Fully generalized Fibonacci series modulo *n* as music sequence generators

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ABSTRACT

In this paper we introduce fully generalized Fibonacci sequences as useful tools for the generation of integers that can be interpreted in the musical context for algorithmic composition. In particular, we take into account the modulo operation on Fibonacci sequences resulting in various periodic behaviors that we interpret in the pitch class domain. First we introduce Fibonacci sequences and generalized Fibonacci sequences, then we discuss the modulo operator applied to sequences. We propose various interpretations of the resulting sequences in terms of pitches and pitch classes, and describe some possible operations. Finally, we introduce fully generalized Fibonacci sequences and describe a possible implementation in an algorithmic composition environment.

1. INTRODUCTION

Use of the Fibonacci sequence has a long tradition in 20th Century Music. Kramer [1] provides an early account of references including Bartók, Stockhausen and Nono (on the latter see also [2], on Ferneyhough see [3]). In these cases, Fibonacci sequences are used to define time proportion, like in [4]. Mathematical and musicological discussions mostly deal with the relation between Fibonacci sequences and golden ratio (see e.g. [5, 6]). Mongoven [6] also reports contemporary uses while proposing an application to tuning systems.
In this paper we are mostly interested in pitch and pitch class interpretation of Fibonacci sequences, e.g. when numbers are mapped onto pitches and pitch classes. First, we discuss some general features of Fibonacci numbers, then we consider some musical applications to pitch domain, finally we generalize Fibonacci sequences and describe possible music developments.

2. MATHEMATICAL ASPECTS OF FIBONACCI SEQUENCES

The Fibonacci numbers are the integer numbers in the sequence (the Fibonacci sequence) defined by the recurrence relation

$$F : \left[\begin{array}{lcl} F_0 & = & 0 \\ F_1 & = & 1 \\ F_i & = & F_{i-1} + F_{i-2}. \end{array} \right. \quad (1)$$

The Fibonacci numbers have been widely studied and satisfy many and varied identities. Many of the identities involve both addition and multiplication and so the full ring structure of the integers is required to prove them.

In matrix representation, given

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

then

$$U^i = \begin{pmatrix} F_{i-1} & F_i \\ F_i & F_{i+1} \end{pmatrix}.$$

It is worth to mention two other well-known sequences.

- The Pell sequence

$$P : \left[\begin{array}{lcl} P_0 & = & 0 \\ P_1 & = & 1 \\ P_i & = & 2P_{i-1} + P_{i-2}. \end{array} \right. \quad (2)$$

- The Lucas sequence

$$L : \left[\begin{array}{lcl} L_0 & = & 2 \\ L_1 & = & 1 \\ L_i & = & L_{i-1} + L_{i-2}. \end{array} \right. \quad (3)$$

Hence the first 9 steps of the three sequences are

i	0	1	2	3	4	5	6	7	8	9	...
F_i	0	1	1	2	3	5	8	13	21	34	...
P_i	0	1	2	5	12	29	70	169	408	985	...
L_i	2	1	3	4	7	11	18	29	47	76	...

(4)

Fibonacci, Pell and Lucas sequences are respectively sequences A000045, A000129 and A000032 in the On-Line Encyclopedia of Integer Sequences (OEIS) [1].

3. PISANO PERIODS

In the following we consider the sequence $F(\mathrm{mod}\, n)$ obtained by taking the remainders of the Fibonacci sequence modulo n, where n is an integer. For example, if $n = 3$ one has

i	0	1	2	3	4	5	6	7	8	9	...
F_i	0	1	1	2	3	5	8	13	21	34	...
$F(\mathrm{mod}\, 3)$	0	1	1	2	0	2	2	1	0	1	...

(5)

[1] https://oeis.org/

$\pi(n)$	**0+**	**1+**	**2+**	**3+**	**4+**	**5+**
+1	1	10	16	30	40	72
+2	3	24	30	48	48	84
+3	8	28	48	40	88	108
+4	6	48	24	36	30	72
+5	20	40	100	80	120	20
+6	24	24	84	24	48	40
+7	16	36	72	76	32	72
+8	12	24	48	18	24	42
+9	24	18	14	56	112	58
+10	60	60	120	60	300	120

Table 1. Pisano periods

$F(\bmod n)$ is a bi-infinite sequence, that is, given any two consecutive terms, we can find the terms preceding and following those terms. In other terms any pair of consecutive terms of $F(\bmod n)$ determines the entire sequence both forward and backward.

Note that there are only n^2 pairs of possible terms hence the serie repeats, and the recurrence of a pair results in recurrence of all following terms. By the recursive term in (1) and the definition of $F(\bmod n)$ we have that if $F_{t+1} \equiv F_{s+1} \bmod n$ and $F_t \equiv F_s \bmod n$ then

$$F_{t+1} \equiv F_{s+1}, \cdots, F_{t-s+1} \equiv F_1 \bmod n$$

and $F_{t-s} \equiv F_0 \bmod n$. Hence we have the following

Theorem 3.1 ([7]). *$F(\bmod n)$ forms a simply periodic series. That is, the series is periodic and repeats by returning to its starting values.*

Denote by $\pi(n)$ the least positive integer k such that $F_k \equiv 0 \bmod n$ and $F_{k+1} \equiv 1 \bmod n$. Thus, $\pi(n)$ denotes the period of $F(\bmod n)$. For example, according to (5), $F(\bmod 3)$ has period 8, so $\pi(3) = 8$.

Definition 3.1.1. *$\pi(n)$ is called* Pisano period *of the sequence $F(\bmod n)$.*

The existence of periodic functions in Fibonacci numbers was noted by Joseph Louis Lagrange in 1774 [8]. In Table 1, the first 60 Pisano periods are shown. Here the rows represent units while columns represent decimals. Hence, to check the Pisano period of, for example, 43, it is enough to control the intersection between the column **4+** and the row **+3**. In Figure 1 there is the plot of the first 10,000 Pisano periods [2].

We can notice that $U^k \equiv I \bmod n$ precisely when $\pi(n)|k$, where I is the identity matrix [3].

We now collect some results about Pisano periods.

Theorem 3.2 ([7]). *For $n > 2$, $\pi(n)$ is even.*

Proposition 3.2.1. *$\pi(lcm(n_1, n_2)) = lcm(\pi(n_1), \pi(n_2))$, where lcm is the lowest common multipole.*

[2] A clear visualization of Pisano periods in relation to periodicity has been proposed by YouTube user Jacob Yatsko here: `https://www.youtube.com/watch?v=o1eLKODSCqw`

[3] We remind that | stands for "divides".

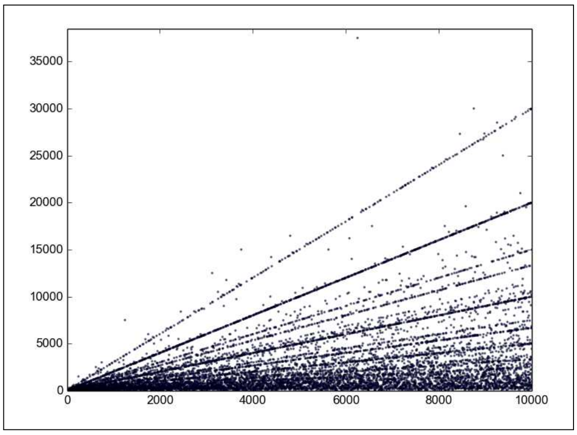

Figure 1. Plot of the first 10,000 Pisano periods;
Source: Robodile, CC BY-SA 4.0 https://creativecommons.org/licenses/by-sa/4.0, via Wikimedia Commons.

As a corollaries we get the following ones.

Corollary 3.2.1. *if $k|n$ then $\pi(k)|\pi(n)$.*

For example, $17|51$ and $\pi(17) = 36$ divides $\pi(51) = 72$. Observe, however that $51/17 = 3$ but $\pi(51)/\pi(17) = 2$. Similarly, $11|22$ and $\pi(11) = 10$ divides $\pi(22) = 30$, but this time, one has $22/11 = 2$ but $\pi(22)/\pi(11) = 3$.

Corollary 3.2.2. *if n has prime factorization*

$$n = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n},$$

then

$$\pi(n) = lcm(\pi(p_1^{e_1}), \pi(p_2^{e_2}), \ldots, \pi(p_n^{e_n})).$$

The following propositions give bounds for $\pi(n)$.

Proposition 3.2.2. *$\pi(n) \leq 6n$, with equality holds if and only if $n = 2 \cdot 5^n$.*

In Figure 1 we can notice that $\pi(250) = 1500$ and $\pi(1250) = 7500$ stand out.

Proposition 3.2.3 ([9]). *Given n, let $t > 0$ such that $L_t \leq n$ where L_t is the $t-th$ Lucas number. Then $\pi(n) \geq 2t$.*

Since 0 represents the initial step, it is important to make some considerations about the appearance of 0.

Note that 0 is in the sequence (as $F_0 = 0$) and since $F(\bmod n)$ is periodic, we get that, for any integer n, infinitely many Fibonacci numbers are divisible by n.

Proposition 3.2.4. *The zeros of the sequence $F(\bmod n)$ are evenly spaced, that is if F_s and F_t are congruent to 0 modulo n, then F_{s+t} and F_{s-t} are congruent to 0 modulo n.*

This follows from the well-know equalities

$$F_{s+t} = F_{s-1}F_t + F_sF_{t+1} \tag{6}$$

and

$$F_{s-t} = (-1)^t(F_sF_{t+1} - F_{s+1}F_t). \tag{7}$$

3.1 Generalized Fibonacci sequences

A first kind of generalization of the Fibonacci sequence consists in considering different initial steps. To this aim we define the sequences $G_{[a,b]}$ of the form

$$G_{[a,b]}: \left[\begin{array}{lcl} G_0 &=& a \\ G_1 &=& b \\ G_i &=& G_{i-1} + G_{i-2}. \end{array} \right. \quad (8)$$

for integers a, b. Clearly $G_{[0,1]}$ is the usual Fibonacci sequence and $G_{[2,1]}$ is the Lucas sequence.

Again, we can consider the sequence $G_{[a,b]}(\bmod n)$. In the following table we have the first steps of $G_{[a,b]}$ and $G_{[a,b]}(\bmod n)$ for $a = 3$, $b = 2$ and $n = 3$:

i	0	1	2	3	4	5	6	7	8	$\cdots$
G_i	3	2	5	7	12	19	31	50	81	$\cdots$
$G_{[3,2]}(\bmod 3)$	0	2	2	1	0	1	1	2	2	$\cdots$

(9)

Proposition 3.2.5. *Let a, b, n be integers with $m \geq 2$.*

(a) $G_{[a,b]}$ *satisfies* $G_i = F_{i-1}a + F_i b$.

(b) $G_{[a,b]}(\bmod n)$ *is periodic.*

Part (a) of the previous proposition shows an important link between $G_{[a,b]}$ sequences and Fibonacci sequence F and part (b) tells us that sequences $G_{[a,b]}$ have Pisano period.

We denote by $\pi(a, b, n)$ the Pisano periods of the sequences $G_{[a,b]}(\bmod n)$. We have the following important result.

Proposition 3.2.6. *The Pisano period of $G_{[a,b]}(\bmod n)$ divides the Pisano period of $F(\bmod n)$, that is*

$$\pi(a, b, n) | \pi(n).$$

Also for $\pi(a, b, n)$ we have the same results of Proposition 3.2.1 and Corollaries 3.2.1 and 3.2.2 (see [7]).

Let $D = b^2 - ab - a^2$. The value D plays an important role to determine $\pi(a, b, n)$.

Theorem 3.3. *If $LCD(D, n) = 1$ then $\pi(a, b, n) = \pi(n)$.*

For example, the Lucas sequence has $D = -5$. Thus, for any n that is not a multiple of 5, the Pisano period of the Lucas sequence mod n is the same of $F(\bmod n)$.

Many papers on sequences $G_{[a,b]}$ mainly focus on the case $LCD(a, b) = 1$, that is a and b are coprime. However, for the topics of the following sections, also the case in which b divides a is of particular interest. In this context, we establish the following result.

Proposition 3.3.1. *If $a = kb$ and b divides n then all sequences $G_{[kb,b]}(\bmod n)$ give the same set of numbers, for all non-negative integers k.*

Proof. Write $n = t \cdot b$. Given any k, we have $k = qn + r$ for suitable integers q and r. Thus the sequence $G_{[kb,b]}$ can be written as

$$(qn+r)b, b, (qn+r+1)b, (qn+r+2)b, (qn+r+3)b, \ldots$$

Thus, when we consider this sequence mod n we get

$$rb \bmod n, b \bmod n, (r + 1)b \bmod n, \ldots$$

This shows that all elements of the sequence are all the multiple of b modulo n, independently of the value of r or, equivalently, of the value of k. □

4. FIBONACCI SEQUENCES AND PITCH CLASSES

Pitch classes are defined in relation to Forte's set theory for atonal music, where each pitch class (e.g. C representing the class of all possible Cs, regardless of octave) is indexed by an integer number (e.g. 0 for C) (see [10, 11, 12], in other contexts defined as chroma values). In this context, we assume: equal temperament; octave equivalence; and enharmonic equivalence. The first assumption says that the octave range ($[f, 2f]$, where f is frequency) can be divided into 12 equal steps where $f_{i+1} = \sqrt[12]{2} f_i$ (12 equal temperament, 12-ET)), the second that each corresponding step in different octaves shares a perceptual "sameness" feature, the third that, in relation to Western harmony, alterations are not relevant, so there is properly no difference among e.g. B♯, C, D♭♭, as they are all represented by pitch class index 0. These three constraints allow for integer notation, that is, each step in an equally tempered octave can be indexed by an integer in the range 0-11. As the pattern repeats itself, there are only 12 pitch classes [12].

The first application of Fibonacci numbers to the pitch domain can be traced back to Joseph Schillinger [13]. In the chapter devoted to *Theory of Melody* of his *System* he suggests to use Fibonacci sequences as a way to generate pitches, like in Figure 2 (omitting $0, 1$). Here Schillinger is also proposing a specific interpretation of Fibonacci sequences as incremental positive intervals rather than pitch classes. This means that $[0, 1, 1, 2, 3]$ can be interpreted not as C, D♭, D♭, D, E♭ but rather as C, D♭, D, E, G. It can be easily seen that, due to the accumulation process, numbers mapped onto pitches very soon exceed the available piano range. As a solution to this problem, Schillinger suggests to transpose each pitch to a viable octave, by means of "readjustment of the range" [13, p. 334] (Figure 3). This operation is indeed equivalent to apply mod 12 to the number. This modulo operation has relevant implications. Much more recently, Haek [14] has proposed an application of Fibonacci mod n to serial composition. In a 2023 YouTube video, Evanstein presents the application *Fibonacci Music Box* that deals with modulo application to Fibonacci series so to obtain pitch sequences [4]. In the following we discuss some more general features with number sequences interpreted as pitch class indices.

5. FIBONACCI MOD *N* AND PITCH CLASSES

In this section we propose an application to the pitch domain of Fibonacci sequences. Generalized Fibonacci sequences provide a very simple yet powerful formalism to

[4] https://www.youtube.com/watch?v=_aIf4WUCNZU

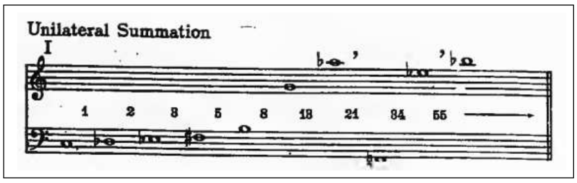

Figure 2. "Unilateral symmetry of Fibonacci series". From [13, p. 334]

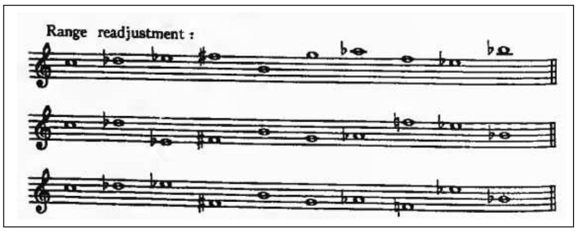

Figure 3. "Readjustment of the range". From [13, p. 334]

generate infinite sequences of integers. Thus, they are rightful candidates to enter the algorithmic composer's toolbox. As already observed by Schillinger, Fibonacci sequences rapidly grow outside a useful pitch range. Modulo application solves this issue but at the same time provides a new feature. When modulo n is applied, the result is a cyclic pattern that can be interpreted in relation to pitch domain. In this case, pitch sequences are interesting as they provide a non-uniform distribution of pitches while ensuring a variable periodicity. We can start by considering three cases:

1. $n < 12$
2. $n = 12$
3. $n > 12$

5.1 *n* <12

While discussing the first case, we also introduce some general aspects.
If $n < 12$, then pitch classes $\geq n$ are necessarily missing from the resulting cycle. Figure 4 shows the pitch class pattern for $G_{[2,1]}$ with $n = 11$ (Lucas Sequence mod 11) (Figure 5 shows a line visualization for the melodic profile). It can be observed that pitch class 11 (= B) is missing. But while this holds true by definition, not all the other 11 pitch classes are present. It is possible to study various features of this pattern. Pitch classes in the sequence are only 7, the set being $\{0, 1, 2, 3, 4, 7, 10\}$. The Pisano period is 10. Finally, the number of occurrences for each pitch class is the following (in which the items in the array indicate occurrences for pitch classes $0 - 11$): $[1, 1, 1, 2, 1, 0, 0, 3, 0, 0, 1, 0]$. This means that there are single occurrences of C, C♯, D, E, B♭, 2 occurrences of D♯, 3 occurrences of G. Of course, there are no occurrences for B, but neither for F, F♯, G♯, A (Figure 6).

How to interpret the sequence in Figure 4? First, it can be seen as a specific melodic form. Second, it can be taken

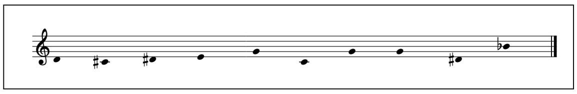

Figure 4. Pitch class sequence for $a = 2$, $b = 1$, $n = 11$.

into account as a generative cell, such as those used e.g. by Stravinskij ([15, 16]). Third, it can be exploited as a sort of weighted pitch class sequence: in the case of Figure 4 pitch class 7 (G) has more occurrences than all the other pitch classes, so it acts like a "modal" pivot for the whole sequence. By taking into account the resulting pitch class set, many analytical features can be explored. Ian Ring's website provides an extensive analytical approach to scale patterns [5]. Ring has automatically computed an extensive set of analytical features from existing literature. In this context, a scale is an ordered sequence of pitch classes starting from 0 and including no intervals < 4. A scale can be represented in bit form with the lowest bit (representing pitch class 0) at the right. The bit form of the set $\{0, 1, 2, 3, 4, 7, 10\}$ is thus $[0, 1, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1]$. This allows to compute the decimal form, that is, 1183. As a scale starts by definition from 0, the decimal is always odd. This number can be used as a pointer to a page in Ring's website, so to explore the resulting scale pattern in relation e.g. to Forte number, Tonnetz, prime form, interval vector, chirality and many others [6].

In short, generalized Fibonacci sequences mod n allows to generate simultaneously a scale *and* a repeating melody. A "Sequence Graph" like the one in Figure 7 shows in a compact way this twofold nature for the previously discussed $G_{[2,1]}$ with $n = 11$. The pitch sequence is $[2, 1, 3, 4, 7, 0, 7, 7, 3, 10]$. Nodes represent pitch classes (integers as labels), their radius being proportional to the number of occurrences (node 7 is the largest). Edges represent the sequencing order, as specified by each edge label. The graph is by definition directed and cyclic [17]. The dotted edge indicates the cycle loop and can also be used to quickly identify the starting node. Its label indicates the Pisano period.

5.2 *n* = 12

All the previous considerations apply to $n = 12$, with the difference that it is indeed possible to create a full chromatic pitch set (decimal notation: 4095). While this is theoretically possible, by considering all combinations of generalized Fibonacci sequences with $0 < a < 100$ and $1 < b < 100$ this does not happen, the best approximation being the pitch class $\{0, 1, 2, 3, 4, 5, 7, 8, 9, 10, 11\}$, that is, the chromatic set without F♯ (decimal: 4031), in all cases with $\pi(n) = 24$. Figure 8 shows the patterns with pitch class set 4031 generated by lowest a (= 0) and b, respectively 1 (classic Fibonacci sequence) and 5 ($G_{[0,1]}$ and $G_{[0,5]}$ with $n = 12$). Even in case of the same pitch class set, still pitch sequences and occurrence counts are different.

[5] https://ianring.com/musictheory/scales/
[6] https://ianring.com/musictheory/scales/1183

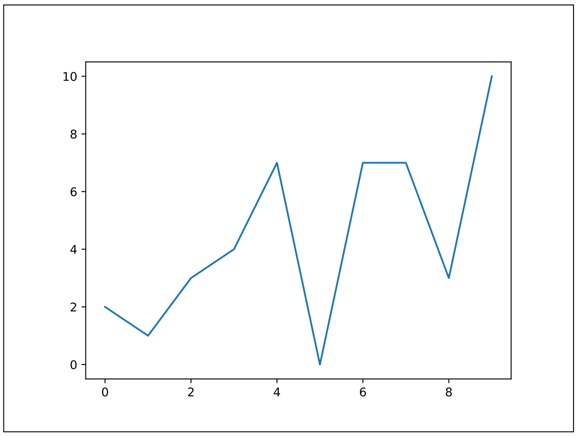

Figure 5. Melodic profile in $a = 2$, $b = 1$, $n = 11$.

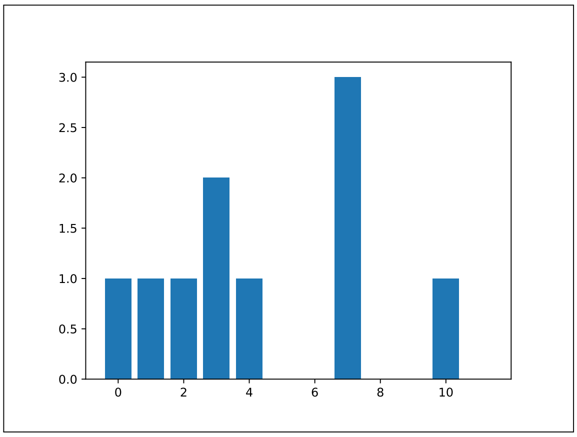

Figure 6. Occurrences for pitch classes in $a = 2$, $b = 1$, $n = 11$.

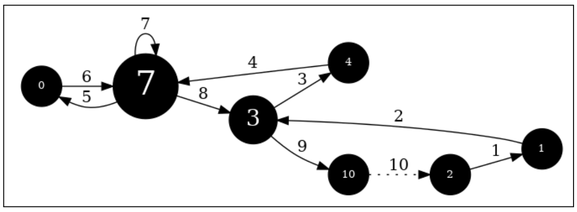

Figure 7. Sequence Graph for in $a = 2$, $b = 1$, $n = 11$.

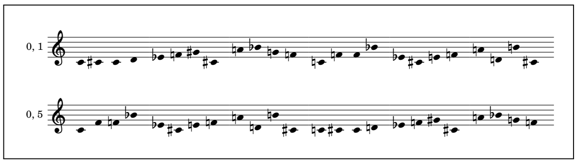

Figure 8. Patterns for $a = 0$ and , $b = 1$ (top) and $b = 5$ (bottom).

Figure 9. Patterns for $a = 2$ and , $b = 1$ and $n = 12, 15, 18, 21, 24$.

If $b = 2$, then all sequences with $a = kb$ (where k is an integer), result in the pitch class set $\{0, 2, 4, 6, 8, 10\}$, with $\pi(n) = 24$, that is, a whole tone scale. Yet, pitch sequences are different, with different occurrence counts. In short, various hexatonic patterns can be generated. When $b = 4$, then all sequences with $a = kb$ result in the pitch class set $\{0, 4, 8\}$, with $n = 8$, that is, an augmented triad. Similarly, with $b = 3$ and $a = kb$, the pitch class set is $\{0, 3, 6, 9\}$, a diminished 7^{th} chord.

In general, by Proposition 3.3.1, if b is a divisor of n, then all sequences $G_{[kb,b]}(\bmod n)$ result in the same set of pitch classes.

5.3 n >12

If n is greater than 12 all the previous considerations apply but the obtained remainders result in a range of more than one octave. This feature can be exploited in order to distribute pitches over a larger span, in this sense privileging a pitch interpretation rather than a pitch class one. Figure 9 shows Lucas sequence ($G_{[2,1]}$) with five moduli ranging from $n = 12$ (top) to $n = 24$ (bottom). Meter is kept in the music notation so that different Pisano periods are clearly visible (respectively: $24, 8, 24, 16, 24$). Extension of pitch range is clearly visible, and in particular top and bottom sequences ($n = 12$ and $n = 24$) show how the same pitches are distributed over 1 and 2 octaves respectively.

This approach leads to interesting results in terms of pitch sequences but does not allow to take into account the analytical features discussed in subsections 5.1 and 5.2, as those features apply not to pitches but to pitch classes. A second approach can thus be proposed so to exploit the richness of variable n while still placing the discussion at the pitch class level. The algorithm is the following:

1. Generate a generalized Fibonacci sequence G
2. Apply modulo $n > 12$ to G so to get G_1
3. Apply modulo $n_1 = 12$ to G_1 so to get G_2

G_2 represents a sequence of pitches in the range $[0, 1, ..., 11]$, that is one octave, and actual pitches can be interpreted as pitch classes. Figure 10 shows the further application of modulo $n = 12$ to the patterns from Figure 9. Pitch patterns with $n = 12$ and $n = 24$ (top and bottom) result in the same pitch class sequence.

Figure 10. Patterns from Figure 9 with further $n_1 = 12$.

6. MORE GENERALIZATIONS

The classic Fibonacci sequence allows for several generalizations. An interesting case is given by the so-called $k-$Fibonacci sequence [18]:

$$F_k : \left[\begin{array}{lcl} F_{k,0} & = & 0 \\ F_{k,1} & = & 1 \\ F_{k,i} & = & kF_{k,i-1} + F_{k,i-2}. \end{array} \right. \quad (10)$$

If $k = 2$, the Pell sequence appears. If $k = 3$ we get the following sequence

i	0	1	2	3	4	5	6	7	8	...
$F_{3,i}$	0	1	3	10	33	109	360	1189	3927	...

(11)

The $k-$Fibonacci sequences satisfy nice properties. For example the following ones show useful relationship among the terms of the sequence:

- Catalan's identity:

$$F_{k,n-r}F_{k,n+r} - F_{k,n}^2 = (-1)^{n+1-r}F_{k,n}^2;$$

- Simson's identity:

$$F_{k,n-1}F_{k,n+1} - F_{k,n}^2 = (-1)^n;$$

- d'Ocagne's identity:

$$F_{k,m}F_{k,n+1} - F_{k,m+1}F_{k,n} = (-1)^n F_{k,m-n}.$$

More properties, with details of the proofs can be found in [19, 20, 21].

Also for this sequences we can talk of Pisano period thanks to the following result.

Theorem 6.1 ([18]). *$F_k(\bmod n)$ is a simple periodic sequence.*

In particular, the results of Corollaries 3.2.1 and 3.2.2 are valid also for $k-$Fibonacci sequences.

Taking a step further, as a new contribution, we define *fully generalized* Fibonacci as in the following:

$$H_{[a,b,k_1,k_2,s]} : \left[\begin{array}{lcl} H_0 & = & a \\ H_1 & = & b \\ H_i & = & k_1 H_{i-1} + k_2 H_{i-2} + s. \end{array} \right. \quad (12)$$

where a, b, k_1, k_2, s are integers. Parameters a and b provide the initial steps, while k_1 and k_2 determine the size of the recursive steps and s is called the shift of the sequence. Obviously the Fibonacci sequence F corresponds to $H_{[0,1,1,1,0]}$, the sequences $G_{[a,b]}$ are obtained for $H_{[a,b,1,1,0]}$ Finally, with $H_{[0,1,k,1,0]}$ we get the $k-$fibonacci sequences F_k.

A larger amount of different sequences can be obtained by changing some parameters in the definition of Fibonacci recursion laws. By means of generalization a very large palette of numerical sequences becomes easily available to the music composer. Moreover, use of moduli introduces further richness. The previous discussion has shown how it may be complex to choose in advance the generator parameters that lead to a certain sequence without empirically verifying the results. In this sense, an experimental approach is the most viable approach to explore fully generalized Fibonacci sequences, as slight modifications of available parameters may have dramatic effects. As an example, let us consider the fully generalized Lucas sequence $H_{[2,1,3,5,3]}$, with $n = 23$ and $n_1 = 12$. The Pisano period is very large ($\pi(n) = 176$) and the sequence graph particularly complex, as it can be seen in Figure 11. In a computational environment for algorithmic composition, visualization and sonification displays allow to navigate these kinds of datascapes.

7. IMPLEMENTATION

Implementation of fully generalized Fibonacci sequences is straightforward. Listing 1 shows two Python functions `fiboR` and `fiboI` that compute a sequence of length `ln` by passing `a, b, k1, k2, s` as arguments. The first is recursive, the second iterative. Length must be empirically large enough to comprise Pisano period (here, 1000). Modulo can be applied by making use of list comprehension on the resulting `seq` sequence (line 9) by applying n and then $n_1 = 12$ so to get pitch classes. Here we are computing $H_{[2,1,3,5,3]}$ with $n = 23$ and $n_1 = 12$ as discussed in section 6.

```
1  def fiboR(a=0, b=1, k1=1, k2=1, s=0, seq = [], i
     = 0, ln = 1000):
2      if i < ln:
3          c = (a*k1) + (b*k2) + s
4          seq.append(c)
5          fiboR(b, c, k1, k2, s, seq, i+1, ln)
6      return [a,b]+seq
7
8  def fiboI(a=0, b=1, k1=1, k2=1, s=0, ln=1000):
9      seq = [a, b]
10     for x in range(ln):
11         a, b = b, (a*k1) + (b*k2) + s
12         seq.append(b)
13     return seq
14
15 n = 23
16 seq = [(x % n) % 12 for x in fiboI(2,1,3,5,3)]
```

Listing 1. Python example.

Functions for period estimation and for data analysis (e.g. occurrence count) and conversion (e.g. decimal notation of pitch class set) are similarly trivial. Jupyter Notebook [7] provides a web-based computing platform that offers an interactive environment for Python development [8]. In the context of algorithmic composition, Jupyter Notebook allows to exploit the large ecosystem of Python libraries. As an example, after computing the decimal representation of a pitch class set, by means of the `webbrowser` module it is possible to automatically access Ian Ring's pages dedicated to that scale. Figures 5 and 6 have been generated using the standard `matplotlib` module for plotting. The sequence graph in Figure 7 has been generated with the `pydot` module, a Python interface for Graphviz [22]. The same module has been used to generate the sequence graph for $H_{[2,1,3,5,3]}$ with $n = 23$ and $n_1 = 12$ computed in Listing 1, shown in Figure 11. The Python Music21package provides an integrated bridge towards music computation and notation [23]. It includes functionalities to analyse, plot and notate music data. In relation to music notation, it allows to generate MusicXML that can be further manipulated in music notation softwares (e.g. Musescore [9]). Like the other previously mentioned modules, Music21 can be directly accessed via Jupyter Notebook, so that notation output can be displayed in the Notebook itself (via Musescore backend) and results can be heard (via MIDI playback). All music examples (Figg. 4, 8, 9, 10), have been generated via Music21 and Musescore. Further data sonification can be obtained by the direct usage of the SuperCollider [10] environment into Jupyter Notebook by means of the *sc3nb* Python-interface [24].

8. CONCLUSIONS AND FUTURE DEVELOPMENTS

Fully generalized Fibonacci sequences are simple yet powerful generative devices that results in complex, unpredictable patterns that, once the modulo operation is applied, nevertheless lead to repeating organizations. These sequences can be interpreted in the music domain in various ways. In our proposal, we mapped integers onto indices for pitch and pitch class description. In the latter case, modulo 12 is the key operation that enables to lead back the obtained results to the vast music theory literature dedicated to the topic. It is worth mentioning that obtained sequences can indeed be manipulated further. As an example, transposition results in adding a factor t to all the integers in a sequence. As the operation is obvious, we have focused on sequence properties rather than on pitch classes in themselves (e.g. we have not taken into account that Fibonacci starts on C while Lucas on D, as pitch organization can be transposed at will). Fully generalized Fibonacci sequences favors an experimental, empirical approach, as through computational environments it is possible for each sequence to explore pitch class set composition, occurrence distribution, Pisano period. In short, fully generalized Fibonacci sequences are useful tools for algorithmic composition.

The previous discussion focused on pitch classed based on 12-ET. We can think of extending the approach by taking into account non-integer numbers. As an example, $4 \mod 2.5 = 1.5$. Fractional parts can be interpreted as semitone fractions, like in MIDI notation, where e.g. 60.5 indicates a middle C raised of a quarter tone. Indeed, other fractions can be obtained. Non-integers can be applied in fully generalized Fibonacci both to parameters a, b, k_1, k_2, s and to $\mod n$. If holding the octave constraint, by further applying $\mod n = 12$ it is possible to create non-12-ET pitch class sets. In relation to this, a different approach to be explored is to define a specific mapping function that maps the output integers from H onto specific sets of non-integers values. Finally, while we consider pitch and pitch class interpretation as particularly promising, fully generalized Fibonacci $\mod n$ can be used to generate values to control arbitrary cycling parameters for sound and music computing. As a suggestion, periodic sequences can be used to fill looping wavetables for audio synthesis. In this sense, one can think of Figure 5 as a wavetable plot. An interesting feature is that these wavetables are amplitude limited, as the amplitude range is by definition in the range $[0, n-1]$, where n is the last applied modulo.

[7] `https://jupyter.org/`

[8] See the implementation on colab: `https://github.com/vanderaalle/FullyGeneralizedFibonacciSequences/blob/main/FibonacciModuloN.ipynb`

[9] `https://musescore.org/`

[10] `https://supercollider.github.io/`

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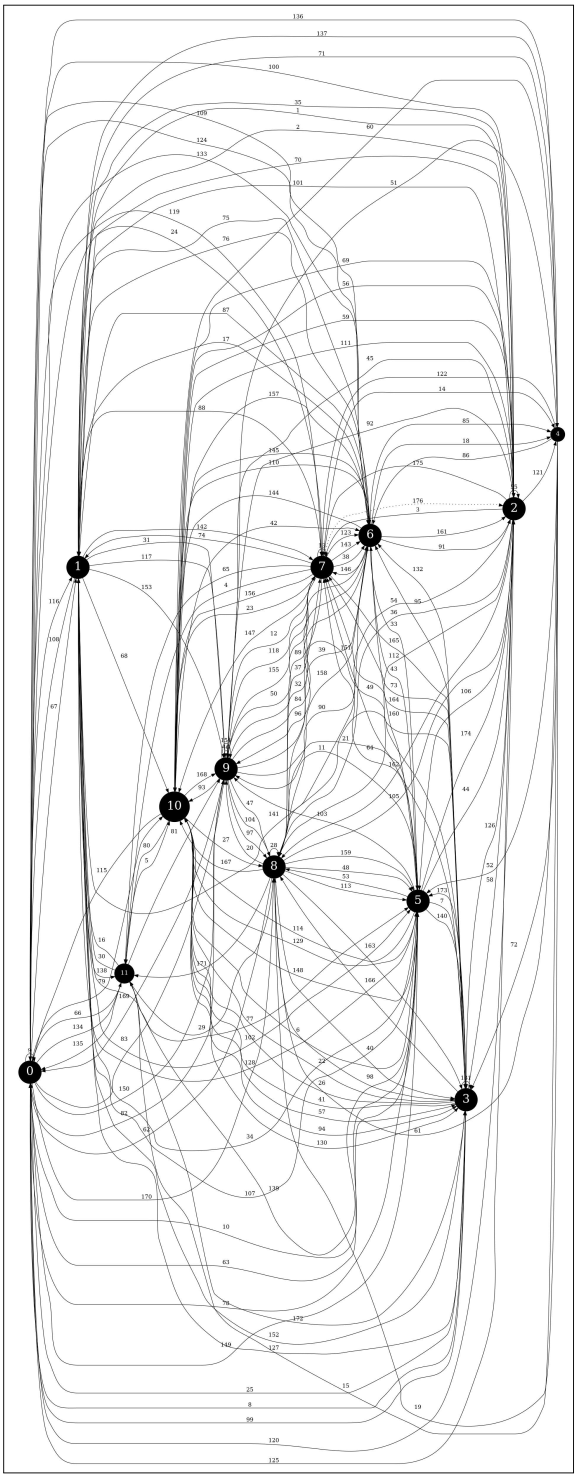

Figure 11. Graph for $H_{[2,1,3,5,5,3]}$ with $n = 23$ and $n_1 = 12$.

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