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Efficient Local Reductions to Basic Modal Logic*

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Abstract. We present novel reductions of the propositional modal logics KB, KD, KT, K4 and K5 to Separated Normal Form with Sets of Modal Levels. The reductions result in smaller formulae than the well-known reductions by Kracht and allow us to use the local reasoning of the prover K$_S$P to determine the satisfiability of modal formulae in these logics. We show experimentally that the combination of our reductions with the prover K$_S$P performs well when compared with a specialised resolution calculus for these logics and with the built-in reductions of the first-order prover SPASS.

1 Introduction

The main motivation for reducing problems in one logic (the source logic) to 'equivalent' problems in another logic (the target logic) is to exploit results and tools for the target logic to solve theoretical or practical problems in the source logic. For propositional modal logics this approach has been researched extensively for reductions of the satisfiability problem in these logics to the satisfiability problem in 'stronger' logics such as first-order logic [10,20], the second-order theory of n successors [6], simple type theory [4], and regular grammar logics [19].

An alternative approach is to reduce propositional modal logics to a 'weaker' logic, in particular, the basic modal logic K. For extensions of K with one of the axioms B, D, alt_1, T, and 4, Kracht [12] defines reduction functions of their global and local satisfiability problem to the corresponding problem in K and proves their correctness. He also defines a reduction function for K5, the extension of K with 5, to K4, but this reduction is incorrect as not all theorems of K4 are theorems of K5. Several features of Kracht's approach are relevant to our work. First, as is not uncommon in modal logic, he treats the modal operator $\Diamond$ as abbreviation for $\neg\Box\neg$, that is, $\Box$ is the only modal operator occurring in modal formulae. Second, the basic idea underlying his reduction functions

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is for a given modal formula φ to generate sufficiently many instances Δ of a modal axiom Λ so that φ is $\mathsf{K}\Lambda$-satisfiable iff $\varphi \wedge \Delta$ is K-satisfiable. Third, Kracht is only concerned with preservation of the computational complexity of the satisfiability problem under consideration, as well as the preservation of other theoretical properties. For instance, the local satisfiability problem in the modal logics covered by Kracht is PSPACE-complete. So, it is sufficient to ensure that Δ is polynomial in size with respect to φ. As Kracht himself concludes, his method offers a uniform way of transferring results about one modal logic to another, but may not be as useful for practical applications.

In [16,15] we have introduced a new normal form for basic multi-modal logic, called Separated Normal Form with Modal Levels, SNF_{ml}, that uses labelled modal clauses. These labels refer to the level within a tree Kripke structure at which a modal clause holds. This can be seen as a compromise between approaches that label formulae with worlds at unspecified level [1,3] and approaches that label formulae with paths [5,23]. A combination of a normal form transformation for modal formulae and a resolution-based calculus for labelled modal clauses can then be used to decide local and global satisfiability in basic modal logic. In [17,18] we have presented K$_S$P, an implementation of that calculus, together with an experimental evaluation that indicates that K$_S$P performs well if propositional variables are evenly spread across a wide range of modal levels within the formulae one wants to decide.

A feature of SNF_{ml} is its use of additional propositional symbols as 'surrogates' for subformulae of a modal formula φ. In the following we take advantage of the availability of those surrogates to provide a novel transformation from extensions of K with a single one of the axioms B, D, T, $\mathsf{4}$ and $\mathsf{5}$ to SNF_{ml}. Another novel aspect is that we modify the normal form so that it uses sets of modal levels as labels instead of a single modal level. In K we only need a definition of a surrogate at the modal level at which the corresponding subformula occurs in φ. But in KB, KT, $\mathsf{K4}$ and $\mathsf{K5}$, we need a definition at every reachable modal level, of which there can be many. We call the resulting normal form, *Separated Normal Form with Sets of Modal Levels*, SNF_{sml}.

The structure of the paper is as follows. In Section 2 we recap common concepts of propositional modal logic including its syntax and semantics. Section 3 defines SNF_{sml} and the reductions of K, KB, KD, KT, $\mathsf{K4}$ and $\mathsf{K5}$ to SNF_{sml}. Correctness is proved in Section 4. Related work is discussed in Section 5. In Section 6 we compare the performance of a combination of our reductions and the modal-layered resolution calculus implemented in prover K$_S$P with resolution calculi specifically designed for the logics under consideration and with translation-based approaches built into the first-order theorem prover SPASS.

2 Preliminaries

The language of modal logic is an extension of the language of propositional logic with a unary modal operator $\Box$ and its dual $\Diamond$. More precisely, given a denumerable set of *propositional symbols*, $P = \{p, p_0, q, q_0, t, t_0, \ldots\}$ as well as propositional *constants* **true** and **false**, *modal formulae* are inductively defined as follows: Constants and propositional symbols are modal formulae. If φ and ψ are modal formulae, then so are $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $\Box\varphi$, and $\Diamond\varphi$. We also assume that $\wedge$ and $\vee$ are associative and commutative operators and consider, e.g., $(p \vee (q \vee r))$ and $(r \vee (q \vee p))$ to be identical formulae. We often omit parentheses if this does not cause confusion. By $\mathsf{var}(\varphi)$ we denote the set of all propositional symbols occurring in φ. This function straightforwardly extends to finite sets of modal formulae. A *modal axiom (schema)* is a modal formula ψ representing the set of all instances of ψ.

A *literal* is either a propositional symbol or its negation; the set of literals is denoted by L. We denote by $\neg l$ the *complement* of the literal $l \in L$, that is, $\neg l$ denotes $\neg p$ if l is the propositional symbol p, and $\neg l$ denotes p if l is the literal $\neg p$. A *modal literal* is either $\Box l$ or $\Diamond l$, where $l \in L$.

A *(normal) modal logic* is a set of modal formulae which includes all propositional tautologies, the axiom schema $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, called the *axiom* K, is closed under modus ponens (if $\vdash \varphi$ and $\vdash \varphi \rightarrow \psi$ then $\vdash \psi$) and the rule of necessitation (if $\vdash \varphi$ then $\vdash \Box\varphi$).

K is the weakest modal logic, that is, the logic given by the smallest set of modal formulae constituting a normal modal logic. By $\mathsf{K}\Sigma$ we denote an *extensions* of K by a set Σ of axioms.

The standard semantics of modal logics is the *Kripke semantics* or *possible world semantics*. A *Kripke frame* F is an ordered pair $\langle W, R\rangle$ where W is a non-empty set of *worlds* and R is a binary (accessibility) relation over W. A *Kripke structure* M over P is an ordered pair $\langle F, V\rangle$ where F is a Kripke frame and the *valuation* V is a function mapping each propositional symbol in P to a subset $V(p)$ of W. We say $M = \langle F, V\rangle$ is *based on the frame* F. A *rooted Kripke structure* is an ordered pair $\langle M, w_0\rangle$ with $w_0 \in W$. To simplify notation, in the following we write $\langle W, R, V\rangle$ and $\langle W, R, V, w_0\rangle$ instead of $\langle\langle W, R\rangle, V\rangle$ and $\langle\langle\langle W, R\rangle, V\rangle, w_0\rangle$, respectively.

Satisfaction (or truth) of a formula at a world w of a Kripke structure $M = \langle W, R, V\rangle$ is inductively defined by:

$$
\begin{array}{ll}
\langle M, w\rangle \models \textbf{true}; & \langle M, w\rangle \not\models \textbf{false}; \\
\langle M, w\rangle \models p & \text{iff } w \in V(p), \text{ where } p \in P; \\
\langle M, w\rangle \models \neg\varphi & \text{iff } \langle M, w\rangle \not\models \varphi; \\
\langle M, w\rangle \models (\varphi \wedge \psi) & \text{iff } \langle M, w\rangle \models \varphi \text{ and } \langle M, w\rangle \models \psi; \\
\langle M, w\rangle \models (\varphi \vee \psi) & \text{iff } \langle M, w\rangle \models \varphi \text{ or } \langle M, w\rangle \models \psi; \\
\langle M, w\rangle \models (\varphi \rightarrow \psi) & \text{iff } \langle M, w\rangle \models \neg\varphi \text{ or } \langle M, w\rangle \models \psi; \\
\langle M, w\rangle \models \Box\varphi & \text{iff for every } v,\ w\,R\,v \text{ implies } \langle M, v\rangle \models \varphi; \\
\langle M, w\rangle \models \Diamond\varphi & \text{iff there is } v,\ w\,R\,v \text{ and } \langle M, v\rangle \models \varphi.
\end{array}
$$

If $\langle M, w\rangle \models \varphi$ holds then M is a *model* of φ, φ is *true at* w *in* M and M *satisfies* φ. A modal formula φ is *satisfiable* iff there exists a Kripke structure M and a world w in M such that $\langle M, w\rangle \models \varphi$. A modal formula φ is *globally true* or *valid* in a Kripke structure M if it is true at all worlds of M; it is *valid* if it is valid in all Kripke structures.

Name	Axiom	Frame Property	
D	$\Box\varphi \to \Diamond\varphi$	Serial	$\forall v \exists w. v\, R\, w$
T	$\Box\varphi \to \varphi$	Reflexive	$\forall w. w\, R\, w$
B	$\varphi \to \Box\Diamond\varphi$	Symmetric	$\forall vw. v\, R\, w \to w\, R\, v$
4	$\Box\varphi \to \Box\Box\varphi$	Transitive	$\forall uvw. (u\, R\, v \wedge v\, R\, w) \to u\, R\, w$
5	$\Diamond\varphi \to \Box\Diamond\varphi$	Euclidean	$\forall uvw. (u\, R\, v \wedge u\, R\, w) \to v\, R\, w$

Table 1. Modal axioms and relational frame properties

In the following we are interested in extensions of K with the axiom schemata shown in Table 1. Each of these axiom schemata defines a class of Kripke frames where the accessibility relation R satisfies the first-order property stated in the table. Given a normal modal logic L with corresponding class of frames $\mathfrak{F}$, we say a modal formula φ is *L-satisfiable* iff there exists a frame $F \in \mathfrak{F}$, a valuation V and a world $w_0 \in F$ such that $\langle F, V, w_0 \rangle \models \varphi$.

A *path rooted at w of length k*, $k \geq 0$, in a frame $F = \langle W, R \rangle$ is a sequence $\vec{w} = (w_0, w_1, \ldots, w_k)$ where for every i, $1 \leq i \leq k$, $w_{i-1}\, R\, w_i$. We say that the path $(w_0, w_1, \ldots, w_k)$ *connects w_0 and w_k*. For a path $\vec{w} = (w_0, \ldots, w_k)$ and world w_{k+1} with $w_k\, R\, w_{k+1}$, $\vec{w} \circ w_{k+1}$ denotes the path $(w_0, \ldots, w_k, w_{k+1})$. A path (w_0) of length 0 is identified with its root w_0. We denote the set of all paths rooted at a world w_0 in F by $\vec{F}[w_0]$ and the set of all paths by $\vec{F}$. The function $\mathsf{trm} : \vec{F} \to W$ maps every path $\vec{w} = (w_0, \ldots, w_k)$ to its terminal world w_k while the function $\mathsf{len} : \vec{F} \to \mathbb{N}$ maps every path $\vec{w} = (w_0, w_1, \ldots, w_k)$ to its length k.

A rooted Kripke structure $M = \langle W, R, V, w_0 \rangle$ is a *rooted tree Kripke structure* iff R is a tree, that is, a directed acyclic connected graph where each node has at most one predecessor, with *root* w_0. It is a *rooted tree Kripke model* of a modal formula φ iff $\langle W, R, V, w_0 \rangle \models \varphi$. In a rooted tree Kripke structure with root w_0 for every world $w_k \in W$ there is exactly one path $\vec{w}$ connecting w_0 and w_k; the *modal level of w_k (in M), denoted by* $\mathsf{ml}_M(w_k)$, is given by $\mathsf{len}(\vec{w})$.

Let $F = \langle W, R \rangle$ be a Kripke frame with $w \in W$. The *unravelling $F^u[w]$ of F at w* is the frame $\langle \vec{W}, \vec{R} \rangle$ where:

- $\vec{W} = \vec{F}[w]$ is the set of all rooted paths at w in F;
- for all $\vec{v}, \vec{w} \in \vec{W}$, if $\vec{w} = \vec{v} \circ w$ for some $w \in W$, then $\vec{v}\, \vec{R}\, \vec{w}$.

Let $F = \langle W, R \rangle$ and $F' = \langle W', R' \rangle$ be two Kripke frames. A function $f \colon W \mapsto W'$ is a *p-morphism* (or a *bounded morphism*) from F to F' if the following holds:

- if $v\, R\, w$, then $f(v)\, R'\, f(w)$.
- if $f(u)\, R'\, w$, then there exists $v \in W$ s.t. $f(v) = w$ and $u\, R\, v$.

Analogously for Kripke models. For $F = \langle W, R \rangle$, $M' = \langle F, V', w_0 \rangle$, and $M = \langle F^u[w_0], V, (w_0) \rangle$, the function trm is a p-morphism from M to M'.

When considering local satisfiability, the following holds (see, [8]):

Theorem 1. *Let φ be a modal formula. Then φ is K-satisfiable iff there is a finite rooted tree Kripke structure $M = \langle F, V, w_0 \rangle$ such that $\langle M, w_0 \rangle \models \varphi$.*

$\varphi \wedge \varphi \Rightarrow \varphi$	$\varphi \wedge \neg\varphi \Rightarrow \mathbf{false}$	$\Box\mathbf{true} \Rightarrow \mathbf{true}$	$\neg\mathbf{true} \Rightarrow \mathbf{false}$	$\neg\neg\varphi \Rightarrow \varphi$
$\varphi \vee \varphi \Rightarrow \varphi$	$\varphi \vee \neg\varphi \Rightarrow \mathbf{true}$	$\Diamond\mathbf{false} \Rightarrow \mathbf{false}$	$\neg\mathbf{false} \Rightarrow \mathbf{true}$	
$\varphi \wedge \mathbf{true} \Rightarrow \varphi$	$\varphi \wedge \mathbf{false} \Rightarrow \mathbf{false}$	$\varphi \vee \mathbf{false} \Rightarrow \varphi$	$\varphi \vee \mathbf{true} \Rightarrow \mathbf{true}$	

Table 2. Rewriting Rules for Simplification

For the normal form transformation presented in the next section we assume that any modal formula φ has been simplified by exhaustively applying the rewrite rules in Table 2 and is in Negation Normal Form (NNF), that is, a formula where only propositional symbols are allowed in the scope of negations. We say that such a formula is in *simplified NNF*.

3 Layered Normal Form with Sets of Levels

A formula to be tested for satisfiability is first transformed into a normal form called *Separated Normal Form with Sets of Modal Levels*, SNF_{sml}, whose language extends that of modal logic with labels consisting of sets of modal levels. Informally, we write $S : \varphi$, where S is a set of natural numbers, to denote that a formula φ is true at modal levels $ml \in S$. We write $\star : \varphi$ instead of $\mathbb{N} : \varphi$.

We introduce some notation that will be used in the following. Let $S^+ = \{l+1 \in \mathbb{N} \mid l \in S\}$, $S^- = \{l-1 \in \mathbb{N} \mid l \in S\}$, and $S^{\geq} = \{n \mid n \geq \min(S)\}$, where $\min(S)$ is the least element in S. Note that the restriction of the elements being in $\mathbb{N}$ implies that S^- cannot contain negative numbers.

The labels in SNF_{sml} work as a kind of *weak* universal operator, allowing us to talk about formulae that are satisfied at all worlds in a given set of modal levels. Formally, we restrict ourselves to rooted tree Kripke structures $M = \langle W, R, V, w_0 \rangle$ and if S is a set of modal levels, then by $M[S]$ we denote the set of worlds that are at a modal level in S, that is, $M[S] = \{w \in W \mid \mathsf{ml}_M(w) \in S\}$. The satisfaction of labelled formulae in a rooted tree Kripke structure M is then defined as follows:

$$M \models S : \varphi \text{ iff for every world } w \in M[S]\text{, we have } \langle M, w \rangle \models \varphi.$$

If $M \models S : \varphi$, then we say that $S : \varphi$ holds in M. Note that if $S = \emptyset$, then $M \models S : \varphi$ trivially holds. For a set Φ of labelled formulae, $M \models \Phi$ iff $M \models S : \varphi$ for every $S : \varphi$ in Φ, and we say Φ is K*-satisfiable*.

A labelled modal formula is then an SNF_{sml} clause iff it is of one of the following forms:

- Literal clause $\quad S : \bigvee_{b=1}^{r} l_b$
- Positive modal clause $\quad S : l' \to \Box l$
- Negative modal clause $\quad S : l' \to \Diamond l$

where $S \subseteq \mathbb{N}$ and l, l', l_b are propositional literals with $1 \leq b \leq r$, $r \in \mathbb{N}$. Positive and negative modal clauses are together known as *modal clauses*. We regard a literal clause as a set of literals, that is, two clauses are the same if they contain the same set of literals.

We assume that the set P of propositional symbols is partitioned into two infinite sets Q and T such that for every modal formula ψ we have $\mathsf{var}(\psi) \subset Q$ and there exists a propositional symbol $t_\psi \in T$ uniquely associated with ψ.

Given a modal formula φ in simplified NNF and $L \in \{\mathsf{K}, \mathsf{KB}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}, \mathsf{K5}\}$, then we can obtain a set Φ_L of clauses in SNF_{sml} such that φ is L-satisfiable iff Φ_L is K-satisfiable as $\Phi_L = \{\{0\} : t_\varphi\} \cup \rho_L(\{0\} : t_\varphi \rightarrow \varphi)$, where ρ_L is defined as follows:

$$\begin{aligned}
\rho_L(S : t \rightarrow \mathbf{true}) &= \emptyset \\
\rho_L(S : t \rightarrow \mathbf{false}) &= \{S : \neg t\} \\
\rho_L(S : t \rightarrow (\psi_1 \wedge \psi_2)) &= \{S : \neg t \vee \eta(\psi_1), S : \neg t \vee \eta(\psi_2)\} \cup \delta_L(S, \psi_1) \cup \delta_L(S, \psi_2) \\
\rho_L(S : t \rightarrow \psi) &= \{S : \neg t \vee \psi\} \\
&\quad \text{if } \psi \text{ is a disjunction of literals} \\
\rho_L(S : t \rightarrow (\psi_1 \vee \psi_2)) &= \{S : \neg t \vee \eta(\psi_1) \vee \eta(\psi_2)\} \cup \delta_L(S, \psi_1) \cup \delta_L(S, \psi_2) \\
&\quad \text{if } \psi_1 \vee \psi_2 \text{ is not a disjunction of literals} \\
\rho_L(S : t \rightarrow \Diamond\psi) &= \{S : t \rightarrow \Diamond\eta(\psi)\} \cup \delta_L(S^+, \psi) \\
\rho_L(S : t \rightarrow \Box\psi) &= P_L(S : t \rightarrow \Box\psi) \cup \Delta_L(S : t \rightarrow \Box\psi)
\end{aligned}$$

where η and δ_L are defined as follows:

$$\eta(\psi) = \begin{cases} \psi, & \text{if } \psi \text{ is a literal} \\ t_\psi, & \text{otherwise} \end{cases} \qquad \delta_L(S, \psi) = \begin{cases} \emptyset, & \text{if } \psi \text{ is a literal} \\ \rho_L(S : t_\psi \rightarrow \psi), & \text{otherwise} \end{cases}$$

and functions P_L, Δ_L are defined as shown in Table 3. The function η maps a propositional literal ψ to itself while it maps every other modal formula ψ to a new propositional symbol $t_\psi \in T$ uniquely associated with ψ. We call t_ψ the *surrogate* of ψ or simply a surrogate. The functions P_{KB} and P_{K5} introduce additional propositional symbols, called *supplementary propositional symbols*, $t_{\Box\neg t_{\Box\psi}} \in T$ and $t_{\Diamond t_{\Box\psi}} \in T$, respectively, that do not correspond to subformulae of the formula we are transforming.

Intuitively, P_{KB} is based on the following consideration: Take a world w in a Kripke structure M with a symmetric accessibility relation R. If there exists a world v with $w\,R\,v$ such that $\langle M, v\rangle \models \Box\psi$, then $\langle M, w\rangle \models \psi$. Now, take the contrapositive of that statement: If $\langle M, w\rangle \not\models \psi$, then for every world v with $w\,R\,v$, $\langle M, v\rangle \not\models \Box\psi$. Equivalently, $\langle M, w\rangle \models \psi$ or $\langle M, w\rangle \models \Box\neg\Box\psi$. This is expressed by the formula $\eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$. For P_{K5}, the formula $t_{\Diamond t_{\Box\psi}} \rightarrow \Box t_{\Diamond t_{\Box\psi}}$ expresses an instance of axiom schema *5*, $\Diamond\varphi \rightarrow \Box\Diamond\varphi$, with $\varphi = \Box\psi$, i.e., $\Diamond\Box\psi \rightarrow \Box\Diamond\Box\psi$. The contrapositive of axiom schema 5 is $\Diamond\Box\varphi \rightarrow \Box\varphi$, equivalent to $\neg\Diamond\Box\varphi \vee \Box\varphi$. For $\varphi = \psi$ this is expressed by the formula $\neg t_{\Diamond t_{\Box\psi}} \vee t_{\Box\psi}$. For the formula $\neg t_{\Diamond t_{\Box\psi}} \rightarrow \Box\neg t_{\Box\psi}$, consider $\neg\Diamond\Box\psi$. By duality of $\Box$ and $\Diamond$, this is

L	$P_L(S : t_{\Box\psi} \to \Box\psi)$	$\Delta_L(S : t_{\Box\psi} \to \Box\psi)$
K	$S : t_{\Box\psi} \to \Box\eta(\psi)$	$\delta_L(S^+, \psi)$
KT	$S : t_{\Box\psi} \to \Box\eta(\psi), S : \neg t_{\Box\psi} \vee \eta(\psi)$	$\delta_L(S \cup S^+, \psi)$
KD	$S : t_{\Box\psi} \to \Box\eta(\psi), S : t_{\Box\psi} \to \Diamond\eta(\psi)$	$\delta_L(S^+, \psi)$
KB	$S : t_{\Box\psi} \to \Box\eta(\psi)$, $S^- : \eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}, S^- : t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$	$\delta_L(S^- \cup S^+, \psi)$
K4	$S^{\geq} : t_{\Box\psi} \to \Box\eta(\psi), S^{\geq} : t_{\Box\psi} \to \Box t_{\Box\psi}$	$\delta_L((S^+)^{\geq}, \psi)$
K5	$\star : t_{\Box\psi} \to \Box\eta(\psi)$, $\star : \neg t_{\Diamond t_{\Box\psi}} \vee t_{\Box\psi}, \star : t_{\Diamond t_{\Box\psi}} \to \Diamond t_{\Box\psi}$, $\star : \neg t_{\Diamond t_{\Box\psi}} \to \Box\neg t_{\Box\psi}, \star : t_{\Diamond t_{\Box\psi}} \to \Box t_{\Diamond t_{\Box\psi}}$	$\delta_L(\star, \psi)$

Table 3. Transformation of $\Box$-formulae in modal logic L

equivalent to $\neg\neg\Box\neg\Box\psi$ and $\Box\neg\Box\psi$. So, $\neg\Diamond\Box\psi \to \Box\neg\Box\psi$ in every normal modal logic, not only K5. The remaining labelled formulae introduced by P_{KB} and P_{K5} ensure that supplementary propositional symbols are defined. For the remaining logics the additional clauses are also based directly on the axiom schemata.

To simplify presentation in the following, we define a function η_f as follows:

$$\eta_f(\varphi_1 \wedge \varphi_2) = \eta(\varphi_1) \wedge \eta(\varphi_2) \qquad \eta_f(\varphi_1 \vee \varphi_2) = \eta(\varphi_1) \vee \eta(\varphi_2)$$
$$\eta_f(\Box\varphi) = \Box\eta(\varphi) \qquad \eta_f(\Diamond\varphi) = \Diamond\eta(\varphi)$$

and we treat the two clauses $S : \neg t_{\psi_1 \wedge \psi_2} \vee \eta(\psi_1)$ and $S : \neg t_{\psi_1 \wedge \psi_2} \vee \eta(\psi_2)$ resulting from the normal form transformation of $\psi_1 \wedge \psi_2$ as a single 'clause' $S : \neg t_{\psi_1 \wedge \psi_2} \vee \eta_f(\psi_1 \wedge \psi_2)$. We also interchangeably write $S : \neg t_{\Box\psi} \vee \eta_f(\Box\psi)$ for $S : t_{\Box\psi} \to \eta_f(\Box\psi)$ and, analogously, $S : \neg t_{\Diamond\psi} \vee \eta_f(\Diamond\psi)$ for $S : t_{\Diamond\psi} \to \eta_f(\Diamond\psi)$. We then call any clause of the form $S : \neg t_\psi \vee \eta_f(\psi)$ a *definitional clause.*

Definition 1. *Let Φ be a set of* SNF_{sml} *clauses. We say $t_\psi \in T$* occurs at level ml in Φ *iff either*

(a) there exists a clause $S : \vartheta$ in Φ with $ml \in S$ such that ϑ is a propositional formula and t_ψ occurs positively in ϑ, or
(b) there exists a clause $S : t_{\Box\psi} \to \Box t_\psi$ in Φ with $ml - 1 \in S$, or
(c) there exists a clause $S : t_{\Diamond\psi} \to \Diamond t_\psi$ in Φ with $ml - 1 \in S$.

Definition 2. *Let Φ be a set of* SNF_{sml} *clauses. Then Φ is* definition-complete *iff for every $t_\psi \in T$ and every level ml, if t_ψ occurs at level ml in Φ then there exists a clause $S : \neg t_\psi \vee \eta_f(\psi)$ in Φ with $ml \in S$.*

Theorem 2. *Let $L \in \{K, KB, KD, KT, K4, K5\}$. Then $\Phi_L = \{\{0\} : t_\varphi\} \cup \rho_L(\{0\} : t_\varphi \to \varphi)$ is definition-complete.*

Proof. By induction over the computation of Φ_L. It is straightforward to see that the transformation of labelled formulae $S : t \to (\psi_1 \wedge \psi_2)$ and $S : t \to (\psi_1 \vee \psi_2)$ only introduces surrogates at levels in S and Δ_L then adds definitional clauses for those surrogates. The transformation of a labelled formula $S : t_{\Diamond\psi} \to \Diamond\psi$ may introduce a surrogate at levels in S^+ and $\delta_L(S^+, \psi)$ then adds definitional clauses for those surrogates. The transformation of a labelled formula $S : t_{\Box\psi} \to \Box\psi$ depends on the logic L. We can see that for every level at which a new surrogate occurs in $P_L(S : t_{\Box\psi} \to \Box\psi)$, then $\Delta_L(S : t_{\Box\psi} \to \Box\psi)$ contains a definitional clause for it at that level.

4 Correctness

Due to space constraints we only prove the correctness of the transformation for KB. We first state several lemmata that are used in the correctness proofs for all logics.

Lemma 1. *Let Φ be a set of definitional clauses such that every t_ψ occurring in Φ is an element of T and all other propositional symbols occurring in Φ are in Q. Let $M = \langle W, R, V, w_0\rangle$ be a rooted Kripke structure. Let $\langle \vec{W}, \vec{R}\rangle$ be the unravelling of $\langle W, R\rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V_\Sigma}, (w_0)\rangle$ be a Kripke structure such that*

– $\vec{V_\Sigma}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ *for every propositional symbol* $p \in Q$, *and*
– $\vec{V_\Sigma}(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}, \vec{w}\rangle \models \psi\}$ *for every surrogate* $t_\psi \in T \cap \mathsf{var}(\Phi)$.

Then $\vec{M} \models \Phi$.

Lemma 2. *Let φ be a L-satisfiable modal formula in simplified NNF where L is a normal modal logic and let $\Phi = \{\{0\} : t_\varphi\} \cup \rho_{\mathsf{K}}(\{0\} : t_\varphi \to \varphi)$. Let $M = \langle W, R, V, w_0\rangle$ be a rooted K model of φ. Let $\langle \vec{W}, \vec{R}\rangle$ be the unravelling of $\langle W, R\rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V}, (w_0)\rangle$ be a Kripke structure such that*

– $\vec{V}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ *for every propositional symbol* $p \in \mathsf{var}(\varphi)$, *and*
– $\vec{V}(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}, \vec{w}\rangle \models \psi\}$ *for every surrogate* $t_\psi \in T \cap \mathsf{var}(\Phi)$.

Then $\vec{M} \models \Phi$.

Lemma 3. *Let $M = \langle W, R, V, w_0\rangle$ be a rooted Kripke structure. Let $\langle \vec{W}, \vec{R}\rangle$ be the unravelling of $\langle W, R\rangle$ at w_0. Let $\vec{M} = \langle \vec{W}, \vec{R}, \vec{V_\Sigma}, (w_0)\rangle$ where $\vec{V_\Sigma}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ for every propositional symbol $p \in Q$.*

Then for every modal formula ψ over Q and for every world $\vec{w} \in \vec{W}$, $\langle \vec{M}, \vec{w}\rangle \models \psi$ iff $\langle M, \mathsf{trm}(\vec{w})\rangle \models \psi$.

Lemma 4. *Let φ be a modal formula in simplified NNF. Let $\Phi_{\mathsf{K}} = \{\{0\} : t_\varphi\} \cup \rho_{\mathsf{K}}(\{0\} : t_\varphi \to \varphi)$. Let Φ with $\Phi_{\mathsf{K}} \subseteq \Phi$ be a definition-complete set of SNF_{sml} clauses, let $M = \langle W, R, V, w_0\rangle$ be a tree K model of Φ and let $M' = \langle W, R', V, w_0\rangle$ be such that*

(4a) $R \subseteq R'$;
(4b) *for every modal clause* $S : t_{\Box\psi} \to \Box\eta(\psi)$ *in* Φ *and every world* $w \in M[S]$, $\langle M', w\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$;
(4c) *for every modal clause* $S : t_{\Box\psi} \to \Box t_\psi$ *in* Φ *and all worlds* $v, w \in W$, *if (i)* $w \in M[S]$ *and (ii)* $wR'v$ *then (iii) there exists a clause* $S' : \neg t_\psi \vee \eta_f(\psi)$ *in* Φ *with* $v \in M[S']$.

Then $\langle M', w_0\rangle \models \varphi$.

Theorems 3 and 4 now state the correctness of our transformation for KB.

Theorem 3. *Let* φ *be a modal formula in simplified NNF. Let* $\Phi_B = \{\{0\} : t_\varphi\} \cup \rho_{\mathsf{KB}}(\{0\} : t_\varphi \to \varphi)$. *If* φ *is* KB*-satisfiable, then* Φ_B *is* K*-satisfiable.*

Proof. The main idea is to show that given a rooted KB model of φ, then a small variation of its unravelling is a rooted tree K model of Φ_B.

Let $M = \langle W, R, V, w_0\rangle$ be a rooted KB model of φ with $\langle M, w_0\rangle \models \varphi$ and symmetric relationship R. Let $\langle \vec{W}, \vec{R}\rangle$ be the unravelling of $\langle W, R\rangle$ at w_0. Let $\vec{M}_{\mathsf{B}} = \langle \vec{W}, \vec{R}, \vec{V}_{\mathsf{B}}, (w_0)\rangle$ where

- $\vec{V}_{\mathsf{B}}(p) = \{\vec{w} \in \vec{W} \mid \mathsf{trm}(\vec{w}) \in V(p)\}$ for every propositional symbol $p \in \mathsf{var}(\varphi)$,
- $\vec{V}_{\mathsf{B}}(t_\psi) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models \psi\}$ for every surrogate $t_\psi \in \mathsf{var}(\Phi_{\mathsf{B}}) \setminus \mathsf{var}(\varphi)$ introduced by rewriting, and
- $\vec{V}_{\mathsf{B}}(t_{\Box\neg t_{\Box\psi}}) = \{\vec{w} \in \vec{W} \mid \langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models \Box\neg\Box\psi\}$ for every supplementary propositional symbol $t_{\Box\neg t_{\Box\psi}}$ introduced in the normal form transformation of a labelled formula $S : t_{\Box\psi} \to \Box\psi$.

Note that $\vec{V}_{\mathsf{B}}$ is well-defined as for every surrogate $t_\psi \in T$, ψ only contains propositional symbols in Q. Let $\Phi_{\mathsf{K}} = \{\{0\} : t_\varphi\} \cup \rho_{\mathsf{K}}(\{0\} : t_\varphi \to \varphi)$.

We now consider the clauses occurring in Φ_{B} and show that they hold in $\vec{M}_{\mathsf{B}}$. By Lemma 2 it follows that $\vec{M}_{\mathsf{B}} \models \Phi_{\mathsf{K}}$. Also, all definitional clauses in $\Phi_{\mathsf{B}} \setminus \Phi_{\mathsf{K}}$ are true in $\vec{M}_{\mathsf{B}}$ by Lemma 1.

Next consider clauses of the form

$$(1)\ S' : \eta(\psi) \vee t_{\Box\neg t_{\Box\psi}} \qquad\qquad (2)\ S' : t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$$

where $t_{\Box\psi}$ is a surrogate for $\Box\psi$. These are not in Φ_{K}. We show both are true in $\vec{M}_{\mathsf{B}}$. We do so by first considering that $t_{\Box\neg t_{\Box\psi}}$ is true at a world and then that it is false.

Case (a): Let $\vec{w} \in \vec{M}_{\mathsf{B}}[S']$ with $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models t_{\Box\neg t_{\Box\psi}}$. Clearly, $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models \eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$. Also, by definition of $\vec{M}_{\mathsf{B}}$, $\langle M_{\mathsf{B}}, \vec{w}\rangle \models \Box\neg\Box\psi$. So, for every $\vec{v} \in \vec{W}$ with $\vec{w}\ \vec{R}\ \vec{v}$, $\langle \vec{M}_{\mathsf{B}}, \vec{v}\rangle \models \neg\Box\psi$. As $t_{\Box\psi}$ is a surrogate for $\Box\psi$, by definition of $\vec{V}_{\mathsf{B}}$, $\vec{v} \notin \vec{V}_{\mathsf{B}}(t_{\Box\psi})$ and $\langle \vec{M}_{\mathsf{B}}, \vec{v}\rangle \models \neg t_{\Box\psi}$. Thus, $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models \Box\neg t_{\Box\psi}$ and, by the semantics of implication, $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$.

Case (b): Let $\vec{w} \in \vec{M}_{\mathsf{B}}[S']$ with $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \not\models t_{\Box\neg t_{\Box\psi}}$. Clearly, by the semantics of implication, $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$. Also, by definition of $\vec{V}_B$, $\vec{w} \notin \vec{V}_{\mathsf{B}}(t_{\Box\neg t_{\Box\psi}})$ implies $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \not\models \Box\neg\Box\psi$ which in turn implies $\langle \vec{M}_{\mathsf{B}}, \vec{w}\rangle \models \Diamond\Box\psi$. So, there exists $\vec{v} \in \vec{W}$ with $\vec{w}\vec{R}\vec{v}$ and $\langle \vec{M}_{\mathsf{B}}, \vec{v}\rangle \models \Box\psi$. Since trm is a p-morphism from

$\vec{M}_B$ to M, $\mathsf{trm}(\vec{w})\, R\, \mathsf{trm}(\vec{v})$. Since R is symmetric, we also have $\mathsf{trm}(\vec{v})\, R\, \mathsf{trm}(\vec{w})$ and by construction of $\vec{M}_B$, for $\vec{u} = \vec{v} \circ \mathsf{trm}(\vec{w})$ we have $\vec{v}\, \vec{R}\, \vec{u}$. Since $\langle \vec{M}_B, \vec{v}\rangle \models \Box\psi$, $\langle \vec{M}_B, \vec{u}\rangle \models \psi$. As trm is a p-morphism and $\langle M, \mathsf{trm}(\vec{u})\rangle \models \psi$ and since $\mathsf{trm}(\vec{w}) = \mathsf{trm}(\vec{u})$, $\langle M, \mathsf{trm}(\vec{w})\rangle \models \psi$. By Lemma 3, from $\langle M, \mathsf{trm}(\vec{w})\rangle \models \psi$ we obtain $\langle \vec{M}_B, \vec{w}\rangle \models \psi$. If ψ is a literal, then $\eta(\psi) = \psi$ and $\langle M, \vec{w}\rangle \models \eta(\psi)$. If ψ is not a literal, then $\eta(\psi) = t_\psi$ and from $\langle \vec{M}_B, \vec{w}\rangle \models \psi$, by definition of $\vec{V}_B$, $\vec{w} \in \vec{V}_B(t_\psi)$ and $\langle \vec{M}_B, \vec{w}\rangle \models t_\psi$. So, $\langle M, \vec{w}\rangle \models \eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$.

Thus, in both cases, for arbitrary $\vec{w} \in \vec{M}_B[S']$, $\eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$ and $t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$ and therefore Clauses (1) and (2) are true in $\vec{M}_B$.

Theorem 4. *Let φ be a modal formula in simplified NNF. Let $\Phi_B = \{\{0\} : t_\varphi\} \cup \rho_{KB}(\{0\} : t_\varphi \to \varphi)$. If Φ_B is K-satisfiable, then φ is KB-satisfiable.*

Proof. The main idea is to show that given a rooted tree K model of Φ_B, its symmetric closure is a rooted KB model of φ.

Let $M = \langle W, R, V, w_0\rangle$ be a rooted tree K model of Φ_B. Let $M^B = \langle W, R^B, V^B, w_0\rangle$ be a structure such that

(a) R^B is the symmetric closure of R, that is, R^B is the smallest relation on W such that $R \subseteq R^B$ and for every $v, w \in W$, $v\, R^B\, w$ implies $w\, R^B\, v$;
(b) $V^B(p) = V(p)$ for every propositional symbol.

Let $\Phi_K = \{\{0\} : t_\varphi\} \cup \rho_K(\{0\} : t_\varphi \to \varphi)$. We show that $M^B \models \Phi_B$ satisfies the three preconditions of Lemma 4. By Lemma 4 this in turn implies that $M^B \models \varphi$.

- Condition (4a) holds as $R \subseteq R^B$.
- For Condition (4b) let (3) $S : t_{\Box\psi} \to \Box\eta(\psi)$ be a modal clause in Φ_B. Then Φ_B also contains the additional clauses (4) $S^- : \eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$ and (5) $S^- : t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$. Let $w \in M[S]$. We have to show that (6) $\langle M^B, w\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$. Assume $\langle M^B, w\rangle \models t_{\Box\psi}$. As $V^B(t_{\Box\psi}) = V(t_{\Box\psi})$ this implies $\langle M, w\rangle \models t_{\Box\psi}$. Let $v \in W$ such that $w\, R^B\, v$.
  Case (a): Assume $w\, R\, v$. As $\langle M, w\rangle \models t_{\Box\psi}$ and $\langle M, w\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$, we have $\langle M, w\rangle \models \Box\eta(\psi)$. As $w\, R\, v$, $\langle M, v\rangle \models \eta(\psi)$. As $\eta(\psi)$ is a literal and $V^B = V$ we obtain $\langle M^B, v\rangle \models \eta(\psi)$. So, $\langle M^B, w\rangle \models t_{\Box\psi} \to \Box\eta(\psi)$.
  Case (b): Assume v is not reachable from w via R. Then wR^Bv was introduced by the symmetric closure operation on R and we must have $v\, R\, w$. That is, v is a R-predecessor of w and from $w \in M[S]$ it follows that $v \in M[S^-]$. So, (7) $\langle M, v\rangle \models \eta(\psi) \vee t_{\Box\neg t_{\Box\psi}}$ and (8) $\langle M, v\rangle \models t_{\Box\neg t_{\Box\psi}} \to \Box\neg t_{\Box\psi}$. From $v\, R\, w$, $\langle M, w\rangle \models t_{\Box\psi}$ and (8), it follows that $\langle M, v\rangle \models \neg t_{\Box\neg t_{\Box\psi}}$. This together with (7) implies $\langle M, v\rangle \models \eta(\psi)$. As $\eta(\psi)$ is a literal and $V^B = V$ we obtain $\langle M^B, v\rangle \models \eta(\psi)$. So, $\langle M^B, w\rangle \models t_{\Box\psi} \to \Box\eta(t_\psi)$.
  Case (a) and Case (b) together show that Property (6) holds.
- For Condition (4c) let (9) $S : t_{\Box\psi} \to \Box t_\psi$ be in Φ_B, $v, w \in W$, $\mathsf{ml}_M(w) = ml \in S$ (i.e., $w \in M[S]$) and $w\, R^B\, v$. We need to show that there exists a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ_B with $v \in M[S']$.
  As in the previous case $w\, R^B\, v$ implies either $w\, R\, v$ or $v\, R\, w$. In the first case $\mathsf{ml}_M(v) = ml + 1$ while in the second case $\mathsf{ml}_M(v) = ml - 1$.

As Φ_B contains Clause (9), t_ψ occurs at level $ml+1$ in Φ_B. By definition of ρ_{KB}, Φ_B also contains the clause (10) $S^- : t_\psi \vee t_{\Box\neg t_{\Box\psi}}$. As $ml \in S$, $ml-1 \in S^-$ and therefore t_ψ also occurs at level $ml-1$ in Φ_B. By Theorem 2, Φ_B is definition-complete, so there must be a clause $S' : \neg t_\psi \vee \eta_f(\psi)$ in Φ_B such that $ml+1$ and $ml-1$ in S'.

Theorem 5. *Let φ be a modal formula in simplified NNF, $L \in \{K, KB, KD, KT, K4, K5\}$, and $\Phi_L = \{\{0\} : t_\varphi\} \cup \rho_L(\{0\} : t_\varphi \to \varphi)$. Then φ is L-satisfiable iff Φ_L is K-satisfiable.*

5 Comparison With Related Work

The approaches most closely related to ours are Kracht's reductions of normal modal logics to basic modal logic [11,12], the global modal resolution calculus [14], and Schmidt and Hustadt's axiomatic translation principle for translations of normal modal logics to first-order logic [24].

The first significant difference to our approach is that Kracht's reductions and the axiomatic translation exclude the modal operator $\Diamond$ from the language and only consider the modal operator $\Box$.

In order to present Kracht's approach, we need some additional notions. Let $\mathsf{sf}(\varphi)$, $\mathsf{dg}(\varphi)$, and $|S|$ denote the set of all subformulae of φ, the maximum nesting of modal operators in φ, and the cardinality of the set S, respectively. Let $\Diamond^0\psi = \Box^0\psi = \Box^{<1}\psi = \psi$, $\Box^{<n+1}\psi = (\psi \wedge \Box\Box^{<n}\psi)$, $\Box^{n+1}\psi = \Box\Box^n\psi$, and $\Diamond^{n+1}\psi = \Diamond\Diamond^n\psi$. We can then define a reduction function ρ_L^{K} for a normal modal logic L in $\{\mathsf{KB}, \mathsf{KD}, \mathsf{KT}, \mathsf{K4}\}$ as follows:

$$\rho_L^{\mathsf{K}}(\varphi) = \begin{cases} \varphi \wedge \Box^{<|\mathsf{sf}(\varphi)|+1} P_{K4}^{\mathsf{K}}(\varphi), & \text{for } L = \mathsf{K4} \\ \varphi \wedge \Box^{<\mathsf{dg}(\varphi)+1} P_L^{\mathsf{K}}(\varphi) & \text{otherwise} \end{cases}$$

where $P_{KB}^{\mathsf{K}}(\varphi) = \{\neg\psi \to \Box\neg\Box\psi \mid \Box\psi \in \mathsf{sf}(\varphi)\}$ $P_{KD}^{\mathsf{K}}(\varphi) = \{\neg\Box\mathbf{false}\}$

$P_{K4}^{\mathsf{K}}(\varphi) = \{\Box\psi \to \Box\Box\psi \mid \Box\psi \in \mathsf{sf}(\varphi)\}$ $P_{KT}^{\mathsf{K}}(\varphi) = \{\Box\psi \to \psi \mid \Box\psi \in \mathsf{sf}(\varphi)\}$

Kracht shows that φ is L-satisfiable iff $\rho_L^{\mathsf{K}}(\varphi)$ is K-satisfiable. There are three differences to our approach. First, $P_L^{\mathsf{K}}(\varphi)$ will include an axiom instance for every occurrence of a subformula $\neg\Box\psi$, equivalent to $\Diamond\neg\psi$, in φ. In contrast, our approach requires no logic specific treatment of such subformulae. Second, the use of $\Box^{<n}P_L^{\mathsf{K}}(\varphi)$ in ρ_L^{K} means that the axiom instance is available at every modal level. This means, for example, that for $\vartheta_1 = \Diamond^{100}(\neg p \wedge \Box p)$, the formula $\rho_{KT}^{\mathsf{K}}(\vartheta_1)$ contains the axiom instance $\Box p \to p$ over 100 times, although it is only required at the level at which $\Box p$ occurs. Third, this is further compounded if the formula ψ in $\Box\psi$ is itself a complex formula. We try to avoid that by using a surrogate propositional symbol t_ψ instead, but this will only have a positive effect if the definitional clauses for t_ψ do not have to be repeated.

The global modal resolution (GMR) calculus operates on SNF_K clauses, that is, clauses of the form

$$\Box^*(\mathbf{start} \to \bigvee_{b=1}^r l_b) \qquad \Box^*(\mathbf{true} \to \bigvee_{b=1}^r l_b) \qquad \Box^*(l' \to \Box l) \qquad \Box^*(l' \to \neg\Box l)$$

[EUC1] $\Box^*(l_1 \to \neg\Box\neg l)$		[EUC2] $\Box^*(l \to \Box l_2)$	
$\Box^*(\mathbf{true} \to \neg l_1 \vee t_{\Diamond l})$	$\Box^*(t_{\Diamond l} \to \neg\Box\neg l)$	$\Box^*(t_{\Diamond l} \to \Box l_2)$	$\Box^*(t_{\Diamond l} \to \neg\Box\neg l)$
$\Box^*(\neg t_{\Diamond l} \to \Box\neg l)$	$\Box^*(t_{\Diamond l} \to \Box t_{\Diamond l})$	$\Box^*(\neg t_{\Diamond l} \to \Box\neg l)$	$\Box^*(t_{\Diamond l} \to \Box t_{\Diamond l})$

Table 4. Inference rules in [14] for $\mathsf{K5}$ (EUC1 and EUC2).

where l, l', l_b are propositional literals with $1 \leq b \leq r$, $r \in \mathbb{N}$, and $\Box^*$ is the universal operator. The calculus has specific inference rules for normal modal logics such as KB, KD, KT, $\mathsf{K4}$, $\mathsf{K5}$. Table 4 shows the two additional rules for $\mathsf{K5}$, the only logic for which there are rules for both $\Box$ and $\neg\Box\neg$, i.e., $\Diamond$. These inference rules can be seen to perform an 'on-the-fly' computation of a transformation. Note that the clauses produced by P_{K5} differ from those produced by GMR for $\mathsf{K5}$. Implicitly, our results here also show that it should be possible to eliminate EUC1 from the GMR calculus.

For the axiomatic translation, we only present the function P_L^{RS} that computes the logic dependent first-order clausal formulae that are part of the overall translation.

$$P_{\mathsf{KB}}^{\mathsf{RS}}(\Box\psi) = \{\forall x(\neg Q_{\Box\psi}(y) \vee \neg R(x,y) \vee Q_\psi(x)) \mid \Box\psi \in \mathsf{sf}(\varphi)\}$$
$$P_{\mathsf{KD}}^{\mathsf{RS}}(\Box\psi) = \{\forall x(\neg Q_{\Box\psi}(x) \vee Q_{\neg\Box\neg\psi}(x)) \mid \Box\psi \in \mathsf{sf}(\varphi)\}$$
$$P_{\mathsf{KT}}^{\mathsf{RS}}(\Box\psi) = \{\forall x(\neg Q_{\Box\psi}(x) \vee Q_\psi(x)) \mid \Box\psi \in \mathsf{sf}(\varphi)\}$$
$$P_{\mathsf{K4}}^{\mathsf{RS}}(\Box\psi) = \{\forall xy(\neg Q_{\Box\psi}(x) \vee \neg R(x,y) \vee Q_{\Box\psi}(y)) \mid \Box\psi \in \mathsf{sf}(\varphi)\}$$
$$P_{\mathsf{K5}}^{\mathsf{RS}}(\Box\psi) = \{\forall xy(\neg Q_{\Box\psi}(y) \vee \neg R(x,y) \vee Q_{\Box\psi}(x)), \forall xy(\neg Q_{\Box\neg\Box\psi}(y) \vee \neg R(x,y) \vee Q_{\Box\neg\Box\psi}(x)) \mid \Box\psi \in \mathsf{sf}(\varphi)\}$$

The predicate symbols Q_ψ correspond to our surrogate symbols t_ψ. The clausal formulae used in the treatment of KT and $\mathsf{K4}$ are translations of the SNF_{ml} clauses we use (or vice versa). KB and $\mathsf{K5}$ are handled in a different way as the first-order clausal formulae refer directly the accessibility relation and can therefore more easily express the transfer of information to a predecessor world. The universal quantification over worlds also means that the constraints expressed by the formulae hold at all modal levels without the need of any repetition.

In Section 6 we will also use the relational and semi-functional translation of modal logics to first-order logic combined with structural transformation to clause normal form. In both approaches $\Box\psi$ is translated as $\forall xy(\neg Q_{\Box\psi}(x) \vee \neg R(x,y) \vee Q_\psi$, while $\Diamond\psi$ becomes $\forall x\exists y(\neg Q_{\Diamond\psi}(x) \vee R(x,y))$ and $\forall x\exists\alpha(\neg Q_{\Diamond\psi}(x) \vee R(x,[x\alpha]))$ in the relational and semi-functional translation, respectively. Then, depending on the modal logics, further formulae representing the semantic properties of the accessibility R are added. For the relational translation these will simply be the formulae in the fourth column of Table 1. The semi-functional translation uses collections of partial accessibility function in addition to the accessibility relation. A predicate def is used to represent on which worlds a partial accessibility function is defined. For each modal logic there is then again a background theory consisting of formulae over def and R that represents the properties of the underlying accessibility relation which is added to the translation of a formula. For example, for $\mathsf{K5}$ the background theory is: $\forall xy\forall\alpha\beta((\neg\mathrm{def}(x)\vee \mathrm{def}(y))\wedge(\neg\mathrm{def}(\mathrm{w}_0)\vee R(\mathrm{w}_0,[\mathrm{w}_0\alpha]))\wedge(\neg\mathrm{def}(x)\vee\neg\mathrm{def}(y)\vee R([x\alpha],[y\beta])))$, where w_0 is a constant representing the root world in a rooted Kripke structure.

6 Evaluation

We have compared the performance of the following approaches: (i) the combination of our reductions with the modal-layered resolution (MLR) calculus for SNF_{ml} clauses [15] implemented in the modal theorem prover KSP, with three different refinements for resolution inferences on labelled propositional clauses; (ii) the global modal resolution (GMR) calculus, also implemented in KSP, with three different refinements for resolution inferences on propositional clauses; (iii) the combinations of the relational and semi-functional translation of modal logics to first-order logic with ordered first-order resolution implemented in the first-order theorem prover SPASS. In total this gives us eight different approaches to compare. The axiomatic translation is currently not implemented in SPASS. Other provers, such as LEO-III [26], LWB [9], MleanCoP [21], do not have built-in support for the full range of logics considered here. LoTREC 2.0 [7] supports all the logics, but is not intended as automatic theorem prover.

The modal-layered resolution calculus operates on SNF_{ml} clauses, that is, clauses of the form

$$ml : \bigvee_{b=1}^{r} l_b \qquad ml : l' \to \Box l \qquad ml : l' \to \Diamond l$$

where $ml \in \mathbb{N}\cup\{\star\}$ and l, l', l_b are propositional literals with $1 \leq b \leq r$, $r \in \mathbb{N}$. In the implementation of the reductions presented in Section 3, we take a SNF_{sml} clause $S : \psi$ simply as an abbreviation of the set of SNF_{ml} clauses $\{ml : \psi \mid ml \in S\}$. Note that this also means that we will have to repeat similar resolution inferences for different modal levels.

KSP [13] implements the reductions presented in Section 3 as well as a normal form transformation of modal formulae to sets of SNF_K clauses. It implements both the MLR and the GMR calculus. Resolution inferences between (labelled) propositional clauses can either be unrestricted (`cplain` option), restricted by an ordering (`cord` option), that is, clauses can only be resolved on their maximal literals with respect to an ordering chosen by the prover in such a way to preserve completeness, restricted to negative resolution (`cneg` option), that is, one of the premises in an inference has to be a negative clause, or restricted to positive resolution. We do not include the last option in our evaluation as it typically performs worse. KSP also implements a range of simplification rules that are applied to modal formulae before their transformation to normal form. Of those we have enabled pure literal elimination (`early_ple` option), simplification using the Box Normal Form [22] and Prenex Normal Form (`bnfsimp` and `prenex`

Logic	Status	Total	KSP (GMR calculus, cneg)	KSP (GMR calculus, cord)	KSP (GMR calculus, cplain)	KSP (MLR calculus, cneg)	KSP (MLR calculus, cord)	KSP (MLR calculus, cplain)	SPASS (semi-functional)	SPASS (relational)
K	Sat	180	110	139	93	141	**155**	132	92	97
K	Unsat	180	154	**156**	151	154	**156**	153	134	122
KD	Sat	180	125	143	118	141	**155**	133	107	103
KD	Unsat	180	154	**156**	151	154	**156**	153	136	130
KT	Sat	100	53	**60**	37	46	56	26	47	39
KT	Unsat	260	233	236	225	230	**238**	220	222	199
KB	Sat	122	28	35	41	49	**89**	22	31	23
KB	Unsat	238	186	196	197	207	**211**	205	159	169
K4	Sat	161	33	39	38	68	**125**	36	0	0
K4	Unsat	199	124	112	146	**168**	165	163	109	35
K5	Sat	60	**14**	10	9	7	10	4	7	0
K5	Unsat	300	251	246	**259**	255	254	246	255	124
All	Sat	803	363	426	336	452	**590**	353	284	262
All	Unsat	1357	1102	1102	1129	1168	**1180**	1140	1015	779

Table 5. Experimental results on LWB benchmark collection

options) [17]. For clause processing, unit resolution and pure elimination are enabled (`unit`, `lhs_unit`, and `ple` options).

SPASS 3.9 [27,28] supports automated reasoning in extended modal logics, including all logics considered here, PDL-like modal logics as well as description logics. It includes eight different translations of modal logics to first-order logic. In our evaluation we have used the relational translation and the semi-functional translation. For the local satisfiability problem in KB to K5, for the relational translation we have added the first-order frame properties given in Table 1 while for the semi-functional translation we have added the background theories devised by Nonnengart [20]. For the transformation to first-order clausal form, we have enabled renaming of quantified subformulae. The only inference rules used are ordered resolution and ordered factoring, the reduction rules used are condensing, backward subsumption and forward subsumption. For the relational and semi-functional translation for K, KB, KD, and KT we thereby obtain a decision procedure, while for the other logics we do not. For K4 and K5, the fragment of first-order clausal logic corresponding to the semi-functional translation of modal formula and their background theories is decidable by ordered resolution with selection [25]. However, the non-trivial ordering and selection function required is not currently implemented in SPASS.

For our evaluation we have chosen the LWB basic modal logic benchmark collection [2], with 20 formulae in each of 18 parameterised classes. For K, all formulae in 9 classes are satisfiable while all formulae in the other 9 classes are unsatisfiable. In their negation normal form, 63% of modal operators are $\Box$ and

37% are $\Diamond$ operators. We have used the collection for each of the six logics. If a formula is unsatisfiable in K then it remains unsatisfiable in the other five logics, while the opposite is not true. As we move to logics other than K, it is also no longer the case that all formulae in a class have the same satisfiability status.

The third column in Table 5 indicates the total number of satisfiable and unsatisfiable formulae for each logic. In the last two lines of the table we sum up the results for all logics. The last eight columns in the table show how many formulae each of the approaches were able to solve with a time limit of 100 CPU seconds for each formula. Benchmarking was performed on a PC with an AMD Ryzen 5 5600X CPU @ 4.60GHz max and 32GB main memory using Fedora release 33 as operating system.

As we can see, the new reductions combined with the modal-layered resolution (MLR) calculus and ordered resolution refinement (`cord`) perform best, achieving the highest number of solved formulae in 8 out of 12 individual categories in the table, on two of those equal with the global modal resolution (GMR) calculus. On 3 categories, GMR outperfoms MLR. On both satisfiable and unsatisfiable formulae in $\mathsf{K5}$ this can be seen as evidence that 'on-the-fly' transformation offers a (slight) advantage over our approach given that the additional clauses hold universally in both approaches. For SPASS we see a clear advantage of the semi-functional translation over the relational one, on both satisfiable and unsatisfiable formulae.

7 Conclusion and Future Work

We have presented new reductions of propositional modal logics KB, KD, KT, $\mathsf{K4}$, $\mathsf{K5}$ to Separated Normal Form with Sets of Modal Levels. We have shown experimentally that these reductions allow us to reason effectively in these logics.

The obvious next step is to consider extensions of the basic modal logic K with combinations of the axioms B, D, T, $\mathsf{4}$, and $\mathsf{5}$. Unfortunately, a simple combination of the reductions for each of the axioms is not sufficient to obtain a satisfiability-preserving reduction for the such modal logics. An example is the simple formula $\neg p \wedge \Diamond\Diamond\Box p$ which is $\mathsf{KB4}$-unsatisfiable. If we define

$$\begin{aligned} P_{\mathsf{KB4}}(S : t_{\Box\psi} \rightarrow \Box\psi) &= P_{\mathsf{KB}}(S : t_{\Box\psi} \rightarrow \Box\psi) \cup P_{\mathsf{K4}}(S : t_{\Box\psi} \rightarrow \Box\psi) \\ \Delta_{\mathsf{KB4}}(S : t_{\Box\psi} \rightarrow \Box\psi) &= \delta_{\mathsf{KB4}}(\star, \psi), \end{aligned}$$

that is, P_{KB4} is the union of P_{KB} and P_{K4}, then the clause set obtained from $\{\{0\} : t_0\} \cup \rho_{\mathsf{KB4}}(\{0\} : t_0 \rightarrow \neg p \wedge \Diamond\Diamond\Box p)$ is K-satisfiable. The same issue also occurs in the axiomatic translation of modal logics to first-order logic where the translation for $\mathsf{KB4}$ is not simply the combination of the translations for KB and $\mathsf{K4}$ [24, Theorem 5.6]. We are currently exploring solutions to this problem.

Regarding practical applications, it would be advantageous to have an implementation of a calculus that operates directly SNF_{sml} clauses. This would greatly reduce the number of inference steps performed on satisfiable formulae and simplify proof search in general. Again, such an implementation is future work.

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