}+\frac{\sin\frac{\beta}{2}}{2\cos^2\frac{\beta}{2}}\right)$$

$$\frac{\partial x}{\partial\lambda}=\frac{c\cos\beta}{\cos\frac{\beta}{2}}.$$

$$h=\sqrt{c^2\lambda^2\left(\sin\frac{\beta}{2}+\frac{\sin\frac{\beta}{2}}{2\cos^2\frac{\beta}{2}}\right)^2+c^2\cos^2\frac{\beta}{2}}\left(\frac{d\beta}{d\phi}\,\frac{(1-e^2\sin^2\phi)^{3/2}}{a\,(1-e^2)}\right)$$

$$=c\left(\frac{a^2(1-e^2)\cos\phi}{c^2\cos\beta(1-e^2\sin^2\phi)^2}\right)\frac{(1-e^2\sin^2\phi)^{3/2}}{a(1-e^2)}\sqrt{\lambda^2\left(\sin\frac{\beta}{2}+\frac{\sin\frac{\beta}{2}}{2\cos^2\frac{\beta}{2}}\right)^2+\cos^2\frac{\beta}{2}}$$

$$=\frac{a}{c}\left(\frac{\cos\phi}{\cos\beta(1-e^2\sin^2\phi)^{1/2}}\right)\sqrt{\lambda^2\left(\sin\frac{\beta}{2}+\frac{\sin\frac{\beta}{2}}{2\cos^2\frac{\beta}{2}}\right)^2+\cos^2\frac{\beta}{2}}$$

$$=\frac{r_p\lambda}{x}\sqrt{\lambda^2\tan^2\frac{\beta}{2}\left(1+\frac{1}{2\cos^2\frac{\beta}{2}}\right)^2+1}=\frac{r_p\lambda}{x}\operatorname{cosec}\psi$$

$$k=\frac{c\cos\beta}{\cos\frac{\beta}{2}}\frac{\sqrt{1-e^2\sin^2\phi}}{a\cos\phi}=\frac{x}{r_p\lambda}$$

For $\phi=0$, $h=\frac{a}{c}$ and $k=\frac{c}{a}$, hence $hk=1$ and the meridians are perpendicular to the straight line equator. For $\lambda=0$,

$$h=\frac{a\cos\frac{\beta}{2}\cos\phi}{c\cos\beta(1-e^2\sin^2\phi)^{1/2}},\quad k=\frac{c\cos\beta(1-e^2\sin^2\phi)^{1/2}}{a\cos\frac{\beta}{2}\cos\phi}$$

and again $hk=1$.

The parallels are straight lines parallel to the x-axis and spaced at the distance $y=2c\sin\frac{\beta}{2}$ from the origin. The x coordinates are linear in λ and hence are equally spaced, the expression in β being a constant for any given parallel. From the equations of the coordinates we get

$$x^2=c^2\frac{\cos^2\beta}{\cos^2\frac{\beta}{2}}\lambda^2$$

$$\cos^2\frac{\beta}{2}=\frac{4c^2-y^2}{4c^2}$$

$$\cos\beta=2\cos^2\frac{\beta}{2}-1$$

$$=\frac{4c^2-y^2}{2c^2}-1$$

$$=\frac{2c^2-y^2}{2c^2}$$

$$\frac{(4c^2-y^2)x^2}{4c^2}=c^2\lambda^2\frac{(2c^2-y^2)^2}{4c^4}$$

$$(4c^2-y^2)x^2=\lambda^2(2c^2-y^2)^2.$$

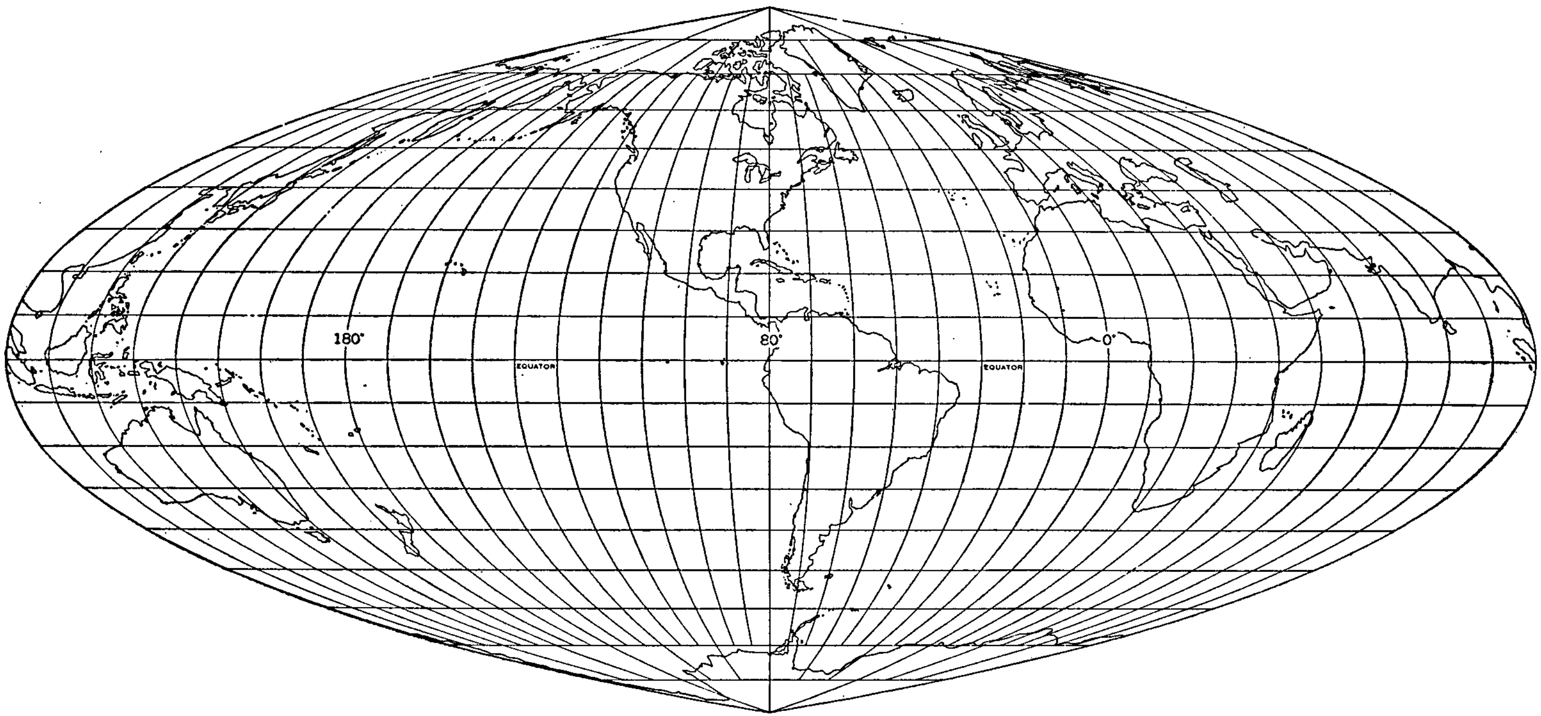

FIGURE 11.—Proposed equivalent projection for the ellipsoid.

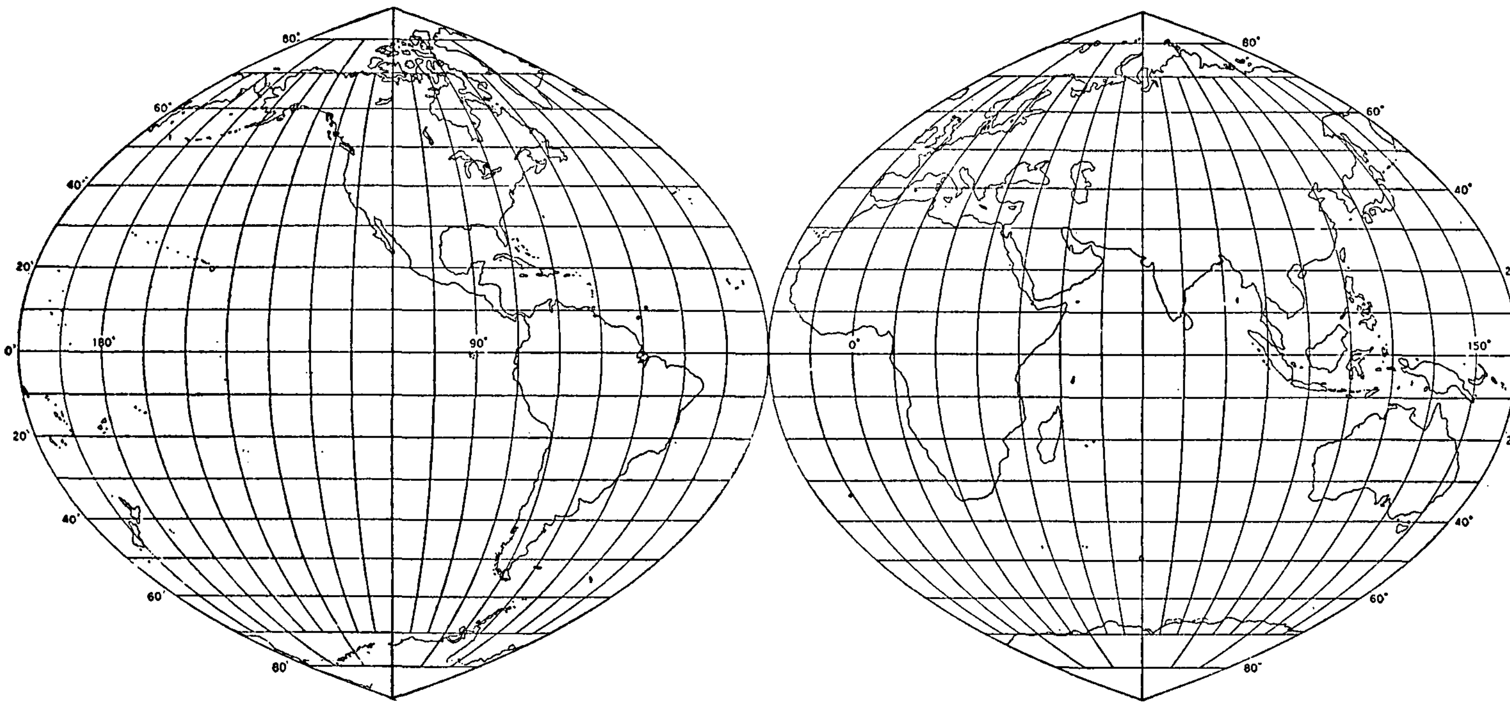

Figure 12.—Proposed equivalent projection for hemispheres.

The meridians are therefore fourth degree curves. Since the projection is equivalent, we have

$$hk \sin \psi = 1 = \sin \psi \sqrt{\lambda^2 \tan^2 \frac{\beta}{2}\left(1+\frac{1}{2 \cos^2 \frac{\beta}{2}}\right)+1}$$

hence

$$\sin \psi = \frac{1}{\sqrt{\lambda^2 \tan^2 \frac{\beta}{2}\left(1+\frac{1}{2 \cos^2 \frac{\beta}{2}}\right)^2+1}}$$

$$\cos \psi = \frac{-\lambda \tan \frac{\beta}{2}\left(1+\frac{1}{2 \cos^2 \frac{\beta}{2}}\right)}{\sqrt{\lambda^2 \tan^2 \frac{\beta}{2}\left(1+\frac{1}{2 \cos^2 \frac{\beta}{2}}\right)^2+1}}$$

$$\cot \psi = -\lambda \tan \frac{\beta}{2}\left(1+\frac{1}{2 \cos^2 \frac{\beta}{2}}\right)$$

DETERMINATION OF THE AXES OF THE TISSOT INDICATRIX

The h and k are conjugate diameters of Tissot's indicatrix, hence we have

$$a^2+b^2=h^2+k^2$$

$$ab=hk \sin \psi=1.$$

From these relations we get

$$a=\frac{1}{2}\sqrt{h^2+k^2+2}+\frac{1}{2}\sqrt{h^2+k^2-2}$$

$$b=\frac{1}{2}\sqrt{h^2+k^2+2}-\frac{1}{2}\sqrt{h^2+k^2-2}$$

$$\tan \delta=\frac{a-b}{2\sqrt{ab}}=\frac{a-b}{2}$$

$$=\frac{1}{2}\sqrt{h^2+k^2-2}$$

$$\tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)=a$$

$$\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)=b.$$

DETERMINATION OF THE DIRECTIONS OF THE AXES

Since the parallels in this projection are straight lines, it is best to reckon the direction from the east in counterclockwise direction. The positive direction on the earth will then agree with the positive direction on the plane. We will count λ positive to the east to agree with the usual positive value for the x.

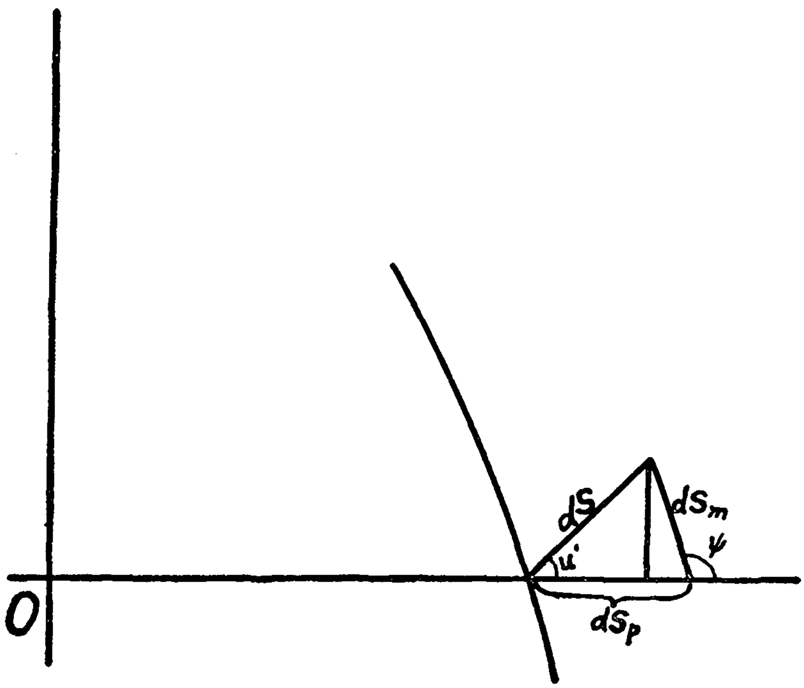

FIGURE 13.—Differential relations for distance and azimuth from the parallel.

From the diagram we have

$$\tan u' = \frac{dS_m \sin \psi}{dS_p + dS_m \cos \psi}, \; \psi > \frac{\pi}{2}$$

$$\overline{dS}^2 = \overline{dS}_m^{\,2} + \overline{dS}_p^{\,2} + 2 dS_m \, dS_p \cos \psi.$$

On the earth we have

$$\tan u = \frac{ds_m}{ds_p}$$

$$\overline{ds}^2 = \overline{ds}_m^{\,2} + \overline{ds}_p^{\,2}.$$

From these two relations we get

$$\tan u' = \frac{\dfrac{dS_m}{ds_m}\dfrac{ds_m}{ds_p}\sin\psi}{\dfrac{dS_p}{ds_p}+\left(\dfrac{dS_m}{ds_m}\right)\left(\dfrac{ds_m}{ds_p}\right)\cos\psi}$$

$$\frac{\overline{dS}^2}{\overline{ds}^2}=K^2=\frac{\left(\dfrac{dS_m}{ds_m}\right)^2\left(\dfrac{ds_m}{ds_p}\right)^2+\left(\dfrac{dS_p}{ds_p}\right)^2+2\left(\dfrac{dS_m}{ds_m}\right)\left(\dfrac{dS_p}{ds_p}\right)\left(\dfrac{ds_m}{ds_p}\right)\cos\psi}{1+\left(\dfrac{ds_m}{ds_p}\right)^2}.$$

or

$$\tan u'=\frac{h\tan u\sin\psi}{k+h\tan u\cos\psi}=\frac{hk\sin\psi\tan u}{k^2+hk\tan u\cos\psi}$$

$$=\frac{\tan u}{k^2+\tan u\cot\psi}.$$

$$K^2=\frac{h^2\tan^2 u+k^2+2hk\tan u\cos\psi}{1+\tan^2 u}.$$

From the equation for $\tan u'$, we see that when $u=\frac{\pi}{2}$

$$\tan u'=+\tan\psi,$$

therefore

$$u'=\psi,\ \psi>\frac{\pi}{2}$$

or u' is the large angle of intersection of the parallel and meridian as it should be.

By differentiating the expression for K^2 with respect to $\tan u$ and equating the result to zero to determine the maximum and minimum value of K^2, we get

$$hk\cos\psi\tan^2 u-(h^2-k^2)\tan u-hk\cos\psi=0$$

$$\tan^2 u-\frac{h^2-k^2}{hk\cos\psi}\tan u-1=0.$$

If $\tan u_1$ and $\tan u_2$ are the roots of this equation, we have

$$\tan u_1\tan u_2=-1.$$

Hence, u_1 and u_2 are orthogonal. By solving this equation, we get

$$\tan u_1=+\frac{h^2-k^2}{2hk\cos\psi}-\sqrt{1+\left(\frac{h^2-k^2}{2hk\cos\psi}\right)^2}$$

$$\tan u_2=+\frac{h^2-k^2}{2hk\cos\psi}+\sqrt{1+\left(\frac{h^2-k^2}{2hk\cos\psi}\right)^2}$$

After $\tan u_1$ and $\tan u_2$ are computed, it is necessary to compute $\tan u_1'$ and $\tan u_2'$ from the equation of relation between $\tan u$ and $\tan u'$. We will not carry through the computation in general terms since the expressions become very complicated as can easily be seen. As a practical matter, it is better to make the computations step by step as it would ultimately have to be done in any case.

EXAMPLE OF THE INDICATRIX

As a concrete example of the working of the formulas, we will make the computations for $\phi=30°$ and $\lambda=60°=\frac{\pi}{3}$. We must first compute h and k for this point.

$$\log \rho_p \text{ at } \phi=30°=6.80506633$$
$$\log \cos \phi=\underline{9.93753063}-10$$
$$\log r=6.74259696$$
$$r=5{,}528{,}368\ m.$$
$$\frac{\pi}{3}r=5{,}789{,}293\ m.$$

$$\text{for } \phi=30°,\ \lambda=\frac{\pi}{3},\ x=5{,}986{,}861\ m.$$

$$k=\frac{5{,}986{,}861}{5{,}789{,}293}=1.03412645$$

$$\cot \psi=-\lambda \tan \frac{\beta}{2}\left(1+\frac{1}{2\cos^2 \frac{\beta}{2}}\right)$$

$$\tan \frac{\beta}{2}=0.26689969;\ \cos \frac{\beta}{2}=0.96617886$$

$$\cos^2 \frac{\beta}{2}=0.93350159$$

$$1+\frac{1}{2\cos^2 \frac{\beta}{2}}=1.53561773$$

$$\frac{\pi}{3}\tan \frac{\beta}{2}=0.27949670$$

$$\cot \psi=-0.42920009$$
$$\tan \psi=-2.32991563$$
$$\psi=113°13'44''.45$$

$$h=\frac{1}{k \sin \psi}$$

$$\sin \psi=0.91893573$$
$$h=1.05230399$$
$$h^2=1.10734369$$
$$k^2=\underline{1.06941751}$$
$$h^2+k^2=2.17676120$$

$$\tan\delta=\frac{1}{2}\sqrt{0.17676120}=0.21021489$$

$$\delta=11°52'17''.655$$

$$\frac{\delta}{2}=5°56'08''.828$$

$$\frac{\pi}{4}+\frac{\delta}{2}=50°56'08''.828$$

$$\frac{\pi}{4}-\frac{\delta}{2}=39°03'51''.172$$

$$\tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)=1.23207119=a$$

$$\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right)=0.81164141=b$$

We can arrive at these values in another way and we will give this method as an illustration.

$$\tan\eta=\frac{k}{h},\ \sin 2\gamma=\sin\psi\sin 2\eta$$

η and γ being auxiliary angles

$$\sin\delta=\tan\left(\frac{\pi}{4}-\gamma\right)$$

$$a=\sqrt{\cot\gamma}\text{ and }b=\sqrt{\tan\gamma}$$

$$\cos^2\eta=\frac{h^2}{h^2+k^2}$$

$$\sin 2\eta=2\tan\eta\cos^2\eta=\frac{2hk}{h^2+k^2}$$

$$\sin 2\gamma=\frac{2hk}{h^2+k^2}\sin\psi$$

$$\frac{2hk}{h^2+k^2}=0.99984821$$

$$\sin 2\gamma=0.91879624$$

$$2\gamma=66°45'02''.63$$

$$\gamma=33°22'31''.315$$

$$\frac{\pi}{4}-\gamma=11°37'28''.685$$

$$\tan\left(\frac{\pi}{4}-\gamma\right)=0.20571864=\sin\delta$$

$$\delta=11°52'17''.655$$

$$\cot\gamma=1.51799942$$

$$a=\sqrt{\cot\gamma}=1.23207119$$

$$b=\sqrt{\tan\gamma}=0.81164141$$

We have thus obtained the same values as by the other method.

We will now proceed to compute u_1 and u_2 and then u_1' and u_2'.

$$\tan u_1=+\frac{h^2-k^2}{2hk\cos\psi}-\sqrt{1+\left(\frac{h^2-k^2}{2hk\cos\psi}\right)^2}$$

$$h^2=1.10734369$$

$$k^2=\underline{1.06941751}$$

$$h^2-k^2=0.03792618$$

$$\cos\psi=-0.39440730$$

$$2hk\cos\psi=-0.85840019$$

$$\frac{h^2-k^2}{2hk\cos\psi}=-0.04418240$$

$$1+\left(\frac{h^2-k^2}{2hk\cos\psi}\right)^2=1.00195208$$

$$\sqrt{1+\left(\frac{h^2-k^2}{2hk\cos\psi}\right)^2}=1.00097556$$

$$\tan u_1=-1.04515796$$

$$\tan u_2=+\frac{h^2-k^2}{2hk\cos\psi}+\sqrt{1+\left(\frac{h^2-k^2}{2hk\cos\psi}\right)^2}$$

$$\tan u_2=+0.95679316$$

$$\tan u_1'=\frac{-1.04515796}{1.06941751+1.04515796\times 0.42920009}$$

$$=-\frac{1.04515796}{1.51799940}=-0.68851013.$$

$$\tan u_2'=\frac{0.95679316}{1.06941751-0.95679316\times 0.42920009}$$

$$=\frac{0.95679316}{0.65876180}=1.45241142.$$

We find

$$\tan u_1' \tan u_2' = -1$$

and u_1' and u_2' are also orthogonal as they should be.

$$\tan u_1 = -1.04515796; \ \tan u_1' = -0.68851013$$

$$u_1 = -46°15'53''.67 \qquad u_1' = -34°32'52''.10.$$

These two correlative angles give the directions of the major axis of the ellipse, since we know that this axis lies in the smaller angle of the intersection of the meridian and parallel at the point. The relation for the ellipse thus becomes

$$\tan [u' + (34°32'52''.10)] = \frac{b}{a} \tan [u + (46°15'53''.67)]$$

$$= b^2 \tan [u + (46°15'53''.67)]$$

$$b^2 = 0.65876178.$$

This equation should be valid for $u = u' = 0$ and it is, as can easily be shown. For $u = \frac{\pi}{2}$, u' should be the larger angle of the intersection of the meridian and parallel.

$$\tan [u' + (34°32'52''.10)] = b^2 \tan (136°15'53''.67)$$

$$= -0.65876178 \times 0.95679316$$

$$= -0.63029877$$

$$u' + (34°32'52''.10) = 147°46'36''.55$$

$$u' = 113°13'44''.45$$

$$= \psi$$

This gives us the correct value for ψ as already computed.

We have given this complete treatment of the indicatrix at this point as an example of the case when the equivalent projection has variation of scale along both the meridian and the parallel. A greater amount of computation is required than is needed for the case of a projection that has true scale either along the meridians or along the parallels. Tissot uses the word automecoic for true scale; thus he would describe the sinusoidal projection as one with the parallels automecoic.

HAMMER-AITOFF PROJECTION

This projection is based on the Lambert azimuthal meridian equal-area projection. If we have such a Lambert projection we can proceed in the following way. Turn the map about the polar axis until it makes an angle of 60° with the horizontal. Then project the map on the horizontal plane by a system of parallel lines all perpendicular to the plane of the map. This will double the length of the Equator and all straight lines parallel to the Equator. Thus, the circular boundary of the Lambert map will be projected into an ellipse with major and minor axis in the ratio of two to one. At the same time all areas will be doubled. Now, if we have a Lambert map computed for every 5° of longitude, we can double the x values for 5°, 10°, etc., and designate them 10°, 20°,

AITOFF'S EQUAL AREA PROJECTION OF THE SPHERE

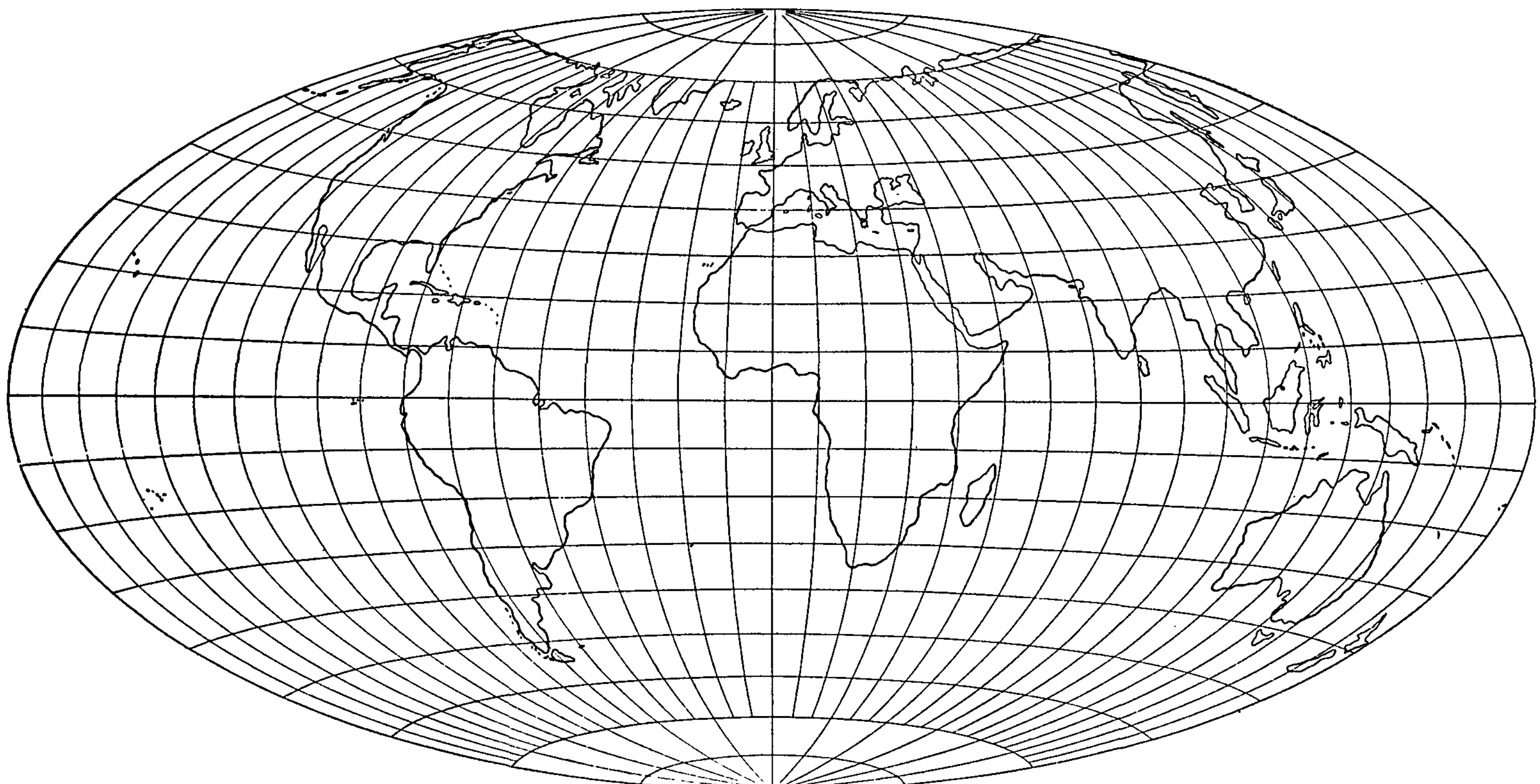

FIGURE 14.—Aitoff's equal-area projection of the sphere.

etc., on the new map. At the same time the y values will remain unchanged. Note that this procedure should apply to the values for 10°, 20°, etc., of latitude since the 5°, 15°, etc., parallels would not be wanted when the meridians on the new map are only given for every 10 degrees. Of course one could compute the 2½°, 7½°, etc., of longitude and then every 5° intersection could be shown on the new map. The resulting map is strictly equal area for the area between meridians is doubled, so it is proper to double the designation of the meridians.

A table for the Lambert azimuthal equivalent projection on a meridian is given in Special Publication No. 67, already referred to. A table for the Hammer-Aitoff projection is given in Special Publication No. 68, "Elements of Map Projection." These tables are both based on the sphere and not on the spheroid. If the tables for the transformation on the authalic sphere are used for a new computation, both of these maps can be based on the spheroid with no extra labor of computation. These tables are given on pages 68–74 and they were computed so that any future computations of equal-area projections can make use of them if it is deemed necessary to take account of the ellipsoid for such purpose.

CURVES OF TRUE SCALE ON EQUIVALENT PROJECTIONS

Since, in the directions of maximum alteration of scale on equivalent projections, the scale is too large in the one direction and too small in an orthogonal direction, there must be some azimuth between these two directions in which there is true scale. In Special Publication No. 68 we called attention to these curves and stated that no detailed study of them had been made. A couple of years ago we received a letter from M. R. MacPhail, Caracas, Venezuela, calling attention to our statement and enclosing a study of the curves that he had made under the interest aroused by our statement. His treatment was an excellent application of differential geometry to the subject in hand. Mr. MacPhail was at that time employed by the Standard Oil Co. of Venezuela. We appreciate such interest in a practical engineer and wish to give him due credit for his work. We will now approach the matter in a slightly different way but will use two diagrams that Mr. MacPhail sent us. His work was sent to us for any use that we might wish to make of it.

As a start, it is evident that there are two sets of such curves forming a network on the projection. We will make use of Tissot's indicatrix in our study of the matter. When the meridians and parallels of the map intersect at right angles, the directions of the axes of the indicatrix ellipse are given by tangents to the curves at their point of intersection. Then the scale along the parallel is reciprocal to the scale along the meridian at the point and they form the a and b semiaxes of the ellipse. The scale in any direction on the earth is then given by the equation

$$a^2 \cos^2 u + b^2 \sin^2 u = K^2.$$

But when the scale is exact K becomes equal to one and we have

$$a^2 \cos^2 u + b^2 \sin^2 u = 1 = ab\ (\sin^2 u + \cos^2 u),$$

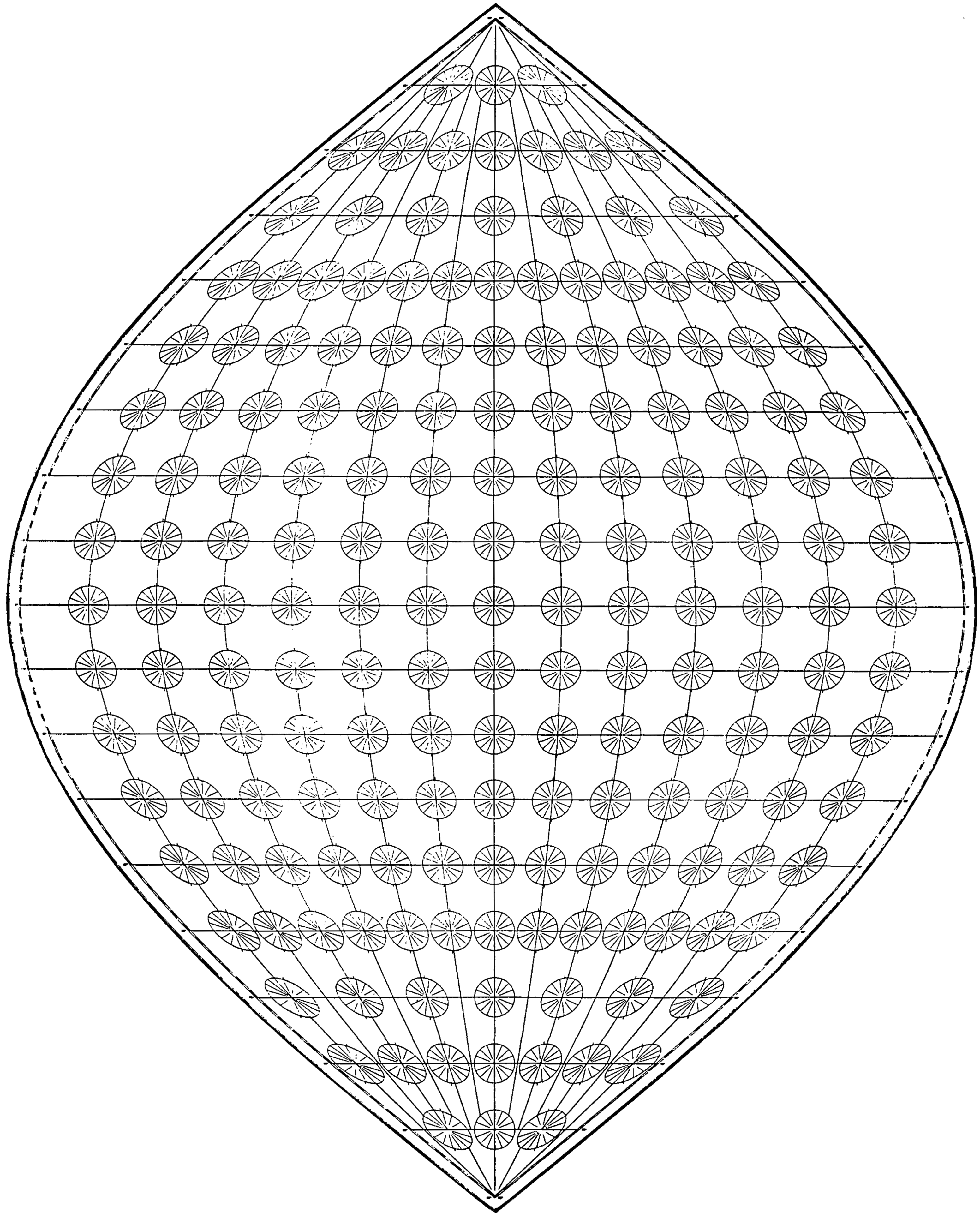

FIGURE 15.—Examples of Tissot's indicatrix on sinusoidal projection.

or

$$(ab-b^2)\sin^2 u=(a^2-ab)\cos^2 u.$$

If a is not equal to b, we get

$$b\sin^2 u=a\cos^2 u.$$

Note that a will not equal b on an equivalent projection except at certain points or sometimes along certain lines. When a does equal b any direction satisfies the condition but we generally consider the curves as then making angles of 45° with the meridians. We should state that these curves are called isoperimetric curves on the projection. Returning to the equation we have

$$\tan u=\pm\sqrt{\frac{a}{b}}=\pm a=\pm\tan\left(\frac{\pi}{4}+\frac{\delta}{2}\right)$$

u being reckoned from the major axis. From the equation of the indicatrix we have

$$\tan u'=\frac{b}{a}\tan u$$

or

$$\tan u'=\pm b=\pm\tan\left(\frac{\pi}{4}-\frac{\delta}{2}\right),$$

hence

$$u'=\pm\left(\frac{\pi}{4}-\frac{\delta}{2}\right).$$

The isoperimetric curves thus run approximately midway between the two axes of the indicatrix. With certain projections that have the meridians and parallels orthogonal, we can easily get the equation of these curves. With the Albers' projection

$$R=f(\phi)\quad \theta=n\lambda.$$

Since the projection is equivalent, the element of area on the map $R\,dR\,d\theta=nR\,dR\,d\lambda$ must equal the corresponding area on the authalic sphere $-c^2\cos\beta\,d\beta\,d\lambda$, the sign being negative because R decreases as β increases.

$$nR\,dR=-c^2\cos\beta\,d\beta$$

since the $d\lambda$ cancels out, being a factor on both sides of the equation. By integration, we get

$$\frac{1}{2}nR^2=c^2(A-\sin\beta),$$

$c^2 A$ being the constant of integration.

$$R=c\sqrt{(A-\sin\beta)\frac{2}{n}}.$$

Since the scale is to be true along the standard parallels, we must have, r_1 and r_2 being the radii of the parallels,

$$R_1\theta=R_1 n\lambda=r_1\lambda$$

$$R_2\theta=R_2 n\lambda=r_2\lambda$$

or dropping the common factor on both sides and squaring we get

$$R_1^2 n^2 = r_1^2 \text{ and } R_2^2 n^2 = r_2^2$$

$$c^2(A - \sin\beta_1)\frac{2}{n}(n^2) = r_1^2$$

$$c^2(A - \sin\beta_2)\frac{2}{n}(n^2) = r_2^2$$

$$c^2(\sin\beta_2 - \sin\beta_1)2n = r_1^2 - r_2^2$$

$$n = \frac{r_1^2 - r_2^2}{2c^2(\sin\beta_2 - \sin\beta_1)}.$$

This agrees with the equation for n in the Albers' projection as ordinarily given. The R's can be computed from either of the expressions

$$R^2 = R_1^2 + \frac{2c^2}{n}(\sin\beta_1 - \sin\beta)$$

or

$$R^2 = R_2^2 + \frac{2c^2}{n}(\sin\beta_2 - \sin\beta).$$

Since the isoperimetric curves intersect a given parallel at a constant angle, such that the angle on the earth and that on the map are complements of each other, we must have dR on the map equal to $\pm ds_p$ on the earth. We have

$$dR = -b\, ds_m$$

$$Rn\, d\lambda = a\, ds_p, \quad \frac{1}{a} = b = \frac{ds_p}{Rn\, d\lambda}$$

$$\frac{dR}{Rn\, d\lambda} = -\frac{b\, ds_m}{a\, ds_p} = -\frac{b^2\, ds_m}{ds_p},$$

but for the perimetric curves,

$$\frac{ds_m}{ds_p} = \pm a$$

$$\frac{dR}{Rn\, d\lambda} = \pm ab^2 = \pm b = \pm\frac{ds_p}{Rn\, d\lambda}$$

$\therefore\ dR = \pm ds_p$, as stated above. But from the differential equation for $R\, dR$, we get

$$dR = -\frac{c^2 \cos\beta\, d\beta}{nR}$$

and

$$ds_p = c \cos\beta\, d\lambda.$$

Substituting these values we get

$$c \cos\beta\, d\lambda = \pm\frac{c^2 \cos\beta\, d\beta}{nR}$$

or

$$d\lambda = \pm\frac{c\, d\beta}{nR}.$$

In the equation for R, let $\beta=2\eta-\frac{\pi}{2}$

then

$$R^2=R_1^2+\frac{2c^2}{n}(\sin\beta_1+\cos 2\eta)$$

$$\cos 2\eta=1-2\sin^2\eta$$

$$R^2=R_1^2+\frac{2c^2}{n}(\sin\beta_1+1-2\sin^2\eta)$$

$$=R_1^2+\frac{2c^2}{n}(\sin\beta_1+1)-\frac{4c^2}{n}\sin^2\eta$$

$$R=\sqrt{R_1^2+\frac{2c^2}{n}(\sin\beta_1+1)}\sqrt{1-\frac{4c^2}{n\left[R_1^2+\frac{2c^2}{n}(\sin\beta_1+1)\right]}\sin^2\eta}.$$

Substituting this value and noting that $d\beta=2\,d\eta$, we have

$$d\lambda=\pm\frac{2c\,d\eta}{n\sqrt{R_1^2+\frac{2c^2}{n}(\sin\beta_1+1)}\sqrt{1-k^2\sin^2\eta}}$$

with

$$k^2=\frac{4c^2}{n\left[R_1^2+\frac{2c^2}{n}(\sin\beta_1+1)\right]}.$$

Let

$$\frac{\sqrt{R_1^2+\frac{2c^2}{n}(\sin\beta_1+1)}}{2c}=t,$$

then

$$nt\,d\lambda=\pm\frac{d\eta}{\sqrt{1-k^2\sin^2\eta}}$$

and

$$R=2\,ct\sqrt{1-k^2\sin^2\eta}.$$

By integration

$$nt(\lambda-\lambda_0)=\pm\int_0^\eta\frac{d\eta}{\sqrt{1-k^2\sin^2\eta}}.$$

This is Legendre's first elliptic integral. By proper choice of λ, we can take $\lambda_0=0$, then

$$\sin\eta=\pm sn(\lambda tn)$$

and $sn\ (\lambda tn)$ is the Jacobian elliptic function. Note that for any given parallel we can choose any meridian from which to reckon λ, since the angle of crossing is constant for that parallel. After starting one curve, the same central meridian must then be used for all other parallels for this curve. When one curve is computed any number of others can be located by their points of crossing the parallels, since they will be symmetrically related to another chosen meridian as the given curve is to its meridian.

637697°—45——5

As η varies from 0 to $\frac{\pi}{2}$, β varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$ and hence covers the full range of latitude. The computation of the λ values is sufficient, for these give us the points of crossing the parallels for any given curve. For $\eta=0$, $\lambda=0$ and we can start from any meridian that we wish on the circle representing the south pole. The curve should cross the standard parallels at a 45° angle.

The equations of the curves on the earth become on substituting the value of

$\eta=\frac{\pi}{4}+\frac{\beta}{2}$

$$\sin\left(\frac{\pi}{4}+\frac{\beta}{2}\right)=\pm sn(nt\lambda).$$

As β varies from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, $nt\lambda$ varies from zero to K, the complete elliptic integral of the first kind.

Let us apply these formulas to a map of the sphere that holds the equator and 30° north latitude as standard parallels. In this case $r_1=a$, $r_2=\frac{\sqrt{3}}{2}a$, $\sin\beta_1=0$, $\sin\beta_2=\frac{1}{2}$ and $c=a$. With these values $n=\frac{1}{4}$ and $R_1^2=16a^2$; these results give $k^2=2/3$; $t=\sqrt{6}.$ Substituting these values in the equation of R, we get

$$R=2\sqrt{6}a\sqrt{1-\frac{2}{3}\sin^2\eta}=2\sqrt{6}a\; dn(\sqrt{6}\theta),\text{ with } k^2=\frac{2}{3}.$$

From the relation

$$\sin\eta=sn\frac{\sqrt{6}}{4}\lambda,$$

when

$$\phi=\frac{\pi}{2},\ \eta=\frac{\pi}{2}$$

and

$$\frac{\sqrt{6}}{4}\lambda=K,$$

in which K is the complete elliptic integral of the first kind,

$$k=0.81649658$$
$$54°44'08''.2$$
$$54°.7356$$

By interpolation in Legendre's Table by use of second differences we get

$$\begin{array}{r} 2.01327 \\ +1578 \\ \hline 2.02905 \\ -9 \\ \hline K=2.02896 \end{array}$$

$$\frac{4}{\sqrt{6}}K=3.3133=\lambda.$$

It is thus seen that the total extent in longitude is a fraction more than π or 180°. Mr. MacPhail has constructed a diagram showing this curve on the projection which we have given as figure 16.

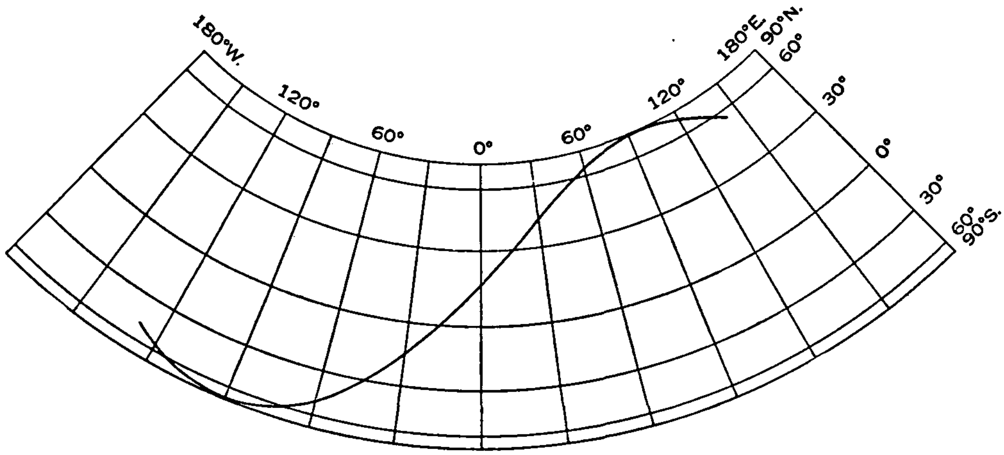

FIGURE 16.—Isoperimetric curves on Albers equivalent projection.

We will now compute the constants for the map of the United States on the Albers' projection. For this projection the following values given in Special Publication No. 68 are: $\log c = 6.8042074$; $\log n = 9.7802478$; R_1 for 29°30′=9,215,188. From Special Publication No. 67, β for 29°30′=29°23′20.″09. With these values we find k^2.

$$k^2 = \frac{4c^2}{n\left[R_1^2 + \frac{2c^2}{n}(\sin\beta_1 + 1)\right]} = 0.94276541$$

$$k = 0.97096108$$

$$t = 1.32639929$$

$$n = 0.6029035.$$

With these values for k, t and n, the longitude of the crossing of any meridian can be computed by means of a table of elliptic functions. The formula for the computation in terms of the authalic latitude is the one to use

$$\sin\left(\frac{\pi}{4} + \frac{\beta}{2}\right) = \pm sn(nt\lambda).$$

The authalic latitudes for any given parallels are given in Special Publication No. 67. For the south pole $\beta = -\frac{\pi}{2}$ and $\lambda = 0$. To compute for the United States map one could start with parallel 25° north. This will give a certain value of λ and we can choose the meridian that we wish to have this value. Suppose it is to apply to the intersection of the meridian 120° with the 25° parallel. Then we compute the value of

λ for 30°; this will be a larger value than that for 25°. Subtract the 25° value from this and that will give the λ east of 120° on parallel 30°; in a similar way we can compute the intersection for any other parallel. We can use the same values starting from the southeast corner and use the negative sign. This gives a member of the other family of isoperimetric curves. After one curve is computed, any number of others of the same family can be constructed by moving each intersection 1°, 2°, 5° or any number of degrees either east or west. The two families of curves are inversely similar and one can be derived from the other by reversing the signs of the intersections. If we started at the central meridian of the map we could lay our values off to the east and to the west on any parallel and thus get the two curves that intersect on the starting parallel. A diagram is given in Special Publication No. 68 with several of these curves shown on it.

THE LAMBERT AZIMUTHAL, EQUAL-AREA, POLAR PROJECTION

This is a special case of a conic projection in which both standard parallels move up to the pole. Then $n=1$ and $R_1=R_2=0$ and $\sin \beta_1=\sin \beta_2=1$. Then

$$R^2=2c^2(1-\sin \beta)=4c^2 \sin^2\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

Let $\beta=\frac{\pi}{2}-p'$, in which p' is the authalic colatitude. This gives us

$$R=2c \sin \frac{p'}{2};\ \alpha=\lambda.$$

$t=1$ and $k=1$ and the elliptic functions become the hyperbolic functions in the following way

$$\begin{aligned} \operatorname{sn} x &\rightarrow \tanh x \\ \operatorname{cn} x &\rightarrow \operatorname{sech} x \\ \operatorname{dn} x &\rightarrow \operatorname{sech} x \end{aligned}$$

The isoperimetric curves are spirals of the form

$$r=2a \operatorname{sech} \alpha=2a \operatorname{sech} \lambda.$$

The curve is shown in figure 17 with the two inversely similar curves shown on it.

On the sinusoidal projection we have shown the Tissot indicatrix in the form

$$\tan\left(u'+\frac{\pi}{4}-\frac{\delta}{2}\right)=\frac{b}{a} \tan\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right)=b^2 \tan\left(u+\frac{\pi}{4}+\frac{\delta}{2}\right).$$

We have shown that the curves of equal scale have the directions $\pm\left(\frac{\pi}{4}+\frac{\delta}{2}\right)$ on the earth and $\pm\left(\frac{\pi}{4}-\frac{\delta}{2}\right)$ on the projection. The plus values correspond to $u=0$ and $u'=0$. These isoperimetric curves are the parallels which were constructed true to scale. For the other curve we must have $u'+\frac{\pi}{4}-\frac{\delta}{2}=-\frac{\pi}{4}+\frac{\delta}{2}$ on the projection or $u'=-\frac{\pi}{2}+\delta\cdot$

Hence

$$\tan u' = -\cot \delta = -\frac{2}{\lambda \sin \phi} = 2 \tan \psi,$$

u' is, of course, a negative angle.

Considering the earth as a sphere, we have seen that u on the earth is either 0 or $-\left(\frac{\pi}{2}+\delta\right)$. For the second value

$$\tan u = -\tan\left(\frac{\pi}{2}+\delta\right) = +\cot \delta = +\frac{2}{\lambda \sin \phi} = \frac{d\phi}{d\lambda \cos \phi}$$

$$\frac{2d\lambda}{\lambda} - \frac{\sin \phi \, d\phi}{\cos \phi} = 0$$

$$2 \log \lambda + \log \cos \phi = \log C$$

log C being the arbitrary constant

$$\lambda^2 \cos \phi = C.$$

By assigning to C a value of $\phi=0$ that will make the isoperimetric curve start at a chosen longitude indicated by λ_0 we should have the equation

$$\lambda^2 \cos \phi = \lambda_0^2.$$

By means of this equation we could compute the longitude of its intersection with the various parallels. The λ's, of course, must be expressed in radian measure and not in degrees. This equation of course applies to the sphere and not to the spheroid, but if the projection is based on the authalic sphere, the curves can be located for the spheroid by using the equation

$$\lambda^2 \cos \beta = \lambda_0^2$$

There still exists the slight variation due to the mapping of the spheroid on the authalic sphere, but for practical purposes this is a negligible quantity.

The same relation is found for the Bonne projection since in it the scale along the parallels is constructed true; the parallels are therefore the one set of isoperimetric curves, and the other set is given in direction by the same relation

$$\tan u' = 2 \tan \psi$$

in which u' is a negative angle measured from the tangent to the parallel at the point.

For the new equivalent projection with the Lambert spacing on the central meridian, it would be necessary first to compute the directions of the axes of the indicatrix ellipse and then get the directions for the isoperimetric curves from them. For the point $\phi=30°$ and $\lambda=60°$ which we have computed we found

$$\tan [u' + (34°32'52''.10)] = 0.65876178 \tan [u + (46°15'53''.67)],$$

$$\pm\left(\frac{\pi}{4}-\frac{\delta}{2}\right) = \pm (39°03'51''.17).$$

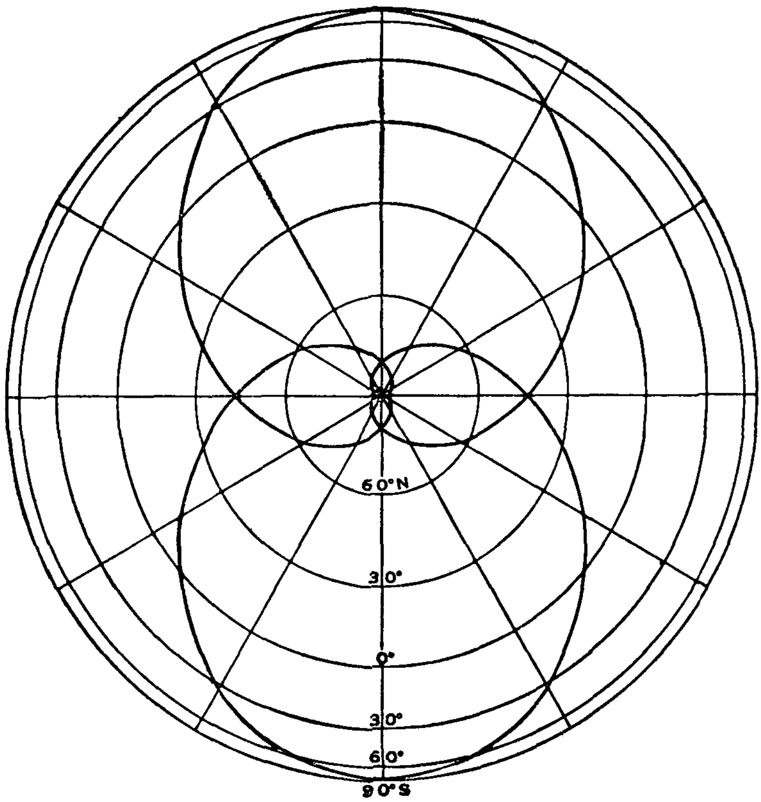

FIGURE 17.—Isoperimetric curves on Lambert's azimuthal equivalent projection.

Hence for one curve

$$u' + (34°32'52''.10) = 39°03'51''.17$$

$$u' = 4°30'59''.07$$

and

$$u' + (34°32'52''.10) = -39°03'51''.17$$

$$u' = -73°36'43''.27$$

This is a sufficient number of examples to indicate the general method of handling such problems in equivalent projections.

It should be noted that there are innumerable other projections of the equivalent class that could be devised but we have given the most important that are in use today. Of those given, various transformations could be devised by transversing the elements in various ways. The conic projections can have the apex of the cone in any desired latitude and longitude, but as a prelude to such a projection the arc distance and azimuths of great circles emanating from this point would have to be computed. These computations could be made on the authalic sphere and in this way take account of the ellipsoid if it is deemed of sufficient importance to do so. We are including such tables computed for a point on the Equator and these could be used for any transverse map with this point serving as the pole. We are including a couple of illustrations of such maps on the Mollweide projection that were adopted from the work of Colonel Close. (See figs. 18 and 19.) These illustrations serve to show what peculiar distortions may be found in a map that is still strictly equivalent in a mathematical sense.

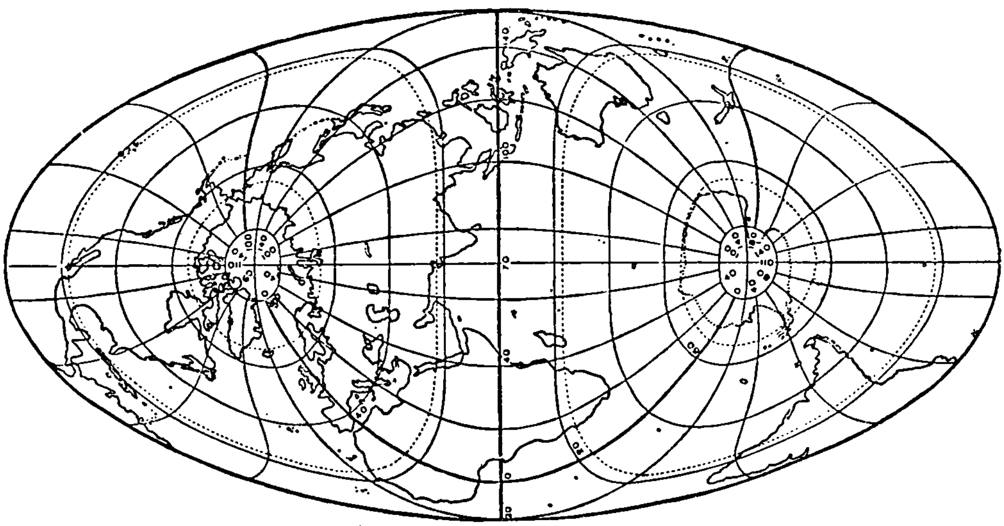

FIGURE 18.—Col. Close's transverse Mollweide projection, after Steers.

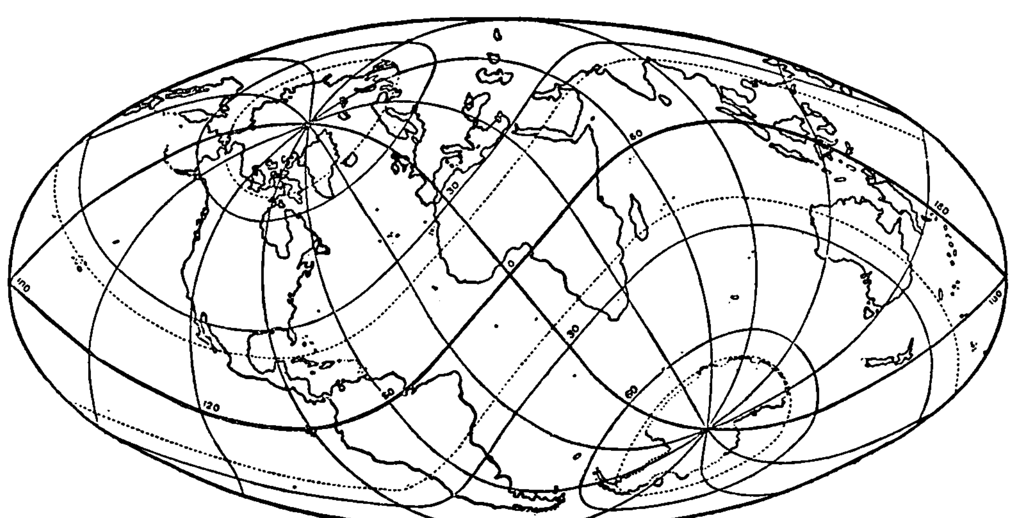

FIGURE 19.—Col. Close's oblique Mollweide projection, after Steers.

TABLES

Transformation from geographic to azimuthal coordinates on the authalic sphere—Center on the Equator

VALUES OF THE AZIMUTH RECKONED FROM THE NORTH, α; TAN α = SIN λ COT β

Latitude	Longitude 0°	Longitude 5°	Longitude 10°	Longitude 15°
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0	0 00 00.00	90 00 00.00	90 00 00.00	90 00 00.00
5	0 00 00.00	45 01 13.97	63 21 50.20	71 24 06.43
10	0 00 00.00	26 24 20.61	44 41 28.35	55 51 18.11
15	0 00 00.00	18 05 40.61	33 03 51.72	44 08 12.20
20	0 00 00.00	13 31 31.02	25 36 23.58	35 32 21.37
25	0 00 00.00	10 38 01.48	20 30 35.38	29 08 22.06
30	0 00 00.00	8 37 22.20	16 43 40.63	24 14 35.22
35	0 00 00.00	7 07 37.43	13 59 19.79	20 22 13.62
40	0 00 00.00	5 57 24.00	11 44 37.23	17 12 55.86
45	0 00 00.00	5 00 12.82	9 53 41.65	14 34 25.96
50	0 00 00.00	4 12 05.92	8 19 37.98	12 18 24.67
55	0 00 00.00	3 30 29.06	6 57 49.33	10 19 03.62
60	0 00 00.00	2 53 37.35	5 45 03.28	8 32 12.41
65	0 00 00.00	2 20 16.25	4 39 00.91	6 54 45.46
70	0 00 00.00	1 49 30.62	3 37 58.10	5 24 20.86
75	0 00 00.00	1 20 37.93	2 40 33.81	3 59 06.26
80	0 00 00.00	0 53 03.99	1 45 42.25	2 37 29.35
85	0 00 00.00	0 26 19.90	0 52 27.60	1 18 10.99
90	0 00 00.00	0 00 00.00	0 00 00.00	0 00 00.00

Latitude	Longitude 20°	Longitude 25°	Longitude 30°	Longitude 35°
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0	90 00 00.00	90 00 00.00	90 00 00.00	90 00 00.00
5	75 42 49.04	78 21 19.77	80 07 08.35	81 21 57.67
10	62 49 56.63	67 26 41.84	70 39 21.09	72 59 04.05
15	52 02 58.82	57 44 29.99	61 55 16.07	65 03 34.21
20	43 20 55.34	49 23 32.51	54 04 15.62	57 43 10.57
25	36 22 57.26	42 18 55.51	47 07 34.85	51 00 59.23
30	30 45 22.52	36 19 39.91	41 01 18.67	44 56 30.75
35	26 08 09.30	31 13 42.52	35 39 08.82	39 26 59.81
40	22 16 00.59	26 50 12.49	30 54 14.23	34 28 33.59
45	18 57 40.71	23 00 10.75	26 40 08.30	29 56 58.81
50	16 04 54.54	19 36 26.40	22 51 11.44	25 48 08.88
55	13 31 36.16	16 33 19.03	19 22 35.01	21 58 16.94
60	11 13 10.68	13 46 19.30	16 10 16.93	18 24 00.28
65	9 06 07.29	11 11 51.87	13 10 54.28	15 02 19.98
70	7 07 40.21	8 47 01.63	10 21 35.61	11 50 38.31
75	5 15 35.33	6 29 23.02	7 39 53.83	8 46 35.09
80	3 28 00.48	4 16 51.64	5 03 40.00	5 48 04.00
85	1 43 18.17	2 07 37.58	2 30 58.03	2 53 08.83
90	0 00 00.00	0 00 00.00	0 00 00.00	0 00 00.00

Transformation from geographic to azimuthal coordinates on the authalic sphere—Center on the Equator—Continued

VALUES OF THE AZIMUTH RECKONED FROM THE NORTH, α; TAN α=SIN λ COT β—Con.

Latitude	Longitude											
	40°			45°			50°			55°		
°	°	′	″	°	′	″	°	′	″	°	′	″
0	90	00	00.00	90	00	00.00	90	00	00.00	90	00	00.00
5	82	17	01.72	82	58	41.99	83	30	49.57	83	55	51.75
10	74	43	34.57	76	03	31.92	77	05	38.90	77	54	19.48
15	67	27	46.63	69	19	56.21	70	48	06.41	71	57	48.02
20	60	35	27.57	62	52	08.94	64	41	12.07	66	08	21.66
25	54	09	51.85	56	42	57.46	58	47	07.33	60	27	37.67
30	48	11	55.92	50	53	43.97	53	07	12.53	54	56	43.31
35	42	40	51.94	45	24	38.22	47	42	00.48	49	36	15.98
40	37	34	44.59	40	14	55.12	42	31	23.91	44	26	25.83
45	32	51	01.91	35	23	12.44	37	34	45.56	39	26	59.76
50	28	26	57.50	30	47	45.71	32	51	01.88	34	37	26.03
55	24	19	44.43	26	26	39.48	28	19	00.71	29	56	58.77
60	20	26	43.21	22	17	54.87	23	57	17.14	25	24	41.88
65	16	45	24.78	18	19	34.27	19	44	22.83	20	59	32.33
70	13	13	31.73	14	29	43.76	15	38	47.86	16	40	22.61
75	9	48	57.26	10	46	34.05	11	39	02.41	12	26	02.51
80	6	29	43.59	7	03	20.36	7	43	37.60	8	15	20.37
85	3	13	59.89	3	33	21.76	3	51	05.71	4	07	03.77
90	0	00	00.00	0	00	00.00	0	00	00.00	0	00	00.00

Latitude	Longitude											
	60°			65°			70°			75°		
°	°	′	″	°	′	″	°	′	″	°	′	″
0	90	00	00.00	90	00	00.00	90	00	00.00	90	00	00.00
5	84	15	26.15	84	30	39.30	84	42	17.28	84	50	52.17
10	78	32	32.06	79	02	20.00	79	25	09.70	79	42	01.74
15	72	52	51.63	73	36	00.56	74	09	10.95	74	33	45.47
20	67	17	48.02	68	12	33.73	68	54	51.94	69	26	18.92
25	61	48	29.29	62	52	44.39	63	42	39.63	64	19	55.90
30	56	25	46.54	57	37	08.01	58	32	56.11	59	14	48.04
35	51	10	12.81	52	26	09.69	53	25	58.00	54	11	04.40
40	46	02	03.65	47	20	04.03	48	21	56.07	49	08	51.27
45	41	01	18.59	42	18	55.40	43	20	55.22	44	08	12.07
50	36	07	43.24	37	22	38.76	38	22	54.61	39	09	07.23
55	31	20	51.56	32	31	00.60	33	27	48.02	34	11	34.39
60	26	40	08.24	27	43	40.18	28	35	24.38	29	15	28.46
65	22	04	50.72	23	00	10.68	23	45	28.27	24	20	41.87
70	17	34	11.10	18	20	00.35	18	57	40.64	19	27	04.89
75	13	07	17.55	13	42	33.59	14	11	39.34	14	34	25.90
80	8	43	15.57	9	07	11.90	9	26	59.86	9	42	31.75
85	4	21	08.85	4	33	14.67	4	43	15.90	4	51	08.14
90	0	00	00.00	0	00	00.00	0	00	00.00	0	00	00.00

Transformation from geographic to azimuthal coordinates on the authalic sphere—Center on the Equator—Continued

VALUES OF THE AZIMUTH RECKONED FROM THE NORTH, α; TAN α=SIN λ COT β—Con.

Latitude	Longitude		
	80°	85°	90°
°	° ′ ″	° ′ ″	° ′ ″
0	90 00 00.00	90 00 00.00	90 00 00.00
5	84 56 45.93	85 00 12.84	85 01 20.94
10	79 53 37.85	80 00 25.29	80 02 39.44
15	74 50 41.56	75 00 36.97	75 03 53.12
20	69 48 02.46	70 00 47.51	70 04 59.75
25	64 45 45.42	65 00 56.61	65 05 57.31
30	59 43 54.62	60 01 03.98	60 06 44.06
35	54 42 33.37	55 01 09.41	55 07 18.56
40	49 41 44.09	50 01 12.72	50 07 39.76
45	44 41 28.20	45 01 13.81	45 07 47.01
50	39 41 46.08	40 01 12.67	40 07 40.07
55	34 42 37.12	35 01 09.32	35 07 19.14
60	29 43 59.67	30 01 03.87	30 06 44.83
65	24 45 51.17	25 00 56.48	25 05 58.20
70	19 48 08.21	20 00 47.38	20 05 00.63
75	14 50 46.61	15 00 36.85	15 03 53.90
80	9 53 41.60	10 00 25.20	10 02 40.02
85	4 56 47.93	5 00 12.80	5 01 21.25
90	0 00 00.00	0 00 00.00	0 00 00.00

VALUES OF THE GREAT CIRCLE DISTANCE FROM THE CENTER, ζ; COS ζ = COS λ COS β

Latitude	Longitude			
	0°	5°	10°	15°
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0	0 00 00.00	5 00 00.00	10 00 00.00	15 00 00.00
5	4 58 39.06	7 03 02.61	11 09 32.44	15 47 10.62
10	9 57 20.56	11 07 45.90	14 04 30.01	17 56 23.32
15	14 56 06.88	15 43 54.81	17 54 37.36	21 02 44.30
20	19 55 00.25	20 30 36.05	22 11 41.17	24 44 55.40
25	24 54 02.69	25 21 58.90	26 42 51.78	28 49 14.79
30	29 53 15.94	30 15 53.83	31 22 08.83	33 07 37.38
35	34 52 41.44	35 11 23.11	36 06 29.20	37 35 17.17
40	39 52 20.24	40 07 57.35	40 54 11.92	42 09 20.37
45	44 52 12.99	45 05 19.95	45 44 17.19	46 47 57.10
50	49 52 19.93	50 03 20.64	50 36 07.25	51 29 54.89
55	54 52 40.86	55 01 52.43	55 29 16.83	56 14 23.94
60	59 53 15.17	60 00 50.10	60 23 27.98	61 00 48.54
65	64 54 01.80	65 00 09.31	65 18 27.24	65 48 41.94
70	69 54 59.37	69 59 46.27	70 14 03.94	70 37 43.31
75	74 56 06.10	74 59 37.34	75 10 09.10	75 27 35.57
80	79 57 19.98	79 59 39.00	80 06 34.90	80 18 04.22
85	84 58 38.75	84 59 47.73	85 03 14.13	85 08 56.36
90	90 00 00.00	90 00 00.00	90 00 00.00	90 00 00.00

Transformation from geographic to azimuthal coordinates on the authalic sphere—Center on the Equator—Continued

VALUES OF THE GREAT CIRCLE DISTANCE FROM THE CENTER, ζ; $\cos \zeta = \cos \lambda \cos \beta$—Con.

Latitude	Longitude											
	20°			25°			30°			35°		
°	°	′	″	°	′	″	°	′	″	°	′	″
0	20	00	00.00	25	00	00.00	30	00	00.00	35	00	00.00
5	20	35	07.61	25	27	33.93	30	22	19.77	35	18	26.67
10	22	14	58.95	26	47	25.91	31	27	43.99	36	12	50.05
15	24	46	36.27	28	52	23.44	33	11	58.20	37	40	33.73
20	27	56	02.16	31	33	32.94	35	29	20.30	39	37	55.09
25	31	31	59.46	34	42	31.87	38	13	52.86	42	00	43.88
30	35	26	26.37	38	12	28.39	41	20	10.38	44	44	53.53
35	39	33	56.30	41	58	08.18	44	43	39.23	47	46	40.09
40	43	50	49.60	45	55	38.77	48	20	39.42	51	02	59.19
45	48	14	36.75	50	02	09.72	52	08	17.94	54	30	41.81
50	52	43	33.73	54	15	35.36	56	04	19.67	58	08	01.27
55	57	16	26.15	58	34	21.74	60	06	58.71	61	52	58.74
60	61	52	19.27	62	57	17.23	64	14	51.16	65	44	03.73
65	66	30	31.36	67	23	25.87	68	26	49.44	69	40	00.85
70	71	10	29.62	71	52	02.80	72	41	58.04	73	39	46.45
75	75	51	47.19	76	22	30.83	76	59	30.10	77	42	25.57
80	80	34	01.28	80	54	18.17	81	18	44.96	81	47	09.75
85	85	16	51.73	85	26	56.58	85	39	06.20	85	53	14.96
90	90	00	00.00	90	00	00.00	90	00	00.00	90	00	00.00

Latitude	Longitude											
	40°			45°			50°			55°		
°	°	′	″	°	′	″	°	′	″	°	′	″
0	40	00	00.00	45	00	00.00	50	00	00.00	55	00	00.00
5	40	15	24.55	45	12	56.40	50	10	51.84	55	09	04.16
10	41	01	03.02	45	51	23.07	50	43	12.64	55	36	06.91
15	42	15	16.08	46	54	15.44	51	36	19.74	56	20	39.80
20	43	55	36.60	48	19	55.05	52	49	06.67	57	21	57.74
25	45	59	10.26	50	06	20.58	54	20	09.02	58	39	02.09
30	48	22	54.68	52	11	19.07	56	07	50.86	60	10	44.25
35	51	03	53.65	54	32	35.51	58	10	30.75	61	55	49.21
40	53	59	25.59	57	07	59.73	60	26	26.64	63	52	58.77
45	57	07	07.36	59	55	30.58	62	53	59.42	66	00	54.10
50	60	24	54.63	62	53	17.91	65	31	35.21	68	18	17.72
55	63	51	00.61	65	59	42.97	68	17	46.40	70	43	54.71
60	67	23	53.69	69	13	17.74	71	11	12.05	73	16	33.41
65	71	02	14.93	72	32	43.72	74	10	37.51	75	55	05.66
70	74	44	55.66	75	56	50.56	77	14	53.87	78	38	26.71
75	78	30	55.17	79	24	34.46	80	22	57.05	81	25	34.85
80	82	19	18.83	82	54	56.84	83	33	46.94	84	15	30.96
85	86	09	16.30	86	27	02.82	86	46	26.34	87	07	17.94
90	90	00	00.00	90	00	00.00	90	00	00.00	90	00	00.00

Transformation from geographic to azimuthal coordinates on the authalic sphere—Center on the Equator—Continued.

VALUES OF THE GREAT CIRCLE DISTANCE FROM THE CENTER, ζ; $\cos\zeta = \cos\lambda\cos\beta$—Con.

Latitude	Longitude 60°	Longitude 65°	Longitude 70°	Longitude 75°
°	° ′ ″	° ′ ″	° ′ ″	° ′ ″
0	60 00 00.00	65 00 00.00	70 00 00.00	75 00 00.00
5	60 07 28.82	65 05 02.58	70 04 43.05	75 03 28.40
10	60 29 48.79	65 24 06.00	70 18 49.37	75 13 51.80
15	61 06 40.88	65 53 57.56	70 42 10.59	75 31 04.86
20	61 57 34.49	66 35 16.62	71 14 32.99	75 54 58.75
25	63 01 49.08	67 27 35.40	71 55 37.86	76 25 21.33
30	64 18 36.05	68 30 19.92	72 45 01.95	77 01 57.29
35	65 47 00.79	69 42 51.11	73 42 18.00	77 44 28.42
40	67 26 04.58	71 04 25.88	74 46 55.27	78 32 33.79
45	69 14 46.34	72 34 18.12	75 58 20.12	79 25 50.11
50	71 12 04.00	74 11 39.64	77 15 56.52	80 23 51.89
55	73 16 55.60	75 55 40.94	78 39 06.54	81 26 11.79
60	75 28 19.99	77 45 31.79	80 07 10.77	82 32 20.84
65	77 45 17.29	79 40 21.70	81 39 28.69	83 41 48.68
70	80 06 49.08	81 39 20.23	83 15 18.97	84 54 03.85
75	82 31 58.39	83 41 37.13	84 53 59.67	86 08 33.93
80	84 59 49.58	85 46 22.47	86 34 48.49	87 24 45.80
85	87 29 28.06	87 52 46.57	88 17 02.84	88 42 05.82
90	90 00 00.00	90 00 00.00	90 00 00.00	90 00 00.00

Latitude	Longitude 80°	Longitude 85°	Longitude 90°
°	° ′ ″	° ′ ″	° ′ ″
0	80 00 00.00	85 00 00.00	90 00 00.00
5	80 02 17.15	85 01 08.05	90 00 00.00
10	80 09 07.54	85 04 31.72	90 00 00.00
15	80 20 28.03	85 10 09.55	90 00 00.00
20	80 36 13.36	85 17 59.10	90 00 00.00
25	80 56 16.29	85 27 56.99	90 00 00.00
30	81 20 27.58	85 39 58.87	90 00 00.00
35	81 48 36.10	85 53 59.49	90 00 00.00
40	82 20 28.92	86 09 52.66	90 00 00.00
45	82 55 51.40	86 27 31.36	90 00 00.00
50	83 34 27.35	86 46 47.71	90 00 00.00
55	84 15 59.07	87 07 33.06	90 00 00.00
60	85 00 07.59	87 29 38.00	90 00 00.00
65	85 46 32.74	87 52 52.50	90 00 00.00
70	86 34 53.35	88 17 05.91	90 00 00.00
75	87 24 47.36	88 42 07.08	90 00 00.00
80	88 15 52.04	89 07 44.44	90 00 00.00
85	89 07 44.12	89 33 46.12	90 00 00.00
90	90 00 00.00	90 00 00.00	90 00 00.00

Functions of authalic latitude for map of world on proposed equal-area projection

Latitude	β	$\frac{\beta}{2}$	$2 \sin \frac{\beta}{2}$	$\cos \frac{\beta}{2}$
°	° ′ ″	° ′ ″		
0	0 00 00.000	0 00 00.0000	0.00000000	1.00000000
10	9 57 20.561	4 58 40.2805	0.17354142	0.99622831
20	19 55 00.252	9 57 30.1260	0.34586512	0.93493366
30	29 53 15.944	14 56 37.9720	0.51574568	0.96617386
40	39 52 20.236	19 56 10.1180	0.68194530	0.94007322
50	49 52 19.929	24 56 09.9645	0.84321448	0.90677854
60	59 53 15.166	29 56 37.5830	0.99829978	0.86651566
70	69 54 59.367	34 57 29.6835	1.14595864	0.81956982
80	79 57 19.983	39 58 39.9915	1.28498084	0.76629372
90	90 00 00.000	45 00 00.0000	1.41421356	0.70710678

Latitude	$\cos \beta$	$\cos \beta / \cos \frac{\beta}{2}$	$\tan \frac{\beta}{2}$
°			
0	1.00000000	1.00000000	0.00000000
10	0.98494169	0.98867065	0.08709923
20	0.94018865	0.95457053	0.17557788
30	0.86700320	0.89735269	0.26689968
40	0.76747532	0.81639951	0.36270861
50	0.64449466	0.71075200	0.46495062
60	0.50169877	0.57898408	0.57604254
70	0.34338038	0.41898735	0 69912203
80	0.17441212	0.22760479	0.83843884
90	0.00000000	0.00000000	1.00000000

Table for an equivalent projection with Lambert, azimuthal, meridional, projection spacing on the central meridian and straight line parallels

TABLE COMPUTED WITH AUTHALIC LATITUDES

Latitude	y	x		
		Long. 0°	Long. 10°	Long. 90°
°	*m*	*m*	*m*	*m*
0	0	0	1,111,949	10,007,539
10	1,105,632	0	1,099,351	9,894,160
20	2,203,506	0	1,061,434	9,552,902
30	3,285,814	0	997,810	8,980,292
40	4,344,672	0	907,794	8,170,150
50	5,372,117	0	790,320	7,112,878
60	6,360,165	0	643,801	5,794,206
70	7,300,899	0	465,392	4,193,032
80	8,186,609	0	253,085	2,277,764
90	9,009,951	0	0	0

TABLE OBTAINED BY MULTIPLYING THE ABOVE VALUES BY 1.5 X 10^{-8} FOR A CONSTRUCTION TABLE IN CENTIMETERS (WITH ADDITIONAL COLUMNS)

Latitude	y	x				
		Long. 0°	Long. 10°	Long. 30°	Long. 90°	Long. 180°
°	cm.	cm.	cm.	cm.	cm.	cm.
0	0	0	1, 668	5, 004	15, 011	30, 023
10	1, 658	0	1, 649	4, 947	14, 841	29, 682
20	3, 305	0	1, 592	4, 776	14, 329	28, 659
30	4, 929	0	1, 497	4, 490	13, 470	26, 941
40	6, 517	0	1, 362	4, 085	12, 255	24, 510
50	8, 058	0	1, 185	3, 556	10, 669	21, 339
60	9, 540	0	966	2, 897	8, 691	17, 383
70	10, 951	0	699	2, 096	6, 290	12, 579
80	12, 280	0	380	1, 139	3, 417	6, 833
90	13, 515	0	0	0	0	0

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