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Mean-variance-skewness model for portfolio selection with fuzzy returns

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ABSTRACT

Numerous empirical studies show that portfolio returns are generally asymmetric, and investors would prefer a portfolio return with larger degree of asymmetry when the mean value and variance are same. In order to measure the asymmetry of fuzzy portfolio return, a concept of skewness is defined as the third central moment in this paper, and its mathematical properties are studied. As an extension of the fuzzy mean-variance model, a mean-variance-skewness model is presented and the corresponding variations are also considered. In order to solve the proposed models, a genetic algorithm integrating fuzzy simulation is designed. Finally, several numerical examples are given to illustrate the modelling idea and the effectiveness of the proposed algorithm.

1. Introduction

Modern portfolio selection theory is derived from the seminal work of Markowitz [19,20] which considered trade-off between return and risk. Since then, numerous portfolio selection models are developed by considering the return and risk such as mean-variance mode and so on. Several researchers like Sharpe [26,27], Stone [28], Sengupta [25], Best and Grauer [3], etc. have done some articles by using various approximation scheme.

Most of the reasonable works on portfolio selection have been done based on only the first two moments of return distributions. However, there is a controversy over the issue of whether higher moments should be considered in portfolio selection. Many researchers (e.g. Arditti [1], Samuelson [24], Kraus and Litzenberger [10], Konno et al. [18], Konno and Suzuki [9], Liu et al. [17], Prakash et al. [21]) argued that the higher moments cannot be neglected unless there are reasons to trust that the returns are symmetrically distributed (e.g. normal) or that higher moments are irrelevant to the investors' decisions.

Samuelson [24] also showed that higher moments are relevant for investors to make decisions in portfolio selection and almost all investors would prefer a portfolio with a larger third order moment if first and second moments are same. Chunhachinda et al. [5], Machado-Santos and Fernandes [18] provided evidence of skewness by using the data of stock markets. All the above discussions motivated us to add the third moment of return distribution of a portfolio selection into a general mean-variance model.

All the above literatures assume that the security returns are random variables. However, if there is not enough historical data, it is more reasonable to assume them as fuzzy variables. Fuzzy portfolio selection has been undertaken in the literature such as Parra et al. [2], Terol et al. [4], Tanaka and Guo [29,30] and Vercher et al. [31]. In 2002, Liu and Liu [15] defined the expected value and variance for measuring the portfolio return and the risk, respectively. Within the framework of credibility theory, several models for fuzzy portfolio selection were proposed such as, mean-semivariance model [7] and cross-entropy minimization model [23] and so forth. In addition, Qin and Li [22] considered option pricing problem in fuzzy environment which is another hottest area in finance.

In fuzzy environment, investors also face to construct a portfolio selection from the potential securities with asymmetric returns. Similar to stochastic approaches, Huang [7] employed semivariance to describing asymmetry of fuzzy returns. Different from Huang's approach, we used skewness of fuzzy returns to characterize the corresponding asymmetry as alternative approach. The purpose of this paper is to establish and analyze fuzzy mean-variance-skewness models.

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The remainder of this paper is organized as follows. In Section 2, a concept of skewness is defined for fuzzy variable as the third central moment and some important properties are proved. Section 3 proposes three mean-variance-skewness models and proves an equivalent form for the first model. Section 4 briefly introduces fuzzy simulation-based genetic algorithm to solve the proposed models and Section 5 gives several numerical examples and finally some conclusions are listed. In addition for better understanding of the paper, some basic definitions and useful results of fuzzy variables are given in Appendix.

2. Skewness of fuzzy variables

In this section, we define a concept of skewness for fuzzy variables and discuss its basic properties.

Definition 1. Let ξ be a fuzzy variable with finite expected value. The skewness of ξ is defined as

$$S[\xi] = E[(\xi - E[\xi])^3]. \tag{1}$$

Example 1. Let $\xi = (a, b, c)$ be a triangular fuzzy variable. Then it is easy to prove that

$$S[\xi] = \frac{(c-a)^2}{32}[(c-b)-(b-a)],$$

which implies that if $c - b \geqslant b - a$, then $S[\xi] \geqslant 0$ and if $c - b \leqslant b - a$, then $S[\xi] \leqslant 0$. Especially, if ξ is symmetric, then we have $b - a = c - b$ and $S[\xi] = 0$. Furthermore, for fixed a and c, if $b = a$, then $S[\xi]$ obtains its maximum value $(c-a)^3/32$; and if $b = c$, then $S[\xi]$ obtains its minimum value $-(c-a)^3/32$ (see Fig. 1)

Example 2. Let ξ be a normally distributed fuzzy variable with membership function $\mu(x) = 2\left[1 + \exp\left(\pi|x-e|/\left(\sqrt{6}\sigma\right)\right)\right]^{-1}$ (see Appendix). For any real number r, it follows from the credibility inversion theorem that

$$\mathrm{Cr}\{\xi \leqslant r\} = \left(1 + \exp\left(\frac{\pi(e-r)}{\sqrt{6}\sigma}\right)\right)^{-1}, \quad \mathrm{Cr}\{\xi \geqslant r\} = \left(1 + \exp\left(\frac{\pi(r-e)}{\sqrt{6}\sigma}\right)\right)^{-1}.$$

It follows from Definition 1 that

$$\begin{aligned} S[\xi] = E[(\xi-e)^3] &= \int_0^{+\infty} \mathrm{Cr}\{(\xi-e)^3 \geqslant r\}\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{(\xi-e)^3 \leqslant r\}\mathrm{d}r = 3\int_0^{+\infty} r^2\mathrm{Cr}\{\xi \geqslant r+e\}\mathrm{d}r - 3\int_{-\infty}^0 r^2\mathrm{Cr}\{\xi \leqslant r+e\}\mathrm{d}r \\ &= 3\int_0^{+\infty} r^2\left(1+\exp\left(\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1}\mathrm{d}r - 3\int_{-\infty}^0 r^2\left(1+\exp\left(-\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1}\mathrm{d}r \\ &= 3\int_0^{+\infty}\left(r^2\left(1+\exp\left(\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1} - r^2\left(1+\exp\left(\frac{\pi r}{\sqrt{6}\sigma}\right)\right)^{-1}\right)\mathrm{d}r = 0. \end{aligned}$$

Example 3. Let ξ be an exponentially distributed fuzzy variable with membership function $\mu(x) = 2\left[1 + \exp\left(\pi x/\left(\sqrt{6}m\right)\right)\right]^{-1}$ (see Appendix). For any $r \geqslant 0$, it follows from the credibility inversion theorem that

$$\mathrm{Cr}\{\xi \leqslant r\} = 1 - \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}m}\right)\right)^{-1}, \quad \mathrm{Cr}\{\xi \geqslant r\} = \left(1 + \exp\left(\frac{\pi r}{\sqrt{6}m}\right)\right)^{-1}.$$

Then we have $E[\xi] = e$. It follows from Definition 1 that

$$\begin{aligned} S[\xi] = E[(\xi - E[\xi])^3] &= 3\int_{E[\xi]}^{+\infty}(u-E[\xi])^2\ \mathrm{Cr}\{\xi \geqslant r\}\mathrm{d}r - 3\int_0^{E[\xi]}(u-E[\xi])^2\ \mathrm{Cr}\{\xi \leqslant r\}\mathrm{d}r \\ &= \frac{18\sqrt{6}m^3}{\pi^3}\left(\int_{\ln 2}^{+\infty}\frac{(r-\ln 2)^2}{1+\exp(r)}\mathrm{d}r - \int_0^{\ln 2}\frac{(r-\ln 2)^2\exp(r)}{1+\exp(r)}\mathrm{d}r\right) = \frac{18\sqrt{6}m^3}{\pi^3}\int_0^{+\infty}\frac{(r-\ln 2)^2}{1+\exp(r)}\mathrm{d}r - \frac{6\sqrt{6}m^3\ln 8}{\pi^3} = \alpha m^3, \end{aligned}$$

where $\alpha = 3\sqrt{6}(9\zeta(3) + 12\ln 2 - \pi^2\ln 2)/\pi^3 \approx 2.914$. Note that $\zeta(w) = \sum_{k=1}^{\infty} k^{-w}$.

Theorem 1. *Let ξ be a fuzzy variable with finite expected value. For any real numbers a and b, we have*

$$S[a\xi + b] = a^3 S[\xi].$$

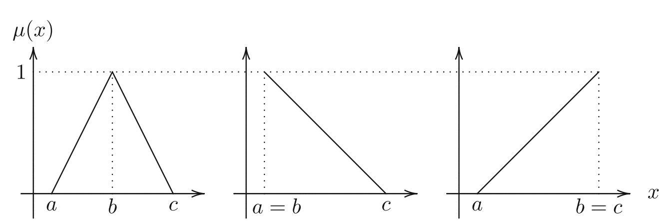

Fig. 1. Membership functions of several particular triangular fuzzy variables.

Proof. It can easily shows that $E[a\xi+b]=aE[\xi]+b$ (see Appendix). It follows from Definition 1 that

$$S[a\xi+b]=E[(a\xi+b-(aE[\xi]+b))^3]=E[a^3(\xi-E[\xi])^3]=a^3E[(\xi-E[\xi])^3]=a^3S[\xi].$$

The proof is complete. □

Theorem 2. *Let ξ be a symmetric fuzzy variable with finite expected value. Then we have*

$$S[\xi]=0.$$

Proof. Let μ be the membership function of ξ. Since ξ is symmetric, there is a real number e such that

$$\mu(e+r)=\mu(e-r),\quad \forall r\in\Re.$$

Furthermore, it is obtained that

$$\sup_{s\geqslant r+e}\mu(s)=\sup_{s\geqslant r}\mu(s+e)=\sup_{s\geqslant r}\mu(e-s)=\sup_{s\leqslant e-r}\mu(s).$$

It follows from the credibility inversion theorem that

$$\mathrm{Cr}\{\xi\geqslant r+e\}=\frac{1}{2}\left(\sup_{s\geqslant r+e}\mu(s)+1-\sup_{s<r+e}\mu(s)\right)=\frac{1}{2}\left(\sup_{r\leqslant e-x}\mu(r)+1-\sup_{r>e-x}\mu(r)\right)=\mathrm{Cr}\{\xi\leqslant e-x\}.$$

First, we prove that $E[\xi]=e$. In fact, according to the definition of expected value, we have

$$E[\xi]=\int_0^{+\infty}\mathrm{Cr}\{\xi\geqslant r\}\mathrm{d}r-\int_{-\infty}^0\mathrm{Cr}\{\xi\leqslant r\}\mathrm{d}r=\int_{-e}^{+\infty}\mathrm{Cr}\{\xi\geqslant r+e\}\mathrm{d}r-\int_e^{+\infty}\mathrm{Cr}\{\xi\leqslant e-r\}\mathrm{d}r=\int_{-e}^0\mathrm{Cr}\{\xi\geqslant r+e\}\mathrm{d}r+\int_0^{+\infty}\mathrm{Cr}\{\xi \geqslant r+e\}\,\mathrm{d}r-\int_0^{+\infty}\mathrm{Cr}\{\xi\leqslant e-r\}\mathrm{d}r+\int_0^e\mathrm{Cr}\{\xi\leqslant e-r\}\mathrm{d}r=\int_{-e}^0\mathrm{Cr}\{\xi\geqslant r+e\}\mathrm{d}r+\int_0^e\mathrm{Cr}\{\xi\leqslant e-r\}\mathrm{d}r=\int_0^e(\mathrm{Cr}\{\xi \geqslant e-r\}+\mathrm{Cr}\{\xi\leqslant e-r\})\mathrm{d}r=e.$$

Furthermore, it follows from the definition of skewness that

$$S[\xi]=\int_0^{+\infty}\mathrm{Cr}\{(\xi-e)^3\geqslant r\}\mathrm{d}r=\int_0^{+\infty}3r^2\mathrm{Cr}\{\xi-e\geqslant r\}\mathrm{d}r-\int_{-\infty}^0 3r^2\mathrm{Cr}\{\xi-e\leqslant r\}\mathrm{d}r=\int_0^{+\infty}3r^2\mathrm{Cr}\{\xi-e\leqslant -r\}\mathrm{d}r-\int_0^{+\infty}3r^2\mathrm{Cr}\{\xi-e\leqslant -r\}\mathrm{d}r=0.$$

The proof is complete. □

3. Mean-variance-skewness models

Let ξ_i be a fuzzy variable representing the return of the ith security, and let x_i be the proportion of the total capital invested in security i. In general, ξ_i is given as $(p_i'+d_i-p_i)/p_i$ where p_i is the closing price of the ith security at present, p_i' is the estimated closing price in the next year, and d_i is the estimated dividends during the coming year.

When minimal expected return and maximal risk are given, the investors interested in the use of skewness prefer a portfolio with large skewness. Therefore, we proposed the following mean-variance-skewness model:

$$\begin{cases}\text{maximize} & S[\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n],\\ \text{subject to}: & E[\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n]\geqslant\alpha,\\ & V[\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n]\leqslant\gamma,\\ & x_1+x_2+\cdots+x_n=1,\\ & x_i\geqslant 0,\quad i=1,2,\ldots,n.\end{cases}\tag{2}$$

The first constraint ensures the expected return is no less than some target value α, and the second one assures that risk does not exceed some given level γ the investor can bear. The last two constraints imply that all the capital will be invested to n securities and short-selling is not allowed.

The first variation of model (2) is the following,

$$\begin{cases}\text{minimize} & V[\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n],\\ \text{subject to}: & E[\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n]\geqslant\alpha,\\ & S[\xi_1x_1+\xi_2x_2+\cdots+\xi_nx_n]\geqslant\beta,\\ & x_1+x_2+\cdots+x_n=1,\\ & x_i\geqslant 0,\quad i=1,2,\ldots,n.\end{cases}\tag{3}$$

The aim of this model is to minimize risk when expected return and skewness are both no less than some given target values α and β, respectively. If the second constraint does not exist, then the above model degenerates to mean-variance model proposed by Huang [6].

The second variation of model (2) is the following,

$$\begin{cases} \text{maximize} & E[\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n], \\ \text{subject to}: & S[\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n] \geqslant \beta, \\ & V[\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n] \leqslant \gamma, \\ & x_1+x_2+\cdots+x_n=1, \\ & x_i \geqslant 0, \quad i=1,2,\ldots,n. \end{cases} \tag{4}$$

The aim of this model is to maximize the expected return. Similarly, if the first constraint does not exist, then it degenerates to the other mean-variance model considered by Huang [6].

The final variation of model (2) is the following multi-objective nonlinear programming,

$$\begin{cases} \text{maximize} & E[\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n], \\ \text{minimize} & V[\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n], \\ \text{maximize} & S[\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n], \\ \text{subject to}: & x_1+x_2+\cdots+x_n=1, \\ & x_i \geqslant 0, \quad i=1,2,\ldots,n. \end{cases} \tag{5}$$

When the membership functions of $\xi_1,\xi_2,\ldots,\xi_n$ are symmetric, it follows from Theorem 2 that $S[\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n]=0$ for any $x_i \geqslant 0, i=1,2,\ldots,n$, which implies the third objective vanishes. Model (5) degenerates a bi-objective mean-variance model.

Theorem 3. *Suppose that $\xi_i=(a_i,b_i,c_i)$ are independent triangular fuzzy variables for $i=1,2,\ldots,n$. Then model (2) degenerates to the following deterministic programming,*

$$\begin{cases} \max & \left(\sum_{i=1}^n x_i(c_i-a_i)\right)^2 \cdot \sum_{i=1}^n x_i(c_i+a_i-2b_i), \\ \text{s.t.} & \sum_{i=1}^n x_i(a_i+2b_i+c_i) \geqslant 4\alpha, \\ & \quad 11\left(\sum_{i=1}^n x_i(c_i-a_i)\right)^2 \left|\sum_{i=1}^n x_i(2b_i-a_i-c_i)\right| \\ & \quad +2\left(8\sum_{i=1}^n x_i(c_i-a_i)+3\left|\sum_{i=1}^n x_i(2b_i-a_i-c_i)\right|\right)\left(\left(\sum_{i=1}^n x_i(c_i-b_i)\right)^2+\left(\sum_{i=1}^n x_i(b_i-a_i)\right)^2\right) \\ & \quad \leqslant 192\gamma\left(\sum_{i=1}^n x_i(c_i-a_i)+\left|\sum_{i=1}^n x_i(2b_i-c_i-a_i)\right|\right), \\ & x_1+x_2+\cdots+x_n=1, \\ & x_i \geqslant 0, \quad i=1,2,\ldots,n. \end{cases} \tag{6}$$

Proof. Since $\xi_i=(a_i,b_i,c_i)$ are all triangular fuzzy variables, it follows from *Extension Principle of Zadeh* that

$$\sum_{i=1}^n \xi_i x_i = \left(\sum_{i=1}^n x_i a_i, \sum_{i=1}^n x_i b_i, \sum_{i=1}^n x_i c_i\right),$$

which is also a triangular fuzzy variable. Furthermore, we obtain $E[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]=\sum_{i=1}^n(a_i+2b_i+c_i)x_i/4$ and $S[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]=\left(\sum_{i=1}^n x_i(c_i-a_i)\right)^2\cdot\sum_{i=1}^n x_i(c_i+a_i-2b_i)$. Meanwhile, we have

$$V[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]=\frac{11\left(\sum_{i=1}^n x_i(c_i-a_i)\right)^2\left|\sum_{i=1}^n x_i(2b_i-a_i-c_i)\right|}{192\left(\sum_{i=1}^n x_i(c_i-a_i)+\left|\sum_{i=1}^n x_i(2b_i-c_i-a_i)\right|\right)} + \frac{2\left(8\sum_{i=1}^n x_i(c_i-a_i)+3\left|\sum_{i=1}^n x_i(2b_i-a_i-c_i)\right|\right)\left(\left(\sum_{i=1}^n x_i(c_i-b_i)\right)^2+\left(\sum_{i=1}^n x_i(b_i-a_i)\right)^2\right)}{192\left(\sum_{i=1}^n x_i(c_i-a_i)+\left|\sum_{i=1}^n x_i(2b_i-c_i-a_i)\right|\right)}.$$

Substituting these equations into model (2), the theorem is proved. □

Remark 1. When security returns are all independent triangular fuzzy variables, models (3) and (4) are also converted into deterministic mathematical programming problems using a similar way to Theorem 3.

4. Genetic algorithm

If the security returns are general fuzzy variables, then it is difficult to obtain the exact values of the expected value, variance and skewness of the portfolio return. Therefore, we employ fuzzy simulation to calculate these values. Fuzzy simulation technique was first introduced by Liu and Iwamura [14], and then was successfully applied to solving fuzzy optimization problems by Liu [12]. In addition, Liu [16] proved the convergence of fuzzy simulation, which shows its effectiveness in approximating exact values.

Assume that ξ_i are fuzzy variables with membership functions μ_i, and x_i are decision variables for all $1 \leqslant i \leqslant n$. In order to calculate the expected value, variance and skewness of $\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n$, we must calculate the value of $\text{Cr}\{\xi_1 x_1+\xi_2 x_2+\cdots+\xi_n x_n \geqslant r\}$ where r is a nonnegative real number. Randomly generate real numbers w_{ji} such that $\mu_j(w_{ji}) \geqslant \varepsilon, j=1,2,\ldots,k, i=1,2,\ldots,N$, respectively, where ε is a sufficiently small number, and N is a sufficiently large integer. Then, the value of $\text{Cr}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \geqslant r\}$ can be estimated by the formula

$$\frac{1}{2}\left(\max_{1\leqslant i\leqslant N}\left\{\min_{1\leqslant j\leqslant n}\mu_j(w_{ji})\middle|\sum_{j=1}^{n} w_{ji}x_j \geqslant r\right\} + 1 - \max_{1\leqslant i\leqslant N}\left\{\min_{1\leqslant j\leqslant n}\mu_j(w_{ji})\middle|\sum_{j=1}^{n} w_{ji}x_j < r\right\}\right).$$

In addition, we write $\rho = E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$ which may be calculated by fuzzy simulation [12].

The following algorithm is used to calculate $S[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$.

Step 1. Set $\beta = 0$.
Step 2. Randomly generate v_{jk} such that $\mu_j(v_{jk}) \geqslant \varepsilon$ for $j = 1, 2, \ldots, n,\ k = 1, 2, \ldots, K$, where ε is a sufficiently small number.
Step 3. Set two numbers

$$a = \min_{1\leqslant k\leqslant K}(v_{1k}x_1 + v_{2k}x_2 + \cdots + v_{nk}x_n - \rho)^3, \quad b = \max_{1\leqslant k\leqslant K}(v_{1k}x_1 + v_{2k}x_2 + \cdots + v_{nk}x_n - \rho)^3.$$

Step 4. Randomly generate a real number r from $[a, b]$.
Step 5. If $r \geqslant 0$, then $\beta \leftarrow \beta + \text{Cr}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \geqslant r\}$.
Step 6. If $r < 0$, then $\beta \leftarrow \beta - \text{Cr}\{\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \leqslant r\}$.
Step 7. Repeat the fourth to sixth steps for K times.
Step 8. Return $a \vee 0 + b \wedge 0 + \beta(b - a)/K$ as the target value.

If we want to calculate $V[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n]$, then we only need replace a, b in the above algorithm by

$$a = \min_{1\leqslant k\leqslant K}(v_{1k}x_1 + v_{2k}x_2 + \cdots + v_{nk}x_n - \rho)^2, \quad b = \max_{1\leqslant k\leqslant K}(v_{1k}x_1 + v_{2k}x_2 + \cdots + v_{nk}x_n - \rho)^2.$$

4.1. Genetic algorithm

Genetic algorithm is an adaptive heuristic search algorithm premised on the evolutionary ideas of natural selection and genetic. Since Holland first proposed it in 1975, genetic algorithm has been widely studied, experimented and applied in many fields such as industrial engineering, finance, operations research and so on. Especially, Liu [12] has successfully applied genetic algorithm to solve many optimization problems with fuzzy parameters.

In this work, a solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is encoded by a chromosome $\mathbf{c} = (c_1, c_2, \ldots, c_n)$, where the genes $c_1, c_2, \ldots, c_n$ are restricted as nonnegative numbers. Then the decoding processes are determined by the link $x_i = c_i/(c_1 + c_2 + \cdots + c_n)$, which ensures that $x_1 + x_2 + \cdots + x_n = 1$ always holds. In addition, a chromosome is called feasible if it satisfies the corresponding constraint conditions.

In GA, we employ the rank-based-evaluation function to measure the likelihood of reproduction for each chromosome. The rank-based evaluation function is defined by $Eval(\mathbf{c}_i) = v(1 - v)^{i-1},\ i = 1, 2, \ldots, pop\_size$ where $v \in (0, 1)$. Especially, $i = 1$ indicates the best individual, and $i = pop\_size$ indicates the worst one.

The procedures of the genetic algorithm is summarized as follows:

Step 1. Initialize *pop_size* feasible chromosomes, in which fuzzy simulation is used to check the feasibility of the chromosomes.
Step 2. Employ fuzzy simulation to compute the objectives for all chromosomes, and then give an order of the chromosomes based on the objective values.
Step 3. Evaluate the evaluation function of each chromosome according to the rank-based-evaluation function. Then calculate the fitness of each chromosome by the evaluation function.
Step 4. Select the chromosomes according to spinning the roulette wheel.
Step 5. Update the chromosomes by crossover operation and mutation operation where fuzzy simulation is utilized to check the feasibility of each child.
Step 6. Repeat Steps 2–5 for a given number of generations.
Step 7. Report the best chromosome, and then decoded into the optimal solution.

5. Numerical examples

In this section, mean-variance-skewness models are applied to the data from Huang [7]. The data is composed of membership functions of 10 security returns, which is shown in Table 1. The returns of first seven securities are triangular fuzzy variables, and the others are fuzzy

Table 1
Fuzzy returns of 10 securities (units per stock).

Security i	Fuzzy return	Security i	Fuzzy return
1	$(-0.3, 1.8, 2.3)$	6	$(-0.8, 2.5, 3.0)$
2	$(-0.4, 2.0, 2.2)$	7	$(-0.6, 1.8, 3.0)$
3	$(-0.5, 1.9, 2.7)$	8	$(1 + (r - 1.6)^4)^{-1}$
4	$(-0.6, 2.2, 2.8)$	9	$(1 + (5r - 7.4)^2)^{-1}$
5	$(-0.7, 2.4, 2.7)$	10	$\exp(-(r - 1.6)^2)$

Table 2
Comparison of results.

	1	2	3	4	5	6	7	Mean	Variance	Semivariance	Skewness
Model (6)	20.00%	–	–	80.00%	–	–	–	1.60	0.7019	0.6141	−0.6823
Model (7)	0	47.06%	–	35.28%	17.66%	–	–	1.60	0.7232	0.6124	−0.7532

Table 3
Investment proportion of 10 securities (%).

Security i	1	2	3	4	5	6	7	8	9	10
Allocation of money	4.04	5.52	8.22	9.47	8.17	0.20	16.55	17.47	21.22	9.14

variables with membership functions $\mu_i, i = 8, 9, 10$. For example, the return of the first security is fuzzy variable $(-0.3, 1.8, 2.3)$ which represents about 1.8 units per stock.

Example 4. Assume that an investor wishes to create a portfolio from the first seven securities. In order to use the proposed models, the investor need to set two parameters: the minimum expected return α and the bearable maximum risk γ. Here, let $\alpha = 1.6$ and $\gamma = 0.8$. Since the first seven returns are all triangular fuzzy variables, we can use model (6) to search for optimal portfolio.

Since the returns are asymmetric, the investor also employs mean-semivariance model to create an optimal portfolio. In order to compare the results of mean-variance-skewness model and mean-semivariance model, we consider model 8 of Huang [7].

$$\begin{cases} \text{maximize} & Sv[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_7 x_7], \\ \text{subject to}: & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_7 x_7] \geqslant 1.6, \\ & x_1 + x_2 + \cdots + x_7 = 1, \\ & x_i \geqslant 0, \quad i = 1, 2, \ldots, 7, \end{cases} \tag{7}$$

where Sv is the semivariance operator of fuzzy variable.

We use MATLAB to solve models (6) and (7) and the computational results are shown in Table 2. The two models obtain different optimal portfolios which have the same mean 1.60 and almost the same semivariance. However, the first portfolio has lower variance, and higher skewness than the second one, which is desired by the investor.

Example 5. If an investor chooses securities from the whole 10 securities, then all the mean-variance-skewness models cannot be converted into deterministic models. Therefore, we use genetic algorithm to solve the proposed models. Assume that the minimum expected return the investor can accept is 1.5 and the bearable maximum risk is 1.2. Based on the optimization model, we obtain the following model,

$$\begin{cases} \text{maximize} & S[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}], \\ \text{subject to}: & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \geqslant 1.5, \\ & V[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \leqslant 1.2, \\ & x_1 + x_2 + \cdots + x_{10} = 1, \\ & x_i \geqslant 0, \quad i = 1, 2, \ldots, 10. \end{cases} \tag{8}$$

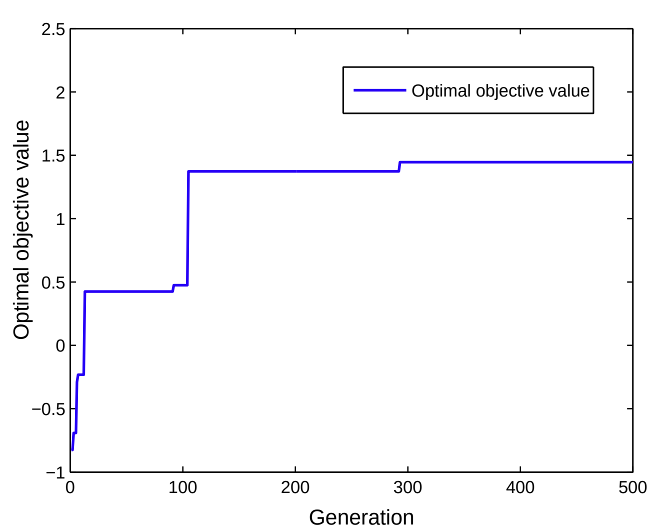

Fig. 2. The convergence of objective value of Example 5.

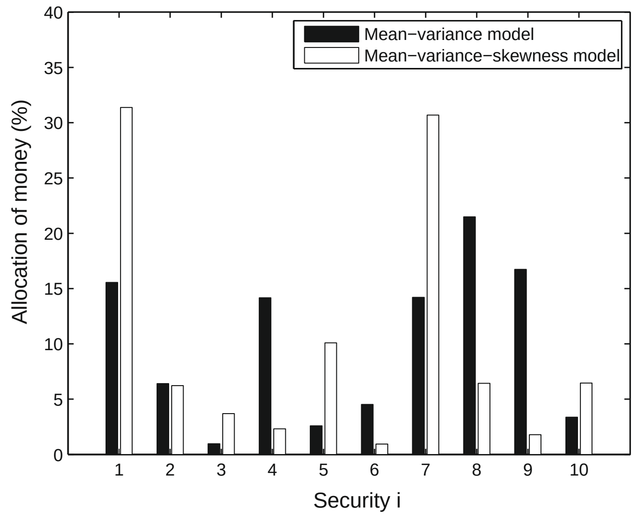

Fig. 3. Allocations of capital for mean-variance model and mean-variance-skewness model.

We choose the following parameters in the GA: $P_c = 0.3, P_m = 0.2, pop\_size = 50$. A run of genetic algorithm (500 generations and 3000 cycles for fuzzy simulation) shows that the allocation of money should be based on Table 3. The corresponding maximum skewness is 1.72. In addition, the convergence of maximum skewness of portfolio is shown in Fig. 2 which indicates genetic algorithm is effective to solve the proposed model.

Example 6. Suppose that an investor wishes that the skewness of his portfolio is at least -1.0, and the minimal expected return is 1.5. If he accepts variance as risk, then the model is formulated as follows,

$$\begin{cases} \text{minimize} & V[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}], \\ \text{subject to}: & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \geqslant 1.5, \\ & S[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \geqslant -1.0, \\ & x_1 + x_2 + \cdots + x_{10} = 1, \\ & x_i \geqslant 0, \quad i = 1, 2, \ldots, 10. \end{cases} \tag{9}$$

First note that if we do not consider skewness of the portfolio, then the model generates mean-variance model. The parameters of GA are chosen as follows: $P_c = 0.4, P_m = 0.3, pop\_size = 60$. Here, we compare the allocation of capital between this model and mean-variance model by Fig. 3. The minimum risk of mean-variance-skewness model is 0.383, and the minimum risk of mean-variance model is 0.285, respectively.

Example 7. Assume that an investor wishes that the skewness of his portfolio is at least -1.0, and the maximal risk does not exceed 1.2. Meanwhile, if the investor wants to maximize the expected return, then the model is formulated as follows,

$$\begin{cases} \text{minimize} & E[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}], \\ \text{subject to}: & S[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \geqslant -1.0, \\ & V[\xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_{10} x_{10}] \leqslant 1.2, \\ & x_1 + x_2 + \cdots + x_{10} = 1, \\ & x_i \geqslant 0, \quad i = 1, 2, \ldots, 10. \end{cases} \tag{10}$$

In order to test the robust of the proposed algorithm, we solve the model by setting the different parameters in the GA. In order to compare the results, we employ the relative error which is defined by (Maximal objective − Actual objective)/Maximal objective $\times$ 100%, where the maximal objective is the maximum of all the computational results obtained. The detailed results are shown in Table 4. Obviously, the

Table 4
Comparison of solutions in Example 6.

No.	*pop_size*	P_c	P_m	Simulation times	Generation	Expected value	Relative error (%)
1	50	0.3	0.2	2500	200	1.6855	0.38
2	30	0.4	0.3	2500	200	1.6856	0.37
3	80	0.8	0.3	2300	150	1.6879	0.24
4	90	0.6	0.7	2500	100	1.6905	0.08
5	110	0.7	0.8	2500	100	1.6812	0.63
6	130	0.5	0.2	2200	100	1.6919	0
7	70	0.2	0.1	2500	200	1.6871	0.28

relative errors do not exceed 1%. That is, the proposed algorithm is robust to set parameters and effective for solving the mean-variance-skewness models.

6. Conclusions

In this paper, a concept of skewness for fuzzy variable was proposed, and several useful theorems were proved. In addition, a mean-variance-skewness model was formulated for fuzzy portfolio selection problem and two variations of this model were also discussed. To solve the proposed model, a genetic algorithm was designed and fuzzy simulation technique was employed. Finally, several numerical examples were illustrated to show the effectiveness of the proposed algorithm. The methodology presented here is quite general and can be extended to the portfolio selection problems in hybrid and uncertain environments.

Acknowledgement

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Appendix. Credibility theory

Credibility theory was founded by Liu in 2004 and refined by Liu [13] as a branch of mathematics for studying fuzzy phenomena. In this part, some main results of credibility theory are recalled for the convenience of reading the paper.

Let ξ be a fuzzy variable with membership function μ. For any $B \subset \Re$, the credibility measure of $\xi \in B$ was defined by Liu and Liu [15] as

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}\left(\sup_{x\in B}\mu(x) + 1 - \sup_{x\in B^c}\mu(x)\right). \tag{11}$$

To rank fuzzy variables, Liu and Liu [15] defined the expected value of ξ as follows,

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \geqslant r\}\,\mathrm{d}r - \int_{-\infty}^0 \mathrm{Cr}\{\xi \leqslant r\}\mathrm{d}r \tag{12}$$

provided that at least one of the two integrals is finite. If fuzzy variables ξ and η are independent, then we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta] \quad \text{for any } a, b \in \Re. \tag{13}$$

Since a fuzzy variable ξ and a constant are clearly independent, we have $E[a\xi + b] = aE[\xi] + b$.

Suppose that ξ is a fuzzy variable with finite expected value. Then its variance was defined by Liu and Liu [15] as

$$V[\xi] = E[(\xi - E[\xi])^2]. \tag{14}$$

It is easy to prove that $V[\xi] = 0$ if and only if $\mathrm{Cr}\{\xi = E[\xi]\} = 1$.

Example 8. A triangular fuzzy variable ξ is fully determined by the triplet (a, b, c) of crisp numbers with $a < b < c$ and its membership function is given by

$$\mu(x) = \begin{cases} (x-a)/(b-a), & \text{if } a \leqslant x \leqslant b, \\ (x-c)/(b-c), & \text{if } b \leqslant x \leqslant c, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

In what follows, we write $\xi = (a, b, c)$ (see Fig. 4). It is easy to prove that $E[\xi] = (a + 2b + c)/4$ and $V[\xi] = (33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3)/(384\alpha)$ where $\alpha = \max\{b - a, c - b\}$ and $\beta = \min\{b - a, c - b\}$. In particular, if $b - a = c - b$, then we have $E[\xi] = b$ and $V[\xi] = (c - a)^2/24$.

Example 9. If ξ is a normally distributed fuzzy variable with the following membership function

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi|x-e|}{\sqrt{6}\sigma}\right)\right)^{-1}, \quad x \in \Re, \tag{16}$$

then Li and Liu [11] proved that $E[\xi] = e$ and $V[\xi] = \sigma^2$ (see Fig. 5).

Example 10. A fuzzy variable ξ is called exponentially distributed if it has the following membership function

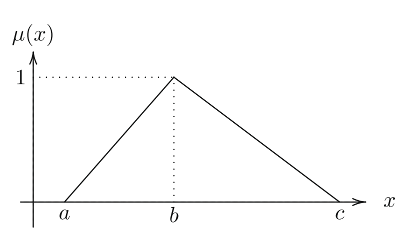

Fig. 4. Membership functions of triangular fuzzy variable $\xi = (a, b, c)$.

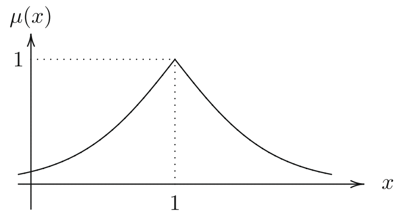

Fig. 5. Membership functions of normally distributed fuzzy variable with $e = 1$ and $\sigma = 1$.

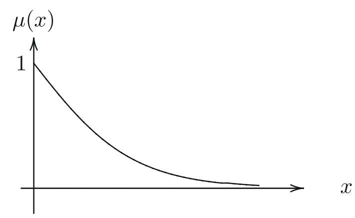

Fig. 6. Membership functions of exponentially distributed fuzzy variable with $m = 1$.

$$\mu(x) = 2\left(1 + \exp\left(\frac{\pi x}{\sqrt{6}m}\right)\right)^{-1}, \quad x \geqslant 0, \tag{17}$$

where $m > 0$. Li and Liu [11] proved that $E[\xi] = \left(\sqrt{6}m \ln 2\right)/\pi$ (see Fig. 6).

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