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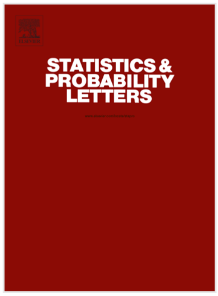

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Asymptotics of Sums of Lognormal Random Variables with Gaussian Copula

Søren Asmussen[1] Leonardo Rojas-Nandayapa[2]

Abstract

Let $(Y_1, \ldots, Y_n)$ have a joint n-dimensional Gaussian distribution with a general mean vector and a general covariance matrix, and let $X_i = \mathrm{e}^{Y_i}$, $S_n = X_1 + \cdots + X_n$. The asymptotics of $\mathbb{P}(S_n > x)$ as $n \to \infty$ is shown to be the same as for the independent case with the same lognormal marginals. In particular, for identical marginals it holds that $\mathbb{P}(S_n > x) \sim n\mathbb{P}(X_1 > x)$ no matter the correlation structure.

Key words: Dependence, subexponential distribution, tail asymptotics, tail dependence, Value-at-Risk.

Tail probabilities $\mathbb{P}(S_n > x)$ of a sum $S_n = X_1 + \cdots + X_n$ of heavy-tailed risks $X_1, \ldots, X_n$ are of major importance in applied probability and its applications in risk management, such as the determination of risk measures like the Value-at-Risk (VaR) for given portfolios of risks, evaluation of credit risk, aggregate claims distributions in insurance, operational risk (Cruz, 2002; Frachot, Moudoulaud and Roncalli, 2004). Under the assumption of independence among the risks, the situation is well understood. In particular, from the very definition of subexponential distributions, given identical marginal distributions, the maximum among the involved risks determines the distribution of the sum and, on the other hand, for non-identical marginals the distribution of the sum is determined by the component(s) with the heaviest tail (Asmussen, 2000, Ch.IX).

Over the last few years, several results in the direction of allowing dependent X_i have been developed. A survey and some new results are given in Albrecher, Asmussen and Kortschak (2005). For regularly varying marginals and $n = 2$, this paper gives bounds for $\mathbb{P}(S_n > x)$ in terms of the tail dependence coefficient

$$\lambda = \lim_{u \to 1^-} \mathbb{P}\big(F_2(X_2) > u \,\big|\, F_1(X_1) > u\big),$$

and it is noted that the asymptotics of $\mathbb{P}(S_n > x)$ is the same as in the independent case when $\lambda = 0$. Alink, Löwe and Wüthrich (2005) and references there contain approximations under the assumption $\lambda > 0$. For general discussion of bounds, see Denuit, Genest and Marceau (1999), Cossete, Denuit and Marceau (2002) and Embrechts and Pucceti (2005).

The overall picture is that, except for some special cases, the situation seems best understood with regularly varying marginals. However, in particular in insurance and finance, lognormal marginals is the more important case (a common folklore states that correlations of log-returns of stock prices etc. are often of the order 0.4). This paper deals with the basic case of lognormal marginals with a multivariate Gaussian copula. That is, we can write $X_k = \mathrm{e}^{Y_k}$ where the random vector $(Y_1, \ldots, Y_n)$ has a multivariate Gaussian distribution with $\mathbb{E}\, Y_k = \mu_k$, $\mathbb{V}\mathrm{ar}\, Y_k = \sigma_k^2$, $\mathbb{C}\mathrm{ov}\,(Y_k, Y_\ell) = \sigma_{k\ell}$ (here $\sigma_{kk} = \sigma_k^2$) and $\mathbb{C}\mathrm{orr}\,(Y_k, Y_\ell) = \rho_{k\ell}$. It is well-known that here $\lambda = 0$ when $\rho_{k\ell} < 1$ so that the results of Alink, Löwe and Wüthrich (2005) do not apply. We write $\underline{\mu} = (\mu_k)_{k=1,\ldots,n}$, $\Sigma = (\sigma_{k\ell})_{k,\ell=1,\ldots,n}$ and $\mathrm{LN}_n(\underline{\mu}, \Sigma)$ for the joint distribution of $(X_1, \ldots, X_n)$. When $n = 1$, we just write F_{μ,σ^2} and $\overline{F}_{\mu,\sigma^2}(x)$ for the tail. Our result is:

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Theorem 1. *Let $(X_1, \ldots, X_n)$ have a $LN(\underline{\mu}, \Sigma)$ distribution with $\rho_{k\ell} < 1$ when $\sigma_k^2 = \sigma_\ell^2$ $(k \neq \ell)$, and let σ^2, μ, m_n be defined by*

$$\sigma^2 \;=\; \max_{k=1,\ldots,n} \sigma_k^2\,, \quad \mu \;=\; \max_{k:\,\sigma_k^2=\sigma^2} \mu_k\,, \quad m_n \;=\; \#\left\{k:\, \sigma_k^2 = \sigma^2,\, \mu_k = \mu\right\}.$$

Then

$$\lim_{x\to\infty} \frac{\mathbb{P}(S_n > x)}{\overline{F}_{\mu,\sigma^2}(x)} \;=\; m_n\,. \tag{1}$$

Remark 1. Note that it is well-known that

$$\overline{F}_{\mu,\sigma^2}(x) \;\sim\; \frac{\sigma}{\sqrt{2\pi}(\log x - \mu)} \exp\Big\{-\frac{(\log x - \mu)^2}{2\sigma^2}\Big\} \tag{2}$$

(this follows, e.g., from the standard asymptotics $\mathbb{P}(Y > x) \sim \varphi(x)/x$ as $x \to \infty$ for a standard normal r.v., where φ is the standard normal density and $\sim$ means that the limit of the ratio is 1). □

Remark 2. For two distributions F_k, F_ℓ, we say that F_k has a lighter tail than F_ℓ if $\overline{F}_k(x)/\overline{F}_\ell(x) \to 0$ as $x \to \infty$. From (2) it follows in particular by easy calculus that $\overline{F}_{\mu_k,\sigma_k^2}(x)$ has a lighter tail than $\overline{F}_{\mu_\ell,\sigma_\ell^2}(x)$ when either $\sigma_k^2 < \sigma_\ell^2$ or $\sigma_k^2 = \sigma_\ell^2$, $\mu_k < \mu_\ell$, a fact that will be used repeatedly It also follows that for any lognormal r.v. and $0 < \beta < 1$,

$$\lim_{x\to\infty} \frac{\mathbb{P}(X > x - x^\beta)}{\mathbb{P}(X > x)} = 1 \tag{3}$$

(just note that $\log^p(x - x^\beta) = \log^p x + \mathrm{o}(1)$ for $p = 1, 2$). □

Remark 3. The hypothesis that $\rho_{k\ell} < 1$ when $\sigma_k^2 = \sigma_\ell^2$ can be dropped at the cost of some technicalities. If $\rho_{k\ell} = 1$ and $\sigma_k^2 = \sigma_\ell^2$ $(k \neq \ell)$, then $X_k = X_\ell\, \mathrm{e}^{\mu_k - \mu_\ell}$, so one can replace $X_k + X_\ell$ by $(1 + \mathrm{e}^{\mu_k - \mu_\ell})X_k$. We omit the rest of the details. □

The idea of the proof of Theorem 1 is to find upper and lower bounds for $\mathbb{P}(S_n > x)$ with equivalent asymptotic behavior. It will be seen that the argument used to prove that the lower bound proposed is asymptotically equivalent to $m_n\overline{F}_{\mu,\sigma}(x)$ is based on the tail independence property of the Gaussian copula. So, if we use any other tail independent copula with subexponential marginal distributions, the argument used remains true. This is not the case for the upper bound; the proof is based on the properties of the marginal distribution and the Gaussian copula. This means that tail independence is not a sufficient condition. Albrecher, Asmussen and Kortschak (2005) present an example of a bivariate and tail independent distribution with lognormal marginals which fails to have the asymptotic behavior in (1). These remarks do not exclude the existence of other tail independent copulas than the Gaussian such that (1) holds for lognormal marginals.

Proof of Theorem 1. We proceed by induction. The case $n = 1$ is straightforward. For the induction step, we assume that the theorem holds for any arbitrary lognormal random vector with Gaussian copula of size n. Next, we will prove that the theorem is true for a random vector of size $n + 1$.

For the proof, the following assumptions (which are made w.l.o.g.) are convenient:

A1. $X_1, \ldots, X_{n+1}$ are ordered in such way that if $\ell < k$ then X_k and X_ℓ either have the same marginal distribution, or X_k has lighter tail than X_ℓ (cf. Remark 2). Thus $\overline{F}_{u,\sigma^2}(x) = \mathbb{P}(X_1 > x)$.

A2. $\mu = 0$. If not, replace X_k and x by $X_k\mathrm{e}^{-\mu}$ and $x\mathrm{e}^{-\mu}$ in $\mathbb{P}(S_{n+1} > x)$.

We also need the following lemma which is proved later:

Lemma 1. *Under the hypothesis of Theorem 1, if* A1 *and the induction hypothesis hold, then there exists* $0 < \beta < 1$ *such that*

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x^\beta, X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)} = 0.$$

Choose β as in Lemma 1 and consider the following relations

$$\begin{aligned}\mathbb{P}(S_{n+1} > x) &= \mathbb{P}(S_{n+1} > x, S_n \le x^\beta) + \mathbb{P}(S_{n+1} > x, S_n > x^\beta, X_{n+1} \le x^\beta) \\ &\quad + \mathbb{P}(S_{n+1} > x, S_n > x^\beta, X_{n+1} > x^\beta) \\ &\le \mathbb{P}(X_{n+1} > x - x^\beta) + \mathbb{P}(S_n > x - x^\beta) + \mathbb{P}(S_n > x^\beta, X_{n+1} > x^\beta).\end{aligned}$$

Using Lemma 1, it follows that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(X_1 > x)} \le \limsup_{x\to\infty} \frac{\mathbb{P}(X_{n+1} > x - x^\beta)}{\mathbb{P}(X_1 > x)} \; + \; \limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x - x^\beta)}{\mathbb{P}(X_1 > x)}.$$

Here Remark 2 and the induction hypothesis guarantee that the two limsup's on the r.h.s. are actually limits, so that the r.h.s. becomes

$$\begin{aligned}&\lim_{x\to\infty} \frac{\mathbb{P}(X_{n+1} > x - x^\beta)}{\mathbb{P}(X_{n+1} > x)} \frac{\mathbb{P}(X_{n+1} > x)}{\mathbb{P}(X_1 > x)} + \lim_{x\to\infty} \frac{\mathbb{P}(X_1 > x - x^\beta)}{\mathbb{P}(X_1 > x)} \frac{\mathbb{P}(S_n > x - x^\beta)}{\mathbb{P}(X_1 > x - x^\beta)} \\ &= \lim_{x\to\infty} \frac{\mathbb{P}(X_{n+1} > x)}{\mathbb{P}(X_1 > x)} + \lim_{x\to\infty} \frac{\mathbb{P}(S_n > x - x^\beta)}{\mathbb{P}(X_1 > x - x^\beta)} \;=\; \lim_{x\to\infty} \frac{\mathbb{P}(X_{n+1} > x)}{\mathbb{P}(X_1 > x)} + m_n^*\end{aligned}$$

where the first step uses Remark 2 and the second step the induction hypothesis with m_n^* denoting the number among $X_1, \ldots, X_n$ in A1 which have the same tail as X_1. From Remark 1 the limit in the r.h.s. is 0 if X_{n+1} has lighter tail than X_1, or 1, if X_{n+1} and X_1 have the same distribution. Observe that by A1, X_1 cannot have lighter tail than X_{n+1}. We have proved

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(X_1 > x)} \le m_{n+1} \tag{4}$$

for the appropriate value of m_{n+1}.

For the second part of the induction step consider the next lower bound of $\mathbb{P}(S_{n+1} > x)$:

$$\begin{aligned}\sum_{i=1}^{n+1} \mathbb{P}(X_i > x, X_j \le x, \text{for all } j \ne i) &\ge \sum_{i=1}^{n+1} \mathbb{P}(X_i > x) - \mathbb{P}\Big(\bigcup_{j\ne i} \{X_i > x, X_j > x\}\Big) \\ &\ge \sum_{i=1}^{n+1} \mathbb{P}(X_i > x) - \sum_{j\ne i} \mathbb{P}(X_i > x, X_j > x).\end{aligned}$$

If A1 holds with $i < j$ then

$$\mathbb{P}(X_i > x, X_j > x) \sim \begin{cases} \mathrm{o}(1)\, \mathbb{P}(X_j > x) & \rho_{ij} < 1 \quad \text{and} \quad \sigma_i^2 \ge \sigma_j^2 \\ \mathbb{P}(X_j > x) & \rho_{ij} = 1 \quad \text{and} \quad \sigma_i^2 > \sigma_j^2. \end{cases}$$

The first asymptotic result comes out from writing the l.h.s. as $\mathbb{P}(X_i > x | X_j > x)\mathbb{P}(X_j > x)$ and using the tail independence of X_i and X_j and the second from the fact that if $\rho_{ij} = 1$ then $X_i = \mathrm{e}^{\mu_i - \mu_j} X_j^{\sigma_j/\sigma_i}$ and therefore

$$\mathbb{P}(X_i > x, X_j > x) = \mathbb{P}(X_j > \min\{x, \mathrm{e}^{\mu_j - \mu_i} x^{\sigma_i/\sigma_j}\}) \sim \mathbb{P}(X_j > x)$$

as $x \to \infty$. Then it follows from Remark 2 that

$$\lim_{x\to\infty} \frac{\mathbb{P}(X_i > x, X_j > x)}{\mathbb{P}(X_1 > x)} = 0 \qquad \forall j \ne i.$$

We have proved that

$$\liminf_{x\to\infty} \frac{\mathbb{P}(S_{n+1} > x)}{\mathbb{P}(X_1 > x)} \ge \liminf_{x\to\infty} \sum_{i=1}^{n+1} \frac{\mathbb{P}(X_i > x)}{\mathbb{P}(X_1 > x)} = m_{n+1}$$

for the appropriate m_{n+1}. Together with (4), this establishes the induction step and completes the proof. □

Proof of Lemma 1. If $\sigma_{n+1} < \sigma$ choose $\sigma_{n+1}/\sigma < \beta < 1$. Then

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x^\beta, X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)} \le \limsup_{x\to\infty} \frac{\mathbb{P}(X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)} = \lim_{x\to\infty} \frac{\mathbb{P}(X_{n+1}^{1/\beta} > x)}{\mathbb{P}(X_1 > x)}.$$

But the last limit is 0 since $X_{n+1}^{1/\beta} \sim \mathrm{LN}\big(\mu_{n+1}/\beta, \sigma_{n+1}^2/\beta^2\big)$ has lighter tail than X_1 because of $\sigma_{n+1}^2/\beta^2 < \sigma^2$.

If $\sigma = \sigma_{n+1}$ define $\gamma = \max_{k=1,\dots,n}\{\sigma_{k(n+1)}/\sigma^2\}$. By A1, we have that $\sigma_k^2 = \sigma^2$ for $k = 1,\dots,n+1$. Then γ is the maximum among the correlations between X_k and X_{n+1} and by the hypothesis of the Theorem 1 it should take values $\gamma \in [-1,1)$. Therefore, we can choose β close enough to 1 to obtain $\max\{1/2,\gamma\} < \beta^2 < 1$ and $(\beta-\gamma/\beta)^2 + \beta^2 > 1$ (observe that $(1-\gamma/1)^2 + 1^2 > 1$). Consider

$$\begin{aligned}\limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x^\beta, X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)} \le& \limsup_{x\to\infty} \frac{\mathbb{P}(X_{n+1} > x^{1/\beta})}{\mathbb{P}(X_1 > x)} \\ &+ \limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x^\beta, x^{1/\beta} > X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)}.\end{aligned}$$

Here the first limit is 0 since $X_{n+1}^{\beta} \sim \mathrm{LN}\big(\beta\mu_n, (\beta\sigma)^2\big)$ has lighter tail than X_1 because of $\beta < 1$. For the second limit, we define

$$X^c(y) = (X_1^c(y), \dots, X_n^c(y)) = (X_1, \dots, X_n | X_{n+1} = y), \quad S_n^c(y) = \sum_{i=1}^n X_i^c(y).$$

So,

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x^\beta, x^{1/\beta} > X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)} = \limsup_{x\to\infty} \frac{1}{\mathbb{P}(X_1 > x)} \int_{x^\beta}^{x^{1/\beta}} \mathbb{P}(S_n^c(y) > x^\beta) f_{X_{n+1}}(y)dy.$$

Standard formulas for the conditional mean vector and conditional covariance matrix in the multivariate normal distribution yield

$$X^c(t) \sim \mathrm{LN}(\{\mu_i + \frac{\sigma_{i(n+1)}}{\sigma^2}(\log t - \mu_{n+1})\}_i, \{\sigma_{ij} - b_{ij}\}_{ij}) \ \text{ where } \ b_{ij} = \frac{\sigma_{i(n+1)}\sigma_{j(n+1)}}{\sigma^2}.$$

We can restrict to consider values of $t > 1$. Since $X_i^c(t)$ has the same distribution as $X_i^c(1)\mathrm{e}^{\frac{\sigma_{i(n+1)}}{\sigma^2}\log t}$, it follows that $X_i^c(t)$ is smaller than $X_i^c(1)\mathrm{e}^{\gamma \log t}$ in stochastic order. So, $S_n^c(t) \le S_n^c(1)t^\gamma$ in stochastic order, and the last limit above can be above bounded by

$$\limsup_{x\to\infty} \frac{1}{\mathbb{P}(X_1 > x)} \int_{x^\beta}^{x^{1/\beta}} \mathbb{P}(S_n^c(1) > x^\beta/y^\gamma) f_{X_{n+1}}(y)dy. \tag{5}$$

If $\gamma \le 0$, the integral in (5) is bounded by $\mathbb{P}(S_n^c(1) > x^\beta)\mathbb{P}(X_{n+1} > x^\beta)$ which by the induction hypothesis is asymptotically equivalent to $m_n^c \mathbb{P}(X_k^c(1) > x^\beta)\mathbb{P}(X_{n+1} > x^\beta)$ with the appropriate integer value m_n^c and index k. Now, from the form of the distribution of $X^c(y)$ and A1 it follows that $X_k^c(1)$ and X_{n+1} have lighter or equivalent tails than X_1 , so bounding m_n^c by n we have proved that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x^\beta, x^{1/\beta} > X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)} \le \limsup_{x\to\infty} \frac{n\mathbb{P}^2(X_1 > x^\beta)}{\mathbb{P}(X_1 > x)}.$$

The last limit is 0 because of the choice $2\beta^2 > 1$, (2) and A2. In the case where $\gamma > 0$, the integral in expression (5) can be bounded by $\mathbb{P}(S_n^c(1) > x^{\beta-\gamma/\beta})\mathbb{P}(X_{n+1} > x^\beta)$. Observe that $\beta - \gamma/\beta > 0$ since we took $\beta^2 > \gamma$; so we can use the same argument as above to conclude that

$$\limsup_{x\to\infty} \frac{\mathbb{P}(S_n > x^\beta, x^{1/\beta} > X_{n+1} > x^\beta)}{\mathbb{P}(X_1 > x)} \le \limsup_{x\to\infty} \frac{n\mathbb{P}(X_1 > x^{\beta-\gamma/\beta})\mathbb{P}(X_1 > x^\beta)}{\mathbb{P}(X_1 > x)}$$

which is again 0 because of the choice $(\beta-\gamma/\beta)^2+\beta^2>1$ and the asymptotic relation

$$\mathbb{P}(X_1 > x^{\beta-\gamma/\beta})\mathbb{P}(X_1 > x^{\beta}) \sim \frac{\sigma^2}{2\pi \log x^{\beta-\gamma/\beta}\log x^{\beta}}\exp\{-\frac{[(\beta-\gamma/\beta)^2+\beta^2]\log^2 x}{2\sigma^2}\}$$

[recall A2]. □

Numerical Examples

Consider a random vector $(X_1, \ldots, X_{10})$ with multivariate Gaussian copula such that $\mu_i = i - 10$, $\sigma_i = i$ and $\sigma_{k\ell} = \rho\sigma_k\sigma_\ell$ for $k \neq \ell$. The two panels in Figure 1 correspond to $\rho = 0.4$, resp. $\rho = 0.95$. The approximation given by (1) is then the r.h.s. of (2) with $\mu = 0$, $\sigma^2 = 10$ (note that $m_n = 1$). This is marked with a solid line with asterisks and a conditional Monte Carlo estimator with a solid line, with the associated 95% confidence limits dotted. For details on the Monte Carlo algorithm, see Asmussen and Rojas-Nandayapa (2006).

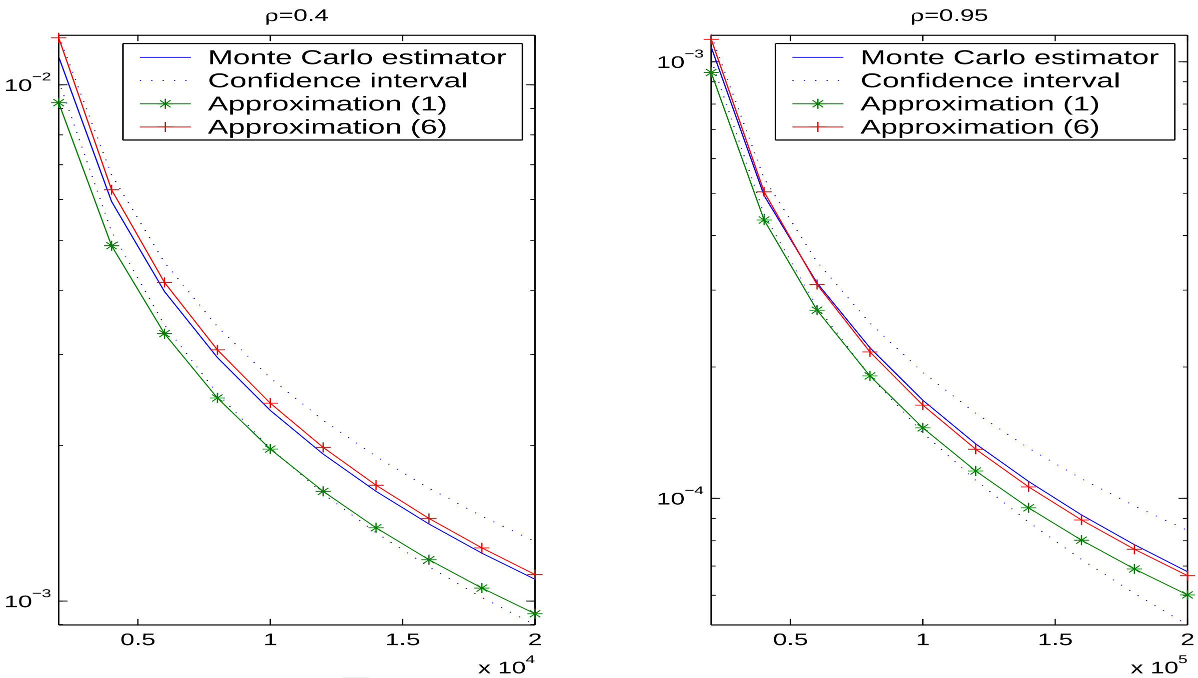

Figure 1: Tail probability $\mathbb{P}(S_{10} > x)$.
Comparison of approximations and simulations.

The numerical results show that the approximation is reasonably accurate but tends to underestimate. This could be explained by some r.v. X_i with tails being only slightly lighter than the heaviest one. This suggests considering the adjusted approximation

$$\mathbb{P}(S_n > x) \sim \sum_{i=1}^{n} \mathbb{P}(X_i > x) \tag{6}$$

which from the proof of Theorem 1 has the same asymptotics and is included in the graphs (solid line with crosses). It is seen that (6) is indeed an improvement, and in fact gives an excellent fit for $\rho = 0.4$. The improvement is less marked for $\rho = 0.95$. However, this also represents an extreme value given that we are approaching comonotonicity where none of the approximations are no longer asymptotically valid, and that, as mentioned earlier, $\rho = 0.4$ is more often argued to be a typical value in financial time series than $\rho = 0.95$.

Figure 2 considers a portfolio which has a similar dispersion of means and variances, but is larger, $n = 50$. We took $\mu_{5i+j} = i - 9$, $\sigma_{5i+j} = i + 1$ for $i = 0, \ldots, 9$, $j = 1, \ldots, 5$ and, as before, $\sigma_{k\ell} = \rho\sigma_k\sigma_\ell$ with $\rho = 0.4$ or $\rho = 0.95$.

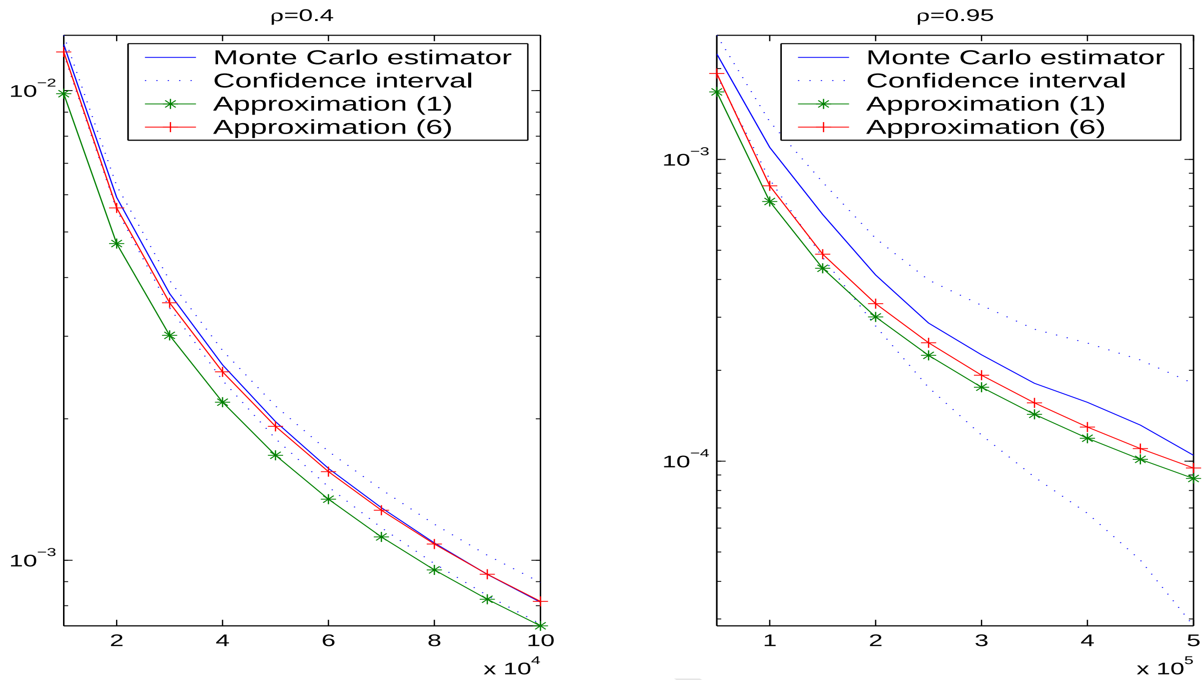

Figure 2: Tail probability $\mathbb{P}(S_{50} > x)$.
Comparison of approximations and simulations.

The final Figure 3 shows relative differences compared to approximation (6) for the $n = 10$ portfolio. The solid line is the difference between (1) and (6) divided by (6). The solid line with asterisks is the same with (1) replaced by simulated values. The graph is compatible with the (obvious) fact that the relative difference between the approximations goes to 0, but shows that the convergence is slow in the present case, as must be expected from the fact that the second largest variance 9 is quite close to the largest one 10. It also indicates that approximation (6) has an excellent fit in the set of parameters used in Figure 3.

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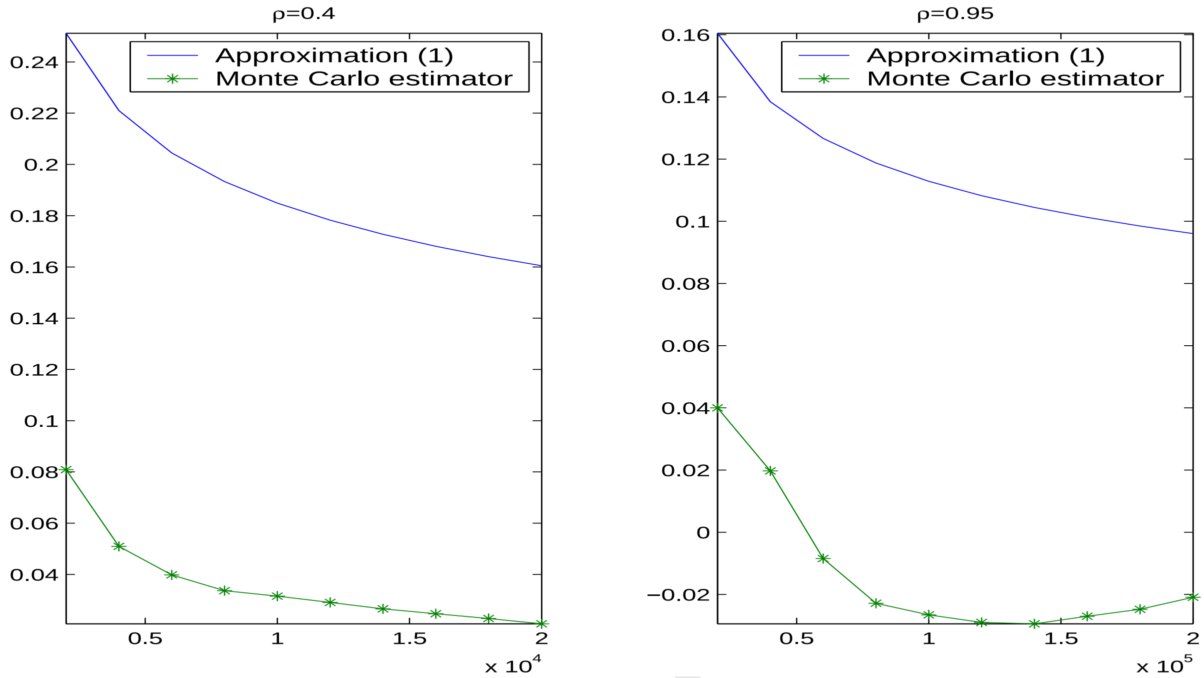

Figure 3: Relative differences
Comparison of approximations and simulations, $n = 10$.

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