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Further generalizations of the Banach contraction principle

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Abstract
We establish a new fixed point theorem in the setting of Branciari metric spaces. The obtained result is an extension of the recent fixed point theorem established in Jleli and Samet (J. Inequal. Appl. 2014:38, 2014).

Keywords: fixed point; Branciari metric space; existence; uniqueness

1 Introduction

The fixed point theorem, generally known as the Banach contraction principle, appeared in an explicit form in Banach's thesis in 1922 [1], where it was used to establish the existence of a solution to an integral equation. Since then, because of its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. This principle states that if (X,d) is a complete metric space and $T:X\to X$ is a contraction map (*i.e.*, $d(Tx,Ty)\leq\lambda d(x,y)$ for all $x,y\in X$, where $\lambda\in(0,1)$ is a constant), then T has a unique fixed point.

The Banach contraction principle has been generalized in many ways over the years. In some generalizations, the contractive nature of the map is weakened; see [2–9] and others. In other generalizations, the topology is weakened; see [10–23] and others. In [24], Nadler extended the Banach fixed point theorem from single-valued maps to set-valued contractive maps. Other fixed point results for set-valued maps can be found in [25–30] and the references therein.

In 2000, Branciari [11] introduced the concept of generalized metric spaces, where the triangle inequality is replaced by the inequality $d(x,y)\leq d(x,u)+d(u,v)+d(v,y)$ for all pairwise distinct points $x,y,u,v\in X$. Various fixed point results were established on such spaces; see [10, 13, 17–20, 22, 31–33] and the references therein.

We recall the following definitions introduced in [11].

Definition 1.1 Let X be a non-empty set and $d:X\times X\to[0,\infty)$ be a mapping such that for all $x,y\in X$ and for all distinct points $u,v\in X$, each of them different from x and y, one has

(i) $d(x,y)=0\iff x=y$;
(ii) $d(x,y)=d(y,x)$;
(iii) $d(x,y)\leq d(x,u)+d(u,v)+d(v,y)$.

Then (X,d) is called a generalized metric space (or for short g.m.s).

Definition 1.2 Let (X,d) be a g.m.s, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is convergent to x if and only if $d(x_n,x) \to 0$ as $n \to \infty$. We denote this by $x_n \to x$.

Definition 1.3 Let (X,d) be a g.m.s and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is Cauchy if and only if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

Definition 1.4 Let (X,d) be a g.m.s. We say that (X,d) is complete if and only if every Cauchy sequence in X converges to some element in X.

The following result was established in [17] (see also [34]).

Lemma 1.1 *Let (X,d) be a g.m.s and $\{x_n\}$ be a Cauchy sequence in (X,d) such that $d(x_n,x) \to 0$ as $n \to \infty$ for some $x \in X$. Then $d(x_n,y) \to d(x,y)$ as $n \to \infty$ for all $y \in X$. In particular, $\{x_n\}$ does not converge to y if $y \neq x$.*

We denote by Θ the set of functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;
(Θ_2) for each sequence $\{t_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;
(Θ_3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = \ell$;
(Θ_4) θ is continuous.

Recently, Jleli and Samet [35] established the following generalization of the Banach fixed point theorem in the setting of Branciari metric spaces.

Theorem 1.1 (Jleli and Samet [35]) *Let (X,d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \theta\big(d(Tx,Ty)\big) \leq \big[\theta\big(d(x,y)\big)\big]^k.$$

Then T has a unique fixed point.

Note that the condition (Θ_4) is not supposed in Theorem 1.1.

The aim of this paper is to extend the result given by Theorem 1.1.

2 Result and proof

Now, we are ready to state and prove our main result.

Theorem 2.1 *Let (X,d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(Tx,Ty) \neq 0 \quad \Longrightarrow \quad \theta\big(d(Tx,Ty)\big) \leq \big[\theta\big(M(x,y)\big)\big]^k, \tag{1}$$

where

$$M(x,y) = \max\big\{d(x,y), d(x,Tx), d(y,Ty)\big\}. \tag{2}$$

Then T has a unique fixed point.

Proof Let $x \in X$ be an arbitrary point in X. If for some $p \in \mathbb{N}$ we have $T^p x = T^{p+1}x$, then $T^p x$ will be a fixed point of T. So, without restriction of the generality, we can suppose that $d(T^n x, T^{n+1}x) > 0$ for all $n \in \mathbb{N}$. Now, from (1), for all $n \in \mathbb{N}$, we have

$$\theta\big(d\big(T^n x, T^{n+1}x\big)\big) \le \big[\theta\big(M\big(T^{n-1}x, T^n x\big)\big)\big]^k, \tag{3}$$

where from (2)

$$\begin{aligned} M\big(T^{n-1}x, T^n x\big) &= \max\big\{d\big(T^{n-1}x, T^n x\big), d\big(T^{n-1}x, TT^{n-1}x\big), d\big(T^n x, TT^n x\big)\big\} \\ &= \max\big\{d\big(T^{n-1}x, T^n x\big), d\big(T^{n-1}x, T^n x\big), d\big(T^n x, T^{n+1}x\big)\big\} \\ &= \max\big\{d\big(T^{n-1}x, T^n x\big), d\big(T^n x, T^{n+1}x\big)\big\}. \end{aligned} \tag{4}$$

If $M(T^{n-1}x, T^n x) = d(T^n x, T^{n+1}x)$, then inequality (3) turns into

$$\theta\big(d\big(T^n x, T^{n+1}x\big)\big) \le \big[\theta\big(d\big(T^n x, T^{n+1}x\big)\big)\big]^k,$$

which implies that

$$\ln\big[\theta\big(d\big(T^n x, T^{n+1}x\big)\big)\big] \le k \ln\big[\theta\big(d\big(T^n x, T^{n+1}x\big)\big)\big],$$

that is a contradiction with $k \in (0,1)$. Hence, from (4) we have $M(T^{n-1}x, T^n x) = d(T^{n-1}x, T^n x)$, and inequality (3) yields

$$\begin{aligned} \theta\big(d\big(T^n x, T^{n+1}x\big)\big) &\le \big[\theta\big(d\big(T^{n-1}x, T^n x\big)\big)\big]^k \\ &\le \big[\theta\big(d\big(T^{n-2}x, T^{n-1}x\big)\big)\big]^{k^2} \le \cdots \le \big[\theta\big(d(x, Tx)\big)\big]^{k^n}. \end{aligned}$$

Thus we have

$$1 \le \theta\big(d\big(T^n x, T^{n+1}x\big)\big) \le \big[\theta\big(d(x, Tx)\big)\big]^{k^n} \quad \text{for all } n \in \mathbb{N}. \tag{5}$$

Letting $n \to \infty$ in (5), we obtain

$$\theta\big(d\big(T^n x, T^{n+1}x\big)\big) \to 1 \quad \text{as } n \to \infty, \tag{6}$$

which implies from (Θ_2) that

$$\lim_{n\to\infty} d\big(T^n x, T^{n+1}x\big) = 0.$$

From condition (Θ_3), there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} = \ell.$$

Suppose that $\ell < \infty$. In this case, let $B = \ell/2 > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} - \ell \right| \le B \quad \text{for all } n \ge n_0.$$

This implies that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} \geq \ell - B = B \quad \text{for all } n \geq n_0.$$

Then

$$n\bigl[d\bigl(T^n x, T^{n+1}x\bigr)\bigr]^r \leq An\bigl[\theta\bigl(d\bigl(T^n x, T^{n+1}x\bigr)\bigr) - 1\bigr] \quad \text{for all } n \geq n_0,$$

where $A = 1/B$.

Suppose now that $\ell = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\theta(d(T^n x, T^{n+1}x)) - 1}{[d(T^n x, T^{n+1}x)]^r} \geq B \quad \text{for all } n \geq n_0.$$

This implies that

$$n\bigl[d\bigl(T^n x, T^{n+1}x\bigr)\bigr]^r \leq An\bigl[\theta\bigl(d\bigl(T^n x, T^{n+1}x\bigr)\bigr) - 1\bigr] \quad \text{for all } n \geq n_0,$$

where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\bigl[d\bigl(T^n x, T^{n+1}x\bigr)\bigr]^r \leq An\bigl[\theta\bigl(d\bigl(T^n x, T^{n+1}x\bigr)\bigr) - 1\bigr] \quad \text{for all } n \geq n_0.$$

Using (5), we obtain

$$n\bigl[d\bigl(T^n x, T^{n+1}x\bigr)\bigr]^r \leq An\bigl(\bigl[\theta\bigl(d(x, Tx)\bigr)\bigr]^{k^n} - 1\bigr) \quad \text{for all } n \geq n_0.$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n\bigl[d\bigl(T^n x, T^{n+1}x\bigr)\bigr]^r = 0.$$

Thus, there exists $n_1 \in \mathbb{N}$ such that

$$d\bigl(T^n x, T^{n+1}x\bigr) \leq \frac{1}{n^{1/r}} \quad \text{for all } n \geq n_1. \tag{7}$$

Now, we shall prove that T has a periodic point. Suppose that it is not the case, then $T^n x \neq T^m x$ for every $n, m \in \mathbb{N}$ such that $n \neq m$. Using (1), we obtain

$$\theta\bigl(d\bigl(T^n x, T^{n+2}x\bigr)\bigr) \leq \bigl[\theta\bigl(M\bigl(T^{n-1}x, T^{n+1}x\bigr)\bigr)\bigr]^k, \tag{8}$$

where from (2)

$$M\bigl(T^{n-1}x, T^{n+1}x\bigr) = \max\bigl\{d\bigl(T^{n-1}x, T^{n+1}x\bigr), d\bigl(T^{n-1}x, T^n x\bigr), d\bigl(T^{n+1}x, T^{n+2}x\bigr)\bigr\}. \tag{9}$$

Since θ is non-decreasing, we obtain from (8) and (9)

$$\theta\big(d\big(T^n x, T^{n+2}x\big)\big) \le \big[\max\big\{\theta\big(d\big(T^{n-1}x, T^{n+1}x\big)\big), \theta\big(d\big(T^{n-1}x, T^n x\big)\big), \theta\big(d\big(T^{n+1}x, T^{n+2}x\big)\big)\big\}\big]^k. \tag{10}$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned} u_n &:= \max\big\{\theta\big(d\big(T^{n-1}x, T^{n+1}x\big)\big), \theta\big(d\big(T^{n-1}x, T^n x\big)\big), \theta\big(d\big(T^{n+1}x, T^{n+2}x\big)\big)\big\} \\ &= \theta\big(d\big(T^{n-1}x, T^{n+1}x\big)\big). \end{aligned}$$

If $|I| < \infty$, then there exists $N \in \mathbb{N}$ such that for every $n \ge N$,

$$\begin{aligned} &\max\big\{\theta\big(d\big(T^{n-1}x, T^{n+1}x\big)\big), \theta\big(d\big(T^{n-1}x, T^n x\big)\big), \theta\big(d\big(T^{n+1}x, T^{n+2}x\big)\big)\big\} \\ &\quad = \max\big\{\theta\big(d\big(T^{n-1}x, T^n x\big)\big), \theta\big(d\big(T^{n+1}x, T^{n+2}x\big)\big)\big\}. \end{aligned}$$

In this case, we obtain from (10)

$$1 \le \theta\big(d\big(T^n x, T^{n+2}x\big)\big) \le \big[\max\big\{\theta\big(d\big(T^{n-1}x, T^n x\big)\big), \theta\big(d\big(T^{n+1}x, T^{n+2}x\big)\big)\big\}\big]^k$$

for all $n \ge N$. Letting $n \to \infty$ in the above inequality and using (6), we get

$$\theta\big(d\big(T^n x, T^{n+2}x\big)\big) \to 1 \quad \text{as } n \to \infty.$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, that we denote also by $\{u_n\}$, such that

$$u_n = \theta\big(d\big(T^{n-1}x, T^{n+1}x\big)\big) \quad \text{for } n \text{ large enough.}$$

In this case, we obtain from (10)

$$\begin{aligned} 1 &\le \theta\big(d\big(T^n x, T^{n+2}x\big)\big) \le \big[\theta\big(d\big(T^{n-1}x, T^{n+1}x\big)\big)\big]^k \\ &\le \big[\theta\big(d\big(T^{n-2}x, T^n x\big)\big)\big]^{k^2} \le \cdots \le \big[\theta\big(d\big(x, T^2 x\big)\big)\big]^{k^n} \end{aligned}$$

for n large enough. Letting $n \to \infty$ in the above inequality, we obtain

$$\theta\big(d\big(T^n x, T^{n+2}x\big)\big) \to 1 \quad \text{as } n \to \infty. \tag{11}$$

Then in all cases, (11) holds. Using (11) and the property (Θ_2), we obtain

$$\lim_{n\to\infty} d\big(T^n x, T^{n+2}x\big) = 0.$$

Similarly, from condition (Θ_3), there exists $n_2 \in \mathbb{N}$ such that

$$d\big(T^n x, T^{n+2}x\big) \le \frac{1}{n^{1/r}} \quad \text{for all } n \ge n_2. \tag{12}$$

Let $\mathcal{N} = \max\{n_0, n_1\}$. We consider two cases.

Case 1. If $m > 2$ is odd, then writing $m = 2L + 1$, $L \geq 1$, using (7), for all $n \geq \mathcal{N}$, we obtain

$$\begin{aligned} d(T^n x, T^{n+m} x) &\leq d(T^n x, T^{n+1} x) + d(T^{n+1} x, T^{n+2} x) + \cdots + d(T^{n+2L} x, T^{n+2L+1} x) \\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+1)^{1/r}} + \cdots + \frac{1}{(n+2L)^{1/r}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Case 2. If $m > 2$ is even, then writing $m = 2L$, $L \geq 2$, using (7) and (12), for all $n \geq \mathcal{N}$, we obtain

$$\begin{aligned} d(T^n x, T^{n+m} x) &\leq d(T^n x, T^{n+2} x) + d(T^{n+2} x, T^{n+3} x) + \cdots + d(T^{n+2L-1} x, T^{n+2L} x) \\ &\leq \frac{1}{n^{1/r}} + \frac{1}{(n+2)^{1/r}} + \cdots + \frac{1}{(n+2L-1)^{1/r}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

Thus, combining all the cases, we have

$$d(T^n x, T^{n+m} x) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/r}} \quad \text{for all } n \geq \mathcal{N}, m \in \mathbb{N}.$$

From the convergence of the series $\sum_i \frac{1}{i^{1/r}}$ (since $1/r > 1$), we deduce that $\{T^n x\}$ is a Cauchy sequence. Since (X, d) is complete, there is $z \in X$ such that $T^n x \to z$ as $n \to \infty$. Without restriction of the generality, we can suppose that $T^n x \neq z$ for all n (or for n large enough). Suppose that $d(z, Tz) > 0$, using (1), we get

$$\theta\big(d(T^{n+1} x, Tz)\big) \leq \big[\theta\big(M(T^n x, z)\big)\big]^k \quad \text{for all } n \in \mathbb{N},$$

where

$$M(T^n x, z) = \max\big\{d(T^n x, z), d(T^n x, T^{n+1} x), d(z, Tz)\big\}.$$

Letting $n \to \infty$ in the above inequality, using (Θ_4) and Lemma 1.1, we obtain

$$\theta\big(d(z, Tz)\big) \leq \big[\theta\big(d(z, Tz)\big)\big]^k < \theta\big(d(z, Tz)\big),$$

which is a contradiction. Thus we have $z = Tz$, which is also a contradiction with the assumption: T does not have a periodic point. Thus T has a periodic point, say z, of period q. Suppose that the set of fixed points of T is empty. Then we have

$$q > 1 \quad \text{and} \quad d(z, Tz) > 0.$$

Using (1), we obtain

$$\theta\big(d(z, Tz)\big) = \theta\big(d(T^q z, T^{q+1} z)\big) \leq \big[\theta(z, Tz)\big]^{k^q} < \theta\big(d(z, Tz)\big),$$

which is a contradiction. Thus, the set of fixed points of T is non-empty, that is, T has at least one fixed point. Now, suppose that $z, u \in X$ are two fixed points of T such that $d(z,u) = d(Tz,Tu) > 0$. Using (1), we obtain

$$\theta\bigl(d(z,u)\bigr) = \theta\bigl(d(Tz,Tu)\bigr) \le \bigl[\theta\bigl(d(z,u)\bigr)\bigr]^k < \theta\bigl(d(z,u)\bigr),$$

which is a contradiction. Then we have one and only one fixed point. □

3 Some consequences

We start by deducing the following fixed point result.

Corollary 3.1 *Let (X,d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exists $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \le \lambda \max\bigl\{d(x,y), d(x,Tx), d(y,Ty)\bigr\} \quad \textit{for all } x,y \in X. \tag{13}$$

Then T has a unique fixed point.

Proof From (13), we have

$$e^{\sqrt{d(Tx,Ty)}} \le \bigl[e^{\sqrt{\max\{d(x,y),d(x,Tx),d(y,Ty)\}}}\bigr]^{\sqrt{\lambda}} \quad \text{for all } x,y \in X.$$

Clearly the function $\theta : (0,\infty) \to (1,\infty)$ defined by $\theta(t) := e^{\sqrt{t}}$ belongs to Θ. So, the existence and uniqueness of the fixed point follows from Theorem 2.1. □

The following fixed point result established in [11] is an immediate consequence of Corollary 3.1.

Corollary 3.2 *Let (X,d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exists $\lambda \in (0,1)$ such that*

$$d(Tx,Ty) \le \lambda d(x,y) \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point.

The following fixed point result established in [34] is an immediate consequence of Corollary 3.1.

Corollary 3.3 *Let (X,d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exist $\lambda, \mu \ge 0$ with $\lambda + \mu < 1$ such that*

$$d(Tx,Ty) \le \lambda d(x,Tx) + \mu d(y,Ty) \quad \textit{for all } x,y \in X.$$

Then T has a unique fixed point.

The following fixed point result is also an immediate consequence of Corollary 3.1.

Corollary 3.4 *Let (X, d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exist $\lambda, \mu, \nu \geq 0$ with $\lambda + \mu + \nu < 1$ such that*

$$d(Tx, Ty) \leq \lambda d(x, y) + \mu d(x, Tx) + \nu d(y, Ty) \quad \text{for all } x, y \in X.$$

Then T has a unique fixed point.

We note that Θ contains a large class of functions. For example, for

$$\theta(t) := 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\alpha}\right), \quad 0 < \alpha < 1, t > 0,$$

we obtain from Theorem 2.1 the following result.

Corollary 3.5 *Let (X, d) be a complete g.m.s and $T : X \to X$ be a given map. Suppose that there exist $\alpha, k \in (0, 1)$ such that*

$$2 - \frac{2}{\pi} \arctan\left(\frac{1}{[d(Tx, Ty)]^\alpha}\right) \leq \left[2 - \frac{2}{\pi} \arctan\left(\frac{1}{[M(x, y)]^\alpha}\right)\right]^k \quad \text{for all } x, y \in X, Tx \neq Ty,$$

where $M(x, y)$ is given by (2). *Then T has a unique fixed point.*

Finally, since a metric space is a g.m.s, from Theorem 2.1 we deduce immediately the following result.

Corollary 3.6 *Let (X, d) be a complete metric space and $T : X \to X$ be a given map. Suppose that there exist $\theta \in \Theta$ and $k \in (0, 1)$ such that*

$$x, y \in X, \quad d(Tx, Ty) \neq 0 \quad \Longrightarrow \quad \theta\big(d(Tx, Ty)\big) \leq \big[\theta\big(M(x, y)\big)\big]^k,$$

where $M(x, y)$ is given by (2). *Then T has a unique fixed point.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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