The Diagnosability of the Matching Composition Network under the Comparison Diagnosis Model

Pao-Lien Lai, Jimmy J.M. Tan, Chang-Hsiung Tsai, and Lih-Hsing Hsu

Abstract—The classical problem of diagnosability is discussed widely and the diagnosability of many well-known networks have been explored. In this paper, we consider the diagnosability of a family of networks, called the Matching Composition Network (MCN); two components are connected by a perfect matching. The diagnosability of MCN under the comparison model is shown to be one larger than that of the component, provided some connectivity constraints are satisfied. Applying our result, the diagnosability of the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n can all be proven to be n, for $n \geq 4$. In particular, we show that the diagnosability of the four-dimensional Hypercube Q_4 is 4, which is not previously known.

Index Terms—Diagnosability, *t*-diagnosable, comparison model, Matching Composition Network, MM* model.

1 Introduction

WITH the rapid development of technology, the need for high-speed parallel processing systems has been continuously increasing. The reliability of the processors in parallel computing systems is therefore becoming an important issue. In order to maintain the reliability of a system, whenever a processor (node) is found faulty, it should be replaced by a fault-free processor (node). The process of identifying all the faulty nodes is called the *diagnosis of the system*. The maximum number of faulty nodes that the system can guarantee to identify is called the *diagnosability of the system*.

In this paper, we consider the diagnosability of the system under the comparison model, proposed by Malek and Maeng [16], [17]. The diagnosability of some well-known interconnection networks under the comparison model has been investigated. For example, Wang [21], [22] showed that the diagnosability of an n-dimensional hypercube Q_n is n for $n \geq 5$ and the diagnosability of an n-dimensional enhanced hypercube is $n+1$ for $n \geq 6$. Fan [12] proved that the diagnosability of an n-dimensional crossed cube is n for $n \geq 4$. Araki and Shibata [1] proposed that the k-ary r-dimensional butterfly network $BF(k,r)$ is $2k$-diagnosable for $k \geq 2$ and $r \geq 5$. Besides, the diagnosability of the Hypercubes, the Crossed cubes, and the Möbius cubes under the PMC diagnostic model were also studied in [2], [10], [11], [14].

We study the diagnosability of a family of interconnection networks, called the Matching Composition Networks (*MCN*), which can be recursively constructed. MCN includes many well-known interconnection networks as special cases, such as the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n. Basically, MCN and these mentioned cubes are all constructed from two graphs G_1 and G_2 with the same number of nodes by adding a perfect matching between the nodes of G_1 and G_2. We shall call these two graphs G_1 and G_2 the *components* of MCN.

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Manuscript received 8 Oct. 2002; revised 7 Nov. 2003; accepted 15 Jan. 2004. For information on obtaining reprints of this article, please send e-mail to: tc@computer.org, and reference IEEECS Log Number 117536.

Our main result is the following: Suppose that the number of nodes in each component is at least $t+2$, the order (which will be defined subsequently) of each node in G_i is t, and the connectivity of G_i is also t, $i = 1, 2$. We prove that the diagnosability of MCN constructed from G_1 and G_2 is $t+1$ under the comparison model, for $t \geq 2$. In other words, the diagnosability of MCN is increased by one as compared with those of the components. Using our result, it is straightforward to see that the diagnosability of the Hypercube Q_n, the Crossed cube CQ_n, the Twisted cube TQ_n, and the Möbius cube MQ_n are n for $n \geq 4$. Some of these particular applications are previously known results [12], [22], using rather lengthy proofs. Our approach unifies these special cases and our proof is much simpler. We would like to point out that the diagnosability of the four-dimensional Hypercube Q_4 is 4, which is not previously known [12], [22]. The diagnosability of the Twisted cube TQ_n and the Möbius cube MQ_n, as far as we know, are not yet resolved until now.

The rest of this paper is organized as follows: Section 2 introduces the comparison model for diagnosis. Section 3 provides preliminaries. In Section 4, we present the Matching Composition Network and discuss its diagnosability. We then discuss the diagnosability of Q_n, CQ_n, TQ_n, and MQ_n in Section 5. Finally, our conclusions are given in Section 6.

2 The Comparison Model for Diagnosis

For the purpose of self-diagnosis of a given system, several different models have been proposed in the literature [16], [17], [18]. Preparata et al. [18] first introduced a model, the so-called PMC-model, for system level diagnosis in multiprocessor systems. In this model, it is assumed that a processor can test the faulty or fault-free status of another processor.

The *comparison model*, called the *MM model*, proposed by Maeng and Malek [16], [17], is considered to be another practical approach for fault diagnosis in multiprocessor systems. In this approach, the diagnosis is carried out by sending the same testing task to a pair $\{u, v\}$ of processors and comparing their responses. The comparison is performed by a third processor w that has direct communication links to both processors u and v. The third processor w is called a *comparator* of u and v.

If the comparator is fault-free, a disagreement between the two responses is an indication of the existence of a faulty processor. To gain as much knowledge as possible about the faulty status of the system, it was assumed that a comparison is performed by each processor for each pair of distinct neighbors with which it can communicate directly. This special case of the MM-model is referred to as the *MM*-model*. Sengupta and Dahbura [20] studied the MM-model and the MM*-model, gave a characterization of diagnosable systems under the comparison approach, and proposed a polynomial time algorithm to determine faulty processors under MM*-model. In this paper, we study the diagnosability of MCN (which will be defined subsequently) under the MM*-model.

In the study of multiprocessor systems, the topology of networks is usually represented by a graph $G = (V, E)$, where each node $v \in V$ represents a processor and each edge $(u, v) \in E$ represents a communication link. The diagnosis by comparison approach can be modeled by a labeled multigraph, called the *comparison graph*, $M = (V, C)$, where V is the set of all processors and C is the set of labeled edges. A labeled edge $(u, v)_w \in C$, with w being a label on the edge, connects u and v, which implies that processors u and v are being compared by w. Under the MM-model, processor w is a comparator for processors u and v only if $(w, u) \in E$ and $(w, v) \in E$. The MM*-model is a special case of the MM model; it is assumed that each processor w such that $(w, u) \in E$ and $(w, v) \in E$ is a comparator for the pair of processors u and v. The comparison graph $M = (V, C)$ of a given system can

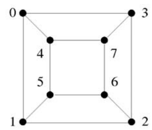


Fig. 1. An example for $T(G,U)$ of Q_3.

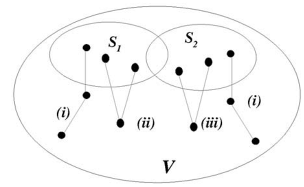


Fig. 2. Description of distinguishability for Theorem 1.

be a multigraph for the same pair of nodes may be compared by several different comparators.

For $(u,v)_w \in C$, the output of comparator w of u and v is denoted by $r((u,v)_w)$, a disagreement of the outputs is denoted by the comparison results $r((u,v)_w)=1$, whereas an agreement is denoted by $r((u,v)_w)=0$.

In this paper, in order to be consistent with the MM model, we have the following assumptions [20]:

1. All faults are permanent;
2. A faulty processor produces incorrect outputs for each of its given testing tasks;
3. The output of a comparison performed by a faulty processor is unreliable; and
4. Two faulty processors with the same input do not produce the same output.

Therefore, if the comparator w is fault-free and $r((u,v)_w)=0$, then u and v are both fault-free. If $r((u,v)_w)=1$, then at least one of u, v, and w must be faulty. The set of all comparison results of a multicomputer system that are analyzed together to determine the faulty processors is called a *syndrome* of the system.

For a given syndrome σ, a subset of nodes $F \subseteq V$ is said to be *consistent* with σ if syndrome σ can be produced from the situation that all nodes in F are faulty and all nodes in $V-F$ are fault-free. Because a faulty comparator can lead to unreliable results, a given set F of faulty nodes may produce different syndromes. Let $\sigma^*(F)=\{\sigma \mid \sigma \text{ is consistent with } F\}$.

Two distinct sets $S_1, S_2 \subset V$ are said to be *indistinguishable* if and only if $\sigma^*(S_1)\bigcap\sigma^*(S_2)\neq\emptyset$; otherwise, S_1, S_2 are said to be *distinguishable*. A system is said to be *t-diagnosable* if, for every syndrome, there is a unique set of faulty nodes that could produce the syndrome, provided the number of faulty nodes does not exceed t.

3 Preliminaries

We need some definitions and previous results for further discussion. Let $G=(V,E)$ be a graph, if there are ambiguities, we shall write the node set V as $V(G)$ and edge set E as $E(G)$. Assume $U\subseteq V(G)$. $G[U]$ denotes the subgraph of G induced by the node subset U of G and $\bar{U}=V(G)-U$.

The *vertex connectivity* (simply abbreviated as *connectivity*) of a network $G=(V,E)$, denoted by $\kappa(G)$ or κ, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. Assume that V_1 and V_2 are two disjoint nonempty subsets of $V(G)$. The *neighborhood set* of V_1 in V_2, denoted by $N(V_2,V_1)$, is defined as $\{x\in V_2 \mid \text{ there exists a node } y\in V_1 \text{ such that } (x,y)\in E(G)\}$. A *vertex cover* of G is a subset $K\subseteq V(G)$ such that every edge of $E(G)$ has at least one end vertex in K. A vertex cover set with the minimum cardinality is called a *minimum vertex cover*.

Given a graph G, let M be the comparison graph of G. For a node $v\in V(G)$, we define X_v to be the set of nodes $\{u \mid (v,u)\in E(G)\}\bigcup\{u \mid (v,u)_w\in E(M) \text{ for some } w\}$ and Y_v to be the set of edges $\{(u,w) \mid u,w\in X_v \text{ and } (v,u)_w\in E(M)\}$. In [20], the *order graph* of node v is defined as $G_v=(X_v,Y_v)$ and the *order* of the node v, denoted by $order_G(v)$, is defined to be the cardinality of a minimum vertex cover of G_v. Let $U\subset V(G)$, we use $T(G,U)$ to denote the set $\{v \mid (u,v)_w\in E(M) \text{ and } w,u\in U, v\in\bar{U}\}$. We observe that $T(G,U)=N(\bar{U},U)$ if $G[U]$ is connected and $|U|>1$. This observation can be extended to the following lemma.

Lemma 1. *Let U be a subset of $V(G)$ and $G[U_i]$, $1\le i\le k$, be the connected components of the subgraph $G[U]$ such that $U=\bigcup_{i=1}^k U_i$. Then, $T(G,U)=\bigcup_{i=1}^k\{N(\bar{U},U_i) \mid |U_i|>1\}$.*

In Fig. 1, taking Q_3 as an example, we have $T(G,U)=\{4,5,6,7\}$, where $U=\{0,1,2,3\}$.

The next lemma follows directly from the definition of connectivity of G.

Lemma 2 [20]. *Let G be a connected graph and U be a subset of $V(G)$. Then, $|N(\bar{U},U)|\ge\kappa(G)$ if $|\bar{U}|\ge\kappa(G)$ and $N(\bar{U},U)=\bar{U}$ if $|\bar{U}|<\kappa(G)$.*

There are several different ways to verify a system to be t-diagnosable under the comparison approach. In this paper, we need three theorems given by Sengupta and Dahbura [20]. The first two are necessary and sufficient conditions for ensuring distinguishability, the third one is a sufficient condition for verifying a system to be t-diagnosable.

Theorem 1 [20]. *For any S_1, S_2 where $S_1,S_2\subset V$ and $S_1\neq S_2$, (S_1,S_2) is a distinguishable pair if and only if at least one of the following conditions is satisfied (see Fig. 2):*

1. *$\exists i,k\in V-S_1-S_2$ and $\exists j\in(S_1-S_2)\bigcup(S_2-S_1)$ such that $(i,j)_k\in C$,*
2. *$\exists i,j\in S_1-S_2$ and $\exists k\in V-S_1-S_2$ such that $(i,j)_k\in C$, or*
3. *$\exists i,j\in S_2-S_1$ and $\exists k\in V-S_1-S_2$ such that $(i,j)_k\in C$.*

Theorem 2 [20]. *A system is t-diagnosable if and only if each node has order at least t and, for each distinct pair of sets $S_1,S_2\subset V$ such that $|S_1|=|S_2|=t$, at least one of the conditions of Theorem 1 is satisfied.*

Theorem 3 [20]. *A system G with N nodes is t-diagnosable if*

1. *$N\ge 2t+1$;*
2. *$order_G(v)\ge t$ for every node v in G;*
3. *$|T(G,U)|>p$ for each $U\subset V(G)$ such that $|U|=N-2t+p$ and $0\le p\le t-1$.*

According to the above three theorems, we observe that Condition 3 of Theorem 3 restricts G, satisfying the first condition of Theorem 1, and ignores Conditions 2 and 3. Hence, we present a hybrid theorem to test whether a system is t-diagnosable.

Theorem 4. *A system G with N nodes is t-diagnosable if*

1. *$N\ge 2t+1$;*
2. *$order_G(v)\ge t$ for every node v in G;*
3. *for any two distinct subsets S_1, $S_2\subset V(G)$ such that $|S_1|=|S_2|=t$ either*
   a. *$|T(G,U)|>p$, where $U=V(G)-(S_1\bigcup S_2)$, and $|S_1\bigcap S_2|=p$ or*
   b. *the pair (S_1,S_2) satisfies Condition 2 or 3 of Theorem 1.*

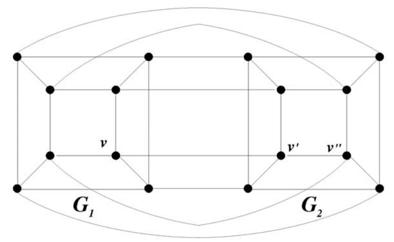


Fig. 3. An example of $G(G_1, G_2; L)$.

Proof. Conditions 1 and 2 are the same as Conditions 1 and 2 of Theorem 3. Consider Condition 3a. S_1 and S_2 are two distinct subsets of $V(G)$ with $|S_1| = |S_2| = t$, $U = V(G) - (S_1 \bigcup S_2)$, and $|S_1 \bigcap S_2| = p$. Then, $0 \leq p \leq t - 1$ and $|U| = N - 2t + p$. If $|T(G, U)| > p$, it implies that the pair (S_1, S_2) satisfies Condition 1 of Theorem 1. Combining Conditions 3a and 3b, by Theorems 1 and 2, this theorem follows. □

4 Diagnosability of Matching Composition Networks

Now, we define the Matching Composition Network (MCN) as follows: Let G_1 and G_2 be two graphs with the same number of nodes. Let L be an arbitrary *perfect matching* between the nodes of G_1 and G_2, i.e., L is a set of edges connecting the nodes of G_1 and G_2 in a one to one fashion; the resulting composition graph is called a *Matching Composition Network* (*MCN*). For convenience, G_1 and G_2 are called the *components* of the MCN. Formally, we use the notation $G(G_1, G_2; L)$ to denote an MCN, which has node set

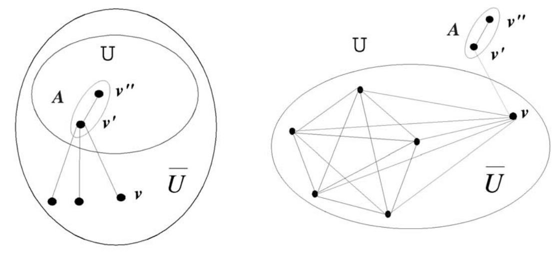
$$V(G(G_1, G_2; L)) = V(G_1) \bigcup V(G_2)$$

and edge set $E(G(G_1, G_2; L)) = E(G_1) \cup E(G_2) \cup L$. See Fig. 3.

What we have in mind is the following: Let G_1 and G_2 be two t-connected networks with the same number of nodes and $order_{G_i}(v) \geq t$ for every node v in G_i, where $i = 1, 2$, and let L be an arbitrary perfect matching between the nodes of G_1 and G_2. Then, the degree of any node v in $G(G_1, G_2; L)$ as compared with that of node v in G_i, $i = 1, 2$, is increased by one. We expect that the diagnosability of $G(G_1, G_2; L)$ is also increased to $t + 1$. For example, the Hypercube Q_{n+1} is constructed from two copies of Q_n by adding a perfect matching between the two and the diagnosability is increased from n to $n + 1$ for $n \geq 5$. Other examples, such as the Twisted cube TQ_{n+1}, the Crossed cube CQ_{n+1}, and the Möbius cube MQ_{n+1}, are all constructed recursively using the same method as above.

Theorem 5. *Let G_1 and G_2 be two networks with the same number of nodes and t be a positive integer. Suppose that $order_{G_i}(v) \geq t$ for every node v in G_i, where $i = 1, 2$. Then, $order_{G(G_1,G_2;L)}(v) \geq t + 1$ for node v in $G(G_1, G_2; L)$.*

Proof. See Fig. 3. Let v be a node of $G(G_1, G_2; L)$. Without loss of generality, we assume that $v \in V(G_1)$, $v' \in V(G_2)$, and $(v, v') \in L$. Of course, node v' is connected to at least one other node v'' in $V(G_2)$. Let $G(v, G_1)$ and $G(v, G(G_1, G_2; L))$ be the order graph of v in graph G_1 and $G(G_1, G_2; L)$, respectively. We observe that $G(v, G_1)$ is a proper subgraph of $G(v, G(G_1, G_2; L))$, both v' and v'' are in the latter, none of them in the former, and (v', v'') is an edge in $G(v, G(G_1, G_2; L))$. Therefore, every vertex cover of the order graph $G(v, G(G_1, G_2; L))$ contains a vertex cover of the order graph $G(v, G_1)$. Besides, any vertex cover of $G(v, G(G_1, G_2; L))$ has to include at least one of v' and v''. Thus, $order_{G(G_1,G_2;L)}(v) \geq order_{G_i}(v) + 1$ for any node v in G_i, $i = 1, 2$. This completes the proof. □



(a) (b)

Fig. 4. An example of the $T(G, U)$ when $|U| = t$. (a) G. (b) v is connected to A.

We need the following lemma later in Theorem 6.

Lemma 3. *Let G be a t-connected network, $|V(G)| \geq t + 2$ and $order_G(v) \geq t$ for every node v in G, where $t \geq 2$. Suppose that U is a subset of nodes of $V(G)$ with $|\bar{U}| \leq t$. Then, $T(G, U) = \bar{U}$.*

Proof. By assumption $|\bar{U}| \leq t$ and $\kappa(G) \geq t$, we prove the lemma by two cases; the first for $|\bar{U}| < \kappa(G)$ and the second for $|\bar{U}| = \kappa(G)$.

If $|\bar{U}| < \kappa(G)$, the induced graph $G[U]$ is connected. By Lemma 1, $T(G, U) = N(\bar{U}, U)$. By Lemma 2, $N(\bar{U}, U) = \bar{U}$. This case holds.

Suppose that $|\bar{U}| = \kappa(G)$. We observe that, adding any node v of $\bar{U}$ to U, the induced subgraph $G[U \bigcup \{v\}]$ forms a connected graph. It implies that every node v of $\bar{U}$ is adjacent to every connected components of $G[U]$. We claim that the subgraph induced by U contains a connected component A with cardinality at least two (see Fig. 4a). Then, the connected component A is adjacent to all nodes in $\bar{U}$ and, so, $T(G, U) = \bar{U}$.

Now, we prove the claim. Suppose, on the contrary, that every connected component of the subgraph induced by U is an isolated node. Let v be an arbitrary node in $\bar{U}$ and let $G_v = (X_v, Y_v)$ be the order graph of v in G. Then, $\bar{U} - \{v\}$ is a vertex cover of G_v because every connected component of $G[U]$ is an isolated node. Since $|\bar{U}| \leq t$, we have $|\bar{U} - \{v\}| \leq t - 1$. Therefore, even if the induced graph $G[\bar{U} - \{v\}]$ is a complete graph (see Fig. 4b), the cardinality of a minimum vertex cover of the order graph G_v is at most $t - 1$. However, this contradicts the hypothesis of $order_G(v) \geq t$ for every node v in G. So, $G[U]$ has a connected component A with cardinality at least two. This proves the claim, and the lemma follows. □

We are now ready to state and prove the following theorem about the diagnosability of Matching Composition Network under the comparison model. As an illustration, the conditions of the following theorem are applicable to some well-known interconnection networks, such as Q_n, CQ_n, TQ_n, and MQ_n for $n = t \geq 3$.

Theorem 6. *For $t \geq 2$, let G_1 and G_2 be two graphs with the same number of nodes N, where $N \geq t + 2$. Suppose that $order_{G_i}(v) \geq t$ for every node v in G_i and the connectivity $\kappa(G_i) \geq t$, where $i = 1, 2$. Then, MCN $G(G_1, G_2; L)$ is $(t + 1)$-diagnosable.*

Proof. Since $|V(G_1)| = |V(G_2)| = N$, $2N \geq 2(t + 2) > 2(t + 1) + 1$. By Theorem 5, $order_{G(G_1,G_2;L)}(v) \geq t + 1$ for any node v in $G(G_1, G_2; L)$. It remains to prove that $G(G_1, G_2; L)$ satisfies Condition 3 of Theorem 4.

Let S_1 and S_2 be two distinct subsets of $V(G)$ with the same number $t + 1$ of nodes and let $|S_1 \bigcap S_2| = p$, then $0 \leq p \leq t$. In order to prove this theorem, we will prove that S_1 and S_2 are distinguishable, i.e., this pair (S_1, S_2) satisfies either Condition 3a or 3b of Theorem 4.

Let $G = G(G_1, G_2; L)$ and $U = V(G) - (S_1 \bigcup S_2)$, then $|U| = 2N - 2(t + 1) + p$. Let $U = U_1 \bigcup U_2$ with $U_i = U \bigcap V(G_i)$ and $\bar{U}_i = V(G_i) - U_i$, $i = 1, 2$. Without loss of generality, we assume that $|U_1| \geq |U_2|$. Let $|\bar{U}_1| = n_1$,

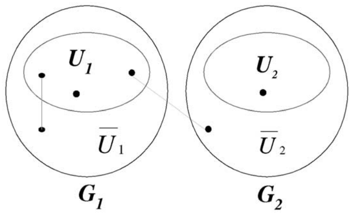


Fig. 5. Illustration of Subcase 2a of Theorem 6.

$|\bar{U}_2| = n_2$, $n_1 + n_2 = 2(t+1) - p$, and $n_1 \leq n_2$. Since $0 \leq n_1 \leq \frac{2(t+1)-p}{2}$, the maximum value of n_1 is equal to $t+1$ when $p = 0$ and $n_2 = t+1$. According to different values of n_1 and n_2, we divide the proof into two cases. The first case is $n_2 \leq t$, which implies $n_1 \leq t$. The second case is $n_2 > t$ and this case is further divided into three subcases $n_1 < t$, $n_1 = t$, and $n_1 > t$.

Case 1: $n_1 \leq t$ and $n_2 \leq t$.

By Lemma 3, we have

$$\begin{aligned} |T(G,U)| &\geq |T(G_1,U_1)| + |T(G_2,U_2)| = |\bar{U}_1| + |\bar{U}_2| \\ &= n_1 + n_2 = 2(t+1) - p. \end{aligned}$$

We know that $0 < p \leq t$, $|T(G,U)| \geq 2(t+1) - p > p$, and Condition 3a of Theorem 4 is satisfied.

Case 2: $n_2 > t$.

We discuss the case according to the following three subcases, 2a) $n_1 < t$, 2b) $n_1 = t$, and 2c) $n_1 > t$.

Subcase 2a: $n_1 < t$.

Since $\kappa(G_1) \geq t$ and $|\bar{U}_1| = n_1 < t$, $G[U_1]$ is connected. By Lemmas 1 and 2, $T(G_1, U_1) = N(\bar{U}_1, U_1) = n_1$. There are n_1 and n_2 nodes in $\bar{U}_1$ and $\bar{U}_2$, respectively, and $n_2 = 2t + 2 - p - n_1$ (see Fig. 5). If all the nodes in $\bar{U}_1$ are adjacent to some n_1 nodes in $\bar{U}_2$, there are still at least $n_2 - n_1 = 2t + 2 - p - 2n_1$ nodes in $\bar{U}_2$ such that each of them is adjacent to some node in U_1 under the matching L. So,

$$|T(G,U)| \geq |T(G_1,U_1)| + (n_2 - n_1) = n_1 + (n_2 - n_1) = n_2.$$

Because $n_2 > t \geq p$, the proof of this subcase is complete.

Subcase 2b: $n_1 = t$.

We know that $n_1 + n_2 = 2(t+1) - p$, $0 \leq p \leq t$, $n_2 > t$, and $n_1 = t$, the only two valid values for n_2 are $t+1$ and $t+2$. $n_2 = t+1$ implies $p = 1$, and $n_2 = t+2$ implies $p = 0$. By Lemma 3, $|T(G_1,U_1)| = |\bar{U}_1| = t \geq 2 > p$ for $p = 0$ or 1. Then, the subcase holds.

Subcase 2c: $n_1 > t$.

Observing that $0 \leq n_1 \leq \frac{2(t+1)-p}{2}$, where $0 \leq p \leq t$ and $n_2 \geq n_1 > t$, so $n_1 = n_2 = t+1$. It also implies $p = 0$. Here, we will prove that the subcase satisfies either Condition 3a or Condition 3b of Theorem 4.

First, if the subgraph induced by U contains a connected component A_1 with cardinality at least two (see Fig. 6), then it must be adjacent to some node in $\bar{U}$. Thus, we know that $|T(G,U)| > 0 = p$ and Condition 3a of Theorem 4 is satisfied.

Otherwise, every connected component of U contains a single node only. By Theorem 1, we know that S_1 and S_2 are distinguishable if there exists a path $\langle u_1 \rightarrow u \rightarrow u_2 \rangle$ such that $u \in U$, and $u_1, u_2 \in S_1 - S_2$ or $u_1, u_2 \in S_2 - S_1$. If $p = 0$, it implies $S_1 \bigcap S_2 = \phi$, any node u in $G[U]$ with degree more than two must be connected to at least two nodes in S_1 or S_2 (see Fig. 6). By Theorem 5, $order_{G(G_1,G_2;L)}(v) \geq t+1$ for every node v in $G(G_1, G_2; L)$, therefore $deg(v) \geq t+1$ for every node v in $G(G_1, G_2; L)$. Since $t \geq 2$, Condition 3b of Theorem 4 is satisfied.

Hence, the subcase holds and the theorem follows. □

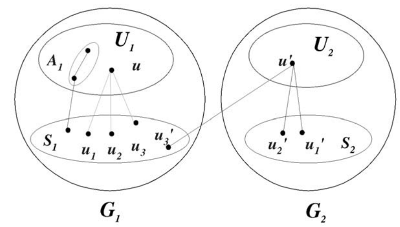


Fig. 6. An example of Subcase 2c of Theorem 6.

By Theorem 3 and Theorem 6, we have the following corollary.

Corollary 1. *Let G_1 and G_2 be two graphs with the same number of nodes N. Suppose that both G_1 and G_2 are t-diagnosable and have connectivity $\kappa(G_1) = \kappa(G_2) \geq t$, where $t \geq 2$. Then, MCN $G(G_1, G_2; L)$ is $(t+1)$-diagnosable.*

5 Applications

In this section, we demonstrate the usefulness of our proposed construction scheme for some well-known networks. For example, the diagnosability of the Hypercube Q_n [19], the Crossed cube CQ_n [6], [7], [8], the Twisted cube TQ_n [9], [13], and the Möbius cube MQ_n [5] can all be proven to be n, for $n \geq 4$.

The Hypercube is a popular topology for interconnection networks. The Crossed cube, the Twisted cube, and the Möbius cube are variations of the Hypercube. For each of these cubes, an n-dimensional cube can be constructed from two copies of $(n-1)$-dimensional subcubes by adding a perfect matching between the two subcubes. The main difference is that each of these cubes has various perfect matching between its subcubes. An n-dimensional cube has 2^n nodes, connectivity n, and each node has the same degree n. In the following, we briefly state the recursive definitions of these cubes and prove that they are all n-diagnosable.

The nodes of these n-dimensional cubes are usually represented by the n-bit binary strings. A binary string u of length n will be written as $u = u_{n-1}u_{n-2}u_{n-3}\ldots u_0$, where $u_i \in \{0,1\}$, $0 \leq i \leq n-1$. The classical n-dimensional Hypercubes Q_n is recursively defined as follows.

Definition 1. *Let $n \geq 1$ be an integer. The Hypercube Q_n of dimension n has 2^n nodes. Q_1 is a complete graph with two nodes labeled by 0 and 1, respectively. For $n \geq 2$, an n-dimensional Hypercube Q_n is obtained by taking two copies of $(n-1)$-dimensional subcubes Q_{n-1}, denoted by Q^0_{n-1} and Q^1_{n-1}. For each $v \in V(Q_n)$, insert a 0 to the front of $(n-1)$-bit binary string for v in Q^0_{n-1} and a 1 to the front of $(n-1)$-bit binary string for v in Q^1_{n-1}. There are 2^{n-1} edges between Q^0_{n-1} and Q^1_{n-1} as follows:*

Let $V(Q^0_{n-1}) = \{0u_{n-2}u_{n-3}\ldots u_0 : u_i = 0 \text{ or } 1\}$ and $V(Q^1_{n-1}) = \{1v_{n-2}v_{n-3}\ldots v_0 : v_i = 0 \text{ or } 1\}$, where $0 \leq i \leq n-2$. A node $u = 0u_{n-2}u_{n-3}\ldots u_0$ of $V(Q^0_{n-1})$ is joined to a node $v = 1v_{n-2}v_{n-3}\ldots v_0$ of $V(Q^1_{n-1})$ if and only if $u_i = v_i$ for $0 \leq i \leq n-2$.

In [22], Wang has proven that the diagnosability of hypercube-structured multiprocessor systems under the comparison model is n when $n \geq 5$. However, the diagnosability of Q_4 is not known to be 4. Using our Theorem 6, we can strengthen the result as follows.

Theorem 7. *The Hypercube Q_n is n-diagnosable for $n \geq 4$.*

Proof. We observe that Q_3 is 3-connected, $order_{Q_3}(v) = 3$ for every node v in Q_3, and the number of nodes of Q_3 is 8, $8 \geq t+2 = 5$ for $t = 3$. It is well-known that Q_4 can be constructed from two copies of Q_3 by adding a perfect matching between these two copies. Therefore, by Theorem 6, Q_4 is 4-diagnosable.

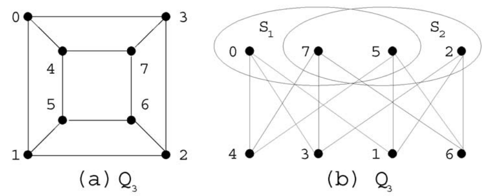


Fig. 7. $S_1 = \{0, 5, 7\}$ and $S_2 = \{2, 5, 7\}$ are not distinguishable.

Then, the proof is by induction on n. We have shown that Q_4 is 4-diagnosable. Assume that it is true for $n = m - 1$. Considering $n = m$, Q_m is obtained from two copies G_1, G_2 of Q_{m-1} by adding a perfect matching joining corresponding nodes in G_1 and G_2. It is well-known that Q_{m-1} is $(m-1)$-connected. By Corollary 1, Q_m is m-diagnosable. This completes the induction proof. □

However, Q_3 is not 3-diagnosable. In Fig. 7, there is a Q_3, let $S_1 = \{0, 5, 7\}$ and $S_2 = \{2, 5, 7\}$. Then, by Theorem 1, S_1 and S_2 are not distinguishable, as shown in Fig. 7.

As we observe, most of the related results on diagnosability of multiprocessors systems [12], [22] are based on a sufficient theorem, namely, Theorem 3. Not satisfying this sufficient condition, such as in the case of Q_4, does not necessarily imply that the network is not 4-diagnosable. Therefore, we propose a hybrid condition, 3a and 3b of Theorem 4, to check the diagnosability of multiprocessor systems under the comparison model. It is more powerful to use. Applying our Theorem 4 and Theorem 6, we show that the diagnosability of Q_4 is indeed 4.

The following is the recursive definition of the n-dimensional Crossed cube CQ_n.

Definition 2 [6]. *The Crossed cube CQ_1 is a complete graph with two nodes labeled by 0 and 1, respectively. For $n \geq 2$, an n-dimensional Crossed cube CQ_n consists of two $(n-1)$-dimensional sub-Crossed cubes, CQ^0_{n-1} and CQ^1_{n-1}, and a perfect matching between the nodes of CQ^0_{n-1} and CQ^1_{n-1} according to the following rule:*

Let $V(CQ^0_{n-1}) = \{0u_{n-2}u_{n-3}\dots u_0 : u_i = 0 \text{ or } 1\}$ and $V(CQ^1_{n-1}) = \{1v_{n-2}v_{n-3}\dots v_0 : v_i = 0 \text{ or } 1\}$. The node $u = 0u_{n-2}u_{n-3}\dots u_0 \in V(CQ^0_{n-1})$ and the node $v = 1v_{n-2}v_{n-3}\dots v_0 \in V(CQ^1_{n-1})$ are adjacent in CQ_n if and only if

1. *$u_{n-2} = v_{n-2}$ if n is even and*
2. *$(u_{2i+1}u_{2i}, v_{2i+1}v_{2i}) \in \{(00, 00), (10, 10), (01, 11), (11, 01)\}$, for $0 \leq i < \lfloor \frac{n-1}{2} \rfloor$.*

Hilbers et al. [13] defined the Twisted cubes using the parity function. Let $u = u_{n-1}u_{n-2}\dots u_0$, where $u_i \in \{0, 1\}$ and $0 \leq i \leq n-1$, the parity function is defined as $P_i(u) = u_i \bigoplus u_{i-1} \bigoplus \cdots \bigoplus u_0$, where $\bigoplus$ is the *exclusive-or* operation.

Definition 3 [13]. *The Twisted cube TQ_1 is a complete graph with two nodes, 0 and 1. Let n be an odd integer and $n \geq 3$. The nodes of an n-dimensional Twisted cube TQ_n are decomposed into four sets $S^{0,0}$, $S^{0,1}$, $S^{1,0}$ and $S^{1,1}$. The set $S^{i,j}$ consists of those nodes $u = u_{n-1}u_{n-2}\dots u_0$ with $u_{n-1} = i$ and $u_{n-2} = j$, where $(i, j) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. The induced subgraph of $S^{i,j}$ in TQ_n is isomorphic to TQ_{n-2}. Edges which connect these four $(n-2)$-dimensional subtwisted cubes can be described as follows: Any node $u_{n-1}u_{n-2}\dots u_0$ with $P_{n-3}(u) = 0$ is connected to $\overline{u}_{n-1}\overline{u}_{n-2}\dots u_0$ and $\overline{u}_{n-1}u_{n-2}\dots u_0$; and to $u_{n-1}\overline{u}_{n-2}\dots u_0$ and $\overline{u}_{n-1}u_{n-2}\dots u_0$, if $P_{n-3}(u) = 1$.*

As stated in [13], for even integer n, the Twisted cube TQ_n can also be defined recursively in a similar way starting from TQ_2, where TQ_2 is isomorphic to Q_2. In order to see the recursive structure of the Twisted cube, we review the classical definition of the *Cartesian product*.

Definition 4. *The Cartesian product of G and H, written $G \times H$, is the graph with vertex $V(G) \times V(H)$ specified by putting $\langle u, v \rangle$ adjacent to $\langle u', v' \rangle$ if and only if 1) $u = u'$ and $(v, v') \in E(H)$, or 2) $v = v'$ and $(u, u') \in E(G)$.*

It is known that the n-dimensional hypercube can be defined as $Q_n = Q_{n-1} \times K_2$ when $n \geq 2$, where K_2 is the complete graph with two nodes.

In additions, the connectivity of the network $G \times H$ is listed as follows:

Lemma 4 [4]. $\kappa(G \times H) \geq \kappa(G) + \kappa(H)$.

In Definition 3, let TQ^0_{n-1} (TQ^1_{n-1}, respectively) be the subgraph of TQ_n induced by $S^{0,0} \bigcup S^{1,0}$ ($S^{1,1} \bigcup S^{0,1}$, respectively). It follows directly from the definition that both TQ^0_{n-1} and TQ^1_{n-1} are isomorphic to the Cartesian product $TQ_{n-2} \times K_2$. Then, TQ_n is constructed from TQ^0_{n-1} and TQ^1_{n-1} by joining them with a particular perfect matching. The connectivity of TQ_n is n [3]. Replacing n by $n-2$, the connectivity of TQ_{n-2} is $n-2$. So, by Lemma 4, both TQ^0_{n-1} and TQ^1_{n-1} are $(n-1)$-connected.

By Theorem 6, we observe that both TQ^0_4 and TQ^1_4 are 4-diagnosable. Then, by Corollary 1, TQ_5 is 5-diagnosable. Applying induction on n, suppose that TQ_{n-2} is $(n-2)$-diagnosable, by Corollary 1, both TQ^0_{n-1} and TQ^1_{n-1} are $(n-1)$-diagnosable. Then, we can prove that TQ_n is n-diagnosable by induction.

Now, we present the definition of the Möbius cubes MQ_n [5]. There are two types of MQ_n, namely, $0 - MQ_n$ and $1 - MQ_n$.

Definition 5 [5]. *$0 - MQ_1$ and $1 - MQ_1$ are both the complete graph on two nodes whose labels are 0 and 1. For $n \geq 2$, both $0 - MQ_n$ and $1 - MQ_n$ contain one 0-type sub-Möbius cube MQ^0_{n-1} and one 1-type sub-Möbius cube MQ^1_{n-1}. The first bit of every node of MQ^0_{n-1} is 0 and the first bit of every node of MQ^1_{n-1} is 1. For two nodes $u = 0u_{n-2}u_{n-3}\dots u_0 \in V(MQ^0_{n-1})$ and $v = 1v_{n-2}v_{n-3}\dots v_0 \in V(MQ^1_{n-1})$,*

1. *u connects to v in $0 - MQ_n$ if and only if $u_i = v_i$, for every i, $0 \leq i \leq n-2$,*
2. *u connects to v in $1 - MQ_n$ if and only if $u_i = \overline{v}_i$, for every i, $0 \leq i \leq n-2$.*

It is known [11], [15] that the Crossed cube CQ_n and the Möbius cube MQ_n are both n-connected. By Theorem 5, we can prove that the order of each node in these two cubes is n. We observe that the two cubes are both constructed recursively using a similar way satisfying the requirements of Theorem 6 and Corollary 1. Therefore, we can prove that CQ_n and MQ_n are both n-diagnosable for $n \geq 4$. Then, we list the following three theorems.

Theorem 8 [12]. *The Crossed cube CQ_n is n-diagnosable for $n \geq 4$.*

Theorem 9. *The Twisted cube TQ_n is n-diagnosable for $n \geq 4$.*

Theorem 10. *The Möbius cube MQ_n is n-diagnosable for $n \geq 4$.*

6 Conclusions

In this paper, we propose a sufficient theorem to verify the diagnosability of multiprocessor systems under the comparison-based model. The conditions of this theorem include all the cases of the original necessary and sufficient condition stated in Theorem 1. Therefore, it is more suitable for verifying the diagnosability of a system. Then, we propose a family of interconnection networks which are recursively constructed, called the Matching Composition Networks.

Each member $G(G_1, G_2; L)$ of this family is constructed from a pair G_1 and G_2 of lower dimensional networks with the same number of nodes, joining by a perfect matching L between the two. Applying Theorem 6 in this paper, we show that the diagnosability of $G(G_1, G_2; L)$ is one larger than those of the G_1 and G_2, provided some regular conditions, as stated in Theorem 6, are satisfied. Many well-known interconnection networks, such as the Hypercubes Q_n, the Crossed cubes CQ_n, the Twisted cubes TQ_n, and the Möbious cubes MQ_n, belong to our proposed family.

We note here that these special cases all satisfy the condition of Theorem 6 for $n \geq 4$. Thus, their diagnosabilities are *n*, for $n \geq 4$. In particular, the diagnosability of the 4-dimensional Hypercube Q_4 is 4. Also, Theorems 9 and 10 are proposed for the first time to describe the diagnosability of the Twisted cube TQ_n and the Möbious cubes MQ_n.

Acknowledgments

This work was supported in part by the National Science Council of the Republic of China under Contract NSC 90-2213-E-009-149.

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