A Minimax Procedure for Electing Committees

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Abstract[4]

A new voting procedure for electing committees, called the *minimax procedure*, is described. Based on approval balloting, it chooses the committee that minimizes the maximum Hamming distance to voters' ballots, where these ballots are weighted by their proximity to other voters' ballots. This *minimax outcome* may be diametrically opposed to the outcome obtained by aggregating approval votes in the usual manner, which minimizes the sum of the Hamming distances and is called the *minisum outcome*. The manipulability of these procedures, and their applicability when election outcomes are restricted in various ways, are also investigated.

The minimax procedure is applied to the 2003 Game Theory Society election of a council of 12 new members from a list of 24 candidates. The composition of the council would have changed by 4 members; there would have been more substantial differences between minimax and minisum outcomes if the number of candidates to be elected had been endogenous rather than being fixed at 12. The minimax procedure, which renders central voters more influential but does not antagonize any voter too much, may produce a committee that better represents the interests of all voters than a minisum committee.

Key words: committees, minimax, minisum

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[4] We thank Eric van Damme, secretary-treasurer of the Game Theory Society (GTS), for providing ballot data for the 2003 council election of the GTS, Peter C. Fishburn for providing his notes analyzing these data (Fishburn, 2004), Tommy Ratliff, and three anonymous reviewers for valuable comments. We are especially grateful to Dan Cao, Juliana Perez, and Shawn Ling Ramirez for research assistance. While we include in the text examples and relatively nontechnical proofs, we put two more technical proofs in the Appendix.

1 Introduction

In this paper we propose a new voting procedure, called the *minimax procedure*, for electing committees. This procedure is based on approval balloting—whereby voters approve of as many candidates as they like (Brams and Fishburn, 1978, 1983)—but votes are not aggregated in the usual manner.[5]

Instead of selecting the candidates that receive the most votes, the minimax procedure selects the set of candidates that minimizes the maximum Hamming distance to voters' ballots, where these ballots are weighted by their proximity to other voters' ballots. This set of candidates constitutes the *minimax outcome*. We define and illustrate Hamming distance in section 2 and show how the proximity weighting of this distance is determined. We also offer a geometric interpretation of minimax outcomes.

We call the set of candidates that minimizes the sum of the Hamming distances to all voters the *minisum outcome*. In fact, this is the usual set of majority winners under approval balloting. We give examples in which tied and nontied minimax and minisum outcomes may be diametrically opposed in section 3.

We argue that when committees of two or more candidates are to be elected, there are good reasons for preferring a minimax outcome. It ensures that no voter is "too far away" from the committee that is elected—based on proximity-weighted Hamming distances—whereas minisum outcomes ensure that voters will, on average, be closer to the committee, even though a few voters may be far away.

In section 4, we discuss the applicability the procedures when there are restrictions on the possible committees to be elected, either in size or in composition. In section 5 we show that while the minisum procedure is not manipulable, the minimax procedure is (when preferences are based on Hamming distance), though in practice the minimax procedure is probably almost as invulnerable as the minisum procedure.

In section 6, we analyze the 2003 Game Theory Society (GTS) election of 12 new members to the GTS council from a list of 24 candidates. There were $2^{24} \approx 16.8$ million possible ballots under approval balloting, because each voter could approve, or not, each of the 24 candidates. Given this huge number, it is hardly surprising that all but two of the 161 GTS members who voted in this election cast different ballots.[6]

[5] Merrill and Nagel (1987) distinguish between a balloting method and a procedure for aggregating voter choices on the ballot. Throughout we assume the balloting method is approval balloting; what we compare are different ways of *aggregating* approval votes.

[6] If all ballots are assumed equiprobable, the probability that no two (of the 161) voters cast identical ballots is $[(s)(s-1)\ldots(s-159)(s-160)]/s^{161}$, where s is the number of possible ballots (16,774,216 in this case). This follows from the fact that the first voter can cast one of s different ballots; for each of these, there are $(s - 1)$ ways for the second voter to cast a different ballot; and so on to the 161st voter. The product of these numbers, divided by the number of possible ballots, s^{161}, gives the probability that no two voters cast identical ballots; the complement of this probability is the probability that at least two

In section 7, we conclude that the minimax procedure is a viable alternative to the minisum procedure for electing committees. Besides professional societies like the GTS, we commend the minimax procedure to colleges, universities, and other organizations that rely substantially on representative committees to make recommendations and decisions.

In other arenas, such as faction-ridden countries like Afghanistan and Iraq, the minimax procedure could facilitate the choice of councils and cabinets that mirror the diversity of interests in the electorate. It could also be used to resolve multi-issue disputes; in fact, a simplified version of this procedure would have led to a different outcome from that achieved in oil-pollution treaty negotiations of 32 countries in 1954 (Brams, Kilgour, and Sanver, 2004).

2 Minisum and Minimax Outcomes

Assume there n voters and k candidates. Under approval balloting, a *ballot* is a binary k-vector, $(p_1, p_2, \ldots, p_k)$, where p_i equals 0 or 1. These binary vectors indicate the approval or disapproval of each candidate by a voter.

To simplify notation, we write ballots such as (1, 1, 0) as 110, which indicates that the voter approves of candidates 1 and 2 but disapproves of candidate 3. (We also use vectors like 110 to represent election outcomes—that is, the committees that are chosen by the voters.) The number of distinct ballots, or possible election outcomes, is 2^k.

To illustrate the selection of representative committees based on the minisum and minimax criteria, consider the following example, in which 4 voters cast three distinct ballots for $k = 3$ candidates:

1 voter: 100

1 voter: 110

2 voters: 101

Under the usual method of aggregating approval votes, we ask whether each of the three candidates wins a majority of votes.

voters cast the same ballot. In the GTS election, the latter probability was only 0.000768, or less than 1 in 1,000, indicating that it was highly improbable that two or more voters would cast the same ballot, given all ballots are equiprobable (also highly unlikely). These calculations are similar to those used to solve the "birthday problem" in probability theory, which asks how many people must be in a room to make the probability greater than 1/2 that at least two people have the same birthday (the answer is 23 or more). In section 5 we define a more "empirical" probability, based on the number of voters voting for different numbers of candidates, which suggests that the probability that some ballots are identical is much higher.

Observe that candidate 1 receives approval from all 4 voters, candidate 2 from 1 voter, and candidate 3 from 2 voters, so candidate 1 is elected and candidate 2 is not elected. Normally, we would say that candidate 3, who is approved by exactly half the voters, would not be elected, but our version of majority voting allows for candidate 3 to be elected or not. That is, outcomes 101 and 100 are both majority-voting outcomes.[7]

The *Hamming distance* between two ballots, *p* and *q*, is $d(p, q)$, the number of components on which they differ. For example, if $k = 3$ and a voter's ballot is 110, the distances, *d*, between it and the eight binary 3-vectors (including itself) are shown below:

Ballot	***d* = 0**	***d* = 1**	***d* = 2**	***d* = 3**
110	110	100 010 111	000 101 011	001

Observe that there are three ballots at Hamming distance $d = 1$, and three more at $d = 2$; ballot 110 is at distance $d = 0$ from itself, and its *antipode*, the ballot on which all components differ, is at $d = 3$.

Define a *majority-voting* (MV) committee to be any subset of candidates that includes all candidates who receive more than $n/2$ voters and none that receive less than than $n/2$ votes, where *n* is the number of voters. Brams, Kilgour, and Sanver (2004, Proposition 4) proved that a committee is an MV committee if and only if the sum of the Hamming distances between all voters and the committee is a minimum. For this reason, we refer to MV committees as *minisum committees*.

As we saw in the 4-voter example, there may be more than one MV committee (100 and 101). In general, an MV committee is not unique if and only if *n* is even and at least one candidate receives exactly $n/2$ votes. (If *n* is odd, MV committees will always be unique since no candidate can receive exactly half the votes.)

Minisum Committees with Count Weights

Following Kilgour, Brams, and Sanver (2006), we focus not on the individual ballots but on the distinct ballots, and the number of times that each was cast. For instance,

[7] In general, if there is a tie between the yes (1) and no (0) votes for a candidate, then there are multiple majority-voting outcomes, both including and excluding this candidate. Defining majority-voting outcomes in this way makes them coincide with minisum outcomes (more on this below).

committees 100 and 101 minimize the sum of the Hamming distances to all voters in our 4-voter example—or, equivalently, the sum of the Hamming distances to all distinct ballots weighted by the numbers of voters who cast each. This is shown by the weighted Hamming distances to the eight possible committees in Table 1. We call the weights *count weights*, because they count the numbers of voters who cast each ballot.

Ballot:	**100**	**110**	**101**	**Sum**	**Max**
Count Weight:	**1**	**1**	**2**		
1. 000	1	2	4	7	4
2. 100	0	1	2	3*	2*
3. 010	2	1	6	9	6
4. 001	2	3	2	7	3
5. 110	1	0	4	5	4
6. 101	1	2	0	3*	2*
7. 011	3	2	4	9	4
8. 111	2	1	2	5	2*

*Minimum in column.

Table 1 : Derivation of Minisum and Minimax Committees Based on Count Weights (4-Voter Example)

The sums of the entries in each row are shown in the Sum column of Table 1. Clearly, the two MV committees, 100 and 101, whose sums of 3 are starred, minimize the sum of the weighted Hamming distances. By Brams, Kilgour, and Sanver (2004, Proposition 4), choosing a committee that minimizes the sum of the weighted Hamming distances, based on count weights, is equivalent to choosing an MV committee. In our example, this committee always includes candidate 1 and may or may not include candidate 3.

Minimax Committees

Following Brams, Kilgour, and Sanver (2004) and Kilgour, Brams, and Sanver (2006), we note that there are other ways to define the most representative committee. Instead of finding a committee that minimizes the sum of the Hamming distances to all ballots, find the committee(s) that minimize the maximum Hamming distance. In our example, these are the three committees that tie with values of 2, which are starred, in the Maximum column of Table 1.

Note that a third committee, 111, ties with minisum committees 100 and 101 as most representative, based on count weights. Because 111 is not a minisum committee, however, it is arguably an inferior choice to 100 and 101. But there is a more fundamental issue regarding minimax committees: The minimax procedure, based on count weights, does not seem as compelling as the same procedure based on a different weighting, as we describe next.

Minimax Committees with Proximity Weights

Proximity weights, like count weights, reflect the number of voters who cast each of the distinct ballots. But they also incorporate information about the closeness of a ballot to all other ballots, based on Hamming distances.

The closer a ballot is to all other ballots, and the more voters who cast it, the more influence it should have on the determination of a committee. The minisum procedure with proximity weights works in this way. The proximity weight of ballot q^j is

$$w_j = \frac{m_j}{\sum_{h=1}^{t} m_h d(q^j, q^h)}, \qquad (1)$$

where m_j is the number of voters who cast ballot $q^j = (q_1^j, q_2^j, \ldots, q_n^j)$ and t is the number of distinct ballots cast. The denominator of the fraction is the sum of the Hamming distances from ballot j to all ballots (including ballot j), weighted by the number of voters who cast each ballot.

To illustrate in our example, the Hamming distances of ballot 100 to itself, 110, and 101 are 0, 1, and 1, respectively. Because these three ballots are cast by 1, 1, and 2 voters, respectively, ballot 100 has weight 1/[(1 × 0) + (1 × 1) + (2 × 1)] = 1/3, with the numerator reflecting the fact that one voter cast this ballot.

Similarly, ballots 110 and 101 have weights of 1/5 and 2/3. As shown in Kilgour, Brams, and Sanver (2006), it is the relative sizes of the weights that matter, so, for convenience, we multiply them by 15 to clear denominators. This yields weights of 5, 3, and 10 for ballots 100, 110, and 101, respectively. Thereby we obtain Table 2, which is the same as Table 1 except that it is based on proximity weights rather than count weights.

Ballot: **Proximity Weight:**	**100** **5**	**110** **3**	**101** **10**	**Sum**	**Max**
1. 000	5	6	20	31	20
2. 100	0	3	10	13	10
3. 010	10	3	30	43	30
4. 001	10	9	10	29	10
5. 110	5	0	20	25	20
6. 101	5	6	0	11*	6*
7. 011	15	6	20	41	20
8. 111	10	3	10	23	10

*Minimum in column

Table 2 : Derivation of Minisum and Minimax Committees Based on Proximity Weights (4-Voter Example)

Notice that only committee 101 minimizes both the sum and the maximum of weighted Hamming distances, based on proximity weights. While committee 101 is also one of the committees singled out by the minisum and minimax criteria, based on count weights, this coincidence will not necessarily be the norm. In fact, we will show in section 3 that the minisum outcome, based on count weights, and the minimax outcome, based on proximity weights, may be antipodes.

A Geometric Interpretation of Minimax Outcomes

Minimax outcomes may be interpreted geometrically, which we illustrate next. Represent the eight possible ballots for three candidates as the vertices of the cube in Figure 1, in which approval (1) or disapproval (0) of each candidate is represented on a different axis (the first candidate on the horizontal axis, the second candidate on the vertical axis, and the third candidate on the planar axis). The three distinct ballots in our example (100, 110, and 101) are circled in Figure 1.

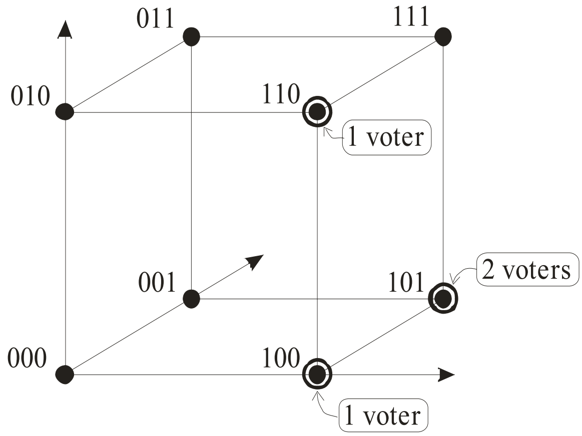

Figure 1 Geometric Representation of Ballots in 4-Voter Example

(Voted-for Ballots Circled)

The proximity weights of (5, 3, 10) for ballots (100, 110, 101) can be thought of as the *inertias* of these ballots: Voters who cast them would depart from them, moving outward the edges of the cube toward other vertices that they would find acceptable, at velocities inversely proportional to these inertias.[8] Thus, after 10 units of time, the two voters who cast ballot 101 would move distance 1 (i.e., traverse 1 edge from their node); the one voter who casts ballot 100 would move distance 2 (i.e., traverse 10/5 = 2 edges from its node); and the one voter who casts ballot 110 would move distance 10/3 = 3 1/3 (i.e., traverse 10/3 = 3 1/3 edges from its node). Moving at these relative rates, it is easy to see that the first committee that *all* voters would reach would be 101 in 6 units of time: The110 voter would find 101 acceptable at time 6; the other voters would find it acceptable sooner (the 100 voter at time 5, and the two 101 voters at time 0, because the latter voters start out at this node).

[8] This interpretation is inspired by a procedure called "fallback bargaining" (Brams and Kilgour, 2001), which can be applied to approval balloting (Brams, Kilgour, and Sanver, 2004; Kilgour, Brams, and Sanver, 2006). Technically, Brams and Kilgour (2001) define fallback bargaining only when preferences form a linear order over all alternatives. We use a straightforward extension of their procedure to allow for weak preferences. Under this procedure, voters fall back, or descend lower and lower, in their preferences until they reach an alternative on which all agree. This alternative minimizes the maximum distance they must traverse in order that their agreement is unanimous. The innovation here is that voters may descend at different rates, depending on the weighting scheme used; a proof that this descent minimizes the maximum *weighted* Hamming distance is given in Kilgour, Brams, and Sanver (2006). If the requirement is that only a majority, not all, voters must agree, the fallback-bargaining outcome is essentially the "majoritarian compromise"; see Hurwicz and Sertel (1999), Sertel and Sanver (1999), and Sertel and Yilmaz (1999).

If count weights rather than proximity weights are used (see Table 1), an analogous argument shows why there are three tied minimax outcomes. The two voters who cast ballots 100 and 110 traverse edges twice as vast as the two voters who cast ballot 101. From Figure 1, it is apparent that the first outcomes on which all four voters will agree will be within one edge of 101, and within two edges of 100 and 110, which are outcomes 100, 101, and 111. These are precisely the minimax outcomes, based on count weights, shown in Table 1.[9]

Henceforth we will use proximity weights, not count weights, to define minimax outcomes. Count weights reflect only the number of voters who cast a particular ballot but not how close this ballot is to other ballots, whereas proximity weights take into account both factors. Thus in our example, with count weights the two 101 voters have twice the inertia of each of voters 100 and 110, even though the 110 voter is not as close to the two 101 voters as the 100 voter is (see Figure 1).

But with proximity weights, the greater closeness of the 100 voter to the two 101 voters increases the 100 voter's inertia, and therefore influence, compared to the 110 voter, in the ratio 5:3. More generally, we think voters whose ballots are close, but not necessarily identical, to the ballots of other voters should add weight to these ballots (i.e., give them greater inertia). Likewise, extreme voters—outliers who are far from other voters—should have reduced influence on the outcome.[10]

A minimax outcome can be visualized as the first outcome that all voters will converge upon as they move along the edges of a hypercube—in all directions from their ballots—at speeds inversely proportional to their proximity weights. Not only may this outcome be very different from the minisum outcome (based on count weights), as we show next, but this difference raises the question of under what circumstances a minimax outcome is preferable to a minisum outcome in the selection of a committee.

3 Minimax Vs. Minisum Outcomes: They May Be Antipodes

Minimax and minisum outcomes may be identical or overlap, as we showed in our previous example. But they may also diverge maximally, as we show next. In each case, we ask which committee—minimax or minisum—better represents the electorate.

[9] It is worth noting that if the count weights were all 1 (if there were one 101 voter rather than two), the minimax outcome would be 100, which is the node exactly "between," and one edge distant from, 110 and 101 (see Figure 1).

[10] Other weighting schemes, of course, are possible, but proximity weights seem to us to balance the need to give representation to outliers, but downgrade this representation according to how far away (disconnected from other voters) they are.

As we will see, the answer depends on which candidates, based on their patterns of support, one thinks should appear on the committee.

Proposition 1. *If there are two or more candidates, tied minisum and tied minimax outcomes may include antipodes.*

Proof. Assume there are $n = 2$ voters who cast ballots 00 and 11 for $k = 2$ candidates. (Geometrically, the four possible committees (00, 10, 01, 11) can be represented by a square.) The minimax outcomes, 01 and 10, are antipodes, each lying at distance one from each of the two ballots. These outcomes, as well as outcomes 00 and 11 that are also antipodes, are all minisum outcomes, whose Hamming distances to the two ballots all sum to 2. The 2-voter example can easily be extended to any larger number of voters or candidates. Q.E.D.

Outcomes 01 and 10 lead to the election of just one person. This is not a committee as this term is usually used, but Proposition 1 holds for larger tied minimax and minisum committees.[11] These examples illustrate not only that minisum and minimax may give antipodes but also that each voting system, by itself, may produce them as well.

Note that there are half as many minimax outcomes as minisum outcomes in the 2-voter example. Whereas the minimax outcomes, 10 and 01, seem reasonable compromises, the additional minisum outcomes, 00 and 11, entirely favor one voter or the other. Manifestly, neither of the latter outcomes well represents *both* voters.

In the examples that follow, we will, for reasons of exposition, use antipodes like 0000 (no candidate elected) and 1111 (all candidates elected). These outcomes can readily be converted into antipodes, like 1100 and 0011, that more plausibly reflect real-world election possibilities.

The next two propositions show that minimax and minisum outcomes may be antipodes when there are as few as 4 candidates (with ties) and 5 candidates (without ties).

Proposition 2. *If there are four or more candidates, a nonunique minimax and a unique minisum outcome may be antipodes.*

Proof. Consider the following example, in which there $n = 11$ voters and $k = 4$ candidates:

1. 3 voters: 0000

[11] Consider the following example comprising 4 voters and 3 candidates: (1) 110; (2) 101; (3) 010; (4) 001. By constructing a table analogous to Tables 1 and 2, it is not difficult to show that there are four minimax outcomes, {000, 100, 011, 111}, which include two antipodal pairs; all eight possible outcomes are minisum. Notice that a larger minimax or minisum committee may not include a smaller committee; for example, 011 does not include 100. This failure of monotonicity—larger committees may not include smaller committees as subsets—is shared with other voting procedures, like the Kemeny rule, that have also been proposed to elect committees (Ratliff, 2003).

2. 2 voters: 0111
3. 2 voters: 1011
4. 2 voters: 1101
5. 2 voters: 1110

Applying equation (1), the proximity weight of ballot #1 is

$3/[(3 \times 0) + (2 \times 3) + (2 \times 3) + (2 \times 3) + (2 \times 3)] = 3/24 = 1/8.$

The proximity weight of ballot #2—and, by symmetry, ballots #3, #4, and #5—is

$2/[(3 \times 3) + (2 \times 0) + (2 \times 2) + (2 \times 2) + (2 \times 2)] = 2/21.$

Multiplying the weights by a factor ($8 \times 21 = 168$) that clears denominators produces a proximity weight of 21 for ballot #1, and a proximity weight of 16 for each of ballots #2, #3, #4, and #5. Thus, the voters who cast ballot #1 are more influential under the minimax procedure than all other voters.

Note that one of the 7 tied minimax outcomes in Table 3 is #1 (0000), whereas the unique minisum (or MV) outcome is the antipode, #16 (1111), as can be calculated directly: 3 of the 5 voters approve of each candidate. This 4-candidate example of antipodal minisum and minimax outcomes can easily be extended to any larger number of candidates. Q.E.D.

In the example in the proof of Proposition 1, there were more minisum outcomes than minimax outcomes (4 minisum and 2 minimax), whereas the opposite is true for the example in the proof of Proposition 2 (1 minisum and 7 minimax). Note that the 3 voters who cast ballot 0000 in the latter example will be totally dissatisfied by minisum outcome 1111, a Hamming distance of 4 away. This seems a good argument for a minimax outcome, which is at maximum distance 3 from the ballot of any voter.

The most stark clash of minimax and minisum outcomes occurs when they are unique and antipodal.

Ballot:	**0000**	**0111**	**1011**	**1101**	**1110**	**Max**
No. of Voters:	**3**	**2**	**2**	**2**	**2**	
Proximity Weight:	**21**	**16**	**16**	**16**	**16**	
1. 0000	0	48	48	48	48	48*
2.	21	64	32	32	32	64

1000						
3. 0100	21	32	64	32	32	64
4. 0010	21	32	32	64	32	64
5. 0001	21	32	32	32	64	64
6. 1100	42	48	48	16	16	48*
7. 1010	42	48	16	48	16	48*
8. 1001	42	48	16	16	48	48*
9. 0110	42	16	48	48	16	48*
10. 0101	42	16	48	16	48	48*
11. 0011	42	16	16	48	48	48*
12. 1110	63	32	32	32	32	63
13. 1101	63	32	32	32	32	63
14. 1011	63	32	32	32	32	63
15. 0111	63	32	32	32	32	63
16. 1111	84	32	32	32	32	84

*Minimum of column.

Table 3 Derivation of Minimax Committees Based on Proximity Weights (11-Voter Example)

Proposition 3. *If there are five or more candidates, a unique minimax and a unique minisum outcome may be antipodes.*

Proof. Consider the following example, in which there are n = 11 voters and k = 5 candidates:

1. 11100
2. 11010
3. 11001
4. 10110
5. 10101
6. 10011
7. 01110
8. 01101
9. 01011
10. 00111
11. 00000

Instead of constructing a table like Table 3, with a row for each of the 32 possible committees, we exploit the example's symmetry by noting that 10 voters approve of exactly 3 candidates in the $\binom{5}{3}$ = 10 different ways that this is possible; voter #11 approves of no candidates.

Applying equation (1), the proximity weight of ballot #1 is

1/[0 + 2 + 2 + 2 + 2 + 4 + 2 + 2 + 4 + 4 + 3] = 1/27;

by symmetry, it is the same for ballots #2 through #10. The proximity weight of ballot #11 is

1/[10(3) + (1 × 0)] = 1/30.

Clearing denominators, the proximity weight of the first 10 ballots is 10, and the proximity weight of ballot #11 is 9. Thus, the voter who casts ballot #11 is slightly less influential than the voters who cast the other 10 ballots.

Because the maximum Hamming distance between any two of the first 10 ballots is 4, the maximum weighted Hamming distance of one of these ballots is 4 × 10 = 40. By contrast, the maximum weighted distance of ballot #11 is 3 × 9 = 27, because this ballot is a Hamming distance of 3 from each of the 10 other ballots (and 0 from itself).

To show that none of the 32 – 11 = 21 other committees (ballots) has a greater maximum weighted Hamming distance than 27, consider (i) the one committee with 5 members (maximum weighted distance of 5 × 9 from 00000), (ii) the five different committees with 4 members (maximum weighted distance of 4 × 9 from 00000), (iii) the ten committees with 2 members (maximum weighted distance of 5 × 10 from one of the 3-member committees), and (iv) the five committees with 1 member (maximum weighted distance of 4 × 10 from one of the 3-member committees). In all these cases, the maximum weighted distances exceed the maximum weighted distance of 3 × 9 = 27 of ballot #11 from all others, so this distance is minimal and, therefore, ballot #1 is the minimax outcome. This 5-candidate example of antipodal minisum and minimax outcomes can easily be extended to any larger number of candidates. Q.E.D.

Once again, a minimax committee (00000) seems better to represent *all* voters than a minisum committee (11111). (Recall that these antipodes might be more plausible 2-member and 3-member committees, such as 11000 and 00111.) Whereas the voter casting ballot #11 would be completely dissatisfied by 11111, the other 10 voters would mildly prefer 11111 to 0000.

These results for antipodes suggest that minimax committees may be more representative of all voters than minisum committees, because they leave no voter too aggrieved, especially not voters whose ballots are relatively close to those of many other voters. To be sure, if the aggrieved voters are only an isolated minority, like voter #11 in the foregoing example, it may be preferable to give better representation to the large majority than to appease the minority.

Our main purpose in this section has been to highlight such a trade-off by posing minimax outcomes as an alternative to minisum outcomes. Whether or not minimax should be used instead of minisum depends on the importance one attaches to the Rawlsian criterion (Rawls, 1971) of making the worst-off voter as well off as possible.

In section 5 we will show that the divergence between minisum and minimax outcomes is not purely theoretical but actually occurred in a real-life election that used approval balloting to elect a committee of 12 members. But first we discuss elections in which not every subset of candidates is a possible outcome.

4 Endogenous Vs. Restricted Outcomes

So far we have assumed that any subset of the candidates can be the minisum or minimax committee elected, whereas it is commonplace to put restrictions on the outcome. For example, one may want to specify the size of the committee to be elected (to ensure that it is neither too small nor too large to function efficiently) or its composition (to ensure that certain groups are at least minimally represented).

We refer to elections as *endogenous* if all outcomes are possible winners; otherwise, they are *restricted*. As shown in Kilgour, Brams, and Sanver (2006), both minimax and minisum procedures apply equally well to restricted and endogenous elections. In a restricted election, one constructs tables, like Table 1, in which only rows representing eligible committees—that is, those not disqualified by the restrictions—appear.

In the election of a committee restricted according to size, the minisum procedure is equivalent to a more familiar procedure, namely plurality voting, as shown by the next proposition.

Proposition 4. *When the size of a committee is restricted to c members, the minisum outcomes are the sets of c candidates receiving the most votes.*

Proof. See Appendix.

The idea behind the proof is the following. We know that when there is no restriction, the minisum outcome is the set of candidates that win a majority of votes (Brams, Kilgour, and Sanver, 2004, Proposition 4). Assume that the number of majority winners is less than the desired committee size c. Then adding to the majority winners those non-majority candidates with the most approvals until the committee size is exactly c minimizes the sum of the weighted distances to these members and, therefore, the sum of weighted distances to these members plus the majority winners. Likewise, if the number of original majority winners is greater than the desired committee size c, subtracting the candidates with the fewest approvals until the committee size is exactly c minimizes the sum of the weighted distances of the candidates who remain.

Unlike the minisum case, we know of no algorithm to find minimax outcomes—short of constructing tables like Table 2. When outcomes are endogenous, we have already shown that minisum and minimax outcomes may be antipodes. Restricting outcomes will not necessarily lead to a common minisum and minimax outcome, as our next example with $n = 4$ voters and $k = 4$ candidates illustrates:

1. 1100
2. 1010
3. 1001
4. 0111

Assume a committee of size $c = 1$ is to be chosen. It is easy to see that 1000 is the unique minisum outcome, because candidate 1 receives 3 votes when the three other candidates receive 2 votes each.[12]

[12] If there were no single-winner restriction, the election of candidate 1 and any one, two, or all three of the other candidates (i.e., outcomes 1100, 1010, 1001, 1110, 1101, 1011, and 1111) are tied minimax outcomes that are, like outcome 1000, also minisum.

To find the minimax outcome, use equation (1) to calculate the proximity weights, which for ballot 1100 is

$1/[0 + 2 + 2 + 3] = 1/7$,

and, by symmetry, is the same for ballots 1010 and 1001. Similarly, the proximity weight of ballot 0111 is

$1/[3 + 3 + 3 + 0] = 1/9$.

Clearing denominators, the proximity weights of the first three ballots are 9 each, and the proximity weight of ballot 0111 is 7. Thus, the voter who casts ballot 0111 is less influential than the voters who cast the other 3 ballots.

As shown in Table 4, the unique minimax outcome is 1111, which does not satisfy the restriction of the committee to one member.[13] (Note that 1111 is not the ballot of any voter, nor is the minimax outcome of 1000.[14]) Surprisingly, of the four possible committees that include one member (see committees #2 – #5 in Table 4), the three tied minimax outcomes—0100, 0010, and 0001, which are committees #3, #4, and #5—do not include the minisum outcome, 1000.

It may seem bizarre not to elect the most approved candidate, especially one approved of by a majority, in a single-winner election. We will revisit this issue in the concluding section, asking whether the minimax criterion is reasonable, especially in single-winner elections.

[13] We show the 16 possible outcomes in Table 4 to illustrate how the restriction to $c = 1$ may alter minimax outcomes, making them, as in this example, disjoint from the unrestricted outcome.
[14] Brams, Kilgour, and Zwicker (1997, 1998) were the first to show that the minisum outcomes need not correspond to the ballot of any voter, which they called the "paradox of multiple elections." Özkal-Sanver and Sanver (2005) show that a voting rules ensures a Pareto-optimal outcome if and only if it never exhibits this paradox.

Ballot:	**1100**	**1010**	**1001**	**0111**	**Max**
No. of Voters:	**1**	**1**	**1**	**1**	
Proximity Weight:	**9**	**9**	**9**	**7**	
1. 0000	18	18	18	21	21
2. 1000	9	9	9	28	28
3. 0100	9	27	27	14	27
4. 0010	27	9	27	14	27
5. 0001	27	27	9	14	27
6. 1100	0	18	18	21	21
7. 1010	18	0	18	21	21
8. 1001	18	18	0	21	21
9. 0110	18	18	36	7	36
10. 0101	18	36	18	7	36
11. 0011	36	18	18	7	36
12. 1110	9	9	27	14	27
13. 1101	9	27	9	14	27
14. 1011	27	9	9	14	27
15. 0111	27	27	27	0	27
16. 1111	18	18	18	7	18*

*Minimum of column.

Table 4 Derivation of Minimax Committees Based on Proximity Weights (4-Voter Example)

5 Manipulability

A voting procedure is *manipulable* if it is possible for a voter, by misrepresenting his or her preferences, to obtain a preferred outcome. To define "preferred," we relate Hamming distance to preferences.

Specifically, we assume that a voter's ballot indicates his or her most-preferred committee, or *top preference*. We further assume that the voter's preference is *spatial* in the sense that outcomes that are farther (as measured by Hamming distance) from the top preference are less preferred. Thus, if p^i is voter *i*'s top preference and *p* and *q* are any outcomes such that $d(p^i, p) < d(p^i, q)$, then voter *i* strictly prefers *p* to *q*. In other words, as distance increases, a voter's preference falls off, reaching a minimum at the antipode of the top preference; note that there is no assumption about the voter's preference among ballots that are at equal Hamming distance from the top preference.[15]

We make an additional assumption about preferences over sets: If outcome *a* is preferred to outcome *b*, then $\{a\}$ is preferred to $\{a, b\}$, which we interpret as a tie in which each of the two outcomes occurs with positive probability. This assumption, which is used in the proof of Proposition 5, seems eminently plausible.

Proposition 5. *The minimax procedure is manipulable, whereas the minisum procedure is not.*

Proof. First consider the minimax procedure. In the example in Table 2, we showed the unique minimax outcome is 101, which is a Hamming distance of 2 from the ballot of the voter who casts ballot 110. But if this voter falsely indicates his or her ballot to be 100, then the situation would appear as the following:

2 voters: 100

2 voters: 101

It is easy to see that the proximity weights according to equation (1) are now equal, so we need only ask which outcome minimizes the maximum Hamming distance of the voters from their ballots. Clearly, the ballots themselves do this, so the minimax outcome is {100, 101}.

The 110 voter who falsely indicated a top preference of 100 prefers this tied outcome, because he or she prefers 100 to 101, and, by our previous assumption, prefers

[15] Like both the minimax and minisum procedures, spatial models of preference could be based on other metrics, such as "root-mean-square," which is essentially Euclidean distance. We think the Hamming metric is particularly well suited for measuring the distance of a voter from an outcome, because it reflects equally the voter's disagreements with the candidates elected and with those not elected. However, spatial models cannot mirror well the preference of a voter who wants a "balanced" committee—say, with an equal number of men and women. For example, assume that eight male (M) and female (F) candidates are listed as follows, FFFFMMMM, and the committee is to have four members. The "balanced" voter's two most-preferred committees might be 11001100 and 00110011, which are antipodes, so voting for either will work to rule out the other, especially under minimax. Of course, the worst case for this voter, 11110000 or 00001111 (all women or all men), can be precluded if it is mandated that the committee must have equal numbers of men and women. Then the male chauvinist who votes for the four males (00001111) will never get his favorite committee but will, instead, support equally 11000011 and 00111100—in fact, *all* balanced committees. Thus, if this voter wants to have some effect on the outcome, it behooves him to vote for some women! For a review of the literature on ranking sets of items, see Barbera, Bossert, and Pattanaik (1998).

{100, 101} to 101. Hence, the minimax procedure is manipulable. The proof that the minisum procedure is not manipulable is given in the Appendix. Q.E.D.

The idea of the proof for the minisum procedure is easy to describe. Because a voter's choices are binary on each candidate, it is always in his or her interest to support those, and only those, candidates of whom he or she approves. Moreover, the voter's decision on each candidate does not affect which other candidates are elected, so each voter cannot be worse off as a consequence of voting truthfully on *all* candidates.

Although the minimax procedure is vulnerable to manipulation in theory, in practice it is probably almost as resilient to manipulation as the minisum procedure. To exploit it would require a manipulative voter to have virtually complete information about the voting intentions of other voters, which is unlikely in most real-world situations. Indeed, merely finding the truthful minimax outcome is computationally hard, as we indicated earlier, reflecting the fact that the number of possible outcomes increases exponentially with the number of candidates.[16]

We next turn to a real-world election. This election renders concrete some of the theoretical and practical issues we have discussed and raises some new questions as well.

6 The Game Theory Society Election

In 2003, the Game Theory Society (GTS) used approval voting for the first time to elect 12 new council members from a list of 24 candidates. (The council comprises 36 members, with 12 elected each year to serve 3-year terms.) We give below the numbers of members who voted for from 1 to all 24 candidates (no voters voted for between 19 and 23 candidates):

Votes cast	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	24
# of voters	3	2	3	10	8	6	13	12	21	14	9	25	10	7	6	5	3	3	1

[16] Although this might be seen as a disadvantage of the minimax procedure, computers make the calculation of minimax outcomes feasible for 30 or more candidates (in section 5 we analyze an election with 24 candidates). For more on the computability of minimax, see Kilgour, Brams, and Sanver (2006).

Casting a total of 1574 votes, the 161 voters, who constitute 45% of the GTS membership, approved, on average, 1574/161 ≈ 9.78 candidates; the median number of candidates approved of, 10, is almost the same.

The modal number of candidates approved of is 12 (by 25 voters), echoing the ballot instructions that 12 of the 24 candidates were to be elected. The approval of candidates ranged from a high of 110 votes (68.3% approval) to a low of 31 votes (19.3% approval). The average approval received by a candidate was 40.7%, though only candidates who received at least 69 votes (42.9 % approval) were elected.

In the GTS election, there were $2^{24} \approx 16.8$ million possible ballots. It turned out that 2 of the 161 voters voted identically. As one might expect, the identical ballot, 111100011001101000000111, was cast by 2 of the 25 modal voters who voted for 12 candidates. If all ballots approving of 12 candidates are assumed equiprobable, the probability that no two of 25 ballots are identical is

$$[t(t-1)(t-2)\ldots(t-24)]/t^{25} \approx 0.999889,$$

where $t = \binom{24}{12} = 2{,}704{,}156$, based on reasoning given in note 3. The complement of this probability, 0.000111, is the probability that at least two voters cast identical ballots.[17]

If there had been no restriction in the GTS election, the 5 candidates approved of by a majority – at least 81 of the 161 voters – would have been elected. Adding the next 7 biggest vote-getters gives the minisum outcome under the restriction that 12 candidates must be elected.

Do these candidates best represent the electorate? In fact, 4 of the 12 minimax winners differ from the minisum winners. Each set of winners is given below—ordered from most popular on the left to least popular on the right—with differences between those elected to each council underscored.[18]

[17] We have made this calculation for each category of voter—from those who cast 1 vote to those who cast 18 votes—excluding only the category containing the one voter who voted for all 24 candidates (because there is only one such ballot). The voters most likely to cast an identical ballot are the three who vote for one candidate; the probability that at least two of them cast the same ballot is 0.121528. To generalize for all voters, let p_i be the probability that no two voters who cast i votes chose an identical ballot. It follows that the probability that no two voters in any category cast an identical ballot is $p_1 p_2 \ldots p_{17} p_{18}$, so the complement of this probability is the probability that at least two voters in one or more categories cast identical ballots. The latter probability in the GTS election is 0.131009; it is far greater than the probability that we calculated in note 3 (0.000768), which did not take into account the 18 categories into which voters sorted themselves empirically. But even this greater probability is likely an underestimate, because it does not reflect the fact that some candidates were far more approved of than others, rendering dubious the assumption that all ballots in each category are equiprobable.

[18] It is worth pointing out that minisum outcomes are always Pareto-optimal; if this were not the case, then there would be some other outcome such that some voter is less distant and no voter is more distant,

Council Restricted to 12 Members

Minisum: 111111111111000000000000

Minimax: 111111011000111000000001

Observe that four of the minisum winners would have been displaced by candidates who received fewer votes, one of whom was the candidate who received the fewest votes.[19]

To exclude this candidate would have put some voters at a greater weighted distance than including him or her. Thereby minimax gives voice to voters who approve of unpopular candidates if they make the council more representative by not leaving some voters "out in the cold."

If the size of the council had not been restricted to 12 winners but instead had been endogenous, the minisum and minimax councils would have differed substantially:

Unrestricted Council (Minisum, 5 Members; Minimax, 8 Members)

Minisum: 111110000000000000000000

Minimax: 111100001010101000000000

As noted earlier, the minisum council would have comprised only the 5 majority winners. By contrast, the minimax outcome includes 8 candidates; four of these are candidates came in 11th, 13th, 15th, and 17th.

We showed in section 5 that it was possible, in theory, for a voter successfully to manipulate a minimax outcome, but we contended that this would be well-nigh impossible in most elections. As a case in point, consider the single voter who voted for all 24 candidates in the GTS election and who, we presume, was indifferent among all the candidates.[20]

Might this voter have influenced the outcome if the size of the council had been endogenous? In fact, if minisum had been the procedure, the 5 biggest vote-getters would still have been elected had this voter not voted. But the minimax outcome would have changed

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contradicting the defining property of minisum. By contrast, minimax outcomes need not be Pareto-optimal. To illustrate, we revisit the example in note 8, in which the top preferences of 4 voters for 3 candidates are as follows: (1) 110; (2) 101; (3) 010; (4) 001. There are four minimax outcomes: (a) 000; (b) 100; (c) 011; (d) 111. Because outcome (c) is at least as good as outcome (a) for all voters, and better for voters (3) and (4), and outcome (b) is at least as good as outcome (d) for all voters, and better for voters (1) and (2), only outcomes (b) and (c) are Pareto-optimal.

[19] Fishburn (2004) shows that the 12 minisum winners tended to be supported somewhat more strongly by voters who voted for few candidates, whereas the reverse was true for the losers. In effect, voters who approved of few candidates were more discriminating, helping to put the minisum winners over the top.

[20] Of course, this voter might simply have relished the role of being an outlier by approving of everybody, even though he or she had no effect on the actual (minisum) outcome under the GTS rules.

From: $111\underline{1}00001010\underline{1}01000\underline{0}0000$ (with voter who approved of all candidates)

To: $111\underline{0}00001010\underline{0}01000\underline{1}0000$ (without this voter).

Thus, the absence of this voter would have reduced the number of winners from 8 to 7, consistent with the reduced approval that all candidates would have received.

This voter's absence would have had no effect on the composition of the 12-member minimax council we gave earlier, however. This suggests that outliers are unlikely to be consequential when the size of a committee is fixed. Thus, fixing the size of the committee may make the minimax procedure less vulnerable to this form of manipulation.

Making the number of candidates to be elected endogenous, and using the minisum procedure, is tantamount to electing only candidates approved of by a majority (5 candidates in the case of the GTS council). In fact, this rule is used to determine who is admitted to certain societies, though the threshold for entry is not always a simple majority.

In the concluding section, we summarize our results and comment on the feasibility of the minimax procedure in different kinds of elections. Not only may this procedure give a dramatically different outcome from the minisum procedure, but this outcome may better reflect diverse views within the electorate.

7 Conclusions

Under approval balloting, each voter approves of a subset of candidates. The minisum and minimax procedures find subsets that are as close as possible to the ballots of all voters, but according to two different senses of "closeness." Whereas the minisum procedure selects the outcome that minimizes the sum of Hamming distances to all voters—or, equivalently, the average Hamming distance—which uses count weights, the minimax procedure finds the outcome that minimizes the maximum weighted Hamming distance, which uses proximity weights.

Geometrically, the latter can be visualized in terms of voters moving from their nodes of a hypercube, which represent their ballots, along the edges at speeds proportional to their proximity weights (or inertias). Minimax outcomes are the node or nodes that *all* voters reach first.

Minimax and minisum may yield diametrically opposed outcomes, or antipodes, if there are as few as four candidates (with ties), five candidates (without ties). If "representation" means not antagonizing any voters—especially those with similar or identical preferences—too much, then minimax outcomes seem more representative of the entire electorate than minisum outcomes.

We analyzed the 2003 election by the Game Theory Society (GTS) of 12 new members to its council. The minimax procedure would have given 4 different winners from the minisum procedure, which was the procedure actually used by the GTS.

There would have been a greater difference if the number of candidates to be elected had been endogenous. The minisum procedure would have elected only the 5 candidates who were approved of by a majority, whereas the minimax procedure would have elected 8 candidates, including 4 relatively unpopular candidates that better represented certain sets of voters.

In single-winner elections, the approval-voting winner (minisum outcome) would seem the normatively most desirable choice. But as we showed in an example in section 4, a different candidate may less antagonize a minority (1 of the 4 voters in this example), so it is not apparent—even in single-winner elections—that the minisum winner should always triumph.

Whereas the minisum procedure is not manipulable, the minimax procedure is. In practice, however, it would be virtually impossible for a voter to induce a preferred outcome because of (i) a lack of information about other voters' intended ballots and (ii) the computational complexity of processing such information, even if it were available.

In the GTS election, the absence of the outlier who voted for all 24 candidates would not have changed the minimax outcome. But if the number of winners had been endogenous, the minimax outcome would have been reduced from 10 to 9 winners. In effect, this voter "pulled" the outcome in the direction of a larger council, which is consistent with his or her approval of all candidates. We do not view this choice as manipulative if this voter was genuinely unconcerned about the composition of the council.

We think that the unanimity rule in fallback bargaining—that the descent continues until all voters approve of an outcome—is plausible in the election of most committees, though less stringent rules are possible (Brams and Kilgour, 2001; Brams, Kilgour, and Sanver, 2004). Whether the size of a committee should be fixed or endogenous (perhaps within a range) will depend, we think, on the importance of electing a committee whose size significantly affects its ability to function.

Even if size is made endogenous, voters should probably be given some guidance as to roughly what size would be appropriate. Without this information, it may be hard for them to gauge how many candidates to approve of in an election.

These practical considerations aside, we believe that more theoretical research on the properties of the minimax procedure is needed. For example, if this procedure is used, is it appropriate to break ties among the minimax winners using minisum? Are there other ways of combining criteria? What effects do the correlated preferences of voters, or perceived similarities in candidates, have on the minimax and minisum outcomes, or on the likelihood of antipodes? How might information (e.g., from polls) affect the manipulability of the procedure?

In addition to these questions, other procedures, especially those that allow for proportional representation (Potthoff and Brams, 1998; Brams and Fishburn, 2002; Ratliff, 2003), should be considered. Just as approval voting in single-winner elections stimulated considerable theoretical and empirical research beginning a generation ago (Weber 1995; Brams and Fishburn, 2002, 2004; Brams and Sanver, 2004), we hope that the minimax procedure generates new research on using approval balloting to elect committees under the minimax procedure.

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Appendix

Proposition 4. *When the size of a committee is restricted to c members, the minisum outcomes are the sets of c candidates receiving the most votes.*

Proof: Assume that there are n voters and k candidates, and that $m_h > 0$ voters cast the ballot $q^h = (q_1^h, q_2^h, \ldots, q_k^h)$, where $h = 1, 2, \ldots, t$. Note that $\sum_{h=1}^{s} m_h = n.$ For an arbitrary binary k-vector $x = (x_1, x_2, \ldots, x_k)$, define

$$d_j(x,q^h) = \begin{cases} 0 & \text{if } x_j = q_j^h \\ 1 & \text{if } x_j \neq q_j^h \end{cases}$$

for $h = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, k$. Then it is clear that the Hamming distance from x to q^h is given by $d(x,q^h) = \sum_{j=1}^{k} d_j(x,q^h)$. The endogenous minisum winner is any x that minimizes $D(x) = \sum_{h=1}^{t} m_h d(x,q^h)$, whereas the restricted minisum winner is any x that minimizes $D(x)$ among all x's containing exactly c 1's. We first find an equivalent representation for $D(x)$.

For any x and j, define $S_j(x) = \sum_{h=1}^{t} m_h d_j(x,q^h)$. $S_j(x)$ represents the number of voters who disagree with k-vector x on candidate j. Note that $D(x) = \sum_{j=1}^{k} S_j(x)$, and that $S_j(x)$ depends only on x_j and not on the other $k - 1$ components of x. Therefore x represents a minisum winner if and only if x minimizes

$$D(x) = \sum_{h=1}^{t} m_h d(x,q^h) = \sum_{j=1}^{k} s_j(x) \quad \text{(A1)}$$

With equation (A1) we can characterize all minisum committees. Define $K_j = \sum_{h=1}^{t} m_h q_j^{\ h}$, so K_j is the number of voters who vote for candidate j; clearly, $n - K_j$ is the number of voters who vote against j. Now[21]

$$S_j(x) = \begin{cases} n - K_j & \text{if } x_j = 1 \\ K_j & \text{if } x_j = 0 \end{cases} \tag{A2}$$

Next consider how to choose $x = (x_1, x_2, \ldots, x_k)$ so that exactly c of the x_j's equal 1 and $D(x)$ is minimized. By equation (A2), $D(x)$ will be the sum of c values of $n - K_j$ (corresponding to the members of the winning committee) and $k - c$ values of K_j (corresponding to the unsuccessful candidates). Clearly, putting the c candidates with the largest values of K_j on the committee minimizes $D(x)$. In other words, the minisum winners under the restriction that a committee of size c is to be elected must correspond to a vector x such that $x_j = 1$ if and only if j belongs to some subset of c candidates that receives the most votes. Q.E.D.

Proposition 5. *The minimax procedure is manipulable, whereas the minisum procedure is not.*

Proof that minisum is non-manipulable: We apply the preference model introduced in the text to equations (A1) and (A2) to show that a voter is best served by voting for his or her top preference. Assume that voter i is one of the m_h voters whose top preference is $q_h = p^i$. We show that i cannot do better than to cast ballot $q^h = p^i$.

Suppose that i's top preference, x_j, satisfies $x_j = 1$ for some j. Then our preference assumptions imply that voter i prefers any committee with $x_j = 1$ to the committee that is otherwise the same but has $x_j = 0$. Consider i's decision to vote truthfully ($p_j^i = 1$) or untruthfully ($p_j^i = 0$) on candidate j. According to equation (A2), selecting $p_j^i = 0$ rather than $p_j^i = 1$ reduces the value of $S_j(x)$ from K_j to $K_j - 1$. The four possibilities implied by equation (A1) for whether candidate j belongs to the minisum winner(s) are set forth in the table below:

	j elected if $p_j^i = 1$?	*j elected if $p_j^i = 0$?*
$K_j < n - K_j$	Never	Never
$K_j = n - K_j$	Sometimes	Never
$K_j = n - K_j + 1$	Always	Sometimes
$K_j > n - K_j + 1$	Always	Always

[21] From equation (A2) it follows that among all possible committees, $x = (x_1, \ldots, x_k)$ minimizes $D(x)$ if and only if, for each j, $x_j = 1$ if $n - K_j < K_j$ and $x_j = 0$ if $K_j < n - K_j$. This minimization proof is different from, and more general than, the proof given in Brams, Kilgour, and Sanver (2004).

In the table, “sometimes” means that the total votes for and against candidate *j* are equal; if such a tie occurs, the set of minisum committees consists of one or more pairs of committees that differ only in that one includes candidate *j* and one does not.)

Because the voter always prefers that candidate *j* be a member of the committee, it follows from the table that voter *i* is never worse off by voting truthfully (i.e., for candidate *j*) and may be better off. The argument is analogous if *i*’s top preference satisfies $x_j = 0$. Q.E.D.