5-list coloring toroidal 6-regular triangulations in linear time

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Abstract. We give an explicit procedure for 5-list coloring a large class of toroidal 6-regular triangulations in linear time. We also show that these graphs are not 3-choosable.

Keywords: List coloring · Toroidal graph · Triangulation · Regular graph · Linear time algorithm

1 Introduction

We will be concerned with the following coloring variant known as *list coloring*, defined independently by Vizing [34] and by Erdős, Rubin, and Taylor [18]. A *list assignment* $\mathcal{L}$ on a graph $G = (V, E)$ is a collection of sets of the form $\mathcal{L} = \{L_v \subset \mathbb{N} : v \in V(G)\}$, where one thinks of each L_v as a *list* of colors available for coloring the vertex $v \in V(G)$. A graph G is *$\mathcal{L}$-choosable* if there exists a function $\mathsf{color}\colon V(G) \to \mathbb{N}$ such that $\mathsf{color}(v) \in L_v$ for every $v \in V(G)$ and $\mathsf{color}(v) \neq \mathsf{color}(w)$ whenever $vw \in E(G)$. A graph G is called *k-choosable* if it is $\mathcal{L}$-choosable for every k-list assignment $\mathcal{L}$ (i.e., an assignment of lists of size at least k, also called k-lists). The least integer k for which G is k-choosable is the *choice number*, or *list chromatic number*, of G and is denoted $\chi_\ell(G)$. If $\chi_\ell(G) = k$, we also say that G is *k-list chromatic*. Notice that the usual notion of graph coloring is equivalent to $\mathcal{L}$-coloring when all the lists assigned by $\mathcal{L}$ are identical. This also shows that $\chi(G) \leq \chi_\ell(G)$ for all graphs G, and in general the inequality can be strict [18,34].

1.1 Motivation

k-choosability is computationally hard. It is well-known that computing the chromatic number is an NP-hard problem [25]. The restricted problem of finding a 4-coloring of a 3-chromatic graph is also NP-hard [24]. Even the problem of 3-colorability of 4-regular planar graphs is known to be NP-complete [14].

Naturally, list coloring is also a computationally hard problem, but much more: for instance, it is well-known [22] that the problem of deciding whether a given planar graph is 4-choosable is NP-hard—even if the 4-lists are all chosen from $\{1, 2, 3, 4, 5\}$ [13]—and so is deciding whether a given planar triangle-free graph is

3-choosable [22]. But, contrast the latter with the fact that every planar triangle-free graph is 3-colorable by Grötzsch's theorem [21], and that a 3-coloring can be found in linear time [15]. In other words, restrictions on graph parameters—such as the girth, as in Grötzsch's theorem—that allow for efficient coloring algorithms need to be strengthened further in order to get list coloring algorithms of a similar flavor.

Note that even proving nontrivial bounds for the choice number is far tougher than the corresponding problem for the chromatic number. Some of the notable instances of such bounds being determined include Brooks's theorem for choosability [34,18], Thomassen's remarkable proof that every planar graph is 5-choosable [32], and Galvin's solution to the famous Dinitz problem [20]. Other interesting examples include the fact that planar bipartite graphs are 3-choosable [5] and that any 4-regular graph decomposable into a Hamiltonian circuit and vertex-disjoint triangles is 3-choosable [19]. However, there is a fundamental difference between the former and latter examples, as we elaborate below.

$\mathcal{L}$-coloring is algorithmically hard. Consider the problem: given a list assignment $\mathcal{L}$ on a graph G, can one efficiently determine whether or not G is $\mathcal{L}$-choosable, and in the case when G is $\mathcal{L}$-choosable can one also efficiently specify a proper coloring from these lists? The theorems of Brooks, Thomassen and Galvin mentioned earlier are some of the few instances where such algorithms are known for a large class of graphs. In the other examples that we mentioned, the proof uses the combinatorial nullstellensatz [4], in particular a powerful application found by Alon and Tarsi [5]. Hence, it does not allow one to extract an efficient algorithmic solution to the problem of $\mathcal{L}$-coloring when the list assignment $\mathcal{L}$ is specified, except in certain special cases. That there is no known efficient algorithm that produces a 3-list coloring from a given list assignment in these examples illustrates the difficulty of the problem of efficiently finding a proper $\mathcal{L}$-coloring even for graphs of small maximum degree. Even just for planar bipartite graphs, an algorithmic determination of a list coloring largely remains open [13].

Hence, efficient $\mathcal{L}$-coloring algorithms for large classes of graphs are interesting. We also place our work within the context of recent results on efficient list coloring algorithms for similar classes of graphs in Section 1.3 below.

1.2 Our work

As noted earlier, in order to find good bounds for the choice number it is natural to place restrictions on certain graph parameters. We shall focus on a certain class of graphs G having bounded *degeneracy number* $d(G)$, defined as $d(G) := \max_{H \leq G}\{\delta(H)\}$, where the maximum is taken over all induced subgraphs H of G, and $\delta(H)$ is the minimum degree of H. A simple inductive argument [3] shows that $\chi_\ell(G) \leq d(G) + 1$ for every simple graph G. This improves the rudimentary upper bound $\chi_\ell(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G. A natural choice for a large collection of graphs with bounded degeneracy number is the class of graphs that are embeddable in a fixed surface, where by a *surface* we mean a compact connected 2-manifold, and a graph is *embeddable* in a surface if, informally speaking, it can be drawn on the surface without any crossing edges. In this paper, we will be concerned only with toroidal graphs, that is, graphs that are embeddable on the torus S_1.

Let $G = (V, E)$ be a toroidal graph, and let F be the set of its faces in an embedding into S_1. The graphs satisfying $\mathsf{degree}(v) = d$ for all $v \in V$ and $\mathsf{degree}(f) = m$ for all $f \in F$, for some $d, m \geq 1$, have been of interest [6,7] especially in the study of vertex-transitive graphs on the torus [30]. A simple calculation using Euler's formula shows that the only possible values of (d, m) are $(3, 6)$, $(4, 4)$ and $(6, 3)$. Our focus will be on the graphs of the last kind, namely the 6-regular triangulations on the torus. Since triangulations have the maximum possible number of edges in any graph with a fixed number of vertices and embeddable on a given surface, one might additionally expect this class of graphs to present a greater obstacle to an efficient solution to the list coloring problem as compared to the others.

The main result of this paper, Theorem 1, is a linear time algorithm for 5-list coloring a large class of these toroidal 6-regular triangulations. Our result is nearly tight for this class in the sense that the list size is at most one more than the choice number for any graph in this family. In fact, in Corollary 1 we find an infinite family of 5-chromatic-choosable graphs for which a list coloring can be specified in linear time.

Here, $T(r, s, t)$ is a triangulation obtained from an $r \times s$ toroidal grid, $r, s \geq 1$ (see Definition 1 for a precise statement):

Theorem 1. *Let G be a simple 6-regular toroidal triangulation. Then, G is 5-choosable under any of the following conditions:*
(1) G is isomorphic to $T(r, s, t)$ for $r \geq 4$;
(2) G is isomorphic to $T(1, s, 2)$ for $s \geq 9$, $s \neq 11$;
(3) G is isomorphic to $T(2, s, t)$ for s and t both even;
(4) G is 3-chromatic.
Moreover, the 5-list colorings can be given in linear time. Furthermore, none of these graphs are 3-choosable. Hence, $\chi_\ell(G) \in \{4, 5\}$ if any of the cases (1) to (4) hold for G.

We are currently unable to comment on the choosability of the excluded graphs, but we note that they consist only of nine nonisomorphic 5-chromatic graphs, as well as a subcollection of triangulations of the specific form $T(1, s, t)$ that are 4-chromatic. For any tuple (r, s, t), there is a simple formula describing each tuple (r', s', t') such that $T(r, s, t)$ is isomorphic to $T(r', s', t')$ (see [6,29]), and there are at most 6 such tuples for any (r, s, t). It is also not difficult to see that the loopless multigraphs $T(r, s, t)$ are all 5-choosable. So, in this sense, Theorem 1 covers the 5-choosability of "most" 6-regular toroidal triangulations. Furthermore, among those graphs covered in Theorem 1, the 5-chromatic ones are precisely those isomorphic to $T(1, s, 2)$ for $s \not\equiv 0 \pmod 4$. Thus, we have:

Corollary 1. *If G is isomorphic to $T(1,s,2)$ for $s \not\equiv 0 \pmod 4$, $s \geq 9$, $s \neq 11$, then G is 5-chromatic-choosable, i.e. $\chi(G) = \chi_\ell(G) = 5$. Moreover, a 5-list coloring can be found in linear time.*

To the best of our knowledge, the method of proof that we employ is novel, in that we develop a framework that allows us to systematically compare the lists on vertices that are not too far apart, and that allows us to compute the list coloring in an efficient manner. By using the differential information between lists on nearby vertices, we reduce the *list configurations* that need to be considered. This kind of "list calculus" differs from other list coloring algorithms in the literature, which instead reduce the possible *graph configurations* by exploiting general structure results on the family of graphs under consideration (minimum girth, edge-width, etc.), while the specific lists on the graphs remain nebulous. Our method of proof could prove fruitful in other areas where a structure theorem—such as Theorem 2 in our case—allows one to shift attention towards the configuration of the lists themselves. We also emphasize that our linear-time algorithm for 5-list coloring these graphs is nearly best possible, since any fixed vertex needs to be "scanned" very few times.

1.3 Related work

Colorability vs. choosability. Note that it follows from Brooks's theorem for choosability that any 6-regular toroidal triangulation not isomorphic to K_7 is 6-choosable. Albertson and Hutchinson [2] showed that there is a unique simple graph in this family that is 6-chromatic, which has 11 vertices, and Thomassen [31] later classified all the 5-colorable toroidal graphs. But a precise characterization of all the 5-chromatic 6-regular toroidal triangulations was completed only recently [12,35,29]. Our results are the first in this line to attempt to characterize the list colorability of the 6-regular triangulations on the torus.

Choosability of grids. The problem of determining the choice number of 4-regular toroidal $m \times n$ grids, for $m, n \geq 3$, has been raised by Cai, Wang and Zhu [10]. These graphs are a special case of those satisfying $(d,m) = (4,4)$. It is easy to show by induction that these grids are all 3-colorable, and the above authors conjecture that they are also 3-choosable. Recent work by Li, Shao, Petrov and Gordeev [26] has nearly determined the choice number of these grids as follows: if mn is even, then the choice number is 3, else it is either 3 or 4. Contrasting this with Theorem 1, we note that both nearly determine the choice number in the sense that the true value of the choice number is either equal to, or one less than, the computed value for each member of the family. However, their result does not a priori give an efficient algorithm for $\mathcal{L}$-coloring the toroidal grids since their proof uses the combinatorial nullstellensatz, whereas our result actually gives a linear time algorithm for $\mathcal{L}$-coloring the toroidal triangulations.

Recent algorithmic advances for list colorings. Dvořák and Kawarabayashi [16] have shown that there exists a polynomial time algorithm for 5-list coloring graphs embedded on a fixed surface. Postle and Thomas [28] have proved that for any surface Σ and every $k \in \{3, 4, 5\}$ there exists a linear time algorithm for determining whether or not an input graph G embedded in Σ and having girth at least $8 - k$ is k-choosable. In particular, when $\Sigma = S_1$ and $k = 5$, this implies that there is a linear time algorithm for determining whether or not any of the 6-regular triangulations under consideration in this paper are 5-choosable. This work was later extended by Postle in [27], wherein he showed that for each fixed surface Σ there exists a linear time algorithm to find a k-list coloring of a graph G with girth at least $8 - k$ for $k \in \{3, 4, 5\}$. Again, when $\Sigma = S_1$ and $k = 5$, this says that there is a linear time algorithm to find a 5-list coloring of a 6-regular triangulation on the torus.

Our results in this paper are stronger than those mentioned above for the class of 6-regular toroidal triangulations. Firstly, the high degree of the polynomial time algorithm in [16] makes it impractical to implement, though the authors suggest that it should likely be possible to reduce the bound enough to make the algorithm practical at least for planar graphs. Secondly, the linear time algorithm in [28] is contingent upon an enumeration of the 6-*list critical* graphs on the torus. Indeed, the authors show that there are only finitely many 6-list critical graphs on the torus, but a full list of these graphs is not explicitly known, and their bound on the maximum number of vertices any 6-list critical graph on the torus can have is far too large to be amenable to a straightforward enumerative check.[1] Also, their linear time algorithm does not specify an $\mathcal{L}$-coloring in the case when the graph is $\mathcal{L}$-choosable for a given list assignment $\mathcal{L}$. Thirdly, the linear time algorithm in [27] first requires a brute-force computation of the list colorings for any such list assignment on graphs of "small" order. However, the bound on the sizes of these small graphs is far too large to be computationally feasible, which makes the algorithm itself of mostly theoretical interest, as noted in a recent work by Dvořák and Postle [17].

This is in contrast with the results in this paper, wherein the 5-choosable graphs identified in Theorem 1 can also be given 5-list colorings in linear time, unlike as in [28]. Furthermore, the non-3-choosability of the 3-chromatic graphs $T(r, s, t)$ is not covered by the results in [28] since these graphs have girth equal to 3, whereas their algorithm for 3-list coloring is applicable only for graphs having girth at least 5. Lastly, our proof of Theorem 1 supplies an implementable algorithm for 5-list coloring all the toroidal graphs under consideration without the need for running a brute-force check on any of them, in contrast with [27].

Structure of this paper. In the rest of this paper, we sketch the proof of Theorem 1. We relegate the full details to the arXiv version [8] due to space constraints.

[1] It is worth contrasting this with the corresponding colorability problem: while Thomassen [33] has shown that for every fixed surface there are only finitely many 6-*critical* graphs that embed on that surface, explicit lists of these 6-critical graphs are known only for the projective plane [1], the torus [31] and the Klein bottle [11,23].

2 Proof of Theorem 1

Definition 1. *For integers $r \geq 1$, $s \geq 1$ and $0 \leq t \leq s-1$, take $V = \{(i,j) : 1 \leq i \leq r, 1 \leq j \leq s\}$ to be the vertex set of the graph $T(r,s,t)$ equipped with the following edges. If $r = 1$, then $(1,j)$ is adjacent to $(1, j \pm 1)$, $(1, j \pm t)$ and $(1, j \pm (t+1))$. If $r > 1$, then: for each $1 < i < r$, (i,j) is adjacent to $(i, j \pm 1)$, $(i \pm 1, j)$ and $(i \pm 1, j \mp 1)$; $(1,j)$ is adjacent to $(1, j \pm 1)$, $(2,j)$, $(2, j-1)$, $(r, j+t+1)$ and $(r, j+t)$; (r,j) is adjacent to $(r, j \pm 1)$, $(r-1, j+1)$, $(r-1, j)$, $(1, j-t)$ and $(1, j-t-1)$.*

Here, addition in the first coordinate is taken modulo r and in the second coordinate is taken modulo s. It is clear that each $T(r,s,t)$ is a 6-regular triangulation of the torus. Conversely:

Theorem 2 (Altshuler [7], 1973). *Every 6-regular triangulation on the torus is isomorphic to $T(r,s,t)$ for some integers $r \geq 1$, $s \geq 1$, and $0 \leq t < s$.*

Also, define the *cylindrical triangulation* $C(r,s)$ to be the graph obtained from $T(r+1, s, 0)$ by deleting the column C_{r+1}. Next, we compile some well-known results (see [18], for instance) on the colorability of paths and cycles:

Lemma 1. *1. An even cycle is 2-list chromatic.*

2. An odd cycle is not 2-colorable, and hence not 2-choosable. However, if $\mathcal{L}$ is a list assignment of 2-lists on an odd cycle such that not all the lists are identical, then the cycle is $\mathcal{L}$-choosable.

3. If $\mathcal{L}$ is a list assignment on an odd cycle having one 1-list, one 3-list, and all the rest as 2-lists, then the cycle is $\mathcal{L}$-choosable.

4. If $\mathcal{L}$ is a list assignment on a path graph having one 1-list, and all the rest as 2-lists, then the path is $\mathcal{L}$-choosable.
Moreover, the $\mathcal{L}$-colorings can all be found in linear time.

The following lemma due to S. Sinha (during an undergraduate research internship with the first author) is in a similar spirit to Thomassen's list coloring of a near-triangulation of the plane [32], and is a key ingredient in the proof of case (1) in Theorem 1.

Lemma 2. *For $r \geq 3$, $s \geq 3$, let $G = C(r,s)$ be a cylindrical triangulation. Suppose that $\mathcal{L}$ is a list assignment on G such that:*
(1) there exists $1 \leq j \leq s$ such that the exterior vertices $(1,j)$ and $(1, j-1)$ have lists of size equal to 4;
(2) every other exterior vertex has a list of size equal to 3;
(3) every interior vertex has a list of size equal to 5.
Then, G is $\mathcal{L}$-choosable. Moreover, an $\mathcal{L}$-coloring can be found in linear time.

Proof (sketch). By inductively coloring the rightmost column using Lemma 1, it suffices to consider the case $r = 3$. Color the vertex $(2,s)$ with $c \in L_{(2,s)} \setminus L_{(1,s)}$, which exists since C_2 has 5-lists. This reduces the sizes of the lists on each of the neighbors of $(2,s)$ by 1, except for $L_{(1,s)}$, which still has size equal to 4. Now, use Lemma 1 to color C_3, and then color the remaining vertices in the order as indicated in Fig. 1. The numbers indicate the reduced list sizes at that step of the coloring algorithm (note that the edges between the top and bottom rows are not shown in this and all subsequent figures). At the final step we are left to color a triangle with lists of sizes 1, 2, and 3, which is easily done.

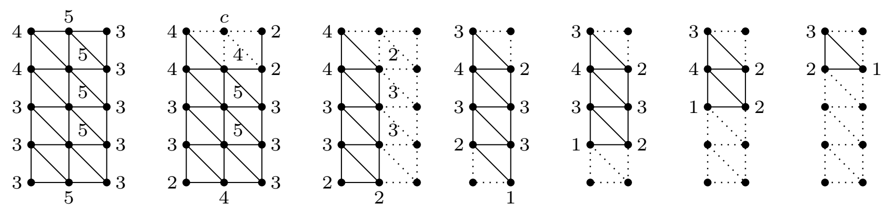

Fig. 1. Illustration of the sizes of the lists on the vertices at each step for $G = C(3, 5)$

2.1 Reductions for the proof of case (1)

Suppose that $T(r, s, t)$, for $r \geq 4$, $s \geq 3$, has a 5-list assignment $\mathcal{L}$. We assume that not all the lists are identical, since these graphs are all 5-colorable in linear time by the results in [12,35,29].

Reduction 1. *For every vertex, its list is contained in the union of its lists on its two left neighbors (as well as of its right neighbors).*

Proof (sketch). If not, choose a color for v that is not in the union of the lists of those two neighboring vertices. Use Lemma 1 to color the entire column of v, and notice that Lemma 2 is now applicable.

Next, focus on a pair of adjacent vertices on the same column that have distinct lists. Applying Lemmas 1 and 2 as before to this pair and their neighbors on an adjacent column (say, the left one), we can cut down to:

Reduction 2. *Whenever (i, j) and $(i, j-1)$ have distinct lists assigned by $\mathcal{L}$, one of the following three configurations holds (and also one of a similar set of configurations obtained by analysing the vertices adjacent on the right column):*
(a) $L_{(i,j)} = L_{(i-1,j+1)}$ *and* $L_{(i,j-1)} = L_{(i-1,j-1)}$;
(b) $L_{(i,j)} = L_{(i-1,j+1)} = L_{(i-1,j)}$ *and* $L_{(i,j-1)} \neq L_{(i-1,j-1)}$;
(c) $L_{(i,j)} \neq L_{(i-1,j+1)}$ *and* $L_{(i,j-1)} = L_{(i-1,j)} = L_{(i-1,j-1)}$.

For the third reduction, focus on a pair of adjacent vertices on adjacent columns that have the same lists. For the fourth reduction, focus on a face in which the two adjacent vertices lying on the same column have the same lists. Using Reduction 2 on the former, and Lemmas 1 and 2 on the latter, we get

Reduction 3. *Whenever u and v are adjacent vertices on distinct columns with $L_u = L_v$, there is a vertex w adjacent to both u and v such that $L_w = L_u = L_v$.*

Reduction 4. *Whenever u, v and w are mutually adjacent vertices having identical lists, with v and w lying on the same column, at least one of the vertical neighbors of u has a list identical to L_u.*

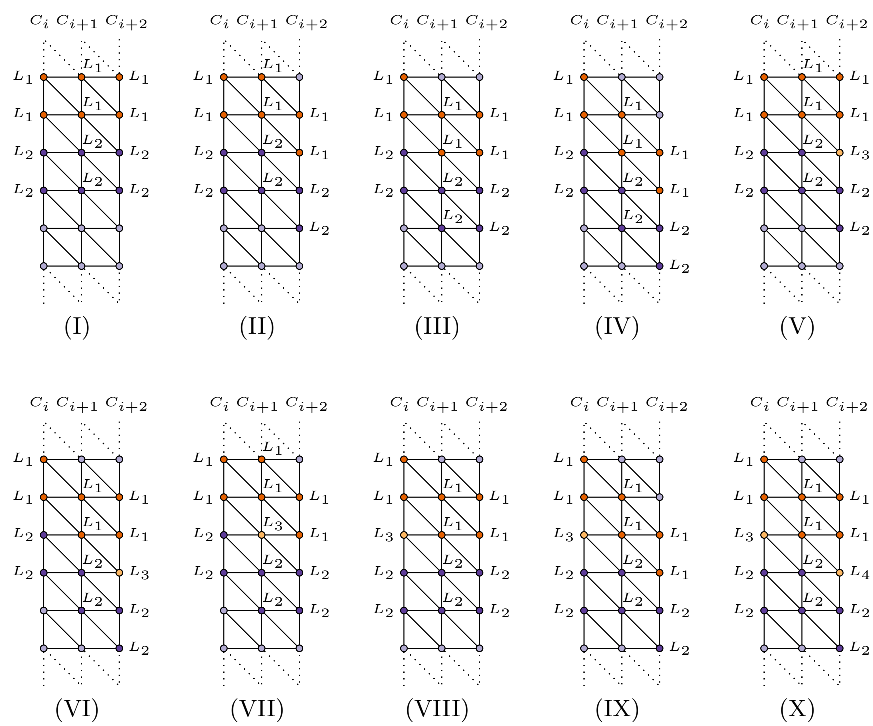

Fig. 2. Illustration of the ten configurations arrived at after reductions

What remains is to exploit the structure of 6-regular triangulations given by Theorem 2 with the rigidity imposed on the list assignment by Reductions 1 to 4. For a list L, define the *list-class* of L in G, denoted $G[L]$, to be induced subgraph of G on those vertices v such that $L_v = L$. Let $L \in \mathcal{L}$ and let H be a (maximal connected) component of $G[L]$. If $V(H)$ is a singleton, we call H an *isolated component*, else we call H a *nonisolated component*.

Lemma 3. *Suppose that $\mathcal{L}$ obeys Reductions 1 to 4.*

(1) Let H be an isolated component, $V(H) = \{(i,j)\}$. Then, there are distinct lists $L', L'' \in \mathcal{L}$ such that $L_{(i-1,j+1)} = L_{(i,j+1)} = L_{(i+1,j+1)} = L_{(i+1,j)} = L'$ and $L_{(i-1,j)} = L_{(i-1,j-1)} = L_{(i,j-1)} = L_{(i+1,j-1)} = L''$.

(2) Let H be a nonisolated component of a list-class $G[L]$, with $v \in V(H)$. Then, at least one vertical neighbor of v also belongs to $V(H)$.

Putting together all of the above, we get a complete description of the list assignment $\mathcal{L}$ from only the information of lists assigned on every four or five consecutive vertices in any one column: the lists propagate across columns in any of precisely ten ways, as shown in Fig. 2. The lists labelled L_3 and L_4 in Fig. 2 belong to isolated components. For the full details for how one arrives at these ten configurations, the reader is requested to see [8].

2.2 Proof of case (1)

Ideally, one would like to complete the proof with another application of Lemma 2. However, an induction argument as in the proof of the lemma does not directly work here, since a naive coloring of C_1 need not give a cylindrical triangulation satisfying the hypothesis (1) of the lemma. Applying a little more discretion in our choices, we use the small set of allowed configurations for $\mathcal{L}$ to arrive at the following two-step coloring scheme (assume $r = 4$ without loss of generality):

1. Properly color C_1 and a set J of alternate vertices in C_3 such that (after reducing the lists) C_2 has one 4-list and the remaining as 3-lists.
2. Properly color C_4, then the remaining vertices in C_3, and finally C_2.

Assuming step 1 is successfully achieved, we complete step 2 as illustrated in Fig. 3.

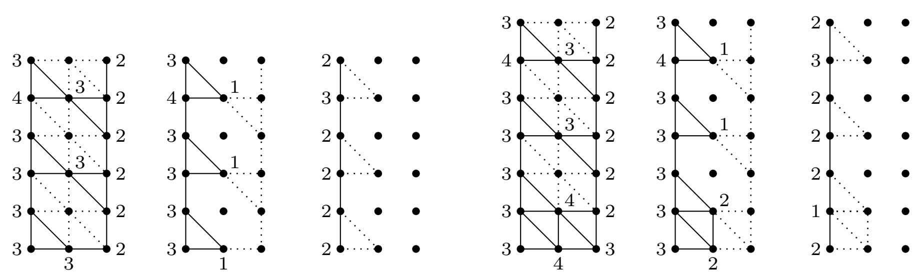

Fig. 3. Sizes of the lists on the columns C_2, C_3 and C_4 in step 2 when $s = 6$ and 7, respectively

Step 1 crucially uses the reduction into the ten cases illustrated in Fig. 2. Indeed, for each of the ten configurations that could appear on the column C_1, we describe an explicit procedure for coloring C_1, as well as for picking out the set J and a coloring for it, so that step 1 is completed. This is a three stage process; for the full details, the reader is again referred to [8].

2.3 Proofs of cases (2) to (4)

Notice that Lemma 2 is not applicable on $C(r,s)$ for $r \leq 2$, so cases (2) and (3) of Theorem 1 require a different line of attack. So, we shall instead use the narrow length of the $r \times s$ grid to place restrictions on the list assignment $\mathcal{L}$. The analysis is therefore shorter in these cases compared to case (1) as discussed above, but we omit the proof here due to space constraints and instead refer the reader to [8].

For case (4), the 5-choosability of the 3-chromatic 6-regular toroidal triangulations was settled in a previous work [9], but a small modification is required to get a linear time algorithm, for which we apply a lemma of Bondy, Bopanna, and Siegel [5] instead of the theorem of Alon and Tarsi [5]. The necessary changes are explicated in [8].

2.4 Proof of non-3-choosability of the graphs in cases (1) to (4)

Note that $T(r,s,t)$ is 3-chromatic if and only if $s \equiv 0 \equiv r - t \pmod 3$. Let $L_1 := \{1,2,3\}$, $L_2 := \{2,3,4\}$, and $L_3 := \{1,3,4\}$. Let $\mathcal{L}$ be the list-assignment that assigns these lists to the columns of $T(r,s,t)$ ($r \geq 4$, $s \geq 3$) as follows: C_1 and C_2 are assigned L_1, C_3 is assigned L_2, and $C_4, \ldots, C_4$ are assigned L_3. Let the vertices $(1,1)$ and $(1,2)$ be properly colored using $\mathcal{L}$ in any manner. This uniquely determines a proper coloring of the induced subgraph on $C_1 \cup C_2$.

Now, there is a unique way to extend this coloring properly to the induced subgraph on $C_2 \cup C_3$ as follows: simply extend the coloring from C_2 to C_3 using the same lists used on C_2, namely $L_1 = \{1,2,3\}$; then, recolor all the vertices in C_3 that have the color 1 with the color 4. In this manner, one can see that the coloring is extended uniquely to the rest of C_3, with 4 occuring in those places where 1 would have occured had C_3 also been colored using $L_1 = \{1,2,3\}$.

Next, repeat the same process to extend the coloring on C_3 to a proper coloring on the induced subgraph on $C_3 \cup C_4 \cup \cdots \cup C_r$ as follows: color the vertices in $C_4 \cup \cdots \cup C_r$ using the colors used on C_3, namely $L_2 = \{2,3,4\}$, and then recolor those vertices in $C_4 \cup \cdots \cup C_r$ that have the color 2 with the color 1.

Now, we note that this coloring cannot be proper on all of $T(r,s,t)$ because this process of successive relabelling has mapped the tuple $(1,2,3)$ to $(2,1,3)$. Thus, for this to be a proper coloring of $T(r,s,t)$, the original coloring on C_1 must arise as the unique extension of the coloring on C_r to the induced subgraph on $C_r \cup C_1$; but, $(2,1,3)$ is not a cyclic permutation of $(1,2,3)$, so this cannot happen for any t.

The rest of the cases (i.e., $r \leq 3$) are handled similarly, and we direct the reader to [8] for the full details.

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