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A NONCOOPERATIVE MODEL OF NETWORK FORMATION

BY VENKATESH BALA AND SANJEEV GOYAL[1]

We present an approach to network formation based on the notion that social networks are formed by individual decisions that trade off the costs of forming and maintaining links against the potential rewards from doing so. We suppose that a link with another agent allows access, in part and in due course, to the benefits available to the latter via his own links. Thus individual links generate externalities whose value depends on the level of decay/delay associated with indirect links. A distinctive aspect of our approach is that the costs of link formation are incurred only by the person who initiates the link. This allows us to formulate the network formation process as a noncooperative game.

We first provide a characterization of the architecture of equilibrium networks. We then study the dynamics of network formation. We find that individual efforts to access benefits offered by others lead, rapidly, to the emergence of an equilibrium social network, under a variety of circumstances. The limiting networks have simple architectures, e.g., the wheel, the star, or generalizations of these networks. In many cases, such networks are also socially efficient.

KEYWORDS: Coordination, learning dynamics, networks, noncooperative games.

1. INTRODUCTION

THE IMPORTANCE OF SOCIAL AND ECONOMIC networks has been extensively documented in empirical work. In recent years, theoretical models have highlighted their role in explaining phenomena such as stock market volatility, collective action, the career profiles of managers, and the diffusion of new products, technologies and conventions.[2] These findings motivate an examination of the process of network formation.

We consider a setting in which each individual is a source of benefits that others can tap via the formation of costly pairwise links. Our focus is on benefits

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[2] For empirical work see Burt (1992), Coleman (1966), Frenzen and Davis (1990), Granovetter (1974), and Rogers and Kincaid (1981). The theoretical work includes Allen (1982), Anderlini and Ianni (1996), Baker and Iyer (1992), Bala and Goyal (1998), Chwe (1998), Ellison (1993), Ellison and Fudenberg (1993), Goyal and Janssen (1997), and Kirman (1997).

that are *nonrival*.[3] We suppose that a link with another agent allows access, in part and in due course, to the benefits available to the latter via his own links. Thus individual links generate externalities whose value depends on the level of decay/delay associated with indirect links. A distinctive aspect of our approach is that the costs of link formation are incurred only by the person who initiates the link. This allows us to model the network formation process as a noncooperative game, where an agent's strategy is a specification of the set of agents with whom he forms links. The links formed by agents define a social network.[4]

We study both one-way and two-way flow of benefits. In the former case, the link that agent i forms with agent j yields benefits solely to agent i, while in the latter case, the benefits accrue to both agents. In the benchmark model, the benefit flow across persons is assumed to be *frictionless*: if an agent i is linked with some other agent j via a sequence of intermediaries, $\{j_1, \ldots, j_s\}$, then the benefit that i derives from j is insensitive to the number of intermediaries. Apart from this, we allow for a general class of individual payoff functions: the payoff is strictly increasing in the number of other people accessed (directly or indirectly) and strictly decreasing in the number of links formed.

Our first result is that *Nash networks are either connected or empty*.[5] Connectedness is, however, a permissive requirement: for example, with one-way flows a society with 6 agents can have upwards of 20,000 Nash networks representing more than 30 different architectures.[6] This multiplicity of Nash equilibria motivates an examination of a stronger equilibrium concept. If an agent has multiple best responses to the equilibrium strategies of the others, then this may make the network less stable as the agent may be tempted to switch to a payoff-equivalent strategy. This leads us to study the nature of networks that can be supported in a strict Nash equilibrium.

We find that the refinement of strictness is very effective in our setting: *in the one-way flow model*, *the only strict Nash architectures are the wheel and the empty network*. Figure 1A depicts a wheel, which is a network where each agent forms exactly one link, represented by an arrow pointing to the agent. (The arrow also indicates the direction of benefit flow). The empty network is one where there are no links. In the *two-way flow model*, *the only strict Nash architectures are the center-sponsored star and the empty network*. Figure 1B depicts a center-sponsored

[3] Examples include information sharing concerning brands/products among consumers, the opportunities generated by having trade networks, as well as the important advantages arising out of social favors.

[4] The game can be interpreted as saying that agents incur an initial fixed cost of forging links with others—where the cost could be in terms of time, effort, and money. Once in place, the network yields a flow of benefits to its participants.

[5] A network is connected if there is a path between every pair of agents. In recent work on social learning and local interaction, connectedness of the society is a standard assumption; see, e.g., Anderlini and Ianni (1996), Bala and Goyal (1998), Ellison (1993), Ellison and Fudenberg (1993), Goyal and Janssen (1997). Our results may be seen as providing a foundation for this assumption.

[6] Two networks have the same architecture if one network can be obtained from the other by permuting the strategies of agents in the other network.

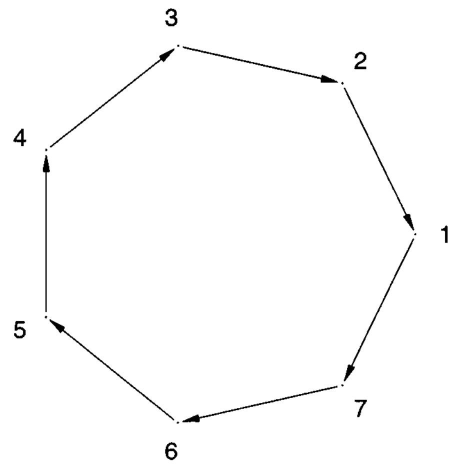

FIGURE 1A.—Wheel network.

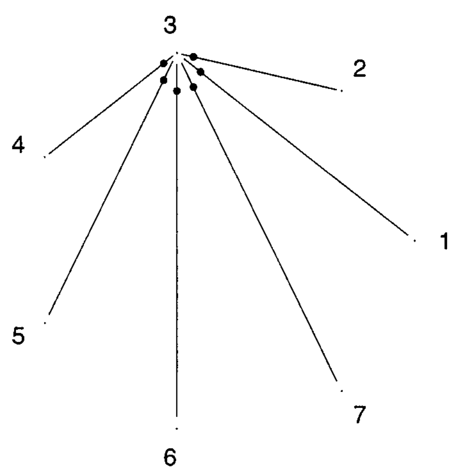

FIGURE 1B.—Center-sponsored star.

star, where one agent forms all the links (agent 3 in the figure, as represented by the filled circles on each link adjacent to this agent).

These results exploit the observation that in a network, if two agents i and j have a link with the same agent k, then one of them (say) i will be indifferent between forming a link with k or instead forming a link with j. We know that Nash networks are either connected or empty. This argument implies that in the one-way flow model a nonempty strict Nash network has exactly n links. Since the wheel is the unique such network, the result follows. In the case of the two-way model, if agent i has a link with j, then no other agent can have a link with j. As a Nash network is connected, this implies that i must be the center of

FIGURE 1C.—Flower and linked star networks.

a star. A further implication of the above observation is that every link in this star must be made or "sponsored" by the center.

While these findings restrict the set of networks sharply, the coordination problem faced by individuals in the network game is not entirely resolved. For example, in the one-way flow model with n agents, there are $(n-1)!$ networks corresponding to the wheel architecture; likewise, there are n networks corresponding to the star architecture. Thus agents have to choose from among these different equilibria. This leads us to study the process by which individuals learn about the network and revise their decisions on link formation, over time.

We use a version of the best-response dynamic to study this issue. The network formation game is played repeatedly, with individuals making investments in link formation in every period. In particular, when making his decision an individual chooses a set of links that maximizes his payoffs given the network of the previous period. Two features of our model are important: one, there is some probability that an individual exhibits *inertia*, i.e., chooses the same strategy as in the previous period. This ensures that agents do not perpetually miscoordinate. Two, if more than one strategy is optimal for some individual, then he *randomizes* across the optimal strategies. This requirement implies, in particular, that a non-strict Nash network can never be a steady state of the dynamics. The rules on individual behavior define a Markov chain on the state space of all networks; moreover, the set of absorbing states of the Markov chain coincides with the set of strict Nash networks of the one-shot game.[7]

Our results establish that the dynamic process converges to a limit network. In the one-way flow model, *for any number of agents and starting from any initial network, the dynamic process converges to a wheel or to the empty network, with probability* 1. The proof exploits the idea that well-connected people generate positive externalities. Fix a network g and suppose that there is an agent i who accesses all people in g, directly or indirectly. Consider an agent j who is not critical for agent i, i.e., agent i is able to access everyone even if agent j deletes

[7] Our rules do not preclude the possibility that the Markov chain cycles permanently without converging to a strict Nash network. In fact, it is easy to construct examples of two-player games with a unique strict Nash equilibrium, where the above dynamic cycles.

all his links. Allow agent j to move; he can form a single link with agent i and access all the different individuals accessed by agent i. Thus if forming links is at all profitable for agent j, then one best-response strategy is to form a single link with agent i. This strategy in turn makes agent j well-connected. We now consider some person k who is not critical for j and apply the same idea. Repeated application of this argument leads to a network in which everyone accesses everyone else via a single link, i.e., a wheel network. We observe that in a large set of cases, in addition to being a limit point of the dynamics, the *wheel is also the unique efficient architecture*.

In the two-way flow model, *for any number of agents and starting from any initial network*, *the dynamic process converges to a center-sponsored star or to the empty network*, *with probability* 1. With two-way flows the extent of the externalities are even greater than in the one-way case since, in principle, a person can access others without incurring any costs himself. We start with an agent i who has the maximum number of direct links. We then show that individual agents who are not directly linked with this agent i will, with positive probability, eventually either form a link with i or vice-versa. Thus, in due course, agent i will become the center of a star.[8] In the event that the star is not already center-sponsored, we show that a certain amount of miscoordination among 'spoke' agents leads to such a star. We also find that *a star is an efficient network for a class of payoff functions*.

The value of the results on the dynamics would be compromised if convergence occurred very slowly. In our setting, there are $2^{n(n-1)}$ networks with n agents. With $n = 8$ agents for example, this amounts to approximately 7×10^{16} networks, which implies that a slow rate of convergence is a real possibility. Our simulations, however, suggest that the *speed of convergence to a limiting network is quite rapid*.

The above results are obtained for a benchmark model with no *frictions*. The introduction of decay/delay complicates the model greatly and we are obliged to work with a linear specification of the payoffs. We suppose that each person potentially offers benefits V and that the cost of forming a link is c. We introduce decay in terms of a parameter $\delta \in [0,1]$. We suppose that if the shortest path from agent j to agent i in a network involves q links, then the value of agent j's benefits to i is given by $\delta^q V$. The model without friction corresponds to $\delta = 1$.

We first show that *in the presence of decay*, *strict Nash networks are connected*. We are, however, unable to provide a characterization of strict Nash and efficient networks, analogous to the case without decay. The main difficulty lies in specifying the agents' best response correspondence. Loosely speaking, in the

[8] It would seem that the center-sponsored star is an attractor because it reduces distance between different agents. However, in the absence of frictions, the distance between agents is not payoff relevant. On the other hand, among the various connected networks that can arise in the dynamics, this network is the only one where a single agent forms all the links, with everyone else behaving as a free rider. This property of the center-sponsored star is crucial.

absence of decay a best response consists of forming links with agents who are connected with the largest number of other individuals. With decay, however, the distances between agents also becomes relevant, so that the entire structure of the network has to be considered. We focus on low levels of decay, where some properties of best responses can be exploited to obtain partial results.

In the one-way flow case, we identify *a class of networks with a flower architecture that is strict Nash* (see left-hand side of Figure 1c). Flower networks trade-off the higher costs of more links (as compared to a wheel) against the benefits of shorter distance between different agents that is made possible by a "central agent." The wheel and the star[9] are special cases of this architecture. In the case of two-way flows, we find that *networks with a single star and linked stars are strict Nash* (see right-hand side of Figure 1c).[10] We also provide a characterization of efficient networks and find that the *star is the unique efficient network for a wide range of parameters*. Simulations of the dynamics for both one-way and two-way models show that convergence to a limit (strict Nash) network is nearly universal and usually occurs very rapidly.

The arguments we develop can be summarized as follows: in settings where potential benefits are widely dispersed, individual efforts to access these benefits lead fairly quickly to the emergence of an equilibrium social network. The limiting networks have simple architectures, e.g., the wheel, the star, or generalizations of these networks. Moreover, in many instances these networks are efficient.

Our paper is a contribution to the theory of network formation. There is a large literature in economics, as well as in computer science, operations research, and sociology on the subject of networks; see, e.g., Burt (1992), Marshak and Radner (1972), Wellman and Berkowitz (1988). Much of this work is concerned with the efficiency aspects of different network structures and takes a planner's viewpoint.[11] By contrast, we consider network formation from the perspective of individual incentives. More specifically, the current paper makes two contributions.

The *first* contribution is our model of link formation. In the work of Boorman (1975), Jackson and Wolinsky (1996), among others, a link between two people requires that both people make some investments and the notion of stable networks therefore rests on pairwise incentive compatibility. We refer to this as a model with *two-sided link formation*. By contrast, in the present paper, link-formation is one-sided and noncooperative: an individual agent can form links with others by incurring some costs. This difference in modelling method-

[9] Star networks can also be defined with one-way flows and should not be confused with the star networks that arise in the two-way flows model.

[10] The latter structure resembles some empirically observed networks, e.g., the communication network in village communities (Rogers and Kincaid (1981, p. 175)).

[11] For recent work in this tradition, see Bolton and Dewatripont (1994) and Radner (1993). Hendricks, Piccione, and Tan (1995) use a similar approach to characterize the optimal flight network for a monopolist.

ology is substantive since it allows the notion of Nash equilibrium and related refinements to be used in the study of network formation.[12]

The difference in formulation also alters the results in important ways. For instance, Jackson and Wolinsky (1996) show that with two-sided link formation the star is efficient but is not stable for a wide range of parameters. By contrast, in our model with noncooperative link formation, we find that the star is the unique efficient network and is also a strict Nash network for a range of values (Propositions 5.3–5.5). To see why this happens, suppose that $V < c$. With two-sided link formation, the central agent in a star will be better off by deleting his link with a spoke agent. In our framework, however, a link can be formed by a 'spoke' agent on his own. If there are enough persons in the society, this will be worthwhile for the 'spoke' agent and a star is sustainable as a Nash equilibrium.

The *second* contribution is the introduction of learning dynamics in the study of network formation.[13] Existing work has examined the relationship between efficient networks and strategically stable networks, in static settings. We believe that there are several reasons why the dynamics are important. One reason is that a dynamic model allows us to study the process by which individual agents learn about the network and adjust their links in response to their learning.[14] Relatedly, dynamics may help select among different equilibria of the static game: the results in this paper illustrate this potential very well.

In recent years, considerable work has been done on the theory of learning in games. One strand of this work studies the myopic best response dynamic; see e.g., Gilboa and Matsui (1991), Hurkens (1995), and Sanchirico (1996), among others. Gilboa and Matsui study the local stability of strategy profiles. Their approach allows for mixing across best responses, but does not allow for transitions from one strategy profile to another based on one player choosing a best response, while all others exhibit inertia. Instead, they require changes in

[12] The model of one-sided and noncooperative link formation was introduced and some preliminary results on the static model were presented in Goyal (1993).

The literature on network games is related to the research in coalition formation in game-theoretic models. This literature is surveyed in Myerson (1991) and van den Nouweland (1993). Jackson and Wolinsky (1996) present a detailed discussion of the relationship between the two research programs. Dutta and Mutuswamy (1997) and Kranton and Minehart (1998) are some other recent papers on network formation. An alternative approach is presented in a recent paper by Mailath, Samuelson, and Shaked (1996), which explores endogenous structures in the context of agents who play a game after being matched. They show that partitions of society into groups with different payoffs can be evolutionary stable.

[13] Bala (1996) initially proposed the use of dynamics to select across Nash equilibria in a network context and obtained some preliminary results.

[14] Two earlier papers have studied network evolution, but in quite different contexts from the model here. Roth and Vande Vate (1990) study dynamics in a two-sided matching model. Linhart, Lubachevsky, Radner, and Meurer (1994) study the evolution of the subscriber bases of telephone companies in response to network externalities created by their pricing policies.

social behavior to be continuous.[15] This difference with our formulation is significant. They show that every strict Nash equilibrium is a socially stable strategy, but that the converse is not true. This is because in some games a Nash equilibrium in mixed strategies is socially stable. By contrast, under our dynamic process, the set of strict Nash networks is equivalent to the set of absorbing networks.

Hurkens (1995) and Sanchirico (1996) study variants of best response learning in general games. They show that if the dynamic process satisfies certain properties, which include randomization across best responses, then it 'converges' to a minimal curb set, i.e., a set that is closed under the best response operation, in the long run. These results imply that weak Nash equilibria are not limit points of the dynamic process. However, in general games, a minimal curb set often consists of more than one strategy profile and there are usually several such sets. The games we analyze are quite large and the main issue here is the nature of minimal curb sets. Our results characterize these sets as well as show convergence of the dynamics.[16]

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 analyzes the case of one-way flows, while Section 4 considers the case of two-way flows. Section 5 studies network formation in the presence of decay. Section 6 concludes.

2. THE MODEL

Let $N = \{1, \ldots, n\}$ be a set of agents and let i and j be typical members of this set. To avoid trivialities, we shall assume throughout that $n \geq 3$. For concreteness in what follows, we shall use the example of gains from information sharing as the source of benefits. Each agent is assumed to possess some information of value to himself and to other agents. He can augment this information by communicating with other people; this communication takes resources, time, and effort and is made possible via the setting up of *pair-wise* links.

A *strategy* of agent $i \in N$ is a (row) vector $g_i = (g_{i,1}, \ldots, g_{i,i-1}, g_{i,i+1}, \ldots, g_{i,n})$ where $g_{i,j} \in \{0, 1\}$ for each $j \in N \setminus \{i\}$. We say agent i has a *link* with j if $g_{i,j} = 1$. A link between agent i and j can allow for either one-way (asymmetric) or two-way (symmetric) flow of information. With one-way communication, the link $g_{i,j} = 1$ enables agent i to access j's information, but not vice-versa.[17] With two-way communication, $g_{i,j} = 1$ allows both i and j to access each other's

[15] Specifically, they propose that a strategy profile s is accessible from another strategy profile s' if there is a continuous smooth path leading from s' to s that satisfies the following property: at each strategy profile along the path, the direction of movement is consistent with each of the different players choosing one of their best responses to the current strategy profile. A set of strategy profiles S is 'stable' if no strategy profile $s' \notin S$ is accessible from any strategy profile $s \in S$, and each strategy profile in S is accessible from every other strategy profile in S.

[16] For a survey of recent research on learning in games, see Marimon (1997).

[17] For example, i could access j's website, or read a paper written by j.

information.[18] The set of all strategies of agent i is denoted by $\mathscr{G}_i$. Throughout the paper we restrict our attention to pure strategies. Since agent i has the option of forming or not forming a link with each of the remaining $n-1$ agents, the number of strategies of agent i is clearly $|\mathscr{G}_i| = 2^{n-1}$. The set $\mathscr{G} = \mathscr{G}_1 \times \cdots \times \mathscr{G}_n$ is the space of pure strategies of all the agents. We now consider the game played by the agents under the two alternative assumptions concerning information flow.

2.1. *One-way Flow*

In the one-way flow model, we can depict a strategy profile $g = (g_1, \ldots, g_n)$ in $\mathscr{G}$ as a *directed network*. The link $g_{i,j} = 1$ is represented by an *edge* starting at j with the arrowhead pointing at i. Figure 2A provides an example with $n = 3$ agents. Here agent 1 has formed links with agents 2 and 3, agent 3 has a link with agent 1 while agent 2 does not link up with any other agent. Note that there is a one-to-one correspondence between the set of all directed networks with n vertices and the set $\mathscr{G}$.

Define $N^d(i; g) = \{k \in N | g_{i,k} = 1\}$ as the set of agents with whom i maintains a link. We say there is a *path* from j to i in g either if $g_{i,j} = 1$ or there exist distinct agents $j_1, \ldots, j_m$ different from i and j such that $g_{i,j_1} = g_{j_1,j_2} = \cdots = g_{j_m,j} = 1$. For example, in Figure 2A there is a path from agent 2 to agent 3. The notation "$j \overset{g}{\to} i$" indicates that there exists a path from j to i in g. Furthermore, we define $N(i; g) = \{k \in N | k \overset{g}{\to} i\} \cup \{i\}$. This is the set of all agents whose information i accesses either through a link or through a sequence of links. We shall typically refer to $N(i; g)$ as the set of agents who are observed by i. We use the convention that $i \in N(i; g)$, i.e. agent i observes himself. Let $\mu_i^d : \mathscr{G} \to \{0, \ldots, n-1\}$ and $\mu_i : \mathscr{G} \to \{1, \ldots, n\}$ be defined as $\mu_i^d(g) \equiv |N^d(i; g)|$ and $\mu_i(g) \equiv |N(i; g)|$ for $g \in \mathscr{G}$. Here, $\mu_i^d(g)$ is the number of agents with whom i has formed links while $\mu_i(g)$ is the number of agents observed by agent i.

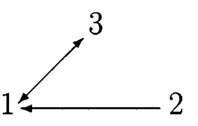

FIGURE 2A

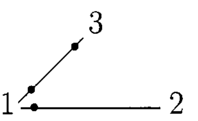

FIGURE 2B

[18] Thus, i could make a telephone call to j, after which there is information flow in both directions.

To complete the definition of a normal-form game of network formation, we specify a class of payoff functions. Denote the set of nonnegative integers by Z_+. Let $\Phi : \mathscr{Z}_+^2 \to \mathscr{R}$ be such that $\Phi(x, y)$ is strictly increasing in x and strictly decreasing in y. Define each agent's payoff function $\Pi_i : \mathscr{G} \to \mathscr{R}$ as

$$\Pi_i(g) = \Phi(\mu_i(g), \mu_i^d(g)). \tag{2.1}$$

Given the properties we have assumed for the function Φ, $\mu_i(g)$ can be interpreted as providing the "benefit" that agent i receives from his links, while $\mu_i^d(g)$ measures the "cost" associated with maintaining them.

The payoff function in (2.1) implicitly assumes that the value of information does not depend upon the number of individuals through which it has passed, i.e., that there is no information decay or delay in transmission. We explore the consequences of relaxing this assumption in Section 5.

A special case of (2.1) is when payoffs are linear. To define this, we specify two parameters $V > 0$ and $c > 0$, where V is regarded as the *value* of each agent's information (to himself and to others), while c is his *cost* of link formation. Without loss of generality, V can be normalized to 1. We now define $\Phi(x, y) = x - yc$, i.e.

$$\Pi_i(g) = \mu_i(g) - \mu_i^d(g)c. \tag{2.2}$$

In other words, agent i's payoff is the number of agents he observes less the total cost of link formation. We identify three parameter ranges of importance. If $c \in (0, 1)$, then agent i will be willing to form a link with agent j for the sake of j's information alone. When $c \in (1, n-1)$, agent i will require j to observe some additional agents to induce him to form a link with j. Finally, if $c > n-1$, then the cost of link formation exceeds the total benefit of information available from the rest of society. Here, it is a dominant strategy for i not to form a link with any agent.

2.2. *Two-way Flow*

In the two-way flow model, we depict the strategy profile $g = (g_1, \ldots, g_n)$ as a *nondirected network*. The link $g_{i,j} = 1$ is represented by an *edge* between i and j: a filled circle lying on the edge near agent i indicates that it is this agent who has initiated the link. Figure 2B below depicts the example of Figure 2A for the two-way model. As before, agent 1 has formed links with agents 2 and 3, agent 3 has formed a link with agent 1 while agent 2 does not link up with any other agent.[19] Every strategy-tuple $g \in \mathscr{G}$ has a unique representation in the manner shown in the figure.

To describe information flows formally, it is useful to define the *closure* of g: this is a nondirected network denoted $\bar{g} = \text{cl}(g)$, and defined by $\bar{g}_{i,j} =$

[19] Since agents choose strategies independently of each other, two agents may simultaneously initiate a two-way link, as seen in the figure.

$\max\{g_{i,j}, g_{j,i}\}$ for each i and j in N.[20] We say there is a *tw-path* (for two-way) in g between i and j if either $\bar{g}_{i,j} = 1$ or there exist agents $j_1, \ldots, j_m$ distinct from each other and i and j such that $\bar{g}_{i,j_1} = \cdots = \bar{g}_{j_m,j} = 1$. We write $i \overset{\bar{g}}{\leftrightarrow} j$ to indicate a tw-path between i and j in g. Let $N^d(i; g)$ and $\mu_i^d(g)$ be defined as in Section 2.1. The set $N(i; \bar{g}) = \{k \mid i \overset{g}{\leftrightarrow} k\} \cup \{i\}$ consists of agents that i observes in g under two-way communication, while $\mu_i(\bar{g}) \equiv |N(i; \bar{g})|$ is its cardinality. The payoff accruing to agent i in the network g is defined as

$$\overline{\Pi}_i(g) = \Phi(\mu_i(\bar{g}), \mu_i^d(g)), \tag{2.3}$$

where $\Phi(\cdot, \cdot)$ is as in Section 2.1. The case of linear payoffs is $\Phi(x, y) = x - yc$ as before. We obtain, analogously to (2.2):

$$\overline{\Pi}_i(g) = \mu_i(\bar{g}) - \mu_i^d(g)c. \tag{2.4}$$

The parameter ranges $c \in (0,1)$, $c \in (1, n-1)$, and $c > n-1$ have the same interpretation as in Section 2.1.

2.3. *Nash and Efficient Networks*

Given a network $g \in \mathcal{G}$, let g_{-i} denote the network obtained when all of agent i's links are removed. The network g can be written as $g = g_i \oplus g_{-i}$ where the '$\oplus$' indicates that g is formed as the union of the links in g_i and g_{-i}. Under one-way communication, the strategy g_i is said to be a *best-response* of agent i to g_{-i} if

$$\Pi_i(g_i \oplus g_{-i}) \geq \Pi_i(g_i' \oplus g_{-i}), \qquad \text{for all } g_i' \in \mathcal{G}_i. \tag{2.5}$$

The set of all of agent i's best responses to g_{-i} is denoted $BR_i(g_{-i})$. Furthermore, a network $g = (g_1, \ldots, g_n)$ is said to be a *Nash network* if $g_i \in BR_i(g_{-i})$ for each i, i.e. agents are playing a Nash equilibrium. A *strict* Nash network is one where each agent gets a strictly higher payoff with his current strategy than he would with any other strategy. For two-way communication, the definitions are the same, except that $\overline{\Pi}_i$ replaces Π_i everywhere. The best-response mapping is likewise denoted by $\overline{BR}_i(\cdot)$.

We shall define our welfare measure in terms of the sum of payoffs of all agents. Formally, let $W: \mathcal{G} \to \mathcal{R}$ be defined as $W(g) = \sum_{i=1}^n \Pi_i(g)$ for $g \in \mathcal{G}$. A network g is *efficient* if $W(g) \geq W(g')$ for all $g' \in \mathcal{G}$. The corresponding welfare function for two-way communication is denoted $\overline{W}$. For the linear payoffs specified in (2.2) and (2.4), an efficient network is one that maximizes the total value of information made available to the agents, less the aggregate cost of communication.

Two networks $g \in \mathcal{G}$ and $g' \in \mathcal{G}$ are equivalent if g' is obtained as a permutation of the strategies of agents in g. For example, if g is the network in Figure 2A, and g' is the network where agents 1 and 2 are interchanged, then g

[20] Note that $\bar{g}_{i,j} = \bar{g}_{j,i}$ so that the order of the agents is irrelevant.

and g' are equivalent. The equivalence relation partitions $\mathscr{G}$ into classes: each class is referred to as an *architecture*.[21]

2.4. *The Dynamic Process*

We describe a simple process that is a modified version of naive best response dynamics. The network formation game is assumed to be repeated in each time period $t = 1, 2, \ldots$. In each period $t \geq 2$, each agent observes the network of the previous period.[22] With some fixed probability $r_i \in (0, 1)$, agent i is assumed to exhibit 'inertia', i.e. he maintains the strategy chosen in the previous period. Furthermore, if the agent does not exhibit inertia, which happens with probability $p_i = 1 - r_i$, he chooses a myopic pure strategy best response to the strategy of all other agents in the previous period. If there is more than one best response, each of them is assumed to be chosen with positive probability. The last assumption introduces a certain degree of 'mixing' in the dynamic process and in particular rules out the possibility that a weak Nash equilibrium is an absorbing state.[23]

Formally, for a given set A, let $\Delta(A)$ denote the set of probability distributions on A. We suppose that for each agent i there exists a number $p_i \in (0,1)$ and a function $\phi_i : \mathscr{G} \to \Delta(\mathscr{G}_i)$ where ϕ_i satisfies, for all $g = g_i \oplus g_{-i} \in \mathscr{G}$:

$$(2.6) \qquad \phi_i(g) \in \text{Interior } \Delta(BR_i(g_{-i})).$$

For $\hat{g}_i$ in the support of $\phi_i(g)$, the notation $\phi_i(g)(\hat{g}_i)$ denotes the probability assigned to $\hat{g}_i$ by the probability measure $\phi_i(g)$. If the network at time $t \geq 1$ is

[21] For example, consider the one-way flow model. There are n possible 'star' networks, all of which come under the equivalence class of the star architecture. Likewise, the wheel architecture is the equivalence class of $(n-1)!$ networks consisting of all permutations of n agents in a circle.

[22] As compared to models where, say, agents are randomly drawn from large populations to play a two-player game, the informational requirements for agents to compute a best response here are much higher. This is because the links formed by a single agent can be crucial in determining a best response. Some of our results on the dynamics can be obtained under somewhat weaker requirements. For instance, in the one-way flow model, the results carry over if, in a network g, an agent i knows only the sets $N(k; g_{-i})$, and not the structure of links of every other agent k in the society. Further analysis under alternative informational assumptions is available in a working paper version, which is available from the authors upon request.

[23] We can interpret the dynamics as saying that the links of the one-shot game, while durable, must be renewed at the end of each period by fresh investments in social relationships. An alternative interpretation is in terms of a fixed-size overlapping-generations population. At regular intervals, some of the individuals exit and are replaced by an equal number of new people. (In this context, p_i is the probability that an agent is replaced by a new agent.) Upon entry an agent looks around and informs himself about the connections among the set of agents. He then chooses a set of people and forms links with them, with a view to maximizing his payoffs. In every period that he is around, he renews these links via regular investments in personal relations. This models the link formation behavior of students in a school, managers entering a new organization, or families in a social setting.

$g^t = g_i^t \oplus g_{-i}^t$, the strategy of agent i at time $t+1$ is assumed to be given by

$$(2.7) \qquad g_i^{t+1} = \begin{cases} \hat{g}_i \in \text{support } \phi_i(g), & \text{with probability } p_i \times \phi_i(g)(\hat{g}_i), \\ g_i^t, & \text{with probability } 1 - p_i. \end{cases}$$

Equation (2.7) states that with probability $p_i \in (0,1)$, agent i chooses a naive best response to the strategies of the other agents. It is important to note that under this specification, an agent may switch his strategy (to another best-response strategy) even if he is currently playing a best-response to the existing strategy profile. The function ϕ_i defines how agent i randomizes between best responses if more than one exists. Furthermore, with probability $1 - p_i$ agent i exhibits 'inertia', i.e. maintains his previous strategy.

We assume that the choice of inertia as well as the randomization over best responses by different agents is independent across agents. Thus our decision rules induce a transition matrix T mapping the state space $\mathcal{G}$ to the set of all probability distributions $\Delta(\mathcal{G})$ on $\mathcal{G}$. Let $\{X_t\}$ be the stationary Markov chain starting from the initial network $g \in \mathcal{G}$ with the above transition matrix. The process $\{X_t\}$ describes the dynamics of network evolution given our assumptions on agent behavior.

The dynamic process in the two-way model is the same except that we use the best-response mapping $\overline{BR}_i(\cdot)$ instead of $BR_i(\cdot)$.

3. THE ONE-WAY FLOW MODEL

In this section, we analyze the nature of network formation when information flow is one-way. Our results provide a characterization of strict Nash and efficient networks and also show that the dynamic process converges to a limit network, which is a strict Nash network, in all cases.

3.1. *Static Properties*

Given a network g, a set $C \subset N$ is called a *component* of g if for every distinct pair of agents i and j in C we have $j \overset{g}{\to} i$ (equivalently, $j \in N(i; g)$) and there is no strict superset C' of C for which this is true. A component C is said to be *minimal* if C is no longer a component upon replacement of a link $g_{i,j} = 1$ between two agents i and j in C by $g_{i,j} = 0$, ceteris paribus. A network g is said to be *connected* if it has a unique component. If the unique component is minimal, g is called *minimally connected*. A network that is not connected is referred to as disconnected. A network is said to be *empty* if $N(i; g) = \{i\}$ and it is called *complete* if $N^d(i; g) = N \setminus \{i\}$ for all $i \in N$. We denote the empty and the complete network by g^e and g^c, respectively. A *wheel* network is one where the agents are arranged as $\{i_1, \ldots, i_n\}$ with $g_{i_2,i_1} = \cdots = g_{i_n,i_{n-1}} = g_{i_1,i_n} = 1$ and there are no other links. The wheel network is denoted g^w. A *star* network has a central agent i such that $g_{i,j} = g_{j,i} = 1$ for all $j \in N \setminus \{i\}$ and no other links.

The (geodesic) distance from agent j to agent i in g is the number of links in the shortest path from j to i, and is denoted $d(i,j;g)$. We set $d(i,j;g)=\infty$ if there is no path from j to i in g. These definitions are taken from Bollobas (1978).

Our first result highlights a general property of Nash networks when agents are symmetrically positioned vis-a-vis information and the costs of access: in equilibrium, either there is no social communication or every agent has access to all the information in the society.

PROPOSITION 3.1: *Let the payoffs be given by* (2.1). *A Nash network is either empty or minimally connected.*

The proof is given in Appendix A; the intuition is as follows. Consider a nonempty Nash network, and suppose that agent i is the agent who observes the largest number of agents in this network. Suppose i does not observe everyone. Then there is some agent j who is not observed by i and who does not observe i (for otherwise j would observe more agents than i). Since i gets values from his links, and payoffs are symmetric, j must also have some links. Let j deviate from his Nash strategy by forming a link with i alone. By doing so, j will observe strictly more agents than i does, since he has the additional benefit of observing i. Since j was observing no more agents than i in his original strategy, j increases his payoff by his deviation. The contradiction implies that i must observe every agent in the society. We then show that every other agent will have an incentive to either link with i or to observe him through a sequence of links, so that the network is connected. If the network is not minimally connected, then some agent could delete a link and still observe all agents, which would contradict Nash.

Figures 3A and 3B depict examples of Nash networks in the linear payoffs case specified by (2.2) with $c \in (0,1)$. The number of Nash networks increases quite rapidly with n; for example, we compute that there are 5, 58, 1069, and in excess of 20,000 Nash networks as n takes on values of 3, 4, 5, and 6, respectively.

A Nash network in which some agent has multiple best responses is likely to be unstable since this agent can decide to switch to another payoff-equivalent

FIGURE 3A.—The star and the wheel (one-way model).

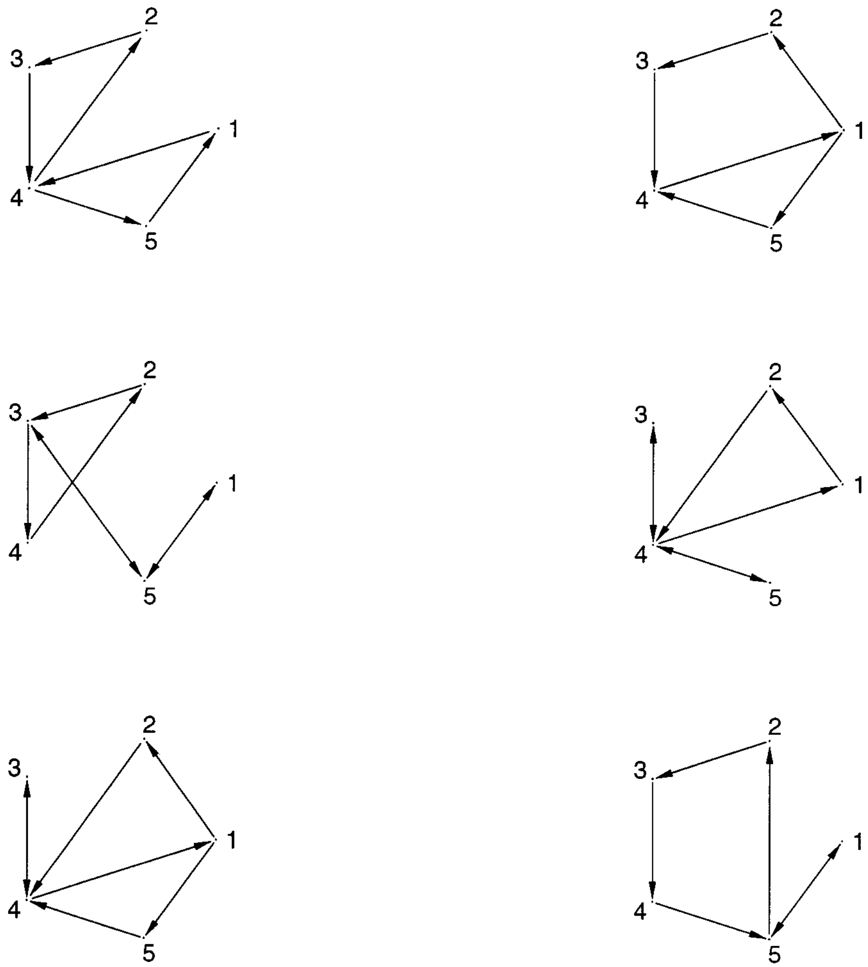

FIGURE 3B.—Other Nash networks.

strategy. This motivates an examination of strict Nash networks. It turns out there are only two possible architectures for such networks.

PROPOSITION 3.2: *Let the payoffs be given by* (2.1). *A strict Nash network is either the wheel or the empty network.* (*a*) *If* $\Phi(\hat{x}+1,\hat{x}) > \Phi(1,0)$ *for some* $\hat{x} \in \{1,\ldots,n-1\}$, *then the wheel is the unique strict Nash.* (*b*) *If* $\Phi(x+1,x) < \Phi(1,0)$ *for all* $x \in \{1,\ldots,n-1\}$ *and* $\Phi(n,1) > \Phi(1,0)$, *then the empty network and the wheel are both strict Nash.* (*c*) *If* $\Phi(x+1,x) < \Phi(1,0)$ *holds for all* $x \in \{1,\ldots,n-1\}$ *and* $\Phi(n,1) < \Phi(1,0)$, *then the empty network is the unique strict Nash.*

PROOF: Let $g \in \mathscr{G}$ be strict Nash, and assume it is not the empty network. We show that for each agent k there is one and only one agent i such that $g_{i,k} = 1$. Since g is Nash, it is minimally connected by Proposition 3.1. Hence there is an agent i who has a link with k. Suppose there exists another agent j such that $g_{j,k} = 1$. As g is minimal we have $g_{i,j} = 0$, for otherwise i could delete the link with k and g would still be connected. Let $\hat{g}_i$ be the strategy where i deletes his link with k and forms one with j instead, ceteris paribus. Define $\hat{g} = \hat{g}_i \oplus g_{-i}$, where $\hat{g} \neq g$. Then $\mu_i^d(\hat{g}) = \mu_i^d(g)$. Furthermore, since $k \in N^d(j; g) = N^d(j; \hat{g})$, clearly $\mu_i(\hat{g}) \geq \mu_i(g)$ as well. Hence i will do at least as well with the strategy $\hat{g}_i$ as with his earlier strategy g_i, which violates the hypothesis that g_i is the unique best response to g_{-i}. As each agent has exactly one other agent who has a link with him, g has exactly n links. It is straightforward to show that the only connected network with n links is the wheel. Parts (a)–(c) now follow by direct verification. *Q.E.D.*

For the linear payoff case $\Pi_i(g) = \mu_i(g) - \mu_i^d(g)c$ of (2.2), Proposition 3.2(a) reduces to saying that the wheel is the unique strict Nash when $c \in (0,1]$. Proposition 3.2(b) implies that the wheel and the empty network are strict Nash in the region $c \in (1, n-1)$, while Proposition 3.2(c) implies that the empty network is the unique strict Nash when $c > n-1$. The final result in this subsection characterizes efficient networks.

PROPOSITION 3.3: *Let the payoffs be given by* (2.1). (*a*) *If* $\Phi(n,1) > \Phi(1,0)$, *then the wheel is the unique efficient architecture, while* (*b*) *if* $\Phi(n,1) < \Phi(1,0)$, *then the empty network is the unique efficient architecture.*

PROOF: Consider part (a) first. Let Γ be the set of values $(\mu_i(g), \mu_i^d(g))$ as g ranges over $\mathscr{G}$. If $\mu_i^d(g) = 0$, then $\mu_i(g) = 1$, while if $\mu_i^d(g) \in \{1, \dots, n-1\}$, then $\mu_i(g) \in \{\mu_i^d(g) + 1, n\}$. Thus, $\Gamma \subset \{1, \dots, n\} \times \{1, \dots, n-1\} \cup \{(1,0)\}$. Given $(x, y) \in \Gamma \setminus \{(1,0)\}$, we have $\Phi(n,1) \geq \Phi(n, y) \geq \Phi(x, y)$ since Φ is decreasing in its second argument and increasing in its first. For the wheel network g^w, note that $\mu_i(g^w) = n$ and $\mu_i^d(g^w) = 1$. Next consider a network $g \neq g^w$: for each $i \in N$, if $\mu_i^d(g) \geq 1$, then $\mu_i(g) \leq n$, while if $\mu_i^d(g) = 0$, then $\mu_i(g) = 1$. In either case,

$$\Pi_i(g^w) = \Phi(n,1) \geq \Phi(\mu_i(g), \mu_i^d(g)) = \Pi_i(g), \tag{3.1}$$

where we have used the assumption that $\Phi(n,1) > \Phi(1,0)$. It follows that $W(g^w) = \sum_{i \in N} \Phi(n,1) \geq \sum_{i \in N} \Phi(\mu_i(g), \mu_i^d(g)) = W(g)$ as well. Thus g^w is an efficient architecture. To show uniqueness, note that our assumptions on Φ imply that equation (3.1) holds with strict inequality if $\mu_i^d(g) \neq 1$ or if $\mu_i(g) < n$. Let $g \neq g^w$ be given; if $\mu_i^d(g) \neq 1$ for even one i, then the inequality (3.1) is strict, and $W(g^w) > W(g)$. On the other hand, suppose $\mu_i^d(g) = 1$ for all $i \in N$. As the wheel is the only connected network with n agents, and $g \neq g^w$, there must be an agent j such that $\mu_j(g) < n$. Thus, (3.1) is again a strict inequality for agent j and $W(g^w) > W(g)$, proving uniqueness.

In part (b), let g be different from the empty network g^e. Then there exists some agent j such that $\mu_j^d(g) \geq 1$. For this agent $\Pi_j(g^e) = \Phi(1,0) > \Phi(n,1) \geq \Phi(\mu_j(g), \mu_j^d(g)) = \Pi_j(g)$ while for all other agents i, $\Pi_i(g^e) = \Phi(1,0) \geq \Phi(\mu_i(g), \mu_i^d(g)) = \Pi_i(g)$. The result follows by summation. *Q.E.D.*

3.2. *Dynamics*

To get a first impression of the dynamics, we simulate a sample trajectory with $n = 5$ agents, for a total of twelve periods (Figure 4).[24] As can be seen, the choices of agents evolve rapidly and settle down by period 11: the limit network is a wheel.

The above simulation raises an interesting question: under what conditions—on the structure of payoffs, the size of the society, and the initial network—does the dynamic process converge? Convergence of the process, if and when it occurs, is quite appealing from an economic perspective since it implies that agents who are myopically pursuing self-interested goals, without any assistance from a central coordinator, are nevertheless able to *evolve* a stable pattern of communication links over time. The following result shows that convergence occurs irrespective of the size of the society or the initial network.

THEOREM 3.1: *Let the payoff functions be given by equation* (2.1) *and let g be the initial network.* (*a*) *If there is some* $\hat{x} \in \{1, \ldots, n-1\}$ *such that* $\Phi(\hat{x}+1, \hat{x}) \geq \Phi(1,0)$, *then the dynamic process converges to the wheel network*, *with probability* 1. (*b*) *If instead*, $\Phi(x+1, x) < \Phi(1,0)$ *for all* $x \in \{1, \ldots, n-1\}$ *and* $\Phi(n,1) > \Phi(1,0)$, *then the process converges to either the wheel or the empty network*, *with probability* 1. (*c*) *Finally*, *if* $\Phi(x+1, x) < \Phi(1,0)$ *for all* $x \in \{1, \ldots, n-1\}$ *and* $\Phi(n,1) < \Phi(1,0)$, *then the process converges to the empty network*, *with probability* 1.

PROOF: The proof relies on showing that given an arbitrary network g there is a positive probability of transiting to a strict Nash network in finite time, when agents follow the rules of the process. As strict Nash networks are absorbing states, the result will then follow from the standard theory of Markov chains. By (2.7) there is a positive probability that all but one agent will exhibit inertia in a given period. Hence the proof will follow if we can specify a sequence of networks where at each stage of the sequence only one (suitably chosen) agent selects a best response. In what follows, unless specified otherwise, when we allow an agent to choose a best response, we implicitly assume that all other agents exhibit inertia.

[24] We suppose that payoffs have the linear specification (2.2) and that $c \in (0,1)$. The initial network (labelled $t = 1$) has been drawn at random from the set of all directed networks with 5 agents. In period $t \geq 2$, the choices of agents who exhibit inertia have been drawn with solid lines, while the links of those who have actively chosen a best response are drawn with dashed lines.

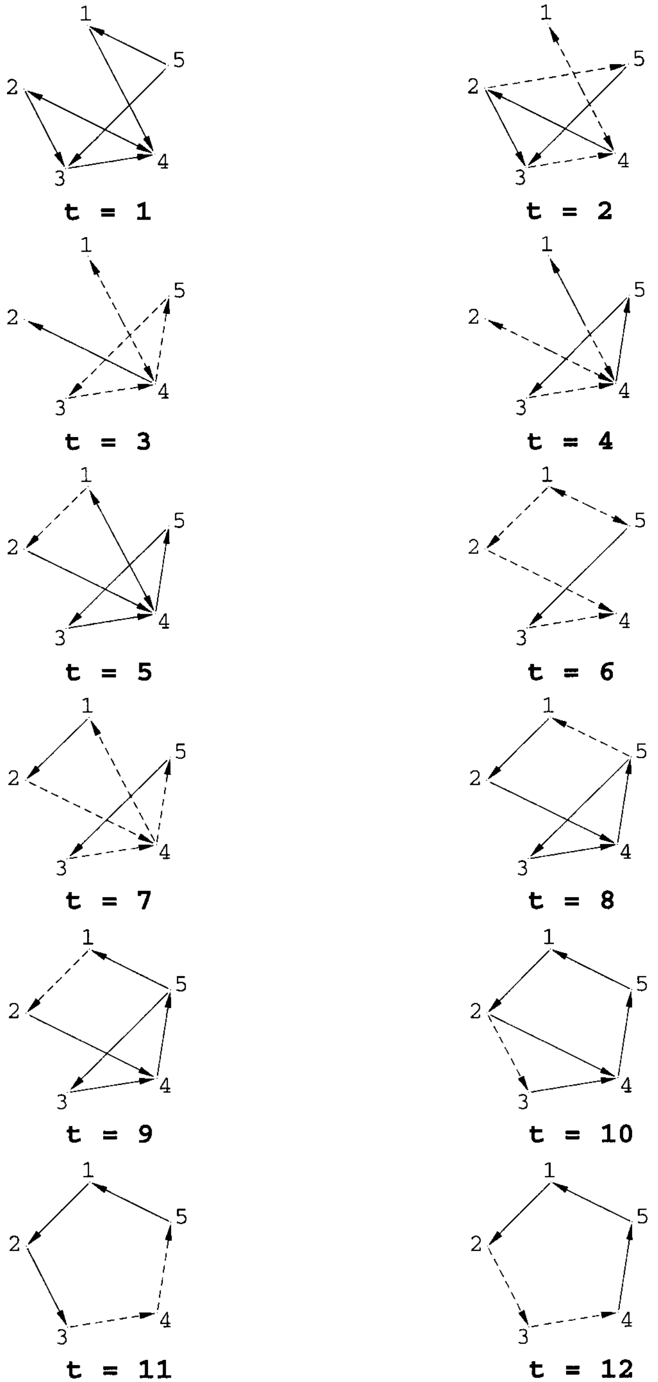

FIGURE 4.—Sample path (one-way model).

We consider part (a) first.[25] Assume initially that there exists an agent j_1 for whom $\mu_{j_1}(g) = n$, i.e. j_1 observes all the agents in the society. Let $j_2 \in \text{argmax}_{m \in N} d(j_1, m; g)$. In words, j_2 is an agent furthest away from j_1 in g. In particular, this means that for each $i \in N$ we have $i \xrightarrow{g_{-j_2}} j_1$, i.e. agent j_1 observes every agent in the society without using any of j_2's links. Let j_2 now choose a best response. Note that a single link with agent j_1 suffices for j_2 to observe all the agents in the society, since $i \xrightarrow{g_{-j_2}} j_1$ for all $i \in N \setminus \{j_1, j_2\}$. Furthermore, as $\Phi(n,1) \geq \Phi(\hat{x}+1,1) \geq \Phi(\hat{x}+1,\hat{x}) \geq \Phi(1,0)$, forming a link with j_1 (weakly) dominates not having any links at all for j_2. Thus, j_2 has a best response $\hat{g}_{j_2}$ of the form $\hat{g}_{j_2,j_1} = 1$, $\hat{g}_{j_2,m} = 0$ for all $m \neq j_1$. Let agent j_2 play this best response. Denote the resulting network as g^1 where $g^1 = \hat{g}_{j_2} \oplus g_{-j_2}$. Note that the property $i \xrightarrow{g^1} j_1$ for all $i \in N$ holds for this network.

More generally, fix s satisfying $2 \leq s \leq n-1$, and let g^{s-1} be the following network: there are s distinct agents $j_1, \ldots, j_s$ such that for each $q \in \{2, \ldots, s\}$ we have $g^{s-1}_{j_q, j_{q-1}} = 1$ and $g^{s-1}_{j_q, m} = 0$ for all $m \neq j_{q-1}$, and furthermore, $i \xrightarrow{g^{s-1}} j_1$ for all $i \in N$. Choose j_{s+1} as follows:

$$j_{s+1} \in \text{argmax}_{m \in N \setminus \{j_1, \ldots, j_s\}} d(j_1, m; g^{s-1}). \tag{3.2}$$

Note that given g^{s-1}, a best response $\hat{g}_{j_{s+1}}$ for j_{s+1} is to form a link with j_s alone. By doing so, he observes $j_s, \ldots, j_1$, and through j_1, the remaining agents in the society as well. Let $g^s = \hat{g}_{j_{s+1}} \oplus g^{s-1}_{-j_{s+1}}$ be the resulting network when j_{s+1} chooses this strategy. Note also that since j_{s+1}'s link formation decision is irrelevant to j_1 observing him, we have $j_{s+1} \xrightarrow{g^s} j_1$, with the same also holding for $j_s, \ldots, j_2$. Thus we can continue the induction. We let the process continue until j_n chooses his best response in the manner above: at this stage, agent j_1 is the only agent with (possibly) more than one link. If agent j_1 is allowed to move again, his best response is to form a single link with j_n, which creates a wheel network g^w. By Proposition 3.2(a), g^w is an absorbing state.

The above argument shows that (a) holds if we assume there is some agent in g who observes the rest of society. We now show that this is without loss of generality. Starting from g, choose an agent i' and let him play a best response $\hat{g}_{i'}$. Label the resulting network $\hat{g}_{i'} \oplus g_{-i'}$ as g'. Note that we can suppose $\mu^d_{i'}(g') \geq 1$. This is because zero links yield a payoff no larger than forming $\hat{x}$ links and observing $\hat{x}+1$ (or more) agents. If $\mu_{i'}(g') = n$ we are done. Otherwise, if $\mu_{i'}(g') < n$, choose $i'' \notin N(i'; g')$ and let him play a best response $\hat{g}_{i''}$. Define $g'' = \hat{g}_{i''} \oplus g'_{-i''}$. As before, we can suppose without loss of generality that $\hat{g}_{i''}$ involves at least one link. We claim that $\mu_{i''}(g'') \geq \mu_{i'}(g') + 1$. Indeed, by forming a link with i', agent i'' can observe i' and all the other agents that i' observes, and thereby guarantee himself a payoff of $\Phi(\mu_{i'}(g') + 1, 1)$. The claim

[25] We thank an anonymous referee for suggesting the following arguments, which greatly simplify our original proof.

now follows because $\Phi(\mu_{i'}(g')+1,1) > \Phi(x,y)$ for any (x,y) pair satisfying $x \leq \mu_{i'}(g')$ and $y \geq 1$. Repeating this argument if necessary, we eventually arrive at a network where some agent observes the entire society, as required.

We now turn to parts (b) and (c). If $\Phi(n,1) < \Phi(1,0)$, it is a dominant strategy for each agent not to form any links. Statement (c) follows easily from this observation. We consider (b) next. Note from Proposition 3.2(b) that the wheel is strict Nash for this payoff regime. Suppose there exists an agent $i \in N$ such that $\mu_i(g) = n$. Then the argument employed in part (a) ensures convergence to the wheel with positive probability. If, instead, $\mu_i(g) < n$, let $\hat{x} \geq 2$ be the largest number such that $\Phi(\hat{x},1) \leq \Phi(1,0)$. Note that $\hat{x} \leq n-1$ since $\Phi(n,1) > \Phi(1,0)$. Suppose there exists $i \in N$ such that $\mu_i(g) \in \{\hat{x}, \ldots, n-1\}$. Then the argument used in the last part of the proof of part (a) can be applied to eventually yield an agent who observes every agent in the society. The last possibility is that for all agents i in g we have $\mu_i(g) < \hat{x}$. Choose an agent i' and consider the network g' formed after he chooses his best response. Suppose $\mu_{i'}^d(g') \geq 1$ and $\mu_{i'}(g') < \hat{x}$. Then $\Pi_{i'}(g') = \Phi(\mu_{i'}(g'), \mu_{i'}^d(g')) < \Phi(\hat{x},1) \leq \Phi(1,0)$ and forming no links does strictly better. Hence, if i' has a best response involving the formation of at least one link, he must observe at least $\hat{x}$ agents (including himself) in the resulting network. Thus we let each agent play in turn—either they will all choose to form no links, in which case the process is absorbed into the empty network, or some agent eventually observes at least $\hat{x}$ agents. In the latter event, we can employ the earlier arguments to show convergence with positive probability to a wheel. *Q.E.D.*

In the case of linear payoffs $\Pi_i(g) = \mu_i(g) - \mu_i^d(g)c$, Theorem 3.1 says that when costs are low ($0 < c \leq 1$) the dynamics converge to the wheel, when costs are in the intermediate range ($1 < c < n-1$), the dynamics converge to either the wheel or the empty network, while if costs are high ($c > n-1$), then the system collapses into the empty network.

Under the hypotheses of Theorem 3.1(b), it is easy to demonstrate *path dependence*, i.e. a positive probability of converging to either the wheel or the empty network from an initial network. Consider a network where agent 1 has $n-1$ links and no other agent has any links. If agent 1 moves first, then $\Phi(x+1,x) < \Phi(1,0)$ for all $x \in \{1,\ldots,n-1\}$ implies that his unique best response is not to form any links, and the process collapses to the empty network. On the other hand, if the remaining agents play one after another in the manner specified by the proof of the above theorem, then convergence to the wheel occurs.

Recall from Proposition 3.3 that when $\Phi(n,1) > \Phi(1,0)$, the unique efficient network is the wheel, while if $\Phi(n,1) < \Phi(1,0)$ the empty network is uniquely efficient. Suppose the condition $\Phi(\hat{x}+1,\hat{x}) \geq \Phi(1,0)$ specified in Theorem 3.1(a) holds. Then as $\Phi(n,1) \geq \Phi(\hat{x}+1,1) \geq \Phi(\hat{x}+1,\hat{x})$ with at least one of these inequalities being strict, we get $\Phi(n,1) > \Phi(1,0)$. Thus we have the following corollary.

COROLLARY 3.1: *Suppose the hypothesis of Theorem* 3.1(*a*) *or Theorem* 3.1(*c*) *holds. Then starting from any initial network, the dynamic process converges to the unique efficient architecture with probability* 1.

Efficiency is not guaranteed in Theorem 3.1(b): while the wheel is uniquely efficient, the dynamics may converge to the empty network instead. However, as the proof of the theorem illustrates, there are many initial networks from which convergence to the efficient architecture occurs with positive probability.

Rates of Convergence: We take payoffs according to the linear model (2.2), i.e. $\Pi_i(g) = \mu_i(g) - \mu_i^d(g)c$. We focus upon two cases: $c \in (0,1)$ and $c \in (1,2)$. In the former case, Theorem 3.1(a) shows that convergence to the wheel always occurs, while in the latter case, Theorem 3.1(b) indicates that either the wheel or the empty network can be the limit.

In the simulations we assume that $p_i = p$ for all agents. Furthermore, let $\hat{\phi}$ be such that it assigns equal probability to all best responses of an agent given a network g. We assume that all agents have the same function $\hat{\phi}$. The initial network is chosen by the method of equiprobable links: a number $k \in \{0, \dots, n(n-1)\}$ is first picked at random, and then the initial network is chosen randomly from the set of all networks having a total of k links.[26] We simulate the dynamic process starting from the initial network until it converges to a limit. Our simulations are with $n = 3$ to $n = 8$ agents, for $p = 0.2$, 0.5, and 0.8. For each (n, p) pair, we run the process for 500 simulations and report the average convergence time. Table I summarizes the results when $c \in (0,1)$ and $c \in (1,2)$. The standard errors are in parentheses.

Table I suggests that the rates of convergence are very rapid. In a society with 8 agents we find that with $p = 0.5$, the process converges to a strict Nash in less than 55 periods on average.[27] Secondly, we find that in virtually all the cases (except for $n = 3$) the average convergence time is higher if $p = 0.8$ or $p = 0.2$

TABLE I

RATES OF CONVERGENCE IN ONE-WAY FLOW MODEL

	$c \in (0,1)$			$c \in (1,2)$		
n	$p = 0.2$	$p = 0.5$	$p = 0.8$	$p = 0.2$	$p = 0.5$	$p = 0.8$
3	15.29(0.53)	7.05(0.19)	6.19(0.19)	8.58(0.35)	4.50(0.17)	5.51(0.24)
4	23.23(0.68)	12.71(0.37)	13.14(0.42)	11.52(0.38)	5.98(0.18)	6.77(0.22)
5	28.92(0.89)	17.82(0.54)	28.99(1.07)	15.19(0.40)	9.16(0.27)	14.04(0.59)
6	38.08(1.02)	26.73(0.91)	55.98(2.30)	19.93(0.57)	12.68(0.41)	28.81(1.16)
7	45.90(1.30)	35.45(1.19)	119.57(5.13)	25.46(0.71)	18.51(0.57)	57.23(2.29)
8	57.37(1.77)	54.02(2.01)	245.70(10.01)	27.74(0.70)	26.24(0.89)	121.99(5.62)

[26] An alternative approach specifies that each network in $\mathcal{G}$ is equally likely to be chosen as the initial one. Simulation results with this approach are similar to the findings reported here.

[27] The precise significance of these numbers depends on the duration of the periods and more generally on the particular application under consideration.

compared to $p = 0.5$. The intuition for this finding is that when p is small, there is a very high probability that the state of the system does not change very much from one period to the next, which raises the convergence time. When p is very large, there is a high probability that "most" agents move simultaneously. This raises the likelihood of miscoordination, which slows the process. The convergence time is thus lowest for intermediate values of p where these two effects are balanced. Thirdly, we find that the average convergence time increases relatively slowly as n increases. So, for instance, as we increase the size of the society from three agents to eight agents, the number of networks increases from 64 to more than 10^{16} networks. Yet the average convergence time (for $p = 0.5$) only increases from around 8 periods to around 54 periods. Finally, we note that the average times are even lower when the communication cost is higher, as seen when $c \in (1, 2)$. This is not simply a reflection of the possibility of absorption into the empty network when $c > 1$: for example, with $n = 8$ this occurred in no more than 3% of all simulations. Instead, it seems to be due to the fact that the set of best responses decreases with higher costs of communication.

4. TWO-WAY FLOW MODEL

In this section, we study network formation when the flow of information is two-way. Our results provide a characterization of strict Nash networks and efficient networks. We also show that the dynamic process converges to a limit network that is a strict Nash network, for a broad class of payoff functions.

4.1. *Static Properties*

Let the network g be given. A set $C \subset N$ is called a *tw-component* of g if for all i and j in C there is a tw-path between them, and there does not exist a tw-path between an agent in C and one in $N \setminus C$. A tw-component C is called *minimal* if (a) there does not exist a *tw-cycle* within C, i.e. $q \geq 3$ agents $\{j_1, \ldots, j_q\} \subset C$ such that $\bar{g}_{j_1, j_2} = \cdots = \bar{g}_{j_q, j_1} = 1$, and (b) $g_{i,j} = 1$ implies $g_{j,i} = 0$ for any pair of agents i, j in C. The network g is called *tw-connected* if it has a unique tw-component C. If the unique tw-component C is minimal, we say that g is *minimally tw-connected*. This implies that there is a unique tw-path between any two agents in N. The *tw-distance* between two agents i and j in g is the length of the shortest tw-path between them, and is denoted by $d(i, j; \bar{g})$. We begin with a preliminary result on the structure of Nash networks.

PROPOSITION 4.1: *Let the payoffs be given by* (2.3). *A Nash network is either empty or minimally tw-connected.*

We make some remarks in relation to the above result. First, by the definition of payoffs, while one agent bears the cost of a link, both agents obtain the benefits associated with it. This asymmetry in payoffs is relevant for defining the

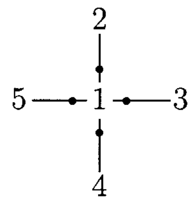

FIGURE 5A.—Center-sponsored.

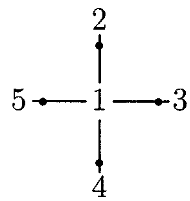

FIGURE 5B.—Periphery-sponsored.

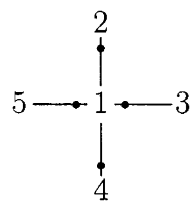

FIGURE 5C.—Mixed-type.

architecture of the network. As an illustration, we note that there are now three types of 'star' networks, depending upon which agents bear the costs of the links in the network. For a society with $n = 5$ agents, Figures 5A–5C illustrate these types. Figure 5A shows a *center-sponsored* star, Figure 5B a *periphery-sponsored* star, and Figure 5C depicts a *mixed-type* star.

Second, there can be a large number of Nash equilibria. For example, consider the linear specification (2.4) with $c \in (0, 1)$. With $n = 3$, 4, 5, and 6 agents there are 12, 128, 2000, and 44352 Nash networks, respectively. Figures 6A and 6B present some examples of Nash networks.

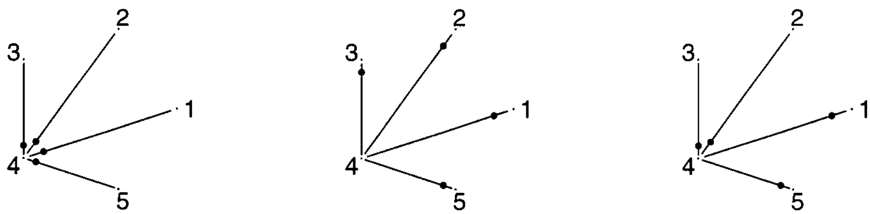

FIGURE 6A.—Star networks (two-way model).

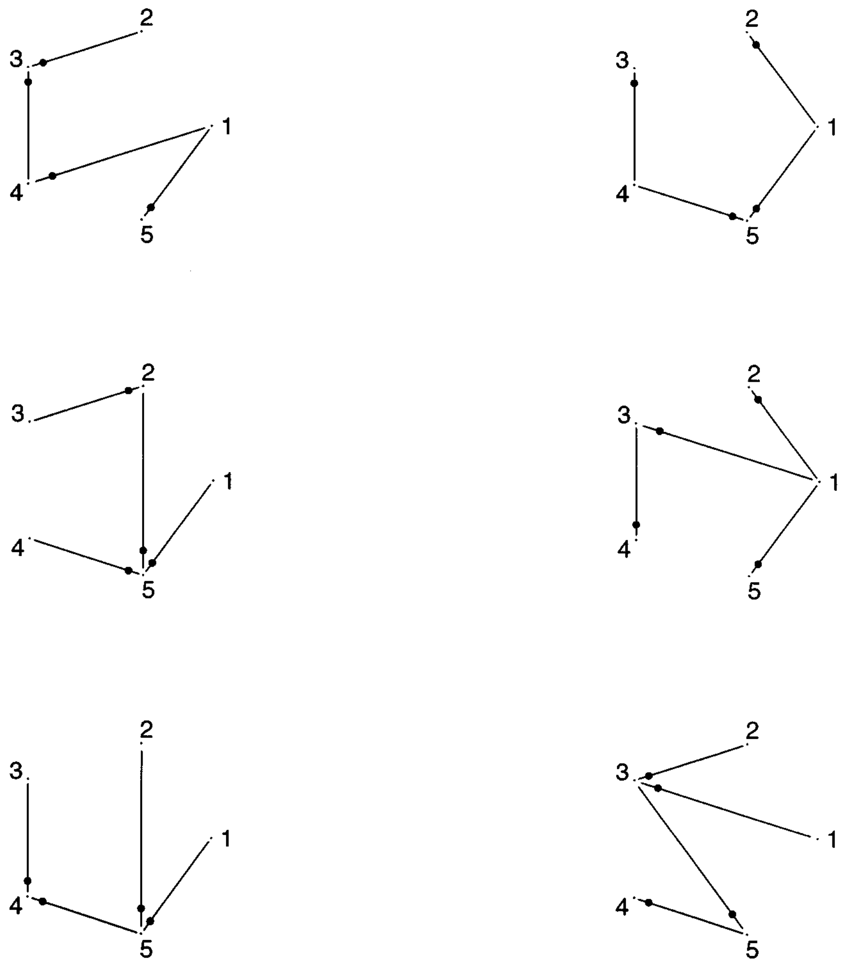

FIGURE 6B.—Other Nash networks.

We now show that the set of strict Nash equilibria is significantly more restrictive.

PROPOSITION 4.2: *Let the payoffs be given by* (2.3). *A strict Nash network is either a center-sponsored star or the empty network.* (*a*) *A center-sponsored star is strict Nash if and only if* $\Phi(n, n-1) > \Phi(x+1, x)$ *for all* $x \in \{0, \ldots, n-2\}$. (*b*) *The empty network is strict Nash if and only if* $\Phi(1,0) > \Phi(x+1, x)$ *for all* $x \in \{1, \ldots, n-1\}$.

PROOF: Suppose g is strict Nash and is not the empty network. Let $\bar{g} = \mathrm{cl}(g)$. Let i and j be agents such that $g_{i,j} = 1$. We claim that $\bar{g}_{j,j'} = 0$ for any $j' \notin \{i, j\}$. If this were not true, then i can delete his link with j and form one with j' instead, and receive the same payoff, which would contradict the assumption that g is strict Nash. Thus any agent with whom i is directly linked cannot have any other links. As g is minimally tw-connected by Proposition 4.1, i must be the center of a star and $g_{j,i} = 0$. If $j' \neq j$ is such that $g_{j',i} = 1$, then j' can switch to j and get the same payoff, again contradicting the supposition that g is strict Nash. Hence, the star must be center-sponsored.

Under the hypothesis in (a) it is clear that a center-sponsored star is strict Nash, while the empty network is not Nash. On the other hand, let g be a center-sponsored star with i as center, and suppose there is some $x \in \{0, \ldots, n-2\}$ such that $\Phi(x+1, x) \geq \Phi(n, n-1)$. Then i can delete all but x links and do at least as well, so that g cannot be strict Nash. Similar arguments apply under the hypotheses in (b). *Q.E.D.*

For the linear specification (2.4), Proposition 4.2 implies that when $c \in (0,1)$ the unique strict Nash network is the center-sponsored star, and when $c > 1$ the unique strict Nash network is the empty network.

We now turn to the issue of efficiency. In general, an efficient network need not be either tw-connected or empty.[28] We provide the following partial characterization of efficient networks.

PROPOSITION 4.3: *Let the payoffs be given by* (2.3). *All tw-components of an efficient network are minimal. If* $\Phi(x+1, y+1) \geq \Phi(x, y)$, *for all* $y \in \{0, \ldots, n-2\}$ *and* $x \in \{y+1, \ldots, n-1\}$, *then an efficient network is tw-connected.*

As the intuition provided below is simple, a formal proof is omitted. Minimality is a direct consequence of the absence of frictions. In the second part, tw-connectedness follows from the hypothesis that an additional link to an unobserved agent is weakly preferred by individual agents; since information flow is two-way, such a link generates positive externalities in addition and therefore increases social welfare.

With two-way flows, the question of efficiency is quite complex. For example, a center-sponsored star can have a different level of welfare than a periphery-sponsored one, since the number of links maintained by each agent is different in the two networks. However, for the linear payoffs given by (2.4), it can easily be shown that if $c \leq n$ a network is efficient if and only if it is minimally tw-connected (in particular, a star is efficient), while if $c > n$, then the empty network is uniquely efficient.

4.2. *Dynamics*

We now study network evolution with the payoff functions specified in (2.3). To get a first impression of the dynamics we present a simulation of a sample

[28] For example, consider a society with 3 agents. Let $\Phi(1,0) = 6.4$, $\Phi(2,0) = 7$, $\Phi(3,0) = 7.1$, $\Phi(2,1) = 6$, $\Phi(3,1) = 6.1$, $\Phi(3,2) = 0$. Then the network $g_{1,2} = 1$, and $g_{i,j} = 0$ for all other pairs of agents (and its permutations) constitutes the unique efficient architecture.

path in Figure 7.[29] The process converges to a center-sponsored star, within nine periods. The convergence appears to rely on a process of agglomeration on a central agent as well as on miscoordination among the remaining agents. In our analysis we exploit these features of the dynamic.

We have been unable to prove a convergence result for all payoff functions along the lines of Theorem 3.1. In the following result, we impose stronger versions of the hypotheses in Proposition 4.2 and prove that the dynamics converge to the strict Nash networks identified by that proposition. The proof requires some additional terminology. Given a network g, an agent j is called an *end-agent* if $\bar{g}_{j,k} = 1$ for exactly one agent k. Also, let $\sigma(i;\bar{g}) = |\{k \mid d(i,k;\bar{g}) = 1\}|$ denote the number of agents at tw-distance 1 from agent i.

THEOREM 4.1: *Let the payoff functions be given by* (2.3) *and fix any initial network* g. (*a*) *If* $\Phi(x+1,y+1) > \Phi(x,y)$ *for all* $y \in \{0,1,\dots,n-2\}$ *and* $x \in \{y+1,\dots,n-1\}$, *then the dynamic process converges to the center-sponsored star, with probability* 1. (*b*) *If* $\Phi(x+1,y+1) < \Phi(x,y)$ *for all* $y \in \{0,1,\dots,n-2\}$ *and* $x \in \{y+1,\dots,n-1\}$, *then the dynamic process converges to the empty network, with probability* 1.

PROOF: As with Theorem 3.1, the broad strategy behind the proof is to show that there is a positive probability of transition to a strict Nash network in finite time. We consider part (a) first. Note that the hypothesis on payoffs implies that $\Phi(n,n-1) > \max_{\{0 \le x \le n-2\}} \Phi(x+1,x)$, which, by Proposition 4.2(a), implies that the center-sponsored star is the unique strict Nash network. Starting from g, we allow every agent to move in sequence, one at a time. Lemma 4.1 in Appendix B shows that after all agents have moved, the resulting network is either minimally tw-connected or is the empty network. Suppose first that the network is empty. Then we allow a single agent to play. As $\Phi(n,n-1) > \max_{\{0 \le x \le n-2\}} \Phi(x+1,x)$, the agent's unique best response is to form links with all the others. This results in a center-sponsored star, and (a) will follow. There is thus no loss of generality in supposing that the initial network g itself is minimally tw-connected.

Let agent $n \in \operatorname{argmax}_{i \in N} \sigma(i;\bar{g})$. Since g is tw-connected, $\sigma(n;\bar{g}) \ge 2$. Furthermore, as g is also minimal, there is a unique tw-path between agent n and every other agent. Thus if $i \ne n$ then either $\bar{g}_{n,i} = 1$ or there exist $\{i_1,\dots,i_q\}$ such that $\bar{g}_{n,i_1} = \cdots = \bar{g}_{i_q,i} = 1$. We shall say that i is *outward-pointing* with respect to n if $g_{i,n} = 1$ in the former case and $g_{i,i_q} = 1$ in the latter case. Likewise, i is *inward-pointing* with respect to n if $g_{n,i} = 1$ in the former case and $g_{i_q,i} = 1$ in the latter case. Suppose that i is an outward-pointing agent and $d(i,n;\bar{g}) \ge 2$. It can be shown that agent i has a best response in which he deletes the link g_{i,i_q}

[29] Here, the payoffs are given by the linear model (2.4) with $c \in (0,1)$. The initial network (labelled $t = 1$) has been drawn at random from the set of all directed networks with 5 agents. In period $t \ge 2$, the choices of agents who exhibit inertia have been drawn with solid lines, while those whose choices are best responses have been drawn using dashed lines.

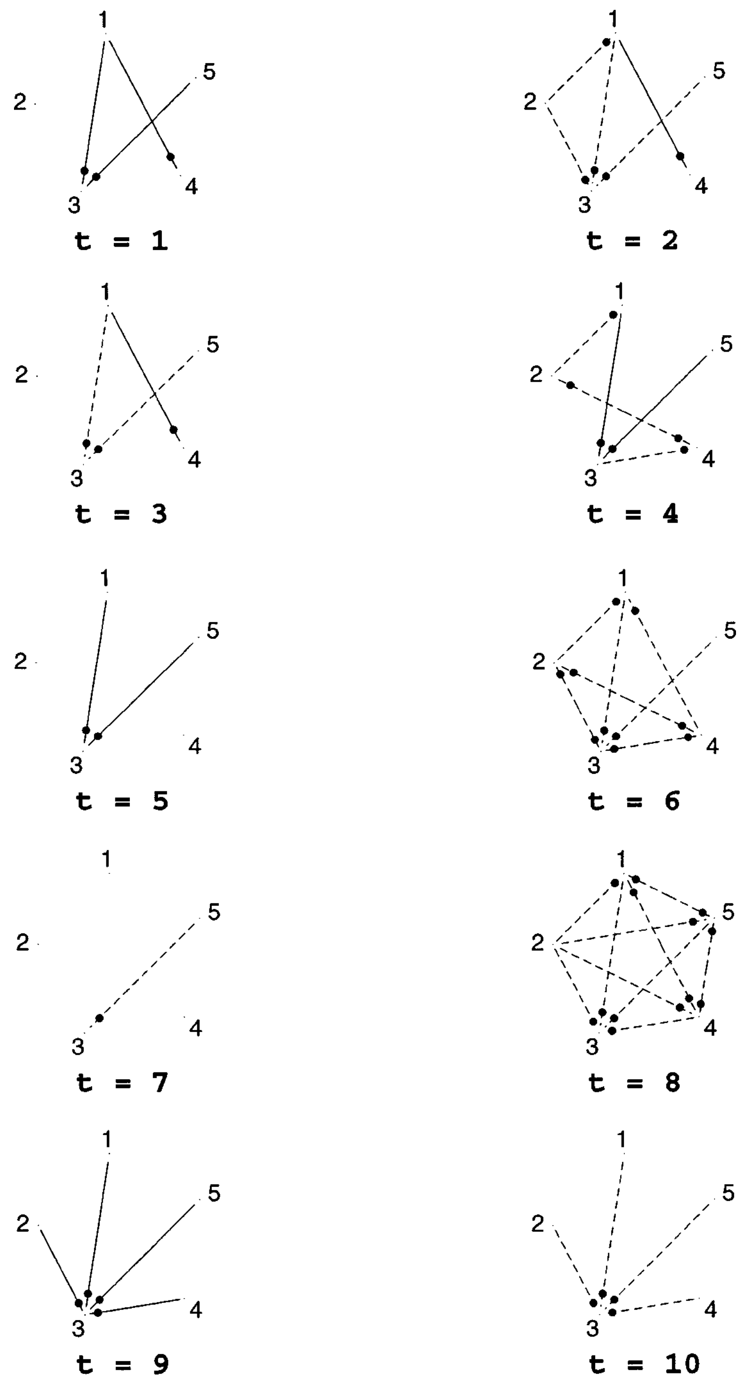

FIGURE 7.—Sample path (two-way model).

and instead forms a link $g_{i,n}=1$ (see Lemma 4.2 and the Remark in Appendix B). Let all such outward-pointing agents move in sequence in this manner and form a link with n. Denote the resulting network by g^1. By construction of g^1, for every outward-pointing agent i vis-a-vis n, it is true that $d(n,i;\bar{g}^1)=1$; thus if $d(n,j;\bar{g}^1)\geq 2$, then j must be an inward-pointing agent with respect to n.

Consider an agent j, with $d(n,j;\bar{g}^1)\geq 3$. Using the argument of Lemma 4.1, it is easily shown that g^1 is minimally tw-connected; thus there is a unique path between n and j and there are at least two agents, j_1 and j_2, on the tw-path between j and n such that $g^1_{j_1,j_2}=g^1_{j_2,j}=1$. From Lemma 4.2 and the Remark in Appendix B, we can infer that it is a best response for j_1 to maintain all links except the link with j_2, and to *switch* the link with j_2 to a link with j instead. Let g' denote the resulting network. Note that j_2 is an outward-pointing agent vis-a-vis n in g'. The arguments above concerning outward-pointing agents apply and it is a best response for agent j_2 to delete his link with j and instead form a link $g_{j_2,n}=1$. Denote the new network by g^2. We have thus shown that $\sigma(n;\bar{g}^2)=\sigma(n;\bar{g}^1)+1$. We use the argument of Lemma 4.1 to deduce that g^2 is minimally tw-connected. Since $\sigma(n;\cdot)$ increases with positive probability as long as the furthest away inward-pointing agent is at distance $q\geq 3$, we eventually arrive at a minimally tw-connected network g^3 such that $\sigma(n;\bar{g}^3)\geq 2$ and $d(n,j;\bar{g}^3)\leq 2$ for all $j\in N$.

As all agents are at a tw-distance no larger than 2 from agent n, it can be seen that there are four possible configurations for an agent i linked with agent n. (a) $g^3_{i,n}=1$ and i has no other links. (b) $g^3_{n,i}=1$ and i has no other links. (c) $g^3_{i,n}=1$ and $g^3_{i,j}=1$ for all $j\in E$, where E is the set of end-agents of i.[30] (d) $g^3_{n,i}=1$ and $g^3_{i,j}=1$ for all $j\in E$, where E is again the set of end-agents of i. We also note that case (d) can be reduced to case (c) by applying the switching argument presented above.

Suppose there is an agent i in configuration (c) with end-agents E so that $g^3_{i,n}=1$ and $g^3_{i,j}=1$ for all $j\in E$. Since $\sigma(n;\bar{g}^3)\geq 2$ there is at least one other agent k at tw-distance 1 from n. Suppose that $g^3_{k,n}=1$. Let agent i and agent k both choose a best response simultaneously. Specifically, it is a best response for i to maintain his links with the agents in E and switch his link from agent n to agent k. Likewise, it is a best response for k to switch his link from n to i. In the resulting network n no longer has a tw-path with either agent: thus k and i *miscoordinate*. We now allow agent n to choose a best response. It is easily checked (using Lemma 4.2 and the Remark in Appendix B), that it is a best response for him to form a link with some agent $j\in E$, ceteris paribus. Now, if i and k again move simultaneously, i can delete his links $g_{i,j}=g_{i,k}=1$ and only form links with the agents in $E\backslash\{j\}$ in addition to forming a link with n. Likewise, k will not form any links (in particular, he will delete his link with i). Finally, if n moves again, he will form a link with k, ceteris paribus. Label the resulting network g^4. Since n now also has a link with j, in addition to links with

[30] These are the set of end-agents in g^3 whose unique link is with agent i.

TABLE II

RATES OF CONVERGENCE IN TWO-WAY FLOW MODEL

n	$p = 0.5$	$p = 0.65$	$p = 0.8$	$p = 0.95$
3	191.12(16.89)	47.78(4.22)	17.34(1.43)	18.19(0.71)
4	318.23(22.93)	71.34(4.93)	17.55(1.02)	14.83(0.53)
5	613.28(36.08)	70.08(4.49)	16.27(0.83)	13.23(0.46)
6	753.88(43.94)	89.84(5.07)	17.90(0.88)	11.89(0.37)
7	1010.64(54.86)	123.44(6.78)	22.11(1.02)	10.28(0.35)
8	1625.63(87.52)	174.62(9.40)	27.87(1.24)	10.34(0.34)

i and k we get $\sigma(n;\bar{g}^4) = \sigma(n;\bar{g}^3) + 1$. The other combinations of cases (a), (b), and (c) can be analyzed with a combination of switching and miscoordination arguments to eventually reach a minimally tw-connected network g^* where $\sigma(n;\bar{g}^*) = n-1$. If g^* is a center-sponsored star, we are done. Otherwise, miscoordination arguments can again be used to show transition to a center-sponsored star.

Part (b) of the result is proved using similar arguments; a sketch is presented in Appendix B. *Q.E.D.*

For linear payoffs (2.4), Theorem 4.1(a) implies convergence to the center-sponsored star when $c \in (0,1)$, while Theorem 4.1(b) implies convergence to the empty network for $c > 1$. In particular, since a star is efficient for $c \leq n$ and the empty network is efficient for $c > n$, the limit network if efficient when $c < 1$ or $c > n$.

Rates of Convergence: We study the rates of convergence for the linear specification in (2.4), i.e. $\Pi_i(g) = \mu_i(\bar{g}) - \mu_i^d(g)c$. We shall suppose $c \in (0,1)$. Our simulations are carried out under the same assumptions as in the one-way model, with 500 simulations for each n and for four different values of p. Table II summarizes the findings.

We see that when $p = 0.5$, average convergence times are extremely high, but come down dramatically as p increases. When $n = 8$ for example, it takes more than 1600 periods to converge when $p = 0.5$, but when $p = 0.95$, it requires only slightly more than 10 periods on average to reach the center-sponsored star. The intuition can be seen by initially supposing that $p = 1$. If we start with the empty network, all agents will simultaneously choose to form links with the rest of society. Thus, the complete network forms in the next period. Since this gives rise to a perfect opportunity for free riding, each agent will form no links in the subsequent period. Thus, the dynamics will oscillate between the empty and the complete network. When p is close to 1, a similar phenomenon occurs (as seen in Figure 7, where $p = 0.75$) except there is now a chance that all but one agent happen to move, leaving that agent as the center of a center-sponsored star. On the other hand, when p is small, few agents move simultaneously. This makes rapid oscillations unlikely, and greatly reduces the speed of convergence.

5. DECAY

In the analysis above, we exploit the assumption that information obtained through indirect links has the same value as that obtained through direct links. This assumption is strong; in general, there will be delays as well as lowering of quality, as information is transmitted through a series of agents. In this section, we study the effects of relaxing the no-decay assumption. Since this is a difficult and voluminous topic, we shall assume a specific functional form for the payoffs, and also largely restrict our study to "small" societies.

5.1. *One-way Flow Model with Decay*

We consider a modification of the linear payoff structure given by (2.2), i.e. where the value of information is $V \equiv 1$ and its cost is $c > 0$. We measure the level of decay by a parameter $\delta \in (0,1]$. Given a network g, it is assumed that if an agent i has a link with another agent j, i.e. $g_{i,j} = 1$, then agent i receives information of value δ from j. More generally if the shortest path in the network from j to i has $q \geq 1$ links, then the value of agent j's information to i is δ^q. The cost of link formation is still taken to be c per link. The payoff to an agent i in the network g is then given by

$$\Pi_i(g) = 1 + \sum_{j \in N(i;g) \setminus \{i\}} \delta^{d(i,j;g)} - \mu_i^d(g)c, \tag{5.1}$$

where $d(i,j;g)$ is the geodesic distance from j to i. The linear model of (2.2) corresponds to $\delta = 1$. Henceforth, we shall always assume $\delta < 1$ unless specified otherwise.

Nash Networks: The trade-off between the costs of link formation and the benefits of having short information channels to overcome transmission losses is central to an understanding of the architecture of networks in this setting. If $c < \delta - \delta^2$, the incremental payoff from replacing an indirect link by a direct one exceeds the cost of link formation; hence it is a dominant strategy for an agent to form links with everyone, and the complete network g^c is the unique (strict) Nash equilibrium. Suppose next that $\delta - \delta^2 < c < \delta$. Since $c < \delta$, an agent has an incentive to directly or indirectly access everyone. Furthermore, $c > \delta - \delta^2$ implies the following: if there is some agent who has links with every other agent, then the rest of society will form a single link with him. Hence a star is always a (strict) Nash equilibrium. Third, it follows from continuity and the fact that the wheel is strict Nash when $\delta = 1$ that it is also strict Nash for δ close to 1. Finally it is obvious that if $c > \delta$, then the empty network is strict Nash. The following result summarizes the above observations and also derives a general property of strict Nash networks.

PROPOSITION 5.1: *Let the payoffs be given by* (5.1). *Then a strict Nash network is either connected or empty. Furthermore*, (*a*) *the complete network is strict Nash if*

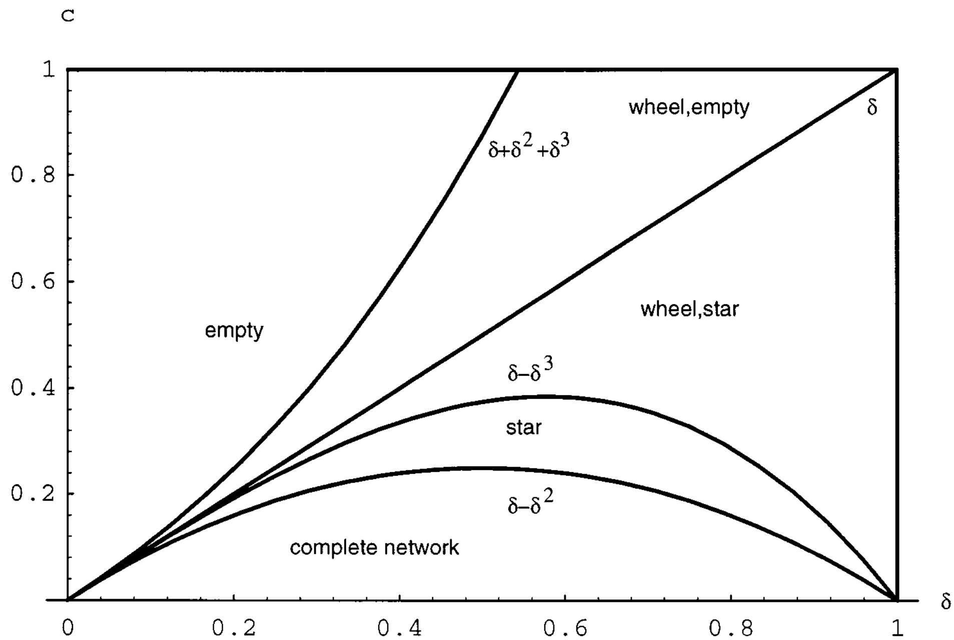

FIGURE 8A.—Strict Nash networks (one-way model, $n = 4$).

and only if $0 < c < \delta - \delta^2$, (*b*) *the star network is strict Nash if and only if* $\delta - \delta^2 < c < \delta$, (*c*) *if* $c \in (0, n-1)$, *then there exists* $\delta(c) \in (0,1)$ *such that the wheel is strict Nash for all* $\delta \in (\delta(c), 1)$, (*d*) *the empty network is strict Nash if and only if* $c > \delta$.

Appendix C provides a proof for the statement concerning connectedness, while parts (a)–(d) can be directly verified.[31] Figure 8A provides a characterization of strict Nash equilibria, for a society with $n = 4$ agents.[32]

Ideally, we would like to have a characterization of strict Nash for all n. This appears to be a difficult problem and we have been unable to obtain such results. Instead, we focus on the case where information decay is "small" and identify an important and fairly general class of networks that are strict Nash. To motivate this class, consider the networks depicted in Figures 9A–9C. Assume that $c \in (0,1)$ and consider the network in Figure 9A. Here, agent 5 has formed three links, while all others have only one. Thus, agent 5's position is similar to that of a "central coordinator" in a star network. When $\delta = 1$, agent 1 (say) does

[31] In the presence of decay, a nonempty Nash network is not necessarily connected. Suppose $n = 6$. Let $\delta + \delta^2 < 1$ and $\delta + \delta^2 - \delta^3 < c < \delta + \delta^2$. Then it can be verified that the network given by the links, $g_{1,2} = g_{2,4} = g_{4,3} = g_{3,2} = g_{5,2} = g_{6,5} = g_{2,6} = 1$ is Nash. It is clearly nonempty and it is disconnected since agent 1 is not observed by anyone.

[32] To show that the networks depicted in the different parameter regions are strict Nash is straightforward. Incentive considerations in each region (e.g. that the star is not strict Nash when $c > \delta$) rule out other architectures.

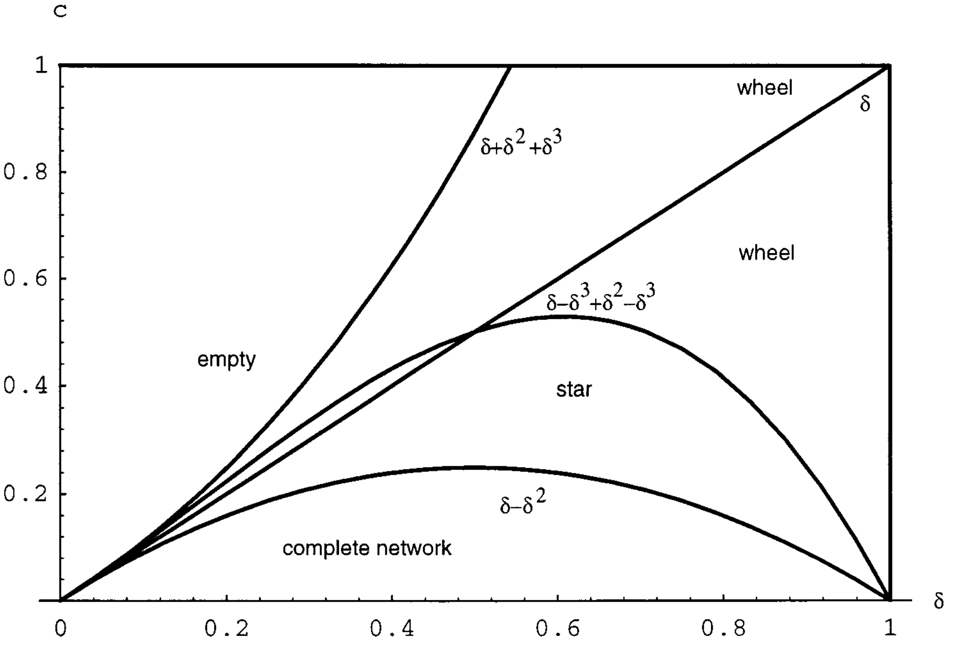

FIGURE 8B.—Efficient networks (one-way model, $n = 4$).

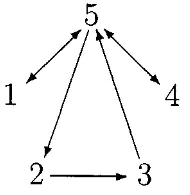

FIGURE 9A

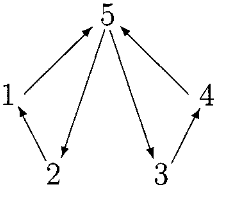

FIGURE 9B

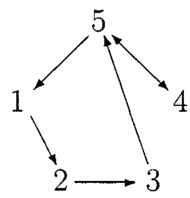

FIGURE 9C

not receive any additional benefit from a link with agent 5 as compared to a link with agent 2 or 3 or 4 instead. Hence this network cannot be strict Nash. However, when δ falls below one, agent 1 strictly benefits from the link with agent 5 as compared to a link with any other agent, since agent 5 is at a shorter distance from the rest of the society. Similar arguments apply for agent 2 and agent 4 to have a link with agent 5. Thus, decay creates a role for "central" agents who enable closer access to other agents. At the same time, the logic underlying the wheel network—of observing the rest of the society with a single link—still operates. For example, under low decay, agent 3's unique best response will be to form a single link with agent 2. The above arguments suggest that the network of Figure 9A can be supported as strict Nash for low levels of decay. Analogous arguments apply for the network in Figure 9B. More generally, the trade-off between cost and decay leads to strict Nash networks where a central agent reduces distances between agents, while the presence of small wheels enables agents to economize on the number of links.

Formally, a *flower* network g partitions the set of agents N into a central agent (say agent n) and a collection $\mathscr{P} = \{\mathscr{P}_1, \ldots, \mathscr{P}_q\}$ where each $P \in \mathscr{P}$ is nonempty. A set $P \in \mathscr{P}$ of agents is referred to as a *petal*. Let $u = |P|$ be the cardinality of petal P, and denote the agents in P as $\{j_1, \ldots, j_u\}$. A flower network is then defined by setting $g_{j_1, n} = g_{j_2, j_1} = \cdots = g_{j_u, j_{u-1}} = g_{n, j_u} = 1$ for each petal $P \in \mathscr{P}$ and $g_{i,j} = 0$ otherwise. A petal P is said to be a *spoke* if $|P| = 1$. A flower network is said to be of *level* $s \geq 1$ if every petal of the network has at least s agents and there exists a petal with exactly s agents. Note that a star is a flower network of level 1 with $n - 1$ spokes, while a wheel is a flower network of level $n - 1$ with a single petal.

We are interested in finding conditions under which flower networks can be supported as strict Nash. However, we first exclude a certain type of flower network from our analysis. Figure 9C provides an example. Here agent 5 is the central agent, and there are exactly two petals. Moreover, one petal is a spoke, so that it is a flower network of level 1. Note that agent 4 will be indifferent between forming a link with any of the remaining agents, since their position is completely symmetric. Thus, this network can never be strict Nash. In what follows, a flower network g with exactly two petals, of which at least one is a spoke, will be referred to as the "exceptional case."

PROPOSITION 5.2: *Suppose that the payoffs are given by* (5.1). *Let* $c \in (s-1, s)$ *for some* $s \in \{1, 2, \ldots, n-1\}$ *and let* g *be a flower network* (*other than the exceptional case*) *of level* s *or higher*. *Then there exists a* $\delta(c, g) < 1$ *such that*, *for all* $\delta \in (\delta(c, g), 1)$, g *is a strict Nash network*. *Furthermore*, *no flower network of a level lower than* s *is Nash for any* $\delta \in (0, 1)$.

The proof is given in Appendix C. When $s > 1$ the above proposition rules out any networks with spokes as being strict Nash. In particular, the star cannot be supported when $c > 1$.

Finally, we note the impact of the size of the society on the architecture of strict Nash networks. As n increases, distances in the wheel network become larger, creating greater scope for central agents to reduce distances. This suggests that intermediate flower networks should become more prominent as the society becomes larger. Our simulation results are in accord with this intuition.

Efficient Networks: The welfare function is taken to be $W(g) = \sum_{i=1}^{n} \Pi_i(g)$, where Π_i is specified by equation (5.1). Figure 8B characterizes the set of efficient networks when $n = 4$.[33] The trade-off between costs and decay mentioned above also determines the structure of efficient networks. If the costs are sufficiently low, efficiency dictates that every agent should be linked with every other agent. For values of δ close to one, and/or if the costs of link formation are high, the wheel is still efficient. For intermediate values of cost and decay, the star strikes a balance between these forces.

A comparison between Figures 8A and 8B reveals that there are regions where strict Nash and efficient networks coincide (when $c < \delta - \delta^2$ or $c > \delta + \delta^2 + \delta^3$). The figures suggest, however, that the overall relationship is quite complicated.

Dynamics: We present simulations for low values of decay, i.e., δ close to 1, for a range of societies from $n = 3$ to $n = 8$.[34] This helps to provide a robustness check for the convergence result of Theorem 3.1 and also gives some indication about the relative frequencies with which different strict Nash networks emerge. For each n, we consider a 25×25 grid of (δ, c) values in the region $[0.9, 1) \times (0, 1)$, but discard points where $c \leq \delta - \delta^2$ or $c \geq \delta$. For the remaining 583 grid values, we simulate the process for a maximum of 20,000 periods, starting from a random initial network. We also set $p = 0.5$ for all the agents.

[33] The assertions in the figure are obtained by comparing the welfare levels of all possible network architectures to obtain the relevant parameter ranges. We used the list of architectures given in Harary (1972).

[34] For $n = 4$ it is possible to prove convergence to strict Nash in all parameter regions identified in Figure 8a. The proof is provided in an earlier working paper version. For general n, it is not difficult to show that, from every initial network, the dynamic process converges almost surely to the complete network when $c < \delta - \delta^2$ and to the empty network when $c > \delta + (n-2)\delta^2$.

Figure 10 depicts some of the limit networks that emerge. In many cases, these are the wheel, the star, or other flower networks. However, some variants of flower networks (left-hand side network for $n = 6$ and right-hand side network for $n = 7$) also arise. Thus, in the $n = 7$ case, agent 2 has an additional link with agent 6 in order to access the rest of the society at a closer distance. Since $c = 0.32$ is relatively small, this is worthwhile for the agent. Likewise, in the $n = 6$ example, two petals are "fused," i.e. they share the link from agent 6 to agent 3. Other architectures can also be limits when c is small, as in the left-hand side network for $n = 8$.[35]

Table III (below) provides an overall summary of the simulation results. Column 2 reports the average time and standard error, *conditional upon convergence to a limit network in* 20,000 *periods*. Columns 3–6 show the relative likelihood of different strict Nash networks being the limit, while the last column shows the likelihood of a limit cycle.[36] With the exception of $n = 4$, the average convergence times are all relatively small. Moreover, the chances of eventual convergence to a limit network are fairly high. The wheel and the star become less likely, while other flower networks as well as nonflower networks become more important as n increases. This corresponds to the intuition presented in the discussion on flower networks. We also see that when $n = 8$, 56.6% of the limit networks are not flower networks. In this category, 45.7% are variants of flower networks (e.g. with fused petals, or with an extra link between the central agent and the final agent in a petal) while the remaining 10.9% are networks of the type seen in the left-hand side network for $n = 8$. Thus, flower networks or their variants occur very frequently as limit points of the dynamics.

5.2. *Two-way Flow Model with Decay*

This section studies the analogue of (5.1) with two-way flow of information. The payoffs to an agent i from a network g are given by

$$\overline{\Pi}_i(g) = 1 + \sum_{j \in N(i;\bar{g}) \setminus \{i\}} \delta^{d(i,j,\bar{g})} - \mu_i^d(g)c. \tag{5.2}$$

The case of $\delta = 1$ is the linear model of (2.4). We assume that $\delta < 1$ unless otherwise specified.

Nash Networks: We begin our analysis by describing some important strict Nash networks.

PROPOSITION 5.3: *Let the payoffs be given by* (5.2). *A strict Nash network is either tw-connected or empty. Furthermore,* (*a*) *if* $0 < c < \delta - \delta^2$, *then the tw-complete network is the unique strict Nash,* (*b*) *if* $\delta - \delta^2 < c < \delta$, *then all three types of stars* (*center-sponsored, periphery-sponsored, and mixed*) *are strict Nash,* (*c*) *if*

[35] Due to space constraints, we do not investigate such networks in this paper.

[36] We assume that the process has entered a limit cycle if convergence to a limit network does not occur within the specified number of periods.

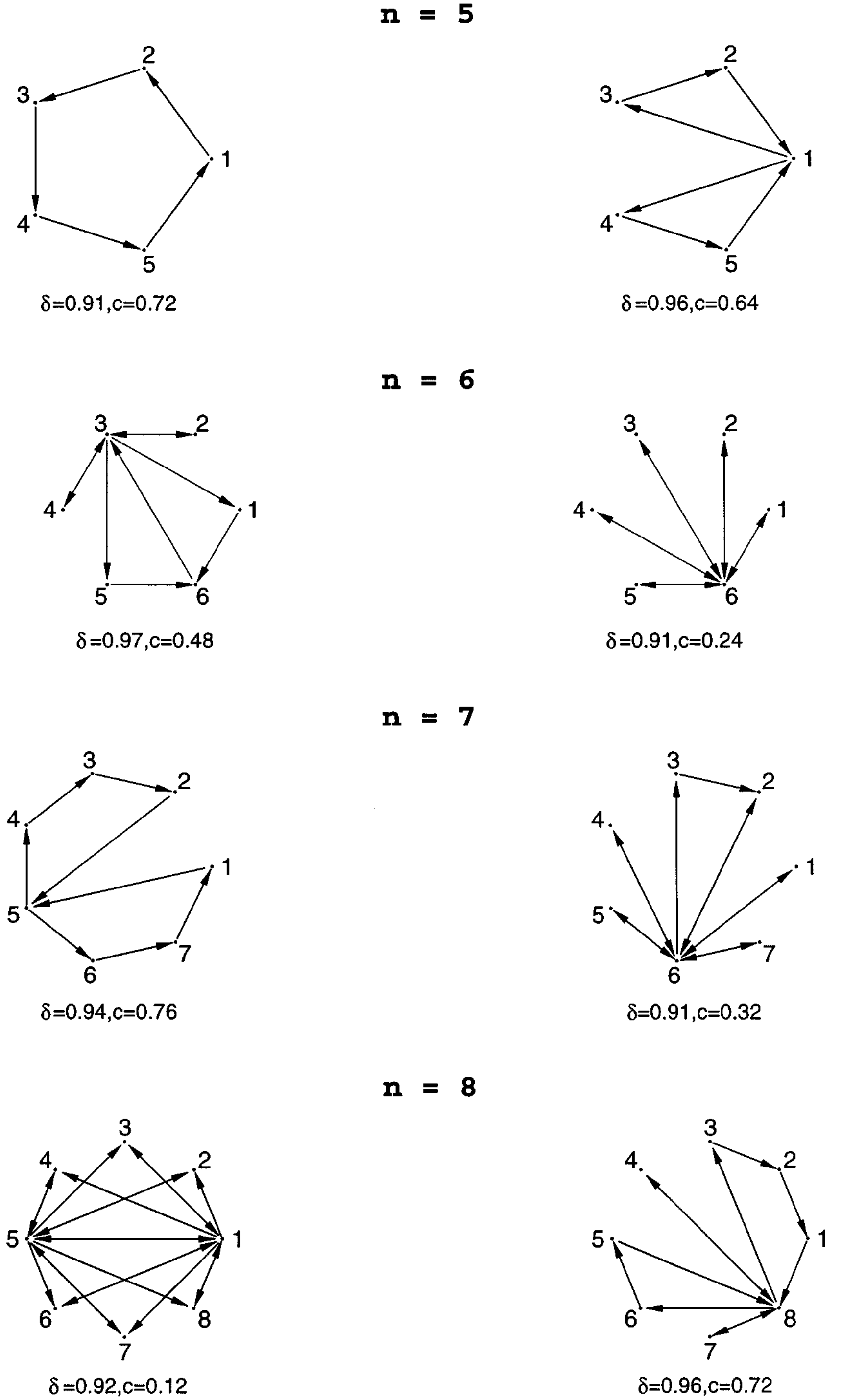

FIGURE 10.—Limit networks (one-way model).

TABLE III

DYNAMICS IN ONE-WAY FLOW MODEL WITH DECAY

n	Avg. Time (Std. Err.)	Flower Networks			Other Networks	Limit Cycles
		Wheel	Star	Other		
3	6.5(0.2)	100.0%	0.0%	0.0%	0.0%	0.0%
4	234.2(61.7)	71.9%	27.8%	0.0%	0.0%	0.3%
5	28.1(6.2)	20.6%	11.5%	58.7%	4.6%	4.6%
6	26.4(3.6)	3.6%	6.3%	58.8%	27.1%	4.1%
7	94.3(14.7)	0.9%	4.1%	56.1%	28.0%	11.0%
8	66.5(8.5)	0.7%	3.8%	37.2%	56.6%	1.7%

$\delta < c < \delta + (n-2)\delta^2$, *then the periphery-sponsored star, but none of the other stars, is strict Nash*, (d) *if* $c > \delta$, *then the empty network is strict Nash.*

Parts (a)–(d) can be verified directly.[37] The proof for tw-connectedness is a slight variation on the proof of Proposition 4.1 (in the case with no decay) and is omitted. Figure 11A provides a full characterization of strict Nash networks for a society with $n = 4$.

Ideally we would like to have a similar characterization for all n. We have been unable to obtain such results; as in the previous subsection, we focus upon

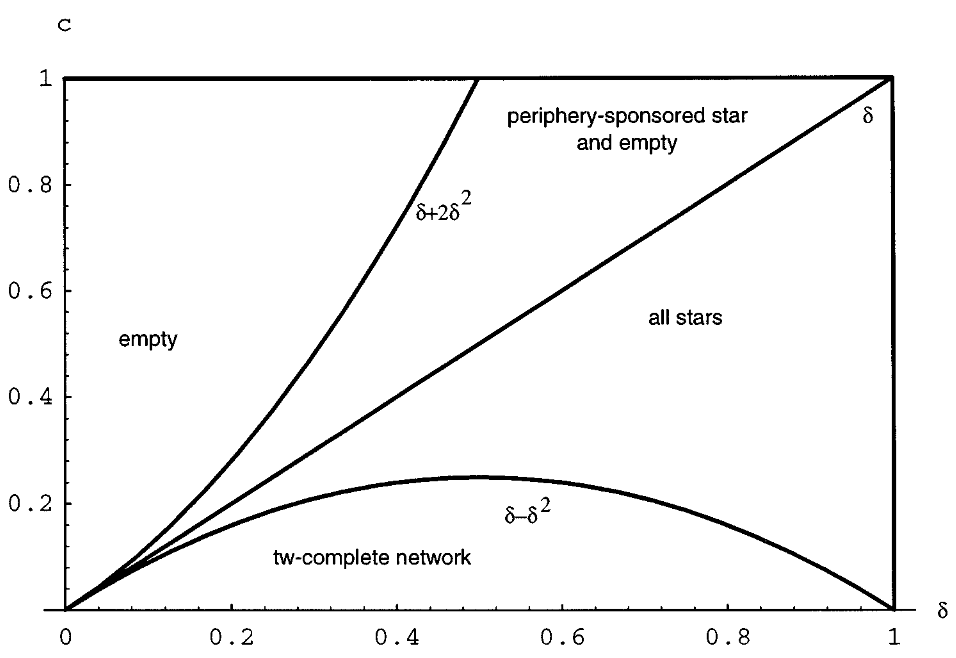

FIGURE 11A.—Strict Nash networks (two-way model, $n = 4$).

[37] A tw-complete network g is one where, for all i and j in N, we have $d(i, j; \bar{g}) = 1$ and $g_{i,j} = 1$ implies $g_{j,i} = 0$.

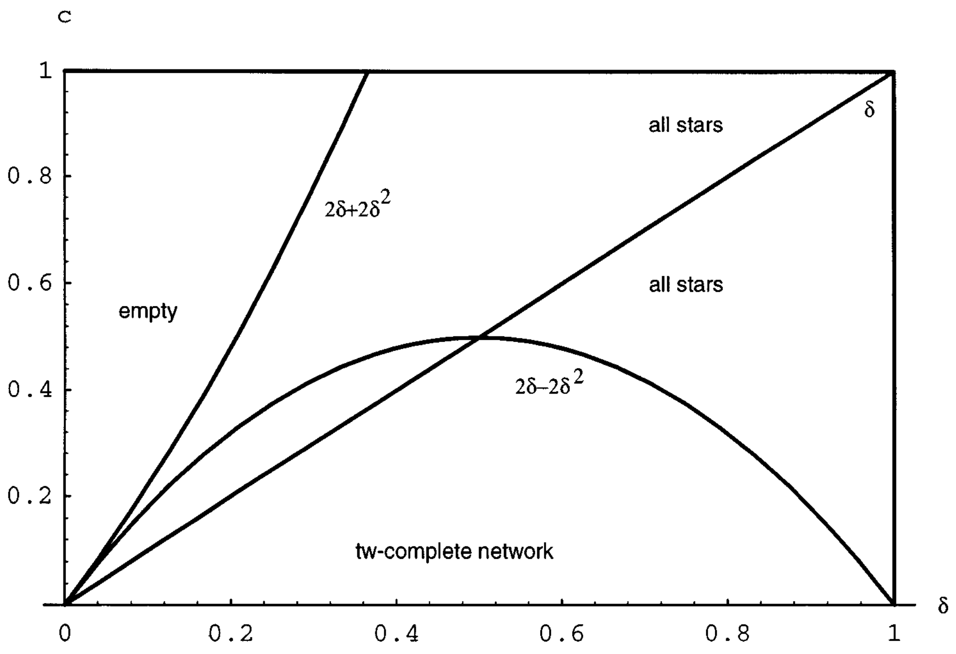

FIGURE 11B.—Efficient networks (two-way model, $n = 4$).

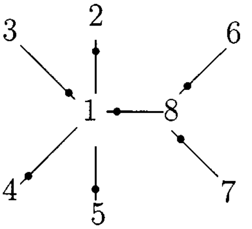

FIGURE 12A.—$|S_1| > |S_2| + 1$.

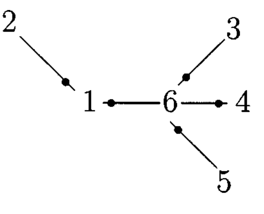

FIGURE 12B.—$|S_1| < |S_2| - 1$.

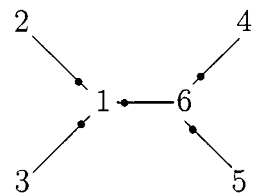

FIGURE 12C.—$|S_1| = |S_2|$.

low levels of decay. When $c \in (0,1)$ we can identify an important class of networks, which we label as *linked stars*. Figures 12A–C provide examples of such networks.

Linked stars are described as follows: Fix two agents (say agent 1 and n) and partition the remaining agents into nonempty sets S_1 and S_2, where $|S_1| \geq 1$ and $|S_2| \geq 2$. Consider a network g such that $g_{i,j} = 1$ implies $g_{j,i} = 0$. Further suppose that $g_{1,n} = 1$. Lastly, suppose one of the three mutually exclusive conditions (a), (b), or (c) holds: (a) If $|S_1| > |S_2| + 1$, then $\max\{g_{1,i}, g_{i,1}\} = 1$ for all $i \in S_1$ and $g_{n,j} = 1$ for all $j \in S_2$. (b) If $|S_1| < |S_2| - 1$, then $\max\{g_{n,j}, g_{j,n}\} = 1$ for all $j \in S_2$ and $g_{1,i} = 1$ for all $i \in S_1$. (c) If $||S_1| - |S_2|| \leq 1$, then $g_{1,i} = 1$ for all $i \in S_1$ and $g_{n,j} = 1$ for all $j \in S_2$.

The agents 1 and n constitute the "central" agents of the linked star. If δ is sufficiently close to 1, a spoke agent will not wish to form any links (if the central agent has formed one with him) and otherwise will form at most one link. Conditions (a) and (b) ensure that the spoke agents of a central agent will not wish to switch to the other central agent.[38]

If $c > 1$ and decay is small, it turns out that there are at most two strict Nash networks. One of them is, of course, the empty network. The other network is the periphery-sponsored star. These observations are summarized in the next result.

PROPOSITION 5.4: *Let the payoffs be given by* (5.2). *Let* $c \in (0,1)$ *and suppose g is a linked star. Then there exists* $\delta(c,g) < 1$ *such that for all* $\delta \in (\delta(c,g), 1)$ *the network g is strict Nash.* (*b*) *Let* $c \in (1, n-1)$ *and suppose that* $n \geq 4$. *Then there exists* $\delta(c) < 1$, *such that if* $\delta \in (\delta(c), 1)$ *then the periphery-sponsored star and the empty network are the only two strict Nash networks.*

[38] Thus, note that in Figure 12A, if $g_{7,8} = 1$ rather than $g_{8,7} = 1$, then agent 7 would strictly prefer forming a link with agent 1 instead, since agent 1 has more links than agent 8. Likewise, in Figure 12B, each link with an agent in S_1 must be formed by agent 1 for otherwise the corresponding 'spoke' agent will gain by moving his link to agent n instead. The logic for condition (c) can likewise be seen in Figure 12C. We also see why $|S_2| \geq 2$. In Figure 12C, if agent 5 were not present, then agent 1 would be indifferent between a link with agent 6 and one with agent 4. Lastly, we observe that since $|S_1| \geq 1$ and $|S_2| \geq 2$, the smallest n for which a linked star exists is $n = 5$.

The proof of Proposition 5.4(a) relies on arguments that are very similar to those in the previous section for flower networks, and is omitted. The proof of Proposition 5.4(b) rests on the following arguments: first note from Proposition 5.3 that any strict Nash network g that is nonempty must be tw-connected. Next observe that for δ sufficiently close to 1, g is minimally tw-connected. Consider a pair of agents i and j who are furthest apart in g. Using arguments from Theorem 4.1(b), it can be shown that if $c > 1$, then agents i and j must each have exactly one link, which they form. Next, suppose that the tw-distance between i and j is more than 2 and that (say) agent i's payoff is no larger than agent j's payoff. Then if i deletes his link and forms one instead with the agent linked with j, his tw-distance to all agents apart from j (and himself) is the same as j, and he is also closer to j. Then i strictly increases his payoff, contradicting Nash. Thus, the maximum tw-distance between two agents in g must be 2. It then follows easily that g is a periphery-sponsored star. We omit a formal proof of this result.

The difference between Proposition 5.4(b) and Proposition 4.2(b) is worth noting. For linear payoffs, the latter proposition implies that if $c > 1$ and $\delta = 1$, then the unique strict Nash network is the empty network. The crucial point to note is that with $\delta = 1$, and $c < n - 1$, the periphery-sponsored star is a Nash but not a strict Nash network, since a 'spoke' agent is indifferent between a link with the central agent and another 'spoke' agent. This indifference breaks down in favor of the central agent when $\delta < 1$, which enables the periphery-sponsored star to be strict Nash (in addition to the empty network).

Efficient Networks: We conclude our analysis of the static model with a characterization of efficient networks.

PROPOSITION 5.5: *Let the payoffs be given by* (5.2). *The unique efficient network is* (*a*) *the complete network if* $0 < c < 2(\delta - \delta^2)$, (*b*) *the star if* $2(\delta - \delta^2) < c < 2\delta + (n-2)\delta^2$, *and* (*c*) *the empty network if* $c > 2\delta + (n-2)\delta^2$.

The proof draws on arguments presented in Proposition 1 of Jackson and Wolinsky (1996) and is given in Appendix C. The nature of networks—complete, stars, empty—is the same, but the range of values for which these networks are efficient is different. This contrast arises out of the differences in the way we model network formation: Jackson and Wolinsky assume two-sided link formation, unlike our framework. Figure 11B displays the set of efficient networks for $n = 4$ in different parameter regions.

Dynamics: We now turn to simulations to study the convergence properties of the dynamics. As in the one-way case, for each n we consider a 25×25 grid of (δ, c) values in the region $[0.9, 1) \times (0, 1)$, with points satisfying $c \leq \delta - \delta^2$ or

$c \geq \delta$ being discarded. As earlier, there are a total of 583 grid values for each n. We also fix $p = 0.5$ as in the one-way model.[39]

Figure 13 depicts some of the limit networks. In most cases, they are stars of different kinds or linked stars. However, as the right-hand side network for $n = 7$ shows, other networks can also be limits. To see this, note that the maximum geodesic distance between two agents in a linked star is 3, whereas agents 5 and 7 are four links apart in this network. We also note that limit cycles can occur.[40]

Table IV provides an overall summary of the simulations. For $n \leq 6$, convergence to a limit network occurred in 100% of the simulations, while for $n = 7$ and $n = 8$ there is a positive probability of being absorbed in a limit cycle. Column 2 reports the average convergence time and the standard error, *conditional upon convergence to a limit network*. Columns 3–8 show the frequency with which different networks are the limits of the process. Among stars, mixed-type ones are the most likely. Linked stars become increasingly important as n rises, while other kinds of networks (such as the right-hand-side network when $n = 7$) may also emerge. Limit cycles are more common when $n = 7$ than when $n = 8$. In contrast to Table II concerning the two-way model without decay, convergence occurs very rapidly even though $p = 0.5$. A likely reason is that under decay an agent has a strict rather than a weak incentive to link to a well-connected agent: his choice increases the benefit for other agents to do so as well, leading to quick convergence. Absorption into a limit network is also much more rapid as compared to Table III for the one-way model, for perhaps the same reason.

6. CONCLUSION

In this paper, we develop a noncooperative model of network formation where we consider both one-way and two-way flow of benefits. In the absence of decay, the requirement of strict Nash sharply delimits the case of networks to the empty network and the one other architecture: in the one-way case, this is a wheel network, where every agent bears an equal share of the cost, while in the two-way case it is a center-sponsored star, where as the name suggests, a single agent bears the full cost. Moreover, in both models, a simple dynamic process

[39] For $n = 4$ convergence to strict Nash can be proved for all parameter regions identified in Figure 11A. For general n, it is not difficult to show that, starting from any initial network, the dynamic process is absorbed almost surely into the tw-complete network when $c < \delta - \delta^2$ and into the empty network when $c > \delta + (n-2)\delta^2$.

[40] To see how this can happen, consider the left-hand side network for $n = 7$ in Figure 13, which is strict Nash. However, if it is agent 3 rather than agent 5 who forms the link between them in the figure, we see that agent 3 can obtain the same payoff by switching this link to agent 1 instead, while all other agents have a unique best response. Thus, the dynamics will oscillate between two Nash networks.

For $n \leq 6$ it is not difficult to show that given $c \in (0, 1)$, the dynamics will always converge to a star or a linked star for all δ sufficiently close to 1. Thus, $n = 7$ is the smallest value for which a limit cycle occurs.

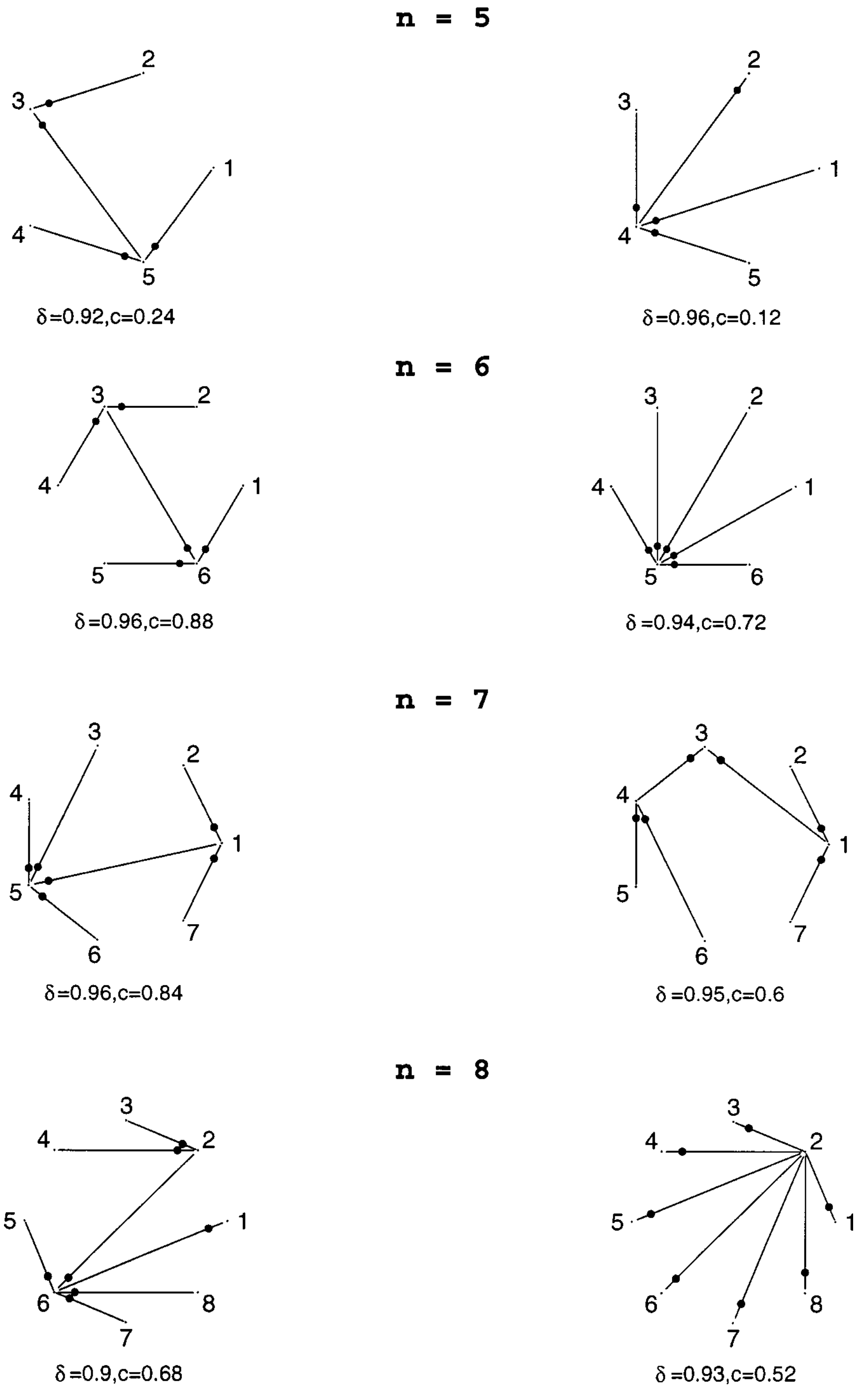

FIGURE 13.—Limit networks (two-way model).

TABLE IV

DYNAMICS IN TWO-WAY FLOW MODEL WITH DECAY

n	Avg. Time (Std. Err.)	Stars Center	Stars Mixed	Stars Periphery	Linked Stars	Other Networks	Limit Cycles
3	166.5(14.2)	100.0%	0.0%	0.0%	0.0%	0.0%	0.0%
4	5.2(0.2)	37.6%	56.9%	5.5%	0.0%	0.0%	0.0%
5	8.9(0.4)	34.0%	53.7%	3.6%	8.7%	0.0%	0.0%
6	8.8(0.3)	26.8%	42.9%	4.3%	26.1%	0.0%	0.0%
7	10.2(0.4)	20.4%	43.4%	3.9%	24.8%	3.8%	3.6%
8	12.3(0.4)	16.6%	34.6%	6.0%	34.5%	7.4%	0.9%

converges to a strict Nash network under fairly general conditions, while simulations indicate that convergence is relatively rapid. For low levels of decay, the set of strict Nash equilibria expands both in the one-way and two-way models. Many of the new strict equilibria are natural extensions of the wheel and the center-sponsored star, and also appear frequently as limits of simulated sample paths of the dynamic process. Notwithstanding the parallels between the results for the one-way and two-way models, prominent differences also exist, notably concerning the kinds of architectures that are supported in equilibrium.

Our results motivate an investigation into different aspects of network formation. In this paper, we have assumed that agents have no "budget" constraints, and can form any number of links. We have also supposed that contacting a well-connected person costs the same as contacting a relatively idle person. Moreover, in revising their strategies, it is assumed that individuals have full information on the existing social network of links. Finally, an important assumption is that the benefits being shared are nonrival. The implications of relaxing these assumptions should be explored in future work.

Dept. of Economics, McGill University, 855 Sherbrooke St. W., Montréal H3A 2T7 Canada; vbala2001@yahoo.com; http://www.arts.mcgill.ca

and

Econometric Institute, Erasmus University, 3000 DR Rotterdam, The Netherlands; goyal@few.eur.nl; http://www.few.eur.nl/few/people/goyal

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APPENDIX A

PROOF OF PROPOSITION 3.1: Let g be a Nash network. Suppose first that $\Phi(n,1) < \Phi(1,0)$. Let $i \in N$. Note that $\mu_i(g) \leq n$. Thus $\mu_i^d(g) \geq 1$ implies $\Pi_i(g) = \Phi(\mu_i(g), \mu_i^d(g)) \leq \Phi(n, \mu_i^d(g)) \leq \Phi(n,1) < \Phi(1,0)$, which is impossible since g is Nash. Hence it is a dominant strategy for each agent to have no links, and g is the empty network. Consider the case $\Phi(n,1) = \Phi(1,0)$. An argument analogous to the one above shows that $\mu_i^d(g) \in \{0,1\}$ for each $i \in N$. Furthermore $\mu_i^d(g) = 1$ can hold if $\mu_i(g) = n$. It is now simple to establish that if g is nonempty, it must be the wheel network, which is connected.[41]

Henceforth assume that $\Phi(n,1) > \Phi(1,0)$. Assume that g is not the empty network. Choose $i \in \text{argmax}_{i' \in N} \mu_{i'}(g)$. Since g is nonempty, $x_i \equiv \mu_i(g) \geq 2$ and $y_i \equiv \mu_i^d(g) \geq 1$. Furthermore, since g is Nash, $\Pi_i(g) = \Phi(x_i, y_i) \geq \Phi(1,0)$. We claim that i observes everyone, i.e. $x_i = n$. Suppose instead that $x_i < n$. Then there exists $j \notin N(i; g)$. Clearly, $i \notin N(j; g)$ either, for otherwise $N(i; g)$ would be a strict subset of $N(j; g)$ and $\mu_j(g) > x_i = \mu_i(g)$, contradicting the definition of i. If $y_j \equiv \mu_j^d(g) = 0$ let j deviate and form a link with i, ceteris paribus. His payoff will be $\Phi(x_i + 1, 1) > \Phi(x_i, 1) \geq \Phi(x_i, y_i) \geq \Phi(1,0)$, so that he can do strictly better. Hence $y_j \geq 1$. By definition of i, we have $x_j \equiv \mu_j(g) \leq x_i$. Let j delete all his links and form a single link with i instead. His new payoff will be $\Phi(x_i + 1, 1) > \Phi(x_i, 1) \geq \Phi(x_j, 1) \geq \Phi(x_j, y_j)$, i.e. he does strictly better. The contradiction implies that $x_i = n$ as required, i.e. there is a path from every agent in the society to agent i.

Let i be as above. An agent j is called *critical* (to i) if $\mu_i(g_{-j}) < \mu_i(g)$; if instead $\mu_i(g_{-j}) = \mu_i(g)$, agent j is called *noncritical*. Let E be the set of noncritical agents. If $j \in \text{argmax}_{i' \in N} d(i, i'; g)$, clearly j is noncritical, so that E is nonempty. We show that $j \in E$ implies $\mu_j(g) = n$. Suppose this were not true. If $y_j \equiv \mu_j^d(g) = 0$, then j can deviate and form a link with i. His new payoff will be $\Phi(n,1) > \Phi(1,0)$. Thus $y_j \geq 1$. If $x_j \equiv \mu_j(g) < n$, let j delete his links and form a single link with i. Since he is noncritical, his new payoff will be $\Phi(n,1) > \Phi(x_j, 1) \geq \Phi(x_j, y_j)$, i.e. he will again do better. It follows that $\mu_j(g) = n$ as required.

We claim that for every agent $j_1 \notin E \cup \{i\}$, there exists $j \in E$ such that $j \in N(j_1; g)$. Since j_1 is critical, there exists $j_2 \in N(j_1; g)$ such that every path from j_2 to i in g involves agent j_1. Hence $d(i, j_2; g) > d(i, j_1; g)$. If $j_2 \in E$ we are done; otherwise, by the same argument, there exists $j_3 \in N(j_2; g)$ such that $d(i, j_3; g) > d(i, j_2; g)$. Since i observes every agent and N is finite, repeating the above process no more than $n-2$ times will yield an agent $j \in E$ such that $j \in N(j_1; g)$. Since we have shown $\mu_j(g) = n$, we have $\mu_{j_1}(g) = n$ as well. Hence g is connected. If g were not minimally connected, then some agent could delete a link and still observe every agent in the society, thereby increasing his payoff, in which case g is not Nash. The result follows. *Q.E.D.*

APPENDIX B

PROOF OF PROPOSITION 4.1: Let g be a nonempty Nash network and suppose it is not tw-connected. Since g is nonempty there exists a tw-component C such that $|C| \equiv x \geq 2$. Choose $i \in C$ satisfying $\mu_i^d(g) \geq 1$. Then we have $\Phi(x,1) \geq \Phi(x, \mu_i^d(g)) = \Phi(\mu_i(\bar{g}), \mu_i^d(g)) = \overline{\Pi}_i(g)$. Note that g_{-i} can be regarded as the network where i forms no links. Since g is Nash, $\overline{\Pi}_i(g) \geq \overline{\Pi}_i(g_{-i}) = \Phi(\mu_i(g_{-i}), 0) \geq \Phi(1,0)$. Thus, $\Phi(x,1) \geq \Phi(1,0)$. As g is not tw-connected, there exists $j \in N$ such that $j \notin C$. If j is a singleton tw-component then the payoff to agent j from a link with i is $\Phi(x+1,1) > \Phi(x,1) \geq \Phi(1,0)$, which violates the hypothesis that agent j is choosing a best response. Suppose instead that j lies in a tw-component D where $|D| \equiv w \geq 2$. By definition there is at least one agent in D who forms links; assume without loss of generality that j is this agent. As with agent i we have $\Phi(w,1) \geq \overline{\Pi}_j(g)$.

Suppose without loss of generality that $w \leq x = |C|$. Suppose agent j deletes all his links and instead forms a single link with agent $i \in C$. Then his payoff is at least $\Phi(x+1,1) > \Phi(w,1) \geq \overline{\Pi}_j(g)$. This violates the hypothesis that agent j is playing a best response. The contradiction implies g is tw-connected. If g is not minimally tw-connected, there exists an agent who can delete a link and still have a tw-path with every other agent, so that g is not Nash. The result follows. *Q.E.D.*

LEMMA 4.1: *Let the payoffs be given by* (2.3). *Starting from any initial network g, the dynamic process* (2.7) *moves with positive probability either to a minimally tw-connected network or to the empty network, in finite time.*

[41] This assertion requires the assumption that $n \geq 3$. If $n = 2$ and $\Phi(2,1) = \Phi(1,0)$, then the disconnected network $g_{1,2} = 1$, $g_{2,1} = 0$ is a Nash network.

PROOF: We first show that the process transits with positive probability to a network all of whose components are tw-minimal. Starting with agent 1, let each agent choose a best response one after the other and let g' denote the network after all agents have moved. Let C be a tw-component of g'. Suppose there is a tw-cycle in C, i.e. there are $q \geq 3$ agents $\{j_1, \ldots, j_q\} \subset C$ such that $\bar{g}'_{j_1, j_2} = \cdots = \bar{g}'_{j_q, j_1} = 1$. Let $S \subset \{j_1, \ldots, j_q\}$ consist of those agents who have formed at least one link within the tw-cycle. Note that S is nonempty. Let j_s be the agent who has played most recently amongst those in S, and assume without loss of generality that $g'_{j_s, j_{s-1}} = 1$. Let g'' be the network prior to agent j_s's move. By definition of j_s we have

$$\text{(B.1)} \qquad \bar{g}''_{j_{s+1}, j_{s+2}} = \cdots = \bar{g}''_{j_q, j_1} = \cdots = \bar{g}''_{j_{s-2}, j_{s-1}} = 1.$$

Consider agent j_s's best response to g''_{-j_s}. There are two possibilities: either $g''_{j_{s+1}, j_s} = 1$, or $g''_{j_{s+1}, j_s} = 0$. In the former case, by virtue of (B.1), agent j_s can get the same information as before without forming the link with j_{s-1}. In the latter case, j_s forms links with both j_{s-1} and j_{s+1} as part of his best response. However, (B.1) again implies that he is strictly better off by forming a link with only one of them. This contradiction shows that C cannot have a tw-cycle. A similar argument shows that $g'_{i,j} = 1$ for two agents i and j in C implies $g'_{j,i} = 0$. Since C is an arbitrary tw-component of g', every such tw-component must be minimal.

Let C_1 be the largest tw-component in g'. If $|C_1| = n$ or $|C_1| = 1$ we are done. Suppose instead that $|C_1| = x$ where $1 < x < n$. Denote the agents in $N \setminus C_1$ as S. There are now two cases (1) and (2).

(1) The unique best response of every agent in S is not to form any links: let all agents in S move simultaneously, with all the agents in C_1 exhibiting inertia. Call the resulting network g^1. Clearly, g^1 has one nonsingleton tw-component C_1 and $|S|$ singleton tw-components. Let $j \in S$. (1a) Suppose j's unique best response is not to form any links. Then $\Phi(x+u, u) \leq \Phi(1, 0)$ for all $u \in \{1, \ldots, |S|\}$ since he has the option of forming links with any subset of the remaining $|S|$ tw-components. If $i \in C_1$ has formed any links, the highest payoff from $u \geq 1$ links is $\Phi(x+u, u) \leq \Phi(1, 0)$ so that to delete all links is a best response. If all the agents in C_1 who have links are allowed to move simultaneously, the empty network results. (1b) Suppose instead that all of j's best responses involve forming one or more links. Since C_1 is the unique nonsingleton tw-component, any best response $\hat{g}_j$ must involve forming a link with C_1. Define $g^2 = \hat{g}_j \oplus g^1_{-j}$. Using above arguments it is easily seen that all tw-components of g^2 are minimal. Let C_2 be the largest tw-component in g^2. Clearly, $C_1 \subset C_2$ with the inclusion being strict. Now proceed likewise with the other singleton tw-components to arrive at a minimal tw-connected network.

(2) There exists an agent j in S all of whose best responses to g' involve forming one or more links: as is (1b), if we let j choose a best response, we obtain a new network g'' where the largest component C_2 satisfies $C_1 \subset C_2$ with the inclusion being strict. Moreover, it can be seen that all tw-components of g'' are minimal. We repeat (1) or (2) with g'' in place of g' and so on until either the empty network or a minimal tw-connected network is obtained. *Q.E.D.*

LEMMA 4.2: *Let g be a minimally tw-connected network. Suppose $\mu_i^d(g) = u \geq 0$. If agent i deletes $s \leq u$ links, then the resulting network has $s+1$ minimal tw-components, $C_1, \ldots, C_{s+1}$, with $i \in C_{s+1}$.*

PROOF: Let g' be the network after i deletes s links, say, with agents $\{j_1, \ldots, j_s\}$. Since g is minimally tw-connected there is a unique tw-path between every pair of agents i and j in g. In particular, if i deletes s links, then each of the s agents $j_1, j_2, j_3, \ldots, j_s$, have no tw-path linking them with agent i as well as no tw-path linking them with each other either. Thus each of the s agents and agent i must lie in a distinct tw-component, implying that there are at least $s+1$ tw-components in the network g'.

We now show that there cannot be more than $s+1$ tw-components. Suppose not. Let $j_1, j_2, \ldots, j_s$ and i belong to the first $s+1$ tw-components and consider an agent k who belongs to the $s+2$th tw-component. Since g is minimally tw-connected there is a unique tw-path between i and k in g; the lack of any such tw-path in g' implies that the unique tw-path between i and k must involve a now deleted link g_{i, j_q} for some $q = 1, 2, \ldots, s$. Thus in g there must be a tw-path between k and j_q, which does not involve agent i. Since only agent i moves in the transition from g to g', there is also a tw-path between k and j_q in g'. This contradicts the hypothesis that k lies in the $s+2$th tw-component. The minimality of each tw-component in g' follows directly from the hypothesis that g' is obtained by deleting links from a minimally tw-connected network. *Q.E.D.*

Lemma 4.2 implies that the following strategy is a best response.

Remark: Suppose $\Phi(x+1, y+1) > \Phi(x, y)$ for all $y \in \{0, \ldots, n-2\}$ and $x \in \{y+1, \ldots, n-1\}$. Let g and $C_1, \ldots, C_{s+1}$ be as in Lemma 4.2 above. Define g_i^* as $g_{i,k}^* = 1$ for one and only one k in each of $C_1, \ldots, C_s$ and $g_{i,k}^* = g_{i,k}$ for all $k \in N \setminus (C_1 \cup \cdots C_2 \cup \{i\})$. Then g_i^* is a best response to g_{-i}.

Proof of Theorem 4.1(b) (Sketch): The hypothesis on the payoffs implies that $\Phi(1,0) > \max_{\{1 \leq x \leq n-1\}} \Phi(x+1, x)$. Proposition 4.2(b) then implies that the empty network is the unique strict Nash network, and hence is an absorbing state for the dynamic process. Next note that if $\Phi(1,0) \geq \Phi(n,1)$, then it is a (weakly) dominant strategy for a player to form no links. In this case convergence to the empty network is immediate. We focus on the case where $\Phi(n,1) > \Phi(1,0)$.

Fix an initial nonempty network g. From Lemma 4.1 we can assume without loss of generality that g is a minimal tw-connected network. Let $n = \operatorname{argmax}_{i \in N} \sigma(i; \bar{g})$ and let $P(n; g)$ and $I(n; g)$ be the set of outward-pointing agents and the set of inward-pointing agents vis-a-vis n, respectively. In addition, define $E(n; g)$ as the end-agents in the network g and let $P^e(n; g) = E(n; g) \cap P(n; g)$ be the set of outward-pointing end-agents. Since $\Phi(n,1) > \Phi(1,0)$, we can apply the argument for outward-pointing agents in part (a) of Theorem 4.1 to have every agent $j \in P^e(n; g)$ form a link $g_{j,n} = 1$. Let g' be the network that results after every $j \in P^e(n; g)$ has moved, and formed a link with n. Define $P^e(n; g')$ analogously, and proceed as before, with every $j \in P^e(m; g')$. Repeated application of this argument leads us eventually to either the periphery-sponsored star or a network in which all end-agents more than one link away from agent n are inward-pointing with respect to n. In the former case a simple variant of the miscoordination argument establishes convergence to the empty network. In the latter case, label the network as g^1 and proceed as follows.

Note that the hypothesis on payoffs implies that if agent i has a link with an end-agent, i's best response must involve deleting that link. Let j be the agent furthest away from n in g^1. Since g^1 is minimally tw-connected, there is a unique path between j and n. Then either $g_{n,j}^1 = 1$ or there is an agent $j_q \neq n$ on the path between n and j, such that $g_{j_q, j} = 1$. In the former case, g^1 must be a star: if n chooses a best response, he will delete all his links, after which a miscoordination argument ensures that the empty network results. In the latter case, let j_q choose a best response and let g^2 denote the resulting network. Clearly j_2 will delete his link with j, in which case j will become a singleton component. Moreover, if j_2 forms any link at all, we can assume without loss of generality that he will form it with n. Let S_2 and S_1 be the set of agents in singleton components in g^2 and g^1, respectively. We have $S_1 \subset S_2$ where the inclusion is strict. Repeated application of the above arguments leads us to a network in which either an agent is a singleton component or is part of a star. If every agent falls in the former category, then we are at the empty network while in the latter case we let agent n move and delete all his links. Then a variant of the miscoordination argument (applied to the periphery-sponsored star) leads to the empty network. *Q.E.D.*

APPENDIX C

Proof of Proposition 5.1 (Sketch): If $c < \delta$, then it is immediate that a Nash network is connected. In the proof we focus on the case $c \geq \delta$. The proof is by contradiction. Consider a strict Nash network g that is nonempty but disconnected. Then there exists a pair of agents i_1 and i_2 such that $g_{i_1, i_2} = 1$. Moreover, since $c \geq \delta$ and g is strict Nash, there is an agent $i_3 \neq i_1$ such that $g_{i_2, i_3} = 1$. The same property must hold for i_3; continuing in this way, since N is finite, there must exist a cycle of agents, i.e. a collection $\{i_1, \ldots, i_q\}$ of three or more agents such that $g_{i_1,i_2} = \cdots = g_{i_q,i_1} = 1$. Denote the component containing this cycle as C. Since g is not connected there exists at least one other component D. We say there is a path from C to D if there exists $i \in C$ and $j \in D$ such that $i \overset{g}{\to} j$. There are two cases: (1) there is no path from C to D or vice-versa, and (2) either $C \overset{g}{\to} D$ or $D \overset{g}{\to} C$.

In case (1), let $i \in C$ and $j \in D$. Since g is strict Nash we get

$$\Pi_i(g_i \oplus g_{-i}) > \Pi_i(g_i' \oplus g_{-i}), \qquad \text{for all } g_i' \in \mathscr{G}_i, \text{ where } g_i' \neq g_i, \tag{C.1}$$

$$\Pi_j(g_j \oplus g_{-j}) > \Pi_j(g_j' \oplus g_{-j}), \qquad \text{for all } g_j' \in \mathscr{G}_j, \text{ where } g_j' \neq g_j. \tag{C.2}$$

Consider a strategy g_i^* such that $g_{i,k}^* = g_{j,k}$ for all $k \notin \{i, j\}$ and $g_{i,j}^* = 0$. The strategy g_i^* thus "imitates" that of agent j. By hypothesis, $j \notin N(i; g)$ and $i \notin N(j; g)$. This implies that the strategy of agent i has no bearing on the payoff of agent j or vice-versa. Hence, i's payoff from g_i^* satisfies

$$\Pi_i(g_i^* \oplus g_{-i}) \geq \Pi_j(g_j \oplus g_{-j}). \tag{C.3}$$

Likewise, the payoff to agent j from the corresponding strategy g_j^* that imitates i satisfies

$$\Pi_j(g_j^* \oplus g_{-j}) \geq \Pi_i(g_i \oplus g_{-i}). \tag{C.4}$$

We know that C is not a singleton. This immediately implies that the strategies g_i and g_i^* must be different. Putting together equations (C.2)–(C.4) with g_i^* in place of g_i' and g_j^* in place of g_j' yields

$$\Pi_i(g_i \oplus g_{-i}) > \Pi_i(g_i^* \oplus g_{-i}) \geq \Pi_j(g_j \oplus g_{-j}) > \Pi_j(g_j^* \oplus g_{-j}) \geq \Pi_i(g_i \oplus g_{-i}). \tag{C.5}$$

The contradiction completes the argument for case (1). In case (2) we choose an $i' \in N(i; g)$ who is furthest away from $j \in D$ and apply a similar argument to that in case (1) to arrive at a contradiction. The details are omitted. The rest of the proposition follows by direct verification. *Q.E.D.*

Proof of Proposition 5.2: Consider the case of $s = 1$ and $c \in (0, 1)$ first. Let g be a flower network with central agent n. Let $M = \max_{i,j \in N} d(i, j; g)$. Note that $2 \leq M \leq n - 1$ by the definition of a flower network. Choose $\delta(c, g) \in (c, 1)$ such that for all $\delta \in [\delta(c, g), 1]$ we have $(n - 2)(\delta - \delta^M) < c$. Henceforth fix $\delta \in [\delta(c, g), 1]$. Suppose $P = \{j_1, \ldots, j_u\}$ is a petal of g. Since $c < \delta$ and no other agent has a link with j_u, agent n will form a link with him in his best response. If n formed any more links than those in g, an upper bound on the additional payoff he can obtain is $(n - 2)(\delta - \delta^M) - c < 0$; thus, n is playing a best response in g. The same argument ensures that agents $j_2, \ldots, j_u$ are also playing their best response. It remains to show the same for j_1. If there is only a single petal (i.e. g is a wheel) symmetry yields the result. Suppose there are two or more petals. For j_1 to observe all the other agents in the society, it is necessary and sufficient that he forms a link with either agent n or some agent $j' \in P'$, where $P' \neq P$ is another petal. Given such a link, the additional payoff from more links is negative, by the same argument used with agent n. If he forms a link with j' rather than n, agent j_1 will get the same total payoff from the set of agents $P' \in \{n\}$ since the sub-network of these agents is a wheel. However, the link with j' means that to access other petals (including the remaining agents in P, if any) agent j_1 must first go through all the agents in the path from n to j', whereas with n he can avoid these extra links. Hence, if there are at least three petals, forming a link with j' will make j strictly worse compared to forming it with n, so that g is a strict Nash network as required. If g contains only two petals P and P', both of level 2 or higher, j_1's petal will contain at least one more agent, and the argument above applies. Finally, if there are two petals P and P' and g is of level 1, then g is the exceptional case, and it is not a strict Nash. Thus, unless g is the exceptional case, it is a strict Nash for all $\delta \in [\delta(c, g), 1)$.

Next, consider $c \in (s - 1, s)$ for some $s \in \{1, \ldots, n - 1)\}$. If g is a flower network of level less than s, there is some petal $P = \{j_1, \ldots, j_{s'}\}$ with $s' \leq s - 1$. Clearly the central agent n can increase his payoff by deleting his link with $j_{s'}$, ceteris paribus. Hence, a flower network of level smaller than s cannot be Nash.

Let g now be a flower network of level s or more. Let $M = \max_{i, j \in N} d(i, j; g)$. Choose $\delta(c, g)$ to ensure that for all $\delta \in [\delta(c, g), 1]$ both (1) $(n-2)(\delta - \delta^M) < \delta$ and (2) $\sum_{q=1}^{s} \delta^q - c > 0$ are satisfied. Let $P = \{j_1, \ldots, j_u\}$ be a petal with $u \geq s$. The requirement (2) ensures that agent n will wish to form a link with j_u. The requirement (1) plays the same role as in $s = 1$ above to ensure that n will not form more than one link per petal. If g has only one petal (i.e. it is a wheel) we are done. Otherwise, analogous arguments show that $\{j_2, \ldots, j_p\}$ are playing their best responses in g. Finally, for j_1, note that $u \geq 2$ implies that each petal is not a spoke. In this event, the argument used in part (a) shows that j_1 will be strictly worse off by forming a link with an agent other than agent n. The result follows. *Q.E.D.*

PROOF OF PROPOSITION 5.5: Consider a network g, and suppose that there is a pair of agents i and j, such that $g_{i,j} \neq 1$. If agent i forms a link $g_{i,j} = 1$, then the additional payoffs to i and j will be at least $2(\delta - \delta^2)$. If $c < 2(\delta - \delta^2)$, then this is clearly welfare enhancing. Hence, the unique efficient network is the complete network.

Fix a network g and consider a tw-component C_1, with $|C_1| = m$. If $m = 2$ then the nature of a component in an efficient network is obvious. Suppose $m \geq 3$ and let $k \geq m - 1$ be the number of links in $|C_1|$. The social welfare of this component is bounded above by $m + k(2\delta - c) + [m(m-1) - 2k]\delta^2$. If the component is a star, then the social welfare is $(m-1)[2\delta - c + (m-2)\delta^2] + m$. Under the hypothesis that $2(\delta - \delta^2) < c$, the former can never exceed the latter and is equal to the latter only if $k = m - 1$. It can be checked that the star is the only network with m agents and $m - 1$ links, in which every pair of agents is at a distance of at most 2. Hence the network g must have at least one pair of agents i and j at a distance of 3. Since the number of direct links is the same while all indirect links are of length 2 in a star, this shows that the star welfare dominates every other network with $m - 1$ links. Hence the component must be a star.

Clearly, a tw-component in an efficient network must have nonnegative social welfare. It can be calculated that the social welfare from a network with two distinct components of m and m' agents, respectively, is strictly less than the social welfare from a network where these distinct stars are merged to form a star with $m + m'$ agents. It now follows that a single star maximizes the social welfare in the class of all nonempty networks. An empty network yields the social welfare n. Simple calculations reveal that the star welfare dominates the empty network if and only if $2\delta + (n-2)\delta^2 > c$. This completes the proof. *Q.E.D.*

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