Modified Odd Weibull Family of Distributions: Properties and Applications

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Abstract

In this paper, a new family of continuous distributions, called the modified odd Weibull-G (MOW-G) family, is studied. The MOW-G family has the feature to use the Weibull distribution as main generator and a new modification of the odd transformation, opening new horizon in terms of statistical modelling. Its main theoretical and practical aspects are explored. In particular, for the mathematical properties, we investigate some results in distribution, quantile function, skewness, kurtosis, moments, moment generating function, order statistics and entropy. For the statistical aspect, the maximum likelihood estimation method is used to estimate the model parameters. The performance of this method is evaluated by a Monte Carlo simulation study. Applications to three practical data sets are given to demonstrate the usefulness of the MOW-G model.

Keywords: Weibull distribution, quantile function, moments, order statistics, entropy, maximum likelihood estimation, simulation.

AMS Subject Classification: 60E05, 62E15, 62F10.

1 Introduction

The (probability) distributions have a great importance for data modelling in several areas such as finance, engineering, biology, industry and medical sciences. In particular, for a given data set, a reliable statistical model can be developed from an appropriate standard distribution (normal, exponential, Weibull, Cauchy, Lindley...). However, such a model often lacks of goodness of fit to satisfy the very exigent demand of some modern studies. Indeed, a top degree of precision in the fitting of the data is often required to capture phenomena of interest. For this reason, many statisticians are trying to modify existing classical distributions by adding one or more parameters, with the aim to improve their flexibility. Some recent families of continuous distributions include the Marshall-Olkin-G family by [16], the odd power Cauchy family by [2], the beta-G family by [11], the McDonald-G family by [1], the Kumaraswamy-G family by [3], the Weibull-X family by [4], the gamma-X family by [5], the gamma-G (type 3) family by [21], the logistic-G family by [22], the exponentiated generalized-G family by [9], the Weibull-G family by [8], the transformed-transformer family by [4] and the extended Weibull-G family by [15].

In this paper, we study a new family of distributions based on the Weibull distribution and a new modification of the odd transformation. In order to explain the interest of this family, let us briefly present our main source of inspiration: the general Weibull family derived to [4]. This general family is characterized by the cumulative distribution function (cdf) given by

$$F(x;\lambda,\theta,\xi) = 1 - e^{-\lambda\{W[G(x;\xi)]\}^{\theta}}, \quad x \in \mathbb{R},$$

where $\lambda, \theta > 0$, $W(y)$ is a function satisfying the following conditions: (i) $W(y) > 0$ for $y \in (0,1)$, (ii) $W(y)$ is differentiable and monotonically non-decreasing for $y \in (0,1)$, (iii) $W(y) \to 0$ when $y \to 0$ and (iv) $W(y) \to +\infty$ when $u \to 1$, and $G(x;\xi)$ is the cdf of a continuous distribution, generally well-established, depending on a parameter vector denoting by ξ. Of course, there are as many functions $W(y)$ as there are general families, but few of them demonstrated both mathematical and practical interests. In the literature, the function $W_1(y) = -\log(1-y)$, $y \in (0,1)$, has been considered to define the Weibull-X family by [4] and the odd function defined by $W_2(y) = y/(1-y)$, $y \in (0,1)$ has been used to define the Weibull-G family by [8]. The related models are complementary in terms of fitting data due to the inequality: $W_1(y) < W_2(y)$, $y \in (0,1)$. The advantages of these two Weibull families are numerous, including: (a) the simplicity of the involved functions, (b) the analytical expression for the quantile function is available, (c) it takes benefit of the Weibull distribution to provide a high degree of flexibility to the involved functions and (d) the generated statistical models have strong fitting properties for a wide variety of data sets.

In this paper, we investigate a new choice for $W(y)$: a slight modification of the odd function given by

$$W_3(y) = \frac{y}{1 - y(1+y)/2}, \quad y \in (0,1).$$

One can also express $W_3(y)$ as a weighted version of $W_2(y)$; after some algebra, we can show that $W_3(y) = W_2(y)w(y)$, with $w(y) = 2/(2+y)$. In addition to the fact that $W_3(y)$ satisfies (i), (ii), (iii) and (iv), it has the following merits: it is simple, its inverse has a tractable analytical expression and it offers an intermediary choice between the standard $W_1(y)$ and $W_2(y)$. Indeed, one can show the following hierarchy:

$$W_1(y) < W_3(y) < W_2(y), \quad y \in (0,1). \tag{1}$$

Hence, we introduce the modified odd Weibull-G (MOW-G) family characterized by the cdf given by

$$F(x;\lambda,\theta,\xi) = 1 - e^{-\lambda\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta}}, \quad x \in \mathbb{R}. \tag{2}$$

In view of (1), the MOW-G family is thus complementary to the Weibull-X and Weibull-G families, and deserves all the attentions. The two other crucial functions of the MOW-G family are presented below. By differentiation of (2), the probability density function (pdf) of the MOW-G family is given by

$$f(x;\lambda,\theta,\xi) = \lambda\theta\frac{g(x;\xi)\big[1+G(x;\xi)^2/2\big]G(x;\xi)^{\theta-1}}{\big\{1-G(x;\xi)\big[1+G(x;\xi)\big]/2\big\}^{\theta+1}}e^{-\lambda\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta}}, \quad x \in \mathbb{R}, \tag{3}$$

where $g(x;\xi)$ is the pdf corresponding to $G(x;\xi)$.
The corresponding hazard rate function (hrf) is given by

$$h(x;\lambda,\theta,\xi) = \frac{f(x;\xi)}{1-F(x;\xi)} = \lambda\theta\frac{g(x;\xi)\big[1+G(x;\xi)^2/2\big]G(x;\xi)^{\theta-1}}{\{1-G(x;\xi)[1+G(x;\xi)]/2\}^{\theta+1}}, \quad x \in \mathbb{R}. \tag{4}$$

The rest of the paper is devoted to the complete study of the MOW-G family, exploring the mathematical, inferential and practical aspects. In Section 2, two special members of the MOW-G family are presented, with plots of the corresponding pdf and hrf. In Section 3, useful linear representations for the cdf and pdf and some derivations are determined, with complete proofs. In Section 4, we derive its main mathematical properties such as quantile function, moments, moments generating function, order statistics and general expressions for the Rényi and Shannon entropies. In Section 5, we estimate the model parameters of the MOW-G family by the maximum likelihood method. Then, a simulation study is performed on a special member of the family to illustrate the convergence properties of the estimators. Three applications to real data illustrate the usefulness of the MOW-G family in Section 6. The paper is concluded in Section 7.

2 Special distributions

In this section, we will give two special distributions belonging to the MOW-G family.

2.1 The MOW-gamma distribution

Here, we consider the gamma distribution with shape parameter $\alpha > 0$ and rate parameter $\beta > 0$ as baseline distribution. The corresponding cdf is given by

$$G(x;\alpha,\beta) = \frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}, \quad x > 0,$$

where $\gamma(\alpha,\beta x) = \int_0^{\beta x} t^{\alpha-1}e^{-t}dt$ and $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt$. The corresponding pdf is given by

$$g(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\beta x}, \quad x > 0.$$

Then, based on (2), the MOW-gamma (MOW-Ga) distribution is characterized by the cdf given by

$$F(x;\lambda,\theta,\alpha,\beta) = 1 - e^{-\lambda\left\{\frac{2\gamma(\alpha,\beta x)}{2\Gamma(\alpha)-\gamma(\alpha,\beta x)[1+\gamma(\alpha,\beta x)/\Gamma(\alpha)]}\right\}^\theta}, \quad x > 0.$$

The corresponding pdf is given by

$$f(x;\lambda,\theta,\alpha,\beta) = \lambda\theta\frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)}\left(1+\frac{\gamma(\alpha,\beta x)^2}{2\Gamma(\alpha)^2}\right)\left\{1-\frac{\gamma(\alpha,\beta x)}{2\Gamma(\alpha)}\left[1+\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right]\right\}^{-\theta-1}$$
$$\times\frac{\gamma(\alpha,\beta x)^{\theta-1}}{\Gamma(\alpha)^{\theta-1}}e^{-\beta x-\lambda\left\{\frac{2\gamma(\alpha,\beta x)}{2\Gamma(\alpha)-\gamma(\alpha,\beta x)[1+\gamma(\alpha,\beta x)/\Gamma(\alpha)]}\right\}^\theta}, \quad x > 0.$$

The corresponding hrf is given by

$$h(x;\lambda,\theta,\alpha,\beta) = \lambda\theta\frac{\beta^\alpha x^{\alpha-1}}{\Gamma(\alpha)}\left(1+\frac{\gamma(\alpha,\beta x)^2}{2\Gamma(\alpha)^2}\right)\left\{1-\frac{\gamma(\alpha,\beta x)}{2\Gamma(\alpha)}\left[1+\frac{\gamma(\alpha,\beta x)}{\Gamma(\alpha)}\right]\right\}^{-\theta-1}\frac{\gamma(\alpha,\beta x)^{\theta-1}}{\Gamma(\alpha)^{\theta-1}}e^{-\beta x},$$

$x > 0$.

Figure 1 displays some plots of the MOW-gamma pdf for some specific parameter values. We observe that the MOW-gamma pdf has a wide variety of shapes.

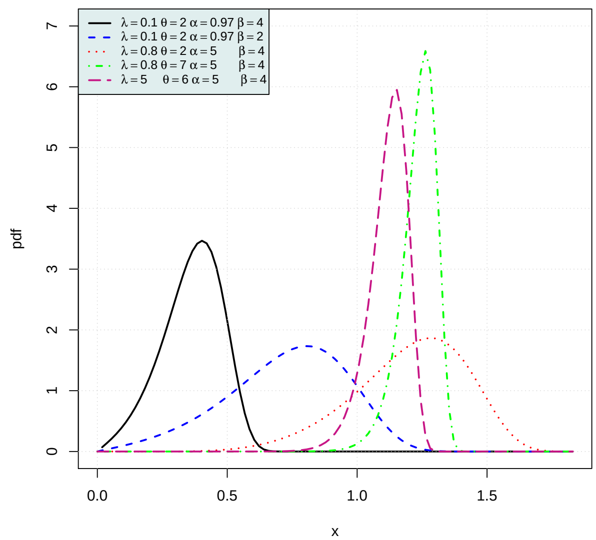

Figure 1: Plots of MOW-gamma pdfs.

2.2 The MOW-normal distribution

For the second special member, we consider the normal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$ as baseline distribution. Hence, the corresponding cdf is given by

$$\Phi(x;\mu,\sigma) = \int_{-\infty}^{x} \phi(t;\mu,\sigma)dt, \quad x \in \mathbb{R},$$

where

$$\phi(t;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}, \quad t \in \mathbb{R}.$$

Then, based on (2), the MOW-normal (MOW-N) distribution is characterized by the cdf given by

$$F(x;\lambda,\theta,\mu,\sigma) = 1 - e^{-\lambda\left\{\frac{\Phi(x;\mu,\sigma)}{1-\Phi(x;\mu,\sigma)[1+\Phi(x;\mu,\sigma)]/2}\right\}^{\theta}}, \quad x \in \mathbb{R}.$$

The corresponding pdf is given by

$$f(x;\lambda,\theta,\mu,\sigma) = \lambda\theta \frac{\phi(x;\mu,\sigma)\left[1+\Phi(x;\mu,\sigma)^2/2\right]\Phi(x;\mu,\sigma)^{\theta-1}}{\{1-\Phi(x;\mu,\sigma)[1+\Phi(x;\mu,\sigma)]/2\}^{\theta+1}} e^{-\lambda\left\{\frac{\Phi(x;\mu,\sigma)}{1-\Phi(x;\mu,\sigma)[1+\Phi(x;\mu,\sigma)]/2}\right\}^{\theta}}, \quad x \in \mathbb{R}.$$

The corresponding hrf is given by

$$h(x;\lambda,\theta,\mu,\sigma) = \lambda\theta \frac{\phi(x;\mu,\sigma)\left[1+\Phi(x;\mu,\sigma)^2/2\right]\Phi(x;\mu,\sigma)^{\theta-1}}{\{1-\Phi(x;\mu,\sigma)[1+\Phi(x;\mu,\sigma)]/2\}^{\theta+1}}, \quad x \in \mathbb{R}.$$

Figure 2 displays some plots of the MOW-normal pdf for some selected values for the parameters. We see that MOW-normal pdf can be unimodal or bimodal, various shapes can be seen.

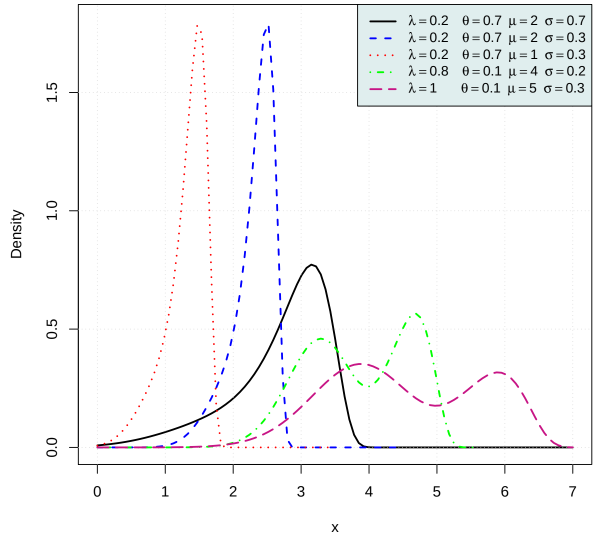

Figure 2: Plots of MOW-normal pdfs.

Remark 1 *Naturally, other special members of the MOW-G family can be expressed. In Section 6, the following ones will be considered:*

- *the MOW-Weibull (MOW-W) distribution with cdf defined by (2) with the baseline cdf of the Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$, i.e.,*

$$G(x;\alpha,\beta) = 1 - e^{-(\frac{x}{\beta})^{\alpha}}, \quad x > 0.$$

- *the MOW Lindley (MOW-L) distribution with cdf defined by (2) with the baseline cdf of the Lindley distribution with parameter $\theta > 0$, i.e.,*

$$G(x;\theta) = 1 - \frac{1+\theta+\theta x}{1+\theta} e^{-\theta x}, \quad x > 0.$$

3 Linear representations

In this section, we present useful linear representations for the cdf and pdf of the MOW-G family, as well as some derivations, in terms of cdfs of the exp-G family. These results are of importance since several mathematical properties of the MOW-G family can be simply deduced from those of the exp-G family.

Hereafter, we suppose that all the conditions are satisfied to differentiate under the sign (infinite) sum and to interchange sum and integral signs. This assumption can be investigated for a given $G(x;\xi)$, but not in full generality as in our study. It is also assumed that $G(x;\xi) < 1$; the limit case is excluded. As a last remark, for a practical purpose, the limit bound $+\infty$ in the coming sums can be replaced by any large positive integer, say 40.

Proposition 1 *Let $F(x;\lambda,\theta,\xi)$ be the cdf given by (2). Then, we have the following linear representation:*

$$F(x;\lambda,\theta,\xi)=\sum_{k=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{l}a_{k,l,m}G(x;\xi)^{\theta k+l+m}, \tag{5}$$

where

$$a_{k,l,m}=\frac{(-1)^{k+l+1}\lambda^k}{2^l k!}\binom{-\theta k}{l}\binom{l}{m}.$$

Proof. By using the power series of the exponential function, we have

$$\begin{aligned}F(x;\lambda,\theta,\xi)&=1-\sum_{k=0}^{+\infty}\frac{(-1)^k\lambda^k}{k!}\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta k}\\&=\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}\lambda^k}{k!}\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta k}.\end{aligned}$$

On the other side, by applying the binomial theorem, we have

$$\begin{aligned}\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta k}&=G(x;\xi)^{\theta k}\sum_{l=0}^{+\infty}\frac{(-1)^l}{2^l}\binom{-\theta k}{l}G(x;\xi)^l[1+G(x;\xi)]^l\\&=\sum_{l=0}^{+\infty}\sum_{m=0}^{l}\frac{(-1)^l}{2^l}\binom{-\theta k}{l}\binom{l}{m}G(x;\xi)^{\theta k+l+m}.\end{aligned} \tag{6}$$

By putting the above equalities together, we obtain the desired linear representation. □

Corollary 1 *By differentiation of (5), the pdf $f(x;\lambda,\theta,\xi)$ given by (3) can be expressed as*

$$f(x;\lambda,\theta,\xi)=\sum_{k=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{l}b_{k,l,m}g(x;\xi)G(x;\xi)^{\theta k+l+m-1},$$

where $b_{k,l,m}=(\theta k+l+m)a_{k,l,m}$.

The result below provides a generalization of Proposition 1.

Proposition 2 *Let υ be a positive integer, $F(x;\lambda,\theta,\xi)$ be the cdf given by (2) and $f(x;\lambda,\theta,\xi)$ be the pdf given by (3). Then, we have the following linear representation:*

$$f(x;\lambda,\theta,\xi)F(x;\lambda,\theta,\xi)^{\upsilon}=\sum_{q=0}^{\upsilon+1}\sum_{k,l=0}^{+\infty}\sum_{m=0}^{l}c_{k,l,m,q}[\upsilon]g(x;\xi)G(x;\xi)^{\theta k+l+m-1},$$

where

$$c_{k,l,m,q}[\upsilon]=\frac{(-1)^{q+k+l}q^k\lambda^k(\theta k+l+m)}{(\upsilon+1)2^l k!}\binom{\upsilon+1}{q}\binom{-\theta k}{l}\binom{l}{m}. \tag{7}$$

Proof. It follows from the binomial theorem that

$$F(x;\lambda,\theta,\xi)^{\upsilon+1}=\sum_{q=0}^{\upsilon+1}\binom{\upsilon+1}{q}(-1)^q e^{-q\lambda\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta}}.$$

The power series of the exponential function gives

$$e^{-q\lambda\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta}}=\sum_{k=0}^{+\infty}\frac{(-1)^k q^k\lambda^k}{k!}\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta k}.$$

Now, by using (6) and combining the equalities above, we obtain the following linear representation:

$$F(x;\lambda,\theta,\xi)^{\upsilon+1}=\sum_{q=0}^{\upsilon+1}\sum_{k,l=0}^{+\infty}\sum_{m=0}^{l}d_{k,l,m,q}[\upsilon]G(x;\xi)^{\theta k+l+m},\tag{8}$$

where

$$d_{k,l,m,q}[\upsilon]=\frac{(-1)^{q+k+l}q^k\lambda^k}{2^l k!}\binom{\upsilon+1}{q}\binom{-\theta k}{l}\binom{l}{m}.$$

We obtain the desired linear representation for $f(x;\lambda,\theta,\xi)F(x;\lambda,\theta,\xi)^{\upsilon}$ by differentiation of (8) and noticing that $c_{k,l,m,q}[\upsilon]=(\theta k+l+m)d_{k,l,m,q}[\upsilon]/(\upsilon+1)$. □

Naturally, by taking $\upsilon=0$, Proposition 2 is reduced to Corollary 1. We end this section by the linear representation for $f(x;\lambda,\theta,\xi)^{\alpha}$.

Proposition 3 *Let $\alpha>0$ and $f(x;\lambda,\theta,\xi)$ be the pdf given by (3). Then, the following linear representation holds:*

$$f(x;\lambda,\theta,\xi)^{\alpha}=\sum_{k,m,s,t=0}^{+\infty}\sum_{l=0}^{k}\sum_{q=0}^{s}u_{k,l,m,s,t}[\alpha]g(x;\xi)^{\alpha}G(x;\xi)^{k+l+2m+\theta t+s+q+\alpha(\theta-1)},$$

where

$$u_{k,l,m,s,t}[\alpha]=\frac{(-1)^{k+t+s}\lambda^{\alpha+t}\alpha^t\theta^{\alpha}}{2^{m+l+s}t!}\binom{\alpha}{m}\binom{-\alpha(\theta+1)}{k}\binom{k}{l}\binom{-\theta t}{s}\binom{s}{q}.$$

Proof. We have

$$f(x;\lambda,\theta,\xi)^{\alpha}=\lambda^{\alpha}\theta^{\alpha}\frac{g(x;\xi)^{\alpha}[1+G(x;\xi)^2/2]^{\alpha}G(x;\xi)^{\alpha(\theta-1)}}{\{1-G(x;\xi)[1+G(x;\xi)]/2\}^{\alpha(\theta+1)}}e^{-\alpha\lambda\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta}}.$$

By using the binomial theorem, it comes

$$[1+G(x;\xi)^2/2]^{\alpha}G(x;\xi)^{\alpha(\theta-1)}=\sum_{m=0}^{+\infty}\frac{1}{2^m}\binom{\alpha}{m}G(x;\xi)^{2m+\alpha(\theta-1)}$$

and

$$\begin{aligned}\{1-G(x;\xi)[1+G(x;\xi)]/2\}^{-\alpha(\theta+1)}&=\sum_{k=0}^{+\infty}\frac{(-1)^k}{2^k}\binom{-\alpha(\theta+1)}{k}G(x;\xi)^k[1+G(x;\xi)]^k\\&=\sum_{k=0}^{+\infty}\sum_{l=0}^{k}\frac{(-1)^k}{2^k}\binom{-\alpha(\theta+1)}{k}\binom{k}{l}G(x;\xi)^{k+l}.\end{aligned}$$

On the other side, proceeding as in (6), we get

$$e^{-\alpha\lambda\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta}} = \sum_{t=0}^{+\infty}\frac{(-1)^t\alpha^t\lambda^t}{t!}\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^{\theta t}$$
$$= \sum_{t=0}^{+\infty}\sum_{s=0}^{+\infty}\sum_{q=0}^{s}\frac{(-1)^{t+s}\alpha^t\lambda^t}{2^s t!}\binom{-\theta t}{s}\binom{s}{q}G(x;\xi)^{\theta t+s+q}.$$

By putting the above equality together, we obtain the desired linear representation. □

Naturally, by taking $\alpha = 1$, Proposition 3 is reduced to Corollary 1.

4 Mathematical properties

In this section, we give some mathematical properties of the MOW-G family such as some results in distribution, quantile function, moments, incomplete moments, moment generating function, order statistics and entropy.

4.1 Some results in distribution

The following results in distribution provide some immediate characterization of the MOW-G family. Let X be a random variable having the cdf given by (2). Then,

- the random variable $Y = G(X;\xi)$ has the cdf given by $F_Y(y) = 1-e^{-\lambda\left\{\frac{y}{1-y(1+y)/2}\right\}^{\theta}}$, $y \in (0,1)$.
- the random variable $Y = \frac{G(X;\xi)}{1-G(X;\xi)[1+G(X;\xi)]/2}$ follows the Weibull distribution with parameters θ and $1/\lambda^{1/\theta}$, i.e., with the cdf given by $F_Y(y) = 1 - e^{-\lambda y^{\theta}}$, $y > 0$.
- the random variable $Y = \left\{\frac{G(X;\xi)}{1-G(X;\xi)[1+G(X;\xi)]/2}\right\}^{\theta}$ follows the exponential distribution with parameter λ, i.e., with the cdf given by $F_Y(y) = 1 - e^{-\lambda y}$, $y > 0$.

4.2 Quantile function

One advantage of the MOW-G family is that the corresponding quantile function has a simple analytical expression. Indeed, by denoting $Q_G(x;\xi)$ the quantile function corresponding to $G(x;\xi)$, the quantile function of the MOW-G family is given by

$$Q(y;\lambda,\theta,\xi) = Q_G\left(-\frac{1}{2}-\left[\frac{-\log(1-y)}{\lambda}\right]^{-\frac{1}{\theta}}+\frac{1}{2}\sqrt{\left(1+2\left[\frac{-\log(1-y)}{\lambda}\right]^{-\frac{1}{\theta}}\right)^2+8};\xi\right),$$
$$y \in (0,1). \tag{9}$$

The median is obtained as

$$M = Q(0.5;\lambda,\theta,\xi) \approx Q_G\left(-\frac{1}{2}-\left(\frac{0.693}{\lambda}\right)^{-\frac{1}{\theta}}+\frac{1}{2}\sqrt{\left(1+2\left(\frac{0.693}{\lambda}\right)^{-\frac{1}{\theta}}\right)^2+8};\xi\right).$$

The other quartiles can be expressed in a similar manner.

One can also use $Q(y;\lambda,\theta,\xi)$ for simulating values for a particular MOW-G distribution: if U is a random variable following the uniform distribution $U(0,1)$, then $X = Q(U;\lambda,\theta,\xi)$ has the cdf given by (2).

4.3 Skewness and kurtosis

We can obtain skewness and kurtosis measures by using the quantile function given by (9). For instance, the Bowley skewness is given by

$$S = \frac{Q(1/4; \lambda, \theta, \xi) + Q(3/4; \lambda, \theta, \xi) - 2Q(1/2; \lambda, \theta, \xi)}{Q(3/4; \lambda, \theta, \xi) - Q(1/4; \lambda, \theta, \xi)}$$

and the Moors kurtosis is given by

$$K = \frac{Q(7/8; \lambda, \theta, \xi) - Q(5/8; \lambda, \theta, \xi) + Q(3/8; \lambda, \theta, \xi) - Q(1/8; \lambda, \theta, \xi)}{Q(6/8; \lambda, \theta, \xi) - Q(2/8; \lambda, \theta, \xi)}$$

Contrary to the skewness and kurtosis measures based on moments, S and K have the merit to always exist and have a clear expression thanks to (9). See [18] and [12] for further details.

By considering the MOW-gamma distribution with fixed parameters $\lambda = 1$ and $\beta = 1$, and varying parameters θ and α, Figures 3 and 4 show the Bowley skewness and Moors kurtosis, respectively. In particular, in Figure 3, we can see that the MOW-gamma distribution can be left or right skewed.

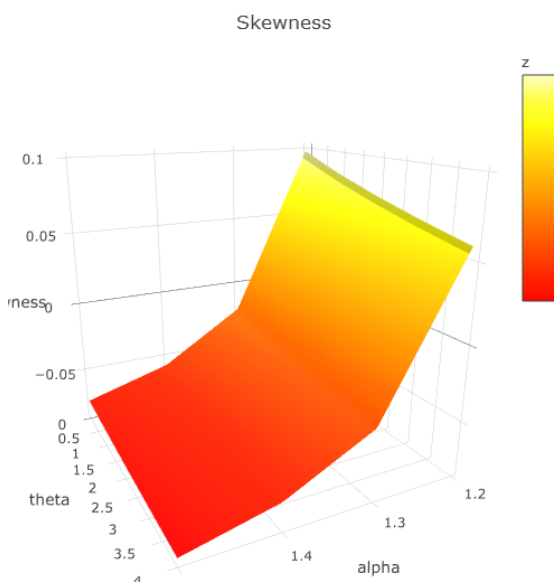

Figure 3: Plot of MOW-gamma skewness.

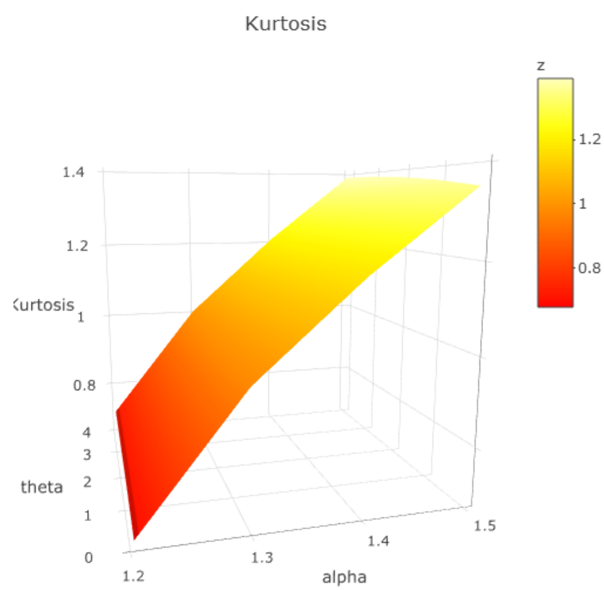

Figure 4: Plot of MOW-gamma kurtosis.

4.4 Moments

Hereafter, let X be a random variable having the cdf of the MOW-G family given by (2).

By using Corollary 1, the r-th moment of X can be obtained as

$$\mu'_r = E(X^r) = \int_{-\infty}^{+\infty} x^r f(x; \lambda, \theta, \xi) dx = \sum_{k=1}^{+\infty} \sum_{l=0}^{+\infty} \sum_{m=0}^{l} b_{k,l,m} \int_{-\infty}^{+\infty} x^r g(x; \xi) G(x; \xi)^{\theta k + l + m - 1} dx.$$

The last integral can be computed for most of the baseline cdfs $G(x;\xi)$. In the next, this will be the case for any integral involving exponentiated $G(x;\xi)$.

The mean of X is given by $\mu = \mu'_1$ and the variance of X is given by $\sigma^2 = \mu'_2 - \mu^2$. The r-th central moment is given by

$$\mu_r = E\left[(X-\mu)^r\right] = \int_{-\infty}^{+\infty} (x-\mu)^r f(x;\lambda,\theta,\xi)dx = \sum_{k=0}^{r} \binom{r}{k} (-1)^k \mu^k \mu'_{r-k}.$$

Assuming that they exist, some skewness and kurtosis measures can be defined from the moments. The most standard ones are the skewness coefficient given by

$$S_*(X) = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\mu'_3 - 3\mu'_2\mu + 2\mu^3}{\sigma^3}$$

and the kurtosis coefficient given by

$$K_*(X) = \frac{\mu_4}{\mu_2^2} = \frac{\mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu'_4}{\sigma^4}.$$

For a given cdf $G(x;\xi)$, they can be calculated.

4.5 Incomplete moments

As for the moments, by using Corollary 1, the r-th incomplete moment of X is obtained as

$$m_r(y) = \int_{-\infty}^{y} x^r f(x;\lambda,\theta,\xi)dx = \sum_{k=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{l} b_{k,l,m} \int_{-\infty}^{y} x^r g(x;\xi)G(x;\xi)^{\theta k+l+m-1}dx. \quad (10)$$

It is a crucial ingredient to define important measures, as the mean deviation of X about μ given by

$$\delta_1 = E(|X-\mu|) = 2\left[\mu F(\mu) - m_1(\mu)\right]$$

or the mean deviation of X about M given by

$$\delta_2 = E(|X-M|) = \mu - 2m_1(M).$$

One can also express equations of very useful curves as the Bonferroni and Lorenz curves.

4.6 Moment generating function

The moment generating function of X can be determined from Corollary 1 as

$$\begin{aligned} M(t) = E(e^{tX}) &= \int_{-\infty}^{+\infty} e^{tx} f(x;\lambda,\theta,\xi)dx \\ &= \sum_{k=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{l} b_{k,l,m} \int_{0}^{+\infty} e^{tx} g(x;\xi)G(x;\xi)^{\theta k+l+m-1}dx. \end{aligned}$$

It is defined for t such that $M(t)$ exists. We also can define it from moments as

$$M(t) = \sum_{k=0}^{+\infty} \frac{t^k}{k!}\mu'_k.$$

4.7 Order statistics

Order statistics naturally appear in many applied situations. We refer to [10] for the general theory. Some basics on the order statistics for the MOW-G family are given below. Let $X_1, \cdots, X_n$ be a random sample of size n with common cdf $F(x; \lambda, \theta, \xi)$ given by (2). Let $X_{i:n}$ be the i-th order statistic. Then, the pdf of $X_{i:n}$ can be expressed as

$$\begin{aligned} f_{i:n}(x; \lambda, \theta, \xi) &= \frac{n!}{(i-1)!(n-i)!} f(x; \lambda, \theta, \xi) F(x; \lambda, \theta, \xi)^{i-1} \left[1 - F(x; \lambda, \theta, \xi)\right]^{n-i} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(x; \lambda, \theta, \xi) F(x; \lambda, \theta, \xi)^{j+i-1}. \end{aligned}$$

It follows from Proposition 2 with $\upsilon = j + i - 1$ that $f(x; \lambda, \theta, \xi)F(x; \lambda, \theta, \xi)^{j+i-1}$ can be expressed as

$$f(x; \lambda, \theta, \xi)F(x; \lambda, \theta, \xi)^{j+i-1} = \sum_{q=0}^{\upsilon+1} \sum_{k,l=0}^{+\infty} \sum_{m=0}^{l} c_{k,l,m,q}[j+i-1] g(x; \xi) G(x; \xi)^{\theta k + l + m - 1},$$

where $c_{k,l,m,q}[j + i - 1]$ is defined by (7) with $\upsilon = j + i - 1$.

So, we have a linear representation of $f_{i:n}(x; \lambda, \theta, \xi)$ in terms of cdfs of the exp-G family as

$$f_{i:n}(x; \lambda, \theta, \xi) = \sum_{j=0}^{n-i} \sum_{q=0}^{\upsilon+1} \sum_{k,l=0}^{+\infty} \sum_{m=0}^{l} e_{i,j,k,l,m,q} g(x; \xi) G(x; \xi)^{\theta k + l + m - 1},$$

where

$$e_{i,j,k,l,m,q} = \frac{n!}{(i-1)!(n-i)!} (-1)^j \binom{n-i}{j} c_{k,l,m,q}[j + i - 1].$$

As in the previous subsections, this linear representation is useful to determine properties of the distribution of $X_{i:n}$. In particular, the r-th moment of $X_{i:n}$ is given by

$$\begin{aligned} \mu'_{i:n} &= \int_{-\infty}^{+\infty} x^r f_{i:n}(x; \lambda, \theta, \xi) dx \\ &= \sum_{j=0}^{n-i} \sum_{q=0}^{\upsilon+1} \sum_{k,l=0}^{+\infty} \sum_{m=0 \, (k+l+m>0)}^{l} e_{i,j,k,l,m,q} \int_{-\infty}^{+\infty} x^r g(x; \xi) G(x; \xi)^{\theta k + l + m - 1} dx. \end{aligned}$$

4.8 Entropy

The entropy is a measure of uncertainty (or ignorance) of a given probability distribution. Here, we investigate two popular entropy measures, the Rényi entropy and Shannon entropy, for the MOW-G family. See [19] and [20] for the former theory and applications.

First of all, the Rényi entropy is defined by

$$I_\alpha(\lambda, \theta, \xi) = \frac{1}{1-\alpha} \log \left\{ \int_{-\infty}^{+\infty} f(x; \lambda, \theta, \xi)^\alpha dx \right\},$$

where $\alpha > 0$ and $\alpha \neq 1$. By using Proposition 3, we immediately obtain

$$\begin{aligned} &I_\alpha(\lambda, \theta, \xi) = \\ &\frac{1}{1-\alpha} \log \left\{ \sum_{k,m,s,t=0}^{+\infty} \sum_{l=0}^{k} \sum_{q=0}^{s} u_{k,l,m,s,t}[\alpha] \int_{-\infty}^{+\infty} g(x; \xi)^\alpha G(x; \xi)^{k+l+2m+\theta t+s+q+\alpha(\theta-1)} dx \right\}. \end{aligned}$$

Also, the Shannon entropy of a random variable X is defined by

$$\eta(\lambda,\theta,\xi) = E\left\{-\log[f(X;\lambda,\theta,\xi)]\right\}.$$

One can determine it by using the relation: $\eta(\lambda,\theta,\xi) = \lim_{\alpha\to 1} I_\alpha(\lambda,\theta,\xi)$. An alternative approach is is presented below. It follows from the expression of $f(x;\lambda,\theta,\xi)$ that

$$\begin{aligned}\eta(\lambda,\theta,\xi) = &-\log(\lambda) - \log(\theta) - E\left\{\log\left[g(X;\xi)\right]\right\} - E\left\{\log\left[1+G(X;\xi)^2/2\right]\right\}\\ &-(\theta-1)E\left\{\log\left[G(X;\xi)\right]\right\} + (\theta+1)E\left\{\log\left[1-G(X;\xi)(1+G(X;\xi))/2\right]\right\}\\ &+\lambda E\left[\left\{\frac{G(X;\xi)}{1-G(X;\xi)(1+G(X;\xi))/2}\right\}^\theta\right].\end{aligned}$$

By using Corollary 1, we have

$$E\left\{\log\left[g(X;\xi)\right]\right\} = \sum_{k=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{l} b_{k,l,m}\int_{-\infty}^{+\infty}\log\left[g(x;\xi)\right]g(x;\xi)G(x;\xi)^{\theta k+l+m-1}dx.$$

The last integral can be calculated for most of the baseline cdfs $G(x;\xi)$. For the other terms, by using the power series of the logarithmic function and the binomial theorem when necessary, we get

$$E\left\{\log\left[1+G(X;\xi)^2/2\right]\right\} = \sum_{i=1}^{+\infty}\frac{(-1)^{i+1}}{i2^i}E\left[G(X;\xi)^{2i}\right],$$

$$E\left\{\log\left[G(X;\xi)\right]\right\} = -\sum_{i=1}^{+\infty}\sum_{j=0}^{i}\frac{(-1)^j}{i}\binom{i}{j}E\left[G(X;\xi)^j\right]$$

and

$$E\left\{\log\left[1-G(X;\xi)(1+G(X;\xi))/2\right]\right\} = -\sum_{i=1}^{+\infty}\sum_{j=0}^{i}\frac{1}{i2^i}\binom{i}{j}E\left[G(X;\xi)^{i+j}\right].$$

Also, by proceeding as in (6), we obtain

$$E\left[\left\{\frac{G(x;\xi)}{1-G(x;\xi)[1+G(x;\xi)]/2}\right\}^\theta\right] = \sum_{i=0}^{+\infty}\sum_{j=0}^{i}\frac{(-1)^i}{2^i}\binom{-\theta}{i}\binom{i}{j}E\left[G(X;\xi)^{i+j+\theta}\right].$$

Any expectation of exponentiated $G(X;\xi)$ can be expressed by using Corollary 1. Indeed, for any $\kappa\geq 0$,

$$\begin{aligned}E\left[G(X;\xi)^\kappa\right] &= \sum_{k=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{l} b_{k,l,m}\int_{-\infty}^{+\infty} g(x;\xi)G(x;\xi)^{\kappa+\theta k+l+m-1}dx\\ &= \sum_{k=1}^{+\infty}\sum_{l=0}^{+\infty}\sum_{m=0}^{l} b_{k,l,m}\frac{1}{\kappa+\theta k+l+m}.\end{aligned}$$

Hence, by combining the equalities above, we obtain a linear representation of $\eta(\lambda,\theta,\xi)$.

5 Inferential considerations

This section is devoted to some inferential considerations of the MOW-G model.

5.1 Estimation

In this section, we explore the estimation of the unknown parameters of the MOW-G model by the maximum likelihood method. Let $x_1, \ldots, x_n$ be an observed sample of size n of a random variable with pdf $f(x; \lambda, \theta, \xi)$ given by (3). Here, λ, θ and ξ are the unknown parameters of interest. Then, the corresponding log-likelihood function is given by

$$\begin{aligned} l_n(\lambda, \theta, \xi) &= \sum_{i=1}^{n} \log[f(x_i; \lambda, \theta, \xi)] \\ &= n\log(\lambda) + n\log(\theta) + \sum_{i=1}^{n} \log[g(x_i;\xi)] + \sum_{i=1}^{n} \log[1 + G(x_i;\xi)^2/2] + (\theta - 1)\sum_{i=1}^{n} \log[G(x_i;\xi)] \\ &- (\theta+1)\sum_{i=1}^{n} \log\{1 - G(x_i;\xi)[1+G(x_i;\xi)]/2\} - \lambda \sum_{i=1}^{n} \left\{ \frac{G(x_i;\xi)}{1 - G(x_i;\xi)[1+G(x_i;\xi)]/2} \right\}^{\theta}. \end{aligned} \quad (11)$$

The maximum likelihood estimators (MLEs) of λ, θ and ξ are obtained as the maximum of $l_n(\lambda, \theta, \xi)$ according to λ, θ and ξ. Assuming that the first partial derivative of $l_n(\lambda, \theta, \xi)$ exist, the MLEs of λ, θ and ξ can be obtained by solving the following equations: $\partial \ell_n/\partial\lambda = 0$, $\partial \ell_n/\partial\theta = 0$, $\partial \ell_n/\partial\xi = 0$, simultaneously, according to λ, θ and ξ. Then, derivatives of (11) with respect to λ, θ and ξ are given by

$$\frac{\partial l_n(\lambda,\theta,\xi)}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \left\{ \frac{G(x_i;\xi)}{1 - G(x_i;\xi)[1+G(x_i;\xi)]/2} \right\}^{\theta},$$

$$\begin{aligned} \frac{\partial l_n(\lambda,\theta,\xi)}{\partial\theta} &= \frac{n}{\theta} + \sum_{i=1}^{n} \log[G(x_i;\xi)] - \sum_{i=1}^{n} \log\{1 - G(x_i;\xi)[1+G(x_i;\xi)]/2\} \\ &- \lambda \sum_{i=1}^{n} \log\left[\frac{G(x_i;\xi)}{1 - G(x_i;\xi)[1+G(x_i;\xi)]/2} \right] \left\{ \frac{G(x_i;\xi)}{1 - G(x_i;\xi)[1+G(x_i;\xi)]/2} \right\}^{\theta} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial l_n(\lambda,\theta,\xi)}{\partial\xi} &= \sum_{i=1}^{n} \frac{g^{(\xi)}(x_i;\xi)}{g(x_i;\xi)} + \sum_{i=1}^{n} \frac{G^{(\xi)}(x_i;\xi)G(x_i;\xi)}{1+G(x_i;\xi)^2/2} + (\theta-1)\sum_{i=1}^{n} \frac{G^{(\xi)}(x_i;\xi)}{G(x_i;\xi)} \\ &+ (\theta+1)\sum_{i=1}^{n} \frac{G^{(\xi)}(x_i;\xi)[1+2G(x_i;\xi)]}{2 - G(x_i;\xi)[1+G(x_i;\xi)]} \\ &+ \lambda\theta \sum_{i=1}^{n} \frac{G^{(\xi)}(x_i;\xi)G(x_i;\xi)^{\theta-1}[1+G(x_i;\xi)^2/2]}{\{1 - G(x_i;\xi)[1+G(x_i;\xi)]/2\}^{\theta+1}}, \end{aligned}$$

where $g^{(\xi)}(x_i;\xi) = \partial g(x_i;\xi)/\partial\xi$ and $G^{(\xi)}(x_i;\xi) = \partial G(x_i;\xi)/\partial\xi$. The interest of the maximum likelihood method is that, under some conditions of regularity, we have theoretical guaranties on the convergence of the resulting estimators when n is large (consistence, asymptotic normality...). This aspect is illustrated in the next subsection with the MOW-gamma model.

5.2 Simulation study

Here, we present a simulation study to examine the performance of the MLEs of the parameters λ, θ, α and β of the MOW-gamma model (see Subsection 2.1 for the mathematical details). Thus, by using the Monte Carlo simulation method, we generate $N = 100$ times samples of size $n = 45 + 5k$, with $k \in \{1, \ldots, 100\}$ from the quantile function of the MOW-gamma distribution. We use the statistical software R (through the package `stats4`). The performance of the estimators is evaluated through their empirical biases and mean square errors (MSEs) given by, respectively,

$$\widehat{Bias}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon), \quad \widehat{MSE}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon}_i - \epsilon)^2,$$

where $\epsilon \in \{\lambda, \theta, \alpha, \beta\}$ and $\hat{\epsilon}_i$ is the MLE of ϵ, obtained at the i-th repetition of the simulation. The simulated results for the above measures are displayed in Figures 5 and 6, respectively. The plots in Figure 5 indicate that the empirical biases of parameters stabilize to 0 when the sample size n increases. This shows the accuracy of the MLEs. The plots in Figure 6 show that, as the sample size n increases, the MSEs decrease and tend to 0. This shows the consistency of the MLEs.

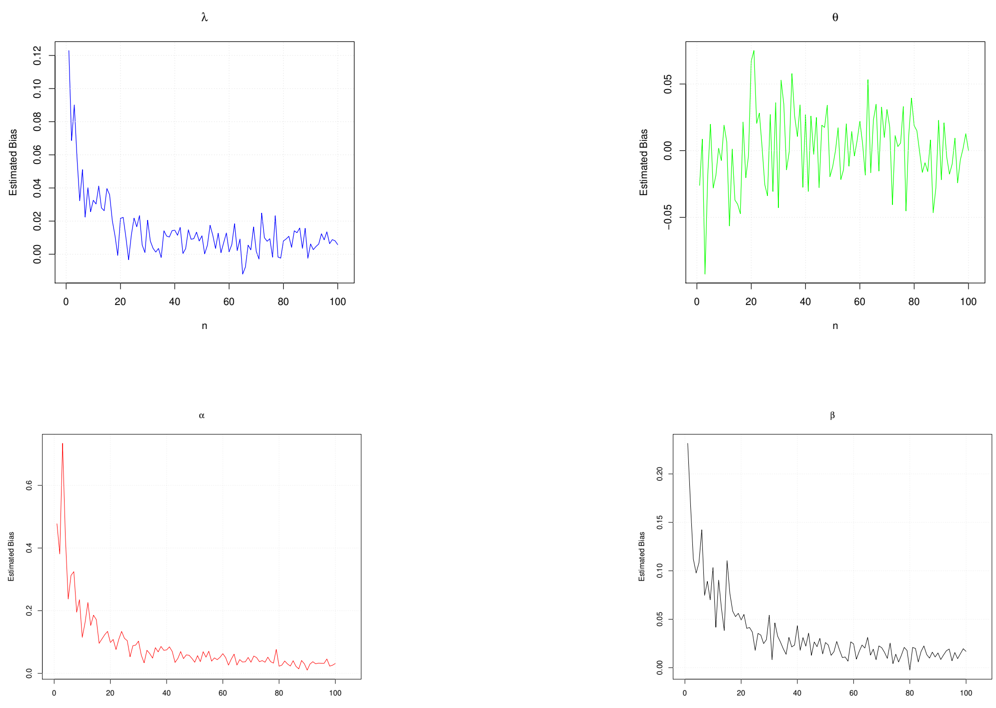

Figure 5: Plots of the empirical biases.

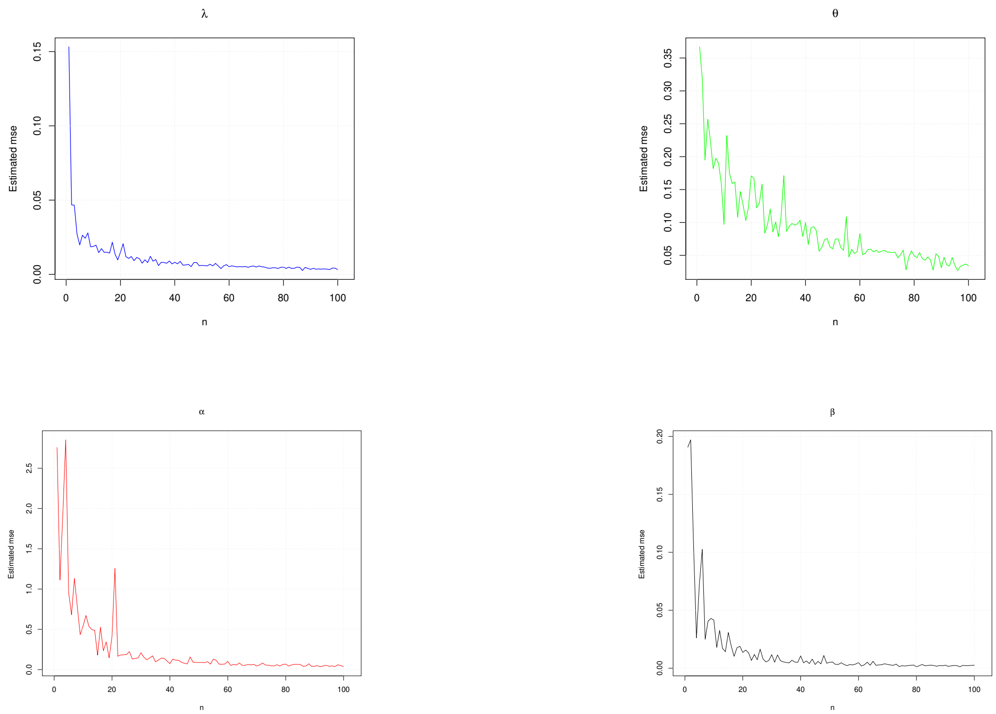

Figure 6: Plots of the empirical MSEs.

6 Applications

In this section, we demonstrate the flexibility and the potentiality of the MOW-G model through three applications on practical data sets having different natures. All the involved parameters are estimated by the maximum likelihood method, as presented in Section 5 for the MOW-G model. The statistical software `R` is used (through the package `AdequacyModel`). In each application, we first compare the MOW-gamma model with other useful competitive models of the literature via the Akaike Information Criterion (AIC). The best model to fit the data is the model with the smallest value of the AIC. Then, we perform an intrinsic investigation of the MOW-G model by comparing the MOW-gamma, MOW-Weibull and MOW-Lindley models (see Remark 1 for more details about the MOW-Weibull and MOW-Lindley distributions). The Kolmogorov-Smirnov (K-S) test is applied to show the pertinence of these models to fit the considered data sets. Then, we compute the AIC, Corrected AIC (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Cramer-von Mises (W^*) and Anderson Darling (A^*) statistics. The best model to fit the data is the model with the smallest value of each these statistics.

6.1 First application

In the first application, we consider the data representing the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by [7]. These data was also studied by [13]. The data set is given by: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

As strong competitors, we focus on the models presented in [13]. By comparing the AIC statistic, Table 1 shows that the MOW-gamma model outperformed the competitors with AIC = 194.404. Then, in order to provide a large view on the applicability of the MOW-G model, we focus our attention on the MOW-gamma, MOW-Weibull and MOW-Lindley models. The MLEs of the parameters are given in Table 2, with the value the log-likelihood functions at this MLEs (ℓ_n) and p-values of the K-S test. Since all the p-values satisfy p-value > 0.05, these models are suitable to fit the considered data set. Table 3 shows a summary of the AIC, BIC, CAIC, HQIC, W^* and A^* statistics for each model, with favorable results for the MOW-gamma model. The histogram of the guinea pigs data and plots of the estimated pdfs are displayed in Figure 7. The cdfs for each model are shown in Figure 8. It is clear that MOW-gamma model is the best to fit the data.

Model	AIC
MOW-gamma	194.404
OGEPF	217.014
EKwPF	224.355
TPF	231.836

Table 1: AIC of the considered models.

Model	MLE	$-l_n$	K-S	p-value
MOW-gamma	$\hat{\lambda}$=3.0951839, $\hat{\theta}$=0.2578448, $\hat{\alpha}$= 12.3690828, $\hat{\beta}$= 3.1644151	93.20229	0.080796	0.7351
MOW-Weibull	$\hat{\lambda}$=32.2341292, $\hat{\theta}$=0.6003577, $\hat{\alpha}$= 3.0375828, $\hat{\beta}$= 13.4027056	95.79146	0.10493	0.4061
MOW-Lindley	$\hat{\lambda}$=31.4349756, $\hat{\theta}$= 1.3388160, $\hat{\alpha}$=0.1576801	97.09702	0.10997	0.3486

Table 2: MLEs of the considered MOW-G parameters, $-\ell_n$ and K-S test.

Model	AIC	BIC	HQIC	CAIC	W^*	A^*
MOW-gamma	194.4046	203.5112	198.03	195.0016	0.06383593	0.4168578
MOW-Weibull	199.5829	208.6896	203.2083	200.1799	0.164807	0.9707013
MOW-Lindley	200.194	207.024	202.9131	200.547	0.20699	1.21183

Table 3: Statistics of the considered MOW-G models.

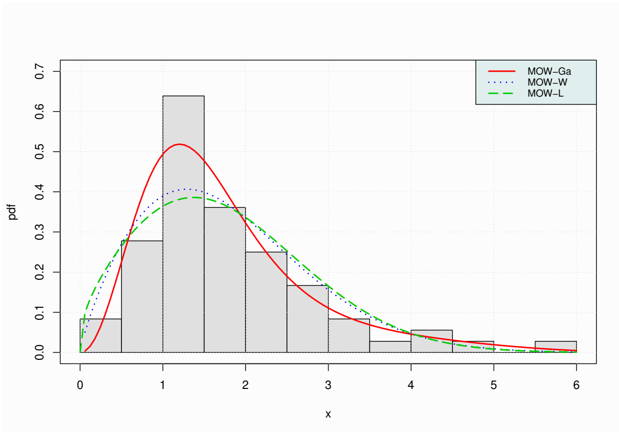

Figure 7: Plots of the estimated pdfs.

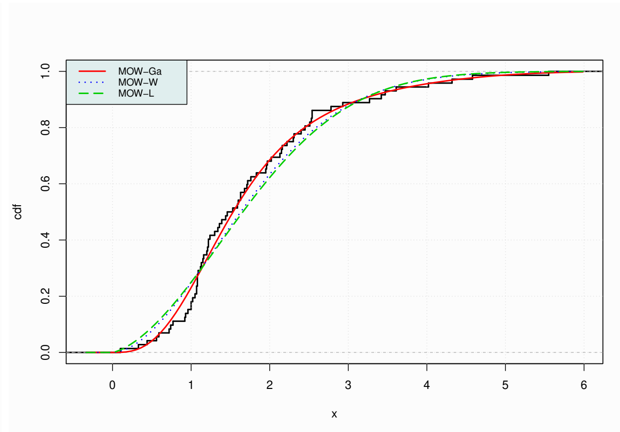

Figure 8: Plots of the estimated cdfs.

6.2 Second application

In the second application, we consider the data set used by [6]. It contains the values of fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps). These data are also studied by [8]. This data set is given by: 70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212.

We consider the competitive models presented in [8]. The measures collected in Table 4 show that the MOW-gamma model is the best by comparing the AIC statistic. Table 5 gives the estimations of the MOW-G models parameters, log likelihood, K-S statistics and the p-value. According these p-values, the MOW-gamma model is the more appropriate to fit these data. Then, Table 6 shows the AIC, BIC, CAIC, HQIC, W^* and A^* statistics for each model. The histogram of the data and plots of the estimated pdfs are displayed in Figure 9. The estimated cdfs for each model are shown in Figure 10. Again, we observe that the MOW-gamma model is the best to fit of the data.

Model	AIC
MOW-gamma	918.1
WBXII	920.6
BBXII	933.2
WLL	924.0

Table 4: AIC of the considered models.

Model	MLE	$-l_n$	K-S	p-value
MOW-gamma	$\hat{\lambda}$=5.7131479, $\hat{\theta}$=0.1670157, $\hat{\alpha}$= 117.7898653, $\hat{\beta}$= 0.5657607	455.0856	0.12476	0.8456
MOW-Weibull	$\hat{\lambda}$=0.002772359, $\hat{\theta}$=0.264697465, $\hat{\alpha}$= 0.954442502, $\hat{\beta}$= 5.517385214	472.5828	0.324817	0.08621
MOW-Lindley	$\hat{\lambda}$= 46.790151334, $\hat{\theta}$= 3.467074486, $\hat{\alpha}$=0.007087977	462.8913	0.10074	0.2569

Table 5: MLEs of the considered MOW-G parameters, $-\ell_n$ and K-S test.

Model	AIC	BIC	HQIC	CAIC	W^*	A^*
MOW-gamma	918.1713	928.6318	922.406	918.588	0.04273115	0.262331
MOW-Weibull	953.1657	963.6262	957.4004	953.5823	0.324817	2.065253
MOW-Lindley	931.7826	939.6279	934.9586	932.03	0.1479253	0.967373

Table 6: Statistics of the considered MOW-G models.

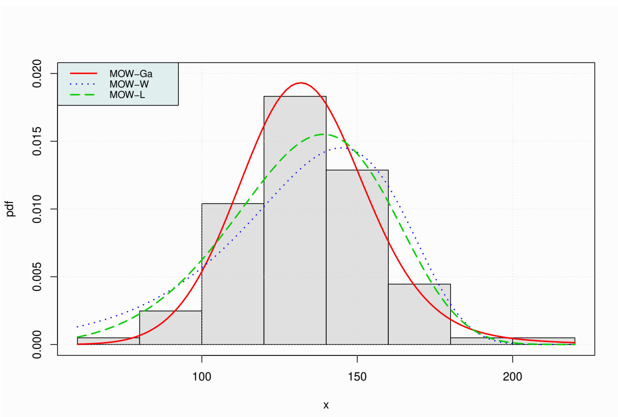

Figure 9: Plots of the estimated pdfs.

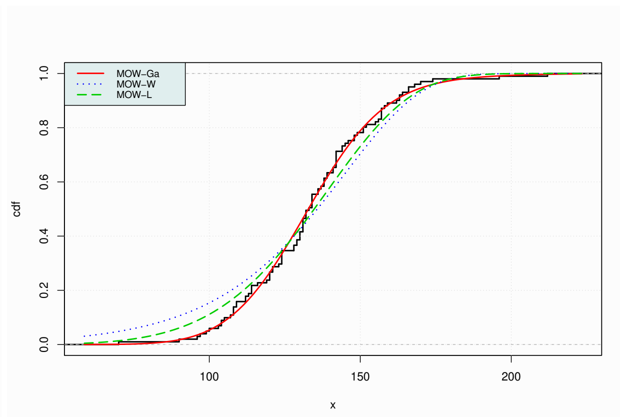

Figure 10: Plots of the estimated cdfs.

6.3 Third application

The third application considers the data set obtained from [17]. It gives the failure and running times of a sample of 30 devices, and are studied by [14], among others. The data set is given by: 2, 10, 13, 23, 23, 28, 30, 65, 80, 88, 106, 143, 147, 173, 181, 212, 245, 247, 261, 266, 275, 293, 300, 300, 300, 300, 300, 300, 300, 300.

We also adopt the models studied in [14] as competitive models. In Table 7, we compare the MOW-gamma model with these competitive models, by comparing their AIC statistics. Again, the MOW-gamma model is the best with AIC = 346.9268. Table 8 gives the estimations of the MOW-G models parameters, log likelihood, K-S statistics and the p-value. All the models satisfy p-value > 0.05, showing that they are adequate to fit these data. Then, Table 9 shows the AIC, BIC, CAIC, HQIC, W^* and A^* statistics for each model. The histogram of the data and plots of the estimated pdfs are displayed in Figure 11. The estimated cdfs for each model are shown in Figure 12.

Model	AIC
MOW-gamma	346.9268
W-g	368.5341
EG	370.9992
EE	374.2259

Table 7: AIC of the considered models.

Model	MLE	$-l_n$	K-S	p-value
MOW-gamma	$\hat{\lambda}$=0.53253335, $\hat{\theta}$=0.03540592, $\hat{\alpha}$= 55.17933057, $\hat{\beta}$= 0.46402242	169.4634	0.14339	0.5681
MOW-Weibull	$\hat{\lambda}$=0.09557056, $\hat{\theta}$=0.21449528, $\hat{\alpha}$=0.95108786, $\hat{\beta}$= 17.60538523	179.372	0.16803	0.3653
MOW-Lindley	$\hat{\lambda}$=0.21711293, $\hat{\theta}$= 0.41644317, $\hat{\alpha}$=0.02661373	175.9652	0.16658	0.3758

Table 8: MLEs of the considered MOW-G parameters, $-\ell_n$ and K-S test.

Model	AIC	BIC	HQIC	CAIC	W^*	A^*
MOW-gamma	346.9268	352.5315	348.7198	348.5268	0.1604515	1.084789
MOW-Weibull	366.744	372.3488	368.537	368.344	0.2185086	1.49234
MOW-Lindley	357.9303	362.1339	359.2751	358.8534	0.1794816	1.206463

Table 9: Statistics of the considered MOW-G models.

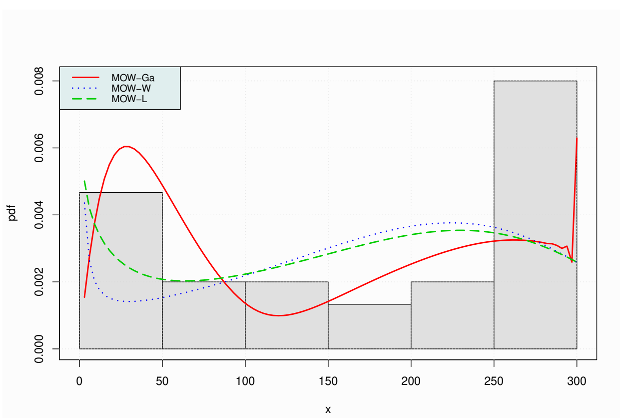

Figure 11: Plots of the estimated pdfs.

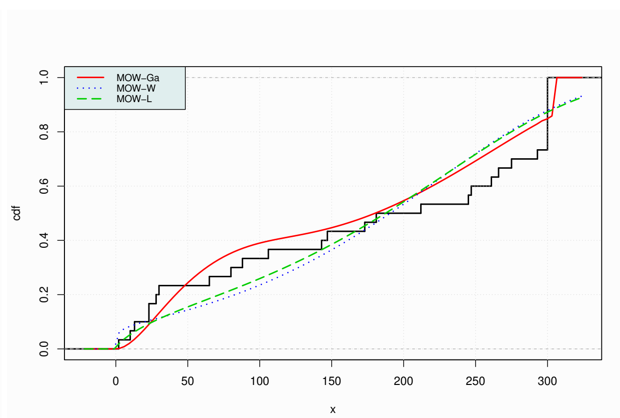

Figure 12: Plots of the estimated cdfs.

In view of these three applications, one can say that the MOW-gamma model is very flexible and give nice fits for a wide variety of data sets with different characteristics (the first data set is clearly right skewed, the second one is nearly symmetric and the third is bimodal).

7 Conclusion

In this paper, we introduce a new family of continuous distributions called the modified odd Weibull-G (MOW-G) family. In particular, it is constructed from the T-X transformation of [4] defined with the Weibull distribution and a promising new alternative of the odd transformation. We study some its mathematical properties including the quantile function, skewness, kurtosis, moments, moment generating function, order statistics and entropy. Then, the inferential aspect of the MOW-G family is explored. The maximum likelihood method is applied to estimate the model parameters and the performance of the MLEs is discussed by the biases and mean squared errors (MSE) using Monte Carlo simulations. A suitable MOW-G model to fit the data is discussed by the use of three practical data sets, the results show that the MOW-gamma model is the best in terms of fit as compared to competitive models. Hence, it is hoped that the new family of distributions will attract wider applications in different fields of study.

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