Serial No. 92

DEPARTMENT OF COMMERCE
U. S. COAST AND GEODETIC SURVEY
E. LESTER JONES, SUPERINTENDENT

CARTOGRAPHY

GENERAL THEORY OF THE LAMBERT CONFORMAL CONIC PROJECTION

BY

OSCAR S. ADAMS
Geodetic Computer
United States Coast and Geodetic Survey

Special Publication No. 53

PRICE, 10 CENTS
Sold only by the Superintendent of Documents, Government Printing Office
Washington, D. C.

WASHINGTON
GOVERNMENT PRINTING OFFICE
1918

National Oceanic and Atmospheric Administration

ERRATA NOTICE

One or more conditions of the original document may affect the quality of the image, such as:

Discolored pages
Faded or light ink
Binding intrudes into the text

This has been a co-operative project between the NOAA Central Library and the Climate Database Modernization Program, National Climate Data Center (NCDC). To view the original document, please contact the NOAA Central Library in Silver Spring, MD at (301) 713-2607 x124 or www.reference@nodc.noaa.gov.

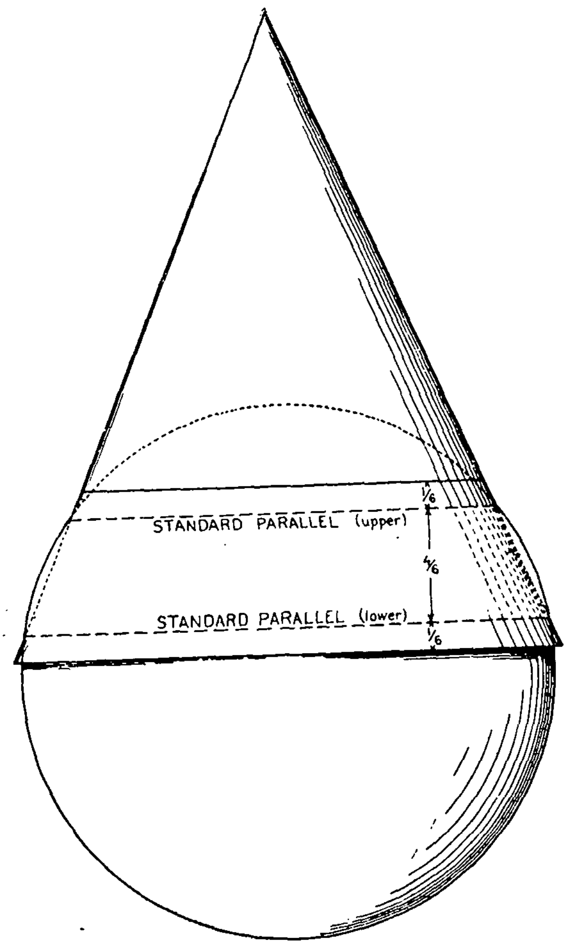
PREFACE.

This publication gives a general development of the theory of the Lambert conformal conic projection. It is intended to supplement the matter found in Special Publication No. 47 entitled, "The Lambert Conformal Conic Projection with Two Standard Parallels." It is also supplementary in a way to Special Publication No. 49, which contains the Lambert projection tables for the region in France, and to Special Publication No. 52, which gives corresponding tables for the United States, since it gives as a whole the mathematical development of the theory upon which they depend.

A short account of Lambert's life and work is given in the introductory paragraphs, followed by a few pages upon the subject of projections in general.

ILLUSTRATIONS.

57341°—18



Frontispiece.—Intersection of a cone and sphere along two standard parallels.

GENERAL THEORY OF THE LAMBERT CONFORMAL CONIC PROJECTION.

By Oscar S. Adams,
Geodetic Computer, U. S. Coast and Geodetic Survey.

Since this publication is to treat of some of Lambert's work, it is altogether fitting that it should be prefaced with a short account of his life, especially with a statement of his significance in the domain of map projections. Johann Heinrich Lambert was born at Mülhausen in Alsace in 1728. He was the son of a poor tailor and his education was entirely the product of his own exertions, due to a systematic course of reading. It was his regular custom to spend 17 hours per day in study and writing. At the early age of 16 he discovered, in computations for the comet of 1744, the so-called Lambert's theorem. During the latter part of his life he resided in Berlin, where he was much honored for his ability. It was in the application of mathematical analysis to the practical problems of life that he especially excelled. His untimely death occurred in 1777 in the forty-ninth year of his age.

His contributions to mathematics were the series which bears his name, the conception of hyperbolic functions, his theorem in conics, and the demonstration of the incommensurability of π. Both Lagrange and Gauss used parts of his work as points of departure for their investigations.

Lambert's work in the field of map projections needs careful consideration. He was the first mathematician to make general investigations upon the subject of projections. Those who preceded him in this work limited themselves to the development of a single method of projection, principally the perspective, but Lambert considered the problem of the representation of a sphere upon a plane from a higher standpoint and he stated certain general conditions that the representation was to fulfil, the most important of these being the preservation of angles or conformality and equal surface or equivalence. These two qualities, of course, can not be obtained in the same projection.

Although Lambert did not fully develop the theory of *these two methods of projection, yet he was the first to* express clearly the ideas regarding them. The former, conformality, has become of the greatest importance to pure mathematics, but both of them are of exceeding importance to the cartographer. It is no more than just, therefore, to date the beginning of a new epoch in the science of map making from the appearance of Lambert's work. What he accomplished is of importance, not only for the generality of the ideas underlying it, but also for his successful application of them in methods of projection. The manner in which he attacks and solves any particular problem is very instructive. He has developed several methods of projection that are not only interesting but that are to-day in use among cartographers, the most important of these being the conformal conic projection.

The initial problem presented to us in map making is the representation upon a plane surface of the relative positions, sizes, and shapes of features that are found upon the curved surface of the earth. A perfect representation is impossible, since the surface of the earth being nondevelopable, can not be spread out in a plane. There are, however, many different ways of obtaining approximate representations, the theory and properties of which constitute the subject of map projections.

The positions of the points upon the earth are usually defined by their latitude and longitude. Hence, if we can devise a suitable method of representing the meridians and parallels upon the sheet, the points can be plotted by their positions relative to these lines and the map can be constructed. The term "projection" is evidently used in a wider sense than that which is given to it in geometry. The majority of map projections are not projections in the geometrical sense—that is, perspective projections, orthogonal projections, etc.—but merely a network of meridians and parallels that makes possible a one-to-one correspondence between the places upon the earth and the points upon the map.

Some of the things to be desired in a map are as follows:

1. Preservation of the shapes of the countries.

2. Preservation of the relative sizes of the countries in their representation upon the map.

3. The distance between places should be in constant ratio to their distances as indicated upon the map.

4. A great circle upon the earth should be represented by a straight line upon the map.

5. The latitude and longitude of any place should be readily found from its position upon the map.

6. The ease with which a projection can be constructed is also to be considered from the practical standpoint.

Only part of these things can be attained by any given method of projection.

The *scale of a map* in any given direction is the ratio which a short distance measured upon the map bears to the corresponding distance upon the surface of the earth. The definition is limited to short distances because the scale of a map generally varies from point to point. It would of course be desirable that the scale of the map should be correct in every direction at every point and constant for all parts of the map. This is impossible, however, since if it were true, the map would be a perfect representation of the spheroidal surface and could be fitted to it. In any given method of projection, therefore, some of the features to be desired must be sacrificed.

The representation of the shapes of countries as nearly correct as possible is one of the most important functions of a map. A large country can not, of course, be represented without some distortion when considered as a whole, but small areas may be mapped by similar figures. A projection that preserves the similarity of small areas is called orthomorphic. The representation of one surface upon another so as to preserve similarity of elements has been called by mathematicians conformal representation. An orthomorphic projection is therefore a conformal representation of the spheroidal surface of the earth upon a plane. Orthomorphic projections are in general not of much use in map making unless the meridians are straight lines.

The Lambert conformal conic projection fulfills this requirement, and it has lately been brought into especial notice by the fact that it is the projection that is used for the battle maps in France. In this projection, as is the case with any conic-projection, the intersecting cone gives a better map than is given by the tangent cone. In the intersecting cone we have a shortening of the scale between the standard parallels and a lengthening of the same at the top and bottom of the map.

The following is a somewhat simplified development of the mathematical theory of the projection, including a determination of the necessary and sufficient conditions for a conformal mapping. Use is made of the tangent cone only in order to determine the two limiting cases of the projection.

It is well known that a plane curve can be expressed in parametric form in such a way that $x=\phi(t)$ and $y=\psi(t)$, ϕ and ψ being some known functions of the variable t. Thus the straight line passing through the point (a, b) with direction cosines α and β can be expressed in the form

$$x=a+\alpha t,\ y=b+\beta t.$$

A circle can be given in the form

$$x=a \cos t,\ y=a \sin t.$$

A space or skew curve can be similarly expressed as

$$x=\phi(t),\ y=\psi(t),\ z=\chi(t).$$

Such a curve may degenerate to a plane curve under certain conditions; the obvious one is that one of the functions is identically zero.

A straight line in space becomes

$$x=a+\alpha t,\ y=b+\beta t,\ z=c+\gamma t.$$

A circular helix is given by the equations

$$x=a \cos t,\ y=a \sin t,\ z=bt.$$

When the coordinates are given in terms of functions of two variable parameters, the locus becomes a surface except under certain conditions that will be specified later. Thus, the tangent surface of a skew curve expressed by the parametric equations $x=\phi(t)$, $y=\psi(t)$, and $z=\chi(t)$ is given by the equations

$$x=\phi(t)+s\phi'(t),\ y=\psi(t)+s\psi'(t),\ z=\chi(t)+s\chi'(t),$$

the primes denoting differentiation with respect to t. By employing s and t as independent variables, the surface given is that swept out by a tangent to the curve as it moves along the curve. The surface of a sphere is given by the equations

$$x=a \sin t \cos s,\ y=a \sin t \sin s, \text{ and } z=a \cos t.$$

To avoid some of the difficult questions connected with the theory of curves and surfaces, the functions that will be considered in this article will be limited to domains in which they are single valued, finite, continuous, and differentiable. These limitations are permissible in the cases to be considered. Thus, in the equations of the sphere the variables are limited to the domains

$$-\frac{\pi}{2}<t<+\frac{\pi}{2} \text{ and } -\pi<s<+\pi.$$

In regard to the functions three cases may occur. In the first place all of the functions may be constant, thus giving a single point. In the second place, the functions may not be constant, but any two of them may be functions of the third; this would give a curve and not a surface since the three functions could then be expressed in terms of some single variable, η, and the equations would become

$$x=X(\eta),\ y=Y(\eta), \text{ and } z=Z(\eta).$$

As the third possibility, at least two of the functions are independent of each other. In this case the locus represented is a surface. Of course any two may be independent, but all that is necessary is that at least two bear this relation to each other.

Let ϕ and ψ be the ones that are thus independent. This requires that their Jacobian is not identically zero, or

$$\begin{vmatrix} \frac{\partial\phi}{\partial u}, & \frac{\partial\phi}{\partial v} \\ \frac{\partial\psi}{\partial u}, & \frac{\partial\psi}{\partial v} \end{vmatrix} \neq 0$$

It should be noted that a parametric representation of a surface is not unique. If such a representation has been found any number of other forms can be obtained by setting the u and v of the first form equal to two arbitrary independent functions of two other variables s and t. Thus the equations may be $u=\lambda(s, t)$ and $v=\mu(s, t)$ with the condition

$$\begin{vmatrix} \frac{\partial \lambda}{\partial s}, & \frac{\partial \lambda}{\partial t} \\ \frac{\partial \mu}{\partial s}, & \frac{\partial \mu}{\partial t} \end{vmatrix} \neq 0$$

If one of the variables is held constant while the other varies, a curve is traced out upon the surface. Thus there is a set of curves for $u=$ constant, and another set for $v=$ constant. These curves are called the parametric curves and u and v are called the curvilinear coordinates. By a change of coordinates as indicated above a set of curves can be determined such that the curves, $u=$ constant, are everywhere perpendicular to the curves, $v=$ constant. In the equations of the sphere, $s=$ constant gives a meridian great circle and $t=$ constant gives a circle of latitude or a parallel. In this case the curves are perpendicular or orthogonal. In a more general way any functional relation between the variables u and v determines a curve upon the surface. This can be expressed in the form of one coordinate as a function of the other, as $v=g(u)$, or in the form $F(u, v)=0$. As a further consideration any ordinary differential equation of the first order between u and v determines a set of ∞^1 curves upon the surface, since the integration of the same would give a functional relation between u and v containing one arbitrary constant.

The element of length upon a surface is given by the equation

$$dS^2=dx^2+dy^2+dz^2.$$

But

$$dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv$$

$$dy=\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv$$

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

hence

$$dS^2=\left[\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2\right]du^2$$
$$+2\left(\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)du\,dv$$
$$+\left[\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2\right]dv^2.$$

To abbreviate the expression it is customary to use the symbols

$$E=\left(\frac{\partial x}{\partial u}\right)^2+\left(\frac{\partial y}{\partial u}\right)^2+\left(\frac{\partial z}{\partial u}\right)^2$$
$$F=\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v}+\frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v}+\frac{\partial z}{\partial u}\,\frac{\partial z}{\partial v}$$
$$G=\left(\frac{\partial x}{\partial v}\right)^2+\left(\frac{\partial y}{\partial v}\right)^2+\left(\frac{\partial z}{\partial v}\right)^2.$$

With these the equation for the element of length becomes

$$dS^2=E\,du^2+2F\,du\,dv+G\,dv^2$$

If v is held constant, the element of length becomes

$$dS^2=E\,du^2,\ \text{since}\ dv=0$$

If the direction is taken positive in which u is increasing, we find

$$dS_v=\sqrt{E}\,du$$

Similarly for $u=$ constant, $dS_u=\sqrt{G}\,dv$

If v is constant $dx=\frac{\partial x}{\partial u}du$, $dy=\frac{\partial y}{\partial u}du$, and $dz=\frac{\partial z}{\partial u}du$.

The direction cosines of the tangent to the curve $v=$ constant are given by the equations $\alpha=\frac{dx_v}{dS_v}$, etc.

These become, $\alpha=\frac{1}{\sqrt{E}}\,\frac{\partial x}{\partial u}$, $\beta=\frac{1}{\sqrt{E}}\,\frac{\partial y}{\partial u}$, and $\gamma=\frac{1}{\sqrt{E}}\,\frac{\partial z}{\partial u}$.

Similarly the direction cosines of the tangent to the curve $u=$ constant are given in the form

$$\alpha'=\frac{1}{\sqrt{G}}\,\frac{\partial x}{\partial v},\quad \beta'=\frac{1}{\sqrt{G}}\,\frac{\partial y}{\partial v},\quad \text{and}\quad \gamma'=\frac{1}{\sqrt{G}}\,\frac{\partial z}{\partial v}$$

If θ is the angle between these two tangents or the angle between the curves, we have from the known relation,

$$\cos\theta = \alpha\alpha' + \beta\beta' + \gamma\gamma'$$

$$= \frac{1}{\sqrt{EG}}\left(\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}\right)$$

$$= \frac{F}{\sqrt{EG}}.$$

It follows, then, that the parametric curves u and v upon a surface on which F is identically zero form an orthogonal system unless the $\sqrt{EG}$ is identically zero at the same time. Since when F is zero neither E nor G is zero for the usual real surfaces, it follows that, when $F=0$, $\theta=\frac{\pi}{2}$.

We shall now determine the conditions that are necessary and sufficient to permit a *conformal representation* of a surface upon a plane. One surface is said to be conformally represented upon another when any infinitely small section of the first surface is represented by a similar figure upon the other.

If a surface is to be represented upon a plane according to some law and x, y, z are the rectangular coordinates of the surface and u, v the rectangular coordinates of the plane, then for every pair of coordinates u, v in the plane a set of coordinates x, y, z of the surface must correspond; that is x, y, and z must be functions of u and v.

We may then consider that the surface with the parameters u, v is to be represented on the plane with the rectangular coordinates u, v. The question then becomes under what conditions such a representation is conformal. Let us suppose that (u, v), $(u+du, v+dv)$ and $(u+\delta u, v+\delta v)$ are three points upon the surface forming an infinitesimal triangle. This will be represented upon the plane by the triangle (u, v), $(u+du, v+dv)$ and $(u+\delta u, v+\delta v)$. These two triangles must be similar but since their positions relative to each other do not come into consideration, no account is to be taken of signs either of sides or angles. Two things are then required, first that the sides of the *triangle extending from the point* (u, v) of the surface must be *in the same ratio* as their representations in the plane extending from the point (u, v) of the plane, and secondly the angle included between the sides of the triangle upon the surface must be equal to the angle included by their images in the plane. That is, the square of the elements of length upon the surface must be proportional to the square of the elements in *the plane and* the cosine of the angle between the lines *upon* the surface must equal *the* cosine *of the corresponding* angle in the plane.

But

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2.$$

$$\delta s^2 = E\,\delta u^2 + 2\,F\,\delta u\,\delta v + G\,\delta v^2.$$

The quantities E, F, and G would be the same in each case. They may be constants or functions of u *and* v but since

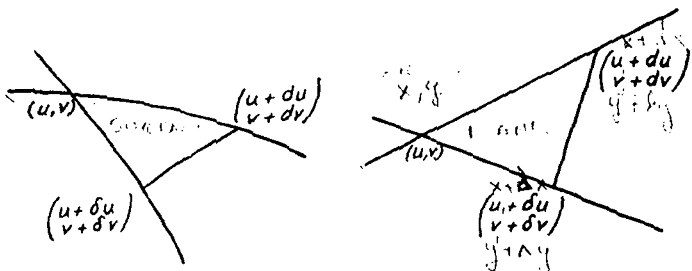


FIG. 1.—Surface triangle and its representation upon a plane.

each curve is *infinitesimal* and starts from the same point (u, v), they would be constants for this point. On the other hand, in the plane, we have

$$ds^2 = du^2 + dv^2$$

$$\delta s^2 = \delta u^2 + \delta v^2.$$

The first condition is *therefore* that the proportion

$$\frac{E\,du^2 + 2\,F\,du\,dv + G\,dv^2}{E\,\delta u^2 + 2\,F\,\delta u\,\delta v + G\,\delta v^2} = \frac{du^2 + dv^2}{\delta u^2 + \delta v^2}$$

may be identically true. This evidently requires that $E = G$ and $F = 0$. Thus the parametric curves must be orthogonal.

The cosine of the angle *between the* curves upon the surface must now *be determined*. The direction cosines of the tangent at the point (u, v) of the curve dS upon the surface are determined by

$$\frac{dx}{dS}, \frac{dy}{dS}, \frac{dz}{dS}$$

or

$$\frac{\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv}{\sqrt{E\, du^2 + 2\, F\, du\, dv + G\, dv^2}}, \text{ etc.}$$

Also the direction cosines of the tangent at the point (u, v) to the curve δS upon the surface are in like manner

$$\frac{\frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v}{\sqrt{E\, \delta u^2 + 2\, F\, \delta u\, \delta v + G\, \delta v^2}}, \quad \frac{\frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v}{\sqrt{E\, \delta u^2 + 2\, F\, \delta u\, \delta v + G\, \delta v^2}}, \text{ etc.}$$

Consequently, if α is the angle between the curves, we have

$$\cos\alpha = \frac{\left\{\begin{matrix}\left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv\right)\left(\frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v\right) + \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv\right) \\ \left(\frac{\partial y}{\partial u}\delta u + \frac{\partial y}{\partial v}\delta v\right) + \left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv\right)\left(\frac{\partial z}{\partial u}\delta u + \frac{\partial z}{\partial v}\delta v\right)\end{matrix}\right\}}{\sqrt{(E du^2 + 2\, F du dv + G dv^2)\, (E\delta u^2 + 2\, F\delta u\, \delta v + G\delta v^2)}}$$

or

$$\cos\ \alpha = \frac{E\, du\, \delta u + F\, (du\, \delta v + \delta u\, dv) + G\, dv\, \delta v}{\sqrt{(E\, du^2 + 2\, F\, du\, dv + G\, dv^2)\, (E\, \delta u^2 + 2\, F\, \delta u\, \delta v + G\, \delta v^2)}}$$

In the plane the corresponding direction cosines are

$$\frac{du}{\sqrt{du^2 + dv^2}}, \quad \frac{dv}{\sqrt{du^2 + dv^2}}$$

and

$$\frac{\delta u}{\sqrt{\delta u^2 + \delta v^2}}, \quad \frac{\delta v}{\sqrt{\delta u^2 + \delta v^2}}$$

If A is the angle between these two lines, we have

$$\cos\ A = \frac{du\ \delta u + dv\ \delta v}{\sqrt{(du^2 + dv^2)\ (\delta u^2 + \delta v^2)}}$$

The second requirement is, therefore, that the equation

$$\frac{E\ du\ \delta u + F\ (du\ \delta v + \delta u\ dv) + G\ dv\ \delta v}{\sqrt{(E\ du^2 + 2\ F\ du\ dv + G\ dv^2)\ (E\ \delta u^2 + 2\ F\ \delta u\ \delta v + G\ \delta v^2)}}$$
$$= \frac{du\ \delta u + dv\ \delta v}{\sqrt{(du^2 + dv^2)\ (\delta u^2 + \delta v^2)}}$$

should be identically true. This requires as before that $E=G$, and $F=0$. Thus of the two conditions only one is necessary and one is also sufficient since each leads to the same results. It is thus seen that the linear element upon the surface must be such that it can be expressed in the form $dS^2=m^2\ (du^2+dv^2)$.

If we have a surface with the linear element $dS^2=Vdu^2+Udv^2$ in which V is a function of v alone and U is a function of u alone, it can be expressed in the required form. Let us put it in the form $dS^2=UV\left(\frac{du^2}{U}+\frac{dv^2}{V}\right)$. By changing the parameters, we may let $\theta=\int\frac{du}{\sqrt{U}}$ and $\eta=\int\frac{dv}{\sqrt{V}}$. The linear element then becomes $dS^2=UV\ (d\theta^2+d\eta^2)$.

When the linear element is in this form, the surface is said to be expressed in terms of an isothermal orthogonal system of parameters and the net of u, v curves is said to form an isothermal orthogonal net. The surface is divided by them into an ∞^2 group of infinitesimal squares.

After the surface has been expressed in this manner in terms of isothermal orthogonal coordinates, the general conformal representation of the surface upon a plane can be determined. This at the same time determines the general conformal mapping of the u, v plane upon another plane with point to point correspondence. This general representation is given by the equation $x+iy=f\ (u+iv)$, in which i denotes as usual $\sqrt{-1}$ and $f\ (u+iv)$ is any analytic function of the complex variable $(u+iv)$. The element of length $dS^2=m^2(du^2+dv^2)$ is represented in the plane by $ds^2=dx^2+dy^2$. Let the curve C be represented by the curve C_1 in the plane. If θ represents the angle which the curve C makes with the curve $v=$constant, we will have $dS_v=mdu$ and

$$\cos\theta=\frac{dS_v}{dS}=\frac{du}{\sqrt{du^2+dv^2}};$$

$$\sin\theta=\frac{dv}{\sqrt{du^2+dv^2}}$$

If θ_1 is the angle that C_1 makes with the axis of X, we have

$$\cos\theta_1 = \frac{dx}{\sqrt{dx^2+dy^2}}$$

$$\sin\theta_1 = \frac{dy}{\sqrt{dx^2+dy^2}}$$

$$e^{i\theta} = \cos\theta + i\sin\theta = \frac{du+i\,dv}{\sqrt{du^2+dv^2}} = \sqrt{\frac{du+i\,dv}{du-i\,dv}}$$

$$\text{or } e^{2i\theta} = \frac{du+i\,dv}{du-i\,dv}$$

Similarly,

$$e^{i\theta_1} = \cos\theta_1 + i\sin\theta_1 = \frac{dx+i\,dy}{\sqrt{dx^2+dy^2}} = \sqrt{\frac{dx+i\,dy}{dx-i\,dy}}$$

$$\text{or } e^{2i\theta_1} = \frac{dx+i\,dy}{dx-i\,dy}$$

Therefore,

$$e^{2i(\theta-\theta_1)} = \frac{du+i\,dv}{du-i\,dv}\cdot\frac{dx-i\,dy}{dx+i\,dy}$$

$$\text{or } e^{2i(\theta-\theta_1)} = \frac{du+i\,dv}{dx+i\,dy}\cdot\frac{dx-i\,dy}{du-i\,dv}$$

But since $x+iy=f(u+iv)$, so also will $x-iy=f(u-iv)$.

By differentiation, we have

$$dx+idy = (du+idv)f'(u+iv)$$

or

$$\frac{dx+idy}{du+idv} = f'(u+iv)$$

and

$$\frac{dx-idy}{du-idv} = f'(u-iv),$$

the primes denoting differentiation with respect to the complex variable. By substituting these values, we have

$$e^{2i(\theta-\theta_1)} = \frac{f'(u-iv)}{f'(u+iv)}.$$

If Γ and Γ_1 are another pair of corresponding curves starting from the same point and their angles are denoted by ϕ and ϕ_1, we should find that $e^{2i(\phi-\phi_1)}$ equaled the same expression. Since this expression is constant for a given point, it follows that

$$e^{2i(\theta-\theta_1)}=e^{2i(\phi-\phi_1)}$$

or

$$\theta-\theta_1=\phi-\phi_1$$

so that

$$\theta-\phi=\theta_1-\phi_1.$$

That is, the angle between the curves upon the surface is preserved in the representation on the plane. If we had made use of the functional relation $x+iy=F\ (u-iv)$ we should find the relation $e^{2i(\theta+\theta_1)}=\dfrac{F'(u-iv)}{F'(u+iv)}$

Therefore in the same way as above

$$\theta+\theta_1=\phi+\phi_1$$

or

$$\theta-\phi=\phi_1-\theta_1.$$

That is, the angles are equal but directed in the inverse sense, a difference that is not taken into account in conformal mapping. The most general conformal mapping is given then by the relations

$$x+iy=f(u+iv)$$

and

$$x+iy=F(u-iv)$$

The two other forms are merely conjugates of these, i. e.,

$$x-iy=f(u-iv)$$

and

$$x-iy=F(u+iv).$$

It is easily shown that these functional relations also preserve the proportionality of lengths. By differentiating the relation $x+iy=f\ (u+iv)$ we find that $dx+idy=(du+idv)\ f'(u+iv)$. By differentiating the conjugate of this, we obtain $dx-idy=(du-idv)f'(u-iv)$.
When these results are multiplied, we obtain

$$(dx)^2-(idy)^2=[(du)^2-(idv)^2]f'(u+iv)f'(u-iv)$$

or

$$dx^2+dy^2=(du^2+dv^2)f'(u+iv)f'(u-iv)$$

But the element of length in the plane has the form

$$ds^2=dx^2+dy^2$$

Also on the surface $dS^2 = m^2 \ (du^2 + dv^2)$
By combining these relations, we find

$$ds^2 = \frac{dS^2}{m^2} f'(u+iv) f'(u-iv)$$

or

$$\frac{ds}{dS} = \frac{\sqrt{f'(u+iv) f'(u-iv)}}{m}$$

The two functions $f'(u+iv)$ and $f'(u-iv)$ are conjugate complex functions and hence their product is real since

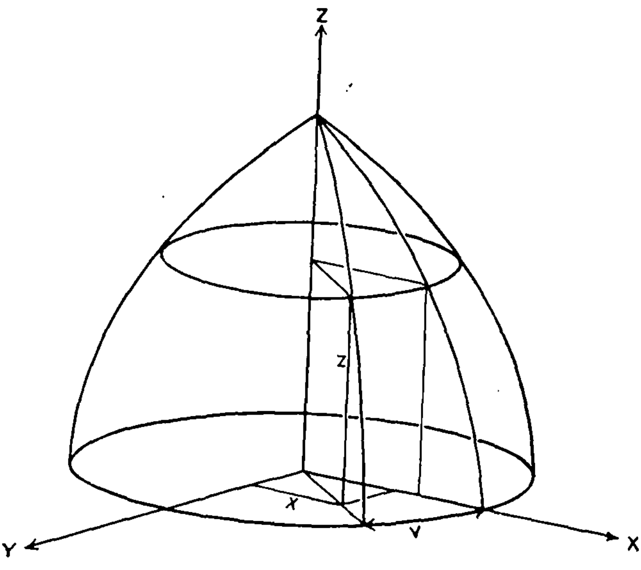


FIG. 2.—Surface of revolution.

both the sum and product of conjugate complex functions are real. The fact that the angle between the curves is held would in itself be sufficient to establish the comformality of the representation, but we shall have need of the above expression which is called the magnification of the representation.

If we have the equation of a plane curve given in the plane in the form $x=u$, $y=0$, $z=f(u)$, a surface of revolution with this curve as a meridian is given by the equations $x=u \cos v$, $y=u \sin v$, $z=f(u)$. For this surface $E=1+[f'(u)]^2$, $F=0$, and $G=u^2$, the prime denoting differentiation with respect to u. The square of the element of length is therefore $dS^2 = \{1 + [f'(u)]^2\}du^2 + u^2dv^2$

or
$$dS^2 = u^2\left\{\frac{1 + [f'(u)]^2}{u^2}du^2 + dv^2\right\}.$$

We can let
$$w = \int \frac{\sqrt{1 + [f'(u)]^2}}{u}du$$

Then $dS^2 = u^2(dw^2 + dv^2)$. The surface is thus expressed in terms of an isothermal orthogonal set of coordinates.

Another test for a conformal mapping is that the functions must satisfy the Cauchy-Riemann partial differential equations. We have

$$x + iy = f(u + iv)$$

$$\frac{\partial x}{\partial u} = f'(u + iv) \quad \text{or} \quad \frac{\partial x}{\partial u} = f'(u + iv)$$

$$\frac{\partial x}{\partial v} = if'(u + iv) \qquad \frac{\partial x}{\partial v} = if'(u + iv)$$

$$i\frac{\partial y}{\partial u} = f'(u + iv) \qquad -\frac{\partial y}{\partial u} = if'(u + iv)$$

$$i\frac{\partial y}{\partial v} = if'(u + iv) \qquad \frac{\partial y}{\partial v} = f'(u + iv).$$

Hence $\dfrac{\partial x}{\partial u} = \dfrac{\partial y}{\partial v}$ and $\dfrac{\partial x}{\partial v} = -\dfrac{\partial y}{\partial u}$.

With the form $x + iy = F(u - iv)$

we obtain
$$\frac{\partial x}{\partial u} = F'(u - iv)$$

$$\frac{\partial x}{\partial v} = -iF'(u - iv)$$

$$i\frac{\partial y}{\partial u} = F'(u - iv).$$

$$i\frac{\partial y}{\partial v} = -iF'(u - iv).$$

Therefore $\dfrac{\partial x}{\partial u} = -\dfrac{\partial y}{\partial v}$ and $\dfrac{\partial x}{\partial v} = \dfrac{\partial y}{\partial u}$.

These are the equations that determine that a given complex function is a function of a complex variable in the accepted sense. The first set preserves the angle between two given arcs in magnitude and sign. The second set results in reversing the angle, or the original angle and the image have opposite signs. Both x and y are functions that satisfy Laplace's equation for two dimensions. Thus with the first set

$$\frac{\partial^2 x}{\partial u^2}=\frac{\partial^2 y}{\partial v \partial u}$$

$$\frac{\partial^2 x}{\partial v^2}=-\frac{\partial^2 y}{\partial u \partial v}.$$

But
$$\frac{\partial^2 y}{\partial v \partial u}=\frac{\partial^2 y}{\partial u \partial v}.$$

Therefore by addition

$$\frac{\partial^2 x}{\partial u^2}+\frac{\partial^2 x}{\partial v^2}=0.$$

Also
$$\frac{\partial^2 y}{\partial u^2}+\frac{\partial^2 y}{\partial v^2}=0.$$

The same results may be found from the second set. From these results we may conclude that the most general conformal representation of one surface upon a plane is given by setting the complex variable in the plane equal to any analytic function of the complex variable formed from the isothermal orthogonal coordinates of the surface, or to any analytic function of the conjugate of this complex variable. In the first case direct equality of angles is found; in the second, the angles are equal but turned in opposite directions.

If the surface to be represented is an ellipsoid of revolution, the parametric equations may be chosen in the following form:

$$x=a \cos M \sin u,\ y=a \sin M \sin u,\ z=b \cos u.$$

a is the semi-major axis, b is the semi-minor axis, M is the longitude and u the complement of the eccentric angle of the generating ellipse or the complement of the reduced latitude.

The element of length upon the spheroid is given by the equation $dS^2 = dx^2 + dy^2 + dz^2$.

But

$$dx = a \cos M \cos u\, du - a \sin M \sin u\, dM;$$
$$dy = a \sin M \cos u\, du + a \cos M \sin u\, dM;$$
$$dz = -b \sin u\, du;$$

hence the element of length becomes $dS^2 = a^2 \sin^2 u\, dM^2 + (a^2 \cos^2 u + b^2 \sin^2 u)\, du^2$. Thus $E = a^2 \sin^2 u$, $F = o$, and $G = a^2 \cos^2 u + b^2 \sin^2 u$. If $\frac{a^2 - b^2}{a^2}$ is put equal to ϵ^2 the equation becomes

$$dS^2 = a^2 \sin^2 u\, [dM^2 + (\cot^2 u + 1 - \epsilon^2)\, du^2].$$

The equation of the generating ellipse is given by

$$x = a \sin u;$$
$$z = b \cos u.$$

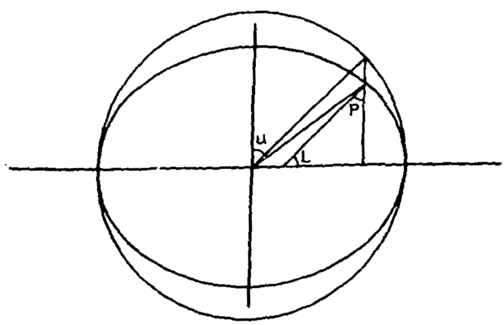


FIG. 3.—Generating ellipse with angles L, p, and u.

The tangent of the colatitude p, is the cotangent of the angle which the normal makes with the X axis. The tangent of the angle which the tangent makes with the X axis is $\frac{dz}{dx}$.

The tangent of the angle which the normal makes with the X axis, or the tangent of the latitude L is equal to $-\frac{dx}{dz}$.

But

$$-\frac{dx}{dz} = \tan L = \frac{a}{b} \cot u$$

or

$$\tan L = \cot p = \frac{a}{b} \cot u$$

hence

$$\frac{b}{a} \tan u = \tan p$$

or

$$\sqrt{1-\epsilon^2} \tan u = \tan p.$$

$$\cos^2 u = \frac{(1-\epsilon^2)\cos^2 p}{1-\epsilon^2 \cos^2 p}, \text{ since } \cos^2 u = \frac{1}{1+\tan^2 u}$$

$$\sin^2 u = \frac{\sin^2 p}{1-\epsilon^2 \cos^2 p}$$

$$\sqrt{1-\epsilon^2} \cdot \frac{du}{\cos^2 u} = \frac{dp}{\cos^2 p}$$

or

$$du = \frac{\sqrt{1-\epsilon^2}.dp}{1-\epsilon^2 \cos^2 p}$$

$$(\cot^2 u + 1 - \epsilon^2)du^2 = \frac{(1-\epsilon^2)^2 dp^2}{(1-\epsilon^2 \cos^2 p)^2 \sin^2 p}$$

hence

$$dS^2 = \frac{a^2 \sin^2 p}{1-\epsilon^2 \cos^2 p}\left[dM^2 + \frac{(1-\epsilon^2)^2 dp^2}{(1-\epsilon^2 \cos^2 p)^2 \sin^2 p}\right]$$

Let

$$d\theta = \frac{(1-\epsilon^2)dp}{(1-\epsilon^2 \cos^2 p)\sin p}$$

then

$$\theta = \int \frac{dp}{\sin p} - \frac{\epsilon}{2}\int \frac{\epsilon \sin p \, dp}{1-\epsilon \cos p} + \frac{\epsilon}{2}\int \frac{-\epsilon \sin p \, dp}{1+\epsilon \cos p}.$$

By integration

$$\theta = \log \tan \frac{p}{2} - \frac{\epsilon}{2} \log (1-\epsilon \cos p) + \frac{\epsilon}{2} \log (1+\epsilon \cos p) + \log G.$$

If the limits of integration are so chosen that G (the constant of integration) becomes equal to unity, we have

$$\theta = \log \left[\tan \frac{p}{2} \cdot \left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\frac{\epsilon}{2}}\right]$$

or

$$e^\theta = \tan \frac{p}{2} \cdot \left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\frac{\epsilon}{2}}$$

The element of length now becomes

$$dS^2 = \frac{a^2 \sin^2 p}{1-\epsilon^2 \cos^2 p}\left(dM^2 + d\theta^2\right).$$

The parameters have been reduced to an isothermal orthogonal system.

We can now determine any number of conformal representations of the spheroid upon the plane. All that is necessary is to make use of the relation

$$x + iy = f\ (M \pm i\theta)$$

f being any analytic function and either combination of signs being used.

With the relation $x+iy=f\ (M-i\theta)$, let $f(v) = Ke^{ilv}$.

We now have

$$x + iy = Ke^{ilM + l \log\left[\tan \frac{p}{2}\cdot\left(\frac{1+\epsilon \cos p}{1-\epsilon\cos p}\right)^{\frac{\epsilon}{2}}\right]}$$

$$= Ke^{\log\left[\tan \frac{p}{2}\cdot\left(\frac{1+\epsilon \cos p}{1-\epsilon\cos p}\right)^{\frac{\epsilon}{2}}\right]^l} \times e^{ilM}$$

$$= K \tan^l \frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{l\epsilon}{2}}\left(\cos lM + i \sin lM\right).$$

By equating the real parts and the imaginary parts, we obtain

$$x = K \tan^l \frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{l\epsilon}{2}} \cos lM$$

$$y = K \tan^l \frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{l\epsilon}{2}} \sin lM.$$

In this projection the parallels become concentric circles. The equation for the radius is

$$r = \sqrt{x^2+y^2} = K \tan^l \frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{l\epsilon}{2}}.$$

The meridians are represented by radii of these concentric circles. This method of projection is the one known as Lambert's conformal conic projection, first developed by John Heinrich Lambert, in his "Beiträge zum Gebrauche der Mathematik," Berlin, 1772. It was later fully discussed by Gauss.

If an angle z is assumed such that

$$\tan\frac{z}{2}=\tan\frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{\epsilon}{2}}$$

this angle will be very nearly equal to the complement of the geocentric latitude.

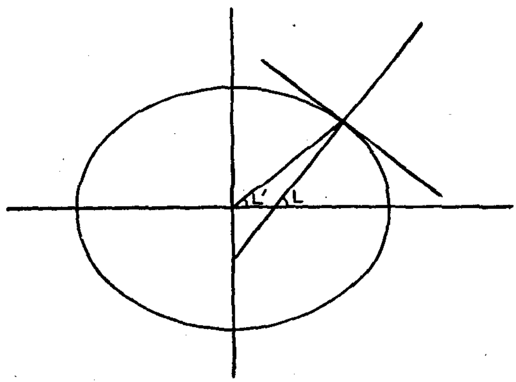


FIG. 4.—Generating ellipse with angles L and L'.

From the equation of the generating ellipse (p. 21), the tangent of the geocentric latitude is given by

$$\frac{z}{x}=\frac{b}{a}\cot u=\tan L'$$

But

$$\tan L=\frac{a}{b}\cot u$$

or

$$\cot u=\frac{b}{a}\tan L,$$

hence

$$\tan L'=\frac{b}{a}\cot u=\frac{b^2}{a^2}\tan L,$$

a and b being the semimajor axis and semiminor axis, respectively.

Then to a sufficient degree of approximation $z=\frac{\pi}{2}-L'$.

The value of z can be computed rigidly very conveniently by assuming an angle q such that $\cos q=\epsilon\cos p$. Then, since

$$\cot\frac{q}{2}=\sqrt{\frac{1+\cos q}{1-\cos q}}$$

$$\tan\frac{z}{2}=\tan\frac{p}{2}\cot^{\epsilon}\frac{q}{2}$$

or

$$\log \tan \frac{z}{2} = \log \tan \frac{p}{2} + \epsilon \log \cot \frac{q}{2}.$$

However, the approximate formula determines z to within a few tenths of a second. By using this auxiliary angle the equations become

$$x = K \tan^l \frac{z}{2} \cos lM$$

$$y = K \tan^l \frac{z}{2} \sin lM$$

$$r = K \tan^l \frac{z}{2}.$$

With these values x is reckoned downward from the center of the concentric circles and y to the right of the central meridian if M is reckoned positive in that direction.

K and l are as yet left arbitrary constants. l may be so determined that the ratio of the lengths of any two arcs of parallels on the map may be equal to the ratio of the lengths of the arcs that they represent. If N is the radius of curvature perpendicular to the meridian, or the length of the normal prolonged to the minor axis, a radian of the parallel L_1 has the length $N_1 \cos L_1$; in the same way the length of a radian of parallel L_2 is $N_2 \cos L_2$. Consequently, the ratio of the lengths of these two arcs is represented by

$$\frac{N_1 \cos L_1}{N_2 \cos L_2}.$$

Since the A factor[1] in the tables for geodetic positions is equal to

$$\frac{1}{N \sin 1''}$$

the ratio becomes

$$\frac{A_2 \cos L_1}{A_1 \cos L_2}.$$

The arc upon the map that represents the radian of parallel L_1 has the length $lr_1 = lK \tan^l \frac{z_1}{2}$. The radian of parallel L_2

[1] See United States Coast and Geodetic Survey Special Publication No. 8, entitled "Formulae and Tables for the Computation of Geodetic Positions."

is likewise represented by $lr_2 = lK \tan^l \frac{z_2}{2}$. The ratio of lengths will be preserved if we have

$$\left(\frac{\tan \frac{z_1}{2}}{\tan \frac{z_2}{2}}\right)^l = \frac{A_2 \cos L_1}{A_1 \cos L_2}$$

or
$$l = \frac{\log \cos L_1 - \log \cos L_2 - \log A_1 + \log A_2}{\log \tan \frac{z_1}{2} - \log \tan \frac{z_2}{2}}.$$

K may now be determined so as to hold not merely the ratio of the arcs of parallels L_1 and L_2 but also to hold the exact length of these parallels. This is an excellent method of determination for mapping such an area as that of the United States. In this way we should have

$$lK \tan^l \frac{z_1}{2} = N_1 \cos L_1 = \frac{\cos L_1}{A_1 \sin 1''};$$

hence
$$K = \frac{\cos L_1}{A_1 \sin 1'' \, l \tan^l \frac{z_1}{2}} = \frac{\cos L_2}{A_2 \sin 1'' \, l \tan^l \frac{z_2}{2}}$$

The twofold determination serves as a check on the computation.

With this determination of l and K, we shall compute the expression for the magnification at any point. We employed as the form of the function f

$$f\,(M - i\theta) = Ke^{ilM + l\theta}$$

hence
$$f'\,(M - i\theta) = i\,Kle^{ilM + l\theta}$$

$$f'\,(M + i\theta) = -i\,Kle^{-ilM + l\theta}$$

$$f'(M - i\theta)f'(M + i\theta) = K^2l^2e^{2l\theta}.$$

But
$$\frac{ds}{dS} = k = \frac{\sqrt{f'(M - i\theta)f'(M + i\theta)}}{m}.$$

From the equation of the linear element on the ellipsoid on page 23

$$m = \frac{a \sin p}{\sqrt{1 - \epsilon^2 \cos^2 p}}$$

$$k = \frac{\sqrt{1 - \epsilon^2 \cos^2 p}}{a \sin p} Kle^{l\theta},$$

But

$$e^{l\theta}=\tan^l \frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{l\epsilon}{2}}$$

therefore

$$k=\frac{Kl\tan^l\frac{p}{2}}{a\sin p}\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{l\epsilon}{2}}\sqrt{1-\epsilon^2\cos^2 p}$$

or

$$k=\frac{lr_n}{\rho_n}=\frac{lr_n}{N_n\cos L_n}=\frac{lr_nA_n\sin 1''}{\cos L_n}$$

r_n being the radius of the circle on the map that represents the parallel of L_n and ρ_n being the radius of the parallel. The last form is obvious from the conditions.

If the parallels to be held are chosen about 1/6 of the distance from the bottom and the top of the area to be mapped, the proper balance will be preserved. The upper and lower part of the map will then be about as much too large in scale as the central part is too small. The scale along L_1 and L_2 will be exactly correct. With this value of K one can tell how much any parallel is in error of scale by computing a radian of the parallel and the length of the arc which represents it on the map. This is just a statement of the equation

$$k=\frac{lr_n}{\rho_n}.$$

With this projection a map could be made of an area such as that of the United States so that it would not be in error of scale in any part of it by more than 1 1/5 per cent. A polyconic projection of the same area is in error of scale by as much as 6½ per cent in some parts. A Lambert projection for the United States to be evenly balanced should hold parallels 29° and 45°. The scale would then be just about 1 per cent short along the 37° parallel and 1 1/5 per cent long along the 49° parallel.

In Special Publication No. 52 of the United States Coast and Geodetic Survey, are given tables of coordinates for a Lambert projection of the United States. The standard parallels were chosen as 33° and 45° to lessen the scale distortion of the middle section of the country. This scheme gives about 2⅓ per cent scale distortion

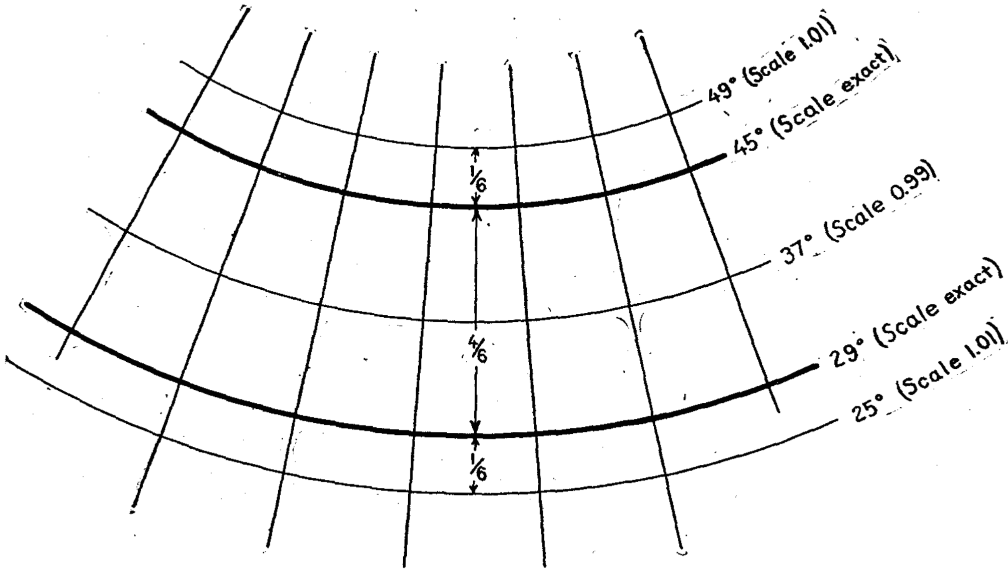


FIG. 5.—Scale distortions with the standard parallels at 29° and 45°.

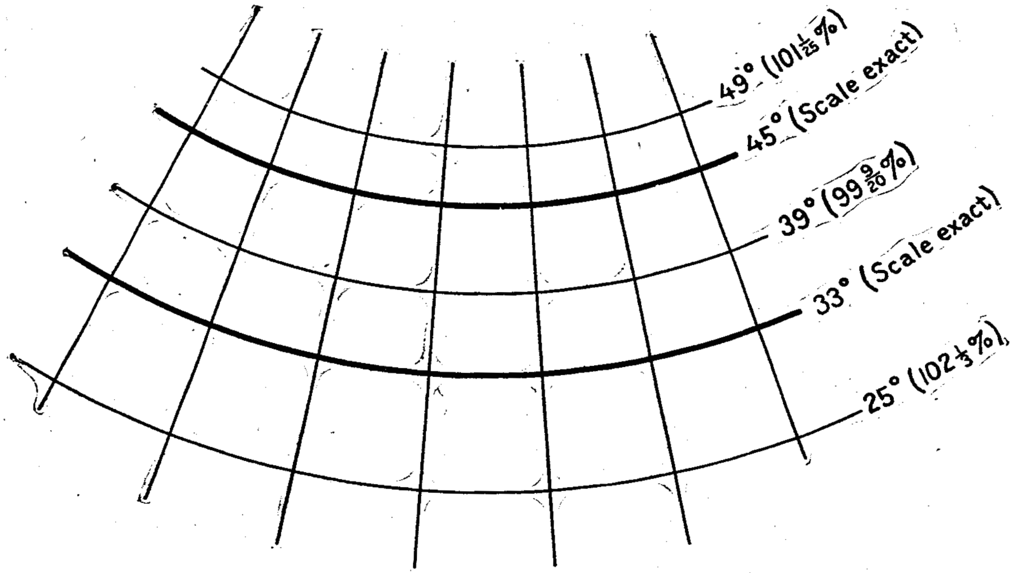


FIG. 6.—Scale distortions with the standard parallels at 33° and 45°.

along the parallel of 25° as the greatest on the whole map. Only a small part of Florida and of Texas are affected by this increase of scale error.

The coordinates for mapping the parallels are most conveniently computed using as origin the point where the parallel crosses the central meridian, the central meridian being the Y axis and a perpendicular to it the X axis. All the formulas required for computation are as follows:

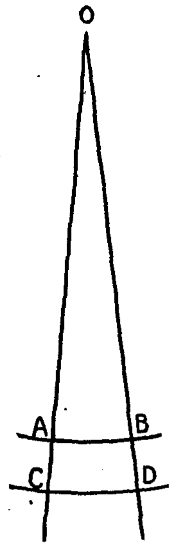


FIG. 7.—Representation of a small geodetic trapezoid.

$$\tan L' = \frac{b^2}{a^2} \tan L$$

$$z = \frac{\pi}{2} - L'$$

$$l = \frac{\log \cos L_1 - \log \cos L_2 - \log A_1 + \log A_2}{\log \tan \frac{z_1}{2} - \log \tan \frac{z_2}{2}}$$

$$K = \frac{\cos L_1}{A_1 \sin 1'' \, l \tan^l \frac{z_1}{2}} = \frac{\cos L_2}{A_2 \sin 1'' \, l \tan^l \frac{z_2}{2}}$$

$$r = K \tan^l \frac{z}{2}$$

$$x = r \sin lM$$

$$y = r\,(1 - \cos lM) = 2r \sin^2 \frac{lM}{2} = x \tan \frac{lM}{2}.$$

If it is desired to make the computation considering the earth as a sphere, it is only necessary to let $\epsilon = 0$. z then equals the polar distance or $z = p = \frac{\pi}{2} - L$; $A_1 = A_2 = \frac{1}{a \sin 1''}$. When these values are inserted in the above formulas, the correct forms are given for the sphere.

The difference of the radii gives the spacing upon the central meridian. If the parallel at the top and the one at the bottom are constructed by determining the coordinates of their intersections with the meridians, the meridians can then be drawn. They can then be subdivided as was done in the case of the central meridian. In this way the coordinates of the other parallels can be determined without computation.

If the two parallels to be held approach each other indefinitely, we shall have to determine the limiting value of l. This can be done in a number of different ways, and in fact the computation was made in six different ways with the same result in each case. The value that was obtained is the same as that obtained by Gauss and cited by Forsyth in his "Theory of Functions of a Complex Variable." Especial pains were taken with this, because Germain in his "Traité des Projections" obtains an erroneous result. The value is here determined by the *same method* that Germain said that he used, not because it is the simplest but to illustrate the fact that the correct result can be so obtained.

The problem is to so determine l that CD may have the same ratio to BD that AB has, or to determine the place where two successive parallels are held at the same ratio. This is the same as saying that the magnification shall be a minimum at this point and can be true only at this point.

$$OB=r$$

length of BD on the map is equal to dr

length of BD on the earth $=\dfrac{a(1-\epsilon^2)dp}{(1-\epsilon^2\cos^2 p)^{\frac{3}{2}}}$

length of CD on the map $=l(r+dr)dM$

length of CD on the earth

$$=\left[\frac{a\sin p}{\sqrt{1-\epsilon^2\cos^2 p}}+d\left(\frac{a\sin p}{\sqrt{1-\epsilon^2\cos^2 p}}\right)\right]dM$$

$$=\left[\frac{a\sin p}{\sqrt{1-\epsilon^2\cos^2 p}}+\frac{a\cos p\,dp}{\sqrt{1-\epsilon^2\cos^2 p}}-\frac{a\epsilon^2\sin^2 p\cos p dp}{(1-\epsilon^2\cos^2 p)^{\frac{3}{2}}}\right]dM$$

$$=\left[\frac{a\sin p}{\sqrt{1-\epsilon^2\cos^2 p}}+\frac{a(1-\epsilon^2)\cos p dp}{(1-\epsilon^2\cos^2 p)^{\frac{3}{2}}}\right]dM.$$

To meet the required conditions, we must have the proportion

$$\frac{l(r+dr)dM}{dr}=\frac{\left[\dfrac{a\sin p}{\sqrt{1-\epsilon^2\cos^2 p}}+\dfrac{a(1-\epsilon^2)\cos p dp}{(1-\epsilon^2\cos^2 p)^{\frac{3}{2}}}\right]dM}{\dfrac{a(1-\epsilon^2)dp}{(1-\epsilon^2\cos^2 p)^{\frac{3}{2}}}}$$

or

$$\frac{lr}{dr}+l=\frac{(1-\epsilon^2\cos^2 p)\sin p}{(1-\epsilon^2)dp}+\cos p.$$

Therefore, since

$$\frac{dr}{lr}=\frac{(1-\epsilon^2)\,dp}{(1-\epsilon^2\cos^2 p)\sin p},$$

$$l=\cos p.$$

The magnification is least at

$$p=\cos^{-1} l \text{ or at } L=\sin^{-1} l.$$

If this least value of the magnification is denoted by k_1 we have

$$k_1=\frac{lK}{a\sqrt{1-l^2}}\left(\frac{1-l}{1+l}\right)^{\frac{l}{2}}\cdot\left(\frac{1+\epsilon l}{1-\epsilon l}\right)^{\frac{l\epsilon}{2}}\cdot\sqrt{1-\epsilon^2 l^2}$$

If l is taken equal to $\cos p$, and K is determined so as to hold the length of the parallel of colatitude p, we shall have the case of a tangent cone.

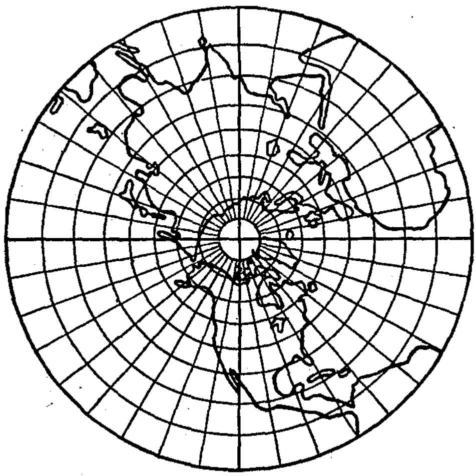


FIG. 8.—Stereographic projection of the northern hemisphere.

Germain obtained the erroneous value

$$l=\frac{\cos p-\epsilon^2\cos p(1-3\sin^2 p)}{1-\epsilon^2}.$$

If l becomes equal to unity, we have $p_0=o$ and the tangent cone becomes a tangent plane at the north pole. The equations now become

$$r = K \tan \frac{z}{2}$$
$$x = r \cos M$$
$$y = r \sin M.$$

This gives a projection for the spheroid analogous to the stereographic projection for the sphere. With the sphere the value of ϵ is zero and z becomes the polar distance. We then have a perspective projection from the south pole upon the plane tangent at the north pole.

If l becomes equal to zero the tangent cone becomes a cylinder tangent at the equator, and p_0 equals $\frac{\pi}{2}$. To determine the values it becomes necessary to evaluate some limits. K becomes infinite but so that Kl is finite and $K-x$ finite:

$$K = \frac{N_0 \cos L_0}{l \tan^l \frac{z_0}{2}}$$
$$Kl = \frac{N_0 \cos L_0}{\tan^l \frac{z_0}{2}}$$

When

$$p_0 \doteq z_0 \doteq \frac{\pi}{2}$$

then

$$\lim Kl = N_0 = a$$

In the general formulas

$$x = K \tan^l \frac{p}{2} \cdot \left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\frac{l\epsilon}{2}} \cos lM$$

$$y = K \tan^l \frac{p}{2} \cdot \left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\frac{l\epsilon}{2}} \sin lM.$$

To evaluate the limit, let us write x in this form

$$x = K \cos lM\, e^{l \log \left[\tan \frac{p}{2} \cdot \left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\frac{\epsilon}{2}}\right]}$$

or on developing the exponential

$$x = K \cos lM \left\{1 + l \log\left[\tan \frac{p}{2} \cdot \left(\frac{1+\epsilon \cos p}{1-\epsilon \cos p}\right)^{\frac{\epsilon}{2}}\right] + O(l^2)\right\}$$

$O(l^2)$ denoting terms with l^2 as the lowest power of l. But

$$x_o = K$$

hence

$$x_o - x = K(1-\cos lM) - Kl\cos lM \log\left[\tan\frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{\epsilon}{2}}\right]$$
$$- K\cos lM\, O(l^2).$$

Taking the limit when $p_o \doteq \frac{\pi}{2}$, $l \doteq o$

$$x_o - x = a\log\left[\cot\frac{p}{2}\cdot\left(\frac{1-\epsilon\cos p}{1+\epsilon\cos p}\right)^{\frac{\epsilon}{2}}\right].$$

Denoting $x_o - x$ by x and substituting for p its value $\frac{\pi}{2} - L$, we have

$$x = a\log_n\left[\tan\left(\frac{L}{2}+\frac{\pi}{4}\right)\cdot\left(\frac{1-\epsilon\sin L}{1+\epsilon\sin L}\right)^{\frac{\epsilon}{2}}\right]$$

$\log_n$ denoting the Naperian logarithm.

$$y = Kl\tan^l\frac{p}{2}\cdot\left(\frac{1+\epsilon\cos p}{1-\epsilon\cos p}\right)^{\frac{l\epsilon}{2}}\frac{\sin lM}{l}.$$

Taking the limit when $p_o \doteq \frac{\pi}{2}$, $l \doteq o$

$$y = aM,$$

M being of course expressed in arc. Interchanging x and y to give the coordinates as usually plotted, we have

$$x = aM$$
$$y = a\log_n\left[\tan\left(\frac{L}{2}+\frac{\pi}{4}\right)\cdot\left(\frac{1-\epsilon\sin L}{1+\epsilon\sin L}\right)^{\frac{\epsilon}{2}}\right].$$

This is a projection of the spheroid analogous to the Mercator projection for the sphere. If ϵ becomes zero we have the Mercator projection of the sphere.

We thus find that the stereographic and the Mercator are special cases of the Lambert projection and are therefore conformal.

Certain points in a conformal projection may be singular points at which the conformality fails. This is the condition at the pole in the Lambert conformal conic projection. The angles between the meridians are not preserved, since the angle between two meridians is lM instead of M as upon the earth. At such points, if $w=f(z)$ is the complex relation, $\frac{dw}{dz}$ is equal to zero or to infinity. The conformality fails at the poles in a Mercator projection but is preserved at the central pole in a stereographic polar projection. Since the angle at the center of the system of circles in the Lambert projection is equal to lM, the 360° in longitude is mapped on a sector of a circle with central angle equal to $360° \times l$. Since l is in the usual cases less than unity, the central angle will be less than 360°. If l

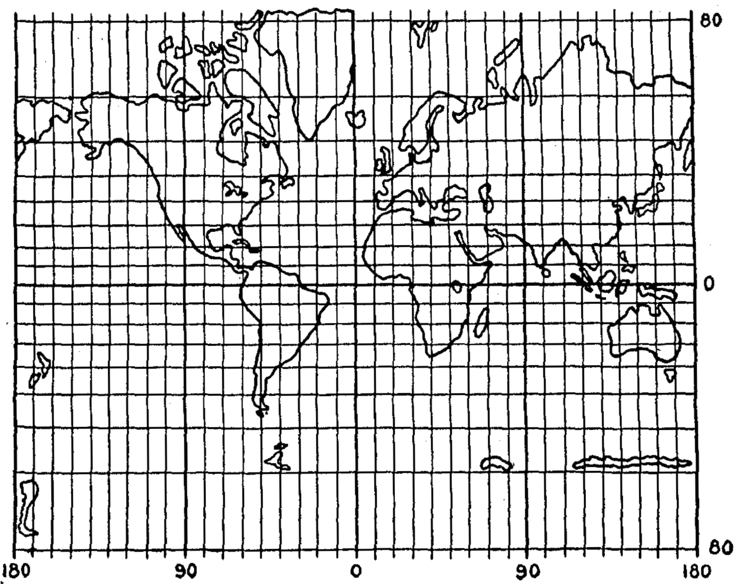


FIG. 9.—Mercator projection.

is equal to ¾, the angle of the sector would be 270°; for l equal to ⅔ the angle would be 240°.

In the war zone in France, the maps are made in the following manner. Instead of using the exact formulas that have been developed, approximations are employed. A cone tangent at 55 grades (49° 30′) is first determined with the parallel of 55^g as the central parallel of the map. Along the central meridian the parallels are spaced from the formula

$$\Delta r = \beta + \frac{\beta^3}{6\rho_0^{\,2}}$$

β being the distance along the meridian on the earth and ρ_0 being the mean radius of curvature at 55^g. This formula for Δr is a Taylor series development of Δr from the rigid formula for r, correct to the third power of β.[1]

The radius for the parallel of 55^g is taken as N_0 cot 55^g, N_0 being the radius of curvature perpendicular to the meridian. The values of Δr being added to or subtracted from this radius give the radii of the other parallels. ΔM sin 55^g gives the arc along the parallel corresponding to

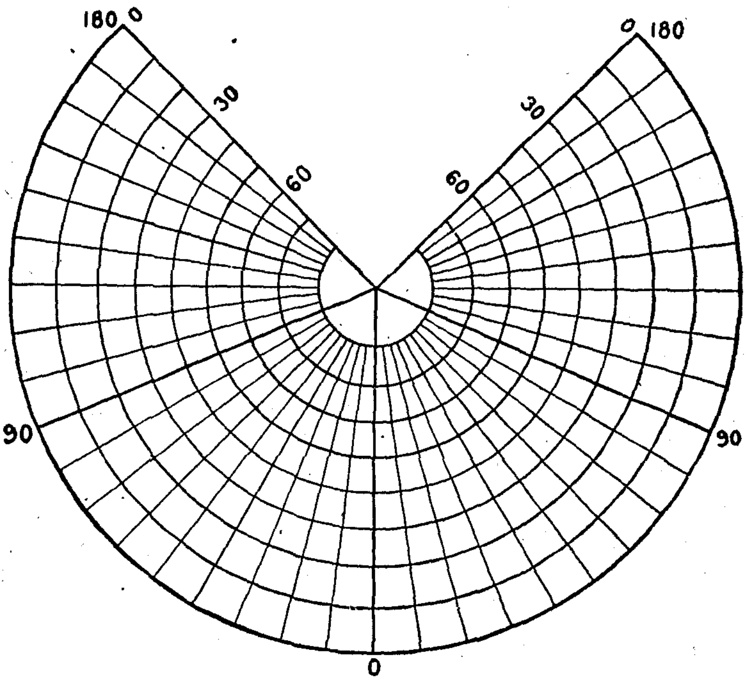


FIG. 10.—Graticule for the Lambert projection of the northern hemisphere with $l=\frac{3}{4}$.

the difference of longitude ΔM reckoned from the central meridian.

After these values are computed the whole scale is reduced by 1 part in 2037. This gives us approximately a cone intersecting at the parallels of 53^g and 57^g. The whole map is then covered with a system of kilometer squares with origin at latitude 55^g and longitude 6^g east

[1] See United States Coast and Geodetic Survey Special Publication No. 47, p. 13.

of Paris. The lines north and south are all parallel to the meridian of 6^g and the east and west lines are perpendicular to the same. A great circle for a limited region is a straight line within the limits of scaling. Since the map is con-

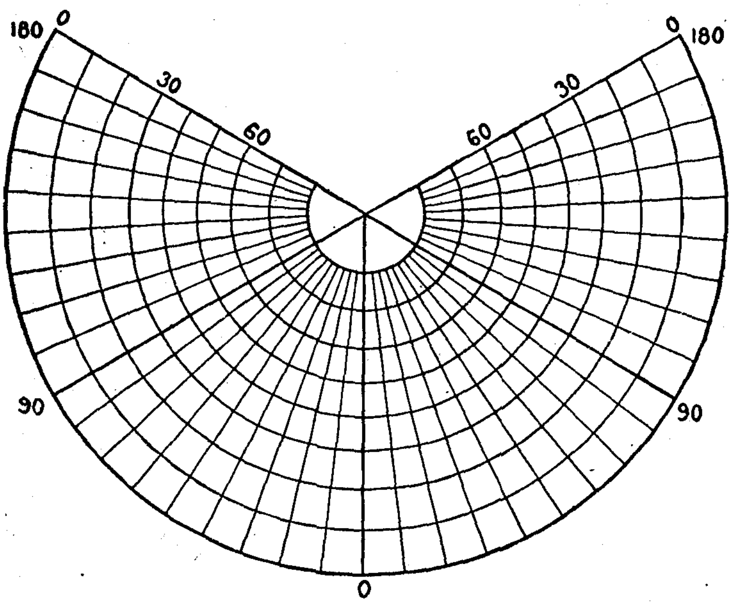


FIG. 11.—Graticule for the Lambert projection of the northern hemispheae with $l=2/3$.

formal, a chart made upon this projection is of great use in determining the direction for gunfire. The scale is also preserved constant within the error of scaling, so that the projection is excellent for the determination of both direction and distances.

BIBLIOGRAPHY.

Adams, Mary. A Little Book on Map Projections. London, 1914.

Bludau, Alois. Die Kartenprojectionen in Elementarer Behandlung. Düsseldorf, 1913.

Craig, James I. The Theory Of Map Projections. Cairo, 1910.

Craig, Thomas. A Treatise on Map Projections. U. S. Coast and Geodetic Survey, 1882.

Euler, Leonhard. Drei Abhandlungen über Kartenprojection. Reprint in Ostwald's Klassiker der Exakten Wissenschaften, No. 93.

Fiorini, Matteo. Le Projezioni delle Carte Geografiche. Bologna, 1881.

Frischauf, Johannes. Die Mathematischen Grundlagen der Landesaufnahme und Kartographie des Erdsphäroids. Stuttgart, 1913.

Gelcich, Eugen. Cartografia: Manuale Teorico-pratico. Milan, 1894.

Germain, Adrien. Traité des Projections des Cartes Géographiques. Paris, 1866?

Gretschel, Heinrich. Lehrbuch der Kartenprojection. Weimar, 1873.

Herz, Norbert. Lehrbuch der Landkartenprojectionen. Leipzig, 1885.

Hinks, Arthur. Map Projections. Cambridge University Press, 1912.

Lambert, Johann Heinrich. Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten, 1772. Reprint in Ostwald's Klassiker der Exakten Wissenschaften, No. 54.

Otti, Hans. Hauptfragen und Hauptmethoden der Kartenentwurfslehre. Aarau, 1911.

Scheffers, Georg. Anwendung der Differential- und Integralrech nung auf Geometrie. Leipzig, 1902.

Tissot, Auguste. Mémoire sur la Représentation des Surfaces et les Projections des Cartes Géographiques. Paris, 1881.

Vital, Arthur. Die Kartenentwurfslehre. Leipzig und Wien, 1903.

Wangerin, Albert. Über Kartenprojection. Abhandlungen von Lagrange und Gauss. Ostwald's Klassiker der Exakten Wissenschaften, No. 55.

Zöppritz, Karl Jacob. Leitfaden der Kartenentwurfslehre. Leipzig und Berlin, 1912.