5-LIST-COLORING TOROIDAL 6-REGULAR TRIANGULATIONS IN LINEAR TIME

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Abstract. We give an explicit procedure for 5-list-coloring a large class of toroidal 6-regular triangulations in linear time. We also show that these graphs are not 3-choosable, so the choice number of each of these graphs is either 4 or 5.

1. Introduction

We shall denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$. For each $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For a graph $G = (V, E)$, the notation $v \to w$ shall indicate that the edge $vw \in E$ is oriented from the vertex v to the vertex w. A graph G is *directed* (or, is a *digraph*) if every edge of G is oriented.

A *(vertex) coloring* of a graph $G = (V, E)$ is an assignment of a "color" to each vertex, that is, an assignment $v \mapsto \mathsf{color}(v) \in \mathbb{N}$ for every $v \in V(G)$. A coloring of G is *proper* if adjacent vertices receive distinct colors. G is *k-colorable* if there exists a proper coloring of the vertices using k colors. The least integer k for which G is k-colorable is called the *chromatic number* of G and is denoted $\chi(G)$. If $\chi(G) = k$, we also say that G is *k-chromatic*.

A variation of k-colorability called *k-choosability* was defined independently by Vizing [39] in 1976 and Erdős, Rubin and Taylor [16] in 1979. A *list assignment* $\mathcal{L}$ on G is a collection of sets of the form $\mathcal{L} = \{L_v \subset \mathbb{N} : v \in V(G)\}$, where one thinks of each L_v as a *list* of colors available for coloring the vertex $v \in V(G)$. G is *$\mathcal{L}$-choosable* if there exists a proper coloring of the vertices such that $\mathsf{color}(v) \in L_v$ for every $v \in V(G)$. A *k-list* is a list of size greater than or equal to k, and a *k-list-assignment* is an assignment of k-lists. G is *k-choosable* if it is $\mathcal{L}$-choosable for every k-list-assignment $\mathcal{L}$. The least integer k for which G is k-choosable is called the *choice number* or *list chromatic number* of G and is denoted $\chi_\ell(G)$. If $\chi_\ell(G) = k$, we also say that G is *k-list-chromatic*.

Generally, the computation of $\chi_\ell(G)$ is more difficult than the corresponding problem for $\chi(G)$. For instance, it is known [20] that the problem of deciding whether a given planar graph is 4-choosable is NP-hard, and so is deciding whether a given planar triangle-free graph is 3-choosable. But, contrast the latter with the fact that every planar triangle-free graph is 3-colorable by Grötzsch's theorem [19], and that in fact a 3-coloring can be found in linear time [15].

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In fact, even proving nontrivial bounds for the choice number is far tougher than the corresponding problem for the chromatic number. Some of the notable instances of such bounds being determined include Brooks's theorem for choosability [9, 16, 39], Thomassen's remarkable proof that every planar graph is 5-choosable [36], and Galvin's solution to the famous Dinitz problem [18].

One powerful and frequently used technique for proving bounds on the choice number is the following theorem due to Alon and Tarsi. Recall that a subgraph H of a directed graph G is said to be *Eulerian* if $\mathsf{indegree}_H(v) = \mathsf{outdegree}_H(v)$ for every $v \in V(H)$. An *even* (resp. *odd*) Eulerian graph is one with an even (resp. odd) number of edges.

Theorem 1.1 (Alon–Tarsi [4], 1992)**.** *If the edges of a graph $G = (V, E)$ can be oriented in such a way that the number of even Eulerian subgraphs is different from the number of odd Eulerian subgraphs, then G is $\mathcal{L}$-choosable for every list assignment $\mathcal{L}$ such that $|L_v| \geq \mathsf{outdegree}_G(v) + 1$ for every $v \in V(G)$.*

One easy consequence of Theorem 1.1 is that planar bipartite graphs are 3-choosable [4]. A more spectacular result—originally conjectured by Erdős and proved by Fleischner and Stiebitz [17]—is that any 4-regular graph on $3n$ vertices that can be decomposed into a Hamiltonian cycle and n pairwise vertex-disjoint triangles is 3-choosable.†

Another important perspective on list-coloring comes from an algorithmic viewpoint: given a list assignment $\mathcal{L}$ on a graph G, can we efficiently determine whether or not G is $\mathcal{L}$-choosable, and in the case when G is $\mathcal{L}$-choosable can we also efficiently specify a proper coloring from these lists? The three results mentioned just before the statement of Theorem 1.1 are some of the few instances where such algorithms are known for a large class of graphs. Thomassen has also used similar ideas as in [36] to prove a weaker version of Grötzsch's theorem for choosability [37, 38], which says that any planar graph of girth at least 5 is 3-choosable, and his proof can be algorithmized to give a proper 3-list-coloring efficiently. But, in most other interesting cases, an algorithmic determination of a list-coloring remains open even when the choice number is known.

Indeed, the proof of Theorem 1.1 uses the combinatorial nullstellensatz [3], and as such it does not allow one to extract an efficient algorithm for specifying a proper $\mathcal{L}$-coloring given a list assignment $\mathcal{L}$, except in certain special cases (for instance, see Lemma 2.3 below). In fact, in both the examples mentioned after the statement of Theorem 1.1, there is no known efficient algorithm that produces a 3-list-coloring from a given list assignment, thereby illustrate the difficulty of the problem of efficiently finding a proper $\mathcal{L}$-coloring even for graphs of small maximum degree.

Now, to find upper bounds for the choice number, Brooks's theorem [9, 16, 39] mentioned earlier gives the rudimentary upper bound of $\Delta(G)$, the maximum degree, for any connected graph G that is not an odd cycle or a complete graph. An improvement on this is the following well-known result (for instance, see [2]). The *degeneracy number*, $d(G)$, of a graph G is defined as $d(G) \coloneqq \max_{H \leq G}\{\delta(H)\}$, where the maximum is taken over all induced subgraphs H of G, and $\delta(H)$ is the minimum degree of H. A straightforward inductive argument shows that $\chi_\ell(G) \leq d(G) + 1$ for every simple graph G. A natural choice for a large collection of graphs with bounded degeneracy number is offered by the class of graphs that are embeddable in a fixed surface. For the sake of clarity, we now define these and other associated terms. By a *surface* we mean a compact connected 2-manifold, and a graph is

† In fact, Erdős only conjectured that such graphs are 3-colorable, but Fleischner and Stiebitz used Theorem 1.1 to prove that they are 3-choosable.

embeddable in a surface if, informally speaking, it can be drawn on the surface without any crossing edges. By the classification of surfaces theorem, every orientable surface is homeomorphic to a sphere with $g \geq 0$ handles attached, denoted S_g, and every nonorientable surface is homeomorphic to a sphere with $k \geq 1$ crosscaps attached, denoted N_k. The *genus* of the surface S_g (resp. N_k) is defined to be g (resp. k).

In this paper, we will be concerned with toroidal graphs, that is, graphs that are embeddable on the torus S_1. Let $G = (V, E)$ be a toroidal graph, and let F be the set of its faces in an embedding into S_1. The graphs satisfying $\mathsf{degree}(v) = d$ for all $v \in V$ and $\mathsf{degree}(f) = m$ for all $f \in F$, for some $d, m \geq 1$, have been of interest [5,6] especially in the study of vertex-transitive graphs on the torus [34]. A simple calculation using Euler's formula shows that the only possible values of (d, m) are $(3, 6)$, $(4, 4)$ and $(6, 3)$. Our focus will be on the graphs of the last kind, which are 6-regular triangulations on the torus. The structure of these graphs is completely characterized by a theorem due to Altshuler [6], which we describe shortly.

It is worth mentioning at this point that the problem of determining the choice number of 4-regular toroidal $m \times n$ grids, for $m, n \geq 3$, has been raised by Cai, Wang and Zhu [10]. These graphs are a special case of graphs of the second kind above. It is easy to show by induction that these grids are all 3-colorable, and the above authors conjecture that they are also 3-choosable. Recent work by Li, Shao, Petrov and Gordeev [23] has nearly determined the choice number of these grids as follows: if mn is even, then the choice number is 3, else it is either 3 or 4. However, their proof heavily involves the combinatorial nullstellensatz, so it does not a priori provide an efficient algorithm for list-coloring the toroidal grids.

Another reason for focussing on triangulations is that the problem is ostensibly harder in this case. Existing results on the choosability of large classes of graphs hold only when the girth is "large"; see, for instance, [10, 11, 37]. These methods do not generalize to graphs with girth 3, and, in particular, to triangulations, so this calls for a different perspective on the problem.

Now, Altshuler [6] showed that every 6-regular toroidal triangulation G can be described as a regular triangulation obtained from an $r \times s$ toroidal grid in which the edges between the first and last column are connected by a shift of t vertices. Concretely, for integers $r \geq 1$, $s \geq 1$ and $0 \leq t \leq s-1$, take $V = \{(i, j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ to be the vertex set of the graph $T(r, s, t)$ equipped with the following edges:

- For each $1 < i < r$, (i, j) is adjacent to $(i, j \pm 1)$, $(i \pm 1, j)$ and $(i \pm 1, j \mp 1)$.
- If $r > 1$, $(1, j)$ is adjacent to $(1, j \pm 1)$, $(2, j)$, $(2, j - 1)$, $(r, j + t + 1)$ and $(r, j + t)$.
- If $r > 1$, (r, j) is adjacent to $(r, j \pm 1)$, $(r - 1, j + 1)$, $(r - 1, j)$, $(1, j - t)$ and $(1, j - t - 1)$.
- If $r = 1$, $(1, j)$ is adjacent to $(1, j \pm 1)$, $(1, j \pm t)$ and $(1, j \pm (t + 1))$.

Here, addition in the first coordinate is taken modulo r and in the second coordinate is taken modulo s. Figure 1 depicts the graph $G = T(5, 6, 2)$; note that the edges between the top and bottom rows are not shown.

It is clear that each $T(r, s, t)$ is a 6-regular triangulation of the torus. Altshuler's theorem says that these are all the 6-regular triangulations on the torus up to isomorphism (similar constructions also appear in [25, 34]).

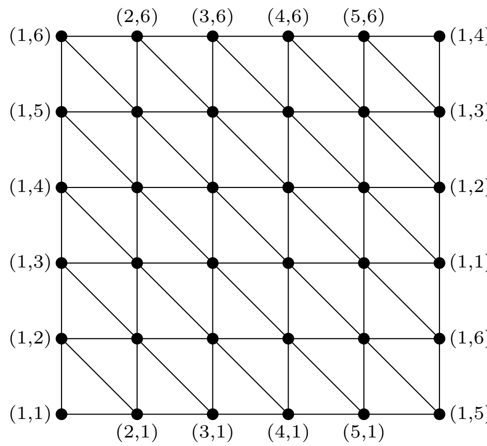


FIGURE 1. $G = T(5, 6, 2)$

Theorem 1.2 (Altshuler [6], 1973). *Every 6-regular triangulation on the torus is isomorphic to $T(r, s, t)$ for some integers $r \geq 1$, $s \geq 1$, and $0 \leq t < s$.*

The focus of this paper is to consider the problem of computing the choice number of the 6-regular toroidal triangulations. In fact, even the ostensibly simpler problem of computing the chromatic number of these graphs has received considerable attention. A classical result due to Heawood [21] shows that every toroidal graph is 7-colorable, and that this is best possible since K_7 is embeddable in the torus. Furthermore, Dirac's map color theorem [14] implies that K_7 is the only 7-chromatic connected toroidal graph. Albertson and Hutchinson [1] showed that there is a unique simple 6-regular toroidal triangulation that is 6-chromatic, which has 11 vertices and which they denote by J. So, every simple 6-regular toroidal triangulation except for J and K_7 is 5-colorable. Thomassen [35] later classified all the 5-colorable toroidal graphs, and posed the question of determing the 5-chromatic 6-regular toroidal triangulations. This was completed subsequently by the combined work of several authors; see [13, 30, 42].

We now state the main result of this paper.

Theorem 1.3. *Let G be a simple 6-regular toroidal triangulation. Then, G is 5-choosable under any of the following conditions:*

(1) G is isomorphic to $T(r, s, t)$ for $r \geq 4$;

(2) G is isomorphic to $T(1, s, 2)$ for $s \geq 9$, $s \neq 11$;

(3) G is isomorphic to $T(2, s, t)$ for s and t both even.

(4) G is 3-chromatic;

Moreover, the 5-list-colorings can be given in linear time.

(5) Furthermore, if G is 3-chromatic, then G is not 3-choosable, except possibly when $G = T(3, 9, 3)$.

Hence, $\chi_\ell(G) \in \{4, 5\}$ if any of the cases (1) to (4) hold for G, except possibly when $G = T(3, 9, 3)$.

The above result contrasts with the one for 4-regular $m \times n$ toroidal grids by Li, Shao, Petrov and Gordeev [23]: both results nearly determine the choice number in the sense that the true value of the choice number is either equal to, or one less than, the computed value. On the other hand, the result in [23] does not provide an algorithm for $\mathcal{L}$-coloring the toroidal grids, whereas our result actually gives a linear time algorithm for $\mathcal{L}$-coloring the toroidal triangulations.

We are currently unable to comment on the choosability of the simple 6-regular triangulations not covered in this theorem, viz. a small, finite set of 5-chromatic graphs as well as an infinite set of 4-chromatic graphs, both of the form $T(1, s, t)$, which follows from the results of Yeh and Zhu [42]. In this sense, Theorem 1.3 covers the 5-choosability of "most" 6-regular toroidal triangulations.

Furthermore, by the results of Yeh and Zhu [42], we know that the 5-chromatic simple 6-regular toroidal triangulations are precisely those isomorphic to $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$, barring the small, finite set of graphs alluded to earlier. So, as a corollary to Theorem 1.3, we obtain an infinite family of 6-regular toroidal triangulations that are 5-*chromatic-choosable*, that is $\chi(G) = \chi_\ell(G) = 5$.

Corollary 1.4. *If G is isomorphic to $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$, $s \geq 9$, $s \neq 11$, then G is 5-chromatic-choosable. Moreover, a 5-list-coloring can be found in linear time.*

Lastly, we mention a recent work by Postle and Thomas [27], wherein it is proved that for any surface Σ and every $k \in \{3, 4, 5\}$ there exists a linear time algorithm for determining whether or not an input graph G embedded in Σ and having girth (that is, the length of a shortest cycle) at least $8 - k$ is k-choosable. In particular, when $\Sigma = S_1$ and $k = 5$, this implies that there is a linear time algorithm for determining whether or not any of the 6-regular triangulations under consideration are 5-choosable. However, this algorithm is contingent upon an enumeration of the 6-list-critical graphs on the torus.[‡] Indeed, Postle and Thomas show that there are only finitely many 6-list-critical graphs on the torus, but a full list of these graphs is not explicitly known. Also, while their linear time algorithm can output whether or not a given simple toroidal graph is $\mathcal{L}$-choosable for a 5-list-assignment $\mathcal{L}$, it does not specify a proper $\mathcal{L}$-coloring in the case when the graph is $\mathcal{L}$-choosable. This is in contrast with the results in this paper, wherein the 5-choosable graphs identified in Theorem 1.3 can also be given proper 5-list-colorings in linear time. Furthermore, the non-3-choosability of the 3-chromatic graphs $T(r, s, t)$ is not covered by their results since these graphs have girth equal to 3, whereas the algorithm in [27] for 3-list-coloring is applicable only for graphs having girth at least 5.

Structure of this paper. In Section 2, we setup the necessary preliminary material; in particular, Lemmas 2.4 and 2.5 will be used at several points in the rest of the paper. In Section 3, we prove a succession of technical lemmas to prepare the proof of Theorem 1.3 in case (1). The proof of this case is completed in Section 4. In Section 5, we prove cases (2) and (3) of Theorem 1.3. In Section 6, we prove cases (4) and (5) of Theorem 1.3. In Section 7, we analyze the remaining cases not covered by Theorem 1.3 and conclude with some conjectures concerning their choosability.

[‡] A graph is *k-list-critical* if it is not $(k-1)$-choosable, but every proper subgraph is $(k-1)$-choosable.

2. Preliminaries

As shown by Altshuler [5, 6], through every vertex v of $T(r,s,t)$ there are three *normal circuits*, which are the simple cycles obtained by traversing through v along each of the three directions (vertical, horizontal, and diagonal) in the natural fashion. These normal circuits have lengths s, $n/\gcd(s,t)$, and $n/\gcd(s,r+t)$, respectively, where $n = rs$ is the order of $T(r,s,t)$.

By picking a different normal circuit to be represented as the vertical cycle, one can see that there exist $0 \le t_1 < n/\gcd(s,t)$ and $0 \le t_2 < n/\gcd(s,r+t)$ such that $T(r,s,t)$ is isomorphic to $T\big(\gcd(s,t), n/\gcd(s,t), t_1\big)$ as well as to $T\big(\gcd(s,r+t), n/\gcd(s,r+t), t_2\big)$. Similarly, by swapping the horizontal and diagonal normal circuits, one can see that $T(r,s,t)$ is isomorphic to $T(r,s,t')$ for $0 \le t' < s$ such that $t' \equiv -r-t \pmod{s}$.

We shall use the notation C_i, for $1 \le i \le r$, to denote the induced subgraph of $T(r,s,t)$ on the ith column of $T(r,s,t)$, that is, on the set of vertices $\{(i,j) : 1 \le j \le s\}$. Note that each C_i is a cycle of length s.

The following lemma is useful in simplifying arguments through the use of symmetry:

Lemma 2.1. *Let $r \ge 1$, $s \ge 1$ and $0 \le t \le s-1$. The map $(i,j) \mapsto (r-i+1, s-j+1)$ on $V\big(T(r,s,t)\big)$ induces an automorphism of $T(r,s,t)$.*

In particular, this automorphism reverses the ordering of the rows (as well as of the columns).

For integers $r, s \ge 3$, define the *cylindrical triangulation* $C(r,s)$ to be the graph obtained from $T(r+1,s,0)$ by deleting the column C_{r+1}. More formally, let $V\big(C(r,s)\big) := \{(i,j) : 1 \le i \le r, 1 \le j \le s\}$ and let $E\big(C(r,s)\big)$ contain the following edges:

- For $1 < i < r$, let (i,j) be adjacent to $(i, j \pm 1)$, $(i \pm 1, j)$ and $(i \pm 1, j \mp 1)$.
- Let $(1,j)$ be adjacent to $(1, j \pm 1)$, $(2,j)$ and $(2, j-1)$.
- Let (r,j) be adjacent to $(r, j \pm 1)$, $(r-1, j+1)$ and $(r-1, j)$.

Again, addition in the second coordinate is taken modulo s. Note that every *interior vertex* of $C(r,s)$, that is, any vertex (i,j) with $1 < j < r$, has degree 6 and every *exterior vertex* of $C(r,s)$, that is, any vertex (i,j) with $j = 1$ or $j = r$, has degree 4.

By an abuse of notation, we shall use C_i to denote the induced subgraph on the ith column of $C(r,s)$, too. Note that if we delete any column of the graph $T(r+1,s,t)$ for any $0 \le t \le s-1$, we still get a graph isomorphic to $C(r,s)$.

We will need the following theorem on finding matchings in regular bipartite graphs.

Definition. A *matching* in a graph $G = (V,E)$ is a subset M of E such that no two edges in M have a common vertex. A matching is said to be *perfect* if every vertex $v \in V$ belongs to some edge in the matching.

Theorem 2.2 (Cole–Ost–Schirra [12], 2001)**.** *Let $G = (V,E)$ be a regular bipartite graph. Then, a perfect matching M of G can be found in $O(|E|)$ time.*

We make the following definition which will simplify some of the terminology in the proofs that follow.

Definition. Let $\mathcal{L}$ be a list assignment on a graph $G = (V, E)$. If, in a partial $\mathcal{L}$-coloring of G, a vertex $v \in V$ is colored with $c \in L_v$, then the color c is no longer available for use on the uncolored neighbors of v. So, the color c is removed from the lists of the neighbors of v, and we do this for each vertex colored in this partial $\mathcal{L}$-coloring of G. The new lists on G are also called the *residual* lists on G, and we shall say that the list on an uncolored vertex u *reduces by* k if the residual list on u is a $(|L_u| - k)$-list.

The following well-known lemma, due to Bondy–Bopanna–Siegel [4, Remark 2.4], gives an algorithmic proof of Theorem 1.1 in the case when G is given an orientation containing no odd directed cycle. For the sake of completeness, we provide a proof of this lemma along the lines in [41].

Definition. Let G be a digraph, and $v \to w$ be an edge of G. We also call w the *successor* of v, and v the *predecessor* of w. A *kernel* of G is an independent set S such that every $v \notin S$ has a successor in S.

Lemma 2.3 (Bondy–Bopanna–Siegel [4, Remark 2.4], 1992)**.** *Let $G = (V, E)$ be a simple graph that is given an orientation containing no odd directed cycle. Suppose $\mathcal{L}$ is a list assignment on G such that L_v is an $(\mathsf{outdegree}(v) + 1)$-list for all $v \in V$. Then, G is $\mathcal{L}$-choosable.*

Proof. If $|V| = 1$, then the statement is trivial, so suppose that $|V| = n > 1$, and that the lemma is true for all graphs with fewer than n vertices. Let c be a color that occurs in some list assigned by $\mathcal{L}$. Consider the induced subgraph H on the set $U \coloneqq \{v \in V : c \in L_v\}$. Clearly, the induced orientation on H also does not have any odd directed cycle. Now, Richardson's theorem [28] says that any digraph without odd directed cycles has a kernel, so let S be a kernel in H. Assign the color c to every vertex in S, and now consider $G' \coloneqq G - S$. Notice that the residual lists have reduced in size by 1 for every list on $U - S$, but every vertex in $U - S$ also has a successor in S. Thus, G' satisfies the induction hypothesis, and we are done. □

One can find in the literature [26, 33] proofs of Richardson's theorem that output a kernel in polynomial time. However, the graphs that we consider (cf. Lemmas 6.1 and 2.4) have enough structure that they permit straightforward linear time algorithms for finding a kernel.

We will often need to color paths and cycles in $T(r, s, t)$ and $C(r, s)$, so we compile well-known results (see [16], for instance) on the colorability of these graphs in the following lemma. Moreover, from Lemma 2.3 and the above comments, we can give linear time algorithms for $\mathcal{L}$-coloring these graphs.

Lemma 2.4.

(1) *An even cycle is 2-list-chromatic.*

(2) *An odd cycle is not 2-colorable, and hence not 2-choosable. However, if $\mathcal{L}$ is a list assignment of 2-lists on an odd cycle such that not all the lists are identical, then the cycle is $\mathcal{L}$-choosable.*

(3) *If $\mathcal{L}$ is a list assignment on an odd cycle having one* 1*-list, one* 3*-list, and all the rest as* 2*-lists, then the cycle is $\mathcal{L}$-choosable.*

(4) *If $\mathcal{L}$ is a list assignment on a path graph having one* 1*-list, and all the rest as* 2*-lists, then the path is $\mathcal{L}$-choosable.*

Moreover, the $\mathcal{L}$-colorings can all be found in linear time.

The following lemma, due to S. Sinha (during an undergraduate research internship with the first author), is in a similar spirit to Thomassen's list-coloring of a near-triangulation of the plane [36], and it will be repeatedly invoked in the proof of case (1) in Theorem 1.3.

Lemma 2.5 (Sinha [31], 2014)**.** *For $r \geq 3$, $s \geq 3$, let $G = C(r, s)$ be a cylindrical triangulation. Suppose that $\mathcal{L}$ is a list assignment on G such that:*

(1) *there exists $1 \leq j \leq s$ such that the exterior vertices $(1, j)$ and $(1, j-1)$ have lists of size equal to* 4*;*

(2) *every other exterior vertex has a list of size equal to* 3*;*

(3) *every interior vertex has a list of size equal to* 5*.*

Then, G is $\mathcal{L}$-choosable. Moreover, an $\mathcal{L}$-coloring can be found in linear time.

Proof. By Lemma 2.4, there is a proper coloring of C_r since it is assigned 3-lists under $\mathcal{L}$. Since every vertex of C_{r-1} is adjacent to exactly two vertices of C_r, a proper coloring of C_r reduces the 5-lists on C_{r-1} to 3-lists.

Thus, by inductively coloring the columns of $C(r, s)$ from the right, we may assume without loss of generality that $r = 3$. We also assume without loss of generality that the lists of size equal to 4 are on the vertices $(1, s-1)$ and $(1, s)$ in the column C_1. Now, color $(2, s)$ with $c \in L_{(2,s)} \setminus L_{(1,s)}$, which exists since C_2 has 5-lists. This reduces the sizes of the lists on each of the neighbors of $(2, s)$ by 1, except for $L_{(1,s)}$, which still has size equal to 4. Now, C_3 has 3-lists on every vertex, except for $(3, s)$ and $(3, s-1)$, which have 2-lists. So, properly color C_3 using Lemma 2.4. Then, color the remaining vertices in a zigzag fashion from the bottom row, coloring $(1, s)$ last, in the following order: $(1, 1)$, $(2, 1)$, $(1, 2)$, $(2, 2)$, $\ldots$, $(1, j)$, $(2, j)$, $\ldots$, $(1, s-2)$, $(2, s-2)$, $(2, s-1)$, $(1, s-1)$, $(1, s)$.

A proper coloring can always be found by coloring the vertices in the above sequence for the following reason. After coloring C_3, the list sizes on the remaining vertices are as follows: $(1, s)$ and $(1, s-1)$ have 4-lists, $(1, 1)$, $(2, 1)$ and $(2, s-1)$ have 2-lists, and all other vertices have 3-lists. So, color the vertex $(1, 1)$ using a color from its list, and the list sizes then are as follows: $(1, s-1)$ has a 4-list, $(1, 2)$ and $(2, s-1)$ have 2-lists, $(2, 1)$ has a 1-list, and all other vertices have 3-lists. Next, color $(2, 1)$ using a color from its list, and observe that the next vertex that is to be colored in the sequence always has at least one color left in its list. The last three vertices left to be colored are in a 3-cycle, with lists of sizes at least 1, 2 and 3. This cycle is properly colorable by Lemma 2.4, so this completes the proof.

It is clear from the proof that this algorithm produces an $\mathcal{L}$-coloring in linear time. Figure 2 illustrates the sizes of the lists at each step of the above coloring sequence for the graph $G = C(3, 5)$. $\square$

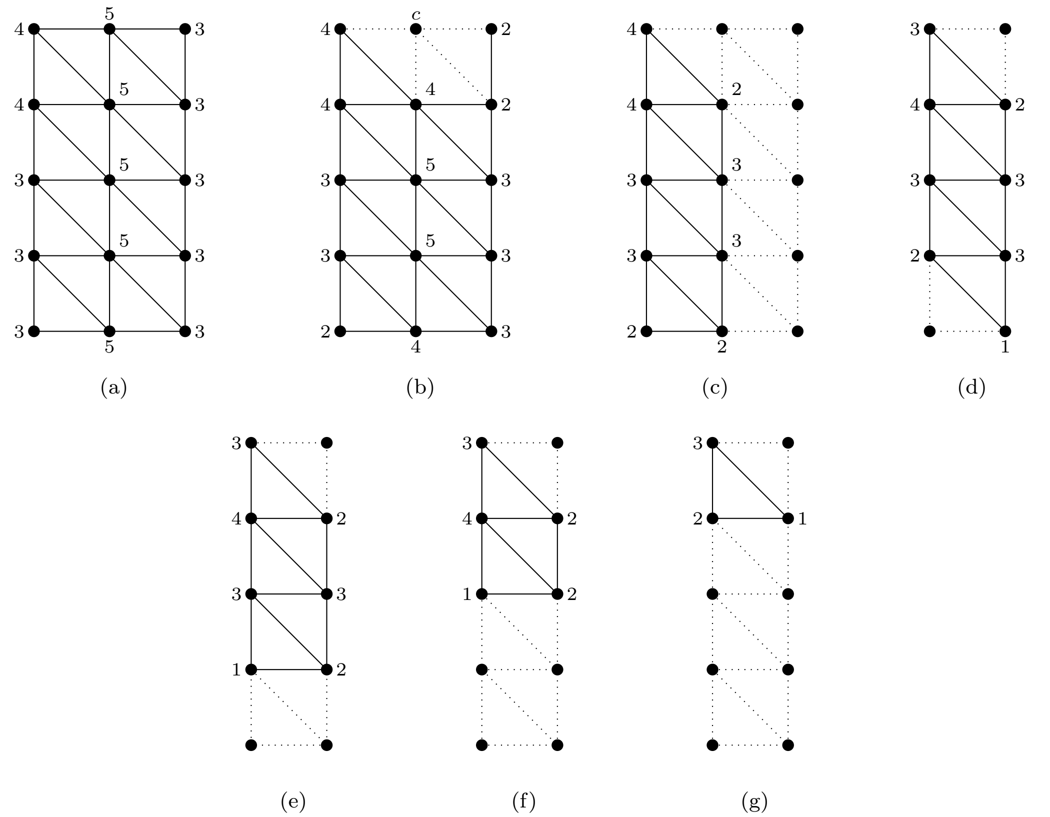


FIGURE 2. Illustration of the sizes of the lists on the vertices at each step for $G = C(3, 5)$

3. Preparation for the proof of case (1) in Theorem 1.3

For $r \geq 4$, $s \geq 3$ and $0 \leq t \leq s-1$, let $G \coloneqq T(r, s, t)$. Fix $\mathcal{L}$ to be a list assignment on G of lists of size equal to 5. We start be eliminating the trivial case: if all the lists of $\mathcal{L}$ are identical, then G is $\mathcal{L}$-choosable because G is 5-colorable [35]. Moreover, a 5-coloring can be found in linear time: see [13, 30, 42].

For a vertex $(i, j) \in C_i$, let its *left neighbors* be the two adjacent vertices in C_{i-1}, its *right neighbors* be the two adjacent vertices in C_{i+1}, and its *vertical neighbors* be the two adjacent vertices in C_i. We shall repeatedly invoke Lemma 2.5 to cut down on the possible choices for the lists assigned by $\mathcal{L}$, until it becomes simple enough to directly specify a proper coloring.

Lemma 3.1. *Suppose that not all the lists in $\mathcal{L}$ are identical. If there is a vertex $v \in V(G)$ such that its list is not contained in the union of the lists of its two left neighbors, then G is $\mathcal{L}$-choosable in linear time.*

Proof. Choose a color for v that is not in the list of either left neighbor of v, and extend the coloring to the cycle C_i containing v by Lemma 2.4. Then, we are left to color a graph isomorphic to $C(r-1,s)$ equipped with lists whose sizes satisfy the hypotheses of Lemma 2.5. Hence, the coloring on C_i extends to a proper coloring of G in linear time by Lemma 2.5, and so we are done. □

Note that by Lemma 2.1 the above lemma is also true when "left neighbors" is replaced by "right neighbors" in the statement. Thus, it suffices to assume that the list assignment $\mathcal{L}$ satisfies the following criterion:

(i) Not all the lists in $\mathcal{L}$ are identical, and for every vertex $(i,j) \in V(G)$, $L_{(i,j)} \subseteq L_{(i-1,j)} \cup L_{(i-1,j+1)}$ and $L_{(i,j)} \subseteq L_{(i+1,j)} \cup L_{(i+1,j-1)}$.

In particular, we may assume that no column has identical lists, for if C_i has identical lists, then so do C_{i-1} and C_{i+1} by criterion (i), so all the lists in $\mathcal{L}$ are identical by induction, a contradiction.

Lemma 3.2. *Suppose that $\mathcal{L}$ satisfies criterion (i).*

(1) Let $(i,j),(i,j-1) \in V(G)$ have distinct lists. Suppose one of the following conditions holds:

(a) $L_{(i,j)} \neq L_{(i-1,j+1)}$ and $L_{(i,j-1)} \neq L_{(i-1,j-1)}$;

(b) $L_{(i,j)} \neq L_{(i-1,j+1)}$ and $L_{(i,j-1)} \neq L_{(i-1,j)}$;

(c) $L_{(i,j)} \neq L_{(i-1,j)}$ and $L_{(i,j-1)} \neq L_{(i-1,j-1)}$.

Then, G is $\mathcal{L}$-choosable in linear time.

(2) Suppose $u,v \in V(G)$ are adjacent vertices lying on distinct columns such that $L_u = L_v$. If for every vertex $w \in V(G)$ that is adjacent to both u and v we have $L_w \neq L_u$, then G is $\mathcal{L}$-choosable in linear time.

(3) Let $(i,j) \in V(G)$ be a vertex such that both its left neighbors have lists identical to $L_{(i,j)}$. Suppose that $L_{(i,j)} \neq L_{(i,j+1)}$ and $L_{(i,j)} \neq L_{(i,j-1)}$. Then, G is $\mathcal{L}$-choosable in linear time.

Proof.

(1) (a) Choose a color for $(i-1,j+1)$ from $L_{(i-1,j+1)} \setminus L_{(i,j)}$, for $(i-1,j-1)$ from $L_{(i-1,j-1)} \setminus L_{(i,j-1)}$, and extend this to a proper coloring of C_{i-1} by Lemma 2.4. Then, we are in the scenario of Lemma 2.5, and so we are done.

(b) Choose a color c for $(i-1,j)$ from $L_{(i-1,j)} \setminus L_{(i,j-1)}$. By criterion (i), $c \in L_{(i,j)}$, so there exists a color $d\ (\neq c) \in L_{(i-1,j+1)} \setminus L_{(i,j)}$. Color $(i-1,j+1)$ with d and extend this to a proper coloring of C_{i-1} by Lemma 2.4. Then, we are in the scenario of Lemma 2.5, and so we are done.

(c) This is similar to the proof of Lemma 3.2(1b) above. Choose a color c for $(i-1,j)$ from $L_{(i-1,j)} \setminus L_{(i,j)}$. By criterion (i), $c \in L_{(i,j-1)}$, so there exists a color $d\ (\neq c) \in L_{(i-1,j-1)} \setminus L_{(i,j-1)}$. Color $(i-1,j-1)$ with d and extend this to a proper coloring of C_{i-1} by Lemma 2.4. Then, we are in the scenario of Lemma 2.5, and so we are done.

(2) Suppose $u=(i,j)$ and $v=(i-1,j)$, Then, neither $(i-1,j+1)$ nor $(i,j-1)$ has a list identical to L_u. But then we are in the scenario of Lemma 3.2(1b), so G is $\mathcal{L}$-choosable in linear time. Similarly, let $u=(i,j-1)$ and $w=(i-1,j)$, Then, neither $(i-1,j-1)$ nor

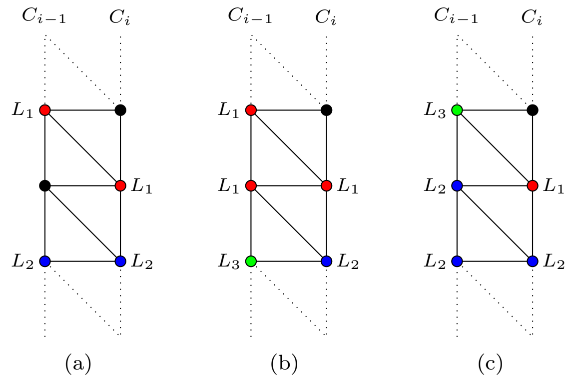


FIGURE 3. Illustrations of configurations (a) to (c) in criterion (ii)

(i,j) has a list identical to L_u. But then we are in the scenario of Lemma 3.2(1c), so again G is $\mathcal{L}$-choosable in linear time.

(3) Choose a color for $(i,j+1)$ from $L_{(i,j+1)} \setminus L_{(i-1,j+1)}$, for $(i,j-1)$ from $L_{(i,j-1)} \setminus L_{(i-1,j)}$, and extend this to a proper coloring of C_i by Lemma 2.4. Then, we are in the scenario of Lemma 2.5, so we are done.

□

Note that, by Lemma 2.1, Lemma 3.2(1) is also true when the list assignment $\mathcal{L}$ instead satisfies one of three analogous conditions relating the lists on (i,j) and $(i,j-1)$ with their right neighbors, and Lemma 3.2(3) is also true when "left neighbors" is replaced by "right neighbors" in the statement.

Thus, in addition to criterion (i), we may also assume the following criteria:

(ii) Whenever (i,j) and $(i,j-1)$ have distinct lists assigned by $\mathcal{L}$, one of the following three configurations holds:

(a) $L_{(i,j)} = L_{(i-1,j+1)}$ and $L_{(i,j-1)} = L_{(i-1,j-1)}$;

(b) $L_{(i,j)} = L_{(i-1,j+1)} = L_{(i-1,j)}$ and $L_{(i,j-1)} \neq L_{(i-1,j-1)}$;

(c) $L_{(i,j)} \neq L_{(i-1,j+1)}$ and $L_{(i,j-1)} = L_{(i-1,j)} = L_{(i-1,j-1)}$.

(iii) whenever u and v are adjacent vertices on distinct columns with $L_u = L_v$, there is a vertex w adjacent to both u and v such that $L_w = L_u = L_v$.

(iv) whenever u, v and w are mutually adjacent vertices having identical lists, with v and w lying on the same column, at least one of the vertical neighbors of u has a list identical to L_u.

The configurations (a) to (c) in criterion (ii) are illustrated in Figure 3. By Lemma 2.1, we also assume one of three analogous configurations holds for the lists on the right neighbors of (i,j) and

$(i, j-1)$ under the hypothesis of criterion (ii), but for the sake of brevity we avoid listing them explicitly.

We now make the following definitions. For a list L, define the *list-class* of L in G, denoted $G[L]$, to be induced subgraph of G on those vertices v such that $L_v = L$. Let $L \in \mathcal{L}$ and let H be a (maximal connected) component of $G[L]$. If $V(H)$ is a singleton, we call H an *isolated component*, else we call H a *nonisolated component*.

Lemma 3.3. *Suppose that $\mathcal{L}$ satisfies criteria (i) to (iv).*

(1) Let H be an isolated component of a list-class $G[L]$, with $V(H) = \{(i, j)\}$. Then, there are distinct lists $L', L'' \in \mathcal{L}$ such that $L_{(i-1,j+1)} = L_{(i,j+1)} = L_{(i+1,j+1)} = L_{(i+1,j)} = L'$ and $L_{(i-1,j)} = L_{(i-1,j-1)} = L_{(i,j-1)} = L_{(i+1,j-1)} = L''$.

(2) Let H be a nonisolated component of a list-class $G[L]$, with $v \in V(H)$. Then, at least one vertical neighbor of v also belongs to $V(H)$.

Proof.

(1) Since (i, j) belongs to an isolated component of $G[L]$, the lists on its vertical neighbors are distinct from L. Let $L_{(i,j+1)} = L'$ and $L_{(i,j-1)} = L''$. By applying criterion (ii) on the vertices (i, j) and $(i, j-1)$ with respect to their left neighbors, we see that only configuration (b) can hold, else (i, j) will not belong to an isolated component. Thus, $L_{(i-1,j)} = L_{(i-1,j-1)} = L''$ and $L_{(i,j+1)} \neq L$.

Now, if $L_{(i-1,j)} \neq L'$, then we can apply criterion (ii) on the vertices $(i-1, j+1)$ and $(i-1, j)$ with respect to their right neighbors, and we see that none of the analogues of configurations (a) to (c) hold, a contradiction. Hence, $L_{(i-1,j)} = L'$.

Next, by Lemma 2.1, we also get $L_{(i+1,j+1)} = L_{(i+1,j)} = L'$ and $L_{(i,j-1)} = L''$.

Lastly, if $L' = L''$, then we will also have $L = L'$ by criterion (i), a contradiction.

(2) Since v is assumed to belong to a nonisolated component H of some list-class $G[L]$, let $u \in V(H)$ with u adjacent to v. If u lies in the same column as v, then we are done, so assume that u and v lie in distinct columns. Then, by criterion (iii), there is a vertex w adjacent to both u and v such that $w \in V(H)$. If w lies in the same column as v, then we are done, so assume that v and w lie in distinct columns. Then, u and w lie on the same column, so by criterion (iv) at least one of the vertical neighbors of v also belongs to $V(H)$.

□

The configuration in Lemma 3.3(1) is illustrated in Figure 4. Note that Lemma 3.3(2) implies that for every $v \in V(H)$, where H is a nonisolated component of some list-class $G[L]$, at least one left neighbor and one right neighbor of v also belongs to $V(H)$, by criterion (i). Hence, Lemma 3.3(2) can be applied inductively on vertices across columns, starting from any $v \in V(H)$. Thus, if (i, j) and $(i, j-1)$ are adjacent vertices in the column C_i with distinct lists, then using Lemma 3.3, we can pin down the possible list configurations on the nearby vertices in the columns C_{i+1} and C_{i+2} to a manageable number, as follows.

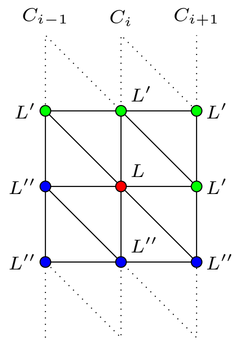


FIGURE 4. The configuration of an isolated component in Lemma 3.3(1)

Lemma 3.4. *Suppose that $\mathcal{L}$ satisfies criteria (i) to (iv). Let $(i, j+1), (i, j) \in V(G)$ have distinct lists L_1, L_2, respectively, and suppose that neither vertex belongs to an isolated component. Then, one of the following configurations holds:*

(I) The vertices (i, k), $(i+1, k)$ and $(i+2, k)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(II) The vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(III) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(IV) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-2)$ have lists identical to L_1 for $k = j+2, j+1$, and have lists identical to L_2 for $k = j, j-1$.

(V) The vertices (i, k), $(i+1, k)$ and $(i+2, k)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_2 for $k = j, j-1$, and the vertex $(i+2, j)$ belongs to an isolated component of some list-class $G[L_3]$, where $L_3 \neq L_1$ and $L_3 \neq L_2$.

(VI) The vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k-1)$ and $(i+2, k-2)$ have lists identical to L_2 for $k = j, j-1$, and the vertex $(i+2, j-1)$ belongs to an isolated component of some list-class $G[L_3]$, where $L_3 \neq L_1$ and $L_3 \neq L_2$.

(VII) The vertices (i, k), $(i+1, k)$ and $(i+2, k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i, k), $(i+1, k-1)$ and $(i+2, k-1)$ have lists identical to L_2 for $k = j, j-1$, and the vertex $(i+1, j)$ belongs to an isolated component of some list-class $G[L_3]$, where $L_3 \neq L_1$ and $L_3 \neq L_2$.

Proof. We start with $L_{(i,j+1)} = L_1$ and $L_{(i,j)} = L_2$. By Lemma 3.3(2), this implies that $L_{(i,j+2)} = L_1$ and $L_{(i,j-1)} = L_2$. By criterion (i), $L_{(i+1,j+1)} = L_1$ and $L_{(i+1,j-1)} = L_2$. Now, again by Lemma 3.3(2), we have three cases:

(1) $L_{(i+1,j+2)} = L_1$ and $L_{(i+1,j)} = L_2$;

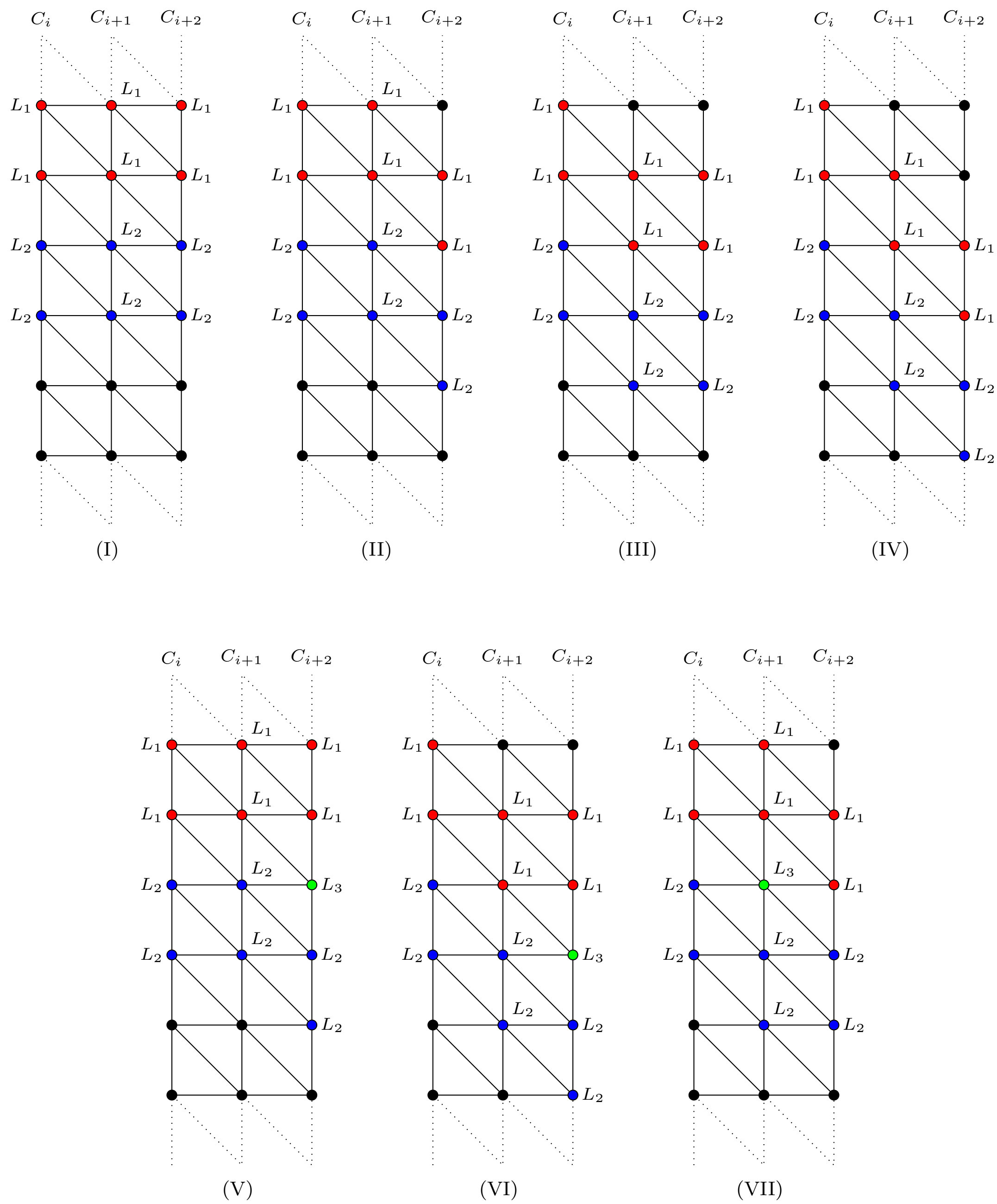


FIGURE 5. Illustration of the configurations of Lemma 3.4

(2) $L_{(i+1,j)} = L_1$ and $L_{(i+1,j-2)} = L_2$;

(3) $L_{(i+1,j+2)} = L_1$, $L_{(i+1,j-2)} = L_2$ and $L_{(i+1,j)} = L_3$ where $L_1 \neq L_3$ and $L_2 \neq L_3$. In particular, by Lemma 3.3(2), $(i+1,j)$ must belong to an isolated component of the list-class $G[L_3]$.

We consider each of these cases in turn.

First, suppose case (1) holds. Then, by criterion (i), $L_{(i+2,j+1)} = L_1$ and $L_{(i+2,j-1)} = L_2$. Then, again by Lemma 3.3(2), we have three cases:

- $L_{(i+2,j+2)} = L_1$ and $L_{(i+2,j)} = L_2$. This is configuration (I).
- $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. This is configuration (II).
- $L_{(i+2,j+2)} = L_1$, $L_{(i+2,j-2)} = L_2$ and $L_{(i+2,j)} = L_3$ where $L_1 \neq L_3$ and $L_2 \neq L_3$. In particular, by Lemma 3.3(2), $(i+2,j)$ must belong to an isolated component of the list-class $G[L_3]$. This is configuration (V).

Next, suppose case (2) holds. Then, by criterion (i), $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. Again by Lemma 3.3(2), we have three cases:

- $L_{(i+2,j+1)} = L_1$ and $L_{(i+2,j-1)} = L_2$. This is configuration (III).
- $L_{(i+2,j-1)} = L_1$ and $L_{(i+2,j-3)} = L_2$. This is configuration (IV).
- $L_{(i+2,j+1)} = L_1$, $L_{(i+2,j-3)} = L_2$ and $L_{(i+2,j-1)} = L_3$ where $L_1 \neq L_3$ and $L_2 \neq L_3$. In particular, by Lemma 3.3(2), $(i+2,j-1)$ must belong to an isolated component of the list-class $G[L_3]$. This is configuration (VI).

Lastly, suppose case (3) holds. Then, by Lemma 3.3(1), $L_{(i+2,j+1)} = L_{(i+2,j)} = L_1$ and $L_{(i+2,j-1)} = L_2$. By Lemma 3.3(2), we also have $L_{(i+2,j-2)} = L_2$. This is configuration (VII). □

Lemma 3.5. *Suppose that $\mathcal{L}$ satisfies criteria (i) to (iv). Let $(i,j+1)$, (i,j) and $(i,j-1)$ have mutually distinct lists L_1, L_3 and L_2, respectively, and suppose that (i,j) corresponds to an isolated component in $G[L_3]$. Then, one of the following configurations holds:*

(VIII) The vertices (i,k), $(i+1,k-1)$ and $(i+2,k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i,k), $(i+1,k)$ and $(i+2,k)$ have lists identical to L_2 for $k = j-1, j-2$.

(IX) The vertices (i,k), $(i+1,k-1)$ and $(i+2,k-2)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i,k), $(i+1,k)$ and $(i+2,k-1)$ have lists identical to L_2 for $k = j, j-1$.

(X) The vertices (i,k), $(i+1,k-1)$ and $(i+2,k-1)$ have lists identical to L_1 for $k = j+2, j+1$, the vertices (i,k), $(i+1,k)$ and $(i+2,k-1)$ have lists identical to L_2 for $k = j-1, j-2$, and the vertex $(i+2,j-1)$ belongs to an isolated component of some list-class $G[L_4]$, where $L_1 \neq L_4$ and $L_2 \neq L_4$, but L_4 may be identical to L_3.

Proof. We start with $L_{(i,j+1)} = L_1$, $L_{(i,j)} = L_3$ and $L_{(i,j-1)} = L_2$, with (i,j) belonging to an isolated component of the list-class $G[L_3]$. By Lemma 3.3(2), we have $L_{(i,j+2)} = L_1$ and $L_{(i,j-2)} = L_2$. By Lemma 3.3(1), we have $L_{(i+1,j+1)} = L_{(i+1,j)} = L_1$ and $L_{(i+1,j-1)} = L_2$. By Lemma 3.3(2), we also have $L_{(i+1,j-2)} = L_2$. By criterion (i), this implies that $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. Now, again by Lemma 3.3(2), we have the following three cases:

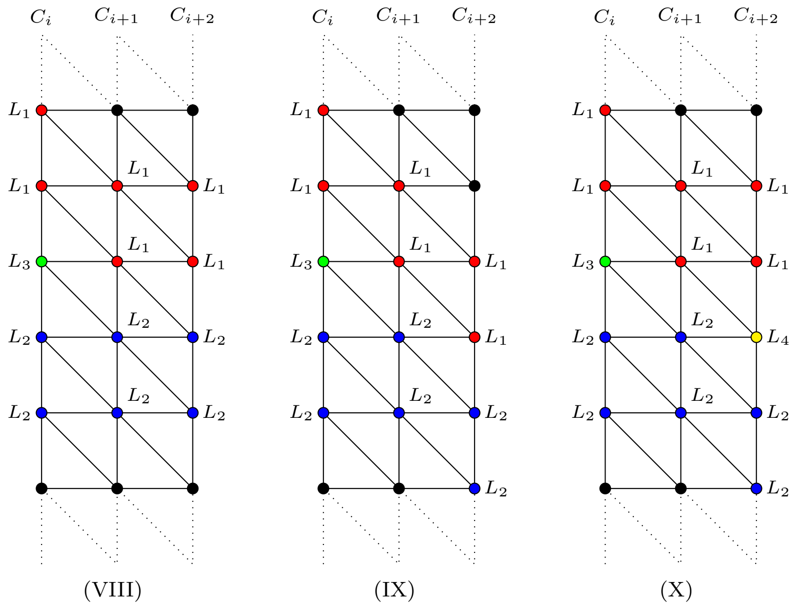


FIGURE 6. Illustration of the configurations of Lemma 3.5

- $L_{(i+2,j)} = L_1$ and $L_{(i+2,j-2)} = L_2$. This is configuration (VIII).
- $L_{(i+2,j-1)} = L_1$ and $L_{(i+2,j-3)} = L_2$. This is configuration (IX).
- $L_{(i+2,j+1)} = L_1$, $L_{(i+2,j-3)} = L_2$ and $L_{(i+2,j-1)} = L_4$ where $L_1 \neq L_4$ and $L_2 \neq L_4$. In particular, by Lemma 3.3(2), $(i+2, j-1)$ must belong to an isolated component of the list-class $G[L_4]$. This is configuration (X). Note that L_4 may be identical to L_3.

□

These configurations are listed in Figures 5 and 6.

We are now in a position to complete the proof of case (1) in Theorem 1.3.

4. Proof of case (1) in Theorem 1.3

By the results in Section 3, it suffices to assume that the list assignment $\mathcal{L}$ on G satisfies criteria (i) to (iv), and that, in particular, Lemma 3.5 holds. Suppose $(i, j+1)$, (i, j) and $(i, j-1)$ are three vertices in the column C_i that satisfy the hypotheses of Lemma 3.5. Then, the vertices $(i+1, j)$ and $(i+1, j-1)$ in the column C_{i+1}, as well as the vertices $(i-1, j+1)$ and $(i-1, j)$ in the column C_{i-1}, satisfy the hypotheses of Lemma 3.4. Thus, there always exists a column that has a pair of adjacent vertices that satisfies the hypotheses of Lemma 3.4, which we shall now take to be C_1 without loss of generality. Furthermore, without loss of generality, let $(1, s)$ and $(1, s-1)$ satisfy the hypotheses of Lemma 3.4.

Now, the first step of our algorithm to find an $\mathcal{L}$-coloring—which we elaborate on below—is to properly color C_1. Then, the lists on C_r all reduce to 3-lists, so C_r can be properly colored by

Lemma 2.5. This in turn causes the lists on C_{r-1} to reduce to 3-lists. Thus, we can inductively color the columns from the right using Lemma 2.5 until we are only left to color the columns C_2, C_3 and C_4. Thus, it suffices to assume without loss of generality that $r = 4$.

Fix $1 \le j \le s$. We start with a few straightforward observations:

(A) If $L_{(1,j)} = L_{(2,j)} = L_{(2,j-1)}$, then any choice of color for $(1, j)$ will reduce the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ by 1 each. Similarly, if $L_{(1,j)} = L_{(1,j-1)} = L_{(2,j-1)}$, then any proper coloring of $(1, j)$ and $(1, j-1)$ will reduce the size of $L_{(2,j-1)}$ by 2.

(B) Consider the vertices $(1, j)$, $(2, j)$, $(2, j-1)$ and $(3, j-1)$. Suppose that a color $c \in L_{(1,j)}$ has been chosen for $(1, j)$, so the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ have potentially reduced by 1 each. Now, if $c \in L_{(3,j-1)}$ too, then coloring $(3, j-1)$ with the color c does not reduce the sizes of the residual lists on $(2, j)$ and $(2, j-1)$ any further.

(C) Suppose that $L_{(1,j)} \cap L_{(3,j-1)} = \emptyset$. If $(1, j)$ is an isolated vertex, then $(1, j+1)$, $(1, j)$, and $(1, j-1)$ satisfy the hypotheses of Lemma 3.5, and moreover these vertices must be in configuration (X). If $(1, j)$ is not an isolated vertex, then either $(1, j)$ and $(1, j-1)$ satisfy the hypotheses of Lemma 3.4, or $(1, j+1)$ and $(1, j)$ satisfy the hypotheses of Lemma 3.4, or $L_{(1,j+1)} = L_{(1,j)} = L_{(1,j-1)}$. If the first case holds, then $(1, j)$ and $(1, j-1)$ must be in configuration (I); if the second case holds, then $(1, j+1)$ and $(1, j)$ must be in configuration (IV); the third case is impossible, since by repeated application of criterion (i) we must have $L_{(1,j)} = L_{(3,j-1)}$.

Furthermore, in the case when $(1, j)$ is an isolated vertex, choosing a color for $(1, j)$ from $L_{(1,j)} \setminus L_{(2,j)}$ and for $(3, j-1)$ from $L_{(3,j-1)} \setminus L_{(2,j-1)}$ will reduce the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ by 1 each. Note that such choices are possible by criterion (i) and Lemma 3.3(1). In the other cases, any choice of color for $(1, j)$ and for $(3, j-1)$ will reduce the sizes of $L_{(2,j)}$ and $L_{(2,j-1)}$ only by 1 each.

These observations are crucial for step 1 of the following two-step coloring algorithm:

1. Properly color C_1 and a set J of alternate vertices in C_3 such that the reduced list sizes on C_2 are as follows: one vertex in C_2 has a 4-list and every other vertex in C_2 has a 3-list.
2. Properly color C_4, then the remaining vertices in C_3, and finally C_2.

Assume for the moment that step 1 has been completed. Then, step 2 can be completed by repeatedly invoking Lemma 2.4 as follows.

As we shall see when we elaborate on step 1, we may assume that the 4-list in the column C_2 is on the vertex $(2, s-1)$, and that the rest of the vertices in C_2 have 3-lists. Also, the set J will turn out to be either $I \coloneqq \{(3, s-2k+2) : k = 1, \ldots, \lfloor s/2 \rfloor\}$ or $I' \coloneqq \{(3, s-2k+1) : k = 1, \ldots, \lfloor s/2 \rfloor\}$.

Now, the sizes of the lists on the remaining vertices of C_3 after the completion of step 1 are as follows: the vertices of C_3 that remain to be colored all have 3-lists; moreover, when s is odd, the vertices $(3, 1)$ and $(3, 2)$ each have a 4-list when the set I is colored in step 1, and the vertices $(3, 1)$ and $(3, s)$ each have a 4-list when the set I' is colored in step 1.

Next, the sizes of the lists on the column C_4 are as follows: each vertex in C_4 has a 2-list; moreover, when s is odd, the vertex $(4, 1)$ has 3-list when the set I is colored in step 1, and the vertex $(4, s)$ has a 3-list when the set I' is colored in step 1.

Thus, regardless of the parity of s, properly color the column C_4 using Lemma 2.4. This reduces the sizes of each of the remaining lists on C_3 by 2. Again by Lemma 2.4, regardless of the parity of s, properly color the remaining vertices in the column C_3. This reduces the list sizes on C_2 as follows. When s is even, each list on C_2 is reduced in size by 1. When s is odd, each list is reduced in size by 1, but for the following exception: if I is colored in step 1, then the list on $(2,2)$ is reduced in size by 2, and if I' is colored in step 1, then the list on $(2,1)$ is reduced in size by 2. In either case, properly color C_2 using Lemma 2.4. This completes step 2.

Figures 7 and 8 illustrate the sizes of the lists in step 2 when s is even and odd, respectively, assuming that the set I is colored in step 1. The edges between the top and bottom rows are not shown in these figures.

We now describe step 1. If $(1,s)$ and $(1,s-1)$ are in any configuration other than (IV) and (VI), then take $J = I$, and if $(1,s)$ and $(1,s-1)$ are in configuration (IV) or (VI), then take $J = I'$, where the sets I and I' are as defined earlier in the description of step 2. From observations (A) to (C), every vertex in C_2 can have a 3-list at the end of step 1 if for every $(3,j) \in J$, either $L_{(1,j+1)} \cap L_{(3,j)} = \emptyset$, or $(1,j+1)$ and $(3,j)$ are assigned the same color. Clearly, if the lists on $(1,j+1)$ and $(3,j)$ are identical, then for any assignment of a color on $(1,j+1)$ we can pick the same color for $(3,j)$. On the other hand, if the lists on $(1,j+1)$ and $(3,j)$ are distinct but not disjoint, then we need to ensure that the color assigned on $(1,j+1)$ belongs to $L_{(1,j+1)} \cap L_{(3,j)}$.

So, call the pair of vertices $(1,j+1),(3,j)$ to be a *good pair* if either $L_{(1,j+1)} = L_{(3,j)}$, or $L_{(1,j+1)} \cap L_{(3,j)} = \emptyset$ and $(1,j+2)$, $(1,j+1)$, and $(1,j)$ are not in configuration (X). Define A to be the set of all pairs $(1,j+1),(3,j)$ that are not good pairs. We now carry out step 1 in three stages. In the first stage, we shall color the vertices in A. In the second stage, we color the vertices $(1,s)$ and $(1,s-1)$ in such a way that the list on $(2,s-1)$ reduces to a 4-list. Finally, we color the remaining vertices of the column C_1, followed by the remaining vertices in J.

Now, for the first stage. Suppose $(3,j) \in A$. If the lists on $(1,j+1)$ and $(3,j)$ are distinct but not disjoint, then choose a common color for $(1,j+1)$ and $(3,j)$ from $L_{(1,j+1)} \cap L_{(3,j)}$. Otherwise, we have that the lists on $(1,j+1)$ and $(3,j)$ are disjoint and the vertices $(1,j+2)$, $(1,j+1)$ and $(1,j)$ are in configuration (X). In this case, choose a color for $(1,j+1)$ from $L_{(1,j+1)} \setminus L_{(2,j+1)}$ and for $(3,j)$ from $L_{(3,j)} \setminus L_{(2,j)}$.

Note that our choice of J ensures that the vertices $(1,1)$, $(1,s)$, $(1,s-1)$ and $(1,s-2)$ are not colored in the first stage above (cf. Figure 5). So, for the second stage, pick colors for $(1,s)$ and $(1,s-1)$ as follows:

- If $(1,s)$ and $(1,s-1)$ are in any of the configurations (I) to (IV), then choose a color for $(1,s)$ from $L_{(1,s)} \setminus L_{(1,s-1)}$ and for $(1,s-1)$ from $L_{(1,s-1)} \setminus L_{(1,s)}$.

- If $(1,s)$ and $(1,s-1)$ are in configuration (V), choose a color for $(1,s)$ from $L_{(1,s)} \cap L_{(3,s-1)} \setminus L_{(1,s-1)}$ (this can be done because of criterion (i) and Lemma 3.3(1)), and choose a color for $(1,s-1)$ from $L_{(1,s-1)} \setminus L_{(1,s)}$.

- If $(1,s)$ and $(1,s-1)$ are in configuration (VI), choose a color for $(1,s)$ from $L_{(1,s)} \setminus L_{(1,s-1)}$, and for $(1,s-1)$ from $L_{(1,s-1)} \cap L_{(3,s-2)} \setminus L_{(1,s)}$ (this can be done because of criterion (i) and Lemma 3.3(1)).

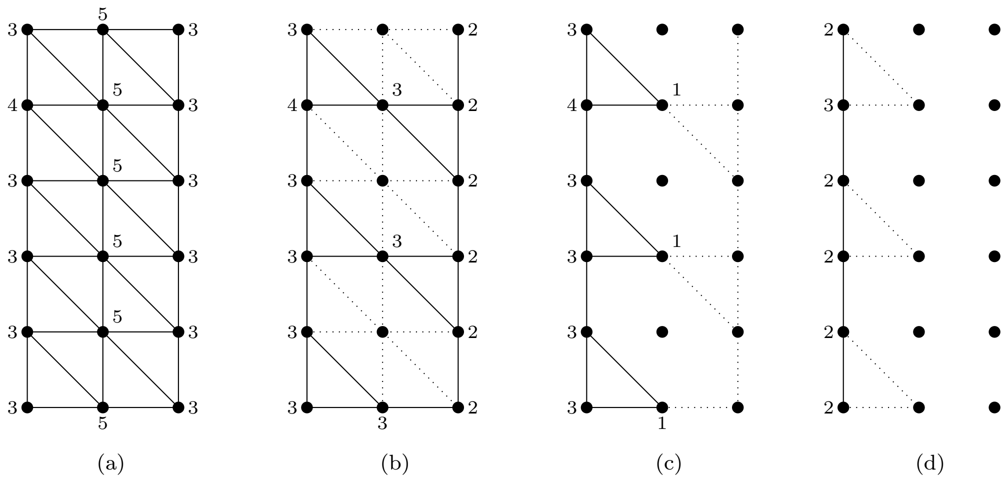


FIGURE 7. Illustration of the sizes of the lists on the columns C_2, C_3 and C_4 in step 2 when $s = 6$ and I is colored in step 1

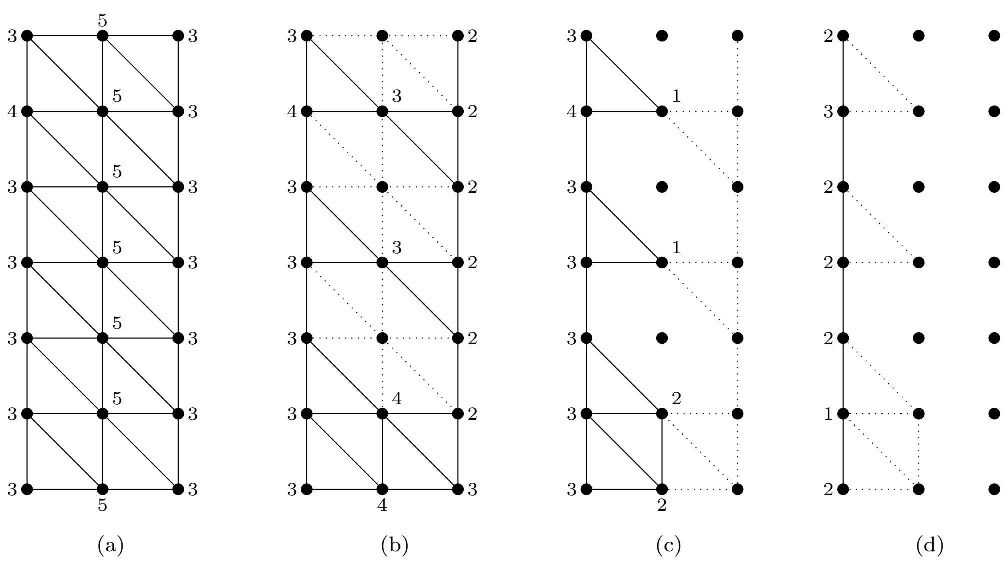


FIGURE 8. Illustration of the sizes of the lists on the columns C_2, C_3 and C_4 in step 2 when $s = 7$ and I is colored in step 1

- If $(1, s)$ and $(1, s-1)$ are in configuration (VII), choose a color for $(1, s)$ from $L_{(1,s)} \setminus L_{(2,s-1)}$, and for $(1, s-1)$ from $L_{(1,s-1)} \setminus L_{(1,s)}$.

This coloring ensures that the vertex $(2, s-1)$ now has a 4-list.

Finally, for the third stage. Color the remaining vertices in the column C_1 using Lemma 2.4. Then, color the remaining vertices in J as follows. Suppose $(3, j) \in J$ was uncolored in the first stage. If the lists on $(3, j)$ and $(1, j+1)$ assigned by $\mathcal{L}$ were identical, choose the same color on $(3, j)$ as that assigned on $(1, j+1)$. If the lists on $(3, j)$ and $(1, j+1)$ are disjoint, then choose any color for $(3, j)$ from its list.

Notice that at the end of this procedure the vertex $(2, s-1)$ still has a 4-list, and that all the other vertices in the column C_2 have 3-lists. So, this completes step 1. Combined with step 2, this completes the proof.

It is also clear from the above description that the coloring can be found in linear time.

5. Proofs of cases (2) and (3) in Theorem 1.3

Proof of case (2) in Theorem 1.3. Let $G = T(1, s, 2)$ for $s \geq 9$, $s \neq 11$. Then, every four successive vertices $(1, j)$, $(1, j+1)$, $(1, j+2)$, $(1, j+3)$ induce a K_4. Suppose that $\mathcal{L}$ is a list assignment on G with lists of size equal to 5. Since G is 5-colorable in linear time by the results in [13, 30], it suffices to assume that not all the lists assigned by $\mathcal{L}$ are identical. Without loss of generality, suppose that $L_{(1,1)} \neq L_{(1,s)}$. Choose a color for $(1, s)$ from $L_{(1,s)} \setminus L_{(1,1)}$. Next, one can properly color the vertices $(1, s-1), (1, s-2), \dots, (1, 7)$ in that order by successively picking a color for each vertex from its (reduced) list. Then, the lists on the remaining vertices are as follows: $(1, 6)$ has a 2-list; $(1, 1)$, $(1, 2)$ and $(1, 5)$ have 3-lists; $(1, 3)$ and $(1, 4)$ have 4-lists. There are two special cases that can be easily dealt with.

Case I: $L_{(1,2)} \cap L_{(1,6)} \neq \emptyset$.

Choose a common color for $(1, 2)$ and $(1, 6)$ from $L_{(1,2)} \cap L_{(1,6)}$. Then, $(1, 1)$ and $(1, 5)$ have 2-lists, and $(1, 3)$ and $(1, 4)$ have 3-lists. If we can pick a color for $(1, 1)$ that does not belong to both $L_{(1,3)}$ and $L_{(1,4)}$, then we will be done by Lemma 2.4, so assume that $L_{(1,1)} \subset L_{(1,3)} \cap L_{(1,4)}$. Then, for any choice of color for $(1, 1)$, the remaining 3-cycle will have 2-lists, so it will have a proper coloring only when the 2-lists are not identical, by Lemma 2.4. But, if picking $a \in L_{(1,1)}$ results in identical 2-lists being present on the remaining 3-cycle, then we instead pick the other color $a' \in L_{(1,1)} \setminus \{a\}$ for $(1, 1)$ to get non-identical 2-lists on the remaining 3-cycle.

So, it suffices to assume that $L_{(1,2)} \cap L_{(1,6)} = \emptyset$.

Case II: $L_{(1,1)} \cap L_{(1,5)} \neq \emptyset$.

Choose a common color c for $(1, 1)$ and $(1, 5)$ from $L_{(1,1)} \cap L_{(1,5)}$. If $c \notin L_{(1,2)}$, then the remaining vertices form a K_4^- with three 3-lists and one 1-list, and one can see that a proper coloring can always be found from this configuration of lists. If $c \in L_{(1,2)}$, then we have a K_4^- with three 2-lists and one 3-list. Since $L_{(1,2)} \cap L_{(1,6)} = \emptyset$ by assumption, we can choose a color for either $(1, 2)$ or $(1, 6)$ that does not belong to $L_{(1,3)}$. Then, we can properly color the rest of the vertices using Lemma 2.4.

So, we additionally assume that $L_{(1,1)} \cap L_{(1,5)} = \emptyset$.

Now, choose a color for $(1,3)$ from $L_{(1,3)} \setminus L_{(1,6)}$. Then, the lists are now as follows: $(1,4)$ has a 3-list; $(1,2)$ and $(1,6)$ have 2-lists; lastly, either $(1,1)$ has a 2-list and $(1,5)$ has a 3-list, or vice-versa, since the color chosen for $(1,3)$ can belong to at most one of $L_{(1,1)}$ and $L_{(1,5)}$. In either case, a color can be chosen for $(1,4)$ such that both $(1,1)$ and $(1,5)$ end up with 2-lists. The lists on $(1,2)$ and $(1,6)$ are now a 1-list and a 2-list, not necessarily in that order, since the color chosen for $(1,4)$ can belong to at most one of $L_{(1,2)}$ and $L_{(1,6)}$. In any case, the remaining four vertices form a path graph with three 2-lists and one 1-list, so a proper coloring can be found using Lemma 2.4.

This completes the proof. It is also clear that the list-coloring can be found in linear time. $\square$

The following lemma will be repeatedly invoked in the proof of case (3) in Theorem 1.3.

Lemma 5.1. *Let $G = K_4^-$ be the complete graph on four vertices with an edge removed, where $V(G) = \{a, b, x, y\}$ and $\{x, y\}$ is an independent set. Suppose that $\mathcal{L}$ is a list assignment on G such that $|L_a| + |L_b| = |L_x| + |L_y|$. Then, one can choose colors for x and y such that the sizes of the lists on a and b reduce by 1 each.*

Proof. If $L_x \cap L_y \neq \emptyset$, then choose a common color for x and y from $L_x \cap L_y$. Clearly, this reduces the sizes of the lists on a and b by 1 each.

So, suppose $L_x \cap L_y = \emptyset$. If $L_x \not\subset L_a \cup L_b$, then we can choose a color for x from $L_x \setminus (L_a \cup L_b)$, and any color for y, to reduce the sizes of the lists on a and b by 1 each. Similarly, when $L_y \not\subset L_a \cup L_b$.

So, suppose that $L_x, L_y \subset L_a \cup L_b$. But then there are k distinct available colors for x and y together, where $k = |L_x| + |L_y|$, as well as for a and b together. Thus, any color in L_x can belong to at most one of L_a and L_b, and similarly for any color in L_y. Moreover, it is not possible that all of the k available colors on x and y belong to a single list (say, L_a), because the other list (L_b) will then be empty, a contradiction. Therefore, it is possible to choose colors for x and y from their respective lists such that each color belongs to a different list between L_a and L_b. This reduces the sizes of the lists on a and b by 1 each. $\square$

Proof of case (3) in Theorem 1.3. Let $G = T(2, s, t)$ for even $s \geq 4$ and even $t \neq 0,\ s-2$. By the remarks following Theorem 1.2, it follows that the graphs $T(2, s, t)$ and $T(2, s, s-t-2)$ are isomorphic, so without loss of generality we assume that $2 \leq t \leq \frac{s}{2} - 1$.

Our strategy is to properly color the column C_2 in such a way that the lists on the column C_1 are all reduced to 2-lists, so that C_1 can then be properly colored using Lemma 2.4. Note that, for every j, the vertices $(1, j)$, $(1, j-1)$, $(2, j-1)$ and $(2, j+t)$ form a K_4^- with the vertices on the column C_2 forming an independent set. This suggests the following scheme of coloring.

Fix $1 \leq j \leq s$. Using Lemma 5.1, we can color $(2, j-1)$ and $(2, j+t)$ such that the lists on $(1, j)$ and $(1, j-1)$ reduce in size by 1 each. Then, regardless of how the rest of the neighbors of $(1, j)$ and of $(1, j-1)$ in the column C_2 are colored, the end result is that the lists on these two vertices reduce to 2-lists, as required. If we can do this for each even j, then we will have colored C_2 in such a way that the lists on C_1 all reduce to 2-lists. A little bit of care is required to ensure that this can be done for every even j, while also ensuring that the coloring on C_2 is proper. The details now follow.

Suppose that $t < \frac{s}{2} - 1$.

- For each $j \in \{2, 4, \ldots, t\}$, we may use Lemma 5.1 to color the vertices of the form $(2, j-1)$ and $(2, j+t)$.
- For $j = t+2$, we need to color $(2, t+1)$ and $(2, 2t+2)$, but notice that the list on $(2, t+1)$ has been reduced to a 4-list, since $(2, t+2)$ has already been colored (when $j = 2$). However, the list on $(1, t+2)$ has also been reduced to a 4-list, so Lemma 5.1 is still applicable.
- For each $j \in \{t+4, t+6, \ldots, s-t-2\}$, notice that the list on $(2, j-1)$ has been reduced to a 3-list, but the lists on $(1, j)$ and $(1, j-1)$ have also been reduced to 4-lists each, so Lemma 5.1 is still applicable.
- For $j = s-t$, notice that the list on $(2, s-t-1)$ is reduced to a 3-list and the list on $(2, s)$ is reduced to a 4-list, but the list on $(1, s-t)$ is reduced to a 3-list and the list on $(1, s-t-1)$ is reduced to a 4-list. So, Lemma 5.1 is still applicable.
- Lastly, for each $j \in \{s-t+2, s-t+4, \ldots, s\}$, notice that the lists on $(2, j-1)$ and $(2, j+t)$ are reduced to 3-lists each, but the lists on $(1, j)$ and $(1, j-1)$ have also been reduced to 3-lists each, so Lemma 5.1 is still applicable.

Next, consider the case when $t = \frac{s}{2} - 1$. Since $t + 2 = s - t$, some of the cases above reduce to a single degenerate case, as explained below:

- As before, for each $j \in \left\{2, 4, \ldots, \frac{s}{2} - 1\right\}$, we may use Lemma 5.1 to color the vertices of the form $(2, j-1)$ and $(2, j+t)$.
- For $j = \frac{s}{2} + 1$, we need to color $\left(2, \frac{s}{2}\right)$ and $(2, s)$, but notice that the lists on $\left(2, \frac{s}{2}\right)$ and $(2, s)$ have been reduced to 4-lists. However, the list on $\left(1, \frac{s}{2} + 1\right)$ has also been reduced to a 3-list, so Lemma 5.1 is still applicable.
- Lastly, just as before, for each $j \in \left\{\frac{s}{2} + 3, \frac{s}{2} + 5, \ldots, s\right\}$, notice that the lists on $(2, j-1)$ and $(2, j+t)$ are reduced to 3-lists each, but the lists on $(1, j)$ and $(1, j-1)$ have also been reduced to 3-lists each, so Lemma 5.1 is still applicable.

Thus, the coloring scheme outlined in the beginning can be implemented over all even j to get a proper coloring of C_2 so that the lists on C_1 are all reduced to 2-lists. The proof is completed by using Lemma 2.4 to properly color C_1. Clearly, the list-coloring is found in linear time. ☐

Figure 9 illustrates the coloring algorithm for $G = T(2, 10, 4)$.

6. Proofs of cases (4) and (5) in Theorem 1.3

Lemma 6.1. *Let G be a 3-regular bipartite graph. Let $V(G)$ be partitioned into two independent sets A and B. Let $\mathcal{L}$ be a list assignment that assigns a list of size 3 to each vertex in A and a list of size 2 to each vertex in B. Then, G is $\mathcal{L}$-choosable. Moreover, such a list-coloring can be found in linear time.*

Proof. Find a perfect matching M in G and consider the graph $G - M$. Since $G - M$ is a 2-regular bipartite graph, it breaks up into a union of disjoint even cycles. Place an orientation on the edges of G such that each of these cycles becomes a directed cycle, and such that the edges in M are oriented from A to B. Then, $\mathsf{outdegree}(v) = 2$ for every $v \in A$ and $\mathsf{outdegree}(w) = 1$ for every

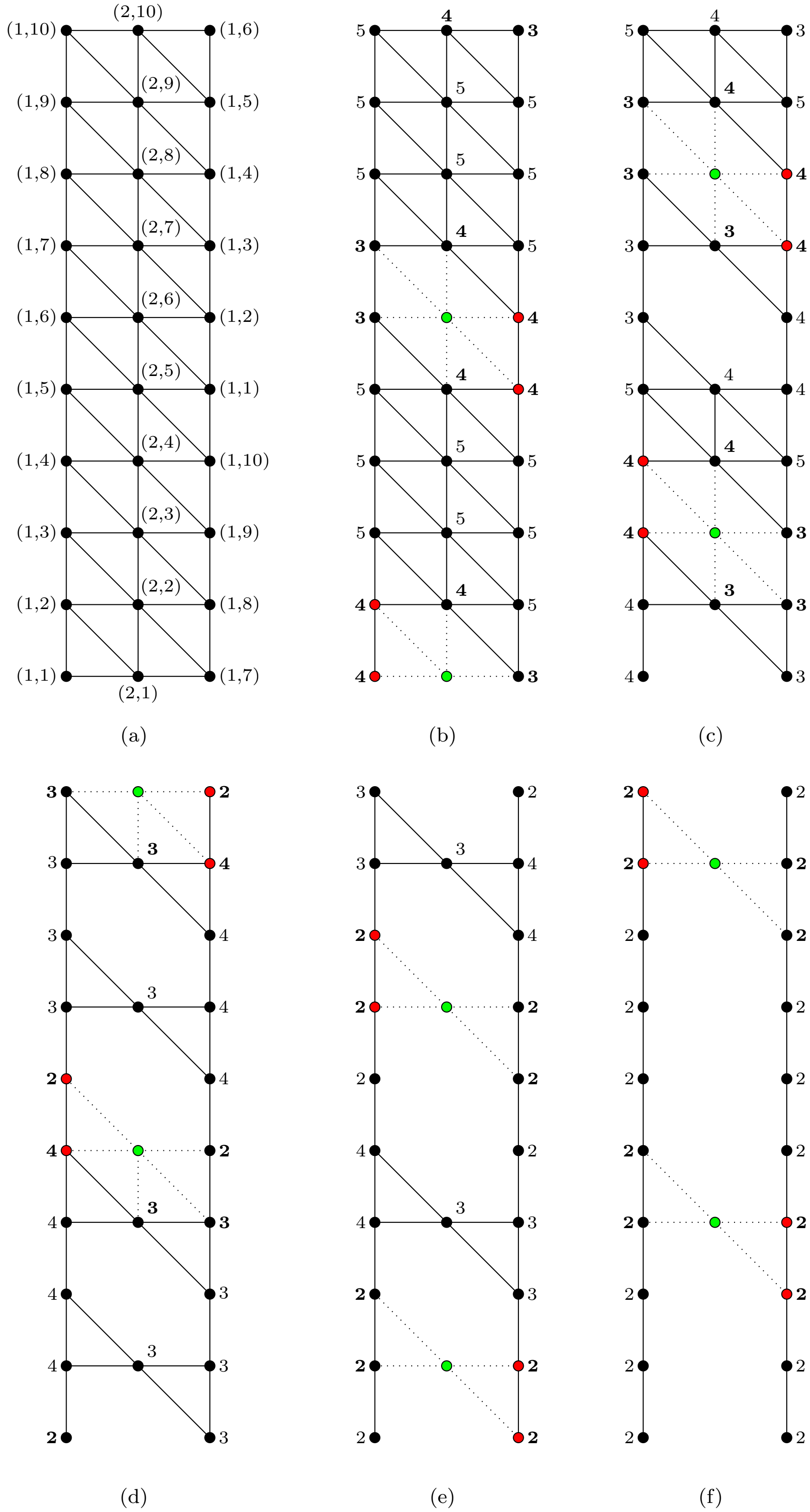


FIGURE 9. Illustration of the proof of case (3) in Theorem 1.3 for $G = T(2, 10, 4)$

$w \in B$. Note that there are no odd directed cycles in this orientation, since G is bipartite. Thus, by Lemma 2.3, G is $\mathcal{L}$-choosable for any list assignment that assigns a list of size 3 to each vertex in A and a list of size 2 to each vertex in B.

Furthermore, Theorem 2.2 shows that M can be found in $O(|E|)$ time, which is also $O(|V|)$ time, since G is of bounded degree. Thus, the list-coloring can be found in linear time. $\square$

In an earlier paper [7], we proved the following theorem:

Theorem 6.2 (Balachandran–Sankarnarayanan [7], 2021)**.** *Let G be a simple 6-regular toroidal triangulation. If G is 3-chromatic, then G is 5-choosable.*

The proof of this theorem is entirely algorithmic, except for one use of the theorem of Alon and Tarsi (Theorem 1.1) to show that a toroidal 3-regular bipartite graph is $\mathcal{L}$-choosable for a list assignment $\mathcal{L}$ as in the hypothesis of Lemma 6.1. Using the proof of Lemma 6.1 in its place, we obtain the proof of case (4) in Theorem 1.3 as a corollary:

Corollary 6.3. *Every simple 3-chromatic 6-regular toroidal triangulation is 5-choosable. Moreover, a 5-list-coloring can be found in linear time.*

Lastly, we show that all the simple 3-chromatic graphs $T(r, s, t)$ (except possibly $T(3, 9, 3)$) are not 3-choosable. Note that $T(r, s, t)$ is 3-chromatic if and only if $s \equiv 0 \equiv r - t \pmod 3$. Let

$$L_1 \coloneqq \{1, 2, 3\}, \quad L_2 \coloneqq \{2, 3, 4\}, \quad L_3 \coloneqq \{1, 3, 4\}.$$

6.1. **The graphs $T(r, s, t)$ for $r \geq 4$, $s \geq 3$.** Let $\mathcal{L}$ be the list-assignment that assigns the above lists to the columns of $T(r, s, t)$ as follows:

$$\begin{aligned} &L_1 : C_1, C_2;\\ &L_2 : C_3;\\ &L_3 : C_4, \ldots, C_r. \end{aligned}$$

Let the vertices $(1, 1)$ and $(1, 2)$ be properly colored using $\mathcal{L}$ in any manner. This uniquely determines a proper coloring of the induced subgraph on $C_1 \cup C_2$.

Now, there is a unique way to extend this coloring properly to the induced subgraph on $C_2 \cup C_3$ as follows: simply extend the coloring from C_2 to C_3 using the same lists used on C_2, namely $L_1 = \{1, 2, 3\}$; then, recolor all the vertices in C_3 that have the color 1 with the color 4. The reason behind this is as follows:

- whenever there exist two adjacent vertices in C_2 that are colored using $\{2, 3\} \subset L_1 \cap L_2$, the common neighbor of these two vertices in C_3 must receive the color 4;
- in any proper coloring of $C_1 \cup C_2$, there will be $s/3$ pairs of vertices in C_2 that are colored using $\{2, 3\}$, and no two of these pairs are adjacent in C_2;
- if a vertex in C_3 has its color fixed to be 4 as above, then the colors of its two vertical neighbors are also fixed.

In this manner, one can see that the coloring is extended uniquely to the rest of C_3, with 4 occuring in those places where 1 would have occured had C_3 also been colored using $L_1 = \{1, 2, 3\}$.

Next, repeat the same process to extend the coloring on C_3 to a proper coloring on the induced subgraph on $C_3 \cup C_4 \cup \cdots \cup C_r$ as follows: color the vertices in $C_4 \cup \cdots \cup C_r$ using the colors used on C_3, namely $L_2 = \{2, 3, 4\}$, and then recolor those vertices in $C_4 \cup \cdots \cup C_r$ that have the color 2 with the color 1.

Now, we note that this coloring cannot be proper on all of $T(r, s, t)$ because this process of successive relabelling has mapped the tuple $(1, 2, 3)$ to $(2, 1, 3)$. Thus, for this to be a proper coloring of $T(r, s, t)$, the original coloring on C_1 must arise as the unique extension of the coloring on C_r to the induced subgraph on $C_r \cup C_1$; but, $(2, 1, 3)$ is not a cyclic permutation of $(1, 2, 3)$, so this cannot happen for any t.

6.2. **The graphs $T(2, s, t)$ for $s \geq 6$.** First, consider the case $s \geq 12$. Since $T(2, s, t)$ is assumed to be simple, we ignore the case $t = s - 1$. Next, by the remarks in Section 2, $T(2, s, t)$ is isomorphic to $T(2, s, s - t - 2)$. Furthermore, $t \equiv 2 \pmod 3$ since $T(2, s, t)$ is assumed to be 3-chromatic. Hence, it suffices to assume that either $t = s - 4$, or t lies in the range $5 \leq t \leq \lfloor s/2 \rfloor - 1$.

Now, let $R_1, \ldots, R_s$ denote the s rows of $T(2, s, t)$. Let $\mathcal{L}$ be the list assignment that assigns the lists L_1, L_2, L_3 to the rows of $T(2, s, t)$ as follows, where $s = 3\ell$:

$$\begin{aligned} L_1 &: R_1, \ldots, R_\ell; \\ L_2 &: R_{\ell+1}, \ldots, R_{2\ell}; \\ L_3 &: R_{2\ell+1}, \ldots, R_s. \end{aligned}$$

Figure 10(a) illustrates this list assignment for the graph $T(2, 12, 5)$; the vertices colored red, green, and blue are assigned the lists L_1, L_2, and L_3, respectively.

Let the vertices $(1, 1)$ and $(1, 2)$ be properly colored using $\mathcal{L}$ in any manner. This uniquely determines a proper coloring of the first ℓ rows. Now, we can find a vertex v in one of the remaining blocks of size ℓ such that the residual list on v has size equal to 1, as follows:

- If the colors on $(1, \ell)$ and $(2, \ell)$ are both present in L_2, then take $v = (1, \ell + 1)$.
- If the color on $(1, \ell)$ is 1, then the pairs $(1, \ell - 1), (1, \ell - 2)$ and $(2, \ell), (2, \ell - 1)$ are colored using $\{2, 3\}$. Additionally, if the color on $(2, \ell)$ is 2, then the pair $(2, \ell - 1), (2, \ell - 2)$ is colored using $\{1, 3\}$, and if the color on $(2, \ell)$ is 3, then the pair $(1, \ell), (1, \ell - 1)$ is colored using $\{1, 3\}$.

  From this data, one choice for v is given by:

$$v = \begin{cases} (2, \ell - 1 + t), & 5 \leq t \leq \ell + 1; \\ (1, \ell - 2 - t), & \ell + 2 \leq t \leq \lfloor s/2 \rfloor - 1 \text{ and } (2, \ell) \text{ is colored } 2; \\ (2, \ell + t), & \ell + 2 \leq t \leq \lfloor s/2 \rfloor - 1 \text{ and } (2, \ell) \text{ is colored } 3; \\ (1, \ell + 3), & t = s - 4. \end{cases}$$

- A similar analysis can be done when $(2,\ell)$ is given the color 1. In that case, one choice for v is:

$$v = \begin{cases} (2, \ell-2+t), & 5 \le t \le \ell+1; \\ (1, \ell-1-t), & \ell+2 \le t \le \lfloor s/2 \rfloor - 1 \text{ and } (1,\ell) \text{ is colored } 2; \\ (2, \ell+t), & \ell+2 \le t \le \lfloor s/2 \rfloor - 1 \text{ and } (1,\ell) \text{ is colored } 3; \\ (1, \ell+2), & t = s-4. \end{cases}$$

This allows one to extend the coloring to the entire block of ℓ vertices in which v belongs. Finally, we can repeat the process to color the third block of ℓ vertices. But, this is not a proper coloring of $T(2, s, t)$ for the same reason as in the previous case.

The remaining cases when $r = 2$ are $T(2, 6, 2)$, $T(2, 9, 2)$ and $T(2, 9, 5)$. Since the last two are isomorphic to each other, we shall only consider $T(2, 9, 5)$. In this case, we use the same list assignment as in the case when $s \ge 12$, and we note that after the first block of ℓ vertices is colored we can choose v to be either $(1, 6)$ or $(1, 5)$, depending on whether the color 1 is given to $(1, 3)$ or $(2, 3)$, respectively. The only graph left to consider is $T(2, 6, 2)$, which we handle in an ad hoc manner in the Appendix.

6.3. **The graphs** $T(3, s, t)$ **for** $s \ge 3$**.** First, consider the case $s \ge 12$ and $t \ne 0$. By the remarks in Section 2, $T(3, s, t)$ is isomorphic to $T(3, s, s-t-3)$, and $t \equiv 0 \pmod 3$ since $T(3, s, t)$ is assumed to be 3-chromatic. Hence, it suffices to assume that t lies in the range $3 \le t \le \lfloor (s-3)/2 \rfloor$. We use the same list assignment on these graphs as in the case $T(2, s, t)$ for $s \ge 12$. A similar analysis shows that $T(3, s, t)$ is not 3-choosable in these cases, so we omit the details.

The only remaining cases are $T(3, s, 0)$ for $s \ge 3$, $T(3, 6, 3)$, $T(3, 9, 3)$ and $T(3, 9, 6)$. It is easy to see that the same list assignment as above also works for $T(3, s, 0)$ for $s \ge 6$. Also, the graph $T(3, 3, 0)$ is isomorphic to $K_{3,3,3}$, which is known to be 4-list-chromatic [22]. Lastly, the graphs $T(3, 6, 0)$ and $T(3, 6, 3)$ are isomorphic to each other, and so are $T(3, 9, 0)$ and $T(3, 9, 6)$. So, the only graph left to consider is $T(3, 9, 3)$. We believe that this graph is also not 3-choosable, but we are presently unable to prove this.

6.4. **The graphs** $T(1, s, t)$ **for** $s \ge 9$**.** Suppose that the 3-chromatic graph $T(1, s, t)$ is not isomorphic to $T(r', s', t')$ for any $r' > 1$. This happens if and only if $\gcd(s, t) = 1 = \gcd(s, t+1)$, so we must have $s \equiv 3 \pmod 6$. By the remarks in Section 2, $T(1, s, t)$ is isomorphic to $T(1, s, s-t-1)$, so we can assume $0 \le t \le \lfloor (s-1)/2 \rfloor$. Moreover, $T(1, s, t)$ has loops when $t = 0$ and has multiple edges when $t = 1, \lfloor (s-1)/2 \rfloor$, so we ignore these cases. Since we assume that $T(1, s, t)$ is 3-chromatic, we also have $t \equiv 1 \pmod 3$. Thus, it suffices to consider only those t in the range $4 \le t \le (s-7)/2$. Note that the least value of s for which there exists some t in the above range and for which $\gcd(s, t) = 1 = \gcd(s, t+1)$ is $s = 21$.

For simplicity, we label the vertex $(1, j)$ with the integer j (recall that j is taken modulo s). We shall use the following modifications of the above coloring scheme.

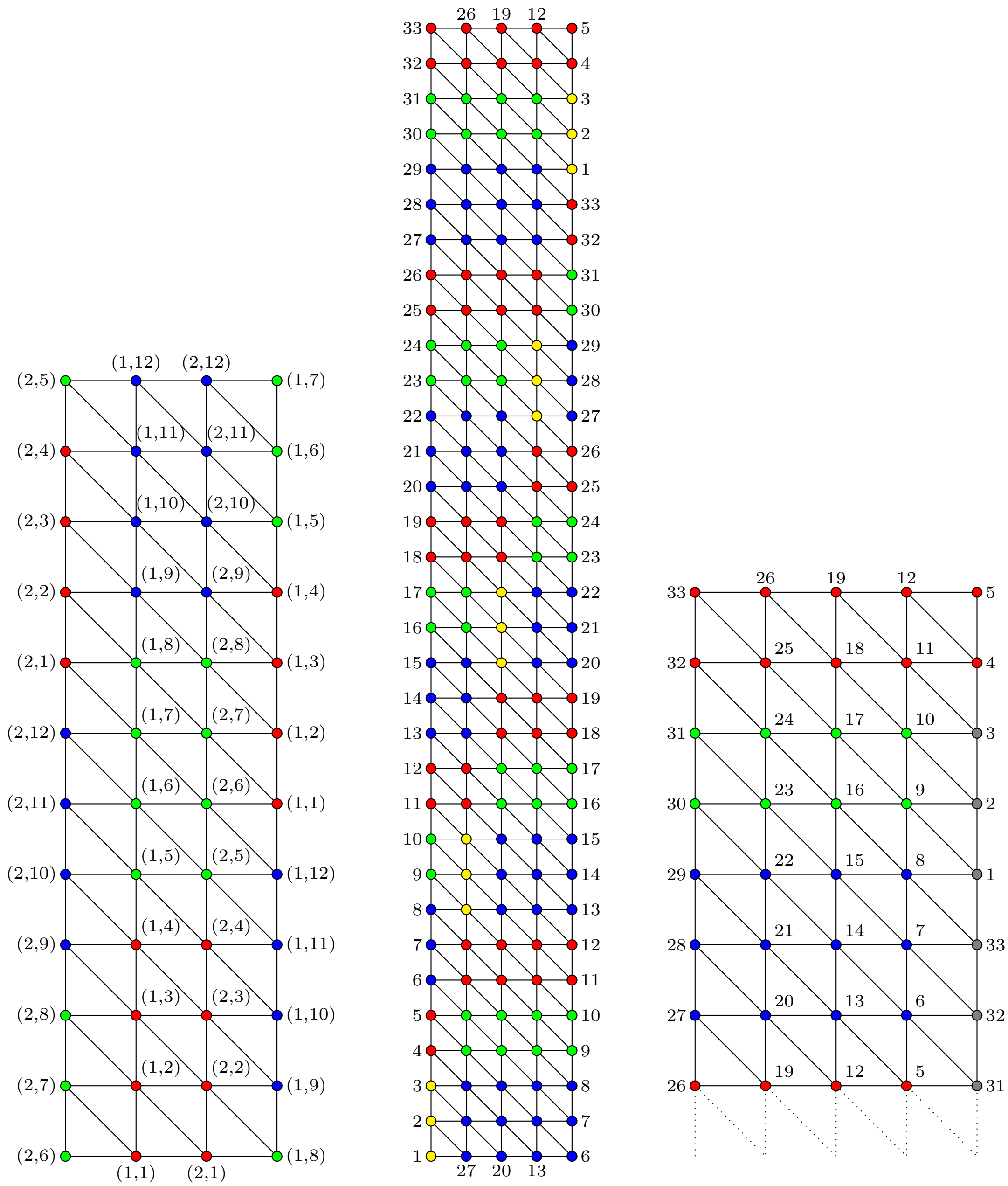
First, suppose that $7 \le t < (s+1)/4$. Fix L_0 to be an arbitrary 3-list. Let $\mathcal{L}$ be the list assignment on $T(1, s, t)$ that assigns the lists L_0, L_1, L_2, L_3 as follows:

$$L_1 : \{s-kt, s-1-kt : k = 0, 1, 2, 3, 4\},$$
$$L_2 : \{s-2-kt, s-3-kt : k = 0, 1, 2, 3\},$$

$$L_3 : \{s-4-kt, \ldots, s-t+1-kt : k = 0, 1, 2, 3\},$$

and any remaining vertices are assigned the list L_0. Figure 10(b) illustrates this list assignment for the graph $T(1, 33, 7)$; the vertices colored yellow, red, green, and blue are assigned the lists L_0, L_1, L_2, and L_3, respectively.

Essentially the same arguments as before work in this case as well, so we omit the details from here onward. Figure 10(c) shows a selected portion of Figure 10(b) on which a similar argument as in the previous cases can be applied to show that $T(1, 33, 7)$ is not 3-choosable.



(a) $T(2, 12, 5)$ (b) $T(1, 33, 7)$ (c) Close-up of $T(1, 33, 7)$

FIGURE 10. Illustration of non-$\mathcal{L}$-colorable 3-list-assignments $\mathcal{L}$ on 3-chromatic graphs $T(r, s, t)$. Distinct colors denote distinct lists among L_0, L_1, L_2, L_3.

Next, suppose that $(s+6)/4 \leq t < (s-3)/3$. Define $\mathcal{L}$ as follows:

$$\begin{aligned}
L_1 &: \{s-kt, s-1-kt : k=0,1,2,3,4\},\\
L_2 &: \{s-2-kt, s-3-kt : k=0,1,2,3,4\},\\
L_3 &: \{s-4-kt, s-5-kt : k=0,1,2,3,4\}\\
&\qquad \cup \{s-6-kt, \ldots, 2s-4t+1-kt : k=0,1,2\}\\
&\qquad \cup \{2s-4t-kt, \ldots, 2s-4t-5-kt : k=1,2\}\\
&\qquad \cup \{2s-4t-6-kt, \ldots, s-t+1-kt : k=0,1,2\},
\end{aligned}$$

and any remaining vertices are assigned the list L_0.

Next, suppose that $(s+3)/3 < t \leq (s-7)/2$. Define $\mathcal{L}$ as follows:

$$\begin{aligned}
L_1 &: \{s-kt, s-1-kt : k=0,1,2,3,4\},\\
L_2 &: \{s-2-kt, s-3-kt : k=0,1,2,3,4\},\\
L_3 &: \{s-4-kt, s-5-kt : k=0,1,2,3,4\}\\
&\qquad \cup \{s-6-kt, \ldots, 2s-3t+1-kt : k=0,1\}\\
&\qquad \cup \{2s-3t-2-kt, \ldots, s-t+1-kt : k=0,1\},
\end{aligned}$$

and any remaining vertices are assigned the list L_0.

The remaining cases are when $t = 4, (s+1)/4, (s-3)/3, (s+3)/3$. These are handled by modifying these list assignments appropriately, as we show in the Appendix. This completes the proof of case (5) in Theorem 1.3.

The proofs of Theorem 1.3 and Corollary 1.4 are now complete from the above results in Sections 4 to 6.

7. Concluding remarks and further questions

In the remarks following the statement of Theorem 1.3 in Section 1, we noted that Theorem 1.3 excludes only a small, finite set of 5-chromatic graphs, as well as an infinite subset of 4-chromatic graphs, both of the form $T(1, s, t)$. We shall elaborate on these details now.

7.1. **The simple graphs $T(r, s, t)$ for $r < 4$ or $s < 3$.** We first look at the graphs that are not covered by case (1) in Theorem 1.3; these are the graphs $T(r, s, t)$ with $r < 4$ or $s < 3$. But, in particular, we need only be concerned with the choosability of the simple graphs among these, because if $T(r, s, t)$ is a loopless multigraph, then we may remove the duplicated edges to get a simple d-regular graph for $d \leq 5$, which is 5-choosable by Brooks's theorem [9, 16, 39], except when the graph is isomorphic to K_6 (but this happens only when $(r, s) \in \{(1, 6), (2, 3), (3, 2)\}$). Moreover, these graphs can be 5-list-colored in linear time [32].

Now, one can check that the graphs $T(r, s, t)$ with $s < 3$ either contain loops or multiple edges, so it suffices to assume $r < 4$ and $s \geq 3$.

For $r = 3, s \geq 3$, there are no graphs $T(3, s, t)$ with loops or multiple edges.

For $r = 2, s \geq 3$, there are no graphs with loops, and the loopless multigraphs are precisely those with $t = 0$, $s-2$, $s-1$, so we assume that $1 \leq t \leq s-3$. In particular, it suffices to assume that $s \geq 4$ in this case.

For $r = 1, s \geq 3$, the graph $T(1, s, t)$ is isomorphic to $T(1, s, s - t - 1)$, so it suffices to consider the values of t in the range $0 \leq t \leq \lfloor (s-1)/2 \rfloor$. Then, the graphs $T(1, s, t)$ with loops are precisely those with $t = 0$, and the loopless multigraphs are precisely those with $t = 1$, $\lfloor (s-1)/2 \rfloor$. So, when $r = 1$, we need only consider the graphs $T(1, s, t)$ with $2 \leq t \leq \lfloor (s-1)/2 \rfloor - 1$. In particular, it suffices to assume that $s \geq 7$ in this case.

7.1.1. *The graphs* $T(3, s, t)$ *for* $s \geq 3$. The normal circuits for $T(3, s, t)$ have lengths s, $3s/\gcd(s, t)$ and $3s/\gcd(s, t + 3)$. Again, if either $3s/\gcd(s, t)$ or $3s/\gcd(s, t + 3)$ is at least 4, then $T(3, s, t)$ is isomorphic to $T(r', s', t')$ for some $r' \geq 4$, and we are done by case (1) in Theorem 1.3. So, assume that both are at most 3. If either one equals 3, then so does the other, and $T(3, s, t)$ is in fact 3-chromatic in this case, so it is 5-choosable by case (4) in Theorem 1.3. So, assume that both are at most 2. But, it is not possible that both $\gcd(s, t)$ and $\gcd(s, t + 3)$ equal 2. Note that all the above 5-list-colorings can be found in linear time as well. The only case left is when $T(3, s, t)$ is isomorphic to $T(1, 3s, t'')$ and it does not satisfy any of the cases (1) to (4).

7.1.2. *The graphs* $T(2, s, t)$ *for* $s \geq 4$. The normal circuits of $T(2, s, t)$ have lengths s, $2s/\gcd(s, t)$ and $2s/\gcd(s, t+2)$. If either $2s/\gcd(s, t)$ or $2s/\gcd(s, t+2)$ is at least 4, then $T(2, s, t)$ is isomorphic to $T(r', s', t')$ for some $r' \geq 4$, and so it is 5-choosable by case (1) in Theorem 1.3. So, suppose that both $2s/\gcd(s, t)$ and $2s/\gcd(s, t + 2)$ are at most 3. Note that both cannot be equal to 3 simultaneously. If any one equals 2, then so does the other, and this case is covered by case (3) in Theorem 1.3. All the above 5-list-colorings can clearly be found in linear time as well. The only case left is when $T(2, s, t)$ is isomorphic to $T(1, 2s, t'')$ and it does not satisfy any of the cases (1) to (4).

7.1.3. *The graphs* $T(1, s, t)$ *for* $s \geq 7$. When $s = 7$, we have to only consider the case $t = 2$, and $T(1, 7, 2)$ is isomorphic to K_7, which is both 7-chromatic and 7-list-chromatic. When $s = 11$, we have to consider the cases $t = 2$, 3, 4, but in each case the graph is isomorphic to the 6-chromatic triangulation J of Albertson and Hutchinson [1] mentioned in Section 1. By Dirac's map color theorem for choosability [8], the graph J is also 6-list-chromatic. So, assume that $s \geq 8$ and $s \neq 11$.

As shown by Yeh and Zhu [42], other than a small, finite list of exceptions, the simple 5-chromatic 6-regular toroidal triangulations are those isomorphic to $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$, $s \geq 9$. Case (2) of Theorem 1.3 shows that the graphs $T(1, s, 2)$ for $s \geq 9$, $s \neq 11$ are all 5-choosable in linear time. Thus, we obtain an infinite class of 5-chromatic-choosable simple toroidal triangulations, proving Corollary 1.4.

Thus, the only graphs that are not covered by Theorem 1.3 are of the form $T(1, s, t)$. Moreover, these consist only of the finitely many 5-chromatic graphs not of the form $T(1, s, 2)$, as well as the 4-chromatic graphs not covered by cases (1) to (3). We are presently unable to comment on the choosability of these remaining graphs.

7.2. **Further questions.** As shown by Yeh and Zhu [42], there is a small, finite set of 5-chromatic graphs of the form $T(1, s, t)$ that are not isomorphic to $T(1, s, 2)$, with the largest among them having 33 vertices. This suggests the following natural question:

Question 7.1. *Is* $\chi_\ell(G) = 5$ *for the finitely many* 6*-regular toroidal triangulations that are* 5*-chromatic and not isomorphic to* $T(1, s, 2)$ *for any* s*?*

In our earlier paper [7], we asked whether there any of the 3-chromatic 6-regular toroidal triangulations are 5-list-chromatic. In light of the results in this paper, we consider a similar question for the 4-chromatic triangulations not covered in Theorem 1.3.

Question 7.2. *Is $\chi_\ell(G) \in \{4,5\}$ for every 4-chromatic 6-regular toroidal triangulation? That is, does there exist a 4-chromatic 6-list-chromatic graph $T(r,s,t)$?*

In our earlier paper [7], we defined the *jump* of a graph G, $\mathsf{jump}(G)$, to be $\chi_\ell(G) - \chi(G)$. There we showed that every loopless 6-regular toroidal triangulation satisfies $\mathsf{jump}(G) \leq 2$. The largest jump for any toroidal graph (which we defined as $\mathsf{jump}(g)$ for $g = 0$) is at least 2 since there exist 3-chromatic planar graphs that are 5-list-chromatic [24, 40]. However, we do not have any "legitimate" example of a nonplanar toroidal graph G that satisfies $\mathsf{jump}(G) = 2$.

Question 7.3. *Does there exist a nonplanar toroidal graph G with $\mathsf{jump}(G) \geq 2$ and such that any planar subgraph H of G has $\mathsf{jump}(H) < 2$?*

We note that such "legitimate" examples must exist as the genus g increases: we have shown [7] that for connected graphs embeddable on an orientable surface with genus $g > 0$, the largest jump among the r-chromatic graphs is of the order $o(\sqrt{g})$ when r is of the order $o(\sqrt{g}/\log_2(g))$, so graphs with small chromatic number (in particular, any planar graph) cannot be the sole examples of graphs attaining a large jump on a surface of large genus $g > 0$.

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Appendix

We describe 3-list assignments on the 3-chromatic graphs not discussed in Section 6, in order to show that they are not 3-choosable. Similar arguments as described in Section 6.4 will show that these graphs are not 3-choosable for the given list assignments, so we omit the details.

First, consider $T(1,s,4)$ for $s \geq 21$. Let $\mathcal{L}$ be the list assignment that assigns the lists L_0, L_1, L_2, L_3 as follows:

$$\begin{aligned}
&L_1 : \{s-kt, s-1-kt : k = 0,1,2,3,4\},\\
&L_2 : \{s-2-kt : k = 0,1,2,3\},\\
&L_3 : \{s-3-kt : k = 0,1,2,3\},
\end{aligned}$$

and any remaining vertices are assigned the list L_0. Figure 12 illustrates this list assignment for the graph $T(1,21,4)$.

Next, consider $T(1,s,t)$ for $t = (s+1)/4$. Define $\mathcal{L}$ as follows:

$$\begin{aligned}
&L_1 : \{s-kt, s-1-kt : k = 0,1,2,3\},\\
&L_2 : \{s-2-kt, s-3-kt : k = 0,1,2,3\},\\
&L_3 : \{s-4-kt, s-5-kt : k = 0,1,2,3\},\\
&\qquad \cup \{s-6-kt, \ldots, s-t+1 : k = 0,1,2\},
\end{aligned}$$

and any remaining vertices are assigned the list L_0. Figure 13 illustrates this list assignment for the graph $T(1,27,7)$.

Next, consider $T(1,s,t)$ for $t = (s+3)/3$, $s > 27$. Define $\mathcal{L}$ as follows:

$$\begin{aligned}
&L_1 : \{s-kt, s-1-kt : k = 0,1,2,3,4\},\\
&L_2 : \{s-2-kt : k = 1,2,3,4\},\\
&\qquad \cup \{s-3-kt : k = 2,3,4\}\\
&\qquad \cup \{s-4-kt, \ldots, s-6-kt : k = 2,3\}\\
&L_3 : \{1, s-2\} \cup \{s-7-kt, \ldots, s-9-kt : k = 1,2,3\}\\
&\qquad \cup \{s-10-kt, \ldots, s-12-kt : k = 1,2\}\\
&\qquad \cup \{s-13-kt, \ldots, s-t+1-kt : k = 0,1\},
\end{aligned}$$

and any remaining vertices are assigned the list L_0. Figure 14 illustrates this list assignment for the graph $T(1,45,16)$. The only remaining case for $t = (s+3)/3$ is $T(1,27,10)$, which is isomorphic to $T(1,27,4)$, so this case is completed.

Lastly, consider $T(1,s,t)$ for $t = (s-3)/3$. By the remarks in Section 2, we can choose the horizontal normal circuit of $T(1,s,t)$ as the vertical normal circuit to get that $T(1,s,t)$ is isomorphic to a graph $T(1,s,t')$ for some $0 \leq t' \leq \lfloor (s-1)/2 \rfloor$. A simple calculation shows that either $t' \equiv -(1+t^{-1}) \pmod{s}$ or $t' \equiv t^{-1} \pmod{s}$. Using $s = 3t+3$ and the fact that $t-1 \equiv 0 \equiv s \pmod{3}$, we see that $t' \neq t$. Thus, the graph $T(1,s,t)$ is isomorphic to one of the cases considered earlier, so it is also not 3-choosable.

This covers the 3-chromatic graphs of the form $T(1, s, t)$. The only case left is $T(2, 6, 2)$, which requires an ad hoc list assignment $\mathcal{L}$ as follows:

$$L_1 : \{(1, 1), (1, 3), (1, 5), (2, 1)\}$$
$$L_2 : \{(1, 2), (1, 4), (1, 6), (2, 6)\}$$
$$L_3 : \{(2, 2), (2, 3), (2, 4), (2, 5)\}.$$

Figure 11 illustrates this list assignment.

This completes the proof of case (5) in Theorem 1.3.

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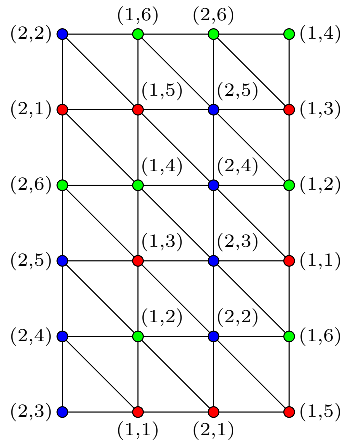
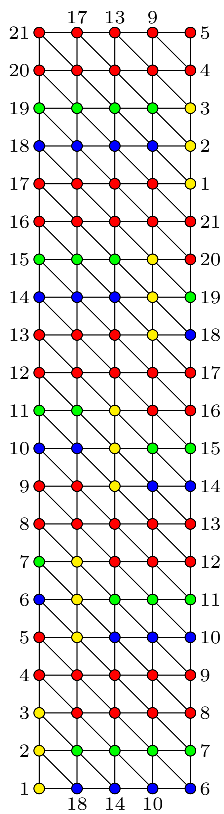


FIGURE 11. A non-$\mathcal{L}$-colorable 3-list-assignment on $T(2,6,2)$



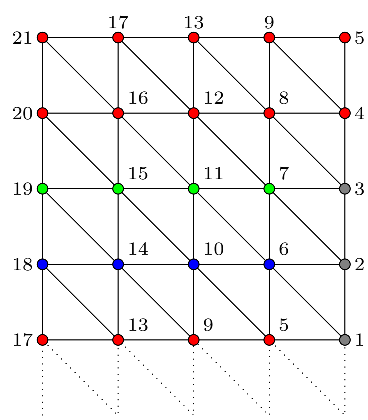


FIGURE 12. A non-$\mathcal{L}$-colorable 3-list-assignment on $T(1,s,4)$ for $s = 21$.

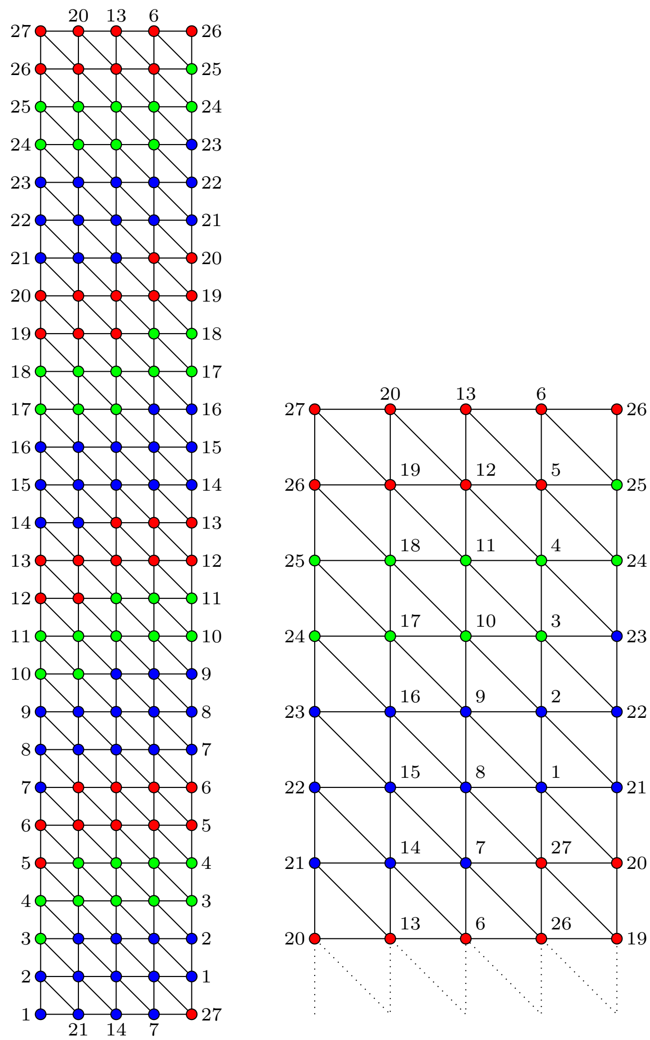


FIGURE 13. A non-$\mathcal{L}$-colorable 3-list-assignment on $T(1, s, (s+1)/4)$ for $s = 27$.

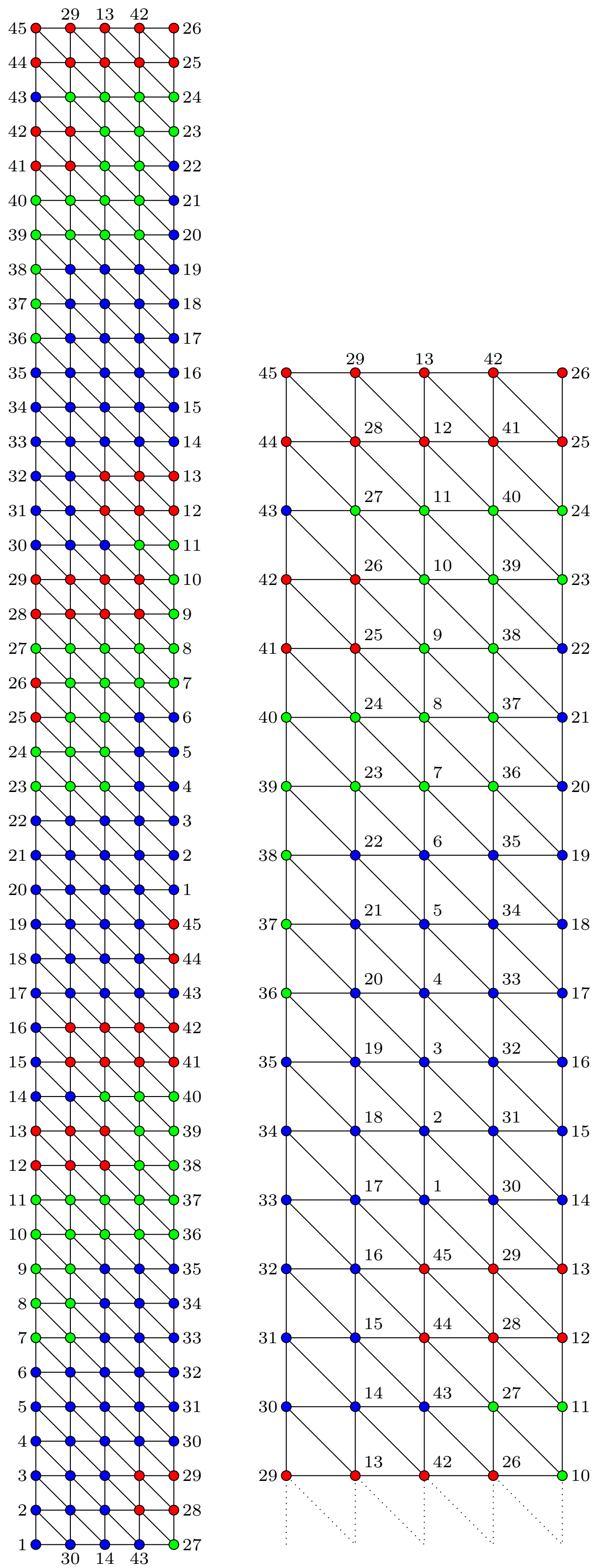


FIGURE 14. A non-$\mathcal{L}$-colorable 3-list-assignment on $T(1, s, (s+3)/3)$ for $s = 45$.