

# Vertex colouring and forbidden subgraphs – a survey

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**Abstract** There is a great variety of colouring concepts and results in the literature. Here our focus is to survey results on vertex colourings of graphs defined in terms of forbidden induced subgraph conditions.

*Thus, one who wishes to obtain useful results from a graph coloring formulation of his problem must do more than just show that the problem is equivalent to the general problem of coloring a graph. If there is to be any hope, one must also obtain information about the structure of the graphs that need to be colored (D. S. Johnson [65]).*

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## 1 Introduction

We consider finite, undirected, and simple graphs  $G$  with vertex set  $V(G)$  and edge set  $E(G)$ . For  $A \subseteq V(G)$  let  $G[A]$  be the subgraph induced by  $A$ . Moreover, a graph  $H$  is an induced subgraph of a graph  $G$ , briefly denoted

by  $H \subset G$ , if there exists a set  $A \subseteq V(G)$  such that  $G[A]$  is isomorphic to  $H$ . A cycle is *odd* or *even* according to the parity of the order of the cycle. An induced subgraph of a graph, which is a cycle of order at least four, is called a *hole*. An induced subgraph of a graph, which is the complement of a cycle of order at least five, is called an *antihole*. Here, the *complement*  $\bar{G}$  of a graph  $G$  is the graph with  $V(\bar{G}) = V(G)$  and  $E(\bar{G}) = \{uv \mid u \text{ is not adjacent to } v \text{ in } G\}$ . For convenience a graph  $G$  without an induced subgraph  $H$  is called *H-free*.  $N(x) = N_G(x)$  denotes the set of vertices adjacent to the vertex  $x$  and  $N[x] = N_G[x] = N(x) \cup \{x\}$ . For terminology and notation not defined here we refer to [127].

A *k-colouring* of a graph  $G$  is an assignment of  $k$  different colours to the vertices of  $G$  such that adjacent vertices receive different colours. The minimum  $k$  for which  $G$  has a  $k$ -colouring is called the *chromatic number* of  $G$  and is denoted by  $\chi(G)$ . The maximum value  $k$  for which a graph  $G$  has a complete subgraph of order  $k$  is called the *clique number* of  $G$  and is denoted by  $\omega(G)$ . Obviously,  $\omega(G) \leq \chi(G)$  holds for every graph  $G$ . However, the computation of both graph parameters  $\omega(G)$  and  $\chi(G)$  is NP-hard.

It is not difficult to colour the vertices of a graph in polynomial time using at most  $\Delta(G) + 1$  colours, where  $\Delta(G)$  denotes the maximum vertex degree of a given graph  $G$ . Moreover, the classical theorem of Brooks [15] states that  $\chi(G) \leq \Delta(G)$  unless  $G$  is a complete graph or an odd cycle. Reed [104] conjectured that the chromatic number is bounded by the average of the trivial upper and lower bound, i. e. for any graph  $G$  of maximum degree  $\Delta$  and clique number  $\omega$ ,  $\chi(G)$  is at most  $\lceil \frac{\Delta+1+\omega}{2} \rceil$ .

By a classical result of Erdős [40] we know that the difference  $\chi(G) - \omega(G)$  can be arbitrarily large. A graph  $G$  is called *perfect* if  $\chi(H) = \omega(H)$  for each induced subgraph  $H \subset G$ . Berge [6] conjectured that a graph  $G$  is perfect if and only if neither  $G$  nor its complement  $\bar{G}$  contains an induced odd cycle of order at least five. This famous conjecture known as *Strong Perfect Graph Conjecture* has recently been solved by Chudnovsky, Robertson, Seymour and Thomas [20].

Gyárfás [54] has introduced the concept of  $\chi$ -bound functions thereby extending the notion of perfectness. Here, a family  $\mathcal{G}$  of graphs is called  *$\chi$ -bound* with  *$\chi$ -binding function*  $f$ , if  $\chi(G') \leq f(\omega(G'))$  holds whenever  $G'$  is an induced subgraph of  $G \in \mathcal{G}$ .

Various sufficient conditions for graphs  $G$  satisfying  $\chi(G) \leq \omega(G) + 1$  and characterizations in terms of forbidden induced subgraphs have been obtained recently by the first author [97]. Only a few results are known so far about graphs  $G$  satisfying  $\chi(G) \leq \omega(G) + k$  for fixed  $k \geq 2$ .

Bollobás and Erdős [41] conjectured  $\chi(G) \leq 2r + 2$  for every graph  $G$

which contains at most  $r$  different odd cycle lengths. This was proved by Gyárfás [55]. Mihók and the second author [91] showed the surprising analogue bound  $\chi(G) \leq 2s + 3$  for every graph  $G$  which contains at most  $s$  different even cycle lengths. The related question for given induced cycle lengths was considered in [99].

The  $k$ -colourability problem is a well-known NP-complete problem. It remains NP-complete even for special graph classes, e. g. triangle-free graphs. While it can be decided in polynomial time for other special graph classes. Edwards [38] has introduced the following approach: If a graph  $G = (V, E)$  contains a dominating set  $D \subset V$  of small size, say of logarithmic size with respect to  $|V|$ , then 3-colourability can be decided in polynomial time for the considered graph by usage of a 2-SAT reformulation of the problem. Based on this approach 3-colourability can be decided in polynomial time for the class of  $P_5$ -free graphs [103].

## 2 Perfect graphs

In this section we will describe the class of perfect graphs containing five basic graph classes as ingredients and additionally four structural faults in order to decompose perfect graphs. This characterization has recently been found by Chudnovsky, Robertson, Seymour and Thomas [20] thereby solving a longstanding famous conjecture of Claude Berge on perfect graphs.

More than four decades ago Berge [5] introduced the concept of perfect graphs motivated by Shannon's notion of the zero-error capacity of a graph which has been applied in Shannon's work on communication theory. Berge defined two kinds of perfectness: A graph  $G$  is called  $\alpha$ -*perfect* if the stability number  $\alpha(H)$  equals the clique covering number  $\Theta(H)$  for every induced subgraph  $H$  of  $G$ . Moreover, a graph  $G$  is called  $\chi$ -*perfect* if the chromatic number  $\chi(H)$  equals the clique number  $\omega(H)$  for every induced subgraph  $H$  of  $G$ . He also mentioned a natural superclass of the families of  $\alpha$ - and  $\chi$ -perfect graphs, namely the class of (in honour to Berge [25]) Berge graphs. A graph  $G$  is *Berge* if  $G$  contains neither an odd cycle of length greater than 3 nor its complement as an induced subgraph.

In 1963 in a booklet on a lecture at the Research and Training School of the Indian Statistical Institute of Calcutta [6], Berge published his famous two conjectures. The *Weak Perfect Graph Conjecture* states that the family of  $\alpha$ -perfect graphs and of  $\chi$ -perfect graphs are identical. The second one is the *Strong Perfect Graph Conjecture* (SPGC) asserting that every Berge graph is likewise  $\alpha$ -perfect. The Weak Perfect Graph Conjecture was first settled by Lovász [82] in 1972. An important role in this proof plays the *Lovász Replication Lemma*, stating that the addition of a new vertex  $v'$  joined to all vertices of the closed neighbourhood of a vertex  $v$  of a perfect

graph  $G$  preserves perfectness. A good candidate for a 'book' proof of the *Perfect Graph Theorem* is due to Gasparyan [48]. The solution of the first conjecture made it superfluous to distinguish between the notions of  $\alpha$ - and  $\chi$ -perfect and these graphs are just referred to as *perfect* graphs. Moreover, the Perfect Graph Theorem states that if a graph  $G$  is perfect, then also its complement  $\bar{G}$  is perfect.

Similar to the class of planar graphs the family of perfect graphs has become one of the famous special graph classes. Not only the two intriguing conjectures of Berge led to its popularity but also its impact to the modelling of real world applications. E. g. certain perfect graphs like interval graphs occur in the investigation of the fine structure of a gene due to Benzer or in archaeology in the sequence-dating problem due to Petrie or threshold graphs occur in the synchronization of parallel processes. More examples can be found in the excellent book of Golubic [49] on algorithmic graph theory and perfect graphs.

Efforts to solve Berge's conjectures revealed importance of perfect graphs for communication theory, polyhedral combinatorics, relation to integrality of polyhedra, geometric algorithms, semi-definite programming, radio channel assignment problem and sorting. More examples can be found in the recent book of Ramirez-Alfonsin and Reed [96] on perfect graphs.

### 2.1 The Strong Perfect Graph Conjecture

For a long time the SPGC seemed to be intractable and initiated a variety of research - a search in e. g. AMS-MathSciNet leads to more than 120 papers about the SPGC. Many researchers contributed important steps on the way to prove the SPGC. Especially, in the recent years Conforti, Cornuéjols, Vušković, Zambelli, Chudnovsky, Robertson, Seymour and Thomas have worked on a decomposition theorem for perfect graphs (*find all basic perfect graph classes & find all possible structural faults leading to a decomposition of 'larger' perfect graphs in question*) which implies a solution of the SPGC. That a structural property of Berge graphs implies the SPGC was conjectured in 2001 by Conforti, Cornuéjols and Vušković (see [34]).

In the following we will briefly motivate this 'paradigm of primitive objects and structural faults' (Chvátal) by the example of triangulated graphs. A graph  $G$  is *triangulated*, if  $G$  contains no induced cycle of length larger than 3. The perfectness of triangulated graphs can easily be proved by the following decomposition result of Dirac [37].

**Theorem 1** [37] *Every connected triangulated graph  $G$  is either a clique or contains a clique cutset.*

Hence, for the class of triangulated graphs complete graphs are the primitive objects and every triangulated graph, which is not a primitive object has a structural fault, e. g. a clique cutset.

### 2.1.1 Basic perfect graphs

One class of basic perfect graphs are the bipartite graphs. A graph  $G = (V, E)$  is *bipartite*, if  $V$  can be partitioned into at most two independent vertex sets, i. e.  $\chi(G) = \omega(G) \leq 2$ . All graph classes mentioned in this subsection are *hereditary*, i. e. every induced subgraph of the graph in consideration is likewise a member of the same graph family. Thus, bipartite graphs are perfect. Unaware of the SPGC the first contribution to this conjecture has been made 1916 and is due to König [76]: A graph  $G$  is bipartite if and only if  $G$  contains no (induced) cycle of odd length.

The least number of colours needed to colour the edges of a graph  $G$  is called the *chromatic index*  $\chi'(G)$ . The problem of colouring the edges of a graph  $G$  is equivalent to the colouring of the vertices of its line graph  $L(G)$ . For a graph  $G$ , the *line graph*  $L(G)$  has the edges of  $G$  as its vertices and distinct edges of  $G$  are adjacent in  $L(G)$  if they are adjacent in  $G$ . In 1916, König [76] proved that a bipartite graph  $G$  satisfies  $\Delta(G) = \chi'(G)$ . Observe, that a graph  $G$  being the line graph of a bipartite graph satisfies  $\omega(G) = \chi(G)$ . Hence, these graphs are also perfect.

Due to the Perfect Graph Theorem the complements of bipartite graphs and line graphs of bipartite graphs are also perfect. This could also be proved directly by usage of other results of König [77]. In summary, for Berge graphs we already mentioned four basic graph classes: bipartite graphs, complements of bipartite graphs, line graphs of bipartite graphs and complements of line graphs of bipartite graphs.

A fifth basic perfect graph class, the double split graphs, was defined by Chudnovsky, Robertson, Seymour and Thomas [20]. A graph  $G = (Q \cup S, E)$  is called a *split graph*, if  $V$  can be decomposed into an independent vertex set  $S$  and a vertex set  $Q$  inducing a clique. It is an easy exercise to show that split graphs are perfect.

**Definition 2** [20] *A graph  $G^*$  is a double split graph, if it can be constructed in the following way. Take a split graph  $G = (Q \cup S, E)$ ; replace a vertex  $q$  in the clique  $Q$  by two non-adjacent vertices  $q'$  and  $q''$ ; replace a vertex  $s$  in the stable set  $S$  by two adjacent vertices  $s'$  and  $s''$ ; add for every such quadruple  $(s', s'', q', q'')$  additional edges s. t. the vertices induce a  $P_4$ . The resulting graph is  $G^*$*

A slight extension of the proof of perfectness of split graphs yields that every double split graph is likewise perfect.

### 2.1.2 Structural faults

In this subsection we deal with structural faults of perfect graphs. We already met one example, the clique cutset. A graph is *minimal imperfect* if it is not perfect but all its proper induced subgraphs are perfect. Hence, the SPGC claims that odd holes and odd antiholes are the only minimal imperfect graphs. Equivalently, there exist no minimal imperfect Berge graphs. An interesting example is due to Chvátal: A *star-cutset* of a graph  $G$  denotes a non-empty set  $C$  of vertices s. t.  $G - C$  is disconnected and there exists at least one vertex in  $C$  which is adjacent to all the remaining vertices of  $C$ .

**Lemma 3** (*Star-Cutset Lemma*, [24])

*No minimal imperfect graph contains a star-cutset.*

*Sketch of the proof:* Let  $G$  be minimal imperfect. Then

- (1) every proper induced subgraph of  $G$  is  $\omega(G)$ -colourable and
- (2) for every stable set  $S$  in  $G$  we have  $\omega(G - S) = \omega(G)$ .

Now suppose  $G$  contains a star-cutset  $C$ . Then  $G - C$  splits into non-empty parts  $V_1$  and  $V_2$  s. t.

- (3) no vertex of  $V_1$  is adjacent to a vertex of  $V_2$ .

Let  $G_i = G[V_i \cup C]$  for  $i = 1, 2$ , then by (1) there exists a colouring  $f_i$  of  $G_i$  using  $\omega(G)$  colours. Since  $C$  is a star-cutset, there exists  $w \in C$  adjacent to all remaining vertices of  $C$ ; write  $v \in S_i$  if  $v \in G_i$  and  $f_i(v) = f_i(w)$ . Obviously, each  $S_i$  is a stable set and  $S_i \cap C = \{w\}$ . Now (3) implies that  $S = S_1 \cup S_2$  is also a stable set. Finally, let  $Q$  be a clique in  $G - S$ . Again by (3)  $Q$  is fully contained in  $G_1 - S_1$  or in  $G_2 - S_2$ . Since each of these two graphs is coloured by  $\omega(G) - 1$  colours, we have  $|Q| \leq \omega(G) - 1$  contradicting (2). ■

A common generalization of a star-cutset and a star-cutset in the complement is a skew-partition. A graph  $G = (V, E)$  has a *skew-partition* if  $V$  can be partitioned into four non-empty sets  $A, B, C, D$  s. t. every vertex in  $A$  and every vertex of  $B$  are adjacent and no vertex in  $C$  is adjacent to a vertex of  $D$ . Chvátal [24] conjectured that no minimal imperfect graph has a skew-partition. A skew-partition is *even*, if every induced path with both ends in  $A \cup B$  and its interior in  $C \cup D$  contains an even number of edges and every induced path in the complement of the graph with both ends in  $C \cup D$  and its interior in  $A \cup B$  likewise contains an even number of edges. Chudnovsky, Robertson, Seymour, Thomas [20] proved for minimal imperfect Berge graphs of minimum order the next result.

**Theorem 4** [20] *No minimal imperfect Berge graph of minimum order admits an even skew-partition.*

Now we present further examples of structural faults. A graph  $G = (V, E)$  has a *2-join* if  $V$  can be partitioned into  $V_1$  and  $V_2$  each of cardinality at

least three with non-empty disjoint subsets  $A_1, B_1 \subseteq V_1$  and  $A_2, B_2 \subseteq V_2$  s. t. all vertices of  $A_1$  are adjacent to all vertices of  $A_2$  and all vertices of  $B_1$  are adjacent to all vertices of  $B_2$  and these are the only edges between  $V_1$  and  $V_2$ . Cornuéjols and Cunningham [35] proved that

**Theorem 5** [35] *No minimal imperfect Berge graph has a 2-join.*

A slight variation of homogeneous sets are  $M$ -joins. A graph  $G = (V, E)$  has an  $M$ -join if  $V$  can be partitioned into six non-empty sets  $A, B, C, D, E, F$  s. t. every vertex in  $A$  has a neighbour and a non-neighbour in  $B$ , and vice versa; for all pairs  $(C, A), (A, F), (F, B), (B, D)$  every vertex of the first set is adjacent to any vertex of the second set, and for all pairs  $(D, A), (A, E), (E, B), (B, C)$  there are no edges joining a vertex of the first set with one of the second set. Chudnovsky, Robertson, Seymour and Thomas [20] used a result of Chvátal and Shibi [25] on homogeneous sets in order to prove that

**Theorem 6** [20], [25] *No minimal imperfect Berge graph has an  $M$ -join.*

In 2002 Chudnovsky, Robertson, Seymour and Thomas [20] proved the following powerful decomposition theorem for Berge graphs.

**Theorem 7** [20] *For every Berge graph  $G$ , either  $G$  or its complement*

- (1) *is bipartite, or*
- (2) *is the line graph of a bipartite graph, or*
- (3) *is a double split graph, or*
- (4) *has an even skew partition, or*
- (5) *has a 2-join, or*
- (6) *has an  $M$ -join.*

In the long remarkable proof of this result the authors make use of an interesting tool of Roussel and Rubio [105]. In [20] the authors called it *The Wonderful Lemma*. Together with the already mentioned properties of minimal imperfect Berge graphs this decomposition theorem implies an affirmative answer to the SPGC.

**STRONG PERFECT GRAPH THEOREM** [20]

*Every Berge graph is perfect.*

Polynomial time recognition algorithms for Berge graphs have recently be announced by Chudnovsky and Seymour and Cornuéjols, Liu and Vušković (see [21], [22], [36]).

## 2.2 Miscellaneous

A related concept to perfectness is the  $\beta$ -perfectness. Here, for a graph  $G$  the *colouring number*  $\beta$  satisfies  $\beta(G) = \max_{H \subseteq G} \{\delta(H) + 1\}$ , where  $\delta(H)$  is the minimum degree of a subgraph  $H$  of  $G$ . If we recursively remove vertices

of minimum degree in the current graph, then we can colour greedily along the reverse order the vertices of the graph  $G$  using at most  $\beta(G)$  colours. Thus we have  $\chi(G) \leq \beta(G)$  for every graph  $G$ . Now a graph is called  $\beta$ -perfect if the chromatic number equals the colouring number for all of its induced subgraphs. The concept of  $\beta$ -perfectness was introduced 1996 by Markossyan, Gasparyan and Reed [88]. There they reveal the parallels of properties of these graphs to the ones of perfect graphs, e. g. they are even-hole-free and they proved an interesting analogue of the Strong Perfect Graph Theorem.

**Theorem 8** [88] *The graphs  $G$  and  $\bar{G}$  are  $\beta(G)$ -perfect if and only if  $G$  and  $\bar{G}$  are even-hole-free.*

Moreover, they demonstrate that the greedy colouring algorithm provides a performance guarantee for the related graph class of even hole-free graphs  $G$ , more precisely  $\chi(G) \geq \beta(G)/2 + 1$ . Furthermore, in [88] a structural characterization of all triangle-free and even-hole-free graphs is given. De Figueiredo and Vušković [44] extended the proof idea for the structural result on triangle-free and even hole-free graphs in order to prove that even-hole-free, diamond-free and short-chorded6-cycle-free graphs are  $\beta$ -perfect. Here, a diamond is a complete graph on four vertices with one missing edge, a short-chorded cycle is a cycle containing a chord forming a triangle with two edges of the cycle. Moreover, they stated the following conjecture.

**Conjecture 9** [44] *Every even-hole- and diamond-free graph is  $\beta$ -perfect.*

An extension of the result of de Figueiredo and Vušković [44] by replacing the diamond by four different supergraphs of the diamond and the short-chorded6-cycle by two different supergraphs of this graph was recently been obtained by Keijsper and Tewes [67]. Moreover they showed that a  $\beta$ -perfect graph does not contain any induced regular graph, except perhaps odd holes and cliques.

The class of even-hole-free graphs is very interesting since it also follows the paradigm of primitive objects and structural faults, i. e. there also exists a decomposition theorem. This theorem is due to Conforti, Cornuéjols, Kapoor and Vušković [27]. Moreover, the same authors [28] use this decomposition theorem to develop a recognition algorithm for even-hole-free graphs. More details on even-hole-free graphs can be found in the survey on *Forbidding Holes and Antiholes* by Hayward and Reed [58]. For instance there they state the challenging conjecture that

**Conjecture 10** [58] *Every even-hole-free graph contains a vertex whose neighbourhood can be partitioned into two cliques.*

A further special graph class with interesting structural properties is the family of  $P_4$ -free graphs, i. e. graphs without an induced path on four vertices. While we will take a closer look on this special graph class in Section 5, we only study its perfectness here. Observe that the family of  $P_4$ -free graphs is the only subclass of perfect graphs defined in terms of one forbidden induced subgraph. For further information on special graph classes



defined in terms of forbidden induced subgraphs we refer to [49] and [11]. The following result was first obtained by Seinsche [112].

**Theorem 11** [112] *Every  $P_4$ -free graph is perfect.*

*Sketch of the proof:* The result is obviously true for  $P_4$ -free graphs of small order and for cliques. W. l. o. g. let  $G = (V, E)$  be connected. Then  $G$  can be represented as the sum of two graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$ , i. e. every vertex of  $G_1$  is adjacent to any vertex of  $G_2$ . If  $G$  contains a cutvertex  $x$ , then the  $P_4$ -freeness forces that  $x$  is adjacent to all vertices of  $G - \{x\}$ . Thus, we have  $G_1 = G[\{x\}]$  and  $G_2 = G - \{x\}$ . If  $G$  is 2-connected, then let  $X$  be a minimal cutset of  $G$ . Now, since every vertex  $x \in X$  is a cutvertex of  $G - (X - \{x\})$ , we know that  $x$  is adjacent to every vertex of  $G - X$ . Therefore,  $G_1 = G[X]$  and  $G_2 = G - X$ . With this property we obtain inductively

$$\chi(G) = \chi(G_1) + \chi(G_2) = \omega(G_1) + \omega(G_2) = \omega(G).$$

Since the family of  $P_4$ -free graphs is hereditary, the last equality implies the result. ■

We close this section with a surprising observation due to Corneil [33]. Here, a graph  $G$  is *self-complementary*, if  $G = \bar{G}$ .

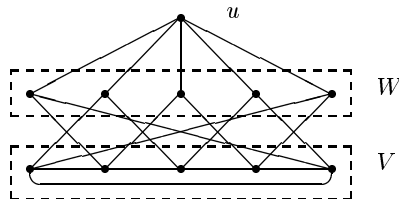
**Observation 12** [33] *If the SPGC is true for self-complementary graphs, then the SPGC is true in general.*

Otherwise, suppose there exists a minimal imperfect Berge graph  $G$ . Then take two copies  $G_1$  and  $G_4$  of  $G$  and two copies  $G_2$  and  $G_3$  of  $\bar{G}$ . Furthermore, add edges such that  $G_1$  and  $G_2$ ,  $G_2$  and  $G_3$  and also  $G_3$  and  $G_4$  are sums of the corresponding two graphs. This new graph  $G^*$  is self-complementary and not perfect.

### 3 $\chi$ -bound graphs

In the previous section we have considered perfect graphs. A natural extension of the family of perfect graphs are  $\chi$ -bound classes of graphs. This concept was introduced by Gyárfás [54]. Here, a family  $\mathcal{G}$  of graphs is called  *$\chi$ -bound* with  *$\chi$ -binding function  $f$* , if  $\chi(G') \leq f(\omega(G'))$  holds whenever  $G'$  is an induced subgraph of  $G \in \mathcal{G}$ . Thus, the class of perfect graphs is precisely the  $\chi$ -bound family of graphs admitting the identity  $f(x) = x$  as  $\chi$ -binding function.

The concept of  $\chi$ -boundness is well defined: On the one hand for any graph the chromatic number  $\chi$  is at least as large as the clique number  $\omega$ . On the other hand a classical result of Erdős [40] (asserting that for any two integers  $g \geq 3$  and  $k \geq 3$ , there exists a graph with girth  $g$  and chromatic number  $k$ .) illustrates that the difference  $\chi - \omega$  of the chromatic number



**Fig. 1** Mycielski/Grötzsch graph  $G_4$

and the clique number of a graph can be arbitrarily large. He proved this result by means of his non-constructive probabilistic method. An elegant construction of a triangle-free graph  $G_k$  with chromatic number  $k$  (for any  $k$ ) is due to Mycielski [94]. In fact, let  $G_1$  be the  $K_1$ ,  $G_2$  be the  $K_2$ ,  $G_3$  be the  $C_5$  and suppose that  $G_k$  with  $k \geq 3$  has the vertex set  $\{v_1, v_2, \dots, v_n\}$ . Form  $G_{k+1}$  by adjoining for each  $i = 1, 2, \dots, n$  a new vertex  $w_i$  with  $w_i$  being adjacent to every vertex of  $N_G(v_i)$  and attaching a new vertex  $u$  adjacent to each vertex  $w_i$ . Note that every graph of the resulting sequence  $(G_k)_{k \in \mathbb{N}}$  of graphs is triangle-free. Moreover,  $G_k$  is  $k$ -chromatic. The graph  $G_4$  is quite often referred to as Grötzsch graph or Mycielski graph (see Figure 1).

In [54] Gyárfás posed four meta problems:

- Does there exist a  $\chi$ -binding function  $f$  for a given family  $\mathcal{G}$  of graphs?
- What is the smallest  $\chi$ -binding function  $f^*$  for  $\mathcal{G}$ ?
- Does there exist a linear  $\chi$ -binding function  $f$  for  $\mathcal{G}$ ?
- Does there exist a polynomial  $\chi$ -binding function  $f$  for  $\mathcal{G}$ ?

For perfect graphs there are classes which can be characterized by forbidden induced subgraphs, e. g.  $P_4$ -free graphs, chordal graphs, split graphs, threshold graphs. What choices of forbidden induced subgraphs guarantee that a family of graphs is  $\chi$ -bound? Since there are graphs with arbitrarily large chromatic number and girth, at least one forbidden subgraph has to be acyclic. Gyárfás [53] and independently Sumner [116] conjectured that this necessary condition is also a sufficient condition for a  $\chi$ -bound family of graphs defined in terms of forbidden induced subgraphs. Partial solutions to this conjecture have been made by Gyárfás, Szemerédi and Tuza [56], Kierstead and Penrice [70], and Scott [110].

### 3.1 $\chi$ -bound graphs with one forbidden induced subgraph

If we consider  $\chi$ -bound families of graphs defined in terms of only one forbidden induced subgraph  $T$ , then we already know that  $T$  is acyclic. Moreover, the following sequence of graphs  $(H_i)$  implies a nice observation. Starting with  $H_1 = \bar{C}_7$ , the complement of the 7-cycle, we define  $H_{i+1} = \bar{C}_7[H_i]$ , the

lexicographic product of the graphs  $\bar{C}_7$  and  $H_i$ . Here, the *lexicographic product* of two graphs  $G$  and  $H$  is the graph  $G[H]$  with vertex set  $V(G) \times V(H)$  and with edges joining  $(u, v)$  and  $(u', v')$  if and only if  $uu'$  is an edge of  $G$  or  $u = u'$  and  $vv'$  is an edge of  $H$ . Note that  $\omega(H_{i+1}) = 3\omega(H_i)$ . Furthermore, in any colouring of  $H_{i+1}$  we need for each copy of  $H_i$  at least  $\chi(H_i)$  different colours. With  $\alpha(\bar{C}_7) = 2$  we then observe that every colour of a colouring of  $H_{i+1}$  appears in at most two different copies of  $H_i$ . Hence,  $H_i$  has the order  $n(H_i) = 7^i$ , independence number  $\alpha(H_i) = 2^i$  and clique number  $\omega(H_i) = 3^i$ . Therefore, its chromatic number  $\chi(H_i)$  is at least  $(7/2)^i = (7/6)^i \omega(H_i)$ . Furthermore, if a member of the sequence  $(H_i)$  contains an acyclic induced subgraph  $T$ , then  $T$  has to be a subgraph of the path  $P_4$ . Hence, we observe the following surprising result:

**Observation 13** *Let  $\mathcal{G}$  be a  $\chi$ -bound family of graphs defined in terms of only one forbidden induced subgraph  $T$ . Then  $T$  is acyclic. Furthermore, if  $T \subset P_4$  then  $\mathcal{G}$  has the (smallest)  $\chi$ -binding function  $f^*(x) = x$ , or otherwise there exists no linear  $\chi$ -binding function  $f$  for  $\mathcal{G}$ .*

Let  $K_{1,n}$  denote the star with  $n$  branches. The special case  $K_{1,3}$  is called a *claw*. Let  $r(p, q)$  be the Ramsey number, that is the smallest integer  $n$  such that every graph  $G$  of order at least  $n$  contains an independent set with  $p$  vertices or a clique with  $q$  vertices. In 1981, Sumner [116] observed that the class of claw-free graphs is  $\chi$ -bound with  $\chi$ -binding function  $f(x) = r(3, x)$ . Since the ratio of the order and the independence number of a graph provides a well-known lower bound for its chromatic number and every graph with an independence number of at most two is obviously also a claw-free graph, it is not difficult to observe that for every  $\chi$ -binding function  $f$  of the class of claw-free graphs we have  $f(x) \geq \frac{r(3, x+1)}{2}$  (cf. [55]). Consequently, by the dependency on the Ramsey number and Kim's famous result [74], that the Ramsey number  $r(3, x)$  has order of magnitude  $x^2/\log x$ , the smallest  $\chi$ -binding function  $f^*$  has also this magnitude. Combining these results implies the following observation.

**Observation 14** [116], [55] & [74] *There exists no linear  $\chi$ -binding function  $f$  for the class of claw-free graphs. More precisely, for the family of claw-free graphs the smallest  $\chi$ -binding function  $f^*$  satisfies  $f^*(x) = O(\frac{x^2}{\log x})$ .*

More general, the next result of Gyarfas [54] shows that the magnitude of the smallest  $\chi$ -binding function for the class of  $K_{1,n}$ -free graphs is strongly related to Ramsey numbers.

**Theorem 15** [54] *The family of  $K_{1,n}$ -free graphs is  $\chi$ -bound and its smallest  $\chi$ -binding function  $f^*$  satisfies  $\frac{r(n, x+1)-1}{n-1} \leq f^*(x) \leq r(n, x)$ .*

Also for the family of  $P_n$ -free graphs, where  $P_n$  is a path on  $n$  vertices, a  $\chi$ -binding function is known.

**Theorem 16** [54] *The family of  $P_n$ -free graphs is  $\chi$ -bound and its smallest  $\chi$ -binding function  $f_n^*$  satisfies  $\frac{r(\lceil n/2 \rceil, x+1)-1}{\lceil n/2 \rceil - 1} \leq f_n^*(x) \leq (n-1)^{x-1}$ .*

It is noteworthy that even  $f_5^*(x) \leq 2^x$  as shown in [71]. The idea of the proof of Theorem 16 in order to establish the upper bound will be illustrated for the special case of triangle-free graphs. Thereby, we slightly improve the upper bound.

**Corollary 17** *Let  $G$  be a triangle-free and  $P_n$ -free graph. Then  $\chi(G) \leq n - 2$ , i. e.  $f_n^*(2) \leq n - 2$ .*

*Proof:* Assume to the contrary that there exists a triangle-free and  $P_n$ -free graph  $G_1$  with  $\chi(G_1) \geq n - 1$ . Say,  $G_1$  is connected. Let  $v_1$  be a vertex of  $G_1$ . Since all neighbours of  $v_1$  form an independent set of  $G_1$ , we deduce that the induced subgraph  $G_1 - N_{G_1}[v_1]$  contains a component  $G_2$  with  $\chi(G_2) \geq n - 2$ . Since  $G_1$  is connected there exists a neighbour  $v_2$  of  $v_1$  such that  $v_2$  is also adjacent to at least one vertex of  $G_2$ . Now proceed iteratively until we receive a sequence of nested graphs  $(G_i)_{i \in \{1, \dots, n-3\}}$ . Observe that the vertex set  $\{v_1, v_2, \dots, v_{n-3}\}$  induces a path  $P_{n-3}$  and  $\chi(G_{n-3}) \geq 3$ . Because  $G_{n-3}$  is triangle-free and not bipartite there exists an induced odd cycle  $C$  containing at least five vertices. Now it is not very difficult to extend the path  $P_{n-3}$  along  $C$  in order to produce an induced path of  $G_1$  containing at least  $n$  vertices, a contradiction. ■

For the class of  $mK_2$ -free graphs, Wagon [125] provides an  $O(x^{2(m-1)})$   $\chi$ -binding function. In particular, for  $m = 2$  he obtained the  $\chi$ -binding function  $f(x) = \binom{x+1}{2}$ . In [54] the lower bound  $r(C_4, K_{x+1})$  for the smallest  $\chi$ -binding function  $f^*$  for the family of  $2K_2$ -free graphs is mentioned. Here,  $r(C_4, K_{x+1})$  denotes the smallest integer  $n$  such that every graph  $G$  of order at least  $n$  contains the complement of  $C_4$ , a cycle with four vertices, or a clique with  $x + 1$  vertices. This lower bound is non-linear, since this special Ramsey number has magnitude  $O(x^{1+\epsilon})$  for some  $\epsilon > 0$  as shown in [23]. Whereas  $f^*(2) = 3$  is trivial, the problem to determine  $f^*(3)$  is far from being trivial. Erdős offered 20\$ to decide whether  $f^*(3) = 4$ . In [54] the (so far unpublished) solution  $f^*(3) = 4$  is contributed to Nagy and Szentmiklóssy.

In Gyárfás' study of the smallest  $\chi$ -binding function  $f_T^*$  for the class of  $T$ -free graphs, where  $T$  is a forest with four vertices, there are three subcases left over. As shown in [54]  $f_{P_3 \cup K_1}^*$  behaves asymptotically like  $\frac{r(3, x+1)}{2}$ . Thus, analogously to the case of claw-free graphs, the magnitude of  $f_{P_3 \cup K_1}^*$  is  $O(\frac{x^2}{\log x})$ . The graph  $P = \overline{P_3 \cup K_1}$  is also known as *paw*. Note that by a result of Olariu [95] on the class of paw-free graphs, asserting that every paw-free graph is either triangle-free or a complete multipartite graph, it is not difficult to obtain that  $f_{P_3 \cup K_1}^*(x) = f_{3K_1}^*(x)$ . Here,  $f_{3K_1}^*$  is the smallest  $\chi$ -binding function for the class of  $3K_1$ -free graphs. Also in [54] the asymptotic behaviour  $\frac{r(4, x+1)}{3}$  of  $f_{4K_1}^*$  is mentioned. Finally, Gyárfás established  $\frac{r(3, x+1)-1}{2} \leq f_{K_2 \cup 2K_1}^*(x) \leq \binom{x+1}{2} + x - 1$ . With a slight modification (e. g. see the sketch of proof of Theorem 18) it is not very difficult to establish the upper bound  $\binom{x+1}{2}$  for  $f_{K_2 \cup 2K_1}^*$ .

Now we turn our attention to the smallest  $\chi$ -binding function  $f_T^*$  of the family of  $T$ -free graphs, where  $T$  is a forest of order 5. By Observation 13 we know that there exists no linear function  $f_T^*$ . If  $T = P_5$ , then easily we obtain  $f_{P_5}^*(2) = 3$  and the example  $G^{P_5}$  of Figure 3 shows  $f_{P_5}^*(3) \geq 5$ . Improving the already mentioned upper bound  $2^x$  for  $f_{P_5}^*$  seems to be a challenging task. If  $T$  is a chair (for a definition see Figure 3), then again we have  $f_{chair}^*(2) = 3$  and in [97] the first author proved  $f_{chair}^*(3) = 4$ . Moreover, we expect  $f_{chair}^*(x) \leq \binom{x+1}{2}$ . The next theorem can be easily proved by induction on  $\omega$  and the observation that for every vertex  $v$  of a  $P_4 \cup K_1$ -free graph  $G$ , the induced subgraph  $G - N_G[v]$  is  $P_4$ -free, therefore perfect and  $N_G(v)$  induces a subgraph  $G'$  such that  $\omega(G') \leq \omega(G) - 1$ . Since the family of  $P_4 \cup K_1$ -free graphs contain all  $P_3 \cup K_1$ -free graphs we easily deduce the lower bound of Theorem 18.

**Theorem 18** *The family of  $P_4 \cup K_1$ -free graphs is  $\chi$ -bound and its smallest  $\chi$ -binding function  $f_{P_4 \cup K_1}^*$  satisfies  $\frac{r(3, x+1)}{2} \leq f_{P_4 \cup K_1}^*(x) \leq \binom{x+1}{2}$ .*

Analogously we obtain the following results:

- (i)  $\frac{r(5, x+1)-1}{4} \leq f_{5K_1}^*(x) \leq r(5, x)$ ;
- (ii)  $\frac{r(3, x+1)}{2} \leq f_{K_2 \cup 3K_1}^*(x) \leq \binom{x+2}{3}$ ;
- (iii)  $O(x^{1+\epsilon}) \leq r(4, x+1) \leq f_{2K_2 \cup K_1}^*(x) \leq \binom{x+2}{3}$  for some  $\epsilon > 0$ ;
- (iv)  $O\left(\frac{x^2}{\log x}\right) = \frac{r(3, x+1)}{2} \leq f_{K_1 \cup K_{1,3}}^*(x) \leq \binom{x+2}{3}$ .

The remaining open case for an acyclic graph of order 5, the  $P_3 + K_2$ , seems to be tractable.

### 3.2 Miscellaneous

There are also interesting  $\chi$ -bound families of graphs with more than one forbidden induced subgraph. For instance Fouquet, Giakoumakis, Maire and Thuillier [45] achieved the upper bound  $\binom{x+1}{2}$  for the smallest  $\chi$ -binding function of the family of  $P_5$ -free and  $\bar{P}_5$ -free graphs. Moreover, they proved the upper bound  $\lfloor \frac{3\omega}{2} \rfloor$  for the smallest  $\chi$ -binding function of the family of  $2K_2$ -free and  $\bar{P}_5$ -free graphs. The same bound is valid for the smallest  $\chi$ -binding function of the family of  $P_3 \cup K_1$ -free and  $C_5$ -free graphs as proven by Hoàng and McDiarmid [59].

A nice byproduct of Observation 14 is an affirmative answer to a question due to Kierstead. In 1989, Kierstead [69] examined a subclass of claw-free graphs and proved that every claw-free graph  $G$ , which does not contain a complete graph  $K_{2s+3}$  minus an edge as an induced subgraph, satisfies  $\chi(G) \leq \max\{\omega(G) + s, r(3, 4s - 1)\}$ . Kierstead [69] asked, whether it is possible to drop the dependency on the Ramsey number  $r(3, 4s - 1)$  in this upper bound. He also noted that a positive answer would imply that for every claw-free graph  $G$  we have  $\chi(G) \leq \frac{3\omega(G)}{2}$ . But there exists no linear  $\chi$ -binding function and Kierstead's question has a negative answer.

We close this section with several conjectures posed by Gyárfás [54], which are related to the Strong Perfect Graph Conjecture.

**Conjecture 19** [54] *There exists a  $\chi$ -binding function for the class of odd-hole-free graphs.*

**Conjecture 20** [54] *For all integers  $l \geq 2$  there exists a  $\chi$ -binding function for the class of graphs without induced cycles of length  $r \in \{2l+1, 2l+3, \dots\}$ .*

**Conjecture 21** [54] *For all integers  $l \geq 2$  there exists a  $\chi$ -binding function for the class of graphs without induced cycles of length  $r \in \{l, l+1, l+2, \dots\}$ .*

Partial solutions to these conjectures are due to Scott [111]. The class of graphs without induced cycles of length  $l \in \{5, 6, \dots\}$  will be defined in Subsection 6.2 as  $\mathcal{G}^I(3, 4)$ . Obviously, the graph sequence  $(H_i)$  constructed at the beginning of the latter subsection is contained in  $\mathcal{G}^I(3, 4)$ .

**Observation 22** [99],[59] *No linear  $\chi$ -binding function for  $\mathcal{G}^I(3, 4)$  exists.*

## 4 The theorems of Brooks and Vizing

In this section we consider two classical colouring results due to Brooks and Vizing. First we reformulate these results on graphs defined in terms of forbidden induced subgraphs and then we discuss whether these reformulated results can be extended.

### 4.1 On Brooks' Theorem

An important result in graph colouring theory is the Theorem of Brooks [15], asserting that every graph  $G$  is  $\Delta(G)$ -colourable unless  $G$  is isomorphic to an odd cycle or a complete graph. A very nice algorithmic proof of Brooks' Theorem is due to Lovasz [83]. Bryant [16] simplified this proof with the following characterization of cycles and complete graphs. Thereby he highlights the exceptional role of the cycles and complete graphs in Brooks' Theorem. An elementary proof of this characterization was obtained in [102].

**Proposition 23** [16] *Let  $G$  be a 2-connected graph. Then  $G$  is a cycle or a complete graph if and only if  $G - \{u, v\}$  is not connected for every pair  $(u, v)$  of vertices of distance two.*

**Brooks' Theorem 24** [15] *If  $G$  is neither complete nor an odd cycle, then  $G$  is  $\Delta(G)$ -colourable.*

Brooks' Theorem states that  $\chi(G) \leq \Delta(G)$  for a graph  $G$  whenever  $3 \leq \omega(G) \leq \Delta(G)$ . Borodin and Kostochka [10] conjectured that  $\omega(G) < \Delta(G)$  implies  $\chi(G) < \Delta(G)$  if  $\Delta(G) \geq 9$ . Reed [104] proved that this is true when  $\Delta(G) \geq 10^{14}$ .

A nice generalization of Brooks' Theorem within the scope of this survey was achieved by Gallai [46]. A graph is *colour-critical*, if we have  $\chi(G) > \chi(H)$  for every proper subgraph  $H$  of  $G$ . If  $\chi(G) > \chi(H)$  holds for every proper induced subgraph  $H$  of  $G$ , then  $G$  is *vertex-critical*. Moreover, if a vertex-critical or colour-critical graph  $G$  has chromatic number  $k$ , then  $G$  is *k-colour-critical* or *k-vertex-critical*, respectively. A generalization of Brooks' Theorem and a very useful tool in the study of vertex-critical graphs is a reformulation of a deep structural result on colour-critical graphs which is due to Gallai [46]. For a short proof of this result see for example [107]. Note that most of the different proofs of this result only require properties of  $k$ -colour-critical graphs, for instance  $\delta \geq k - 1$ , which also hold for  $k$ -vertex-critical graphs.

**Theorem 25** [46] *Let  $G$  be a  $k$ -vertex-critical graph and  $Low(G)$  denotes the low-vertex graph of  $G$  induced by the vertices of degree  $k - 1$  of  $G$ . Then every 2-connected induced subgraph of  $Low(G)$  is either an odd hole (odd cycle of length greater than 3) or complete.*

In the book of Toft and Jensen [63] (Problem 4.6, p.83) the problem of improving Brooks' Theorem for the class of triangle-free graphs is stated or, more generally provided that the graph contains no  $K_{r+1}$ . The problem has its origin in a paper of Vizing [124]. The best known improvement of Brooks' Theorem in terms of the maximal degree for the class of triangle-free or, more generally  $K_{r+1}$ -free graphs is due to Borodin, Kostochka [10], Catlin [17], Kostochka [78]. They proved that if  $3 \leq r \leq \Delta(G)$  and  $G$  contains no  $K_{r+1}$ , then  $\chi(G) \leq \frac{r}{r+1}(\Delta(G) + 2)$ . Kostochka [78] proved  $\chi(G) \leq 2/3(\Delta(G) + 3)$  for every triangle-free graph  $G$ . The remaining authors independently proved that  $\chi(G) \leq 3/4(\Delta(G) + 2)$  for every triangle-free graph  $G$ . For the class of triangle-free graphs Brooks' Theorem can be restated in terms of forbidden induced subgraphs, since triangle-free graphs  $G$  satisfy  $G[N_G[x]] \cong K_{1,d_G(x)}$  for every vertex  $x$  of  $G$ .

**Theorem 26** [15] *(Triangle-free version of Brooks' Theorem)*

*Let  $G$  be a triangle-free and  $K_{1,r+1}$ -free graph. Then  $G$  is  $r$ -colourable unless  $G$  is isomorphic to an odd cycle or a complete graph with at most two vertices.*

The following theorem will extend this triangle-free version of Brooks' Theorem. An  $r$ -sunshade (with  $r \geq 3$ ) is a star  $K_{1,r}$  with one branch subdivided once. The 3-sunshade is sometimes called *chair* and the 4-sunshade *cross*.

**Observation 27** *Let  $G$  be a triangle- and chair-free graph, then  $\chi(G) \leq 3$ . Moreover if  $G$  is connected, then equality holds iff  $G$  is an odd hole.*

**Theorem 28** [97] *If  $G$  is a triangle- and cross-free graph, then  $\chi(G) \leq 3$ .*

In [102], the last theorem is extended.

**Theorem 29** [102] *Let  $G$  be a connected, triangle-free and  $r$ -sunshade-free graph with  $r \geq 3$ , which is not an odd cycle. Then*

- (a)  $G$  is  $r$ -colourable;
- (b)  $G$  is bipartite, if  $\Delta(G) \geq 2r - 3$ ;
- (c)  $G$  is  $(r-1)$ -colourable, if  $r = 3, 4$  or if  $\Delta(G) \leq r - 1$ .

**Problem 30** Let  $\mathcal{G}$  be the class of all connected, triangle-free and  $r$ -sunshade-free graphs with  $5 \leq r \leq \Delta(G) \leq 2r - 4$ . Does there exist an  $r$ -chromatic member  $G^* \in \mathcal{G}$ ?

Using Kostochka's result that  $\chi(G) \leq 2/3(\Delta(G) + 3)$  for every triangle-free graph  $G$ , it is not very difficult for  $r \geq 9$  to reduce Problem 30 to the range  $3/2(r - 3) \leq \Delta(G) \leq 2r - 4$ .

An intriguing improvement of Brooks' Theorem by bounding the chromatic number of a graph by a convex combination of its clique number  $\omega$  and its maximum degree  $\Delta$  plus 1 is suggested by Reed [104].

**Conjecture 31** [104] For any graph  $G$  of maximum degree  $\Delta$  and clique number  $\omega$ ,  $\chi(G)$  is at most  $\lceil \frac{\Delta+1+\omega}{2} \rceil$ .

Even in the special case of triangle-free graphs no affirmative answer is known so far.

**Conjecture 32** [104] Any triangle-free graph  $G$  satisfies  $\chi(G) \leq \frac{\Delta(G)}{2} + 2$ .

Asymptotically even the smaller upper bound  $\Delta(G)/\log(\Delta(G))$  is valid as shown by Johansson [66] and independently by Kim [73]. If the last conjecture is true then it is not very difficult to reduce Problem 30 to the range  $2r - 5 \leq \Delta(G) \leq 2r - 4$ , which seems to be tractable. Moreover, an affirmative answer to this special case of Reeds conjecture on triangle-free graphs, would imply that there exists no 5-regular, 5-chromatic or 6-regular, 6-chromatic triangle-free graph. These negative results would settle the remaining cases of Grünbaum's girth problem ([52], see also [63]).

We close this part with a result of Stacho [115] related to Brooks' Theorem. In [115] the invariant  $\Delta_2(G) = \max_{u \in V(G)} \max_{v \in N(u), d(v) \leq d(u)} d(v)$  of a given graph  $G$  was introduced and the following result was presented.

**Theorem 33** [115] Let  $G$  be a graph. Then  $\chi(G) \leq \Delta_2(G) + 1$ . Moreover, if  $\Delta(G) \geq 3$ , then to determine whether  $\chi(G) \leq \Delta_2(G)$  is an NP-complete problem.

Observe that  $\chi(G) \leq \Delta_2(G) + 1 \leq \Delta(G) + 1$  and  $\Delta_2(G) = \Delta(G)$ , if  $G$  contains two adjacent vertices  $u, v \in V(G)$  with  $d(u) = d(v) = \Delta(G)$ .

#### 4.2 On Vizing's Theorem

In this section let us briefly consider the problem of colouring the edges of a graph, instead of the vertices, in such a way that no two adjacent edges receive the same colour. The *chromatic index*  $\chi'(G)$  is the least number of colours required to colour the edges of a graph  $G$  in this sense. Obviously



every edge colouring of a simple graph  $G$  uses at least  $\Delta(G)$  colours. In 1916 König [76] proved that for bipartite graphs the chromatic index equals the maximum degree. Almost fifty years later in 1964 Vizing proved with the help of a sophisticated recolouring technique a fundamental and nowadays classical result in graph theory.

**Vizing's Theorem 34** [122] *Let  $G$  be a graph, then  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + \mu(G)$ . Here  $\mu(G)$  denotes the maximum number of edges joining two vertices in  $G$ . In particular,  $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$  for simple graphs  $G$ .*

It is noteworthy that Vizing's proof of this theorem provides an algorithm with a polynomial worst case running time to colour the edges of a given graph  $G$  using  $\Delta(G) + \mu(G)$  different colours.

The problem of colouring the edges of a graph  $G$  is equivalent to the colouring of the vertices of its *line graph*  $L(G)$  having the edges of  $G$  as its vertices, and two distinct edges of  $G$  are adjacent in  $L(G)$  if they are adjacent in  $G$ . Clearly, every edge colouring of a graph  $G$  is a vertex colouring of  $L(G)$  and vice versa. Moreover, the maximum degree  $\Delta(G)$  of a graph  $G$ , which is non-isomorphic to a triangle, is equal to the clique number  $\omega(L(G))$ . Therefore, Vizing's Theorem can be reformulated in the language of line graphs asserting that for simple graphs  $G$  the bound  $\chi(L(G)) \leq \omega(L(G)) + 1$  is satisfied. Because of Vizing's result, line graphs satisfy this bound  $\omega + 1$  for the chromatic number in a quite natural way. Therefore we call this special upper bound for the chromatic number the *Vizing bound*. Thus, in the following we are interested in graph classes defined in terms of forbidden induced subgraphs admitting the special  $\chi$ -binding function  $f(x) = x + 1$ . Finally, an elegant characterization of line graphs in terms of nine forbidden induced subgraphs (see Figure 2 the so-called Beineke graphs) proven by Beineke [4] in 1968 transfers Vizing's Theorem into a result in the scope of this survey. In fact, the reformulated Vizing Theorem asserts that if a graph  $G$  does not admit one of the Beineke graphs (see Figure 2) as induced subgraph then we have  $\chi(G) \leq \omega(G) + 1$ .

In 1977, Choudom [19] was the first to examine an intriguing question: Does the Vizing bound for the chromatic number also hold for superclasses of line graphs? In particular, these superclasses should be defined by forbidding only a selection of Beineke's nine graphs as induced subgraphs. Choudom determined two superclasses of line graphs for which the Vizing bound for the chromatic number is also true. Both superclasses are defined by only four forbidden induced subgraphs of Beineke's nine graphs. Both sets of forbidden induced subgraphs contain the  $K_{1,3}$  and the  $K_5 - e$ . Thus, Choudom's results extend Vizing's classical result concerning edge colourings. In 1980, Javdekar [61] conjectured that the Vizing bound for the chromatic number holds even for the class of graphs not containing  $K_{1,3}$  and  $K_5 - e$  as an induced subgraph. This superclass of line graphs contains both of Choudom's enlarged graph classes. Kierstead and Schmerl

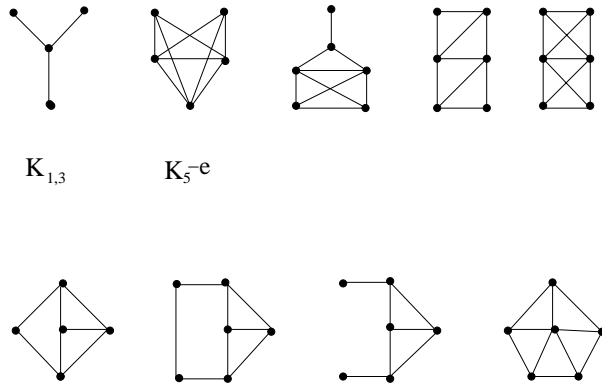


Fig. 2 Beineke graphs

[72] showed 1983 that this conjecture can be reformulated in terms of edge colourings of special multigraphs. Finally, Kierstead [68] proved this conjecture in 1984. This result forms a partial solution to the problem of *finding all pairs of connected forbidden induced subgraphs which imply the Vizing bound for the chromatic number*. A pair  $(A, B)$  of connected forbidden induced subgraphs, which imply the Vizing bound for the chromatic number, such that neither forbidding  $A$  nor forbidding  $B$  is superfluous, is briefly called a *good Vizing-pair*. Moreover, a good Vizing-pair is *saturated*, if for every good Vizing-pair  $(A', B')$  with  $A \subset A'$  and  $B \subset B'$  we have  $A \cong A'$  and  $B \cong B'$ . Based on Erdős' already cited result and a number of certain graphs  $G$  with chromatic number  $\omega(G) + 2$  it is not difficult to obtain the following result.

**Theorem 35** [97] *If  $(A, B)$  is a good Vizing-pair, then  $A$  has to be a tree with  $A \not\subset P_4$  and  $B \in \{K_5 - e, HVN, K_4, K_3, P, D\}$ . (see Figure 3)*

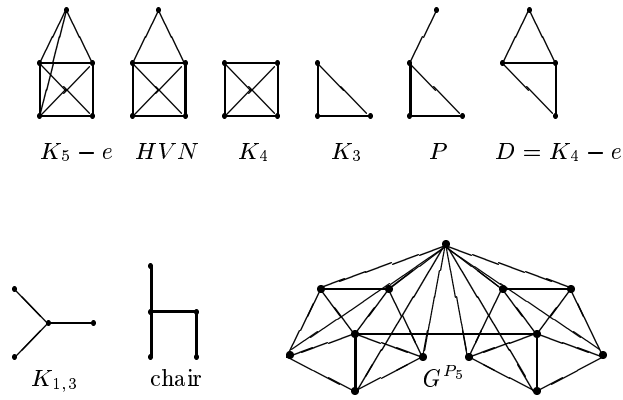
**Observation 36** [97] *The graph  $G^{P_5}$  (see Figure 3) is 5-chromatic,  $K_4$ -free and  $P_5$ -free.*

**Theorem 37** [97] *Let  $A$  be a connected graph such that every  $A$ -free graph  $G$  with  $\omega(G) \leq 3$  satisfies  $\chi(G) \leq \omega(G) + 1 \leq 4$ . Then  $A$  is an induced subgraph of the chair.*

The next result extends Kierstead's generalization of Vizing's Theorem. The proof is very tedious and was carried out in [97].

**Theorem 38** [97] *Let  $B$  be an induced subgraph of the HVN or the  $K_5 - e$  and  $G$  be a  $B$ -free and chair-free graph, then  $\chi(G) \leq \omega(G) + 1$ .*

A subproblem is to determine all pairs  $(A, B)$  of connected graphs such that a graph  $G$  is 3-colourable, if  $G$  does not admit either  $A$  or  $B$  as an induced



**Fig. 3** Extremal graphs

subgraph. Obviously, a necessary condition for a graph to be 3-colourable is the absence of the  $K_4$ . Therefore, w. l. o. g.  $A$  must be an induced subgraph of the  $K_4$  and we have two non-trivial cases  $A \cong K_4$  and  $A \cong K_3$ .

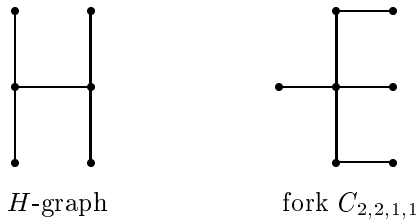
Due to Theorem 35 the  $A$  accompanying graph  $B$ , in order to force  $(A, B)$  to be a good pair, has to be a tree. Applying Seinsche's result [112] that every  $P_4$ -free graph is perfect, we easily deduce that  $(K_4, P_4)$  is a good pair, i. e. if a graph  $G$  does not admit  $K_4$  and  $P_4$  as an induced subgraph, then  $G$  is 3-colourable. The well-known 5-wheel  $W_5 := K_1 \vee C_5$ , the join of a 5-cycle and an isolated vertex, is an example for a 4-chromatic and  $K_4$ -free graph. Here, the *join* of graphs  $G$  and  $H$ , written  $G \vee H$ , is the graph obtained from the disjoint union of  $G$  and  $H$  by adding the edges  $\{uv | u \in V(G), v \in V(H)\}$ . Recall that the companion graph  $B$  to  $K_4$  has to be a tree. Now obviously every acyclic induced subgraph of  $W_5$  is also an induced subgraph of the  $P_4$ . Therefore,  $(K_4, P_4)$  is a saturated pair and the case  $A \cong K_4$  is also settled.

Thus, it remains to study triangle-free graphs which are 3-colourable. More precisely, we want to determine all trees  $T$  such that every triangle-free and  $T$ -free graph is 3-colourable. The starting point is again the reduction of possible trees  $T$ .

**Proposition 39** [97] *The 4-chromatic Mycielski graph  $G_4$  is  $(3K_2)$ -free and  $(K_2 \cup P_3)$ -free.*

In 1955, Gleason and Greenwood [51] determined the Ramsey number  $r(5, 3) = 14$ . Hence, there exists a graph  $G^*$  of order 13 with  $\alpha(G^*) \leq 4$  and  $\omega(G^*) \leq 2$ . Thus, obviously we have  $\chi(G^*) \geq n(G^*)/\alpha(G^*) \geq 13/4$  and  $G^*$  is not 3-colourable.

**Proposition 40** [51] *There exists a triangle-free graph  $G^*$  with  $\alpha(G^*) \leq 4$ , which is not 3-colourable.*



For the next theorem we need to define two special trees. The  $H$ -graph is a tree with six vertices which can be drawn like the capital letter  $H$ . The *fork* is the tree  $C_{2,2,1,1}$ , i. e. a star with four branches with exactly two branches subdivided once. Every tree  $T$ , such that every triangle-free and  $T$ -free graph is 3-colourable, fulfills the inequality  $\alpha(T) \leq 4$ , because of the Proposition 40 of this section. With the Proposition 39 of this section we deduce that  $T$  is  $(3K_2)$ -free and  $(K_2 \cup P_3)$ -free. An easy analysis yields the following theorem.

**Theorem 41** [97] *Let  $T$  be a tree, such that every triangle-free and  $T$ -free graph  $G$  satisfies  $\chi(G) \leq \omega(G) + 1 \leq 3$ . Then  $T \cong H$  or  $T$  is an induced subgraph of the fork.*

Now we are able to present the next results:

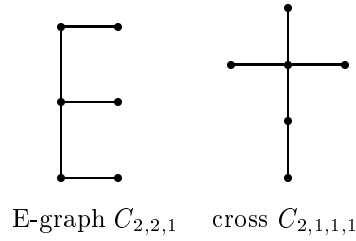
**Theorem 42** [97] *Let  $(A, B)$  be a saturated pair of connected forbidden induced subgraphs implying 3-colourability. Then  $A \in \{K_3, K_4\}$  and  $B \subset B' \in \{P_4, H, \text{fork}\}$ . Moreover, if  $A \cong K_4$ , then  $B \cong P_4$ . In case that  $A \cong K_3$ , then  $B \cong H$  or  $B$  is an induced subgraph of the fork.*

Together with the following theorem, which will be discussed in this subsection, we almost achieve a complete characterization of all saturated pairs  $(A, B)$  implying 3-colourability.

**Theorem 43** [97] *In the following we list up some good pairs  $(A, B)$  of connected forbidden induced subgraphs implying 3-colourability:*

1.  $(K_4, P_4)$ , i. e. with no  $K_4$  and no induced path of order four;
2.  $(K_3, H)$ , where  $H$  is a six-vertex graph drawn like the capital letter  $H$ ; ;
3.  $(K_3, E)$ , where  $E$  is a six-vertex graph drawn like the capital letter  $E$ ;
4.  $(K_3, \text{cross})$ , where the cross is a  $K_{1,4}$  with exactly one edge subdivided once.

*For all above mentioned pairs  $(A, B)$  we also can determine an algorithm that 3-colours every  $A$ -free and  $B$ -free graph.*



In order to complete the characterization, we have to settle the case  $(K_3, fork)$ . For a triangle- and fork-free graph  $G$ , it is an open question, whether  $G$  can be 3-coloured.

**Conjecture 44** [97] *Let  $G$  be a triangle-free and fork-free graph. Then  $\chi(G) \leq \omega(G) + 1 \leq 3$ .*

In the study of the class of triangle-free and  $H$ -free graphs two well-known 3-regular graphs [7], the odd prism and the even Möbius ladder, will play a key role. The *prism*  $Pr_n$  with  $n \geq 3$  consists of two disjoint cycles  $C_1 = v_1v_2 \dots v_nv_1$  and  $C_2 = w_1w_2 \dots w_nw_1$  and the remaining edges are of the form  $v_iw_i$  for every  $i \in \{1, 2, \dots, n\}$ . A prism  $Pr_n$  is *odd*, if  $n$  is odd. The *Möbius ladder*  $Ml_n$  with  $n \geq 2$  is constructed from the cycle  $C = u_1u_2 \dots u_{2n}u_1$  by adding the edges  $u_iu_{i+n}$  for every  $i \in \{1, 2, \dots, n\}$  joining each pair of opposite vertices of  $C$ . The Möbius ladder  $Ml_n$  is *even*, if  $n$  is even. Prisms  $Pr_n$  and Möbius ladders  $Ml_n$  with  $n \geq 4$  are triangle-free. Moreover, prisms  $Pr_n$  and Möbius ladders  $Ml_n$  are  $H$ -free. We give two results that settle the question of 3-colourability for triangle- and  $H$ -free graphs. If  $G$  is a triangle-free and  $H$ -free graph, then our next theorem - a structural result - will enable us to determine a  $\chi(G)$ -colouring of  $G$  with  $\chi(G) \leq \omega(G) + 1 \leq 3$ .

**Theorem 45** [97] *Let  $G$  be a connected triangle-free and  $H$ -free graph, which is not an even Möbius ladder  $Ml_{2l}$  or an odd prism  $Pr_{2l+1}$  with  $l \geq 2$ . Then one of the following properties holds:*

- (i)  $G$  is bipartite;
- (ii)  $\delta \leq 2$ .

The next theorem will reveal that the odd prism  $Pr_{2l+1}$  and the even Möbius ladder  $Ml_{2l}$  with  $l \geq 2$  are saturated graphs in the class of triangle-free and  $H$ -free graphs.

**Theorem 46** [97] *If a connected, triangle-free and  $H$ -free graph  $G$  contains a subgraph  $T$ , which is isomorphic to an odd prism  $Pr_{2l+1}$  or an even Möbius ladder  $Ml_{2l}$  for some  $l \geq 2$ , then  $G \cong T$ .*

Let  $G$  be a connected, triangle-free and  $H$ -free graph of order  $n$ . If  $G$  is isomorphic to an odd prism  $Pr_{2l+1}$  or an even Möbius ladder  $Ml_{2l}$  for some  $l \geq 2$ , then obviously we have  $\chi(G) = 3$ . Furthermore, it is not very difficult to present a 3-colouring for an odd prism  $Pr_{2l+1}$  or an even Möbius ladder

$Ml_{2l}$  for every  $l \geq 2$ . Suppose that  $G$  is not isomorphic to an odd prism  $Pr_{2l+1}$  or an even Möbius ladder for some  $l \geq 2$ . If  $\delta(G) \geq 3$ , then with Theorem 45 we obtain that  $G$  is bipartite. Now assume that  $\delta(G) \leq 2$ . Then again our structural theorem implies the existence of some integer  $r \in \{1, \dots, n\}$  and vertices  $v_1, \dots, v_r$ , such that  $d_{G_1}(v_1) \leq 2$  with  $G_1 := G$ ,  $d_{G_2}(v_2) \leq 2$  with  $G_2 := G_1 - v_1, \dots, d_{G_r}(v_r) \leq 2$  with  $G_r := G_{r-1} - v_{r-1}$  and  $G^* := G_r - v_r$  is the empty graph, if  $n = r$  and otherwise  $G^*$  is (because of the last theorems) a bipartite graph with  $\delta(G^*) \geq 3$ . In the case  $r = n$ , this property is called *2-degenerate*. Hence, we can colour the bipartite induced subgraph  $G^*$  with two colours  $\alpha$  and  $\beta$  and furthermore we can colour the remaining vertices of  $G$  with at most three colours  $\alpha, \beta$  and  $\gamma$  along the inverse ordering  $v_r v_{r-1} \dots v_2 v_1$ . Thus, we have easily constructed a 3-colouring of  $G$ .

**Corollary 47** [97] *Let  $G$  be a triangle-free and  $H$ -free graph, then  $\chi(G) \leq \omega(G) + 1 \leq 3$ .*

Before we study the whole class of triangle-free and fork-free graphs, we will examine two of its subclasses. The first subclass contains the triangle-free and  $E$ -free graphs. A well known class of graphs is the family of nearly bipartite graphs. Here, a graph  $G$  is *nearly bipartite*, if for every vertex  $w$  of  $G$  the graph  $G_w := G - N_G[w]$  is bipartite.

**Theorem 48** [97] *Every triangle-free and  $E$ -free graph is nearly bipartite.*

Note that for every vertex  $w$  of a triangle-free and  $E$ -free graph  $G$  it is very easy to construct a 3-colouring  $\phi$  of  $G$  such that all vertices of  $N_G(w)$  receive the same colour of  $\phi$ .

**Corollary 49** [97] *Let  $G$  be a triangle-free and  $E$ -free graph. Then  $\chi(G) \leq \omega(G) + 1 \leq 3$ .*

Recall Theorem 28 that a triangle-free and cross-free graph  $G$  satisfies  $\chi(G) \leq \omega(G) + 1 \leq 3$ .

**Remark 50** *The proof of Theorem 28 has algorithmic impact likewise to the proof of Brooks' Theorem. Therefore, if  $G$  is a triangle-free and cross-free graph, it is not very difficult to construct a 3-colouring of  $G$ .*

The results on triangle-free and  $E$ -free or cross-free graphs give some evidence that Conjecture 44 is true, i. e. every triangle-free and fork-free graph is 3-colourable. Thus, applying Theorem 41, Corollary 47, Corollary 49 and Theorem 28, we obtain that the following set  $(K_3, B)_3$  contains the saturated graphs  $B$ , such that a triangle-free and  $B$ -free graph  $G$  is 3-colourable. If Conjecture 44 is true, we have:

$$(K_3, B)_3 = \{H; fork\},$$

and otherwise, if Conjecture 44 is not true, we have:

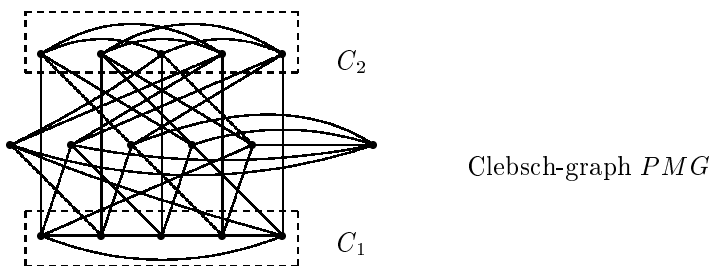
$$(K_3, B)_3 = \{H; E; cross\}.$$

Olariu [95] discovered that the connected, paw-free and non-triangle-free graphs are exactly the complete multipartite graphs. Hence, we can transform the results concerning triangle-free graphs to those concerning paw-free graphs. Recall that it is proved in [97] that the pairs  $(K_5 - e, chair)$ ,  $(HVN, chair)$  and  $(P, H)$  are saturated. Finally, motivated by the result that  $(K_4 - e)$ -free and  $P_5$ -free graphs also satisfy the Vizing bound, we conjectured in [97] that the following set  $(A, B)_{\omega+1}$  contains all saturated pairs  $(A, B)$  implying the Vizing bound:

$$(A, B)_{\omega+1} = \{(K_5 - e, chair); (HVN, chair); (P, H); (P, fork); (K_4 - e, H); (K_4 - e, fork)\}.$$

## 5 $P_l$ -free graphs

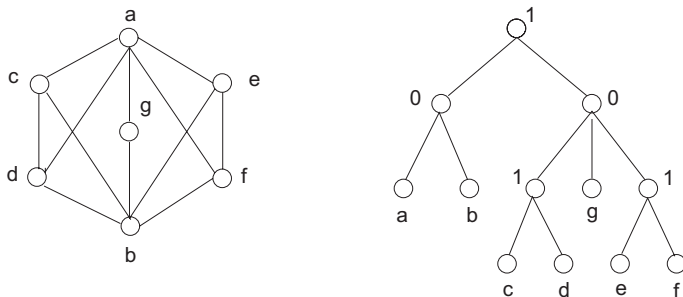
This section is devoted to  $P_l$ -free graphs. In the latter sections we already mentioned this class partially. E. g. the family of  $P_l$ -free graphs admits the  $\chi$ -binding function  $f(\omega(G)) = (l - 1)^{\omega(G)-1}$ . Furthermore, with a slight modification of Gyárfás' proof of the latter result we obtained in Section 3 the  $\chi$ -binding function  $f(2) = l - 2$  for the family of  $P_l$ -free and triangle-free graphs. This guarantees that every triangle-free and  $P_6$ -free graph is 4-colourable. However, not every triangle-free and  $P_6$ -free graph is 3-colourable: two exceptions are the well-known 4-chromatic Mycielski-Grötzsch graph  $G_4$  and the Clebsch graph  $PMG$  (cf. the figure below).



In [103] we extended the theorem of Gyárfás in this special case and also extended a result due to Sumner asserting that every  $P_5$ -free and triangle-free graphs is 3-colourable. Two vertices  $u$  and  $v$  of a graph  $G$  are called *similar*, if  $N_G(u) \subseteq N_G(v)$  or  $N_G(v) \subseteq N_G(u)$  holds.

**Theorem 51** [103] *Let  $G$  be a connected triangle-free and  $P_6$ -free graph which is not 3-colourable and contains no similar vertices. Then  $G$  contains the Mycielski-Grötzsch graph  $G_4$  as induced subgraph and is an induced subgraph of the 16-vertex Clebsch-graph  $PMG$ .*

Thus, we can easily decide whether a given triangle-free and  $P_6$ -free graph can be 3-coloured. This motivates the question whether  $k$ -colourability can be decided in polynomial time for the family of  $P_l$ -free graphs. Moreover,



**Fig. 4** Example of a cograph and its cotree representation

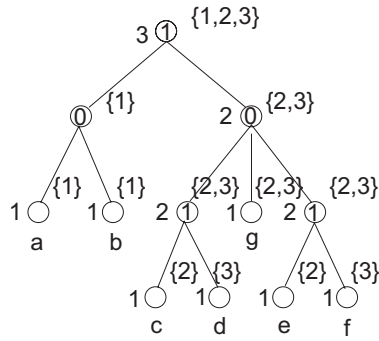
in case of an affirmative answer, it is also of interest, whether a  $k$ -colouring can also be determined in polynomial time. For small values  $l \in \{1, 2, 3\}$  chromatic aspects of  $P_l$ -free graphs are trivial, since the only  $P_1$ -free graph is the empty graph,  $P_2$ -free graphs are edgeless and  $P_3$ -free graphs only contain disjoint cliques. Thus, every  $P_3$ -free graph  $G$  can be easily coloured with  $\omega(G)$  colours.

The first non-trivial case are  $P_4$ -free graphs, a subclass of perfect graphs. Therefore, we have  $\chi(G) = \omega(G)$  for every  $P_4$ -free graph (and every  $P_4$ -free graph  $G$  is  $\omega(G)$ -colourable). In the following we want to sketch an efficient algorithm to  $\omega(G)$ -colour a given  $P_4$ -free graph. The key to obtain this algorithm is the possibility to represent a given  $P_4$ -free graph by a data structure called a cotree. A graph is called a *cograph*, if for every induced subgraph  $H$  of  $G$  with at least two vertices either  $H$  or  $\bar{H}$  is disconnected. Seinsche [112] proved that a graph  $G$  is  $P_4$ -free if and only if  $G$  is a cograph. With this result it is not very difficult to deduce perfectness of  $P_4$ -free graphs. In [30] Corneil, Lerchs and Burlingham discovered a recursive definition of cographs:

- An isolated vertex is a cograph.
- If  $G_1, \dots, G_r$  are cographs, then the disjoint union  $G_1 \cup G_2 \cup \dots \cup G_r$  is likewise a cograph.
- If  $G$  is a cograph, then likewise  $\bar{G}$  is a cograph.

This recursive characterization provides a canonical decomposition scheme for cographs: a disconnected cograph is decomposed into its components and a connected cograph into the components of its complement. This can be iterated until only isolated vertices are left. Such a decomposition can be represented by a tree - the *cotree*. Each of these operations corresponds to an inner vertex of the tree and the obtained isolated vertices, the vertices of the cograph, are now the leaves of the cotree. The next figure gives an example. Building up a cotree for a given cograph can be done in linear time





**Fig. 5** Example of an  $\omega$ -colouring of a cograph

[31]. Very recently Corneil [32] announced a simple linear time procedure to build up a cotree for a given cograph based on lexicographic breath first search.

Using the cotree representation of a  $P_4$ -free graph  $G$  it is not very difficult to determine  $\omega(G)$  and an  $\omega(G)$  colouring. Here we just give a 3-colouring for our example in the above figure. For small values  $l \in \{1, 2\}$  the question, whether an arbitrarily graph is  $l$ -colourable, is trivial. For  $l = 1$  it remains to check whether the input graph is edgeless; for  $l = 2$  it has to be clarified whether the graph in consideration is bipartite. Thus 3-colourability is the first non-trivial case. A necessary condition for a graph  $G$  to be 3-colourable is that for every vertex of  $G$  its neighbourhood induces a bipartite graph. The question whether a given graph  $G = (V, E)$  is 3-colourable can be reformulated as an instance of the special satisfiability problem 3-SAT, which was proved to be NP-complete by Cook [29].

**3-SAT.** *Let  $C$  be a collection of  $m$  clauses over the set  $V$  of  $n$  Boolean variables such that every clause has exactly three literals. Is there a truth assignment for  $C$  that satisfies all clauses?*

Let  $V = \{1, \dots, n\}$  be the vertex set of  $G$  and  $E = \{e_{ij} | i \text{ adjacent to } j\}$  be the edge set of  $G$ . Now we introduce  $3n$  Boolean variables:  $x_i^{(l)}$  for every  $i \in \{1, \dots, n\}$  and  $l \in \{1, 2, 3\}$  with  $x_i^{(l)} = \text{true}$  corresponding to the statement that the vertex  $i$  receives the colour  $l$ . Now each edge  $e_{ij}$  of  $G$  is represented by three clauses, namely  $(\bar{x}_i^{(l)} \vee \bar{x}_j^{(l)})$  for  $l \in \{1, 2, 3\}$  and each vertex  $x_i$  of  $G$  is represented by five clauses, namely  $(x_i^{(1)} \vee x_i^{(2)} \vee x_i^{(3)})$ ,  $(\bar{x}_i^{(1)} \vee \bar{x}_i^{(2)} \vee \bar{x}_i^{(3)})$ ,  $(x_i^{(1)} \vee \bar{x}_i^{(2)} \vee \bar{x}_i^{(3)})$ ,  $(\bar{x}_i^{(1)} \vee x_i^{(2)} \vee \bar{x}_i^{(3)})$ ,  $(\bar{x}_i^{(1)} \vee \bar{x}_i^{(2)} \vee x_i^{(3)})$ . Now it is not very difficult to check that a satisfying truth assignment of this 3-SAT instance corresponds to a 3-colouring of  $G$ .

Our interest on this reformulation is justified by its 2-SAT impact, if we consider a 3-precoloured dominating set of the graph. Here, a subset  $D \subseteq V(G)$  of a graph  $G$  is called a *dominating set*, if every vertex  $x \in V(G) - D$  is adjacent to at least one vertex in  $D$ . The following approach for 3-colourability was used in [103] (see also [38]). The basic idea of the approach states the following: Let  $D$  be a dominating set in a graph  $G = (V, E)$ . Then we can test whether a 3-colouring of  $D$  can be extended to a 3-colouring of  $G$  by constructing a corresponding 2-satisfiability formula with at most  $3|V|$  variables and  $3|E| + 5|V|$  clauses. Since 2-satisfiability is solvable in linear time  $O(3|E| + 5|V|)$ , we deduce the next result.

**Corollary 52** [103] *For a graph  $G = (V, E)$  with a dominating set  $D$ , we can decide 3-colourability and determine a proper 3-colouring in time  $O(3^{|D|} * |E|)$ .*

A complete subgraph of  $G$  is called a dominating clique if the vertices comprise a dominating set. In [1], Bacsó and Tuza showed that every  $P_5$ -free, connected graph contains a dominating clique or a dominating set inducing a  $P_3$ . Having tested, whether a  $P_5$ -free, connected graph  $G$  is  $K_4$ -free, the result of Bacsó and Tuza guarantees the existence of a dominating set of  $G$  of size at most three. Together with Corollary 52 we then obtain the following result.

**Proposition 53** [103] *3-colourability can be decided and, if so, a proper 3-colouring can be determined in polynomial time for  $P_5$ -free graphs.*

An accompanying algorithm (see [89]) has a running time  $O(|V|^\alpha)$  for  $G = (V, E)$ . Here,  $2 < \alpha < 2,36$  is the exponent given by the fast matrix multiplication.

**Theorem 54** [100] *Let  $G$  be a  $P_6$ -free graph. Then the 3-colourability of  $G$  can be decided and, if so, a proper 3-colouring can be determined in time  $O(|E| * |V|^\alpha)$ .*

*Sketch of the proof:* Since  $G$  is  $P_6$ -free there exists no hole of length  $\geq 7$  in  $G$ . Thus, if  $G$  contains an odd hole  $C$ , then  $C$  is a 5-hole. Checking whether a given graph  $G = (V, E)$  contains a 5-hole can be performed in time  $O(|E| * |V|^\alpha)$ ; this is the most expensive part of the algorithm. In case that there exists a 5-hole  $C$ , we analyze the structure of  $G$ . We will extend the approach based on Corollary 52 and consider a fixed precolouring of  $C$  with three colours, say  $v_1$  and  $v_3$  with colour 1,  $v_2$  and  $v_4$  with colour 2, and  $v_5$  with colour 3. Furthermore, we extend this precolouring of  $C$  until no uncoloured vertex of  $G$  is adjacent to two differently precoloured vertices and until there exists no diamond twin such that one of its vertices is a precoloured vertex and the other vertex is uncoloured. Here, a *diamond twin* is the pair of non-adjacent vertices of a diamond  $K_4 - e$ . Observe that in any feasible 3-colouring of a graph, diamond twins have to receive the same colour. If there occurs a colour conflict, then  $G$  is not 3-colourable. Otherwise, let  $PC$  denote the set of precoloured vertices of

$G$ . If  $PC$  is a dominating set, then we can apply Corollary 52 in order to test in running time  $O(|E|)$ , whether the remaining graph is 3-colourable. Therefore, assume that  $PC$  is not a dominating set of  $G$ . Hence, there exist vertices of  $G$  not being dominated by vertices of  $PC$ . Thus, the encoding of the question whether our given graph  $G = (V, E)$  is 3-colourable as an instance of the special satisfiability problem 3-SAT does not reduce to a special satisfiability problem 2-SAT by the precolouring of  $PC$ . Especially, the vertices not dominated by  $PC$  correspond to the remaining 3-clauses. Observe that  $V(G) = \{v \in V(G) \mid \exists \bar{v} \in V(C) : \text{dist}(v, \bar{v}) \leq 3\}$ . Let  $Q$  be the set  $V - N_G[PC]$  of vertices not dominated by a precoloured vertex. Obviously,  $Q \subset N_2 \cup N_3$ . A tedious analysis of vertices of  $Q$  yields on one hand the remaining 2-clauses and on the other hand a degenerated search tree in order to settle this subcase within the above running time bound.

Since  $G$  is 5-hole-free in the remaining subcase we easily deduce that  $G$  is perfect and due to its  $K_4$ -freeness hence 3-colourable. A 3-colouring can be obtained within the time bound by an algorithm of Tucker [118],[119].■

Very recently Sgall and Woeginger [113] also studied chromatic aspects of graphs without long induced paths. They were able to proof that the 4-colourability decision problem is NP-complete for  $P_{12}$ -free graphs and the 5-colourability decision problem is NP-complete for  $P_8$ -free graphs. Finally, we summarize the results and open cases for the  $k$ -colourability problem in the following table with  $n = |V|$ ,  $m = |E|$  for a given  $P_l$ -free graph  $G = (V, E)$ .

$l \setminus k$	3	4	5	6	7	8	...	12	...
3	$O(m)$	$O(m)$	$O(n^\alpha)$	$O(mn^\alpha)$	?	?	?	?	...
4	$O(m)$	$O(m)$	?	?	?	?	?	NPc	...
5	$O(m)$	$O(m)$	?	?	?	NPc	NPc	NPc	...
6	$O(m)$	$O(m)$	?	?	?	NPc	NPc	NPc	...
7	$O(m)$	$O(m)$	?	?	?	NPc	NPc	NPc	...
...	...	...	...	...	...	...	...	...	...

Table 1: Computational complexity of  $k$ -COL for  $P_l$ -free graphs.

At the end of this section we want to highlight a few of these open problems. The first one seems to be a tractable one, whereas the second and third problem will probably require a new approach.

**Problem 55** *Is  $4 - \text{COL}(P_5)$  solvable in polynomial time?*

Recall Observation 13 that there exists no linear  $\chi$ -binding function for the class of  $P_5$ -free graphs.

**Problem 56** *Is  $3 - \text{COL}(P_7)$  solvable in polynomial time?*

An intriguing problem is to establish for a non-trivial  $k$  (maybe a function of the order of the graphs in consideration, e. g. a sublinear bound  $\log(n)$ ) that  $3 - \text{col}(P_k)$  remains NP-complete.

**Problem 57** *Find (the smallest) an integer  $k \geq 7$  such that  $3 - \text{COL}(P_k)$  remains NP-complete!*

## 6 Graphs with prescribed cycle lengths

The most famous result relating induced (cycle) subgraphs to vertex colouring is the Strong Perfect Graph Theorem as already mentioned in Section 2. In this context related conjectures and results by Gyárfás were presented in Sections 2 and 3. Also the excellent survey of Hayward and Reed [58] on graph classes defined in terms of forbidden holes and antiholes conditions is a valuable source. In this section we focus on chromatic aspects of graphs with prescribed cycle lengths rather than considering forbidden subgraph conditions. Starting with prescribed (non-induced!) cycle lengths constraints we proceed with prescribed induced cycle lengths constraints.

### 6.1 Cycle lengths

In [41], Bollobás and Erdős asked the following: *Let us denote by  $C_o(G)$  the set of odd cycle lengths in a graph  $G$ , i. e.  $C_o(G) = \{2m + 1 : G \text{ contains a cycle of length } 2m + 1, \text{ for } m \geq 1\}$ . If  $|C_o(G)| = k$ , is it true that  $\chi(G) \leq 2k + 2$ , with equality if and only if  $G$  contains a complete graph  $K_{2k+2}$ ?* An affirmative answer to this problem was given by Gyárfás. Since every  $k$ -degenerate graph is  $(k + 1)$ -colourable, he even proved a stronger result:

**Theorem 58** [55] *Every graph  $G$  with  $|C_o(G)| = k$  is  $(2k + 1)$ -degenerate. If  $G$  is a 2-connected graph with minimum degree at least  $2k + 1$ , then  $|C_o(G)| = k \geq 1$  implies  $G = K_{2k+2}$ .*

Also, we will give an affirmative answer to the analogous problem for even cycle lengths: *Let us denote by  $C_e(G)$  the set of even cycle lengths in a graph  $G$ , i. e.  $C_e(G) = \{2m : G \text{ contains a cycle of length } 2m, \text{ for } m \geq 2\}$ . If  $|C_e(G)| = k$ , is it true that  $\chi(G) \leq 2k + 3$ , with equality if and only if  $G$  contains a complete graph  $K_{2k+3}$ ?* In [91] a polynomial vertex-colouring algorithm called MAXBIP was presented. Based on MAXBIP the following results can be obtained:

**Theorem 59** [91] *Let  $G$  be a 2-connected graph with  $|C_o(G)| = r$  and  $|C_e(G)| = s$ . Then the algorithm MAXBIP finds a proper vertex  $k$ -colouring of  $G$  with  $k \leq \min\{2r + 2, 2s + 3\}$  colours.*

This leads to the following unexpected and surprising corollary:

**Corollary 60** [91] *Let  $G$  be a graph with  $|C_o(G)| = r$  and  $|C_e(G)| = s$ , then  $\chi(G) \leq \min\{2r + 2, 2s + 3\} \leq r + s + 2$ .*

This result is best possible in the following sense.

**Theorem 61** [91] *Let  $G$  be a graph with  $|C_e(G)| = s$ , then  $\chi(G) \leq 2s + 3$ . Furthermore, if  $\chi(G) = 2s + 3$ , then  $G$  contains a  $K_{2s+3}$ . Let  $G$  be a graph with  $|C_o(G)| = r$ , then  $\chi(G) \leq 2r + 2$ . Furthermore, if  $\chi(G) = 2r + 2$ , then  $G$  contains a  $K_{2r+2}$ .*

We now describe the vertex-colouring algorithm MAXBIP. Let us assume in the sequel that  $G$  is 2-connected. The following algorithm will construct a sequence of vertex disjoint MAXimal (induced) BIPartite subgraphs  $B_1, B_2, \dots, B_m, m \geq 1$ , of  $G$  such that  $V(G) = \bigcup_{i=1}^m V(B_i)$ .

**Algorithm MAXBIP**

INPUT: a 2-connected graph  $G$

- STEP 1 Choose an arbitrarily vertex  $x_1 \in V(G)$  and add successively vertices  $x_2, x_3, \dots$  to obtain a connected maximal bipartite subgraph  $G[V(B_1)]$ . Let  $S := V(B_1)$  and  $T := N(B_1) \setminus S$ .
- STEP 2 Successively place every vertex of  $T$  in the smallest  $B_i$  such that  $G[V(B_i)]$  is bipartite. Let  $S := V(B_1) \cup N(B_1)$  and  $R := V(G) \setminus S$ .
- STEP 3 If  $R = \emptyset$ , then STOP. If  $R \neq \emptyset$ , then let  $j$  be the smallest integer such that  $N(V(B_j)) \setminus S \neq \emptyset$ . Extend the components of  $G[V(B_j)]$  by successively adding vertices to obtain a maximal bipartite subgraph  $B_j^*$  in  $G[V(B_j) \cup (N(V(B_j)) \setminus S)]$ . Set  $B_j := B_j^*$ . Let  $B_j$  play the role of  $B_1$  in STEP 1. Let  $S := S \cup V(B_j)$  and  $T := N(B_j) \setminus S$ . As in STEP 2 now successively place every vertex of  $T$  in the smallest  $B_i, i \geq j$ , such that  $G[V(B_i)]$  is bipartite. Let  $S := S \cup N(B_j)$  and  $R := V(G) \setminus S$  and repeat STEP 3.

OUTPUT  $B_1, B_2, \dots, B_m, m \geq 1$

Let us colour the vertices of  $B_i, 1 \leq i \leq m$ , so that the first vertex which is placed in some  $B_i, 1 \leq i \leq m$ , will be coloured with the smallest available colour  $2q - 1$  according to FIRST-FIT. Then all vertices of this subgraph  $B_i$  will be coloured properly with colours  $2q - 1$  and  $2q$ .

For given sets  $C_o(G)$  and  $C_e(G)$  it may be possible to improve the upper bounds for the chromatic number  $\chi(G)$ . If  $|C_o(G)| = 1$ , then Theorem 61 is leading to the following corollary.

**Corollary 62** [91] *Let  $G$  be a 2-connected graph with  $|C_o(G)| = 1$ . Then*

- (i)  $\chi(G) = 4$  if  $G$  contains  $K_4$  and
- (ii)  $\chi(G) = 3$  if  $G$  contains no  $K_4$ .

In [126], the chromatic number of graphs with  $C_o(G) = \{3, 5\}$  were characterized. Recall the wheel  $W_5$  of order 6 (one center vertex which is adjacent to all vertices of a cycle of order 5).

**Theorem 63** [126] *Let  $G$  be a 2-connected graph with  $C_o(G) = \{3, 5\}$ . Then*

- (i)  $\chi(G) = 6$  if  $G$  contains a  $K_6$ ,
- (ii)  $\chi(G) = 5$  if  $G$  contains a  $K_5$  but no  $K_6$ ,
- (iii)  $\chi(G) = 4$  if  $G$  contains a  $K_4$  but no  $K_5$ ,
- (iv)  $\chi(G) = 4$  if  $G$  contains a  $W_5$  but no  $K_4$  and
- (v)  $\chi(G) = 3$  if  $G$  contains no  $W_5$  and no  $K_4$ .

## 6.2 Induced cycle lengths

For a given simple graph we can also consider its set of induced cycle lengths. In [99], colouring algorithms and upper bounds for the chromatic number of some classes were obtained in terms of given induced cycle lengths. Sumner [116] showed that triangle-free and  $P_5$ -free or triangle-free,  $P_6$ -free and  $C_6$ -free graphs are 3-colourable.

For  $t \geq 5$  define  $\mathcal{G}_t$  as the class of all triangle-free graphs which are  $P_t$ -free and  $C_i$ -free for  $6 \leq i \leq t$ . For  $k \geq 1$  and  $3 \leq n_1 < n_2 < \dots < n_k$  let  $\mathcal{G}^I(n_1, n_2, \dots, n_k)$  be the class of all graphs whose induced cycle lengths are equal to one of  $n_1, n_2, \dots, n_k$ . Thus

$$\mathcal{G}_5 \subset \mathcal{G}_6 \subset \mathcal{G}_7 \subset \dots \subset \mathcal{G}^I(4, 5)$$

and all graphs  $G$  of  $\mathcal{G}_5$  and  $\mathcal{G}_6$  are 3-colourable by Sumner's result. Note that all graphs of  $\mathcal{G}_t$  have diameter at most  $t - 2$  whereas graphs of  $\mathcal{G}^I(4, 5)$  can have arbitrary diameter. Does 3-colourability still hold for these superclasses of  $\mathcal{G}_5$  and  $\mathcal{G}_6$ ? The next theorem obtained in [99] states that all graphs of  $\mathcal{G}^I(4, 5)$  are 3-colourable. Moreover, we can guarantee a certain 3-colouring with some additional properties.

For a fixed integer  $p \geq 2$  we call a graph  $G \in \mathcal{G}^I(4, 2p + 1)$  *3\*-colourable with root  $v$* , if there is a 3-colouring of  $G$  such that  $G[N_G^p(v)]$  is coloured with two colours, where  $N_G^p(v)$  is the set of vertices having distance  $p$  from  $v$ . Observe that this definition implies the following useful property: If  $G$  is 3\*-colourable with root  $v$ , then we can choose a 3-colouring s. t.  $G[N_G^i(v)]$  is coloured monochromatic for every  $1 \leq i < p$  and  $G[N_G^p(v)]$  is coloured with two colours. Furthermore, if this property holds for every vertex of  $G \in \mathcal{G}^I(4, 2p + 1)$ , then we call  $G$  *3\*-colourable*. This definition is motivated by the following observation.

If  $G_1, G_2 \in \mathcal{G}^I(4, 2p + 1)$  and  $v_i \in G_i$  for  $i = 1, 2$ , then the new graph  $G^*$  with vertex set  $V(G^*) = V(G_1 - v_1) \cup V(G_2 - v_2)$  and edge set  $E(G^*) = E(G_1 - v_1) \cup E(G_2 - v_2) \cup \{u_1 u_2 \mid u_i \in N_{G_i}(v_i) \text{ for } i = 1, 2\}$  is likewise a member of  $\mathcal{G}^I(4, 2p + 1)$ . The invariance of  $\mathcal{G}^I(4, 2p + 1)$  concerning this graph operation is the reason for the equivalence of 3\*- and 3-colourability for the class  $\mathcal{G}^I(4, 2p + 1)$ .

**Theorem 64** [99] *Every graph of  $\mathcal{G}^I(4, 2p + 1)$  with  $p \geq 2$  is 3\*-colourable.*

The proof of this theorem is based on decomposition and provides a polynomial time algorithm to 3\*-colour a given graph  $G \in \mathcal{G}^I(4, 2p + 1)$ . The class  $\mathcal{G}^I(4, 5)$  is a canonical extension of  $\mathcal{G}^I(4)$ , which are the well-known chordal bipartite graphs (e. g. see [11]). Very recently Brandt [12] examined the maximal (with respect to edge addition) triangle-free members of the class  $\mathcal{G}^I(4, 5)$  with emphasis on graph homomorphisms. Moreover, he introduced for members of  $\mathcal{G}^I(4, 5)$  the terminology of *chordal triangle-free graphs*.

Theorem 64 and Theorem 17 form partial solutions for the following conjecture of Hoàng and McDiarmid [59].

**Conjecture 65** [59] *If  $G$  is a triangle-free graph with at least one hole and  $h$  is the length of its longest hole then  $\chi(G) \leq h - 2$ .*

Motivated by Theorem 64 we also considered in [99] the classes  $\mathcal{G}^I(2q, 2p+1)$  and  $\mathcal{G}^I(2p'+1, 2q')$  for  $q, q' \geq 3$  and  $p, p' \geq 2$ , which are contained in the larger class  $\mathcal{G}^I(n_1, n_2, \dots, n_k)$  with  $n_1 \geq 5$ . Recall that a graph  $G$  is  $r$ -degenerate, if there exists an ordering  $(v_1, \dots, v_n)$  of  $V(G)$  s. t.  $d_{G[\{v_i, \dots, v_n\}]}(v_i) \leq r$  for all  $1 \leq i \leq n$ .

**Theorem 66** [99] *Every graph of  $\mathcal{G}^I(n_1, n_2, \dots, n_k)$  with  $k \geq 1$  and  $n_1 \geq 5$  is  $(k+1)$ -degenerate. Especially, every vertex  $v$  of  $G$  being an endvertex of a longest induced path of  $G$  satisfies  $d_G(v) \leq k+1$ .*

**Corollary 67** [99] *Every graph of  $\mathcal{G}^I(n_1, n_2, \dots, n_k)$  with  $k \geq 1$  and  $n_1 \geq 5$  is  $(k+2)$ -colourable.*

Obviously, Corollary 67 is best possible for  $k = 1$ . But for  $k = 2$  we are able to improve Corollary 67. For the next theorem we need to recall the definition of the famous Petersen graph  $P^*$ . This 3-regular, non-bipartite graph  $P^*$  of order 10 is a member of the class  $\mathcal{G}^I(5, 6)$ . The Petersen graph  $P^*$  consists of two disjoint induced 5-cycles  $C^1 = a_0a_1a_2a_3a_4a_0$  and  $C^2 = b_0b_1b_2b_3b_4b_0$  and the additional edges  $a_0b_0, a_1b_3, a_2b_1, a_3b_4$  and  $a_4b_2$ . Obviously  $P^*$  is 3-colourable.

**Theorem 68** [99] *Every graph  $G$  of  $\mathcal{G}^I(2q, 2p+1)$  or  $\mathcal{G}^I(2p'+1, 2q')$  with  $q, q' \geq 3$  and  $p, p' \geq 2$  fulfills at least one of the following properties:*

1.  $G$  is bipartite;
2.  $G$  satisfies  $\delta(G) \leq 2$ ;
3.  $G \in \mathcal{G}^I(5, 6)$  and one of the following properties holds:
  - (a)  $G \cong P^*$ ;
  - (b)  $G$  contains a clique cutset, i. e. a  $K_1$  or a  $K_2$  clique cutset.

Testing whether  $G$  is bipartite, has minimal degree two or contains a complete cutset of size at most two can be done very efficiently. Moreover, if  $G \in \mathcal{G}^I(5, 6)$  is non-bipartite,  $\delta(G) \geq 3$  and contains no complete cutset, then  $G \cong P^*$ , which obviously is 3-colourable. Hence, the last theorem provides a polynomial time algorithm to 3-colour a given graph  $G \in \mathcal{G}^I(2q, 2p+1)$  or  $G \in \mathcal{G}^I(2p'+1, 2q')$  with  $q, q' \geq 3$  and  $p, p' \geq 2$ . This algorithm (recursively) applies the fact that the graph (in question) is bipartite, has a vertex of degree at most two, is isomorphic to the Petersen graph or can be decomposed into two smaller graphs according to a complete cutset of size at most two.

**Corollary 69** [99] *Every graph  $G$  of  $\mathcal{G}^I(2q, 2p+1)$  or  $\mathcal{G}^I(2p'+1, 2q')$  with  $q, q' \geq 3$  and  $p, p' \geq 2$  is 3-colourable.*

Now we consider the related problem of finding a smallest  $\chi$ -binding function  $f^*$  for  $\mathcal{G}^I(n_1, n_2)$  and for completeness also for its subclasses  $\mathcal{G}^I(n_i)$  with  $i = 1, 2$ .

$\begin{array}{c} \rightarrow \\ n_1, n_2 \\ \downarrow \end{array}$	3	4	odd $\geq 5$	even $\geq 6$
3	$f^*(\omega) = \omega$ chordal	$\nexists$ linear $f^*$ Theorem 22 Conj.[54] : $\exists f^*$	$f^*(\omega) \geq \left(\frac{n_2+1}{n_2-1}\right)\omega$ Conj.: " = "	$f^*(\omega) = \omega$ Rusu [106]
4		$f^*(\omega) = \omega \leq 2$ chordal bipartite	$f^*(\omega) = \omega + 1 \leq 3$ Theorem 64	$f^*(\omega) = \omega \leq 2$ $\subset$ bipartite
odd $\geq 5$			$f^*(\omega) = \omega + 1 \leq 3$ Markossyan,... [88]	$f^*(\omega) = \omega + 1 \leq 3$ Corollary 69
even $\geq 6$				$f^*(\omega) = \omega \leq 2$ $\subset$ bipartite

Table 2:  $\chi$ -binding function  $f^*$  for  $\mathcal{G}^I(n_1, n_2)$ .

For convenience we drop the condition that  $n_1$  is always smaller than  $n_2$  in the definition of  $\mathcal{G}^I(n_1, n_2)$ .

(I)  $n_1, n_2$  even :

For even  $n_1$  and  $n_2$  all graphs of  $\mathcal{G}^I(n_i)$  with  $i = 1, 2$  and  $\mathcal{G}^I(n_1, n_2)$  are bipartite and thus perfect with  $f^*(\omega) = \omega \leq 2$ .

(II)  $n_1$  even,  $n_2$  odd : (A)  $n_2 \geq 5$  :

By our previous results every graph of  $\mathcal{G}^I(n_2)$  and  $\mathcal{G}^I(n_1, n_2)$  is 3-colourable, i. e. with  $\omega \leq 2$  we have  $f^*(\omega) = \omega + 1 \leq 3$ .

(II)  $n_1$  even,  $n_2$  odd : (B1)  $n_2 = 3$  and  $n_1 \geq 6$  :

Due to Rusu [106] all members of a superclass of  $\mathcal{G}^I(3, 2q)$  are perfect for any  $q \geq 3$ . Hence, we also have  $f(\omega) = \omega$ . A well-known subclass of  $\mathcal{G}^I(3, 2q)$  is  $\mathcal{G}^I(3)$  containing the chordal graphs.

(II)  $n_1$  even,  $n_2$  odd : (B2)  $n_2 = 3$  and  $n_1 = 4$  :

In Observation 22 we already considered the class  $\mathcal{G}^I(3, 4)$  and we demonstrated that there exists no linear  $\chi$ -binding function for  $\mathcal{G}^I(3, 4)$ . It is noteworthy that  $\mathcal{G}^I(3, 4)$  contains all weakly triangulated graphs.

(III)  $n_1, n_2$  odd : (C1)  $n_1, n_2 \geq 5$  :

Markossyan, Gasparyan and Reed [88] showed that all triangle-free and even-hole-free graphs are 2-degenerate and thus are 3-colourable. Hence,  $f^*(\omega) = \omega + 1 \leq 3$  is a  $\chi$ -binding function for  $\mathcal{G}^I(n_1)$  and  $\mathcal{G}^I(n_1, n_2)$ .

(III)  $n_1, n_2$  odd : (C2)  $n_1 = 3$  :

It is an open problem, whether there exists a linear  $\chi$ -binding function for  $\mathcal{G}^I(3, n_2)$ . The graph-sequence  $G_r = C_{n_2}[K_r]$ , the lexicographic product



of the odd cycle  $C_{n_2}$  and the complete graph  $K_r$ , reveals that we have  $f^*(\omega) \geq ((n_2 + 1)/(n_2 - 1))\omega$  for every  $\chi$ -binding function. We expect that  $f^*(\omega) = ((n_2 + 1)/(n_2 - 1))\omega$ .

**Problem 70** *Does there exist a linear  $\chi$ -binding function for  $G^I(3, n_2)$  with  $n_2$  odd  $\geq 5$ ?*

## 7 Computational complexity

In this part of the survey we summarize related computational complexity results. The basic problems can be defined as follows:

COLOURABILITY (for short COL)

Input: A graph  $G$  and a positive integer  $k$ .

Question: Is  $G$   $k$ -colourable?

$k$ -COLOURABILITY (for short  $k$ -COL)

Input: A graph  $G$ .

Question: Is  $G$   $k$ -colourable?

The  $k$ -COL problem is solvable in polynomial time for  $k \leq 2$ , i. e. 1-COL is the trivial decision problem, whether the input graph is edgeless and 2-COL, whether the input graph is bipartite. But  $k$ -COL remains *NP*-complete for  $k = 3$  [47]. Observe that for each positive integer  $k$  the  $k$ -COL problem reduces to the COL problem. In addition for every  $k$  the  $k$ -COL problem reduces to the  $(k + 1)$ -COL problem. Conversely, a theorem of Lovász [84] asserts that COL reduces to 3-COL. Thus, all problems 3-COL, 4-COL, ..., COL are equivalent. Recently, Bodlaender [8] established a direct proof of the NP-completeness of a variant of the COL-problem, i. e. a generic proof similar to Cook's proof of the NP-completeness of SATISFIABILITY. In Section 6 we already collected computational complexity results on graphs without long induced paths. There is a variety of special graph classes for which the 3-COL problem remains NP-complete:

- planar graphs with maximum degree four [47];
- triangle-free and  $K_{1,5}$ -free graphs [87],
- even for triangle-free, 4-regular graphs [87];
- 4-regular, 2-connected, diamond- and claw-free graphs [60].

In [38] Edwards has shown that the  $k$ -COL problem can be solved in polynomial time for dense graphs. For each fixed integer  $k$  and rational number  $\alpha$  with  $0 \leq \alpha < 1$ , we define problem  $\Pi(k, \alpha)$  as follows:

$\Pi(k, \alpha)$

Input: A graph  $G = (V, E)$  with  $|V| = n$  and  $\delta(G) \geq \alpha n$ .

Question: Is  $G$   $k$ -colourable?

Based on the *domination approach* already described in Section 6 Edwards proved the following result.

**Theorem 71** [38] *Let  $k \geq 3$  be an integer. Then,*

- if  $0 \leq \alpha \leq \frac{k-3}{k-2}$ ,  $\Pi(k, \alpha)$  is NP-complete;
- if  $\alpha > \frac{k-3}{k-2}$ ,  $\Pi(k, \alpha) \in P$ .

In case that  $k = 3$  the first part of Theorem 71 is just the statement that 3-COL is NP-complete. However, Edwards obtained the following strengthening.

**Lemma 72** [38] *Let  $c, \beta > 0$  be fixed. Then the following problem is NP-complete:*

*Input: A graph  $G = (V, E)$  with  $|V| = n$  and  $\delta(G) \geq cn^{1-\beta}$ .*

*Question: Is  $G$  3-colourable?*

In 1907 Mantel [86] has shown that every graph  $G$  with  $n$  vertices and more than  $n^2/4$  edges has a triangle. Hence, every triangle-free graph  $G$  with  $n$  vertices satisfies  $\delta(G) \leq n/2$ . We already demonstrated with the Mycielski-construction in Section 3 that there exist triangle-free graphs of arbitrarily large chromatic number. Hajnal used Kneser graphs of order  $n$  to show that such graphs may have minimum degree close to  $n/3$ . Erdős and Simonovits [42] conjectured that this is best possible. For the history of dense triangle-free graphs and open questions we refer to [14].

**Problem 73** *Do triangle-free graphs  $G$  with  $n$  vertices and  $\delta(G) > n/3$  have bounded chromatic number  $c$ ?*

Erdős and Simonovits [42] conjectured  $c = 3$ , but that was disproved by Häggkvist [57]; replacing a vertex of degree  $i$  by  $i - 1$  independent vertices in the Grötzsch-Mycielski graph he constructed a 10-regular and 4-chromatic graph of order 29. Jin [64] conjectured there is no upper bound for the chromatic number, whereas Brandt [13] conjectured that  $c = 4$ . With the additional property of regularity  $c = 4$  is true, as proven in [13]. In fact a more general statement is satisfied.

**Theorem 74** [13] *Let  $G$  be a regular maximal triangle-free graph of order  $n$  with degree exceeding  $n/3$ . Then  $G$  contains a dominating star  $K_{1,t}$  with  $t \leq 3$*

By usage of the *domination approach* described in Section 5, we easily derive the next corollary.

**Corollary 75** *Let  $G$  be a regular maximal triangle-free graph of order  $n$  with degree exceeding  $n/3$ . Then we can decide 3-COL and determine a proper 3-colouring in linear time.*

Recently, Thomassen [117] has proven the following:

**Theorem 76** [117] *For each natural number  $t$ , let  $c_t$  be the smallest number such that every triangle-free graph with  $n$  vertices and minimum degree  $> c_t n$  has chromatic number  $< t$ . Then  $c_t \rightarrow 1/3$  as  $t \rightarrow \infty$ .*

By Brooks theorem [15] every connected graph with maximum degree at most three has a 3-colouring or is isomorphic to a complete graph on four vertices. On the other hand it is NP-complete to decide, whether a given graph  $G$  with maximum degree four admits a 3-colouring, even if the graphs in consideration are triangle-free. In other words, if all graphs induced by the neighbourhood of a vertex are isomorphic to an edgeless graph, this problem still remains NP-complete. Thus, it is natural to ask for the computational complexity of deciding, whether there exists a 3-colouring in a graph having maximal degree at most four and such that all neighbourhoods of cardinality four are isomorphic to a given graph  $H$ . In [75] Kochol, Lozin and the first author gave a complete answer to this question showing that - with respect to a given graph  $H$  - the problem is either

- (1) NP-complete, or
- (2) it can be solved in linear time,

i. e. dichotomy holds. Subject to the assumption  $P \neq NP$  both cases exclude each other. For the polynomial cases the designed algorithms not only decide existence but actually find a 3-colouring in linear time, if there exists a 3-colouring. It is noteworthy that a slightly stronger result is satisfied. Let  $\mathcal{H}'$  be a subset of the set  $\mathcal{H}$ , where  $\mathcal{H}$  contains all possible graphs induced by four vertices. Then for the corresponding decision problem, where every graph induced by the neighbourhood of a vertex of degree four has to be a member of  $\mathcal{H}'$ , likewise dichotomy holds.

Let  $\mathcal{H}'$  be a fixed subset of  $\mathcal{H}$ .

#### **$\mathcal{H}'$ -ISOM-3-COL**

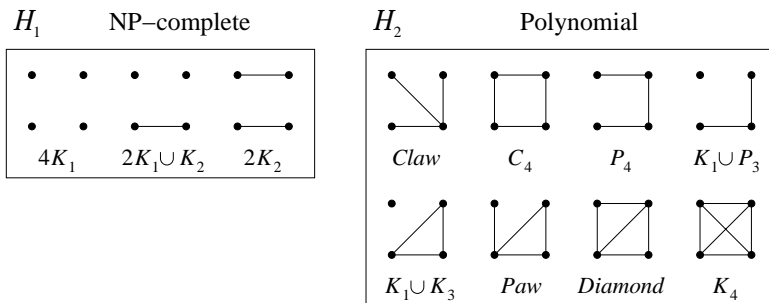
*Input:* A graph  $G$  having maximal degree at most four and such that every neighbourhood of cardinality four of  $G$  induces a graph, which is isomorphic to a member  $H \in \mathcal{H}'$ .

*Question:* Does there exist a 3-colouring of  $G$ ?

**Theorem 77** [75] *The problem  $\mathcal{H}'$ -ISOM-3-COL is NP-complete if  $\mathcal{H}' \cap \mathcal{H}_1 \neq \emptyset$ ; in all other cases the problem is solvable in polynomial time.*

The following result contributes another example of a dichotomy result in this area. In [109] this observation was used to enhance the exponential time 3-colouring algorithm. Explicitly, this result was mentioned recently by Zverovich [128]. Here, a graph  $G$  is *locally connected* if for every vertex  $v$  the neighbourhood  $N_G(v)$  induces a connected graph.

**Observation 78** *The decision problem, whether a locally connected graph is  $k$ -colourable is NP-complete for  $k \geq 4$  and can be decided in linear time for  $k \leq 3$ . Moreover, to 3-colour the YES-instances of the decision problem can be done in linear time as well.*



**Fig. 6** Computational complexity of  $\mathcal{H}'$ -ISOM-3-COL

*Sketch of the proof:* Since the 3-colourability decision problem is NP-complete, even for connected, planar graphs, introducing a vertex adjacent to all remaining vertices, easily yields the result for  $k = 4$  and locally connected graphs, even for apex graphs. Here, an *apex graph* is the join of a planar graph and an isolated vertex. On the other hand a necessary condition for a graph to be 3-colourable is that for every vertex its neighbourhood has to induce a bipartite graph. Furthermore if we have the additional property that the instance graph is locally connected, then for every vertex  $v$  its neighbourhood has to induce a connected bipartite graph with a unique bipartition  $(A, B)_v$ . Moreover, in any 3-colouring of the instance graph all vertices of  $A$  and likewise all vertices of  $B$  have to receive the same colour. Therefore we can contract  $(A, B)_v$  to adjacent vertices  $a$  and  $b$  forming the new neighbourhood of  $v$ . Observe that this contraction preserves the property of being locally connected. In a further simplification phase of a 3-colouring algorithm we reduce vertices of degree less or equal than 2. This briefly sketched algorithm terminates and decides 3-colourability for locally connected graphs and with a bookkeeping of the contraction and reduction operations we can 3-color the YES-instances. With an appropriate data structure this algorithm can be implemented in linear time. ■

Another dichotomy result has been obtained by Král, Kratochvíl, Tuza and Woeginger [79]. Let  $H$  be a given graph.

#### $H$ -free COLOURING

*Input:* A  $H$ -free graph  $G$  and a positive integer  $k$ .

*Question:* Does there exist a  $k$ -colouring of  $G$ ?

**Theorem 79** [79] *The problem  $H$ -free COLOURING is solvable in polynomial time if  $H$  is an induced subgraph of  $P_4$  or of  $K_1 \cup P_3$ , and NP-complete for any other  $H$ .*

Furthermore, in [79] the authors pose the following meta-problem: *Given a finite set  $A$ , what is the computational complexity of deciding the chromatic number of  $A$ -free graphs?*

In the final part of this section we focus on exact algorithms for  $k$ -COL. The first exact algorithm for  $k$ -COL was presented by Christofides, which is based on the following ideas: *An independent set  $I$  of vertices of a graph is called maximal if it is not a proper subset of any independent set  $I'$ . It is well-known that if a graph  $G$  is  $k$ -colourable then there is a partition of its vertex set into  $k$  independent sets where at least one of these sets is maximal. Now by computing all maximal independent sets of a given graph  $G$  and repeating this computation recursively for the remaining graphs the chromatic number of  $G$  can be determined.* Lawler [80] has shown that the Christofides algorithm has a worst-case running time of  $O(mn(1 + 3^{1/3})n)$ , where  $n$  is the order and  $m$  the size of the input graph. Thus  $k$ -COL can be solved within  $O(mn\beta_k^n)$ , where  $\beta \leq 1 + 3^{1/3} \sim 2,4422$  for all  $k \geq 3$ . Recently, Eppstein [39] has improved this complexity to  $O(2,415^n)$ . As suggested by Lawler [80], to test a graph for 3-COL, one can generate all maximal independent sets in  $O(mn3^{1/3}n)$  and then check the induced subgraph on each complementary set of vertices for bipartiteness. It follows that such a test can be performed within  $O(mn3^{1/3}n)$  time, where  $3^{1/3} \sim 1,4422$ . This bound is sharp since graphs on  $n$  vertices may have up to  $3^{1/3}$  maximal independent sets as shown by Moon and Moser [93]. Significant improvements  $O(1,3289^n)$  and  $O(1,398^n)$  are due to Beigel and Eppstein [3] and the second author [109]. Applying the famous planar separator theorem of Lipton and Tarjan [81] it is not very difficult to design an algorithm verifying in subexponential time  $2^{O(\sqrt{n})}$  for a given planar graph on  $n$  vertices, whether  $G$  is 3-colorable.

## 8 Concluding Remarks

The results presented in this survey indicate that there is a lot of ongoing research in the field of *vertex colouring and forbidden subgraphs*. Further publications, not only on perfect graphs, will appear in the next years.

We have already mentioned several references (both survey papers and books) for further reading at specific places in the previous sections. Furthermore, there are overlaps of *vertex colourings and forbidden subgraphs* and other topics of graph theory, which could not be addressed here in more details. We refer the interested reader to *graph colorings with local constraints - a survey* by Tuza [121], *graphs on surfaces* by Mohar and Thomassen [90], *perfect graphs* by Ramirez-Alfonsin and Reed [96], *graph colourings and the probabilistic method* by Molloy and Reed [92] and *graph colouring problems* by Jensen and Toft [63].

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