Exposition of Two-Player Zero-Sum Poker Games

Master's Thesis of

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Thursday, April 22, 2010

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1 Introduction

Poker games, the betting games with incomplete information for two or more players are interesting problems to investigate. Not only chance but also psychology, strategy, deception, and audacity are involved in decision making process. If it is only a game of luck like rock, paper, scissors, or a game without favorable plays like tic-tac-toe, we should have not seen all familiar faces at the high stakes poker World Bracelet final tables year after year.

In particular, poker games can be understood systematically and rigorously by game theoretical and probabilistic models. Therefore, they yield a wealth of problem for the mathematicians and game theorists. John von Neumann and Oskar Morgenstern analyzed two simple poker games in *Theory of Games and Economic Behavior*. Thereafter, papers generalized the results, extended ideas to simulate real Poker games more closely, or help to reveal certain aspects of the game. Increasingly, poker players who implement mathematical point of view to the game have emerged, beating players adhering to experience and psychological manipulation. Intuition and experience are superseded by a group of mathematical and rational plays by the PhDs. Chris Ferguson, a computer science PhD from UCLA and Bill Chen, a mathematics PhD from UC Berkeley are the most representative players. Their works in theoretical poker are discussed here (Ferguson and Ferguson 2003; Ferguson, Ferguson, and Garaway 2007; Chen and Ankenman 2006). The simplified models developed by themselves have helped to play optimally.

There are many advantages modeling the poker games with statistical and game theoretical tools. Simplified models help us to concentrate on certain aspects of the game. For example, when the model only specifies one round of betting, it reflects the effects of bluffing. However, when more rounds of betting are mixed into the model, what kinds of hands should be played can be found, and some results may even be counterintuitive though optimal. The ultimate goal for the players is to optimize. Though one may already have a favorable pure or mixed strategies, it might not be maximizing the profit for a player or it may even be dominated by certain type of strategies. The notions such as Bayesian-Nash and trembling hand equilibria help to achieve desirable plays. Risk management should also be considered.

For those who are unfamiliar with poker, and for those who may have different names, some basic terminology is introduced. In the beginning, each player may need to put in a certain fixed number of chips in order to be dealt a hand. These chips are called **ante** and may be considered dead money, the money you may need to earn back from the **pot** where all the money is aggregated and awarded to the eventual winner. The first player can do nothing and pass the opportunity to the next player by **checking**, or **bet** or **raise** a certain amount for the other players to match in order to stay in the game. Then, player II (or subsequent players if there are more than two players) can either **fold** so that he forfeits his chance of winning and money he has put in, **call** by matching the amount player I has raised, or **re-raise** by calling the amount player I has raised, and raising additional chips on top. Then player I or subsequent players have the choice to call or re-raise further. Someone wins when everyone else folds or everyone else checks and he has the best hand.

In this exposition, we discuss several existing models, and extend upon them by making some fundamental modifications that help to reveal different aspects of the game. They are all two-player zero-sum games with initial hands to both players $U(0,1)$. The von Neumann model in von Neumann and Morgenstern (1953) is the foundation of all the models. It allows just one raise for the first player, as it reveals the importance of bluffing, and its relation to the value of the game. When the bet is allowed to be any positive number, an equilibrium is solved (Newman 1959). Thereafter, additional raises are allowed. Ferguson and Ferguson (2003) solved the game with 1 bet for each player. I then extend the model by allowing one more re-raise for player I.

Previously, all the models have fixed hands for both players. I construct a simple model with changing hands. After a signal, player I's can be changed for better or worse. This is motivated by the situation where a player may possibly have a straight draw or flush draw and the river card (the last card that appears on the table) may appear to have helped him. How the players react to it should be explored.

2 von Neumann's Game: 1 fixed bet for Player I

This section presents the basic model of von Neumann while definitions that are applicable to latter sections are introduced along the way. Two players each contribute an ante of \$1, and are dealt "hands" x_1 and x_2, respectively independent and identically distributed as $U(0,1)$. Player I can check or bet a predetermined amount B, and player II can call or fold if player I bets. The only available information for each player is his own hand and the game structure. Strategies for two players are

Player I: $s_1 : x_1 \rightarrow \{\text{check, bet}\}$;

Player II: $s_2 : x_2 \times s_1(x_1) \rightarrow \{\text{call, fold}\}$.

A mixed strategy can be specified as $b : [0,1] \rightarrow [0,1]$ where if player I is dealt hand x_1, he bets with probability $b(x)$. Similarly, player II's mixed strategies are functions $c : [0,1] \rightarrow [0,1]$, where if player II is dealt x_2 an player I bets, then player II calls with probability $c(y)$.

A player's payoff is dependent of $x_1, x_2, s_1(x_1)$, and $s_2(x_2)$. Extensive form of the game and optimal strategies are presented in Figure 1 with player I's payoffs shown (their reciprocals are player II's because the game is zero-sum).

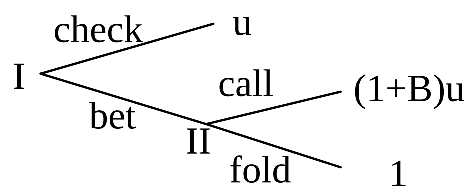

Figure 1: Extensive form of von Neumann's game,
where $u = 1$ if $x_1 \geq x_2; = -1$ otherwise.

We want to investigate how players play in equilibrium. They are going to play a pair of optimal strategies as defined below. A strategy is optimal if given any hand and the other player's strategy, there is no incentive to deviate to any other strategy.

Definition 2.1. *For player i, a strategy s_i^* is* **optimal** *if given any other strategy s_i' and other player's strategy s_j,*

$$\int_{x_j} u_i(x_i, x_j, s_i^*(x_i), s_j(x_j))dx_j \geq \int_{x_j} u_i(x_i, x_j, s_i'(x_i), s_j(x_j))dx_j \quad \forall x_i, x_j \in (0,1). \tag{2.1}$$

If both/all players are playing optimal strategies, then the collection of the strategies are an **equilibrium strategy**.

For two similar hands, payoff from an optimal strategy should be the same. Otherwise, there is an incentive to deviate to the strategy played if given the other hand. Therefore, hands slightly bigger and slightly smaller yield similar payoffs. This idea is embodied in the **indifference condition (IC)**. For example in the optimal strategy above, player II is indifferent between folding and calling when he is dealt c.

Lemma 2.2 (Indifference Condition)**.** *For strategy s_i^*, as $\epsilon \to 0^+$,*

$$\int_{x_j} u_i\big(x_i, x_j, s_i^*\big(x_i - \epsilon\big), x_j\big) \to \int_{x_j} u_i\big(x_i, x_j, s_i^*\big(x_i + \epsilon\big), x_j\big) \quad \forall x_i. \tag{2.2}$$

Theorem 2.3. *An equilibrium strategy of von Neumann's game is presented as follows. Player I checks when $a \leq x_1 \leq b$ and bets amount B otherwise; in case of raise, player II calls if $x_2 > c$ and folds otherwise, where*

$$a = \frac{B}{(B+1)(B+4)}, \quad b = \frac{B^2 + 4B + 2}{(B+1)(B+4)}, \quad c = \frac{B(B+3)}{(B+1)(B+4)}.$$

Two players' optimal strategies are illustrated in Figure 2.

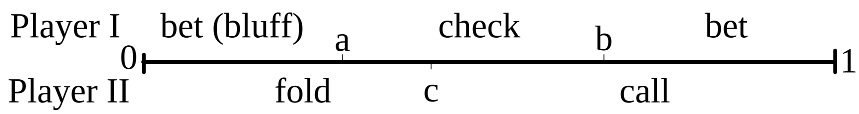

Figure 2: Optimal strategies of both players

Proof. Optimal strategies are found by backward induction. Player II's optimal strategy is found first. When Player I raises, player II calls if his expected payoff is greater than -1, which is his payoff from folding. Since his payoff depends piecewise-linearly on his hand strength x_2, $u_2(x_1, x_2)$, player II's payoff, is a monotonic function of x_2 if he calls. Therefore, player II's optimal strategy is to call when $x_2 > c$ and to fold otherwise.

Given player II's optimal strategy, player I should bet if his expected payoff of betting is greater than of checking (Tie situations need not be considered because the density function is non-atomic). Player I's payoff given x_1

- from checking is $(+1) \cdot (x_1 - 0) + (-1) \cdot (1 - x_1) = 2x_1 - 1$.
- from betting is

$$\int_0^c (+1)dx_2 + \int_c^1 (-1 - B)dx_2 = c + (-1 - B)(1 - c) = (B + 2)c - B - 1, \quad x_1 < c,$$

or

$$\int_0^c (+1)dx_2 + \int_c^{x_1} (1 + B)dx_2 + \int_{x_1}^1 (-1 - B)dx_2 = (B + 1)(2x_1 - 1) - Bc, \quad x_1 \geq c.$$

Then by IC in Equation 2.2,

$$2x_1 - 1 = 2c + Bc - B - 1, \tag{2.3}$$

$$2x_1 - 1 = (B + 1)(2x_1 - c - 1) + c \tag{2.4}$$

Since I's payoff is also piecewise linear with respect to x_1, player I's optimal strategy is to bet if $x_1 < a$ or $x_1 > b$, and to check if $a \leq x_1 \leq b$. Then $x_1 \leq c$ in Equation 2.3, and $x_1 \geq c$ in Equation 2.4,

$$2a - 1 = 2c + Bc - B - 1 \tag{2.5}$$

$$2b - 1 = (B + 1)(2b - c - 1) + c \tag{2.6}$$

Player II's optimal strategy should obey the indifference condition,

$$\Big[a(1+B)+(1-b)(-1-B)\Big]\Big/\Big(a+1-b\Big)=-1 \tag{2.7}$$

Solve Equations 2.5 and 2.6 give values a and b and substitute into Equation 2.7, $c = B(B+3)\big/\big[(B+1)(B+4)\big]$. a and b as functions of B are obtained by re-substitution. □

Remark 2.4. *Note that the only optimal pure strategy for Player I is as described above. However, for player II, even if he varies his strategy between a and b given that he calls with $b-c$ proportion of hands, while still folding $x_2 < a$ and calls $x_2 > b$, there is no better strategy for player I.*

2.1 Payoff Square and Player I's Value

Payoffs of both players can be determined. In addition, given that player I determines his bet amount before hands are assigned, the optimal B that maximizes the expected payoff is of interest. First, **payoff squares** that describe strategies and corresponding payoffs of players are introduced.

Definition 2.5. *A* **payoff square** *is a two-dimensional square diagram with hands of player I in x-axis, and player II's in y-axis. A point in the square indicates a hand pair (x_1, x_2), and the payoff, $u_1(x_1, x_2, s_1(x_1, x_2), s_2(x_1, x_2))$, indicated at the point is resulted from the strategies played corresponding to the hands.*

Remark 2.6. *Payoffs of any strategy set can be depicted by the payoff square. It can also be generalized to n-dimension, which is equivalent to taking multiple integrals of n variables (Besides the payoff square, only a payoff "cube" depicting a three-player game is beneficial).*

Corollary 2.7. *Given that both players follow the optimal strategies described in Theorem 2.3, player I's payoff is $u_1(x_1, x_2, s_1, s_2) = a$. Optimal bet amount is $B^* = 2$.*

Proof. Expected payoff from player I checking all hands is 0, $+1$ below $x_1 = x_2$ and -1 above $x_1 = x_2$. Equivalently, the payoff differential from this strategy is illustrated and expected payoff would be the same overall, so we add 1 above $x_1 = x_2$, subtract 1, and cancel out a square region on the top of $+B$ and $-B$ to get the resulting square on the right. The original, complete payoff square as well as its geometric and algebraic manipulation are shown in Figure 3.

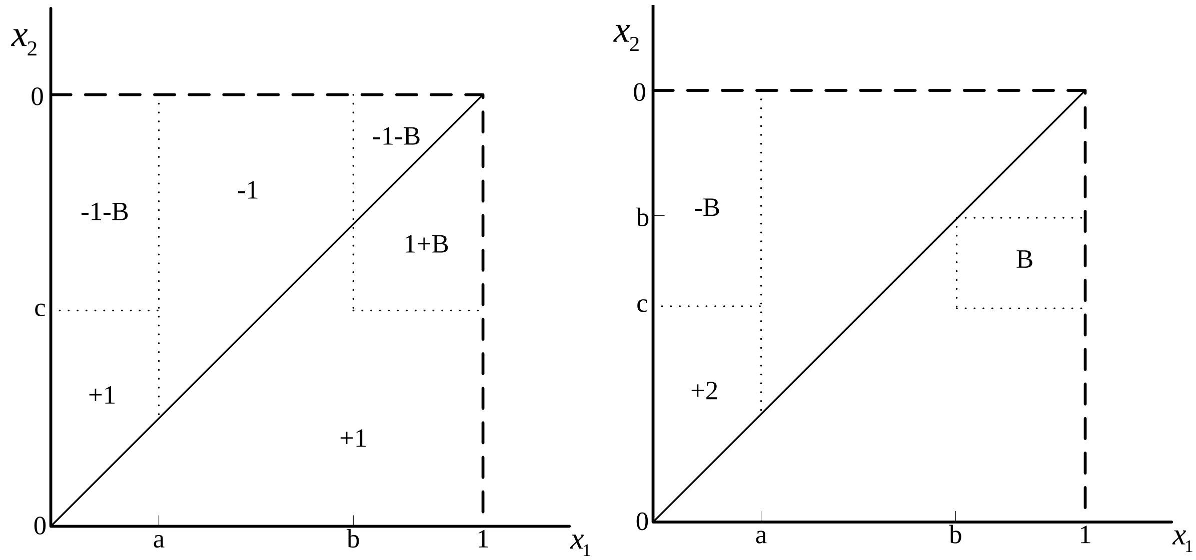

Figure 3: Illustration of Player I's payoff by payoff square.

The expected payoff of player I is

$$\begin{aligned} u_1 &= B(1-b)(b-c) - B(a)(1-c) + (2)(1/2)(c+c-a)(a) \\ &= B\big/\big[(B+1)(B+4)\big] = a. \end{aligned} \tag{2.8}$$

Maximizing $u_1(B)$ with respect to B, $\big(4-B^2\big)\big/\big[(B+1)^2(B+4)^2\big] = 0, B^* = 2$. $u_1(2) = 1/9, a = 1/9, b = 7/9, c = 5/9$. □

The result deserves some discussion. Player I's payoff, $B/(B^2+5B+4)$ is positive for all $B > 0$, and it achieves its maximum at $B^* = 2$, the pot size. This means that the game favors player I who is given the chance to raise, and he maximizes his payoff to be 1/9 by betting pot size every time. Player I has an advantage because he can bluff with his worst hands. More importantly, for real poker perhaps, he must bluff with his *worst* but not mediocre hands.

3 Newman's Game: 1 variable bet for Player I

In contrast to von Neumann's model in which the bet amount is pre-determined and fixed, Donald Newman presents a model that has the same game structure but allows any bet amount Newman (1959). In this game, player I's expected payoff is 1/7 because player I bluffs 1/7 of time. Optimality of the strategy is proven by showing that the given pure strategy is a saddle point of all strategies.

The game tree is the same as above, but any positive real number bet B is allowed. Player I's strategy $B(x_1) : (0,1) \to \mathbb{R}_+ \cup \{0\}$, and Player II calls the amount when $x_2 \in S_{B(x_1)}$. Collection of the intervals S_B is denoted Σ. Expected payoff to A due to the pure strategy is then

$$u_1(x_1,x_2) = \begin{cases} 1 & \text{if } x_2 \notin S_{B(x_1)} \\ u(1+B(x_1)) & \text{if } x_2 \in S_{B(x_1)} \end{cases} = \begin{cases} 1+0 \\ 1+[u(1+B)-1] \end{cases}$$

$$E(\Sigma,B) = 1 + \int\int_{x_2\in S_{B(x_1)}} \big[u(1+B(x_1))-1\big]dx_1dx_2 \tag{3.1}$$

Theorem 3.1. *Let $\xi = 2/(B+2)$, then the* **optimal strategy** *is that I checks if $x_1 \in (1/7, 4/7)$, bets B with hand $1/7 \cdot (1-3\xi^2+2\xi^3) = (4/7)(2+3B)(2+B)^{-3}$ or $1-3/7\cdot\xi^2 = 1-(12/7)(2+B)^{-2}$; II calls if and only if $x_2 > 1-6/7\xi = 1-(12/7)(2+B)^{-1}$ and folds otherwise.*

Proof. We must prove, according to definition of Nash equilibrium, that

Proposition 3.2. *Player II's payoff is maximized: $-E(\Sigma_0,B_0) \geq -E(\Sigma,B_0)\ \forall\Sigma$*

Proposition 3.3. *Player I's payoff is maximized: $E(\Sigma_0,B_0) \geq E(\Sigma_0,B)\ \forall B$*

where Σ_0 and B_0 are asserted solutions. Together, we must prove that

$$E(\Sigma,B_0) \geq E(\Sigma_0,B_0) \geq E(\Sigma_0,B) \tag{3.2}$$

First, we prove assertion 1. Given any S_B, define $S'_B = \big(1-\text{measure}(S_B),1\big)$. Since integrand in equation 3.1 is monotone decreasing in x_2. We have

$$-E(\Sigma',B) \geq -E(\Sigma,B).$$

From poker point of view, this equation states that if player II decides to call proportion p of all hands, then he should call with his best p of the hands to yield higher expected payoff, regardless of player I's strategy. However, it is certainly true that

$$-E(\Sigma',B_0) \geq -E(\Sigma,B_0). \tag{3.3}$$

Now consider B_0. $\forall B_0 > 0$, there are two hands $x_1 < x_2$ correspoding to the amount of bet. Let $S_0^* = (0,1)$, and for $B>0$, define

$$S_B^* = \Big[S'_B \cap (x_1,1)\Big] \cup (x_2,1).$$

Potentially if they are nonempty intervals, S_B^* differs from S_B' by removing (a, x_1) where $E \geq 0$ and adding (x_2, a) where $E \leq 0$. Therefore,

$$-E(\Sigma^*, B_0) \geq -E(\Sigma', B_0) \tag{3.4}$$

S_B^* are intervals of the form

1. $x_1 \leq a^*(B) \leq x_2$ for $B > 0$

2. $a^*(0) = 0$

For player II, this means that he should call with hands $(p, 1)$ with p better than the bluffing hand but worse than player I's better hand.

For such a^*, by strategy B_0, $u(x, a^*) = a^* + u(1 - a^*)(1 + B)$,

$$E(\Sigma^*, B_0) = \int_0^1 a^* + (B+1)[|a^* - x_1| + x_1 - 1]dx \tag{3.5}$$

It can be shown that E is independent of a^* if it satisfies the two conditions above, by splitting E into $\int_0^{1/7} + \int_{1/7}^{4/7} + \int_{4/7}^{1}$ and changing the independent variable into ξ as defined above, a^* cancels out of the expression (in $\int_{1/7}^{4/7}$, $a^* = 0$; intuitively, a^* with corresponding values in each of the two intervals cancel each other out).

This in particular means that

$$E(\Sigma^*, B_0) = E(\Sigma_0, B_0). \tag{3.6}$$

By equations 3.3, 3.4, 3.6, we have

$$-E(\Sigma_0, B_0) = -E(\Sigma^*, B_0) \geq -E(\Sigma', B_0) \geq -E(\Sigma, B_0),$$

verifying assertion 1.

Next by backward induction, given player II's strategy, we need to prove assertion 2. Here we drop x_1 subscript because Player II's payoff is no longer inolved. By eq.3.5,

$$E = \int_0^1 1 - \frac{6}{7}\xi + \frac{2-\xi}{\xi}\Big[\big|1 - \frac{6}{7}\xi - x_1\big| + x_1 - 1\Big]dx. \tag{3.7}$$

Since ξ is a function of x, we verify for the three intervals

1. $x < 1/7$. Integrand is

$$1-\frac{6}{7}\xi+\frac{2-\xi}{\xi}\Big[\big(1-\frac{6}{7}\xi-x\big)+x-1\Big]=1-\frac{6}{7}\xi+\frac{2-\xi}{\xi}(\frac{6}{7}\xi)=1-\frac{12}{7}=-\frac{5}{7}$$

independent of ξ, so the choice $x=\frac{1}{7}(1-3\xi^2+2\xi^3)$ is justified.

2. $1/7 \le x \le 4/7$. The integrand has non-vanishing derivative sign(x), so maximum occurs at an endpoint. $\xi = 0$ produces $-5/7$ and $\xi = 1$, $2x-1$, Since $2x-1 \ge -5/7$ for all x in the interval, $\xi = 1$, i.e. $B = 0$ is the optimal choice.

3. $x > 4/7$. Derivative $= 0$ when $x = 1-\frac{3}{7}\xi^2$. and integrand equals $\frac{1}{7}(19-24\xi+6\xi^2)$. Considering along with endpoints $\xi = 0, 1$,

$$\frac{1}{7}(19-24\xi+6\xi^2)=1-\frac{6}{7}\xi^2+\frac{12}{7}(1-\xi^2)\ge V(1)=1-\frac{6}{7}\xi^2=2x-1\ge V(0)=-\frac{5}{7}.$$

So $x = 1-\frac{3}{7}\xi^2$ is optimal.

Therefore, the strategy presented is optimal for both players, given that the two assertions are proven. □

4 Fergusons' Game: 1 fixed bet for Player I and 1 fixed bet for Player II

The key extension to the previous model is allowing player II to re-raise amount B_2 after calling player I's bet B_1, and then player I either calls or folds. This model is discussed in Ferguson and Ferguson (2003) and Ferguson, Ferguson, and Garaway (2007). The pure strategies given hands $x_1, x_2 \sim U(0,1)$, are

Player I: $s_1 : x_1 \times s_2(s_1) \to$ {check, bet B_1}×{fold, call}={bet-fold, bet-raise, check},

Player II: $s_2 : x_2\times$ {check, bet} $\to$ {bet B_2, check, fold}.

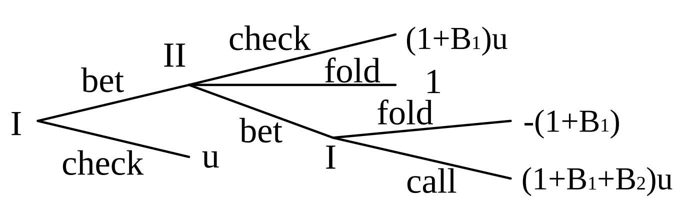

Figure 4: Extensive form of the two players with player I's payoffs.

Theorem 4.1. *Player I bet-folds when $x_1 \in (0, a] \cup (b, c]$, checks when $x_1 \in (a, b]$, and bet-calls when $x_1 > c$. Player II folds when $x_2 \le d$, calls when $x_2 \in (e, f]$, and re-raises when $x_1 \in (d, e] \cup (f, 1]$. $0 \le a \le e \le b \le c \le f \le 1$ and $0 \le d \le e$ (d can be larger or smaller than a), where*

$$
\begin{aligned}
a &= \frac{B_1^2(2+2B_1+B_2)^2}{(1+B_1)\Delta}, b = 1 - \frac{2+B_1}{B_1}a, c = 1 - \frac{2B_1(2+B_1)(2+2B_1+B_2)}{\Delta}, \\
d &= \frac{B_1+2a}{2+B_1}, e = \frac{B_1}{1+B_1} - a, f = 1 - \frac{B_1(2+B_1)(2+2B_1+B_2)}{\Delta}
\end{aligned}
$$

where $\Delta = B_1(4+B_1)(2+2B_1+B_2)^2 + (1+B_1)(2+B_1)^2 B_2$. The optimal strategies are illustrated in Figure 5.

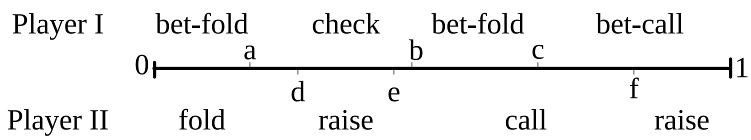

Figure 5: Optimal Strategies of both players.

Proof. First, apply the indifference conditions for

- Player I at a: $-2a + (-2 - B_1)d = B_1$;
- Player I at b: $2B_1 b + (2 + B_1)d + (-2B_1 - 2)e = B_1$;
- Player I at c: $(-2 - 2B_1 - B_2)d + (2 + 2B_1 + B_2)e + B_2 f = B_2$;
- Player II at d: $(2 + B_1)a - (2 + B_1)b + (2 + 2B_1 + B_2)c = B_1 + B_2$;
- Player II at e: $(-2 - 2B_1)b + (2 + 2B_1 + B_2)c = B_2$;
- Player II at f: $-c + 2f = 1$.

Solving six linearly independent equations of six unknowns, we get results as presented in the theorem. Then suppose we ignore the antes contributed by the players by treating them as sunk costs. Given that Player II uses the conjectured optimal strategy, and player I has hand x_1, I's "gain" from

- checking is $2x_1$.
- bet-folding is $2a$ if $0 < x < e$; $2a + 2(1+B)(x-e)$ if $e < x < f$, and $2a + 2(1+B_1)(f-e)$ if $f < x < 1$.

- bet-calling is $2a-2(1+B_1+B_2)(e-d)$ if $0<x<d$; $2a-2(1+B_1+B_2)(e-x)$ if $d<x<e$; $2a+2(1+B_1)(x-e)$ if $e<x<f$; and $2a+2(1+B_1)(f-e)+2(1+B_1+B_2)(x-f)$ if $f<x<1$.

We can verify that at every critical point, the payoffs from two strategies are equal. Then noting that the payoff function is piecewise linear, we verify that player I's strategy is optimal. Similarly given player I's optimal strategy and Player II's hand x_2, Player II's expected payoff from

- folding is 0 if $0<x_2<a$, $2(x_2-a)$ if $a<x_2<b$, and $2(b-a)$ if $b<x_2<1$.
- calling is $-2(1+B_1)(a-x_2)$ if $0<x_2<a$; $2(y-a)$ if $a<x_2<b$; and $2(b-a)+2(1+B_1)(x_2-b)$ if $b<x_2<1$.
- raising is 0 if $0<x_2<a$, $2(x_2-a)$ if $a<x_2<b$; $2(b-a)$ if $b<x_2<c$; and $2(b-a)+(1+B_1+B_2)(x_2-c)$ if $c<x_2<1$.

Then with II's boundary conditions verified, player II is proven to play an optimal strategy as well. Complicated operations were done by Maple 11. □

Corollary 4.2. *Expected payoff of player I is a. Optimal bet for player I,* $B_1^*=1+\sqrt{13}/3$. *Optimal bet for II is* $B_2^*=2B_1^*+2=4+2\sqrt{13}/3$.

Proof. Applying the modified payoff square as illustrated in Figure 6,

$$\begin{aligned}
u_1 &= B_1\big[-a(1-d)-(c-b)(e-d)+(1-c)(e-d)+(1-b)(b-e)\big]+\\
&\quad 2\big[(2d-a)(a/2)-(c-b)(e-d)\big]+B_2\big[(1-c)(e-d)-(f-c)(1-f)\big]\\
&= B_1^2\Big/\{(1+B_1)\big[B_1(4+B_1)+(1+B_1)(2+B_1)^2B_2/(2+2B_1+B_2)^2\big]\}=a.
\end{aligned}$$

Minimizing u_1 with respect to B_2 while B_1 is fixed, is equivalent to maximizing $B_2/(2+2B_1+B_2)^2$,

$$\frac{(2B_1+2+B_2)-2B_2}{(2+2B_1+B_2)^2}=0\Rightarrow B_2^*=2B_1^*+2.$$

Substitute in, $u_1=8B_1^2\Big/\Big[(1+B_1)(9B_1^2+36B_1+4)\Big]$. Then maximizing u_1 with respect to B_1 yields $9B_1^3-40B_1-8=0$. Three roots are $B_1=-2,1+\sqrt{13}/3,1-\sqrt{13}/3$. Clearly,

$$B_1^*=1+\sqrt{13}/3\approx 2.202,\quad B_2^*=4+2\sqrt{13}/3\approx 6.404$$

Substituting in B_1^* and B_2^*, $a=0.0955,b=0.8178,c=0.909,d=0.569,e=0.592,f=0.954$.

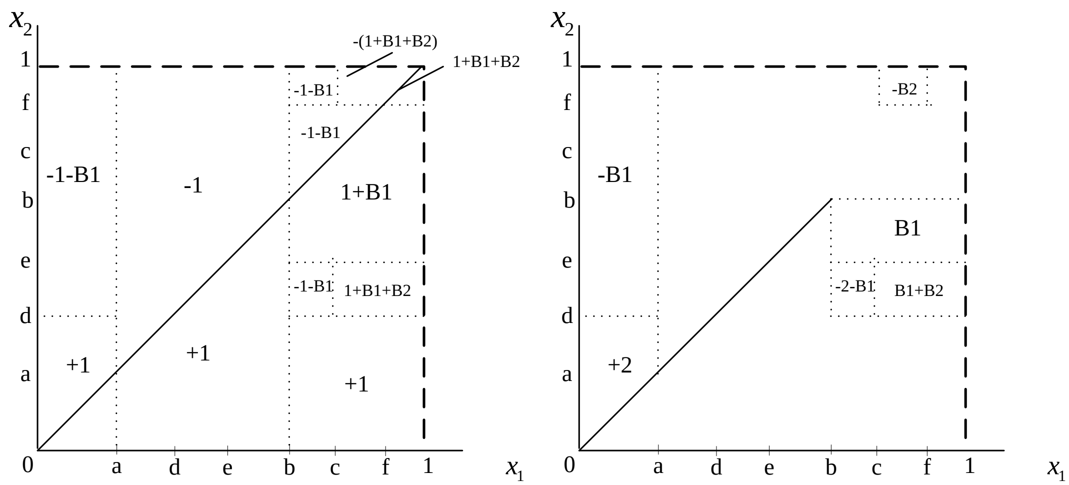

Figure 6: Payoff of Player I by payoff square.

Though the game still favors player I, its payoff (0.0955) is lower than that in the original von Neumann model (1/9). This shows that allowing player II to re-raise restricts player I's audacity of bluffing, thus decreasing his advantage and profit. Obviously player I always has advantage given that he makes a voluntary decision first: he can always checks to yield an expected payoff of 0. Also note that player I's optimal bet is a little over the pot size, and player II is little under the added pot size, however very close. This gives some insight and justification to bet by pot size in real poker.

In addition, player II can bluff with his "good" hands. This is justified as follows. Player I will fold his worst hands because he is caught bluffing, and fold some of his good hands because player II will also raise with his best hands. However, if player II simply calls, he has higher chance of losing to the good hands that player I has raised with. Folding is even more disastrous as he is bluffed by the worst hands of Player I.

In both models, payoffs for player I are both a which corresponds to the initial bluff region by Player I in the first round. This fact needs to be further investigated in order to determine whether value of player I is always a allowing additional re-raises. □

5 My Game: 2 fixed bets for Player I and 1 fixed bet for Player II

This is the simplest model allowing both players the chance to bet but with different numbers of such chances. This model is investigated in order to be compared with von Neumann's and Fergusons' models with $(1,0)$ and $(1,1)$ bets for Players I and II, respectively. How the game favors player I by adding one round of betting compared to Fergusons' and by adding one round of betting for each player compared to von Neumann's are of interest. In addition, with more information, we can conjecture a generalized equilibrium strategy if infinite number of raises is allowed for both players. If we define one round of betting as one bet and one re-raise, then this model is one of "one and half round betting".

Player I can either checks or raises B_1 after each player i is dealt hand x_i. Player II then can decide to fold, call, or re-raise amount B_2 after calling. Player I then can fold, call, or re-raise an additional amount B_3 with player II calling or folding. The strategies are illustrated as

Player I: $[0, 1] \rightarrow$ {check, raise B_1}$\times${fold, call, re-raise B_3}

Player II: $[0, 1] \rightarrow$ {fold, call, raise B_2}$\times${fold, call}

The game tree is illustrated below with payoff of Player I denoted.

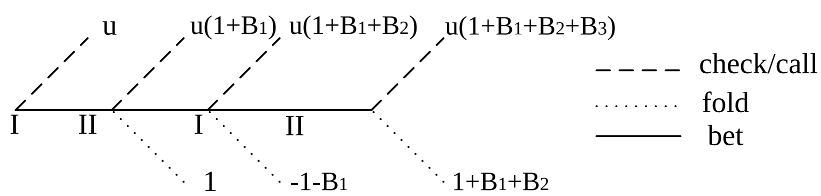

Figure 7: Game Tree of Three Raises

Theorem 5.1. *An optimal strategy is described and illustrated as follows. Player I check for $a \leq x_1 \leq b$, bet B_1 otherwise; player II will fold $x_2 < d$, call with $e \leq x_2 \leq f$, and re-raise B_2 for hands $d \leq x_2 < e$ and $f < x_2 \leq 1$. Then player I would fold $x_1 < a, b \leq x_1 \leq c$, call $g \leq x_1 \leq h$, and bet amount B_3 with $c < x_1 < g, h \leq x_1 \leq 1$. Player II will fold except calling with hands $x_2 \leq i$. Strategies at boundary points can be either one of the two adjacent to it as the probability is non-atomic.*

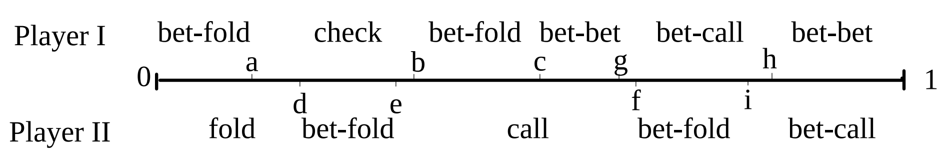

Figure 8: An equilibrium strategy for two players.

Optimal bet sizes are $B_1^* = 1 + \sqrt{4329 + 624\sqrt{13}}\Big/69 \approx 2.176$, $B_2^* = (1 + \sqrt{13}/3)(1 + B_1) \approx 2.2(1 + B_1) \approx 7$, *and* $B_3^* = 2B_2^* + 2B_1^* + 2 \approx 20$, *with*

$$\begin{aligned} &\textit{Player I:} \quad a = 0.09758, b = 0.81272, c = 0.91086, g = 0.91557, h = 0.9906, \\ &\textit{Player II:} \quad d = 0.5678, e = 0.5875, f = 0.96229, i = 0.9811. \end{aligned} \tag{5.1}$$

The expected payoff of player I is again a.

Proof. The optimal strategies are extensions from Fergusons' model just as it is the extension to the von Neumann model. For regions $[c, 1]$ where previously player I bet-calls, it is divided into three regions: bet-bet, bet-call, bet-bet, and previous $[f, 1]$ region is divided into bet-fold and bet-call areas. Instead of six unknowns, nine unknown therefore are involved. The assumption about the relationship is

$$a \leq e \leq b \leq c \leq g \leq f \leq i \leq h; \quad d \leq e \tag{5.2}$$

The indifference conditions are

1. Player I indifferent between bet-folding and checking at a:

$$d + (-1 - B_1)(1 - d) = 2a - 1; \tag{5.3}$$

2. Player I indifferent between checking and bet-folding at b:

$$2b - 1 = d + (2b - 2e + d - 1)(1 + B_1); \tag{5.4}$$

3. Player I indifferent between bet-folding and bet-betting at c:

$$2(e - d + 1 - f)(2 + 2B_1 + 2B_2) = (1 - i)(2 + 2B_1 + 2B_2 + B_3); \tag{5.5}$$

4. Player I indifferent between bet-betting and bet-calling at g:

$$i(1 + B_1 + B_2 + B_3) - B_3 = (2f - i - 1)(1 + B_1 + B_2); \tag{5.6}$$

5. Player I indifferent between bet-calling and bet-betting at h:

$$2h = 1 + i; \tag{5.7}$$

6. Player II indifferent between folding and bet-folding at d:

$$(-1)(a + 1 - b) = (1 + B_1)(a + c - b) - (1 + B_1 + B_2)(1 - c); \tag{5.8}$$

7. Player II indifferent between bet-folding and calling at e:

$$(1 + B_1 + B_2 + B_3)(c - 1) = (1 + B_1)(2b - c - 1); \tag{5.9}$$

8. Player II indifferent between calling and bet-folding at f:

$$(2g - 2c)(1 + B_1 + B_2) = (2f - c - 1)B_2; \tag{5.10}$$

9. Player II indifferent between bet-folding and bet-calling at i:

$$(g - c)(2 + 2B_1 + 2B_2 + B_3) = (1 - h)B_3; \tag{5.11}$$

Solving the nine linear equations yield the nine unknowns. Payoff square is used to derive the expected payoff of player I.

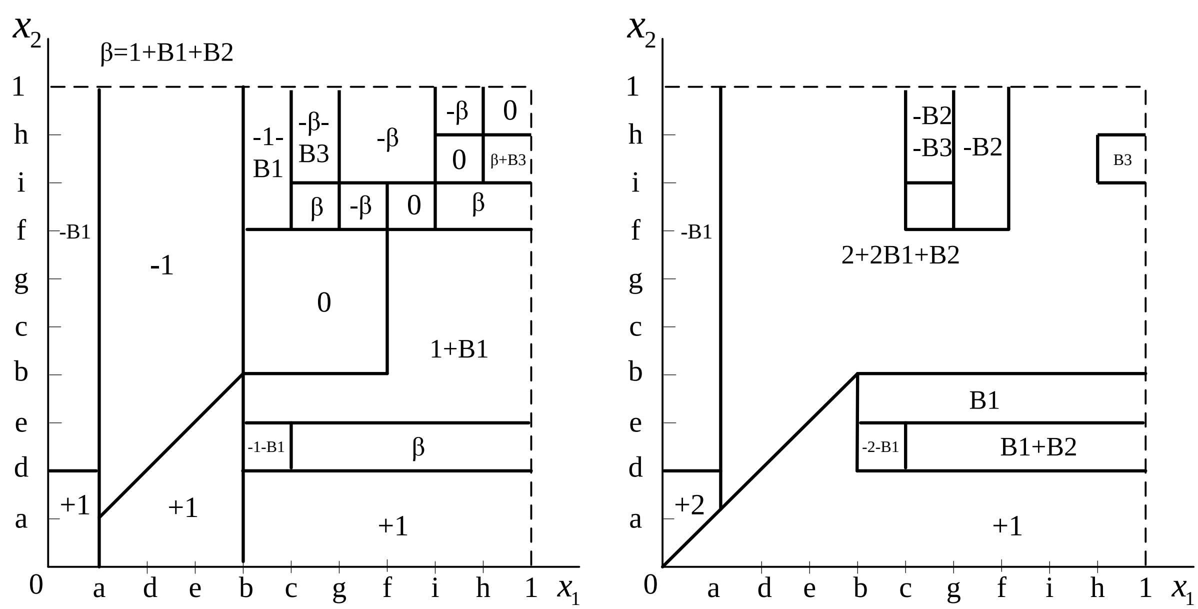

Figure 9: Payoff squares of the One and Half Round Game

$$\begin{aligned}
U_I =& \frac{1}{2}(2d - a)a(+2) - a(1 - d)B_1 + (-2 - B_1)(c - b)(e - d) + B_1(b - e)(1 - b) \\
&+ (B_1 + B_2)(1 - c)(e - d) + (1 - h)(h - i)B_3 - (f - g)(1 - f)B_2 \\
&+ (g - c)(1 - i)(-B_2 - B_3) + (2 + 2B_1 + B_2)(g - c)(i - f) = a
\end{aligned} \tag{5.12}$$

To prove that the given strategy is optimal, we prove that I and II are playing the best strategies given any hand x_1 given that the other player is playing the respective strategy.

Player I's payoff for hand x_1 from

- checking: $2x_1 - 1$

- bet-folding: $x_1 < e : d - (1 + B_1)(1 - d)$; $e \leq x_1 \leq f : d - (1 + B_1)(e - d + 1 - x_1) + (x_1 - e)(1 + B_1)$; $x_1 > f : d + (f - e)(1 + B_1) - (e - d + 1 - f)(1 + B_1)$;

- bet-calling: $x_1 \leq d : d - (1 - d)(B_1 + 1)$; $d < x_1 < ed + (f - 1 + 2x_1 - e - d)(1 + B_1 + B_2) + (e - f)(1 + B_1)$; $e \leq x_1 \leq f : d + (e - d + f - 1)(1 + B_1 + B_2) + (2x_1 - e - f)(1 + B_1)$; $x > f : d + (f - e)(1 + B_1) + (1 + B_1 + B_2)(e - d + 2x_1 - f - 1)$;

- bet-betting: $x_1 < e : d + (e - d + i - f)(1 + B_1 + B_2) + (f - e)(-1 - B_1) + (1 - i)(-1 - B_1 - B_2 - B_3)$; $e \leq x_1 \leq f : d + (e - d + i - f)(1 + B_1 + B_2) + (i - 1)(1 + B_1 + B_2 + B_3) + (2x_1 - e - f)(1 + B_1)$; $f < x \leq i : d + (e - d + i - f)(1 + B_1 + B_2) + (f - e)(1 + B_1) + (i - 1)(1 + B_1 + B_2 + B_3)$; $f < x \leq i : d + (e - d + i - f)(1 + B_1 + B_2) + (f - e)(1 + B_1) + (2x_1 - i - 1)(1 + B_1 + B_2 + B_3)$.

Player II's payoff given hand x_2 from

- folding is 0 if $0 \leq x_2 < a$; $2x_2 - 2a - 1$ if $a \leq x_2 \leq b$; $2b - 2a - 1$ if $x_2 > b$;

- calling is $b - a + (2x_2 - a + b - 1)(1 + B_1)$ if $x_2 < a$; $(a + b - 1)(1 + B_1) + 2x_2 - a - b$ if $a \leq x_2 \leq b$; $(2x_2 - b - 1 + a)(1 + B_1) + b - a$ if $x_2 > b$;

- bet-folding is $(2c - b + a - 1)(1 + B_1 + B_2) + a - b$ if $x_2 < a$; $(2c - b + a - 1)(1 + B_1 + B_2) + 2x_2 - a - b$ if $a \leq x \leq b$; $(c - b + a + g - 1)(1 + B_1 + B_2) + b - a+$ if $b < x_2 < g$; $b - a + (2c - b + a - 2g - 1 + 2x_2)(1 + B_1 + B_2)$ if $g < x_2 < h$; $b - a + (2c - b + a - 2g + 2h - 1)$ if $x_2 > h$;

- bet-calling is $a - b + (a + c - b)(1 + B_1) + (g - h)(1 + B_1 + B_2) + (1 - h + g - c)(1 + B_1 + B_2 + B_3)$ if $x_2 < a$; $2x_2 - a - b + (a + c - b)(1 + B_1) + (g - h)(1 + B_1 + B_2) + (1 - h + g - c)(1 + B_1 + B_2 + B_3)$ if $a \leq x_2 \leq b$; $b - a + (a + c - b)(1 + B_1) + (g - h)(1 + B_1 + B_2) + (1 - h + g - c)(1 + B_1 + B_2 + B_3)$ if $x_2 < c$; $b - a + (c - b + a)(1 + B_1) + (g - h)(1 + B_1 + B_2) + (2x_2 - g - c + h - 1)(1 + B_1 + B_2 + B_3)$ if $c \leq x_2 \leq g$; $b - a + (c - b + a)(1 + B_1) + (2x_2 - g - h)(1 + B_1 + B_2) + (g - c + h - 1)(1 + B_1 + B_2 + B_3)$ if $g < x_2 \leq h$; $b - a + (c - b + a)(1 + B_1) + (h - g)(1 + B_1 + B_2) + (g - c + 2x_2 - h - 1)(1 + B_1 + B_2 + B_3)$ if $x_2 > h$.

Since the nine equations are satisfied at the critical points, and payoff functions are piecewise linear to the hands and strategies, it can be verified that both players are playing the best strategy given a certain hand. $\square$

Let's compare the results to the earlier models. The optimal expected payoff of player I (0.0975) is lower than that in the von Neumann model ($1/9 = 0.11$), and higher than that of Fergusons' (0.0955). This result is because of the betting power given to each player. In von Neumann's basic model, only player I has one chance of betting, Fergusons give equal number of raises to each player but player I the advantage of raising first. Though player I is favored because he makes a decision first, his advantage is reduced because permitting player II to re-raise restricts player I's audacity to bluff thus reducing the proportion of hands he bluffs with and raises with overall.

However, in this game with one and half round of betting, player I gets back some of the edge by being both the first and the last to be able to raise. However, notice that player II folds over half and just a little under two thirds of his hands, and it is the same proportion of hands that have been folded initially compared to the previous model. The loss of payoff for player II comes mainly from two areas: more initial hands folded and additional hands that are simply called in the previous game but now need to be folded after re-raise by player I. However, player I's expected payoff is not greatly affected. We suspect that the expected payoff of player I is converging to a value near 0.097.

6 Discussion: Multiple re-raises

The game tree allowing two raises by each player is illustrated in Figure 10, and model of more raises can extended by adding more "branches".

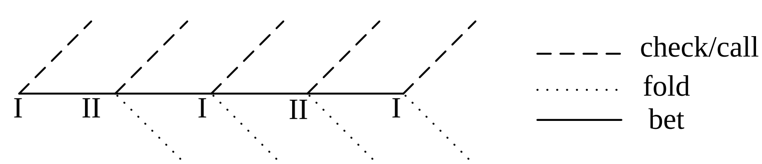

Figure 10: Game tree of two raises.

By forward induction, we conjectured the optimal strategies for both players to be as follows. For player i, $i = 1, 2$, in round j, let $f_{i,j}$ be the folding threshold, $c_{i,j}$ the calling threshold, and $r_{i,j}$ the raising threshold. $f_{i,j} < c_{i,j} < r_{i,j} < 1$, dividing the hands into four intervals. Then player i in round j will

$$\begin{cases} \text{fold} & \text{if } x_i \in (0, f_{i,j}] \\ \text{raise} & \text{if } x_i \in (f_{i,j}, c_{i,j}] \\ \text{call/check} & \text{if } x_i \in (c_{i,j}, r_{i,j}] \\ \text{raise} & \text{if } x_i \in (r_{i,j}, 1] \end{cases}$$

$f_{I,1} = 0$ in this model because no hand will be folded by player I. Furthermore, $0 < f_{i,j}, c_{i,j}, r_{i,j} < f_{i,J}$ for all $J > j$, producing strictly smaller interval of remaining hands. Relationship of inequalities between two players' folding, calling and raising thresholds needs to be investigated. Similarly, we can apply indifference conditions to each of the boundary points in the crossing intervals and solve for equilibrium conditions, but it will only grow more complicated algebraically.

The value of the game, that is, the expected payoff of player I will eventually converge to a certain value close to $0.096 - 0.097$, more crudely $1/10$. The fact that the value of the game is always equal to the initial bluff region of player I should be able to proven rigorously, possibly with help of the cancellations made in payoff square.

(Cutler 1975) discusses a model allowing infinite pot-size raises, but with initial force-in for player I. It is possible to be solved since the bet is restricted to be pot size, the ratio of high raises and bluffs is always 2 to 1. Both players will make bluffs and raises with fewer but the same proportion of remaining hands. The game favors player II because player I is forced to raise in the first round. In addition, the paper shows that the profit from player II being super bluffer and ultraconservative player is virtually the same.

7 The Model with Changing Hands

Players I and II are dealt hands $x_1, x_2 \in U[0, 1)$. Player I is then forced to bet B_1. Player II can decide to call or fold. If called, the game proceeds. A ball marked $+$ or $-$ is shown with equal chance, and player I's hand x_1 is changed accordingly. If $+$ is shown, there is $1/2$ chance that the ball stays unchanged or $1/2$ chance that the ball is changed to $x_1 + 0.5$ if $x_1 < 0.5$ and stays unchanged x_1 for $x_1 \geq 0.5$. If $-$ ball is shown, then there is half chance that all hands stay unchanged and half chance that x_1 is changed to $x_1 - 0.5$ for hands $x_1 \geq 0.5$ and unchanged if $x_1 < 0.5$. Then player I can bet again, with fixed amount B_2. Player II can either call or fold. Game Tree is as below.

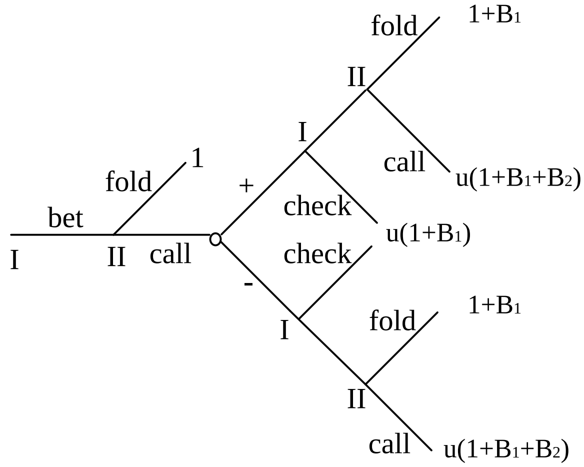

Figure 11: Game of the Changing Hand Game

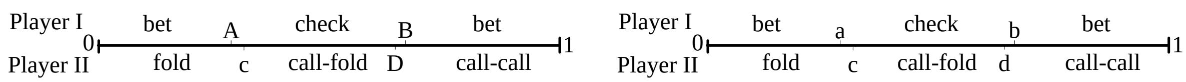

Figure 12: Equilibrium Strategy of The Changing Hand Game where a, b, d are equivalent to a^-, b^-, d^-, and A, B, D are a^+, b^+, d^+

The critical points in the optimal strategies have the order relation:

$$0 < a^+ < c < \frac{1}{2} < d^+ < b^+ < 1 \tag{7.1}$$

$$0 < a^- < c < \frac{1}{2} < d^- < b^- < 1 \tag{7.2}$$

Indifference conditions at the critical points are

1^+) When $+$ is shown, Player I is indifferent between checking and betting at a^+

$$(-1 - B_1) = \frac{2d^+ - c - 1}{1 - c}(1 + B_1 + B_2)$$

$$(c - 1)(1 + B_1) = (2d^+ - c - 1)(1 + B_1 + B_2)$$

$$(2d^+ - 2c)(1 + B_1) + (2d^+ - c - 1)B_2 = 0 \tag{7.3}$$

2^+) When $+$ is shown, Player I is indifferent between checking and betting at b^+

$$\frac{2b^+ - c - 1}{1 - c}(1 + B_1) = \frac{d^+ - c}{1 - c}(1 + B_1) + \frac{2b^+ - d^+ - 1}{1 - c}(1 + B_1 + B_2)$$

$$2b^+ = d^+ + 1 \tag{7.4}$$

3$^+$) When $+$ is shown, Player II is indifferent between folding and calling at d^+

$$(-1-B_1)[\frac{3}{2}(1-b^+)+\frac{1}{2}a^+] = (1+B_1+B_2)[\frac{1}{2}a^+ + \frac{3}{2}(b^+-1)]$$
$$2a^+(1+B_1)+B_2(a^++3b^+-3)=0 \tag{7.5}$$

1$^-$) When $-$ is shown, Player I is indifferent between checking and betting at a^-

$$(2d^- - 2c)(1+B_1)+(2d^- - c+1)B_2 = 0 \tag{7.6}$$

2$^-$) When $-$ is shown, Player I is indifferent between checking and betting at b^-

$$2b^- = d^- + 1 \tag{7.7}$$

3$^-$) When $-$ is shown, Player II is indifferent between folding and calling at d^-

$$(-1-B_1)[\frac{1}{2}(1-b^-)+\frac{3}{2}a^-] = (1+B_1+B_2)[\frac{3}{2}a^- + \frac{1}{2}(b^--1)]$$
$$6a^-(1+B_1)+B_2(3a^-+b^--1)=0 \tag{7.8}$$

4) When confronted with a raise, Player II is indifferent between folding and call-folding at c [1]:

$$\begin{aligned}-1 =& \frac{1}{2}\big[-\frac{3}{4}(1+B_1)-\frac{1}{2}a^+(1+B_1)-(\frac{1}{2}-c)\frac{1}{2}(1+B_1)+\frac{1}{2}(c-a^+)(1+B_1)\big]\\ &+\frac{1}{2}\big[-\frac{1}{4}(1+B_1)-\frac{3}{2}a^-(1+B_1)-(\frac{1}{2}-c)\frac{3}{2}(1+B_1)+\frac{3}{2}(c-a^-)(1+B_1)\big]\\ 2 =& (1+B_1)(2+a^++3a^--4c)\end{aligned} \tag{7.9}$$

Solving the seven equations yield the seven unknowns:

$$\begin{aligned}&a^- = \frac{(x+2)y}{\Delta}, a^+ = \frac{3y(x+2)}{2\Delta}, c = \frac{2x+2x^2+3xy+y}{\Delta}\\ &b^+ = b^- = \frac{9xy+8y+6x^2+10x+4}{2\Delta}, d^+ = d^- = \frac{4xy+3y+2x^2+2x}{\Delta}.\end{aligned} \tag{7.10}$$

where $\Delta = 5xy+4x^2+8x+4+5y$.

[1]Actually, assuming $c > 0.5$ and $c < 0.5$ yields the same indifference condition for c.

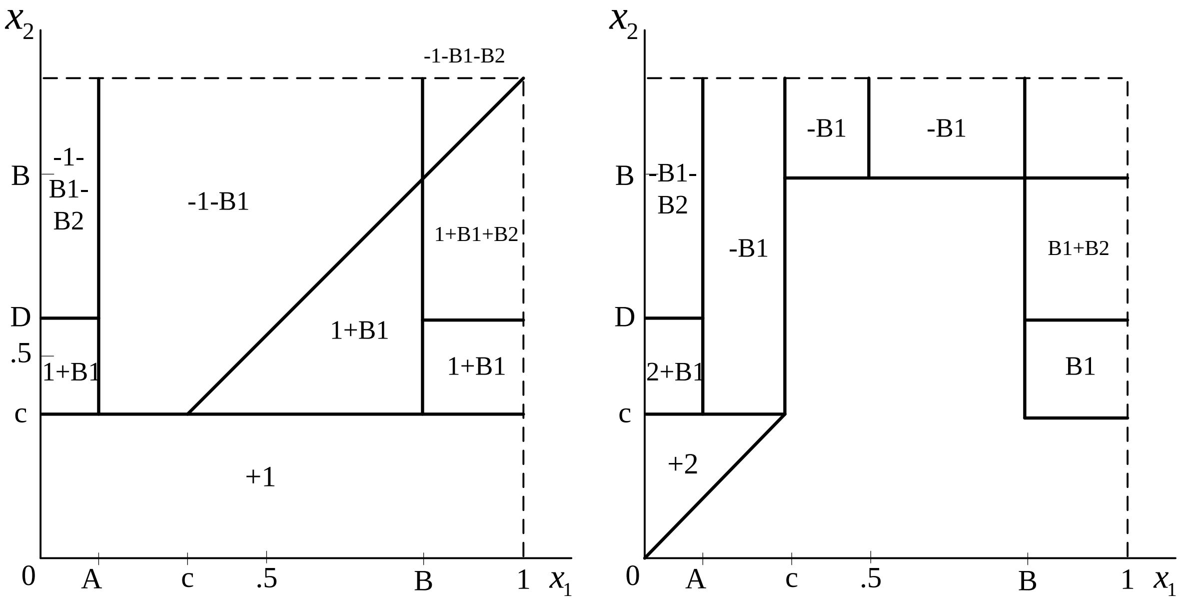

Figure 13: Payoff Square of the + sub-game.

Player I's payoff $V_1 = \left(V^+ + V^-\right)\Big/2$ where V^+ is the payoff when + signal is shown,

$$\begin{aligned}V^+ =&\Big[c^2 + (d^+ - c)A(x+2) + A(1-d^+)(-B_1 - B_2) - (c - A)(1-c)B_1 \\ &- (1/2 - c)(1-B)B_1 - (3(b^+ - 1/2))(1-b^+)B_1 + (3*(1-b^+))(d^+ - c)x \\ &+ (3(b^+ - d^+))(1-b^+)(B_1 + B_2)\Big]\Big/2\end{aligned} \tag{7.11}$$

and V^- is the payoff when $-$ is shown,

$$\begin{aligned}V^- =&\Big[3c^2 + 3(d^- - c)a(B_1+2) + 3a^-(1-d^-)(-B_1 - B_2) - 3(c - a^-)(1-c)B_1 \\ &- 3(\frac{1}{2} - c)(1-b^-)B_1 - (b^- - \frac{1}{2})(1-b^-)B_1 + (1-b^-)(d^- - c)B_1 \\ &+ (b - d^-)(1-b)(B_1 + B_2)\Big]\Big/2\end{aligned} \tag{7.12}$$

Overall,

$$\begin{aligned}V = -&\Big[56B_1^3B_2 + 36B_2B_1^4 + B_1^2B_2^3 + 20B_2^2B_1^3 + 48B_1^4 - 16B_2 + 48B_1^3 + 16B_1^5 \\ &- 12B_2B_1^2 + 4B_2^3 + 16B_1^2 + 4B_2^3B_1 - 48B_1B_2 - 40B_2^2B_1 - 20B_2^2\Big]\Big/\Big[(B_1+1)\Delta\Big]^2\end{aligned} \tag{7.13}$$

When the bets are restricted to the pot size, that is, $B_1 = 2, B_2 = 4$, value of the game to player I is $-7\big/36$, and

$$a^- = \frac{1}{36}, a^+ = \frac{1}{4}, b^- = b^+ = \frac{19}{24}, c = \frac{5}{12}, d^- = d^+ = \frac{7}{12}. \tag{7.14}$$

Therefore, the game favors player II. Payoff squares for the + situation are illustrated. The − situation is the same except that the boundaries are replaced by respective a^-, b^-, and d^-.

8 Conclusion

In this exposition, we discussed some classical models of two-player zero-sum poker games with initial independent and identically distributed uniform $(0, 1)$. The models differ by number of bets allowed, amount of bet allowed each round, and the possibility of change to the hand held. While equilibrium strategies are found, various aspects of the game are revealed. For example, the von Neumann model shows the importance of bluffing that directly affects the value of the game. Subsequent models that extend it by adding more bets, reveal how these additional bet allowances affect the values and form of optimal strategies. It is found that players raise some of the mediocre hands but simply call with better hands. The reason behind it is that, the opposite player would likely fold the hands worse than the player's, but call with better hands, so raising may not necessarily generate more profit but possibly aggravate loss.

The model with the changing hand suggests how players respond to signals. With a signal favorable to player I, he will have more power to bluff. Essentially the model presented distorts the uniform distribution into a "bi-uniform" one which is a combination of two uniform distributions with different probabilities. The force-in can be eliminated, and instead the player should be given the option to check or raise.

In addition, a tool that is used to clearly show the payoffs of players under different strategies, called the payoff square, is used to calculate the payoffs of players. It may be used to understand why the value of the game equals the initial bluff region, and it can be extended to three-player games as well.

Overall, the models presented help us to understand many small parts of the poker game, though it is far from a perfect understanding.

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