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► **To cite this version:**

Manuel Pascal Martin, Stéphane Cordier, Jérôme Balesdent, Dominique Arrouays. Periodic solutions for soil carbon dynamics equilibriums with time-varying forcing variables. *Ecological Modelling*, 2007, XXX (XXX), pp.XXX. 10.1016/j.ecolmodel.2006.12.030 . hal-00124048v2

HAL Id: hal-00124048

<https://hal.science/hal-00124048v2>

Submitted on 16 Feb 2007

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1 **Periodic solutions for soil carbon dynamics equilibriums** 2 **with time-varying forcing variables**

3
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8 **Abstract**

9 Numerical models that simulate the dynamics of carbon in soil are increasingly used to
10 improve our knowledge and help our management of the carbon cycle. Calculation of the long
11 term behavior of these models is necessary in many applications but encounters the difficulty
12 of managing the periodic forcing variables, e.g., seasonal variations, such as carbon inputs
13 and decomposition rates. This calculation is conventionally done by running the model over
14 large time durations or by assuming constant forcing variables. Two methods, which make it
15 possible to rapidly compute periodic solutions taking into account the time variations of these
16 variables, are proposed. The first one works on discrete-time models and the second one on
17 continuous-time models involving Fourier transforms. Both methods were tested on the
18 Rothamsted carbon model (RothC), a discrete-time model which has also been given a
19 continuous approximation, using realistic and unrealistic sets of time-varying forcing functions.
20 Both methods provided an efficient way to compute the periodic solutions of the RothC
21 model within the application domain of the model. Compared to running the discrete model to
22 the equilibrium, reduction in the computational cost was of up to 95% at the expense of a
23 maximum absolute error of 1% for the estimation of carbon stocks. For specific distributions
24 of the forcing variables the use of Fourier transform of zero order, which was equivalent to
25 assume constant forcing variables, led to a maximum absolute error of 55% in the estimation
26 of the long term behavior of the model. There, a Fourier transform of order higher than zero is
27 required.

28 Keywords: soil organic carbon dynamics, discrete formulation, continuous formulation,
29 steady state, periodic solutions, linear model, Fourier series.

30 **Introduction**

31 **1.1. carbon dynamics**

32 The soil organic carbon (SOC) plays an important role in several environmental and land
33 management issues. One of the most important issues is the role that SOC plays as part of the
34 terrestrial carbon and might play as a regulator of the atmospheric CO₂. Many factors are
35 likely, in a near future, to modify the SOC content, including changes in agricultural practices
36 (Betts, 2000; Vleeshouwers and Verhagen, 2002; Bellamy et al., 2005) and global climate
37 changes (Jenkinson et al., 1991; Cao and Woodward, 1998; Cox et al., 2000; Jones et al.,
38 2005; Knorr et al., 2005). Understanding SOC and soil organic matter (SOM) dynamics as a
39 function of soil characteristics, agricultural management and climatic conditions is therefore
40 crucial, and many models have been developed in this perspective. Most models of SOM
41 turnover, excepting a few (Bosatta and Agren, 2003), are compartmental models, exhibiting
42 various degree of complexity. The compartments represent carbon originating from plants or
43 contained in soil and transformed by microorganisms and each one is characterized by a
44 particular decomposition rate representing more labile or more stable forms of soil organic
45 matter. Some models include N turnover and/or plant growth modules (CENTURY) when
46 others only focus on SOC (RothC). Also, most use a linear method of transferring quantities
47 between the different compartments (Baisden and Amundson, 2003) but some models
48 including non-linear dynamics have also been developed more recently (Manzoni et al.,
49 2004).

50 These models are used in a variety of ways and often for long term studies (Coleman et al.,
51 1997; Falloon and Smith, 2002; Franko et al., 2002; Shevtsova et al., 2003; Shirato, 2005;
52 Shirato et al., 2005b; Shirato and Yokozawa, 2005a). The behavior of the SOC system, over a
53 long term and assuming that the environment of the system (inputs of organic carbon, climatic
54 conditions) is stable, is reported to tend toward a steady state. Although many soils under
55 study might not have reached equilibrium, being able to compute and predict the long-term
56 solution is extremely valuable. It gives a synthetic view of the system in given agro-climatic
57 conditions, makes it possible to test if a studied soil has reached an equilibrium or not, to
58 envision what would be the consequences of specific events onto a given soil assuming that a
59 new stable state is reached and to serve as a control case or initial conditions (Thornton and
60 Rosenbloom, 2005). Technically, the equilibrium assumption is also commonly used to solve
61 analytically mathematical systems. Such an analytical solution gives an explicit relationship
62 between model inputs and outputs and may, in turn, be computed without simulating or

63 integrating numerically the system until it reaches a stable state, thus saving computation
64 time. When models cannot be formulated analytically, estimating the steady state solutions
65 still can be useful and generic or model specific efficient numerical methods are available
66 (Thornton et al., 2005). More generally, being able to use analytical forms of the long-term
67 solutions is particularly useful in understanding models behavior and relationships between
68 input and output variables of the model.

69 Some of the SOC models have been formulated mathematically (Parshotam, 1996; Bolker et
70 al., 1998; Yang et al., 2002; Baisden et al., 2003; Manzoni et al., 2004) and approaches, as the
71 development of the ICBM family models (Katterer and Andren, 2001), specifically aim at
72 proposing analytically solved models representing the conventional wisdom of soil C and N
73 modelling. For these models, when studying N and C soil content at steady state, it is usually
74 assumed that forcing variables (typically climatic variables and variables representing inputs)
75 can be set to their average value, calculated over a representative year for instance, which
76 considerably eases the mathematical treatment. There, the long-term behavior of the model
77 truly is a steady state. Consequences of such an assumption have for now been tested only
78 empirically for some models and specific conditions. In some cases, environmental shifts
79 from one stable state to another or brief events are considered and mathematical treatment
80 used to estimate the new stable state after perturbation or the system resilience. We propose
81 here two methods which make it possible to deal with continuously time varying agro-
82 climatic conditions (e.g. forcing variables), when they can be specified as periodic functions.
83 The first one works on discrete-time models and the second one on continuous-time models
84 involving Fourier transforms. We considered the Rothamsted model with crop cultivations, as
85 representative of many models of soil organic matter dynamics, anticipating that the second
86 method could also be applied to models involving non-linear dynamics. We used these
87 methods to test the consequences of assuming yearly constant agro climatic condition instead
88 of considering their intra-annual variability.

89 **Methods**

90 ***1.2. Discrete formulation***

91 The RothC model (Coleman et al., 1997) splits the soil carbon into four active compartments
92 and one inactive. At each time step, the four active compartments, decomposable plant
93 material, DPM, resistant plant material, RPM, the microbial community BIO and the humus,
94 HUM, undergo decomposition as a function of a rate constant, depending on the compartment
95 and on a rate modifier. The rate modifier depends on the clay content of the soil, climatic

96 variables and land cover. Products of the decomposition are CO₂ and carbon feeding the BIO
 97 and HUM compartments. The fraction of the decomposed carbon incorporated into BIO and
 98 HUM increases as a function of the clay content of the soil. Carbon enters the soil through the
 99 DPM and RPM compartments. The fraction input in DPM and RPM respectively is chosen as
 100 a constant which is an estimate of the decomposability of the plant material. It depends on the
 101 cultivation being considered. The model can be formulated as

$$C_{t+1} = F_t C_t + B_t$$

102 Where

$$C_t = \begin{pmatrix} dpm \\ rpm \\ bio \\ hum \end{pmatrix}_t$$

$$F_t = \begin{pmatrix} e^{-\rho_t k_{dpm}} & 0 & 0 & 0 \\ 0 & e^{-\rho_t k_{rpm}} & 0 & 0 \\ \alpha(1 - e^{-\rho_t k_{dpm}}) & \alpha(1 - e^{-\rho_t k_{rpm}}) & \alpha(1 - e^{-\rho_t k_{bio}}) + e^{-\rho_t k_{bio}} & \alpha(1 - e^{-\rho_t k_{hum}}) \\ \beta(1 - e^{-\rho_t k_{dpm}}) & \beta(1 - e^{-\rho_t k_{rpm}}) & \beta(1 - e^{-\rho_t k_{bio}}) & \beta(1 - e^{-\rho_t k_{hum}}) + e^{-\rho_t k_{hum}} \end{pmatrix}$$

103 and

$$B_t = \begin{pmatrix} a_{dpm} \\ a_{rpm} \\ a_{bio} \\ a_{hum} \end{pmatrix} b_t$$

104 The four input coefficients (a_{dpm} , a_{rpm} , a_{bio} and a_{hum}) sum up to 1 and in the most common case
 105 one uses $a_{dpm}=\gamma$, $a_{rpm}=1-\gamma$, $a_{bio}=0$ and $a_{hum}=0$, γ depending on the quality of the plant
 106 material.

107 Here, α and β are fractions of metabolized C incorporated respectively into BIO and HUM. b_t
 108 is the carbon amount (t.ha⁻¹) entering the system at month t , k_i the decomposition rate for
 109 compartment i and ρ_t the rate modifier.

110 *Characterization of the long-term behavior*

111 In case where b_t and ρ_t are constant, one can demonstrate that the (I_4-F) matrix, where the
 112 matrix I_4 is the 4-by-4 identity matrix, has an inverse (see below) and that the system yields a
 113 steady state solution. Assuming that F and B are respectively the time constant carbon flows
 114 and carbon inputs, one can write

$$C^* = (I_4 - F)^{-1} B$$

115 However, usually F_t and B_t vary through time but it can be assumed that they have a periodic
 116 behavior. Typically, if the agronomical practices are cyclic and if the weather conditions can
 117 be considered as periodic, ρ_t , b_t , and consequently F_t , B_t will also behave periodically.
 118 Assuming that the periodicity of these variables is P , one looks for a solution of C such that

$$C_{t+P} = C_t \quad (1)$$

119 For example, considering the common case the case where P is 12 months, we can write
 120 down

$$\begin{aligned} C_{t+1} &= F_1 C_t + B_1 \\ &\dots \\ C_{t+11} &= F_{11} C_{t+10} + B_{11} \\ C_t &= F_{12} C_{t+11} + B_{12} \end{aligned}$$

121 which can be reformulated as:

$$\begin{pmatrix} 0 & I_4 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & I_4 \end{pmatrix} \begin{pmatrix} C_t \\ \vdots \\ C_{t+10} \\ C_{t+11} \end{pmatrix} = \begin{pmatrix} F_1 & 0 & \dots & \dots & 0 \\ 0 & F_2 & \ddots & & \dots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & F_{12} \end{pmatrix} \begin{pmatrix} C_t \\ \vdots \\ C_{t+10} \\ C_{t+11} \end{pmatrix} + \begin{pmatrix} B_1 \\ \vdots \\ B_{11} \\ B_{12} \end{pmatrix} \quad (2)$$

122 and finally yields:

$$\begin{pmatrix} C_t \\ \vdots \\ C_{t+11} \end{pmatrix} = - \begin{pmatrix} F_1 & -I_4 & 0 & \dots & 0 \\ 0 & F_2 & -I_4 & & \dots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -I_4 \\ -I_4 & 0 & \dots & 0 & F_{12} \end{pmatrix}^{-1} \begin{pmatrix} B_1 \\ \vdots \\ B_{12} \end{pmatrix} \quad (3)$$

123 Solving this system (which can be performed *via* matrix inversion by common statistical
 124 packages or spreadsheet programs) yields a vector of dimension $4*12$, which is the sequence

125 of states $C_i=(dpm_i, rpm_i, bio_i, hum_i)^T$, i in $[[1,12]]$ and which characterizes the oscillatory state
 126 of carbon stock in each compartment, keeping track of the temporal variability of the forcing
 127 variables over the period.

128 **1.3. Continuous formulation**

129 In real soil systems, processes involved in the RothC model are continuous in time and thus
 130 one can propose the following continuous formulation, where C' denotes the derivative of C
 131 with respect of time:

$$C'(t) = \rho(t)AC(t) + B(t) \quad (4)$$

132 with

$$A = \begin{pmatrix} -k_{dpm} & 0 & 0 & 0 \\ 0 & -k_{rpm} & 0 & 0 \\ \alpha k_{dpm} & \alpha k_{rpm} & (\alpha - 1)k_{bio} & \alpha k_{hum} \\ \beta k_{dpm} & \beta k_{rpm} & \beta k_{bio} & (\beta - 1)k_{hum} \end{pmatrix} \quad (5)$$

$$B(t) = (a_{dpm} \ a_{rpm} \ a_{bio} \ a_{hum})^T b(t)$$

133 $\rho(t)$ is the decomposition rate modifier. In the current RothC formulation, $\rho(t)$ is a function of
 134 monthly rainfall, temperature, pan open evaporation and land cover, as well as percentage of
 135 clay in the considered soil. When the time varying input variables (climatic and agricultural
 136 variables) are considered as periodic functions of period P , $\rho(t)$ itself is a periodic function
 137 with the same period. $b(t)$ is considered on a periodical basis too, as for the discrete
 138 formulation of the model. Again, the system defined by $C(t)$ is expected to tend toward a
 139 oscillatory state as $t \rightarrow +\infty$. Study of the eigenvalues of $\rho(t)A$ enables to characterize such a
 140 behavior. Let us define $y(t)$ and $\mathcal{A}(t)$.

$$y(t) = e^{-\mathcal{A}(t)}C(t) \quad (6)$$

141 where $\mathcal{A}(t)$ is the primitive of $\rho(t)A$.

142 From Eq.(4) and (6) we obtain

$$y'(t) = e^{-\mathcal{A}(t)}B(t)$$

143 and

$$y(t) = y_0 + \int_0^t e^{-\mathcal{A}(s)}B(s)ds \quad (7)$$

144 Setting $\mathcal{A}(0)$ to the four by four zero matrix, from Eq.(6) and (7) it can be written

$$C(t) = e^{A(t)}C_0 + e^{A(t)} \int_0^t e^{-A(s)} B(s) ds$$

145 We can first observe that if $A(t)$ eigenvalues are negative, $C(t)$ as $t \rightarrow +\infty$ do not depend on
 146 initial conditions C_0 . Secondly, if $C(t)$ has a periodic solution, say C_0 , with period P , it can be
 147 shown using semigroup representations that it satisfies the following Equation,

$$C_0 = e^{A(T)}C_0 + e^{A(T)} \int_0^T e^{-A(s)} B(s) ds$$

148 A solution to this equation exists and is unique if $(Id - e^{-A(T)})$ has an inverse, which can be
 149 demonstrated to be true if A is invertible. A sufficient condition for A to be invertible is that
 150 all its eigenvalues are nonpositive, i.e. using the definition of A (Eq.(5)) when $\alpha + \beta < 1$. This
 151 condition is always true for the RothC model due to the definition of α and β (Coleman and
 152 Jenkinson, 1995) and thus the stock of carbon in each compartment tends towards a periodic
 153 solution in large times whatever the input values are.

154 *Characterization of the long-term behavior*

155 Approximations of this behavior can be made using Fourier series. Setting

$$C_N(t) = \sum_{k=-N}^N C_k e^{ikt}, \quad \rho_N(t) = \sum_{k=-N}^N \rho_k e^{ikt} \quad \text{and} \quad b_N(t) = \sum_{k=-N}^N b_k e^{ikt} \quad (8)$$

156 with

$$\rho_k = \int_0^{2\pi} \rho(s) e^{-iks} ds \quad \text{and} \quad b_k = \int_0^{2\pi} b(s) e^{-iks} ds$$

157 ρ_k and b_k coefficients can be obtained from the monthly input values used in the RothC model.
 158 $C(t)$, $A(t)$ and $B(t)$ can be replaced by their respective Fourier transform in Eq.(4) giving the
 159 following approximation.

$$\sum_{k=-N}^N ikC_k e^{ikt} = A \sum_{j,k / j+k=-N}^N C_j \rho_k e^{i(j+k)t} + \sum_{k=-N}^N b_k e^{ikt}$$

160 Setting N , the order of the Fourier series, to zero, we calculate the C_0 term.

$$C_0 = -A^{-1} B_0 / \rho_0 \quad (9)$$

161 Assuming that $B_0 = (\gamma, 1 - \gamma, 0, 0)^T \cdot b_0$, which means that carbon inputs only to the DPM and
 162 RPM compartments, leads to the following solution:

$$C_0 = \begin{pmatrix} \gamma b_0 / \rho_0 k_{dpm} \\ (1-\gamma)b_0 / \rho_0 k_{rpm} \\ b_0 \alpha / [(1-\alpha-\beta)\rho_0 k_{bio}] \\ b_0 \beta / [(1-\alpha-\beta)\rho_0 k_{hum}] \end{pmatrix} \quad (10)$$

163 b_0 and ρ_0 terms represent averages of $b(t)$ and $\rho(t)$ over the considered period, which is one
 164 year. This solution gives an explicit formulation of the long-term behavior of the system,
 165 which obviously equals what would have been found using the assumption of constant forcing
 166 variables set to the average values. This solution does not enable to takes into account the
 167 temporal variability of the carbon stocks throughout the year. This could have been achieved
 168 by computing the C_1 term which itself is an approximation of the primary oscillations of the
 169 system's long-term behavior. Such a calculation lies beyond the scope of this paper. These
 170 oscillations directly depend on the temporal variability of the forcing variables but their size is
 171 usually small compared to the total SOC. In the following, they will be handled only with the
 172 discrete formulation of the RothC model (Eq.(3)).

173 **1.4. Comparison of the approaches**

174 Parshotam (1996) showed that given some restrictions the continuous formulation (Eq.(4)) is
 175 a good approximation of what would be the continuous formulation of the original discrete
 176 time RothC model. It is possible to turn this the other way round and say that the RothC
 177 model is an approximation of the discretization of the continuous model given in Eq.(4).
 178 Discretization of Eq.(4) leads to, in case of constant inputs during the sampling intervals
 179 (Parshotam, 1996):

$$C_{(k+1)\Delta t} = e^{\rho(k\Delta t)A\Delta t} C_{k\Delta t} + (\rho(k\Delta t)A)^{-1} (e^{\rho(k\Delta t)A\Delta t} - I_4) B(k\Delta t)$$

180 RothC is an approximation of the above equation because

$$F_t \approx e^{\rho(k\Delta t)A\Delta t} \quad \text{and} \quad B_t \approx (\rho(k\Delta t)A)^{-1} (e^{\rho(k\Delta t)A\Delta t} - I_4) B(k\Delta t)$$

181 Thus, the parameters used in the continuous model (e.g. α , β , k_{dpm} , k_{rpm} , k_{bio} and k_{hum}) should
 182 not be equated with those of the RothC model. However, numerically it makes little
 183 difference, and in the following developments, we shall do it.

184 Both approaches (using the discrete or the continuous formulation) can be used to
 185 characterize the periodic long-term behavior of the system. The first approach (Eq.(3)) uses
 186 the discrete formulation of the model which formally reproduces the specification of the
 187 RothC model contrary to the second one (Eq.(9)) which uses a continuous formulation and a
 188 zero order Fourier transform. This latest approach gives a simpler explicit solution, function

189 of the input variables and parameters of the model (α , β , k_{dpm} , k_{rpm} , k_{bio} and k_{hum}). It might also
 190 be more interesting to work with the continuous form of the model because of its greater
 191 generality and since it is usually easier to discretize a continuous model rather than doing the
 192 opposite.

193 To assess the validity of both approaches in characterizing the long-term behavior with time-
 194 varying forcing variables, we first compared both approaches between themselves and with
 195 the discrete model where variability of forcing is leveraged over the year. This was performed
 196 using a weather dataset composed by monthly averages calculated over 12 years (1992-2004)
 197 on a 0.125° grid (4144 cells) covering the French country. For each point of the grid, we
 198 considered a unique crop system, with inputs being 0.50, 0.20, 0.10, 0.10, 0.10, 1.44 tC.ha⁻¹
 199 respectively for each month from March to August, otherwise null with a bare soil (adapted
 200 from Swinnen et al., 1995 and Bolinder et al., 1997 for winter wheat). %Clay was set to 10%.
 201 The computation of the long-term behavior using the discrete formulation resulted in a
 202 periodic solution, i.e. a sequence of twelve C_i states, i in [1..12] (Eq.(3)). The solution given
 203 by a discrete formulation with averaged forcing variables was a single state noted C_{avg} , the
 204 calculation obtained using the continuous formulation was also a single state, noted C_0
 205 (Eq.(9)). We compared C_{avg} , with C_0 and with the average state of the C_i states, noted $\langle C_i \rangle$.

206 To test more systematically the validity of the estimator based on Fourier series introduced in
 207 (Eq.(8)), e.g. C_0 , the periodic solutions were also computed when varying the forcing
 208 variables independently, using in some case extreme and unrealistic values. The precision of
 209 C_0 was assessed using the σ/C_0 ratio, where σ is the standard deviation of the C_i states and
 210 represents the size of the oscillations characterizing the periodic solution. The bias of C_0 was
 211 assessed using the $\text{abs}(\langle C_i \rangle - C_0)/C_0$ ratio. We modeled the distributions of the forcing
 212 variables using Gaussian functions as

$$y_i = y_{\min} + \frac{N_i(d)}{\sum_i N_i(d)} a \quad \text{with} \quad N_i(d) = \frac{1}{d\sqrt{2\pi}} e^{-\frac{(i-6)^2}{2d^2}} \quad (11)$$

213 Where y_i , i in [1..12] is the value of the forcing variable (either $b(t)$ or $\rho(t)$) at month i , a and d
 214 respectively the amplitude and dispersion characterizing the distributions, and y_{\min} a minimum
 215 value for the considered variable. Low values of the d parameter yielded distributions having
 216 a spike around the sixth month, high values uniform distributions. The a parameter
 217 represented a scaling parameter. a and d were varied at once and sampled linearly within their
 218 range of variation. y_{\min} and the range of variation of d and a were, respectively for $b(t)$ and

219 $\rho(t)$, (0.1, [0.1,5] and [0.1,10]) and (0.83, [0.1,5] and [0.1,100]). One forcing variable
220 remained constant whilst the amplitude and dispersion of the other was varied and set to 0.2
221 for b_i , i in [1..12] and set to 0.8, 0.8, 0.8, 0.82, 1.06, 1.10, 1.06, 0.82, 0.8, 0.8, 0.8, 0.8 for ρ_i , i
222 in [1..12].

223 To compute the solutions with both approaches, the calculation of modifiers of the
224 decomposition rates was done using SQL requests under the PostgreSQL DBMS and further
225 calculations using the R Software. All the computations were done on a bi-xeon, 2Go RAM.

226 **Results**

227 Computation times on the standard climatic dataset were 57.9", and 46.9" for respectively the
228 C_i (discrete formulation) and C_θ (continuous formulation) calculations (to be compared to
229 15'10" needed when running the Fortran implementation of RothC available online at the
230 Rothamsted Research website; this time includes for each point of the climatic dataset reading
231 the data files, running the model using the equilibrium mode and writing the results).

232 On the long term, temporal variability of the forcing variables resulted in oscillations of the
233 total SOC which size reached for some points of the standard climatic dataset 11% of the total
234 SOC, as estimated using the discrete formulation. However, over this standard climatic
235 spectrum, all methods gave similar results regarding the average SOC value at equilibrium
236 (maximum absolute error of 1%). The long term values for the different carbon compartments
237 at equilibrium were all similar but the value for the DPM compartment (Figure 1). For this
238 compartment, the effect of the temporal variability of the forcing variable (obtained by
239 comparing $\langle C_i \rangle$ to the other solutions) was important. The error caused by using RothC as
240 the discretization of the continuous formulation (see § 1.4) can be seen when comparing C_{avg}
241 with C_θ . It appears that C_{avg} slightly overestimates the DPM pool compared to C_θ .

242 C_θ became strongly biased and imprecise for extreme distributions of $\rho(t)$ where amplitude of
243 this forcing variable was high and variability over the period was large (for $a=100$ and $c=0.1$
244 yielding maximum ρ values of 100.1, bias reached 0.5 and imprecision 0.27, Figure 2, top
245 diagrams). The imprecision and bias of C_θ also depended on the amplitude and dispersion of
246 $b(t)$ but always remained small (Figure 2, bottom diagrams). We checked (not displayed here)
247 that the bias and imprecision of C_θ compared to C_i was not caused by using RothC as a
248 discretization of the continuous model but by the fact that C_θ leverages the time-variability of
249 the forcing variables when C_i does not. Thus, the domains where C_θ is imprecise and more

250 importantly biased are the domains where the time-variability of the forcing variables greatly
251 determines the behavior of the system.

252 **Conclusion**

253 The continuous formulation using Fourier transforms makes it possible to specify analytically
254 the forcing variables as functions of time, and then to obtain analytical solutions for the
255 mathematical formulation of the model of carbon dynamics under study. Here, we used a zero
256 order transform, which makes the forcing variables constant through time, and studied the
257 validity of such an assumption.

258 We showed in turn that such an approximation resulted in short computation times and is
259 reasonably precise (i) for the common application domain of the RothC model and (ii) in case
260 there is no concern about intra-annual variations of decomposable plant material and to a
261 smaller extent of the microbial community. If these conditions are not met, one may want to
262 use the discrete formulation or higher order Fourier transforms in order to grasp more of the
263 temporal variability of the variables. It is likely that in standard conditions, the use of average
264 agro-climatic conditions for computing steady state solutions of linear models of organic
265 matter dynamics, which is commonly found in literature on the subject (Bolker et al., 1998;
266 Baisden et al., 2003; Manzoni et al., 2004), can be legitimated. The Fourier series approach is
267 not restricted to linear models or to models taking only C dynamics into account. It could be
268 particularly relevant for non-linear systems, where assuming that forcing variable can be
269 averaged could become even more tedious than for linear systems. We also emphasize here
270 the fact that the method proposed to deal with the continuous formulation, since it essentially
271 relies on the use of Fourier series, is most suited to the modeling of periodic functions and to
272 the case where decomposition and input functions can themselves be considered as periodical.

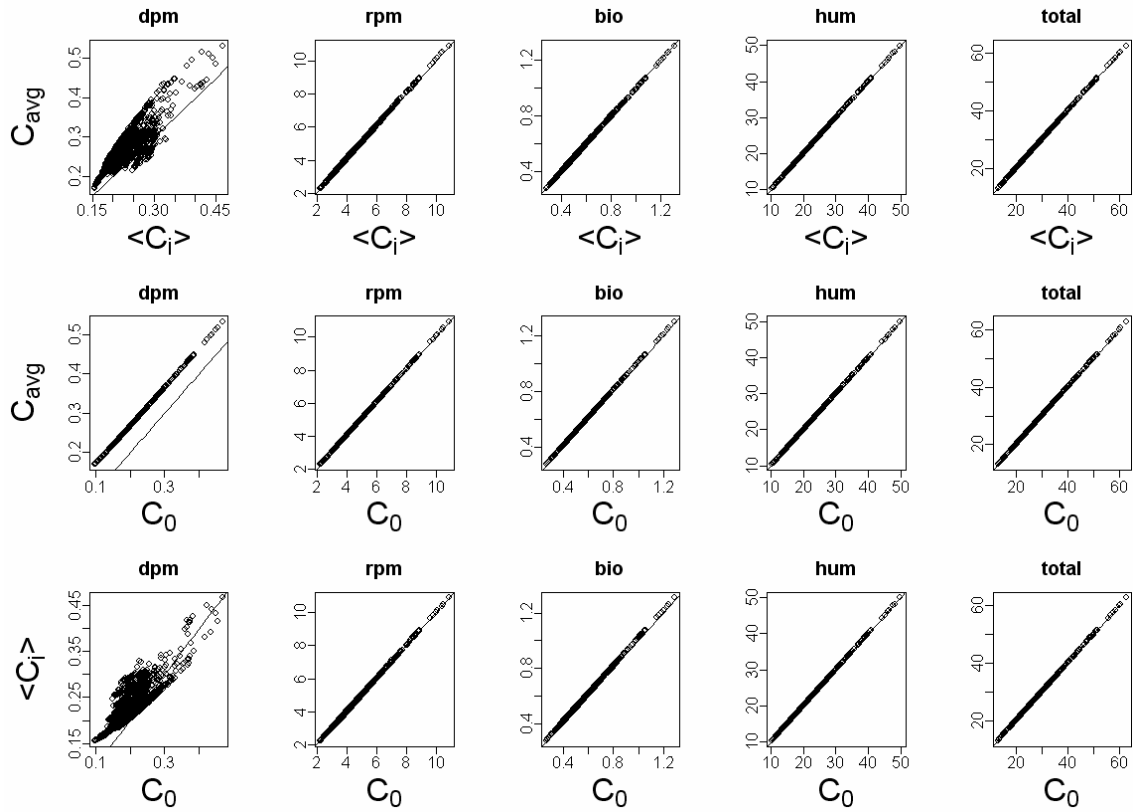
273 The approach concerning discrete-time models was used here as a way to quickly compute the
274 long-term behavior of the discrete-time model RothC, without making any approximations
275 and thus having a larger application domain than the approach using the continuous
276 formulation. It is not likely to result in a simple analytical formulation of the equilibrium of
277 the system, because of the relative complexity of the matrix to be inversed in order to
278 compute the solution. Nevertheless, it speeds up considerably the computation compared to
279 the use of the RothC's implementation. It can be applied to more complex systems and makes
280 it possible to take into account the full time-variability of the discrete forcing variables. There
281 might be constraints on the applicability of the discrete method, aimed at ensuring the
282 solvability of the system described in Eq.(3). Determining these constraints is out of the scope

283 of this paper but results about the Toeplitz matrices might help in this perspective. Indeed, the
284 algebraic structure of system (2) is of circulant form (or Toeplitz) and this allows the use of
285 very efficient methods for solving the eigenvalues problem we are interested in (Gray, 2006).

286 The methods proposed here to compute equilibrium solutions gave, within the RothC
287 application domain, similar results for long term SOC dynamics compared with the Fortran
288 implementation of the RothC model, while being up to 19.4 times faster. This might be
289 critical when applying the model on very large data sets, for instance those produced by
290 combining climatic, landuse and soil characteristics layers within a GIS, in order to spatially
291 compute long term SOC stocks. In addition, working at equilibrium simplifies the analysis of
292 the results as only the long-term solution is considered. Both methods are of course not
293 restricted to one-year periods and could be applied to cycles with much longer periods, for
294 instance, to crop rotations or to large climatic oscillations (with periods). It would be also
295 interesting to consider the case where, while remaining oscillatory, the forcing variables
296 exhibit a drift. This could be applied, for instance, to study the effect of climate change on
297 SOC stocks.

298 **Acknowledgments**

299 The authors thank David Coleman, Rothamsted Research, UK, for giving useful details about
300 the RothC implementation and the referees for their helpful comments.



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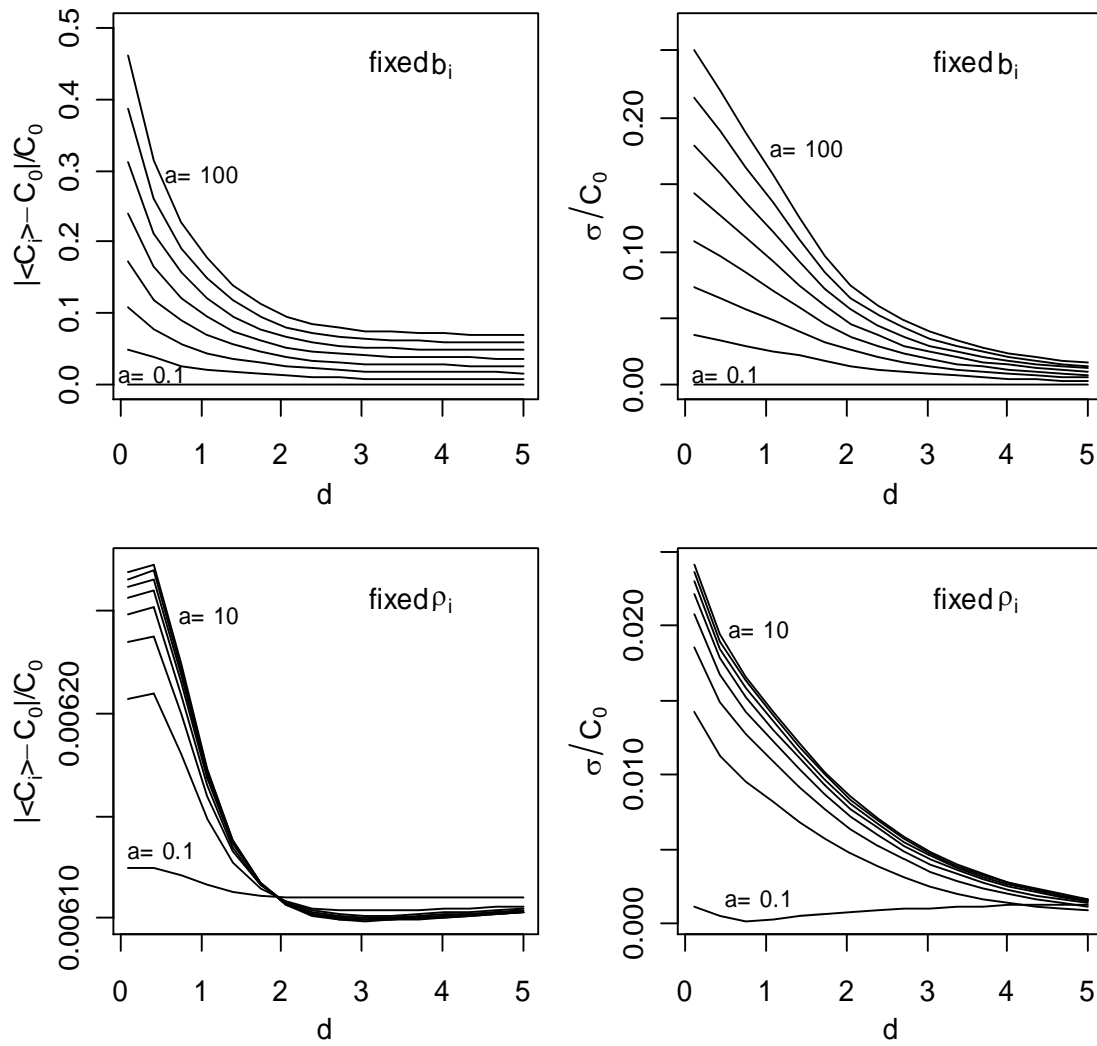
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Figure 1 : long-term solutions for each compartment of the model and for the total carbon content, over the whole set of climatic conditions and with %clay=10%. All values are given in $t.ha^{-1}$. First line of diagrams plots the results of the discrete formulation of RothC ($\langle C_i \rangle$) against the results obtained with the discrete formulation with constant forcing variables (C_{avg}). Line Two gives the results of the continuous form (C_0) against C_{avg} and line three $\langle C_i \rangle$ against C_0 . The plain lines drawn on the charts represent the $y=x$ function.



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Figure 2 : Left hand diagrams give the precision and right hand diagrams bias of the C_0 estimator. Top diagrams are obtained keeping the sequence of monthly inputs constant and varying dispersion and amplitude of the ρ_i sequence. Bottom diagrams are obtained keeping the ρ_i sequence constant and varying dispersion and amplitude of inputs over the period. a and d are the parameters used in Eq.(11).

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