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Fractal river networks, Horton's laws and Tokunaga cyclicity

David G. Tarboton[1]

Utah Water Research Laboratory, Utah State University, Logan, UT 84322-8200, USA

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Abstract

The structure and scaling of river networks characterized using fractal dimensions related to Horton's laws is assessed. The Hortonian scaling framework is shown to be limited in that strict self similarity is only possible for structurally Hortonian networks. Dimension estimates using the Hortonian scaling system are biased and do not admit space filling. Tokunaga cyclicity presents an alternative way to characterize network scaling that does not suffer from these problems. Fractal dimensions are presented in terms of Tokunaga cyclicity parameters.

1. Introduction

River networks have long been recognized as scaling, possessing self similar structures over a considerable range of scales. Without a scale on a map, it would be impossible to distinguish the Amazon network from smaller river networks. This was recognized early and Horton (Horton, 1932, 1945) provided a set of scaling laws, later refined by Strahler (1952), Schumm (1956) and others that characterize river networks. Mandelbrot (1983) used the empirical length–area power law relationship to suggest that rivers were fractal. Since then, considerable work has been done on the fractal dimensions of rivers, river networks and the relationships between fractal dimensions and the more traditional characterizations of scaling such as Horton's laws (Tarboton et al., 1988; La Barbera and Rosso, 1989; Marani et al., 1991; Rosso et al., 1991; Liu, 1992).

Here I review how Horton's laws which characterize the self similarity of river networks can be used to obtain fractal dimensions. I then show that Horton's laws are inadequate in that they do not admit space filling networks. The strictly Hortonian scaling framework is found to be restrictive and I suggest that cyclicity in the river network

[1]Tel.: 801-797-3172; fax: 801-797-3663; e-mail: dtarb@cc.usu.edu.

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branching structure as proposed by Tokunaga (1978) provides a better, more general, framework within which to characterize the scaling and fractal properties of river networks. Fractal dimensions based on Tokunaga parameters are presented.

2. Fractal streams

Mandelbrot (Mandelbrot, 1977, 1983) recognized that the dimensional inconsistency in the empirical relationship between mainstream length and channel area (Hack, 1957; Gray, 1961; Leopold et al., 1964; Eagleson, 1970)

$$L = CA^{\alpha} \tag{1}$$

could be interpreted as the mainstream length being a fractal measure with dimension $D_s = 2\alpha$. The subscript s indicates that this is the fractal dimension due to the sinuosity or meandering of streams. Reported values of α from 0.55 to 0.7 suggest D_s between 1.1 and 1.4 centered around 1.15.

Formally channel length is recognized as a fractal measure (Feder, 1988, p. 14), M_d, determined by

$$M_d = \gamma N(\delta)\delta^d \underset{\delta \to 0}{\longrightarrow} \begin{cases} 0 & d > D \\ M_d & d = D \\ \infty & d < D \end{cases} \tag{2}$$

where $N(\delta)$ is the number of balls (boxes or rulers) size δ required to cover the set (in this case points on the main stream) and γ a geometric factor. The fractal (Hausdorff–Besicovitch) dimension, D, is the critical dimension for which the measure M_d changes from zero to infinity. M_D measures the size of the set in dimension D with units of L^D. The apparent length measured at scale δ is calculated as

$$L(\delta) = N(\delta)\delta = (M_D/\gamma)\delta^{1-D} \sim \delta^{1-D} \tag{3}$$

This length diverges as $\delta \to 0$. This relationship is commonly used to estimate fractal dimension. Length is measured with rulers of different lengths δ (or different map scales) and the slope of the log–log plot of L versus δ gives D. This procedure has been used (Hjelmfelt, 1988; Robert and Roy, 1990; Nikora, 1991; Rosso et al., 1991) to estimate the fractal dimension of various streams to be in the range 1 to 1.2.

The interpretation of Eq. (1) as indicating that streams are fractal requires that Eq. (1) be established using maps of the same scale. It assumes statistical self similarity of river basins of different sizes. It is probable that the length and area data used to establish Eq. (1), assimilated by Eagleson (1970), are from a range of map scales. Measurement of L for large basins from small scale maps would distort the scaling exponent α towards the value of 0.5. If the maps are scaled such that all basins appear the same size on paper, the lengths measured at same resolution with respect to the paper would be proportional to paper size, i.e. $A^{0.5}$. I believe it is this effect in the analysis of Montgomery and Dietrich (1992) that leads them to conclude $L \sim A^{0.5}$.

An alternative interpretation of Eq. (1) is that river basins are allometric, with larger basins tending to be more elongated. This was the interpretation of Hack (1957) and others prior to the discovery of fractals. Fig. 1 illustrates the consequences of this interpretation. Some recent channel network models have predicted elongation as an explanation of Eq. (1) (Inaoka and Takayasu, 1993; Ijjasz-Vasquez et al., 1993). These models impose a numerical grid on the landscape and the predicted allometry is in terms of this grid scale. Thus if the model were applied at a different numerical grid scale to areas the same physical size different width to length ratios would result. This poses a problem in selecting grid size for any practical application of these models and casts doubt on the validity of the allometric prediction they make.

It is not my experience that river basins are on average (statistically) more elongated to the extent shown on Fig. 1 at larger scales. Rather it appears, at least visually, that river networks have statistically self similar shapes over a large range of scales. Therefore I favor the fractal interpretation of Eq. (1) over the allometric interpretation. It may be that $\alpha > 0.5$ is due to some combination of elongation and fractal sinuosity in channels. This would make detection of elongation even more difficult over the range of sizes of channel networks commonly studied.

3. Horton's laws and fractal dimensions

Central to understanding the scaling of river networks is the ordering system used to categorize streams. Here I use the Strahler (1952) ordering system. Stream properties refer to adjoining segments of channel with the same stream order. Horton's ratios are empirical quantities that describe the scaling structure of a river network.

$$R_B = N_{w-1}/N_w$$
$$R_L = L_w/L_{w-1} \quad (4)$$
$$R_A = A_w/A_{w-1}$$

give, respectively, the bifurcation, length and area ratios as ratios of the number of streams, N_w, mean length of streams, L_w, and mean total contributing area, A_w, in streams with successive orders w. Horton (Horton, 1932, 1945) and Schumm (1956) (area ratio) discovered that these ratios are approximately constant across all orders of a river basin. Their applicability with the Strahler ordering system has been widely demonstrated. Horton's ratios quantify the self similarity present in river networks.

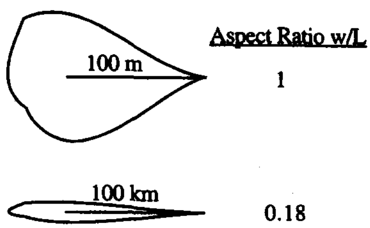

Fig. 1. Allometric interpretation of $L \sim ^{0.57}$

Kirchener (1993) has recently showed that these scaling laws are statistically inevitable in virtually all possible networks, and profoundly indifferent to network structure. He argues that they are a consequence of the hierarchy created by the Strahler (1952) stream ordering rules. Geometric tree structures, of which channel networks are an example, possess an inherent self similarity, which when ordered using Strahler ordering leads inevitably to Horton's laws. With the discovery of fractals, fractal dimensions have become a popular and standard way to quantify self similarity. It is therefore of interest to develop the relationship between fractal dimensions of river networks and the inevitable scaling in river networks reflected in Horton's laws. Relationships between Horton scaling ratios and fractal dimensions lends insight to the structure of river networks despite the inevitability of Horton's laws.

Horton's length and area ratios, which relate mainstream length and channel area at small and large scale, yield the mainstream fractal dimension (Rosso et al., 1991; Liu, 1992; Beer and Borgas, 1993)

$$D_s = 2 \ln R_L / \ln R_A \quad (5)$$

This is derived by restating Eq. (1) in the form $L \sim A^{D_s/2}$ and taking the ratio of it applied to the highest order stream in basins of order w and $w-1$. This derivation takes mainstream length as the length of the highest order stream. It is more conventional to think of mainstream length as the sum of stream lengths over all orders. When this is done to the limit of infinitesimal resolution using geometric series the mainstream length is $L_\Omega/(1-1/R_L)$, a multiplicative constant times length of the highest order stream. Therefore Eq. (5) also applies to this interpretation of mainstream length.

Horton's bifurcation ratio indicates that a network is on average comprised of R_B subnetworks, that by the length ratio are scaled down by a factor $1/R_L$. The similarity dimension due to network branching is (La Barbera and Rosso, 1989):

$$D_b = \ln R_B / \ln R_L \quad (6)$$

The area ratio indicates that drainage area is on average comprised of R_B subareas scaled down in area by length scale factor $1/\sqrt{R_A}$, resulting in similarity dimension for area,

$$D_a = 2 \ln R_B / \ln R_A \quad (7)$$

These similarity arguments neglect the left behind main stream or higher order part of the subbasin present in the fractal generation process. A network of order Ω is comprised of R_B subnetworks of order $\Omega-1$ plus the main stream of order Ω. These dimensions can also be calculated more formally, accounting for the main stream part, in the context of fractal measures. The network is taken as part of an infinite network that in principle could be obtained from refining map resolution indefinitely. Horton's ratios are taken to hold exactly over all orders. The program for calculating dimension is then to sum the measured quantity over all orders. The result is a geometric series (because of the geometric scaling implied by Horton's laws). The limit depends on the limiting resolution δ taken as the average length of lowest order streams. The limit is compared with Eq. (3) to obtain D. When this procedure is applied to the total length of streams with resolution $\delta = L_\Omega/R_L^{\Omega-\lambda}$, $\lambda \rightarrow -\infty$ as the length of the lowest order streams one obtains Eq. (6). Analogously, if one takes $\delta = A_\lambda^{0.5} = (A_\Omega / R_A^{\Omega-\lambda})^{0.5}$, $\lambda \to -\infty$ as the length scale of the lowest order subbasins and sums area over all orders, recognizing that interbasin areas also follow Horton's area law (Beer and Borgas, 1993), one obtains Eq. (7). The limit process results in stream order going to infinity. The absolute value of stream order then loses importance in favor of differences between stream properties as a function of the difference in their orders. The limit process is recognized as a necessary mathematical construct while acknowledging a lower and upper limit of scaling, defining the range of scales over which the fractal description is useful. The wider this range of scales, the more useful the fractal description. The fact that this limit procedure yields the same answers as the self similarity arguments is a reflection of the fact that the lowest order basins dominate the total length and area of river basins.

The derivation of Eq. (6) omitted to consider the fact that streams are themselves fractal measures. The length ratio describes how the ratio of stream length measure scales with order. In the fractal perspective one must assume (because of self similarity) that this also applies to the lowest order streams with length δ. In terms of Eq. (2)

$$\delta = N(\varepsilon)\varepsilon^{D_s} \tag{8}$$

and

$$L_T = N(\delta)\delta^{D_b} \tag{9}$$

where ε is the size of a linear ball (ruler or box) and L_T the total length of streams in a channel network. Combining these we find (Tarboton et al., 1990)

$$L_T = N(\delta)N(\varepsilon)^{D_b}\varepsilon^{D_b D_s} \tag{10}$$

This shows that the total length of streams is a measure with dimension

$$D_a = D_b D_s \tag{11}$$

which is a combination of branching and stream dimensions. It is no coincidence that D_a calculated from using Eq. (5) and Eq. (6) in Eq. (11) is the same as Eq. (7). In the infinitesimal fractal limit the dimension of area comprising the basin is the same as the dimension of all points on the channel network. Marani et al. (1991) call this connectivity: "A point belongs to a drainage basin if there exists a channel connecting that point to the outlet of the basin."

4. Limitations of Horton system

Fig. 2 illustrates a basin ordered according to Strahler's ordering. In this illustrative network there are nine first order streams, three second order streams and one third order stream. Thus R_B is (by construction) exactly three. However, this does not apply to all the subbasins, two of which have $R_B = 2$. Thus for this basin exact self similarity does not hold. The same can be said for practically all natural basins. Horton's rules are statistical descriptors of self similarity and only hold on average. However, one sees here that the first order streams flowing directly into the third order stream introduce a downward bias

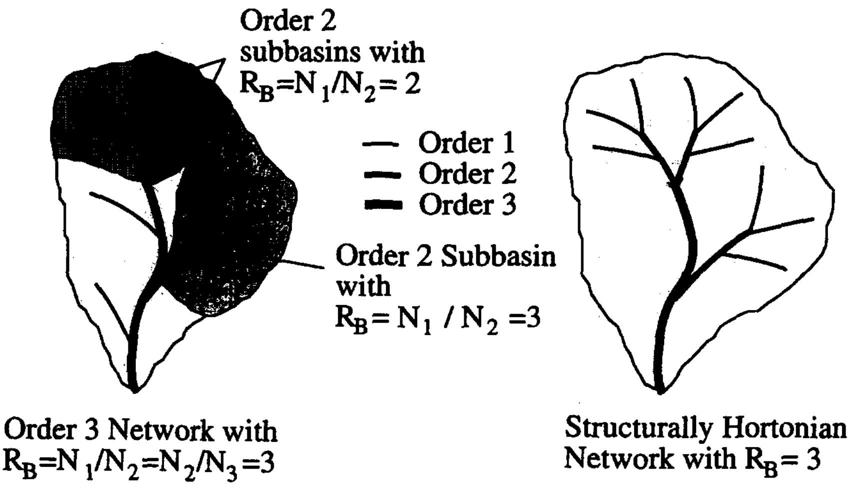

Fig. 2. Strahler ordering and structurally Hortonian network.

into the average bifurcation ratio.

$$\underset{\text{over subbasins}}{\text{Average}}\ (N_1/N_2) \leq \text{Overall}\ (N_1/N_2) \tag{12}$$

This is always the case. The inequality is strictly less than whenever streams flow into streams more than one order higher. This bias can only be avoided if the network is structurally Hortonian, meaning that streams only flow into streams one order higher. The derivations above assumed Horton ratios were constant over all orders. They therefore only apply to structurally Hortonian networks, which are rare in nature. The average R_B for subnetworks is constrained to be less than or equal to the overall R_B. This bias is apparent even where Horton ratios are more loosely interpreted as referring to average values since whenever a low order stream flows into a stream more than one order higher the overall bifurcation ratio is higher than the bifurcation ratio of subbasins. This biases (downwards) fractal dimension estimates based on R_B.

Since the total area draining any stream is larger than the sum of area draining its tributaries, $R_A > R_B$. The area dimension D_a (from Eq. (7)) is therefore strictly less than two. The Hortonian system does not admit a network to be space filling. One can debate as to whether networks in practice are space filling, however it seems desirable to at least work in a theoretical framework that admits space filling.

5. Tokunaga cyclicity

The problem in the description of network branching using Horton's bifurcation ratio was recognized by Smart (1967) and Scheidegger (1968). Tokunaga (1978) gives an alternative description of network branching (illustrated in Fig. 3) that does not suffer from this deficiency. Let the number of streams of order i flowing laterally into a higher order stream of order j be denoted by ${}_j\varepsilon_i$. Tokunaga (1978) suggests that ${}_j\varepsilon_{j-k}$ ($k > 0$) are on

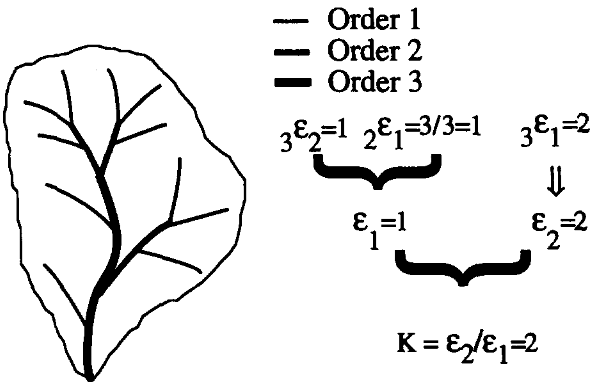

Fig. 3. Order 3 network adhering exactly to Tokunaga cyclicity.

average independent of j, denoted

$$\varepsilon_k = {}_j\varepsilon_{j-k} \tag{13}$$

This is a self similarity property. Furthermore Tokunaga suggests that analogous to Horton's ratios the ε_k are also geometrically dependent on order.

$$K = \varepsilon_k/\varepsilon_{k-1} \tag{14}$$

The two parameters ε_1 and K are analogous to R_B in that they completely describe the network branching structure. The number of streams of each order w within a basin of order Ω is (Tokunaga, 1978)

$$N(\Omega, w) = \frac{Q^{\Omega-w-1} - P^{\Omega-w-1}}{Q-P}(2+\varepsilon_1-P)Q + (2+\varepsilon_1)P^{\Omega-w-1} \tag{15}$$

where P and Q are parameters given by

$$P = \frac{2+\varepsilon_1+K-\sqrt{(2+\varepsilon_1+K)^2-8K}}{2} \tag{16}$$

$$Q = \frac{2+\varepsilon_1+K+\sqrt{(2+\varepsilon_1+K)^2-8K}}{2}$$

Eq. (15) gives a law of stream numbers such that the log of stream numbers plots against order as a slightly concave shape, agreeing qualitatively with this tendency reported by Shreve (1966). Putting $\varepsilon_1 = R_B - 2$ and $K = 0$ one obtains $P = 0$, $Q = R_B$ and $N(\Omega, w) = R_B^{\Omega-w}$. Thus Tokunaga cyclicity generalizes Horton's bifurcation law, retaining it as a special case. The appeal of Tokunaga's formulation is its self similarity. Subnetworks within a network are statistically equivalent, except for a scaling factor, without needing to be structurally Hortonian. In Tokunaga cyclicity the length and area laws can be retained unchanged.

To calculate similarity dimensions for a Tokunaga network of order Ω, let the linear scale reduction factor for each order be r ($= 1/R_L$ for streams or $1/\sqrt{R_A}$ for areas). The network is comprised of $2+\varepsilon_1$ subnetworks order $\Omega-1$ reduced by factor r, $\varepsilon_1 K$ networks order $\Omega-2$ reduced by factor r^2, $\varepsilon_1 K^2$ networks order $\Omega-3$ reduced by factor r_3 and so on. The similarity dimension is then calculated using uneven scale ratios (Mandelbrot, 1983, p. 57; Feder, 1988, p. 63).

$$(2+\varepsilon_1)r^D + \varepsilon_1 K r^{2D} + \varepsilon_1 K^2 r^{3D} + \ldots = 1 \quad (17)$$

This geometric series can be reduced to a quadratic equation for r^{-D}. The larger root gives

$$D = \ln Q/\ln(1/r) \quad (18)$$

which for stream lengths ($r = 1/R_L$) gives

$$D_b = \ln Q/\ln R_L \quad (19)$$

and for stream areas ($r = 1/\sqrt{R_A}$) gives

$$D_a = 2\ln Q/\ln R_A \quad (20)$$

(Eq. (17) can also be reduced to a quadratic equation in r^D which yields different solutions. The largest root overall is the one that yields Eq. (18). I do not know a good reason for selecting the largest root in this procedure to obtain dimension by the similarity approach, except that it corresponds to what is obtained using the infinitesimal limit approach.)

These results can also be obtained by summing over all orders to the infinitesimal limit. If one assumes stream lengths obey Horton's length law one obtains (not surprisingly) Eq. (19). If one assumes link lengths are on average constant (and hence going to 0) throughout the network one obtains (Tarboton, 1989)

$$D_b = \frac{\ln Q}{\ln K} \quad (21)$$

Under the constant average link length assumption (link length = length of lowest order stream L_1) the following length scaling relationship holds (Tarboton, 1989).

$$L_w = \left(1 + \frac{K^{w-1} - 1}{K-1}\right)L_1 \quad (22)$$

This goes asymptotically (w large) to $R_L = K$, showing the similarity between Eq. (21) and Eq. (19). To account for the streams themselves being fractal the argument used above applies. Note that D_b, Eq. (19) times D_s, Eq. (5), gives D_a, Eq. (20).

Since basin areas cannot occupy a space of dimension higher than the plane ($D_a \leq 2$) Eq. (20) suggests that $R_A \geq Q$. Tokunaga (1978) showed that under certain assumptions R_A asymptotically equals Q. The assumptions were: (1) A basin can be divided into infinitesimally small subbasins and interbasin areas; (2) The average area of subbasins of order j, A_j, is larger than the interbasin area $\beta_{\lambda,j}$, defined as the average of interbasin areas draining directly into streams of order λ when the lowest resolved stream is order j; (3) The value of $K/(2+\varepsilon_1)$ is less than 1. These assumptions are required for convergence of geometric series and disappearance of terms containing interbasin areas. It is somewhat theoretically disturbing (at least to me) that the area ratio, a property that relates well defined measures A_w and A_{w-1} which are not dependent on how finely the network is resolved, should depend on the infinitesimal limit for its derivation. Tarboton (1989) shows that under the alternative assumption that link area is on average constant, R_A also asymptotically equals Q. The latter approach can be combined with an assumption of constant average link length to get asymptotically $R_L = K$ and $D_b = \ln Q/\ln K$. This amounts to assuming a constant (spatially uniform) coefficient of channel maintenance, or drainage density. These are two reasonable scenarios under which networks are space filling.

There may also be reasonable alternative scenarios that result in $R_A > Q$ giving networks that do not fill space. The set of points connected to the outlet may form a set with $D_a < 2$. This would imply holes in the network (unchanneled areas) at all scales. Such a scenario would violate the assumptions necessary to get $R_A = Q$ by having large unchanneled interbasin or link areas. One plausible scenario is in basins where the drainage density is dependent on slope. Finely dissected high drainage density headwater basins may flow into open lower drainage density high order basins. Horton's slope law and a physical understanding of the basis for drainage density may hold the key to understanding the scaling of these non space filling networks and the connection between the fractal dimensions of river networks and physical landforming processes.

6. Examples

6.1. Peano network

An example of a network that conforms exactly to Tokunaga cyclicity is the Peano network (Mandelbrot, 1983) analyzed by Marani et al. (1991) illustrated in Fig. 4. It has $\varepsilon_1 = 1$ and $K = 2$ which in Eq. (16) yield $Q = 4$ and $P = 1$. It has $R_L = 2$ and $R_A = 4$. D_b and D_a are both 2 and Eq. (5) gives $D_s = 1$. This is consistent with the Peano network filling space and being comprised of linear, dimension 1 (straight), streams. Horton's bifurcation ratio is not constant for this network. In the lowest order subbasins it is 3, while for the network as a whole it approaches 4.

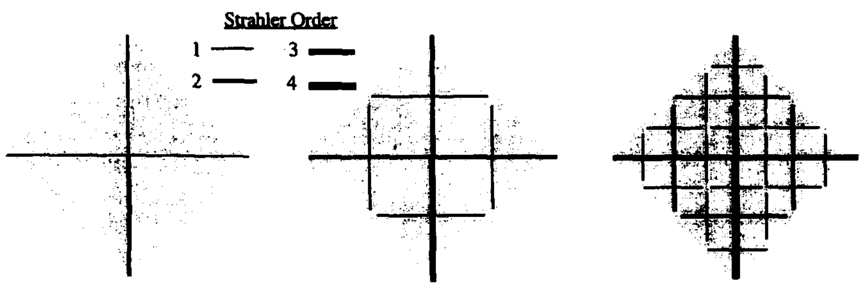

Fig. 4. Peano network (Mandelbrot, 1983; Marani et al., 1991). For every change of ruler scale every link generates four links, two resulting from subdivision in half of the previous link and two new links. This network conforms exactly to Tokunaga cyclicity with $K = 2$ and $\varepsilon_1 = 1$.

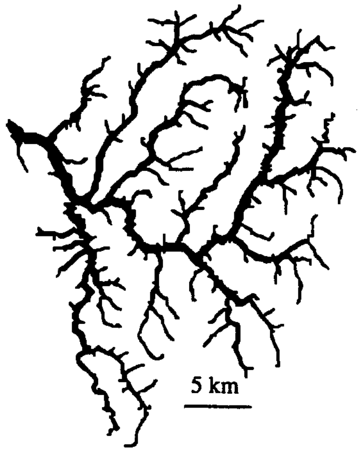

Fig. 5. Buck creek, California, USA.

Table 1
Branching analysis of Buck creek

	Order (w)				
	1	2	3	4	5
Number of streams	164	33	10	2	1
Number of side streams entering order w stream					
Order 1	–	33	30	29	6
Order 2	–	–	7	6	0
Order 3	–	–	–	5	1
Order 4	–	–	–	–	0

	${}_j\varepsilon_{j-1}$	${}_j\varepsilon_{j-2}$	${}_j\varepsilon_{j-3}$	${}_j\varepsilon_{j-4}$
$j = 2$	33/33 = 1	–	–	–
$j = 3$	7/10 = 0.7	30/10 = 3	–	–
$j = 4$	5/2 = 2.5	6/2 = 3	29/2 = 14.5	–
$j = 5$	0/1 = 0	1/1 = 1	0/1 = 0	6/1 = 6
Average[a]	$\varepsilon_1 = 0.98$	$\varepsilon_2 = 2.8$	$\varepsilon_3 = 9.7$	$\varepsilon_4 = 6$

[a] Estimates of ε_k are weighted by the number of receiving streams in the calculation of ${}_j\varepsilon_{j-k}$.

$K = 2.89$ = average of ($\varepsilon_2/\varepsilon_1$, $\varepsilon_3/\varepsilon_2$, $\varepsilon_4/\varepsilon_3$) weighted by the sum of the number of receiving streams used to estimate ε_j and ε_{j-1}.

6.2. Buck creek

Buck creek is a 606 km^2, fifth order basin in Northern California, USA. The stream network (Fig. 5) was extracted from a 30-m USGS digital elevation model (Tarboton, 1989; Tarboton et al., 1991). Horton ratios were estimated as: $R_B = 3.67$, $R_L = 2.2$, $R_A = 4.55$. Table 1 illustrates the procedures for estimating Tokunaga parameters $K = 2.89$ and $\varepsilon_1 = 0.98$. Using these in Eq. (16) gives $Q = 4.62$, $P = 1.25$. With these Eq. (15) is used to calculate the number of streams of each order. This is compared to observations and Horton's bifurcation law in Fig. 6. Based on these parameters one obtains the following fractal dimensions using the Horton system, Eqs. (5)–(7):

$$D_s = 2 \ln R_L / \ln R_A = 1.04$$

$$D_b = \ln R_B / \ln R_L = 1.64$$

$$D_a = 2 \ln R_B / \ln R_A = 1.72$$

By comparison in the system based on Tokunaga's rules, Eq. (5), Eq. (19) and Eq. (20):

$$D_s = 2 \ln R_L / \ln R_A = 1.04$$

$$D_b = \ln Q / \ln R_L = 1.94$$

$$D_a = 2 \ln Q / \ln R_A = 2.02$$

The area dimension D_a for the Horton system is unrealistically low. This network was extracted from a digital elevation model with constant support area used to identify channels. This procedure effectively prescribes constant drainage density and space filling, $D_a = 2$. D_a estimated using Tokunaga cyclicity is closer to 2.

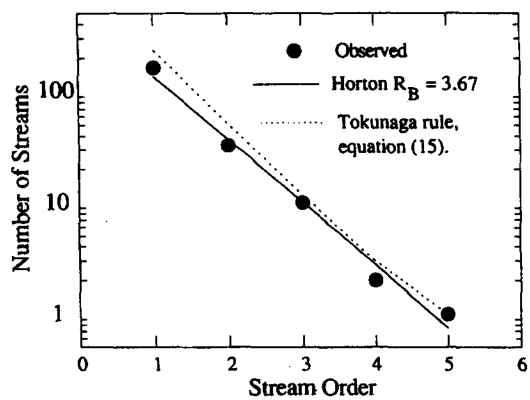

Fig. 6. Buck creek stream number analysis.

7. Conclusions

Fractal dimensions may be used to characterize the branching (D_b) and sinuosity (D_s) of stream networks. The fractal idealization recognizes the self similarity in river networks and extends this to the infinitesimal resolution limit. The river network through branching and sinuosity connects to each point it drains, this connected set having combined dimension $D_a = D_b D_s$ which may be less than or equal to two. In the case where $D_a = 2$ the network is completely connected (to every point) and fills space. Horton's bifurcation and area laws are limited in that they do not admit space filling. Whether or not river networks are space filling, this is a flaw in the theoretical framework founded on Horton's laws. Horton's bifurcation ratio is also biased whenever streams flow into streams more than one order higher. Tokunaga cyclicity generalizes Horton's bifurcation law and overcomes these difficulties. The branching, sinuosity and area fractal dimensions pertaining to river networks in terms of this description of self similarity were presented. It is my opinion that Tokunaga cyclicity offers the possibility for better understanding the geometrical structure and scaling of river networks and their contributing area. Further work is required to refine procedures for estimating Tokunaga cyclicity parameters and check their applicability for estimating fractal dimensions in observed river networks.

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