Complexity of Vehicle Routing and Scheduling Problems

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The complexity of a class of vehicle routing and scheduling problems is investigated. We review known NP-hardness results and compile the results on the worst-case performance of approximation algorithms. Some directions for future research are suggested. The presentation is based on two discussion sessions during the Workshop to Investigate Future Directions in Routing and Scheduling of Vehicles and Crews, held at the University of Maryland at College Park, June 4–6, 1979.

I. INTRODUCTION

In this paper the computational complexity of a class of vehicle routing and scheduling problems is investigated. The problem class is defined in Sec. II. We review known NP-hardness results for the problems in this class in Sec. III, and we compile the results on the worst-case performance of approximation algorithms designed for their solution in Sec. IV. Some directions for future research are suggested in Sec. V.

The results presented in this paper were the subject of two discussion sessions during the Workshop to Investigate Future Directions in Routing and Scheduling of Vehicles and Crews, held at the University of Maryland at College Park, June 4-6, 1979.

II. A CLASS OF PROBLEMS

The general *single vehicle routing problem* (VRP) [26] is defined as follows: Given a strongly connected mixed graph G consisting of a set V of v vertices, a set E of e (undirected) edges and a set A of a (directed) arcs, with specified subsets $V' \subseteq V$, $E' \subseteq E$ and $A' \subseteq A$, and given non-negative weights on the edges and the arcs, find a tour containing V', E' and A' which is of minimum total weight. Various well-known routing problems emerge for specific restrictions on E, A, V', E', and A'; they are defined in Table I.

The *m-vehicle routing problem* (mVRP) is a natural extension of the VRP. The purpose is to find m tours, each containing a common distinguished vertex (the depot) and collectively containing the sets V', E' and A', such that the maximum of the total

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TABLE I. Single vehicle routing problems.

Name	Code	E	A	V'	E'	A'
Traveling salesman problem	TSP	—[a]	ϕ	V	ϕ	ϕ
Directed traveling salesman problem	DTSP	ϕ	—	V	ϕ	ϕ
Chinese postman problem	CPP	—	ϕ	ϕ	E	ϕ
Directed Chinese postman problem	DCPP	ϕ	—	ϕ	ϕ	A
Mixed Chinese postman problem	MCPP	—	—	ϕ	E	A
Rural postman problem	RPP	—	ϕ	ϕ	—	ϕ
Directed rural postman problem	DRPP	ϕ	—	ϕ	ϕ	—
Stacker-Crane problem	SCP	—	—	ϕ	ϕ	A

[a]—: Arbitrary.

weights of the tours is minimized. The resulting special cases are referred to as the mTSP, the mDTSP, etc.

The generic *single depot vehicle scheduling problem* (VSP) is the following: given a depot d and n trips j from b_j to c_j, which have to be completed within specified time intervals $[t_j, u_j]$ ($j = 1, \ldots, n$), and given the traveling times between all pairs (d, b_j), (b_j, c_j), (c_j, b_k) and (c_j, d), find a feasible schedule which requires a minimum number of vehicles. Special cases to be considered correspond to restrictions such as $t_j = u_j$ or $t_j = 0, u_j = u$ for $j = 1, \ldots, n$.

The *l-depot vehicle scheduling problem* (lVSP) is a generalization in which there are l depots d_i, where m_i vehicles are located ($i = 1, \ldots, l$); each vehicle has to return to its depot.

III. NP-HARDNESS RESULTS

The basic results on the computational complexity of vehicle routing and scheduling problems are listed in Table II. For the *easy* problems, which are solvable in polynomial time, the running time of the most efficient known algorithm for their solution is given. The *NP-hard* problems are not solvable in polynomial time, unless $\mathcal{P} = \mathcal{NP}$. We refer to [13, 19, 23] for introductions to the theory of NP-completeness and to [13, A1.3, A2.3] for additional details.

The NP-hardness results for routing problems still apply if G is *planar*; see, e.g., [14, 27]. We also note that even the *geometric* TSP, which is defined by points and distances in the Euclidean plane, is NP-hard [11, 28].

All NP-hardness results mentioned are "strong" in the sense that they hold even with respect to a *unary* encoding of the problem data [12]. However, for any fixed $m \geqslant 2$, the mCPP and the mDCPP are only known to be *binary* NP-hard.

In summary, almost all vehicle routing and scheduling problems are NP-hard and hence unlikely to be solvable in polynomial time. As a means to differentiate further within the class of NP-hard problems, we will consider the worst-case performance of fast approximation algorithms in the next section. A less formal indication of the complexity of routing problems is the number of disconnected components in the graph induced by V', E' and A'. For example, when there are c of such components, the RPP can be solved recursively in $O(v^{2c+1}/c!)$ time [9].

TABLE II. Complexity of vehicle routing and scheduling problems.

Problem	Complexity	Reference
Routing		
VRP	NP-hard	
TSP	NP-hard	[18]
DTSP	NP-hard	[18]
CPP	$O(v^3)$	[7]
mCPP	NP-hard	[10]
DCPP	$O(v^3 \log a)$	[8]
mDCPP	NP-hard	[10]
MCPP	NP-hard	[27]
RPP	NP-hard	[21]
DRPP	NP-hard	[21]
SCP	NP-hard	[10]
Scheduling		
VSP (all $t_j = u_j$)	$O(n^3)$	[6]
VSP (all $t_j = 0$, all $u_j = u$)	NP-hard	[22]
lVSP (all $t_j = u_j$)	open	

IV. WORST-CASE PERFORMANCE OF APPROXIMATION ALGORITHMS

All results on the worst-case performance of specific approximation algorithms for vehicle routing problems that we are aware of are listed in Table III. The performance is usually measured by the maximum ratio ρ of the approximate solution value to the optimum value, over all instances of the problem in question. The table gives global upper bounds on ρ, as well as lower bounds on ρ that can (asymptotically) be achieved for a class of "bad" instances. All terms that tend to zero when v increases have been deleted; log denotes the logarithm to the base 2.

The theory of NP-completeness has been applied to show that, for some NP-hard optimization problems, certain approximation algorithms which guarantee a fixed maximum performance ratio ρ do not exist, unless $\mathcal{P}=\mathcal{NP}$. Results of this type for vehicle routing problems are listed in Table IV. These problems require some comments.

The *capacitated* mTSP (mCPP) is a modification of the mTSP (mCPP), in which each vertex (edge) has a given demand and the total demand in each tour should not exceed a given limit. The objective in this case is to minimize the sum of the total tour weights rather than their maximum.

The *general* TSP is usually defined as the problem of finding a tour of minimum total weight which visits each vertex exactly once. The TSP in our definition allows multiple visits, but can be seen as a special case of the general TSP in which the weights satisfy the *triangle inequality*. Conversely, the TSP with arbitrary weights can be transformed into the TSP for which the triangle inequality holds by adding a suitably large constant to all weights. The distinction between both problem types, however, is justified by the results in Tables III and IV.

Additional results for the general TSP are the following. Local search over polynomial-

TABLE III. Worst-case performance of vehicle routing approximation algorithms.

Problem	Algorithm	Upper Bound	Lower Bound	Complexity	Reference
TSP	nearest neighbor	$\frac{1}{2}\lceil \log v \rceil + \frac{1}{2}$	$\frac{1}{3}\log(v+1) + \frac{4}{9}$	$O(v^2)$	[32]
	sequential Clarke–Wright		$\frac{2}{7}\log v + \frac{5}{21}$	$O(v^2)$	[15]
	insertion	$\lceil \log v \rceil + 1$	4	$O(v^2)$	[32]
	nearest insertion	2	2	$O(v^2)$	[32]
	cheapest insertion	2	2	$O(v^2 \log v)$	[32]
	nearest addition	2	2	$O(v^2)$	[32]
	nearest merger	2	2	$O(v^2 \log v)$	[32]
	k-optimal for all $k \leqslant \frac{v}{4}$		2		[32]
	double spanning tree	2	2	$O(v^2)$	[32]
	spanning tree + matching	$\frac{3}{2}$	$\frac{3}{2}$	$O(v^3)$	[3, 5]
DTSP	repeated assignment	$\lceil \log v \rceil$		$O(v^3)$	[25]
MCPP	Edmonds–Johnson	2	2	$O(v^3 + e^2a + a^3)$	[9]
	reversed Edmonds–Johnson	2	2	$O(v^3 + e^2a + a^3)$	[9]
	mixed strategy	$\frac{5}{3}$	$\frac{3}{2}$	$O(v^3 + e^2a + a^3)$	[9]
planar MCPP	mixed strategy	$\frac{3}{2}$	$\frac{3}{2}$	$O(v^3 + e^2a + a^3)$	[9]
RPP	spanning tree + matching	$\frac{3}{2}$	$\frac{3}{2}$	$O(v^3 + e)$	[9]
SCP	mixed strategy	$\frac{9}{5}$		$O(v^3 + a^3)$	[10]
mTSP	nearest neighbor	$\frac{m}{2}\log v + m$	$\frac{m}{6}\log v$	$O(v^2)$	[10]
	nearest insertion	$2m$	$2m$	$O(v^2)$	[10]
	tour splitting	$\frac{5}{2} - \frac{1}{m}$	$\frac{5}{2} - \frac{1}{m}$	$O(v^3)$	[10]
mCPP	tour splitting	$2 - \frac{1}{m}$		$O(v^3)$	[10]
mSCP	tour splitting	$\frac{14}{5} - \frac{1}{m}$		$O(v^3 + a^3)$	[10]

TABLE IV. Nonexisting vehicle routing approximation algorithms (unless $\mathcal{P}=\mathcal{NP}$).

Problem	Algorithm	ρ	Reference
Any unary NP-hard problem	Algorithm polynomial in problem size and $\frac{1}{\epsilon}$ for all $\epsilon > 0$	$1+\epsilon$	[12]
General TSP	Polynomial-time algorithm	$<\infty$	[33]
	Local search with polynomial time per iteration	$<\infty$	[29]
Capacitated mTSP on a tree	Polynomial-time algorithm	$<\frac{3}{2}$	[16]
Capacitated mCPP on a tree	Polynomial-time algorithm	$<\frac{3}{2}$	[16]

size neighborhoods will never guarantee optimality [34], and instances have been constructed for which local search would be particularly ineffective [30].

Altogether, there appear to be considerable differences in complexity within the class of NP-hard problems. Many of the polynomial transformations between these problems that preserve optimality, clearly do not preserve the performance of approximation algorithms. The transformation of the general TSP to the TSP provides a striking example of this phenomenon. Transformations that preserve the problem structure to a greater extent are the subject of ongoing research [1, 24, 31].

V. CONCLUDING REMARKS

The survey presented in Secs. III and IV bears witness to an impressive research effort in analyzing the inherent complexity of vehicle routing and scheduling problems. It is also clear that more work needs to be done. The complexity status of the lVSP is still open. The worst-case analysis of some of the standard approximation algorithms is nonexistent or incomplete. And for the DTSP, no polynomial-time algorithm is known to guarantee a constant maximum performance ratio.

It should be pointed out that the worst-case approach is pessimistic in the sense that approximation algorithms rarely attain their maximum performance ratio in practice. For example, the TSP algorithm from [3], in which a spanning tree is combined with a matching on its odd-degree vertices, yields a solution value that tends to be much closer to the optimum than the guaranteed 50% deviation. In a clever implementation of this algorithm [4], a spanning tree is found using v subgradient iterations as in [17]; by then, the number of odd-degree vertices is often so small that a matching is found quickly by complete enumeration. This produces both a lower bound and an upper bound on the optimum, which usually differ by no more than a few percent.

Probabilistic analyses of the average-case or almost-everywhere performance of approximation algorithms have to provide a theoretical explanation of these phenomena. For the geometric TSP, such an approach has led to some remarkable results [20].

Finally, we note that there are several developments on the interface of mathematical programming and complexity theory that might ultimately influence the area of routing and scheduling as well. Suffice it to mention the efforts to relate the existence of polynomial-time algorithms to the existence of good characterizations of the polytope of feasible solutions, and the recent development of a polynomial-time algorithm for linear programming [2]. It seems that complexity theory interpreted in a broad sense will continue to have a direct impact on the study of vehicle routing and scheduling problems.

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