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Clausal Presentation of Theories in Deduction Modulo

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Abstract Resolution modulo is an extension of first-order resolution in which rewrite rules are used to rewrite clauses during the search. In the first version of this method, clauses are rewritten to arbitrary propositions. These propositions are needed to be dynamically transformed into clauses. This unpleasant feature can be eliminated when the rewrite system is clausal, i.e., when it rewrites clauses to clauses. We show in this paper how to transform any rewrite system into a clausal one, preserving the existence of cut free proofs of any sequent.

Keywords resolution, deduction modulo, cut free proof, clause

1 Introduction

Deduction modulo[1] is an extension of first-order predicate logic where axioms (for instance $P \Leftrightarrow (Q \Rightarrow R)$) are replaced by rewrite rules (for instance $P \rightarrow (Q \Rightarrow R)$) used to replace a proposition by an equivalent one at any time during a proof. One of its applications is the sequent calculus modulo[1] which is an extension of pure sequent calculus (see Fig.1). The resolution method in deduction modulo, resolution modulo[1], is an extension of first-order resolution[2-4] in which rewrite rules are used to narrow clauses during the search. Resolution modulo is sound and complete, i.e., for any confluent rewrite system $\mathcal{R}$, the sequent $\Gamma \vdash \Delta$ has a cut free proof in sequent calculus modulo $\mathcal{R}$ if and only if the empty clause can be derived from the clauses of $\Gamma, \neg\Delta$ (denoted by $Cl(\Gamma, \neg\Delta)$) with two rules: the usual resolution rule and the narrowing rule that permits to rewrite, or more generally narrow, a clause.

In resolution modulo, rules rewrite clauses to arbitrary propositions, which need to be dynamically transformed into clauses. For instance, the rule $P \rightarrow (Q \Rightarrow R)$ rewrites the clause P to a non-clausal proposition $Q \Rightarrow R$. In the process of deriving the empty clause from the clauses $\{P\}$, $\{Q\}$, $\{\neg R\}$, we first derive $\{Q \Rightarrow R\}$ from $\{P\}$, then we need to transform $\{Q \Rightarrow R\}$ into a clause $\{\neg Q, R\}$. See the left derivation in Fig.2 ($\rightsquigarrow$: resolution, $\rightarrow$: rewriting, $\twoheadrightarrow$: dynamic transformation to clauses). In another example, attempting to derive the empty clause from $\{P\}$, $\{\neg Q(x)\}$ with the rewrite rule $P \rightarrow \exists x Q(x)$, we first derive $\{\exists x Q(x)\}$ from $\{P\}$, then $\{\exists x Q(x)\}$ needs to be transformed into a clause $\{Q(c)\}$ with a new Skolem symbol c (see the right derivation in Fig.2). The problem we address in this paper is to avoid these dynamic transformations.

This unpleasant dynamical transformation can be eliminated when the rewrite system is clausal, i.e., when it rewrites clauses to clauses. This is the idea of polarized resolution modulo[5]. See Fig.3 for a presentation of polarized resolution modulo where unification problems are kept as constraints and $|A|$ denotes the clause of proposition A. See [6] for an efficient implementation of polarized resolution modulo.

Polarized resolution modulo is sound and complete, i.e., for any clausal rewrite system $\mathcal{R}$, the sequent $\Gamma \vdash \Delta$ has a cut free proof in polarized sequent calculus modulo $\mathcal{R}$ if and only if a clause $\Box[\mathcal{C}]$ with $\mathcal{E}$-unifiable constraints $\mathcal{C}$ can be derived from $Cl(\Gamma, \neg\Delta)$ in polarized resolution modulo $\mathcal{R}$. See Fig.4 for an example of polarized resolution module with the clausal rewrite system containing one rule $P \rightarrow_{-} \neg Q \vee R$. Compared with Fig.2, in the process of deriving the empty clause from the clauses $\{P\}, \{Q\}, \{\neg R\}$, we do not need to dynamically transform $Q \Rightarrow R$ to $\{\neg Q, R\}$ in Fig.4. Thus the problem can be reformulated as translating a rewrite system into a clausal one.

In polarized resolution modulo, rewrite systems distinguish rules as positive and negative, with negative rules rewriting atomic propositions to clausal proposi-

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$$\frac{}{A \vdash B} \text{ axiom if } A \rightarrow_-^* P,\ B \rightarrow_+^* P,\ P \text{ atomic} \qquad \frac{\Gamma, B \vdash \Delta \quad \Gamma \vdash C, \Delta}{\Gamma \vdash \Delta} \text{ cut if } A \rightarrow_-^* B,\ A \rightarrow_+^* C$$

$$\frac{\Gamma, B, C \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ contr-left if } A \rightarrow_-^* B,\ A \rightarrow_-^* C \qquad \frac{\Gamma \vdash B, C, \Delta}{\Gamma \vdash A, \Delta} \text{ contr-right if } A \rightarrow_+^* B,\ A \rightarrow_+^* C$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta} \text{ weak-left, } A \text{ atomic} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta} \text{ weak-right, } A \text{ atomic}$$

$$\frac{}{\Gamma \vdash A, \Delta} \top\text{-right if } A \rightarrow_+^* \top \qquad \frac{}{\Gamma, A \vdash \Delta} \bot\text{-left if } A \rightarrow_-^* \bot$$

$$\frac{\Gamma \vdash B, \Delta}{\Gamma, A \vdash \Delta} \neg\text{-left if } A \rightarrow_-^* \neg B \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma \vdash A, \Delta} \neg\text{-right if } A \rightarrow_+^* \neg B$$

$$\frac{\Gamma, B, C \vdash \Delta}{\Gamma, A \vdash \Delta} \wedge\text{-left if } A \rightarrow_-^* (B \wedge C) \qquad \frac{\Gamma \vdash B, \Delta \quad \Gamma \vdash C, \Delta}{\Gamma \vdash A, \Delta} \wedge\text{-right if } A \rightarrow_+^* (B \wedge C)$$

$$\frac{\Gamma, B \vdash \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \vee\text{-left if } A \rightarrow_-^* (B \vee C) \qquad \frac{\Gamma \vdash B, C, \Delta}{\Gamma \vdash A, \Delta} \vee\text{-right if } A \rightarrow_+^* (B \vee C)$$

$$\frac{\Gamma \vdash B, \Delta \quad \Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \Rightarrow\text{-left if } A \rightarrow_-^* (B \Rightarrow C) \qquad \frac{\Gamma, B \vdash C, \Delta}{\Gamma \vdash A, \Delta} \Rightarrow\text{-right if } A \rightarrow_+^* (B \Rightarrow C)$$

$$\frac{\Gamma, C \vdash \Delta}{\Gamma, A \vdash \Delta} \forall\text{-left if } A \rightarrow_-^* \forall x B, (t/x)B \rightarrow_-^* C \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A, \Delta} \langle x, B\rangle\forall\text{-right if } A \rightarrow_+^* \forall x B, x \notin FV(\Gamma\Delta)$$

$$\frac{\Gamma, B \vdash \Delta}{\Gamma, A \vdash \Delta} \exists\text{-left if } A \rightarrow_-^* \exists x B, x \notin FV(\Gamma \quad \Delta) \qquad \frac{\Gamma \vdash C, \Delta}{\Gamma \vdash A, \Delta} \exists\text{-right if } A \rightarrow_+^* \exists x B, (t/x)B \rightarrow_+^* C$$

Fig.1. Polarized sequent calculus modulo.

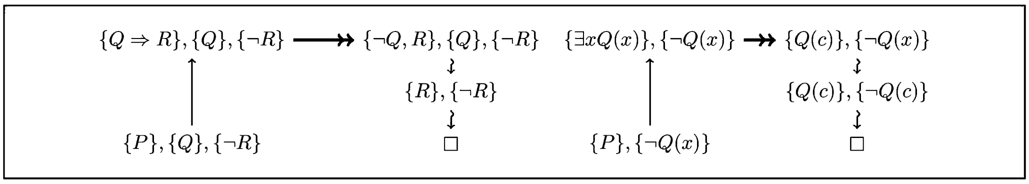

Fig.2. Examples of resolution modulo.

$$\frac{(U, P_1, \ldots, P_n)[\mathcal{C}_1] \quad (V, \neg Q_1, \ldots, \neg Q_p)[\mathcal{C}_2]}{(U \cup V)[\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{P_1 = \ldots = P_n = Q_1 = \ldots = Q_p\}]} \quad \text{Resolution}$$

$$\frac{(U, P)[\mathcal{C}]}{(U \cup |V|)[\mathcal{C} \cup \{P = Q\}]} \text{ if } Q \longrightarrow_- V \text{ is one rule of } \mathcal{R} \quad \text{Extended Narrowing}$$

$$\frac{(U, \neg P)[\mathcal{C}]}{(U \cup |V|)[\mathcal{C} \cup \{P = Q\}]} \text{ if } Q \longrightarrow_+ \neg V \text{ is one rule of } \mathcal{R} \quad \text{Extended Narrowing}$$

Fig.3. Polarized resolution modulo.

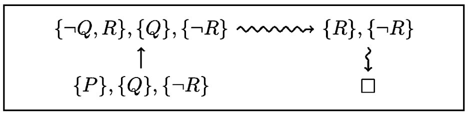

Fig.4. Example of polarized resolution modulo.

tions and positive rules rewriting atomic propositions to negation of clausal propositions. This is needed because the extended narrowing rule with $P \rightarrow \neg Q \vee R$ for example transforms the clause $\{P\}$ to the clause $\{\neg Q, R\}$. But when we have the clauses $\{\neg P\}$ and $\{\neg Q\}$ for example, we cannot use the same rewrite rule, since it would transform $\{\neg P\}$ into $\{\neg(\neg Q \vee R)\}$ which is not a clause. In this case, since $Cl(\{\neg(\neg Q \vee R)\}) = \{\{Q\}, \{\neg R\}\}$, we want to transform $\{\neg P\}$ into clause $\{Q\}$ directly. Instead of $P \rightarrow \neg Q \vee R$, we want to use the positive rule $P \rightarrow \neg Q$. Using this rewrite rule, the extended narrowing of Fig.3 transforms the clause $\{\neg P\}$ into the clause $\{Q\}$ and we can conclude with the resolution rule.

In this paper, we show how to transform any rewrite system into an equivalent clausal one, preserving the existence of cut free proofs of any sequent. In this way, polarized resolution modulo can be applied to the system directly.

The paper is organized as follows. In Section 2 we recall some basic notions in resolution modulo and polarized resolution modulo. In Section 3 we construct the translator (Definition 4). Then we prove that the translator terminates and returns a clausal rewriting system (Theorem 1). In Section 4 we prove that the clausal rewrite system returned by the translator preserves the existence of cut free proofs for any sequent of the original language (Theorem 2). In Section 5 we give a complete example. Finally, we conclude the paper in Section 6.

2 Preliminaries

Definitions and propositions of this section are taken from [1, 4-5].

Definition 1. *A proposition is a literal if it is either atomic or the negation of an atomic proposition. A clause is a set of literals. A proposition is clausal if it is* $\bot$ *or of the form* $L_1 \vee \ldots \vee L_n$ *where* $L_1, \ldots, L_n$ *are literals. If* $A = L_1 \vee \ldots \vee L_n$ *is a clausal proposition, we write* $|A|$ *for the clause* $\{L_1, \ldots, L_n\}$.

Definition 2. *A polarized rewrite system is a triple* $\mathcal{R} = \langle \mathcal{E}, \mathcal{R}_-, \mathcal{R}_+ \rangle$ *where* $\mathcal{E}$ *is a set of equations between terms,* $\mathcal{R}_-$ *and* $\mathcal{R}_+$ *are sets of rewrite rules whose left-hand sides are atomic propositions and right-hand sides are arbitrary propositions. The rules of* $\mathcal{R}_-$ *are called negative rules and those of* $\mathcal{R}_+$ *are called positive rules. A rewrite system is clausal if negative rules rewrite atomic propositions to clausal propositions and positive rules rewrite atomic propositions to negations of clausal propositions. Let* $\mathcal{R} = \langle \mathcal{E}, \mathcal{R}_-, \mathcal{R}_+ \rangle$ *be a polarized rewrite system. We define the equivalence relation* $=_\mathcal{E}$ *as the congruence on terms generated by the equations of* $\mathcal{E}$. *We then define the one-step negative and positive rewriting relations* $\to_-$ *and* $\to_+$ *as follows.*

- *If* $t_i =_\mathcal{E} u$ *then* $P(t_1, \ldots, t_i, \ldots, t_n) \to_- P(t_1, \ldots, u, \ldots, t_n)$ *and* $P(t_1, \ldots, t_i, \ldots, t_n) \to_+ P(t_1, \ldots, u, \ldots, t_n)$.
- *If* $P \to A$ *is a rule of* $\mathcal{R}_s$ *and* σ *is a substitution then* $\sigma P \to_s \sigma A$, *where* s *is either* $-$ *or* $+$.
- *If* $A \to_{\bar{s}} A'$ *then* $\neg A \longrightarrow_s \neg A'$, *where* $\bar{\cdot}$ *swaps* $-$ *and* $+$.
- *If* $(A \to_s A'$ *and* $B = B')$ *or* $(A = A'$ *and* $B \to_s B')$, *then* $A \wedge B \to_s A' \wedge B'$ *and* $A \vee B \to_s A' \vee B'$.
- *If* $(A \to_{\bar{s}} A'$ *and* $B = B')$ *or* $(A = A'$ *and* $B \longrightarrow_s B')$, *then* $A \Rightarrow B \to_s A' \Rightarrow B'$.
- *If* $A \to_s A'$ *then* $\forall x\, A \to_s \forall x\, A'$ *and* $\exists x\, A \to_s \exists x\, A'$.

We define the sequent one-step term rewriting relation $\to$ *as follows*:

- *If* $A \to_- A'$ *then* $(\Gamma, A \vdash \Delta) \to (\Gamma, A' \vdash \Delta)$.
- *If* $A \to_+ A'$ *then* $(\Gamma \vdash A, \Delta) \to (\Gamma \vdash A', \Delta)$.

As usual, if R is any binary relation, we write R^* for its reflexive-transitive closure.

Proposition 1. *Let* $\mathcal{R} = (\mathcal{E}, \mathcal{R}_-, \mathcal{R}_+)$ *be a polarized rewrite system. If* $(\Gamma \vdash \Delta) \to^* (\Gamma' \vdash \Delta')$ *and* $\Gamma' \vdash \Delta'$ *has a cut free proof modulo* $\mathcal{R}$ *then* $\Gamma \vdash \Delta$ *has a cut free proof modulo* $\mathcal{R}$ *of the same size.*

Intuitively, the size of a proof is the number of nodes in the proof tree.

Definition 3. *Let* L *be a language containing an equality predicate in each sort. Let* $\mathcal{R}$ *be a polarized rewrite system in* L. *Let* $\mathcal{U}_\mathcal{R}$ *be the set of axioms containing*:

- *the axioms of equality for* L,
- *for each equational axiom* $t = u$ *of* $\mathcal{E}$, *the universal closure of the proposition* $t = u$,
- *for each rule* $P \to A$ *of* $\mathcal{R}_-$, *the universal closure of the proposition* $P \Rightarrow A$,
- *for each rule* $P \to A$ *of* $\mathcal{R}_+$, *the universal closure of the proposition* $A \Rightarrow P$.

Proposition 2. *Let* $\mathcal{L}$ *be a language and* $\mathcal{R}$ *be a polarized rewrite system in* $\mathcal{L}$. *Let* $\mathcal{L}'$ *be the language obtained by adding an equality symbol in each sort of* $\mathcal{L}$. *Then, a sequent* $\Gamma \vdash \Delta$ *of* L *is provable modulo* $\mathcal{R}$ *if and only if it is provable in* $\mathcal{U}_\mathcal{R}$[①].

3 Translator

In this section we will show how to translate a rewrite system into a clausal one. We first translate each rule into a negative rule and a positive rule, then translate the negative rule into rule(s) rewriting atomic propositions to clausal propositions and translate the positive rule into rule(s) rewriting atomic propositions to negations of clausal propositions (see Fig.5).

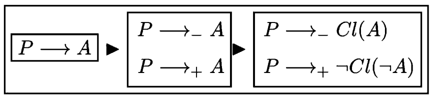

Fig.5. Main idea underlying our translation.

To make our translator simpler, rewrite rules $P \to_- A$, $P \to_+ A$ need to be transformed into $P \to_- A \vee \bot'$ and $P \to_+ \neg(\neg A \vee \bot')$ respectively before put into the translator defined as follows.

We add a symbol $\bot'$ into the language which is just a primed version of $\bot$. This symbol is used to prove the termination of the translator only.

Definition 4 (Translation).

Step 1: *translate the rewrite rule* $P \to_- A$, $P \to_+ A$ *into* $P \to_- A \vee \bot'$ *and* $P \to_+ \neg(\neg A \vee \bot')$ *respectively.*

Step 2: *translate the source rule in Table* 1 *into its target rule(s) and keep on recurring in step* 2 *while it can be applied.*

Step 3: *translate the rewrite rule* $P \to_- \bot' \vee B$ *and* $P \to_+ \neg(\bot' \vee B)$ *into* $P \to_- B$, $P \to_+ \neg B$ *respectively.*

If the rewrite rule r' is one of the target rule(s) of r, we denote this by $r \rhd r'$. The polarized rewrite system $\mathcal{R}$ is one step translated into $\mathcal{R}'$ (denoted by $\mathcal{R} \blacktriangleright \mathcal{R}'$)

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①Dowek G. Simple type theory as a clausal theory. https://who.rocq.inria.fr/Gilles.Dowek/publi.html, April 2013.

Table 1. Translator for the Negative and Positive Rules

Case Number	Source Rule	Target Rule(s)
1	$P \rightarrow_- \bot \vee R$	$P \rightarrow_- R$
2	$P \rightarrow_- Q \vee R$ (Q is atomic)	$P \rightarrow_- R \vee Q$
3	$P \rightarrow_- (Q_1 \wedge Q_2) \vee R$	$P \rightarrow_- (Q_1 \vee R)$ $P \rightarrow_- (Q_2 \vee R)$
4	$P \rightarrow_- (Q_1 \vee Q_2) \vee R$	$P \rightarrow_- Q_1 \vee (Q_2 \vee R)$
5	$P \rightarrow_- (Q_1 \Rightarrow Q_2) \vee R$	$P \rightarrow_- (\neg Q_1 \vee (Q_2 \vee R))$
6	$P \rightarrow_- \forall x Q \vee R$	$P \rightarrow_- (Q \vee R)$
7	$P \rightarrow_- \exists x Q \vee R$	$P \rightarrow_- ((f(l)/x)Q \vee R)$ l free variables of $\exists xQ, R$
8	$P \rightarrow_- \neg\bot \vee R$	Drop this rule
9	$P \rightarrow_- \neg Q \vee R$ (Q is atomic)	$P \rightarrow_- R \vee \neg Q$
10	$P \rightarrow_- \neg(\neg Q) \vee R$	$P \rightarrow_- Q \vee R$
11	$P \rightarrow_- \neg(Q_1 \wedge Q_2) \vee R$	$P \rightarrow_- \neg Q_1 \vee (\neg Q_2 \vee R)$
12	$P \rightarrow_- \neg(Q_1 \vee Q_2) \vee R$	$P \rightarrow_- \neg Q_1 \vee R$ $P \rightarrow_- \neg Q_2 \vee R$
13	$P \rightarrow_- \neg(Q_1 \Rightarrow Q_2) \vee R$	$P \rightarrow_- Q_1 \vee R$ $P \rightarrow_- \neg Q_2 \vee R$
14	$P \rightarrow_- \neg(\forall x Q) \vee R$	$P \rightarrow_- ((f(l)/x)\neg Q) \vee R$ l free variables of $\forall xQ, R$
15	$P \rightarrow_- \neg(\exists x Q) \vee R$	$P \rightarrow_- (\neg Q) \vee R$
16	$P \rightarrow_+ \neg(\bot \vee R)$	$P \rightarrow_+ \neg R$
17	$P \rightarrow_+ \neg(Q \vee R)$ (Q is atomic)	$P \rightarrow_+ \neg(R \vee Q)$
18	$P \rightarrow_+ \neg((Q_1 \wedge Q_2) \vee R)$	$P \rightarrow_+ \neg(Q_1 \vee R)$ $P \rightarrow_+ \neg(Q_2 \vee R)$
19	$P \rightarrow_+ \neg((Q_1 \vee Q_2) \vee R)$	$P \rightarrow_+ \neg(Q_1 \vee (Q_2 \vee R))$
20	$P \rightarrow_+ \neg((Q_1 \Rightarrow Q_2) \vee R)$	$P \rightarrow_+ \neg(\neg Q_1 \vee (Q_2 \vee R))$
21	$P \rightarrow_+ \neg(\forall x Q \vee R)$	$P \rightarrow_+ \neg(Q \vee R)$
22	$P \rightarrow_+ \neg(\exists x Q \vee R)$	$P \rightarrow_+ \neg((f(l)/x)Q \vee R)$ l free variables of $\exists xQ, R$
23	$P \rightarrow_+ \neg(\neg\bot \vee R)$	Drop this rule
24	$P \rightarrow_+ \neg((\neg Q) \vee R)$ (Q is atomic)	$P \rightarrow_+ \neg(R \vee (\neg Q))$
25	$P \rightarrow_+ \neg(\neg(\neg Q) \vee R)$	$P \rightarrow_+ \neg(Q \vee R)$
26	$P \rightarrow_+ \neg(\neg(Q_1 \wedge Q_2) \vee R)$	$P \rightarrow_+ \neg(\neg Q_1 \vee (\neg Q_2 \vee R))$
27	$P \rightarrow_+ \neg(\neg(Q_1 \vee Q_2) \vee R)$	$P \rightarrow_+ \neg(\neg Q_1 \vee R)$ $P \rightarrow_+ \neg(\neg Q_2 \vee R)$
28	$P \rightarrow_+ \neg(\neg(Q_1 \Rightarrow Q_2) \vee R)$	$P \rightarrow_+ \neg(Q_1 \vee R)$ $P \rightarrow_+ \neg(\neg Q_2 \vee R)$
29	$P \rightarrow_+ \neg(\neg(\forall x Q) \vee R)$	$P \rightarrow_+ \neg(((f(l)/x)\neg Q) \vee R)$ l free variables of $\forall xQ, R$
30	$P \rightarrow_+ \neg(\neg(\exists x Q) \vee R)$	$P \rightarrow_+ \neg(\neg Q \vee R)$

if $\mathcal{R}'$ is from $\mathcal{R}$ by translating one rule of $\mathcal{R}$ (replace one rule of $\mathcal{R}$ by its target rule(s)).

For example, given the polarized rewrite system containing only two rules $P \rightarrow_- (Q \Rightarrow R)$ and $P \rightarrow_+ (Q \Rightarrow R)$. Here we start step 2 with $P \rightarrow_- (Q \Rightarrow R) \vee \bot'$ and $P \rightarrow_+ (Q \Rightarrow R) \vee \bot'$. Finally, we get the rules $P \longrightarrow_- \neg Q \vee R$, $P \longrightarrow_+ \neg Q$, $P \longrightarrow_+ \neg(\neg R)$ (see Fig.6).

Notice that variables that are free in the right-hand side but not in the left-hand side are authorized in the rewrite rule. This does not have impact on the correctness of the results. We can universally quantify on the free variables in the right-hand side but not in the left-hand side. For instance, we have $P(x) \rightarrow_- \forall y Q(x,y)$ and $P(x) \rightarrow_+ \neg\forall y Q(x,y)$ by quantifying on $P(x) \rightarrow_- Q(x,y)$ and $P(x) \rightarrow_+ \neg Q(x,y)$, respectively.

We now prove that the translator terminates and returns a clausal rewriting system. We only prove this for the negative rules, as the proofs for the positive rules are similar. Let $P \rightarrow_- A \vee \bot'$ be the input of step 2. During the translation, $\bot'$ always splits A into the part that has been translated and the part that has not been translated. The part that has not been translated becomes smaller and smaller until it disappears.

Definition 5. *Let $P \rightarrow_- A$ be a negative rule with only one occurrence of $\bot'$. We say that $\bot'$ is free in $P \rightarrow_- A$, if*

- *$A = \bot'$ or,*
- *$A = B_1 \vee B_2$ and $\bot'$ is free in $P \rightarrow_- B_1$ or,*
- *$A = B_1 \vee B_2$ and $\bot'$ is free in $P \rightarrow_- B_2$.*

Intuitively, we say that $\bot'$ is free in $P \rightarrow_- A$ when $\bot'$ is not an argument of other connectives than $\vee$, for instance, $A = \exists x B \vee (\bot' \vee C)$. But $A = \bot' \wedge C$ is not an instance of Definition 5, because $\bot'$ is an argument of $\wedge$ in $\bot' \wedge C$.

Lemma 1. *Let $r_1 \triangleright r_2 \triangleright \ldots \triangleright r_n$ be a translation sequence in step 2 such that $\bot'$ is free in r_1. Then $\bot'$ is free in r_i where $1 \leqslant i \leqslant n$.*

Proof. We prove this by induction on the number of translation steps. It is sufficient to show that if $\bot'$ is free in r_i then $\bot'$ is free in r_{i+1}. We prove this for each case of Table 1 used for $r_i \triangleright r_{i+1}$. We only consider the four most complex cases (the proofs for the other cases are similar):

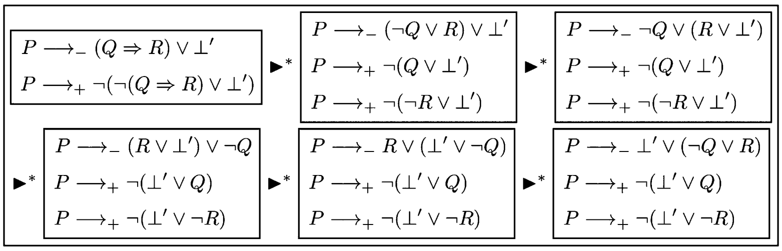

Fig.6. Translation example.

Case 1. By induction hypothesis, $\bot'$ is free in r_i, where r_i has the form $P \to \bot \vee R$. By Definition 5, there exists a $\bot'$ in $\bot \vee R$. Since $\bot \neq \bot'$, $\bot'$ in R, by Definition 5, $\bot'$ is free in $P \to_{-} R$, that is r_{i+1}.

Case 2. By induction hypothesis, $\bot'$ is free in r_i, where r_i has the form $P \to Q \vee R$ where Q is atomic. By Definition 5, there exists a $\bot'$ in $Q \vee R$. Since $Q \neq \bot'$, $\bot'$ in R, by Definition 5, $\bot'$ is free in $P \to R$. So $\bot'$ is free in $P \to R \vee Q$.

Case 3. By induction hypothesis, $\bot'$ is free in r_i, where r_i has the form $P \to (Q_1 \wedge Q_2) \vee R$. By Definition 5, there exists a $\bot'$ in $(Q_1 \wedge Q_2) \vee R$. Moreover $\bot'$ is in R, because if $\bot'$ is in $Q_1 \wedge Q_2$, then $\bot'$ is not free in $P \to (Q_1 \wedge Q_2) \vee R$. By Definition 5, $\bot'$ is free in $P \to R$. So $\bot'$ is free in $P \to Q_1 \vee R$ and $P \to Q_2 \vee R$.

Case 4. By induction hypothesis, $\bot'$ is free in r_i, where r_i has the form $P \to (Q_1 \Rightarrow Q_2) \vee R$. By Definition 5, there exists a $\bot'$ in $(Q_1 \Rightarrow Q_2) \vee R$. Since if $\bot'$ is in $Q_1 \Rightarrow Q_2$, then $\bot'$ is not free in $P \to (Q_1 \Rightarrow Q_2) \vee R$, so $\bot'$ is in R. By Definition 5, $\bot'$ is free in $P \to R$. So $\bot'$ is free in $P \to (\neg Q_1 \vee Q_2) \vee R$. □

Lemma 2. *Let* $(P \to (A \vee \bot')) \rhd (P \to A_1) \rhd \ldots \rhd (P \to A_n)$ *be a translation sequence in step* 2 *such that* $\bot'$ *is free in* $P \to A \vee \bot'$. *Then the symbols occurring on the right-hand side of* $\bot'$ *consist of "(", ")", "$\vee$" and literals in* A_i, *where* $1 \leqslant i \leqslant n$.

Proof. Since $\bot'$ is free in $P \to A \vee \bot'$, by Lemma 1, $\bot'$ is free in $P \to A_i$ where $1 \leqslant i \leqslant n$. We prove this lemma by induction on the number of translation steps. It is sufficient to show if the symbols on the right-hand side of $\bot'$ consist of "(", ")", "$\vee$" and literals in A_i so do the symbols on the right-hand side of $\bot'$ in A_{i+1}. We prove this for each case of Table 1 used for $(P \to A_i) \rhd (P \to A_{i+1})$. We only consider the four most complex cases (the proofs for the other cases are similar):

Case 1. Since $\bot'$ is free in $P \to A_i$ where A_i has the form $\bot \vee R$, so $\bot'$ is in R. By induction hypothesis, the symbols on the right-hand side of $\bot'$ consist of "(", ")", "$\vee$" and literals in R.

Case 2. Since $\bot'$ is free in $P \to A_i$ where A_i has the form $Q \vee R$ where Q is atomic, so $\bot'$ is in R. By induction hypothesis, the symbols on the right-hand side of $\bot'$ consist of "(", ")", "$\vee$" and literals in R. Because Q is literal, the symbols on the right-hand side of $\bot'$ consist of "(", ")", "$\vee$" and literals in $R \vee Q$.

Case 3. Since $\bot'$ is free in $P \to A_i$ where A_i has the form $(Q_1 \wedge Q_2) \vee R$, so $\bot'$ is in R. By induction hypothesis, the symbols on the right-hand side of $\bot'$ consist of "(", ")", "$\vee$" and literals in R. So do the symbols on the right-hand side of $\bot'$ in $Q_1 \vee R$ and $Q_2 \vee R$.

Case 4. By induction hypothesis, the symbols on the right-hand side of $\bot'$ consist of "(", ")", "$\vee$" and literals in r_1 where r_1 has the form $P \to (Q_1 \vee Q_2) \vee R$. So do these symbols in $P \to Q_1 \vee (Q_2 \vee R)$. □

Definition 6. *Let* P *be a proposition. The size of* P (*denoted by* $|P|$) *is defined as follows*:
- *if* P *is atomic, then* $|P| = 1$,
- *if* $P = \bot$, *then* $|P| = 1$,
- *if* $P = \bot'$, *then* $|P| = 1$,
- *if* $P = Q_1 \vee Q_2$, *then* $|P| = |Q_1| + |Q_2| + 1$,
- *if* $P = Q_1 \wedge Q_2$, *then* $|P| = |Q_1| + |Q_2| + 3$,
- *if* $P = Q_1 \Rightarrow Q_2$, *then* $|P| = |Q_1| + |Q_2| + 3$,
- *if* $P = \neg Q_1$, *then* $|P| = |Q_1| + 1$,
- *if* $P = \exists x Q_1$, *then* $|P| = |Q_1| + 1$,
- *if* $P = \forall x Q_1$, *then* $|P| = |Q_1| + 1$.

The next definition introduces the notion of size for the part of a rule that is on the left of $\bot'$, i.e., the size of the part that has not been translated.

Definition 7. *Let* $P \to A$ *be a negative rule with* $\bot'$ *free in it. The size of* $P \to A$ (*denoted by* $\|P \to A\|$) *is defined as follows*:

$$\|P \to A\| = \begin{cases} 1, & \text{if } A = \bot', \\ \|P \to Q_1\|, & \text{if } A = Q_1 \vee Q_2 \text{ with } \bot' \text{ in } Q_1, \\ |Q_1| + \|P \to Q_2\| + 1, & \text{if } A = Q_1 \vee Q_2 \text{ with } \bot' \text{ in } Q_2. \end{cases}$$

According to Definition 5, either $A = \bot'$ or A has the form $Q_1 \vee Q_2$, so Definition 7 is well formed.

Lemma 3. *For any negative rule* $P \to A \vee \bot'$ *with* $\bot'$ *not in* A, *the translation in step* 2 *terminates at* $P \to A_n$ *where* A_n *has the form* $\bot'$ *or* $\bot' \vee B$.

Proof. Suppose the translation proceeds as follows: $(P \to A \vee \bot') \rhd \ldots \rhd (P \to A_m) \rhd \ldots$. By Definition 5, $\bot'$ is free in $P \to A \vee \bot'$. By Lemma 1, $\bot'$ is in $P \to A_i$. That is either $A_i = \bot'$ or $A_i = A_i^1 \vee A_i^2$.

The translation terminates since the pair $\langle \|P \to A_i\|, |A_i^1| \rangle$ decreases according to the lexicographic order after each translation step. We prove this for each case used for translation in Table 1. In general except case 4, the first part of the pair $\langle \|P \to A_i\|, |A_i^1| \rangle$ decreases. For instance, in case 11 the pair $\langle \|P \to_{-} \neg(Q_1 \wedge Q_2) \vee R\|, |\neg(Q_1 \wedge Q_2)| \rangle$ decreases to $\langle \|P \to_{-} \neg Q_1 \vee (\neg Q_2 \vee R)\|, |\neg Q_1| \rangle$. This is because

$$\begin{aligned} &\|P \to_{-} \neg(Q_1 \wedge Q_2) \vee R\| \\ &= |\neg(Q_1 \wedge Q_2)| + \|P \to_{-} R\| + 1 \\ &= |(Q_1 \wedge Q_2)| + 1 + \|P \to_{-} R\| + 1 \\ &= |Q_1| + |Q_2| + 3 + 1 + \|P \to_{-} R\| + 1 \\ &= (|Q_1| + 1) + (|Q_2| + 1) + \|P \to_{-} R\| + 3 \\ &= |\neg Q_1| + |\neg Q_2| + \|P \to_{-} R\| + 3 \\ &= |\neg Q_1| + (|\neg Q_2| + \|P \to_{-} R\| + 1) + 2 \\ &= |\neg Q_1| + \|P \to_{-} \neg Q_2 \vee R\| + 2 \end{aligned}$$

$$= \|P \rightarrow_{-} \neg Q_1 \vee (\neg Q_2 \vee R)\| + 1.$$

In case 4, the pair $\langle \|P \rightarrow (Q_1 \vee Q_2) \vee R\|, |(Q_1 \vee Q_2)|\rangle$ decreases to $\langle \|P \rightarrow (Q_1 \vee Q_2) \vee R\|, |Q_1|\rangle$. The first part is unchanged and the second part decreases. $\square$

By Lemma 3, after using Table 1, every rule is transformed into one of the two following forms: $P \rightarrow \bot'$ or $P \rightarrow \bot' \vee B$. By Lemma 2, symbols occurring in B are "(", ")", "$\vee$" and literals, that is B is clausal. Step 3 finishes the transformation, that is $(P \rightarrow \bot') \rhd (P \rightarrow \bot)$ or $(P \rightarrow \bot' \vee B) \rhd (P \rightarrow B)$ where B is a clausal proposition.

The proofs for positive rules are the same as those for negative rules. We are now able to state the first main result of this paper.

Theorem 1. *For any polarized rewrite system $\mathcal{R}_0$, the translator will eventually stop, producing a rewrite system $\mathcal{R}_f$ such that $\mathcal{R}_0 \blacktriangleright \mathcal{R}_1 \blacktriangleright \ldots \blacktriangleright \mathcal{R}_f$ and $\mathcal{R}_f$ is clausal. We say that $\mathcal{R}_f$ is the final polarized rewrite system of $\mathcal{R}_0$.*

Proof. Here we only prove this for the part of $\mathcal{R}_{0-}$, as the proof for the part of $\mathcal{R}_{0+}$ is similar.

We first deal with each rule in $\mathcal{R}_{0-}$ by step 1. Then $\bot'$ is free in every rule and we process each rule by step 2. The transformation in step 2 terminates because the multi-set (there is a number pair for each rule in $\mathcal{R}_{j\_}$) of pairs $\langle \|P \rightarrow A_i\|, |A_i^1|\rangle$ defined in Lemma 3 decreases according to the multi-set ordering. Finally we deal with each rule by step 3 and obtain $\mathcal{R}_f$. By Lemma 2, the polarized rewrite system $\mathcal{R}_f$ is clausal. $\square$

4 Equivalence

In this section we prove that the clausal rewrite system returned by the translator preserves the existence of cut free proofs for any sequent of the original language. Let $\Gamma \vdash_{\mathcal{R}} \Delta$ (respectively $\Gamma \vdash^{cf}_{\mathcal{R}} \Delta$) denotes that the sequent $\Gamma \vdash \Delta$ has a proof (cut free proof) in polarized sequent calculus modulo $\mathcal{R}$ (Fig.1), $\rightarrow_{+}^{\mathcal{R}*}$ (respectively $\rightarrow_{-}^{\mathcal{R}*}$) denotes the reflexive and transitive closure of $\rightarrow_{+}$ (respectively $\rightarrow_{-}$) in $\mathcal{R}$ and $Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}} \square$ denotes that empty clause can be derived from $Cl(\Gamma, \neg\Delta)$ in polarized resolution modulo $\mathcal{R}$ (Fig.3). Our final goal is to prove that when $\mathcal{R}_0$ has the cut elimination property and $\Gamma \vdash \Delta$ is a sequent in the language of $\mathcal{L}$,

$$\Gamma \vdash_{\mathcal{R}_0} \Delta \Leftrightarrow Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}_f} \square.$$

It has been proved in [5] following the lines of [1, 7] that $\Gamma \vdash^{cf}_{\mathcal{R}_f} \Delta$ if and only if $Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}_f} \square$. So it is sufficient to show $(\Gamma \vdash_{\mathcal{R}_0} \Delta) \Leftrightarrow (\Gamma \vdash^{cf}_{\mathcal{R}_f} \Delta)$. The road map of the proof is depicted in Fig.7, where P. is the abbreviation of proposition.

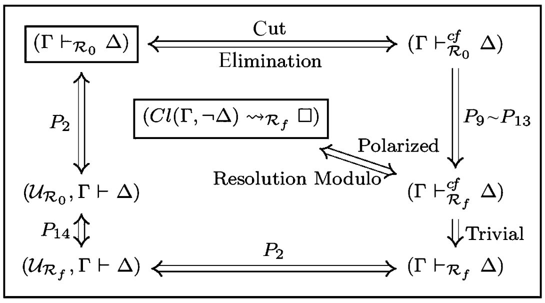

Fig.7. Structure of the equivalence proof.

4.1 Basic Facts

Propositions in this subsection will be used in the next subsection.

Proposition 3 (Substitution). *Let $\mathcal{R} = (\mathcal{E}, \mathcal{R}_{-}, \mathcal{R}_{+})$ be a polarized rewrite system and σ be a capture-avoiding substitution (substitute variables with terms). If $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$, then $\sigma(\Gamma \vdash \Delta)$ has cut free proof modulo $\mathcal{R}$ of the same size, where $\sigma(\Gamma \vdash \Delta)$ is obtained by applying σ to each proposition in $\Gamma \vdash \Delta$.*

Proof. We will prove this by induction on the size of the proof of $\Gamma \vdash \Delta$ in $\mathcal{R}$. We only consider the non-trivial cases: $\forall$ rules.

If the proof has the form:

$$\cfrac{\cfrac{\pi}{\Gamma, C \vdash \Delta}}{\Gamma, A \vdash \Delta}\forall-\text{left}$$

with a proposition B and term u such that $A \rightarrow_{-}^{\mathcal{R}*} \forall x B$ and $(u/x)B \rightarrow_{-}^{\mathcal{R}*} C$, then $\sigma A \rightarrow_{-}^{\mathcal{R}*} \forall w \sigma(w/x)B$ and $(\sigma u/w)\sigma(w/x)B \rightarrow_{-}^{\mathcal{R}*} \sigma C$, where $w \notin domain(\sigma)$. By induction hypothesis, the sequent $\sigma(\Gamma, C \vdash \Delta)$ has a cut free proof modulo $\mathcal{R}$. We conclude with rule $\forall$-left and the proof of $\sigma(\Gamma, A \vdash \Delta)$.

If the proof has the form:

$$\cfrac{\cfrac{\pi}{\Gamma \vdash B, \Delta}}{\Gamma \vdash A, \Delta}\forall-\text{right}$$

with $A \rightarrow_{+}^{\mathcal{R}*} \forall x B$ and $x \notin FV(\Gamma, \Delta)$, then $\sigma A \rightarrow_{+}^{\mathcal{R}*} \forall w \sigma(w/x)B$, where $w \notin domain(\sigma)$ and $w \notin FV(\sigma\Gamma, \sigma\Delta)$. By induction hypothesis, the sequent $\Gamma \vdash (w/x)B, \Delta$ has a cut free proof of the same size as that of $\Gamma \vdash B, \Delta$. By induction hypothesis, the sequent $\sigma\Gamma \vdash \sigma(w/x)B, \sigma\Delta$ has a cut free proof of the same size as that of $\Gamma \vdash (w/x)B, \Delta$. We build the cut free proof of $\sigma(\Gamma \vdash A, \Delta)$ with $\forall$-right applied to $\sigma\Gamma \vdash \sigma(w/x)B, \sigma\Delta$. $\square$

Proposition 4 (Inversion). *Let $\mathcal{R} = (\mathcal{E}, \mathcal{R}_{-}, \mathcal{R}_{+})$ be a polarized rewrite system. If a sequent of the left column in Table 2 has a cut free proof π modulo $\mathcal{R}$,* *then we can build a cut free proof π' modulo $\mathcal{R}$ for corresponding sequent in the right column such that π' is either smaller than π or of the same size as π.*

Table 2. Inversion

$\Gamma \vdash Q_1 \wedge Q_2, \Delta$	$\Gamma \vdash Q_1, \Delta$ and $\Gamma \vdash Q_2, \Delta$
$\Gamma \vdash Q_1 \vee Q_2, \Delta$	$\Gamma \vdash Q_1, Q_2, \Delta$
$\Gamma \vdash Q_1 \Rightarrow Q_2, \Delta$	$\Gamma, Q_1 \vdash Q_2, \Delta$
$\Gamma \vdash \neg Q, \Delta$	$\Gamma, Q \vdash \Delta$
$\Gamma \vdash \forall x Q(x), \Delta$	$\Gamma \vdash Q(t), \Delta$ for all terms t
$\Gamma, Q_1 \wedge Q_2 \vdash \Delta$	$\Gamma, Q_1, Q_2 \vdash \Delta$
$\Gamma, Q_1 \vee Q_2 \vdash \Delta$	$\Gamma, Q_1 \vdash \Delta$ and $\Gamma, Q_2 \vdash \Delta$
$\Gamma, Q_1 \Rightarrow Q_2 \vdash \Delta$	$\Gamma \vdash Q_1, \Delta$ and $\Gamma, Q_2 \vdash \Delta$
$\Gamma, \neg Q \vdash \Delta$	$\Gamma \vdash Q, \Delta$
$\Gamma, \exists x Q(x) \vdash \Delta$	$\Gamma, Q(t) \vdash \Delta$ for all terms t

Proof. All proofs are similar, so we only consider the most complex case: $\Gamma \vdash \forall x Q(x), \Delta$.

We prove it by induction on the size of the proof of $\Gamma \vdash \forall x Q(x), \Delta$.

We consider the last rules in π. In the first series of cases these rules are applied to the proposition $\forall x Q(x)$.

If the proof has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma \vdash B, C, \Delta}}{\Gamma \vdash \forall x Q(x), \Delta}\text{contr-right},$$

then $\forall x Q(x) \rightarrow_+^{\mathcal{R}*} B$, $\forall x Q(x) \rightarrow_+^{\mathcal{R}*} C$. By Proposition 1, the sequent $\Gamma \vdash \forall x Q(x), \forall x Q(x), \Delta$ has a cut free proof of the same size as that of $\Gamma \vdash B, C, \Delta$. We conclude with induction hypothesis (twice) and the rule contr-right.

If the proof has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma \vdash \Delta}}{\Gamma \vdash \forall x Q(x), \Delta}\text{weak-right},$$

then $\Gamma \vdash \Delta$ has a proof smaller than that of $\Gamma \vdash \forall x Q(x), \Delta$. We conclude with the rule weak-right.

If the proof has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma \vdash C, \Delta}}{\Gamma \vdash \forall x Q(x), \Delta}\forall\text{-right}$$

with $\forall x Q(x) \rightarrow_+^{\mathcal{R}*} \forall x C$ and $x \notin FV(\Gamma, \Delta)$, by the definition of one-step rewriting, we have $Q(x) \rightarrow_+^{\mathcal{R}*} C$. By Proposition 1, $\Gamma \vdash Q(x), \Delta$ has a cut free proof which is the same size as that of $\Gamma \vdash C, \Delta$. We conclude with Proposition 3.

In the second series of cases, the last rule in π applies to a proposition different from $\forall x Q(x)$. For example, if the last rule in π is $\exists$-left, then π has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma, B \vdash \forall x Q(x), \Delta}}{\Gamma, A \vdash \forall x Q(x), \Delta}\exists\text{-left}$$

with $A \rightarrow_-^{\mathcal{R}*} \exists y B$ and $y \notin FV(\Gamma, \Delta) \cup FV(\forall x Q(x))$. Since y is a bound variable in $\forall y B$, we suppose $y \notin FV(t)$. By induction hypothesis, the sequent $\Gamma, B \vdash Q(t), \Delta$ has a cut free proof. We conclude with the rule $\exists$-left. □

Proposition 5 ($\forall_1$). *Let $\mathcal{R} = (\mathcal{E}, \mathcal{R}_-, \mathcal{R}_+)$ be a polarized rewrite system with one negative rule $P \rightarrow_- (Q(x) \vee R)$ where x is a variable not free in P nor in R. If sequents $\Gamma, \forall x Q(x) \vdash \Delta$ and $\Gamma, R \vdash \Delta$ have cut free proofs modulo $\mathcal{R}$ so does $\Gamma, P \vdash \Delta$.*

Proof. We will prove a more general property: if the sequents $\Gamma, (\forall x Q(X))^n \vdash \Delta$ and $\Gamma', R \vdash \Delta'$ have cut free proofs modulo $\mathcal{R}$ so does the sequent $\Gamma, \Gamma', P \vdash \Delta, \Delta'$, where $(\forall x Q(x))^n$ is n copies of $\forall x Q(x)$.

The proof is by induction on the size of the cut free proof π of $\Gamma, (\forall x Q(X))^n \vdash \Delta$.

We consider the last rules in π. In the first series of cases these rules are applied to the proposition $\forall x Q(x)$.

If the proof has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma, B, C, (\forall x Q(x))^{n-1} \vdash \Delta}}{\Gamma, \forall x Q(x), (\forall x Q(x))^{n-1} \vdash \Delta}\text{contr-left}$$

with $\forall x Q(x) \rightarrow_-^{\mathcal{R}*} B$, $\forall x Q(x) \rightarrow_-^{\mathcal{R}*} C$, by Proposition 1, the sequent $\Gamma, (\forall x Q(x))^{n+1} \vdash \Delta$ has a cut free proof with the same size as that of $\Gamma, B, C, (\forall x Q(x))^{n-1} \vdash \Delta$. We conclude with induction hypothesis.

If the proof has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma, (\forall x Q(x))^{n-1} \vdash \Delta}}{\Gamma, (\forall x Q(x))^n \vdash \Delta}\text{weak-left},$$

we conclude with induction hypothesis.

If the proof has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma, (\forall x Q(x))^{n-1}, C \vdash \Delta}}{\Gamma, (\forall x Q(x))^n \vdash \Delta}\forall\text{-left}$$

with a proposition B and a term u such that $\forall x Q(x) \rightarrow_-^{\mathcal{R}*} \forall x B$ and $(u/x)B \rightarrow_-^{\mathcal{R}*} C$, then $Q(x) \rightarrow_-^{\mathcal{R}*} B$. So $Q(u) \rightarrow_-^{\mathcal{R}*} (u/x)B \rightarrow_-^{\mathcal{R}*} C$. By Proposition 1, the sequent $\Gamma, (\forall x Q(x))^{n-1}, Q(u) \vdash \Delta$ has a cut free proof with the same size as that of $\Gamma, (\forall x Q(x))^{n-1}, C \vdash \Delta$. By induction hypothesis, $\Gamma, \Gamma', P, Q(u) \vdash \Delta, \Delta'$ has a cut free proof. We obtain the proof of $\Gamma, \Gamma', P, P \vdash \Delta, \Delta'$ by applying the rule $\vee$-left to proofs of $\Gamma, \Gamma', P, Q(u) \vdash \Delta, \Delta'$ and $\Gamma, \Gamma', R \vdash \Delta, \Delta'$. We conclude with the rule contr-left.

In the second series of cases, the last rule in π applies to a proposition different from $\forall x Q(x)$. For example, if the last rule in π is $\forall$-right, then π has the form

$$\cfrac{\cfrac{\pi_1}{\Gamma, (\forall x Q(x))^n \vdash B, \Delta}}{\Gamma, (\forall x Q(x))^n \vdash A, \Delta}\ \forall\text{-right}$$

with $A \rightarrow_+^{\mathcal{R}*} \forall y B$ and $y \notin FV(\Gamma, \Delta, \Gamma', \Delta') \cup FV(P)$. By induction hypothesis, the sequent $\Gamma, \Gamma, P \vdash B, \Delta, \Delta'$ has a cut free proof. We conclude with the rule $\forall$-right. □

The proofs for Propositions 6 $\sim$ 8 are similar to that of Proposition 5.

Proposition 6 ($\forall_2$). *Let $\mathcal{R} = (\mathcal{E}, \mathcal{R}_-, \mathcal{R}_+)$ be a polarized rewrite system with one positive rule $P \rightarrow \neg((Q(x) \vee R))$ where x is a variable not free in P nor in R. If sequents $\Gamma, \forall x Q(x) \vdash \Delta$ and $\Gamma, R \vdash \Delta$ have cut free proofs modulo $\mathcal{R}$ so does $\Gamma \vdash P, \Delta$.*

Proposition 7 ($\exists$). *Let $\mathcal{R} = (\mathcal{E}, \mathcal{R}_-, \mathcal{R}_+)$ be a polarized rewrite system. If the sequent $\Gamma \vdash \exists x Q, \Delta$ has a cut free proof modulo $\mathcal{R}$ so does $\Gamma, \forall x \neg Q \vdash \Delta$.*

Proposition 8 ($\bot$). *Let $\mathcal{R}$ be a polarized rewrite system. If the sequent $\Gamma \vdash \bot, \Delta$ has a cut free proof in $\mathcal{R}$ so does $\Gamma \vdash \Delta$.*

4.2 From $\mathcal{R}_0$ to $\mathcal{R}_f$

We now prove $(\Gamma \vdash^{cf}_{\mathcal{R}_0} \Delta) \Rightarrow (\Gamma \vdash^{cf}_{\mathcal{R}_f} \Delta)$. It is sufficient to show $(\Gamma \vdash^{cf}_{\mathcal{R}_n} \Delta) \Rightarrow (\Gamma \vdash^{cf}_{\mathcal{R}_{n+1}} \Delta)$. We prove this for the steps 1$\sim$3 of the translator.

Proposition 9 (Step 1). *Let $\mathcal{R}$ be a polarized rewrite system and $P \rightarrow_- Q$ a negative rule of $\mathcal{R}$. The sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$ if and only if it has a cut free proof modulo $\mathcal{R}'$ where $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing $P \rightarrow_- Q$ with $P \rightarrow_- Q \vee \bot$.*

Proof. It is proved by induction over the proof size.

If the proof has the form

$$\cfrac{}{A \vdash B}\ \text{axiom}$$

with an atomic proposition C such that $A \rightarrow_-^{\mathcal{R}*} C$ and $B \rightarrow_+^{\mathcal{R}*} C$, then $B \rightarrow_+^{\mathcal{R}'*} C$. There are two cases. In the first case, if $P \rightarrow_- Q$ is used in the derivation $A \rightarrow_-^{\mathcal{R}*} C$, for instance, the derivation $A \rightarrow_-^{\mathcal{R}*} P \rightarrow_-^{\mathcal{R}*} Q \rightarrow_-^{\mathcal{R}*} C$, then $A \rightarrow_-^{\mathcal{R}'*} P \rightarrow_-^{\mathcal{R}'} Q \vee \bot \rightarrow_-^{\mathcal{R}'*} C \vee \bot$. We build the cut free proof of $A \vdash B$ in $\mathcal{R}'$ as follows:

$$\cfrac{\cfrac{}{C \vdash B}\ \text{axiom} \quad \cfrac{}{\bot \vdash B}\ \bot\text{-left}}{A \vdash B}\ \vee\text{-left}\ .$$

In the second case, if $P \rightarrow_- Q$ is not used in the derivation $A \rightarrow_-^{\mathcal{R}*} C$, then $A \rightarrow_-^{\mathcal{R}'*} C$ and $B \rightarrow_+^{\mathcal{R}'*} C$. So we can obtain the cut free proof of $A \vdash B$ by using the rule axiom in $\mathcal{R}'$.

If the proof has the form

$$\cfrac{\cfrac{\pi}{\Gamma, B, C \vdash \Delta}}{\Gamma, A \vdash \Delta}\ \text{contr-left}$$

with $A \rightarrow_-^{\mathcal{R}*} B$, $A \rightarrow_-^{\mathcal{R}*} C$, by Proposition 1, the sequent $\Gamma, A, A \vdash \Delta$ has a cut free proof of the same size as that of $\Gamma, B, C \vdash \Delta$. By induction hypothesis, the sequent $\Gamma, A, A \vdash \Delta$ has a cut free proof in $\mathcal{R}'$. We conclude with the rule contr-left.

If the proof has the form

$$\cfrac{\cfrac{\pi}{\Gamma \vdash \Delta}}{\Gamma, A \vdash \Delta}\ \text{weak-left}$$

by induction hypothesis, the sequent $\Gamma \vdash \Delta$ has a cut free proof in $\mathcal{R}'$. We conclude with rule weak-left.

If the proof has the form

$$\cfrac{}{\Gamma \vdash A, \Delta}\ \top\text{-right}$$

with $A \rightarrow_+^{\mathcal{R}*} \top$, since $P \rightarrow_- Q$ is not used in the derivation $A \rightarrow_+^{\mathcal{R}*} \top$, we have $A \rightarrow_+^{\mathcal{R}'*} \top$. We conclude with rule $\top$-right.

If the proof has the form

$$\cfrac{}{\Gamma, A \vdash \Delta}\ \bot\text{-left}$$

with $A \rightarrow_-^{\mathcal{R}*} \bot$. There are two cases. In the first case, if $P \rightarrow_- Q$ is used in the derivation $A \rightarrow_-^{\mathcal{R}*} \bot$, for instance, the derivation $A \rightarrow_-^{\mathcal{R}*} P \rightarrow_-^{\mathcal{R}*} Q \rightarrow_-^{\mathcal{R}*} \bot$, then $A \rightarrow_-^{\mathcal{R}'*} P \rightarrow_-^{\mathcal{R}'} Q \vee \bot \rightarrow_-^{\mathcal{R}'*} \bot \vee \bot$. We build the cut free proof of $\Gamma, A \vdash \Delta$ in $\mathcal{R}'$ as follows:

$$\cfrac{\cfrac{}{\Gamma, \bot \vdash \Delta}\ \bot\text{-left} \quad \cfrac{}{\Gamma, \bot \vdash \Delta}\ \bot\text{-left}}{\Gamma, A \vdash \Delta}\ \vee\text{-left}\ .$$

In the second case, if $P \rightarrow^{\mathcal{R}} Q$ is not used in the derivation $A \rightarrow_-^{\mathcal{R}*} C$, then $A \rightarrow_-^{\mathcal{R}'*} \bot$. We conclude with the rule $\bot$-left.

If the proof has the form

$$\cfrac{\cfrac{\pi}{\Gamma \vdash B, \Delta}}{\Gamma, A \vdash \Delta}\ \neg\text{-left}$$

with $A \rightarrow_-^{\mathcal{R}*} \neg B$, then $A = \neg A'$ or A is atomic. If $A = \neg A'$, then $A' \rightarrow_+^{\mathcal{R}*} B$. By Proposition 1, the sequent $\Gamma \vdash A', \Delta$ has cut free proof of the same size as that of $\Gamma \vdash B, \Delta$ in $\mathcal{R}$. By induction hypothesis, the sequent $\Gamma \vdash A', \Delta$ has a cut free proof in

$\mathcal{R}'$. We conclude with rule $\neg$-left. If A is atomic, then $A \to_{-}^{\mathcal{R}*} A' \to_{-}^{\mathcal{R}*} \neg B' \to_{-}^{\mathcal{R}*} \neg B$ with $B' \to_{+}^{\mathcal{R}*} B$, where A' is the last atomic proposition in the derivation. By Proposition 1, the sequent $\Gamma \vdash B', \Delta$ has a cut free proof of the same size as that of $\Gamma \vdash B, \Delta$ in $\mathcal{R}$. By induction hypothesis, the sequent $\Gamma \vdash B', \Delta$ has a cut free proof π' in $\mathcal{R}'$. There are two cases. In the first case, if $P \to_{-} Q$ is used in the derivation $A \to_{-}^{\mathcal{R}*} \neg B'$, for instance, $A \to_{-}^{\mathcal{R}*} P \to_{-}^{\mathcal{R}*} Q \to_{-}^{\mathcal{R}*} \neg B'$, then $A \to_{-}^{\mathcal{R}'*} P \to_{-}^{\mathcal{R}'} Q \vee \bot \to_{-}^{\mathcal{R}'*} \neg B' \vee \bot$. We build the cut free proof of $\Gamma, A \vdash \Delta$ in $\mathcal{R}'$ as follows:

$$\dfrac{\dfrac{\dfrac{\pi'}{\Gamma \vdash B', \Delta}}{\Gamma, \neg B' \vdash \Delta}\neg\text{-left} \qquad \dfrac{}{\Gamma, \bot \vdash \Delta}\bot\text{-left}}{\Gamma, A \vdash \Delta}\vee\text{-left}\ .$$

In the second case, if $P \to_{-} Q$ is not used in the derivation $A \to_{-}^{\mathcal{R}*} \neg B'$, then $A \to_{-}^{\mathcal{R}'*} \neg B'$. We conclude with the rule $\neg$-left.

If the last rule is one of the other rules, the argument is analogous. □

The proofs for Propositions 10 ∼ 12 are similar to that of Proposition 9. Propositions 5 ∼ 8 are used to prove Propositions 10 ∼ 12.

Proposition 10 (Step 1). *Let $\mathcal{R}$ be a polarized rewrite system and $P \to_{+} A$ a positive rule of $\mathcal{R}$. The sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$ if and only if it has a cut free proof modulo $\mathcal{R}'$ where $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing $P \to_{+} A$ with $P \to_{+} \neg(\neg A \vee \bot)$.*

Proposition 11 (Step 3). *Let $\mathcal{R}$ be a polarized rewrite system and $P \to_{-} \bot \vee A$ a negative rule of $\mathcal{R}$. The sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$ if and only if it has a cut free proof modulo $\mathcal{R}'$ where $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing $P \to_{-} \bot \vee A$ with $P \to_{-} A$.*

Proposition 12 (Step 3). *Let $\mathcal{R}$ be a polarized rewrite system and $P \to_{+} \neg(\bot \vee \neg A)$ a positive rule of $\mathcal{R}$. The sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$ if and only if it has a cut free proof modulo $\mathcal{R}'$ where $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing $P \to_{+} \neg(\bot \vee \neg A)$ with $P \to_{+} A$.*

Proposition 13 (Step 2). *Let $\mathcal{R}$, $\mathcal{R}'$ be a polarized rewrite system with $\mathcal{R} \blacktriangleright \mathcal{R}'$. If the sequent $\Gamma \vdash \Delta$ has a cut free proof modulo $\mathcal{R}$, then it has a cut free proof modulo $\mathcal{R}'$.*

Proof. We prove this for each case of Table 1. We only consider the most complex cases: case 3, 6, 7, 22. In general, we prove this by induction on the size of the proof of the sequent $\Gamma \vdash \Delta$ in $\mathcal{R}$.

Case 3. Consider the last rule used in the proof of $\Gamma \vdash \Delta$ in $\mathcal{R}$. We only consider the non-trivial case: the rule $\vee$-left.

If the proof has the form

$$\dfrac{\dfrac{\pi}{\Gamma, B \vdash \Delta} \qquad \dfrac{\pi'}{\Gamma, C \vdash \Delta}}{\Gamma, A \vdash \Delta}\vee\text{-left}$$

with $A \to_{-}^{\mathcal{R}*} (B \vee C)$, by the definition of one-step rewriting, either $A = (B' \vee C')$ for some B' and C' or A is atomic.

In the first case we have $B' \to_{-}^{\mathcal{R}*} B$, $C \to_{-}^{\mathcal{R}*} C'$. By Proposition 1, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs of the same size as that of $\Gamma, B \vdash \Delta$ and $\Gamma, C \vdash \Delta$ respectively. By induction hypothesis, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs in $\mathcal{R}'$ and we conclude with the $\vee$-left rule.

In the second case, there exist an atomic proposition A' and two propositions B', C' such that $A \to_{-}^{\mathcal{R}*} A' \to_{-}^{\mathcal{R}} B' \vee C' \to_{-}^{\mathcal{R}*} B \vee C$ with $B' \to_{-}^{\mathcal{R}*} B$ and $C' \to_{-}^{\mathcal{R}*} C$. By Proposition 1, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs of the same size as that of $\Gamma, B \vdash \Delta$ and $\Gamma, C \vdash \Delta$ respectively. By induction hypothesis, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs in $\mathcal{R}'$.

As A' is atomic and $A' \to_{-}^{\mathcal{R}} B' \vee C'$, then either $A' \to_{-}^{\mathcal{R}} B' \vee C'$ is the rule $P \to_{-}^{\mathcal{R}} (Q_1 \wedge Q_2) \vee R$ with $B' = (Q_1 \wedge Q_2)$ and $C' = R$ or $A' \to_{-}^{\mathcal{R}} B' \vee C'$ is not the rule $P \to_{-}^{\mathcal{R}} (Q_1 \wedge Q_2) \vee R$.

If $A' \to_{-}^{\mathcal{R}} B' \vee C'$ is the rule $P \to_{-}^{\mathcal{R}} (Q_1 \wedge Q_2) \vee R$ with $B' = (Q_1 \wedge Q_2)$ and $C' = R$, by Proposition 4, the sequent $\Gamma, Q_1, Q_2 \vdash \Delta$ has a cut free proof in $\mathcal{R}'$. We can build a cut free proof of $\Gamma, P, Q_2 \vdash \Delta$ in $\mathcal{R}'$ with the rule $\vee$-left applied to the proofs of $\Gamma, Q_1, Q_2 \vdash \Delta$ and $\Gamma, C' \vdash \Delta$. We can build a cut free proof of $\Gamma, P, P \vdash \Delta$ in $\mathcal{R}'$ with the rule $\vee$-left applied to the proofs of $\Gamma, P, Q_2 \vdash \Delta$ and $\Gamma, C' \vdash \Delta$. We can build a cut free proof of $\Gamma, P \vdash \Delta$ in $\mathcal{R}'$ with the rule contr-left applied to the proof of $\Gamma, P, P \vdash \Delta$. Since $A \to_{-}^{\mathcal{R}*} P$ and both A and P are atomic, we have $A \to_{-}^{\mathcal{R}'*} P$. We conclude with Proposition 1.

If $A' \to_{-}^{\mathcal{R}} B' \vee C'$ is not the rule $P \to_{-}^{\mathcal{R}} (Q_1 \wedge Q_2) \vee R$, we can build a cut free proof of $\Gamma, A' \vdash \Delta$ in $\mathcal{R}'$ with the rule $\vee$-left applied to the proofs of $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$. Since $A \to_{-}^{\mathcal{R}*} A'$ and both A and A' are atomic, we have $A \to_{-}^{\mathcal{R}'*} A'$. We conclude with Proposition 1.

Case 6. Consider the last rule used in the proof of $\Gamma \vdash \Delta$ in $\mathcal{R}$. We only consider the non-trivial case: the rule $\vee$-left.

If the proof has the form

$$\dfrac{\dfrac{\pi}{\Gamma, B \vdash \Delta} \qquad \dfrac{\pi'}{\Gamma, C \vdash \Delta}}{\Gamma, A \vdash \Delta}\vee\text{-left}$$

with $A \rightarrow_{-}^{\mathcal{R}*} (B \vee C)$, then either $A = (B' \vee C')$ or A is atomic.

In the first case we have $B' \rightarrow_{-}^{\mathcal{R}*} B$, $C \rightarrow_{-}^{\mathcal{R}*} C'$. By Proposition 1, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs of the same size as that of $\Gamma, B \vdash \Delta$ and $\Gamma, C \vdash \Delta$ respectively. By induction hypothesis, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs in $\mathcal{R}'$ and we conclude with the $\vee$-left rule.

In the second case, there exist an atomic proposition A' and two propositions B', C' such that $A \rightarrow_{-}^{\mathcal{R}*} A' \rightarrow_{-}^{\mathcal{R}} B' \vee C' \rightarrow_{-}^{\mathcal{R}*} B \vee C$ with $B' \rightarrow_{-}^{\mathcal{R}*} B$ and $C' \rightarrow_{-}^{\mathcal{R}*} C$. By Proposition 1, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs of the same size as that of $\Gamma, B \vdash \Delta$ and $\Gamma, C \vdash \Delta$ respectively. By induction hypothesis, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs in $\mathcal{R}'$.

As A' is atomic and $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$, then either $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is the rule $P \rightarrow_{-}^{\mathcal{R}} \forall x Q \vee R$ with $B' = \forall x Q$ and $C' = R$ or $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is not the rule $P \rightarrow_{-}^{\mathcal{R}} \forall x Q \vee R$.

If $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is the rule $P \rightarrow_{-}^{\mathcal{R}} \forall x Q \vee R$ with $B' = \forall x Q$ and $C' = R$, by Proposition 5, the sequent $\Gamma, P \vdash \Delta$ has a cut free proof in $\mathcal{R}'$. Since $A \rightarrow_{-}^{\mathcal{R}*} P$ and both A and P are atomic, we have $A \rightarrow_{-}^{\mathcal{R}'*} P$. We conclude with Proposition 1.

If $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is not the rule $P \rightarrow_{-}^{\mathcal{R}} \forall x Q \vee R$, we can build a cut free proof of $\Gamma, A' \vdash \Delta$ in $\mathcal{R}'$ with the rule $\vee$-left applied to the proofs of $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$. Since $A \rightarrow_{-}^{\mathcal{R}*} A'$ and both A and A' are atomic, we have $A \rightarrow_{-}^{\mathcal{R}'*} A'$. We conclude with Proposition 1.

Case 7. Consider the last rule used in the proof of $\Gamma \vdash \Delta$ in $\mathcal{R}$. We only consider the non-trivial case: the rule $\vee$-left.

If the proof has the form

$$\cfrac{\cfrac{\pi}{\Gamma, B \vdash \Delta} \qquad \cfrac{\pi'}{\Gamma, C \vdash \Delta}}{\Gamma, A \vdash \Delta}\ \vee\text{-left}$$

with $A \rightarrow_{-}^{\mathcal{R}*} (B \vee C)$, then either $A = (B' \vee C')$ or A is atomic.

In the first case we have $B' \rightarrow_{-}^{\mathcal{R}*} B$, $C \rightarrow_{-}^{\mathcal{R}*} C'$. By Proposition 1, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs of the same size as that of $\Gamma, B \vdash \Delta$ and $\Gamma, C \vdash \Delta$ respectively. By induction hypothesis, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs in $\mathcal{R}'$ and we conclude with the $\vee$-left rule.

In the second case, there exist an atomic proposition A' and two propositions B', C' such that $A \rightarrow_{-}^{\mathcal{R}*} A' \rightarrow_{-}^{\mathcal{R}} B' \vee C' \rightarrow_{-}^{\mathcal{R}*} B \vee C$ with $B' \rightarrow_{-}^{\mathcal{R}*} B$ and $C' \rightarrow_{-}^{\mathcal{R}*} C$. By Proposition 1, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs of the same size as that of $\Gamma, B \vdash \Delta$ and $\Gamma, C \vdash \Delta$ respectively. By induction hypothesis, the sequents $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$ have cut free proofs in $\mathcal{R}'$.

As A' is atomic and $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$, then either $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is the rule $P \rightarrow_{-}^{\mathcal{R}} \exists x Q \vee R$ with $B' = \exists x Q$ and $C' = R$ or $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is not the rule $P \rightarrow_{-}^{\mathcal{R}} \exists x Q \vee R$.

If $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is the rule $P \rightarrow_{-}^{\mathcal{R}} \exists x Q \vee R$ with $B' = \exists x Q$ and $C' = R$, by Proposition 4, the sequent $\Gamma, (f(l)/x)Q \vdash \Delta$ has a cut free proof in $\mathcal{R}'$. We can build a cut free proof of $\Gamma, P \vdash \Delta$ in $\mathcal{R}'$ with the rule $\vee$-left applied to the proofs of $\Gamma, (f(l)/x)Q \vdash \Delta$ and $\Gamma, R \vdash \Delta$. Since $A \rightarrow_{-}^{\mathcal{R}*} P$ and both A and P are atomic, we have $A \rightarrow_{-}^{\mathcal{R}'*} P$. We conclude with Proposition 1.

If $A' \rightarrow_{-}^{\mathcal{R}} B' \vee C'$ is not the rule $P \rightarrow_{-}^{\mathcal{R}} \exists x Q \vee R$, we can build a cut free proof of $\Gamma, A' \vdash \Delta$ in $\mathcal{R}'$ with the rule $\vee$-left applied to the proofs of $\Gamma, B' \vdash \Delta$ and $\Gamma, C' \vdash \Delta$. Since $A \rightarrow_{-}^{\mathcal{R}*} A'$ and both A and A' are atomic, we have $A \rightarrow_{-}^{\mathcal{R}'*} A'$. We conclude with Proposition 1.

Case 22. Consider the last rule used in the proof of $\Gamma \vdash \Delta$ in $\mathcal{R}$. We only consider the non-trivial case: the rule $\neg$-right.

If the proof has the form

$$\cfrac{\cfrac{\pi}{\Gamma, B \vdash \Delta}}{\Gamma \vdash A, \Delta}\ \neg\text{-right}$$

with $A \rightarrow_{+}^{\mathcal{R}*} \neg B$, then either $A = \neg B'$ or A is atomic.

In the first case we have $B' \rightarrow_{-}^{\mathcal{R}*} B$. By Proposition 1, the sequent $\Gamma, B' \vdash \Delta$ has a cut free proof of the same size as that of $\Gamma, B \vdash \Delta$. By induction hypothesis, the sequent $\Gamma, B' \vdash \Delta$ has a cut free proof in $\mathcal{R}'$ and we conclude with the $\neg$-right rule.

In the second case, there exist an atomic proposition A' and a proposition B' such that $A \rightarrow_{+}^{\mathcal{R}*} A' \rightarrow_{+}^{\mathcal{R}} \neg B' \rightarrow_{+}^{\mathcal{R}*} \neg B$ with $B' \rightarrow_{-}^{\mathcal{R}*} B$. By Proposition 1, the sequent $\Gamma, B' \vdash \Delta$ has a cut free proof of the same size as that of $\Gamma, B \vdash \Delta$. By induction hypothesis, the sequent $\Gamma, B' \vdash \Delta$ has a cut free proof in $\mathcal{R}'$.

As A' is atomic and $A' \rightarrow_{+}^{\mathcal{R}} \neg B'$, then either $A' \rightarrow_{+}^{\mathcal{R}} \neg B'$ is the rule $P \rightarrow_{+}^{\mathcal{R}} \neg((\exists x Q) \vee R)$ with $B' = (\exists x Q) \vee R$ or $A' \rightarrow_{+}^{\mathcal{R}} \neg B'$ is not the rule $P \rightarrow_{+}^{\mathcal{R}} \neg((\exists x Q) \vee R)$.

If $A' \rightarrow_{+}^{\mathcal{R}} \neg B'$ is the rule $P \rightarrow_{+}^{\mathcal{R}} \neg((\exists x Q) \vee R)$ with $B' = (\exists x Q) \vee R$, by Proposition 4, the sequents $\Gamma, (f(l)/x)Q \vdash \Delta$ and $\Gamma, R \vdash \Delta$ have cut free proofs in $\mathcal{R}'$. We can build a cut free proof of $\Gamma, (f(l)/x)Q \vee R \vdash \Delta$ in $\mathcal{R}'$ with the rule $\vee$-left applied to the proofs of $\Gamma, (f(l)/x)Q \vdash \Delta$ and $\Gamma, R \vdash \Delta$. Then we can get a cut free proof of $\Gamma \vdash P, \Delta$ by the rule $\neg$-right. Since $A \rightarrow_{+}^{\mathcal{R}*} A' = P$ and both A and P are atomic, we have $A \rightarrow_{+}^{\mathcal{R}'*} P$. We conclude with Proposition 1.

If $A' \rightarrow_{+}^{\mathcal{R}} \neg B'$ is not the rule $P \rightarrow_{+}^{\mathcal{R}} \neg((\exists x Q) \vee R)$, we can build a cut free proof of $\Gamma \vdash A', \Delta$ in $\mathcal{R}'$ with the rule $\neg$-right applied to the proof of $\Gamma, B' \vdash \Delta$. Since $A \rightarrow_{+}^{\mathcal{R}*} A'$ and both A and A' are atomic, we have $A \rightarrow_{+}^{\mathcal{R}'*} A'$. We conclude with Proposition 1. □

4.3 From $\mathcal{R}_f$ to $\mathcal{R}_0$

In this subsection we will prove $(\Gamma \vdash_{\mathcal{R}_f}^{cf} \Delta) \Rightarrow (\Gamma \vdash_{\mathcal{R}_0}^{cf} \Delta)$. The method to prove this is different from that used in Subsection 4.2.

As $\mathcal{R}_0$ has the cut elimination property, it is sufficient to show $(\Gamma \vdash_{\mathcal{R}_{n+1}} \Delta) \Rightarrow (\Gamma \vdash_{\mathcal{R}_n} \Delta)$. If a sequent $\Gamma \vdash \Delta$ has a cut free proof in $\mathcal{R}_f$, then it has a proof in $\mathcal{R}_f$. If we can prove $(\Gamma \vdash_{\mathcal{R}_{n+1}} \Delta) \Rightarrow (\Gamma \vdash_{\mathcal{R}_n} \Delta)$, then it has a proof in $\mathcal{R}_0$. Using the cut elimination theorem for $\mathcal{R}_0$, we get that it has a cut free proof in $\mathcal{R}_0$.

Proposition 14. *Let $\mathcal{L}$ be a language and $\mathcal{R}$ be a polarized rewrite system in $\mathcal{L}$. Let $\mathcal{R}'$ be polarized rewrite system with $\mathcal{R} \blacktriangleright \mathcal{R}'$. If a sequent in the language $\mathcal{L}$ has a proof in $\mathcal{R}'$, then it has a proof in $\mathcal{R}$.*

Proof. Using Proposition 2, all we need to prove is that the theory $\mathcal{U}_{\mathcal{R}'}$ is a conservative extension of $\mathcal{U}_{\mathcal{R}}$. We will prove this by cases. There are four interesting cases. The other cases are trivial and we omit them.

If $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing the negative rule $P \rightarrow_{-} \perp \vee R$ with $P \rightarrow_{-} R$, then there is only one difference between $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{U}_{\mathcal{R}'}$. The theory $\mathcal{U}_{\mathcal{R}}$ contains the universal closure of $P \Rightarrow \perp \vee R$ while the theory $\mathcal{U}_{\mathcal{R}'}$ contains the universal closure of $P \Rightarrow R$. But they are equivalent in predicate logic.

If $\mathcal{R}'$ is obtained from $\mathcal{R}$ by dropping the negative rule $P \rightarrow_{-} \neg\perp \vee R$, then there is only one difference between $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{U}_{\mathcal{R}'}$. The theory $\mathcal{U}_{\mathcal{R}}$ contains the universal closure of $P \Rightarrow \neg\perp \vee R$ while the theory $\mathcal{U}_{\mathcal{R}'}$ does not contain this axiom. But this axiom is trivially provable in predicate logic.

If $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing the negative rule $P \rightarrow_{-} (Q_1 \wedge Q_2) \vee R$ with $P \rightarrow_{-} Q_1 \vee R$ and $P \rightarrow_{-} Q_2 \vee R$, then there is only one difference between $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{U}_{\mathcal{R}'}$. The theory $\mathcal{U}_{\mathcal{R}}$ contains the universal closure of $P \Rightarrow (Q_1 \wedge Q_2) \vee R$ while the theory $\mathcal{U}_{\mathcal{R}'}$ contains the universal closure of propositions $P \Rightarrow Q_1 \vee R$ and $P \Rightarrow Q_2 \vee R$. But the conjunction of the two axioms of $\mathcal{U}_{\mathcal{R}'}$ is equivalent to that of $\mathcal{U}_{\mathcal{R}}$.

If $\mathcal{R}'$ is obtained from $\mathcal{R}$ by replacing the negative rule $P \rightarrow_{-} \exists x Q \vee R$ with $P \rightarrow_{-} (f(l)/x)Q \vee R$, then there is only one difference between $\mathcal{U}_{\mathcal{R}}$ and $\mathcal{U}_{\mathcal{R}'}$. The theory $\mathcal{U}_{\mathcal{R}}$ contains the universal closure of $P \Rightarrow \exists x Q \vee R$ while $\mathcal{U}_{\mathcal{R}'}$ contains the universal closure of $P \Rightarrow (f(l)/x)Q \vee R$. But the axiom of $\mathcal{U}_{\mathcal{R}'}$ is equivalent to the Skolemization of that of $\mathcal{U}_{\mathcal{R}}$. □

Theorem 2. *Let $\mathcal{R}_0$ be a polarized rewrite system with cut elimination property and $\mathcal{R}_f$ be the final polarized rewrite system of $\mathcal{R}_0$. For a sequent $\Gamma \vdash \Delta$ containing no occurrence of the Skolem symbols that are introduced by the translation of the rewrite system, the following conditions are equivalent*:

1) $\Gamma \vdash_{\mathcal{R}_0}^{cf} \Delta$,
2) $\Gamma \vdash_{\mathcal{R}_f}^{cf} \Delta$,
3) $\Gamma \vdash_{\mathcal{R}_f} \Delta$,
4) $\Gamma \vdash_{\mathcal{R}_0} \Delta$.

Proof. 1) ⇒ 2) is by Propositions 9 ∼ 13; 2) ⇒ 3) is trivial; 3) ⇒ 4) is by Proposition 14; 4) ⇒ 1) is derived by the cut elimination property of $\mathcal{R}_0$. □

5 Example

Consider the rewrite system $\mathcal{R}_0$ containing two rules $P \rightarrow (Q \Rightarrow R)$ and $Q \rightarrow (\exists x G(x))$. With $\mathcal{R}_0$ as input, the translator returns $\mathcal{R}_f$ containing five rules $P \rightarrow_{-} (\neg Q \vee R)$, $Q \rightarrow_{-} G(c)$, $P \rightarrow_{+} \neg Q$, $P \rightarrow_{+} \neg(\neg R)$, $Q \rightarrow_{+} \neg G(x)$. See Fig.8 for the derivation of □ from $\{\neg P\}, \{\neg G(x)\}$ in resolution modulo $\mathcal{R}_0$. See Fig.9 for the derivation of □ from $\{\neg P\}, \{\neg G(x)\}$ in polarized resolution modulo $\mathcal{R}_f$. There is no dynamic transformation in Fig.9.

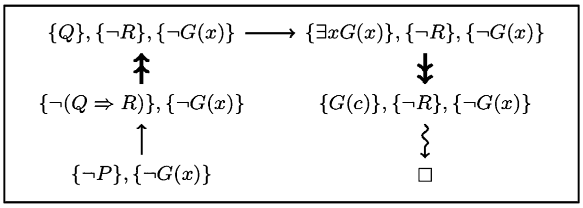

Fig.8. Resolution modulo $\mathcal{R}_0$.

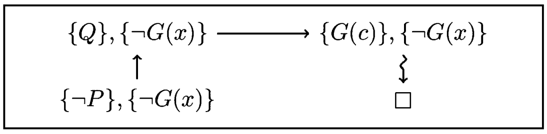

Fig.9. Polarized resolution modulo $\mathcal{R}_f$.

6 Conclusions and Future Work

Notice that as a by-product of our results we have a partial cut elimination property for $\mathcal{R}_f$. Indeed if $\Gamma \vdash \Delta$ is a sequent in the language $\mathcal{L}$ (i.e., it does not contain any Skolem symbol), then $\Gamma \vdash_{\mathcal{R}_f} \Delta \Rightarrow \Gamma \vdash_{\mathcal{R}_f}^{cf} \Delta$. This result does not extend to the full language. For instance, let $\mathcal{R}_0$ be the rewrite system containing one rule $P \rightarrow \forall x Q(x)$. Its final rewrite system $\mathcal{R}_f$ contains two rules $P \rightarrow_{-} Q(x)$ and $P \rightarrow_{+} \neg\neg Q(c)$, where c is a new Skolem symbol. The sequent $Q(c) \vdash \forall x Q(x)$ has a proof in $\mathcal{R}_f$ (see Fig.10), but it does not have a cut free

$$\dfrac{\dfrac{\dfrac{\dfrac{}{P \vdash Q(x)}\ \text{axiom}}{P \vdash \forall x Q(x)}\ \forall\text{-right}}{Q(c), P \vdash \forall x Q(x)}\ \text{weak-left} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{Q(c) \vdash Q(c)}\ \text{axiom}}{Q(c), \neg Q(c) \vdash P}\ \neg\text{-left}}{Q(c) \vdash P}\ \neg\text{-right}}{Q(c) \vdash P, \forall x Q(x)}\ \text{weak-right}}{Q(c) \vdash \forall x Q(x)}\ \text{cut}$$

Fig.10. Proof with cut.

proof. Fortunately, we do not need the cut elimination property of $\mathcal{R}_f$ to prove $\Gamma \vdash_{\mathcal{R}_0} \Delta$ if and only if $Cl(\Gamma, \neg\Delta) \rightsquigarrow_{\mathcal{R}_f}$. $\square$

In this paper, we translated any polarized rewrite system into a clausal one. We proved that the obtained clausal polarized rewrite system preserves the existence of cut free proofs for any sequent of the original language. In this way, polarized resolution modulo can be applied to the system directly. However the obtained clausal polarized rewrite system may lose the cut elimination property. So one of possibilities for the future work is dropping the hypothesis that $\mathcal{R}_0$ has the cut elimination property. We could then try to prove the equivalence between $\mathcal{R}_0$ and $\mathcal{R}_f$ by using *polarized unfolding sequent calculus*[8] which is equivalent to the polarized sequent calculus. There are only two derivation rules with rewriting as side conditions in polarized unfolding sequent calculus, so the proof of equivalence between $\mathcal{R}_0$ and $\mathcal{R}_f$ may be simpler. Another one is fixing $\mathcal{R}_f$ such that $\mathcal{R}_f$ has the cut elimination property. Since the proofs of the paper rely on the particular translator used here, if we use another way of transforming rewriting systems into clausal ones, we have to redo the proof of correctness. So the third possibility for the future work is to use a more abstract way of transforming rewriting rules into clausal ones.

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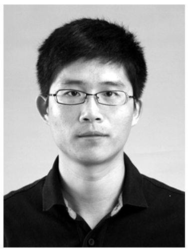

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