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Quotients of Polytopes and C-Groups*

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Abstract. Abstract polytopes are combinatorial and geometrical structures with a distinctive topological flavor, which resemble the convex polytopes. C-groups are generalizations of Coxeter groups and are the automorphism groups of abstract polytopes which are regular. We investigate general properties of quotients of abstract polytopes and C-groups.

1. Introduction

It is a standard method to derive, from a given combinatorial structure, new structures by making suitable identifications. This quotient construction also applies to the general class of abstract polytopes. Abstract polytopes are combinatorial and geometrical structures with a distinctive topological flavor, which resemble the convex polytopes. In recent years much work has been done on the classification by topological type of those abstract polytopes which are regular. For background material, basic results, and more advanced classification results the reader is referred to, for example, [1]–[8].

In this note we investigate quotients of abstract polytopes and C-groups. In our earlier works [4]–[7] we often ran into trouble with constructions of quotients where certain facts were intuitively clear but nevertheless required a tedious, if not complicated, proof. It is the purpose of this note to give a short exposition on general facts about quotients which we could not find in this form in the literature.

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2. Basic Notions

An (*abstract*) *prepolytope of rank n*, or an *n-prepolytope*, is a partially ordered set $\mathcal{P}$ with a strictly monotone rank function with range $\{-1, 0, \ldots, n\}$. The elements of rank i are called the *i-faces* of $\mathcal{P}$, or *vertices*, *edges*, and *facets* of $\mathcal{P}$ if $i = 0$, 1, or $n - 1$, respectively. The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ all contain exactly $n + 2$ faces, including the unique minimal (improper) face F_{-1} and the unique maximal (improper) face F_n of $\mathcal{P}$. Also, if F and G are an $(i - 1)$-face and an $(i + 1)$-face with $F < G$, then there are exactly *two* i-faces H such that $F < H < G$. Equivalently, if $0 \leq i \leq n - 1$ and Φ is a flag of $\mathcal{P}$, exactly one flag exists, the *i-adjacent* flag Φ^i of Φ, which differs from Φ in exactly the i-face. For two faces F and G with $F \leq G$ we call $G/F := \{H | F \leq H \leq G\}$ a *section* of $\mathcal{P}$. We usually safely identify a face with the section F/F_{-1}. For a face F the section F_n/F is called the *coface* of $\mathcal{P}$ at F, or the *vertex-figure at* F if F is a vertex. By $\mathcal{F}(\mathcal{P})$ we denote the set of flags of $\mathcal{P}$.

A prepolytope $\mathcal{P}$ is called *flag-connected* if any two flags Φ and Ψ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_{k-1}, \Phi_k = \Psi$. which are such that Φ_{j-1} and Φ_j are adjacent (differ by exactly one face) for each j. A prepolytope $\mathcal{P}$ is *strongly flag-connected* if each section of $\mathcal{P}$ (including $\mathcal{P}$ itself) is flag-connected; equivalently, for the above sequence we may further assume that $\Phi \cap \Psi \subset \Phi_j$ for each j. By a *polytope of rank n*, or simply an *n-polytope*, we mean a strongly flag-connected n-prepolytope.

In studying quotients we are mainly interested in polytopes. An n-polytope $\mathcal{P}$ is *regular* if its automorphism group $A(\mathcal{P})$ is transitive on the flags. The group $A(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is generated by *distinguished generators* $\rho_0, \ldots, \rho_{n-1}$, where ρ_i is the unique involutory automorphism which keeps all but the i-face of a *base flag* $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ of $\mathcal{P}$ fixed. In particular, $A(\mathcal{P})$ has the *intersection property* (with respect to $\rho_0, \ldots, \rho_{n-1}$), namely,

$$(1) \quad \langle \rho_i | i \in I \rangle \cap \langle \rho_i | i \in J \rangle = \langle \rho_i | i \in I \cap J \rangle \qquad \text{for all} \quad I, J \subset \{0, \ldots, n-1\}.$$

Also, the generators ρ_i satisfy the relations

$$(2) \qquad (\rho_i \rho_j)^2 = \varepsilon \qquad \text{if} \quad 0 \leq i < j - 1 \leq n - 2$$

and

$$(\rho_{i-1} \rho_i)^{p_i} = \varepsilon \qquad \text{if} \quad 0 \leq i \leq n - 1,$$

with the p_i's given by the Schläfli symbol $\{p_1, \ldots, p_{n-1}\}$.

By a *C-group* ("C" in honour of Coxeter) we mean a group which is generated by involutions such that (1) holds. If (2) also holds, the group is called a *string C-group*. Examples of string C-groups are the Coxeter groups with a string diagram; however, there are lots of other examples. It is known that the string C-groups are precisely the groups of the regular polytopes [5]. We remark that, in dealing with C-groups, it is usually the intersection property that causes problems. On the other hand, this property is the crucial one for polytopality.

3. Quotients

Though the quotient construction applies more generally to other kinds of posets, we restrict our attention to abstract polytopes and prepolytopes.

Let $\mathscr{P}$ and $\mathscr{Q}$ be two prepolytopes of the same rank. An incidence-preserving mapping $\varphi: \mathscr{P} \mapsto \mathscr{Q}$ is called a *rap-map* if φ is *r*ank (preserving) and *a*djacency *p*reserving, the latter meaning that adjacent flags of $\mathscr{P}$ are mapped onto (distinct) adjacent flags of $\mathscr{Q}$. Note that φ is automatically surjective if $\mathscr{Q}$ is flag-connected. A surjective rap-map φ is called a *covering*. A covering φ is called a *k-covering* if it maps sections of $\mathscr{P}$ of rank at most k by an isomorphism onto corresponding sections of $\mathscr{Q}$; that is, if $F, G \in \mathscr{P}$ are such that $F < G$ and rank $G -$ rank $F = k + 1$, then φ induces an isomorphism of G/F onto $G\varphi/F\varphi$. If a covering $\varphi: \mathscr{P} \mapsto \mathscr{Q}$ exists, then we also say that $\mathscr{P}$ is a *covering* of $\mathscr{Q}$, or that $\mathscr{P}$ *covers* $\mathscr{Q}$, or that $\mathscr{Q}$ is *covered* by $\mathscr{P}$. We use similar terminology for k-coverings. A particularly interesting case is that of $(n-1)$-coverings; here, φ preserves the structure of the facets and vertex-figures of $\mathscr{P}$.

(3) Theorem. *Let $\mathscr{P}$ be a polytope, let $\mathscr{Q}$ be a prepolytope, and let $\varphi: \mathscr{P} \mapsto \mathscr{Q}$ be a rap-map. Then $\mathscr{Q}$ is a polytope (and φ is a covering) if and only if the image of each section of $\mathscr{P}$ is a section of $\mathscr{Q}$.*

Proof. Suppose first that the image of every section of $\mathscr{P}$ is a section of $\mathscr{Q}$. Let $\hat{F}, \hat{G}$ be faces of $\mathscr{Q}$ with $\hat{F} < \hat{G}$. Then $\hat{G} = G\varphi$ for some face G of $\mathscr{P}$. Since, by assumption, $\hat{F} \in \hat{G}/\hat{F}_{-1} = (G/F_{-1})\varphi$, we have $\hat{F} = F\varphi$ for some face F with $F < G$. By assumption again, $\hat{G}/\hat{F} = (G/F)\varphi$. Since G/F is flag-connected, it follows that $\hat{G}/\hat{F}$ is flag-connected. Since $\hat{F}, \hat{G}$ are arbitrary, we see that $\mathscr{Q}$ is strongly connected, and hence a polytope.

Conversely, suppose that the image of some section of $\mathscr{P}$ is not a section of $\mathscr{Q}$. Let G/F be a minimal such section of $\mathscr{P}$. If $\hat{F} := F\varphi$, $\hat{G} := G\varphi$, we then have $(G/F)\varphi \subset \hat{G}/\hat{F}$. Hence there is a face $\hat{H}$ of $\mathscr{Q}$, such that $\hat{F} < \hat{H} < \hat{G}$, but $\hat{H} \notin (G/F)\varphi$. However, if $\hat{H}$ is any such face, then no flag of $\hat{G}/\hat{F}$ containing $\hat{H}$ can meet $(G/F)\varphi$ (except at $\hat{F}$ and $\hat{G}$), otherwise G/F would fail to be minimal. It follows that $\hat{G}/\hat{F}$ is not flag-connected, so that $\mathscr{Q}$ is not a polytope. □

(4) Corollary. *Let $\mathscr{P}$ be a polytope, let $\mathscr{Q}$ be a prepolytope, and let $\varphi: \mathscr{P} \mapsto \mathscr{Q}$ be a rap-map. Then $\mathscr{Q}$ is a polytope if and only if, whenever $\hat{F}, \hat{G}, \hat{H}$ are faces of $\mathscr{Q}$ with $\hat{F} < \hat{H} < \hat{G}$, and $\hat{F} = F\varphi$, $\hat{G} = G\varphi$ for some faces F, G of $\mathscr{P}$, then $\hat{H} = H\varphi$ for some $H \in G/F$.*

Proof. This really restates Theorem (3), since the condition $\hat{F} = F\varphi$, $\hat{G} = G\varphi$ with $F < G$ really defines $\hat{G}/\hat{F}$. □

A common way to obtain coverings is the construction of quotients. Let $\mathcal{P}$ be an n-polytope and let N be a subgroup of $A(\mathcal{P})$. In most of our applications $\mathcal{P}$ is regular and N is a normal subgroup of $A(\mathcal{P})$. However, we do not generally impose these restrictions on $\mathcal{P}$ and N.

By $\mathcal{P}/N$ we denote the set of orbits of N in $\mathcal{P}$. For a face F of $\mathcal{P}$ we write its orbit in the form $F \cdot N$. On $\mathcal{P}/N$ we introduce a partial order as follows: if $Z_1, Z_2 \in \mathcal{P}/N$, then $Z_1 \leq Z_2$ if and only if $Z_1 = F \cdot N$ and $Z_2 = G \cdot N$ for some faces F and G of $\mathcal{P}$ with $F \leq G$. The set $\mathcal{P}/N$ together with this partial order is called the *quotient of* $\mathcal{P}$ *with respect to* N. The mapping $\pi: F \mapsto F \cdot N$ from $\mathcal{P}$ onto $\mathcal{P}/N$ is called the *canonical projection.*

(5) Proposition. *Let $\mathcal{P}$ be an n-polytope and let N be a subgroup of $A(\mathcal{P})$. Then $\mathcal{P}/N$ is a flag-connected (ranked) poset of rank n with a unique minimal face and a unique maximal face.*

Proof. Clearly, $F_{-1} \cdot N = \{F_{-1}\}$ and $F_d \cdot N = \{F_d\}$. The rank function of $\mathcal{P}/N$ is given by $\operatorname{rank}(F \cdot N) := \operatorname{rank} F$ for $F \in \mathcal{P}$. The canonical projection $\pi: \mathcal{P} \mapsto \mathcal{P}/N$ induces a surjective mapping $\hat{\pi}: \mathcal{F}(\mathcal{P}) \mapsto \mathcal{F}(\mathcal{P}/N)$ between the sets of flags of $\mathcal{P}$ and $\mathcal{P}/N$; here, if $\Psi = \{G_{-1}, G_0, \ldots, G_n\} \in \mathcal{F}(\mathcal{P})$, then

$$\Psi\hat{\pi} := \{G_{-1} \cdot N, G_0 \cdot N, \ldots, G_n \cdot N\} = \Psi\pi.$$

The surjectivity of $\hat{\pi}$ is an immediate consequence of the definition of the partial order on $\mathcal{P}/N$. However, then the flag-connectedness of $\mathcal{P}/N$ follows from the fact that $\hat{\pi}$ maps adjacent flags of $\mathcal{P}$ onto equal or adjacent flags of $\mathcal{P}/N$. □

Similar arguments to those in the proof of Proposition (5) show that all sections of rank $n - 1$ of $\mathcal{P}/N$ are flag-connected. More generally, if F is a face of $\mathcal{P}$, then any two flags of $\mathcal{P}/N$ which contain the face $F \cdot N$ can be joined by a suitable sequence of flags which all contain $F \cdot N$. However, further connectivity properties of $\mathcal{P}/N$ will depend on the choice of N.

To give an example that $\mathcal{P}/N$ need not be a prepolytope, consider the square tessellation $\mathcal{P} = \{4, 4\}$ with vertex set $\mathbb{Z}^2$, and take for N the group generated by the translation τ with translation vector $(1, 1)$. Then the fundamental region of N is an infinite strip and $\mathcal{P}/N$ is a "tessellation" of this strip. If $\Phi = \{F_0, F_1, F_2\}$ is the base flag of $\mathcal{P}$ with $F_0 = (0, 0)$, $F_1 = \{(0, 0), (0, 1)\}$, and $F_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, then there are four faces of rank 1 in the 1-section $F_2 \cdot N/F_0 \cdot N$ of $\mathcal{P}/N$, namely, $H \cdot N$ with H an edge of F_2.

Generally we are interested in quotients $\mathcal{P}/N$ which are again polytopes and have certain kinds of sections isomorphic to those of the original polytope $\mathcal{P}$. This property can be achieved by imposing certain conditions on N.

Let $-1 \leq i < j \leq n$, and let F be an i-face and let G be a j-face of $\mathcal{P}$ such that $F < G$. Then $F\pi$ is an i-face and $G\pi$ is a j-face of $\mathcal{P}/N$ such that $F\pi < G\pi$. Let Γ be a chain of $\mathcal{P}$ of type $\{-1, 0, \ldots, i-1, i, j, j+1, \ldots, n\}$ which contains F and

G. By N_Γ we denote the stabilizer of Γ in N. Then we have the canonical projection $\beta\colon G/F \mapsto (G/F)/N_\Gamma$ as well as the incidence-preserving mapping

$$(6)\qquad \begin{cases} \gamma\colon (G/F)/N_\Gamma \mapsto G\pi/F\pi, \\ \qquad H\cdot N_\Gamma \mapsto H\cdot N \qquad (H\in G/F). \end{cases}$$

In particular, $\beta\gamma = \pi$ (or, more exactly, $\beta\gamma$ is the restriction of π given by $\pi\colon G/F \mapsto G\pi/F\pi$). Further, γ is injective if and only if two faces of G/F are equivalent modulo N_Γ whenever they are equivalent modulo N. Note that if γ is bijective, it must be an isomorphism; in fact, the special nature of γ guarantees that γ^{-1} preserves incidence. Clearly, if γ is an isomorphism, then it induces a bijection

$$\mathscr{F}((G/F)/N_\Gamma) \mapsto \mathscr{F}(G\pi/F\pi).$$

(7) Proposition. *Let $\mathscr{P}$ be an n-polytope and let N be a subgroup of $A(\mathscr{P})$. Then $\mathscr{P}/N$ is an n-polytope if and only if the following two conditions hold:*

(a) *γ of* (6) *is surjective for all F, G, and Γ as above.*
(b) *β, γ are injective if* rank G − rank $F = 2$.

Proof. First, note that γ is surjective if and only if π (restricted to G/F) is surjective. Also, each flag of $\mathscr{P}/N$ has exactly one i-adjacent flag for each i if and only if β, γ are isomorphisms when rank G − rank $F = 2$; in this case, π is a rap-map. However, then the statement follows from Theorem (3) and Proposition (5). □

By $\mathscr{F}(\mathscr{P})/N$ we denote the set of orbits of $\mathscr{F}(\mathscr{P})$ under the action of N. Then we have the mapping

$$(8)\qquad \begin{aligned} &\mu\colon \mathscr{F}(\mathscr{P})/N \mapsto \mathscr{F}(\mathscr{P}/N), \\ &\quad \Psi\cdot N \mapsto \Psi\pi \qquad (\Psi\in\mathscr{F}(\mathscr{P})). \end{aligned}$$

It follows from the proof of Proposition (5) that μ is surjective. Our next theorem answers the question when $\mathscr{F}(\mathscr{P})/N$ and $\mathscr{F}(\mathscr{P}/N)$ can be naturally identified under μ.

(9) Theorem. *Let $\mathscr{P}$ be an n-polytope and let N be a subgroup of $A(\mathscr{P})$ such that $\mathscr{P}/N$ is a polytope. Then μ of* (8) *is a bijection if and only if, for all F, G, and Γ as above, γ of* (6) *is an isomorphism.*

Proof. By Proposition (7), the maps γ are surjective. Now suppose that μ is a bijection, and let F, G, and Γ be as above. To prove that $\gamma\colon (G/F)/N_\Gamma \mapsto G\pi/F\pi$ is injective, let K and L be faces in G/F such that $K\cdot N = L\cdot N$. Choose flags Ψ and Ω of $\mathscr{P}$ such that $\Gamma\subset\Psi$, Ω and $K\in\Psi$, $L\in\Omega$. Then $\hat{\Psi} := \{H\cdot N \mid H\in\Psi\}$ and

$\hat{\Omega} := \{H \cdot N | H \in \Omega\}$ are flags of $\mathscr{P}/N$ which contain the face $K \cdot N = L \cdot N$ as well as the chain $\hat{\Gamma} := \{H \cdot N | H \in \Gamma\}$. Let $k :=$ rank K (=rank L). Since $\mathscr{P}/N$ is a polytope, a sequence

$$\hat{\Psi} = \hat{\Psi}_0, \hat{\Psi}_1, \ldots, \hat{\Psi}_{m-1}, \hat{\Psi}_m = \hat{\Omega}$$

of flags of $\mathscr{P}/N$, all containing $K \cdot N$ and $\hat{\Gamma}$, exists such that $\hat{\Psi}_{l-1}$, $\hat{\Psi}_l$ are adjacent for $l = 1, \ldots, m$; then $\hat{\Psi}_{l-1}$, $\hat{\Psi}_l$ are i_l-adjacent for some i_l with rank $F = i < i_l < j =$ rank G, $i_l \neq k$. Nevertheless μ is a bijection, so that, for all $l = 0, \ldots, m$, a flag Ψ_l of $\mathscr{P}$ exists such that $\hat{\Psi}_l = (\Psi_l \cdot N)\mu$. Here we may assume that $\Psi_0 = \Psi$. However, now, by condition (b) of Proposition (7), adjacent flags of $\mathscr{P}$ are never in the same orbit of N, so that the canonical mapping $\mathscr{F}(\mathscr{P}) \mapsto \mathscr{F}(\mathscr{P})/N$ followed by μ sends adjacent flags of $\mathscr{P}$ to adjacent flags of $\mathscr{P}/N$. It follows that, starting with $\Psi_0 = \Psi$, we can choose the flags Ψ_l of $\mathscr{P}$ in such a way that Ψ_{l-1} and Ψ_l are i_l-adjacent for $l = 1, \ldots, m$; here, i_l is as above. This gives us a sequence

$$\Psi \cdot N = \Psi_0 \cdot N, \Psi_1 \cdot N, \ldots, \Psi_{m-1} \cdot N, \Psi_m \cdot N = \Omega \cdot N$$

of orbits of N, where the sequence $\Psi = \Psi_0, \Psi_1, \ldots, \Psi_m$ in $\mathscr{F}(\mathscr{P})$ has the same adjacency properties as the corresponding sequence in $\mathscr{F}(\mathscr{P}/N)$. Note that here we cannot conclude that $\Psi_m = \Omega$. However, since $i < i_l < j$ and $i_l \neq k$ for all l, each flag Ψ_l of $\mathscr{P}$ must contain both the face K and the chain Γ. Also $\Psi_m \cdot N = \Omega \cdot N$, so that $\varphi \in N$ exists such that $\Psi_m \varphi = \Omega$ and thus $K\varphi = L$. However $\Gamma \subset \Omega$, so that $\varphi \in N_\Gamma$. It follows that $K \cdot N_\Gamma = L \cdot N_\Gamma$. This proves that γ is injective.

Conversely, let all maps γ be isomorphisms. To prove that μ is injective, let $\Psi_1 = \{F_{-1}, G_0, \ldots, G_{n-1}, F_n\}$, $\Psi_2 = \{F_{-1}, H_0, \ldots, H_{n-1}, F_n\}$ be flags of $\mathscr{P}$ such that $\Psi_1 \pi = \Psi_2 \pi$. Then $\Psi_1 \cdot N = \Psi_2 \cdot N$ if we can prove by induction on j that $\varphi \in N$ exists such that $G_i \varphi = H_i$ for $0 \leq i \leq j$. For $j = 0$ this is true by assumption. To prove it for $j + 1$, assume $\varphi \in N$ has been chosen such that $G_i \varphi = H_i$ for $i \leq j$. Then $H_j \leq G_{j+1}\varphi, H_{j+1}$. However, $G_{j+1} \cdot N = H_{j+1} \cdot N$, so that $G_{j+1}\varphi$ and H_{j+1} are faces of the section F_n/H_j which are in the same orbit of N. On the other hand, the map γ defined by $G := F_n$, $F := H_j$, and $\Gamma := \{F_{-1}, H_0, \ldots, H_j, F_n\}$ is an isomorphism, so that $G_{j+1}\varphi$ and H_{j+1} are also in the same orbit of N_Γ; that is, $G_{j+1}\varphi\tau = H_{j+1}$ for some $\tau \in N_\Gamma$. Thus, if $i \leq j$ we also have $G_i \varphi\tau = H_i \tau = H_i$, so that $\varphi\tau$ has the required property with respect to $j + 1$. This completes the proof of Theorem (9). □

Theorem (9) explains why in most applications it is natural to require that the mappings γ of (6) are isomorphisms. We usually impose this condition. For regular polytopes we also give a further justification in Theorem (15). The following theorem identifies the groups of quotients of regular polytopes.

(10) Theorem. *Let $\mathscr{P}$ be a regular n-polytope and let N be a subgroup of $A(\mathscr{P})$. Assume that, for all F, G, and Γ as above, γ of (6) is an isomorphism, and that*

β is injective if rank G − rank $F = 2$. *Then $\mathscr{P}/N$ is an n-polytope with group $A(\mathscr{P}/N) \cong B/N$, where B is the normalizer of N in $A(\mathscr{P})$. In particular, $\mathscr{P}/N$ is regular if and only if N is normal in $A(\mathscr{P})$, in which case $A(\mathscr{P}/N) \cong A(\mathscr{P})/N$.*

Proof. By Proposition (7) and Theorem (9), $\mathscr{P}/N$ is an n-polytope and μ of (8) is a bijection. Now, to prove the statement about $A(\mathscr{P}/N)$, observe that for each $\varphi \in B$ the mapping $\hat{\varphi}: F \cdot N \mapsto F\varphi \cdot N$ defines an automorphism of $\mathscr{P}/N$. Thus there is a homomorphism $\alpha: B \mapsto A(\mathscr{P}/N)$ given by $\varphi\alpha = \hat{\varphi}$. Let $\tau \in \ker(\alpha)$. Then, for any $\Psi \in \mathscr{F}(\mathscr{P})$ we have $\Psi\pi = (\Psi\tau)\pi$. Hence, since μ is injective, $\Psi\tau = \Psi\varphi$ for some $\varphi \in N$, and thus $\tau = \varphi \in N$. It follows that $\ker(\alpha) = N$. It remains to show that α is surjective.

Let $\rho \in A(\mathscr{P}/N)$ and let $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ be the base flag of $\mathscr{P}$. Since $\mathscr{P}$ is regular (by assumption) and ρ preserves flags of $\mathscr{P}/N$, we have $(\Phi\pi)\rho = (\Phi\varphi)\pi$ for some $\varphi \in A(\mathscr{P})$. We shall prove that this implies $((\Phi\psi)\pi)\rho = (\Phi\psi\varphi)\pi$ for each $\psi \in A(\mathscr{P})$.

To see this, join the flags Φ and $\Phi\psi$ of $\mathscr{P}$ by a sequence

$$\Phi = \Phi_0, \Phi_1, \ldots, \Phi_{k-1}, \Phi_k = \Phi\psi,$$

in which any two consecutive flags are adjacent. Then

$$\Phi\varphi = \Phi_0\varphi, \Phi_1\varphi, \ldots, \Phi_{k-1}\varphi, \Phi_k\varphi = \Phi\psi\varphi$$

is a similar such sequence joining $\Phi\varphi$ and $\Phi\psi\varphi$, in which the flags $\Phi_{m-1}\varphi$ and $\Phi_m\varphi$ differ in a face of the same rank as Φ_{m-1} and Φ_m ($m = 1, \ldots, k$). However, the images of adjacent flags in $\mathscr{P}$ under the canonical mapping $\mathscr{F}(\mathscr{P}) \mapsto \mathscr{F}(\mathscr{P}/N)$ cannot coincide, so that the sequences

$$\Phi\pi = \Phi_0\pi, \Phi_1\pi, \ldots, \Phi_{k-1}\pi, \Phi_k\pi = (\Phi\psi)\pi$$

and

$$(\Phi\varphi)\pi = (\Phi_0\varphi)\pi, (\Phi_1\varphi)\pi, \ldots, (\Phi_{k-1}\varphi)\pi, (\Phi_k\varphi)\pi = (\Phi\psi\varphi)\pi$$

have the same adjacency properties as their preimages. Hence, since $(\Phi\pi)\rho = (\Phi\varphi)\pi$ and ρ preserves the type of adjacency, we must have $(\Phi_m\pi)\rho = (\Phi_m\varphi)\pi$ for all m. For $m = k$ this proves $((\Phi\psi)\pi)\rho = (\Phi\psi\varphi)\pi$, as required.

Now, for $\psi \in N$, this equation implies

$$(\Phi\psi\varphi)\pi = ((\Phi\psi)\pi)\rho = (\Phi\pi)\rho = (\Phi\varphi)\pi.$$

However, this in turn shows that $\Phi\psi\varphi = \Phi\varphi\tau$ and thus $\psi\varphi = \varphi\tau$ for some $\tau \in N$. It follows that $\varphi \in B$. Also, since $((\Phi\psi)\pi)\rho = ((\Phi\psi)\varphi)\pi$ for each $\psi \in A(\mathscr{P})$, we have

$$(F \cdot N)\rho = (F\pi)\rho = (F\varphi)\pi = F\varphi \cdot N = (F \cdot N)\hat{\varphi}$$

for each F in $\mathscr{P}$, and thus $\rho = \hat{\varphi}$. This proves that α is surjective, and hence $A(\mathscr{P}/N) = B/N$.

Finally, consider the base flag $\Phi\pi$ of $\mathscr{P}/N$. Then any flag of $\mathscr{P}/N$ is of the form $(\Phi\varphi)\pi$ with $\varphi \in A(\mathscr{P})$. Hence, $\mathscr{P}/N$ is regular if and only if, for each $\varphi \in A(\mathscr{P})$, $\tau \in B$ exists such that $(\Phi\varphi)\pi = (\Phi\pi)\hat{\tau} = (\Phi\tau)\pi$ and thus $(\Phi\varphi) \cdot N = (\Phi\tau) \cdot N$, or, equivalently, $\varphi \in \tau N \subset B$. It follows that $\mathscr{P}/N$ is regular if and only if $B = A(\mathscr{P})$, in which case $B/N = A(\mathscr{P})/N$. This completes the proof. □

Note that in Theorem (10) the hypothesis on β is equivalent to requiring that N does not contain a conjugate of a distinguished generator ρ_i of $A(\mathscr{P})$. The following proposition deals with an interesting special case in which all mappings γ are isomorphisms. See also Theorem (16) for an equivalent version of Theorem (11), in the regular case.

(11) Theorem. *Let $\mathscr{P}$ be an n-polytope and let N be a subgroup of $A(\mathscr{P})$ such that each orbit of N intersects each proper section of $\mathscr{P}$ in at most one face. Then, for all F, G, and Γ as above, the map γ of (6) is an isomorphism (with $N_\Gamma = 1$ if G/F is proper) and the sections $G\pi/F\pi$ of $\mathscr{P}/N$ and G/F of $\mathscr{P}$ are isomorphic. In particular, $\pi\colon \mathscr{P} \mapsto \mathscr{P}/N$ is an $(n-1)$-covering of n-polytopes.*

Proof. Let G/F be a proper section of $\mathscr{P}$. By our assumptions, $N_\Gamma = 1$, so that in (6) we may identify $H \cdot N_\Gamma$ with H. Since $G/F \cap H \cdot N = \{H\}$ for all $H \in G/F$, the map γ is clearly injective. To prove surjectivity let $H \cdot N \in G\pi/F\pi = G \cdot N/F \cdot N$. Here we may assume that $F \leq H$. Also, $\varphi \in N$ exists such that $H \leq G\varphi$. However, if $F \neq F_{-1}$, then G and $G\varphi$ are faces in the proper section F_n/F of $\mathscr{P}$ which are in the same orbit; hence we must have $G = G\varphi$ and $H \in G/F$. If $F = F_{-1}$, then $G \neq F_n$ and similar arguments work with the roles of F and G interchanged. It follows that γ is surjective. Note also that β is injective if G/F is proper. □

Concluding this section we discuss analogues of the intersection property for the groups of regular polytopes.

(12) Proposition. *Let $\mathscr{P}$ be a regular n-polytope with group*

$$A(\mathscr{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle,$$

and let N be a subgroup of $A(\mathscr{P})$ such that N does not contain a conjugate of a generator ρ_i and, for all F, G, and Γ as above, γ of (6) is an isomorphism. Then, for each i, j with $-1 \leq i < j \leq n$, and each $\varphi \in A(\mathscr{P})$, we have

$$\langle \rho_k | k > i \rangle \varphi N \cap \langle \rho_k | k < j \rangle \varphi N = \langle \rho_k | i < k < j \rangle \varphi N.$$

Proof. By assumption, $\mathscr{P}/N$ is a polytope and μ of (8) is a bijection. Let $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ be the base flag of $\mathscr{P}$. Write $\hat{F}_k := F_k\varphi \cdot N$ for $k = -1, \ldots, n$. Consider the orbits $(\Phi\tau\varphi) \cdot N$, with

$$\tau\varphi N \subset \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle \varphi N \cap \langle \rho_0, \ldots, \rho_{j-1} \rangle \varphi N.$$

Their images under μ all contain $\{\hat{F}_{-1}, \hat{F}_0, \ldots, \hat{F}_i, \hat{F}_j, \hat{F}_{j+1}, \ldots, \hat{F}_n\}$, so that their restrictions to sections of $\mathscr{P}/N$ of type $\{i, \ldots, j\}$ form flags of $\hat{F}_j/\hat{F}_i$. However, the map γ with

$$F = F_i\varphi, \qquad G = F_j\varphi, \qquad \Gamma = \{F_{-1}, F_0\varphi, \ldots, F_i\varphi, F_j\varphi, F_{j+1}\varphi, \ldots, F_n\}$$

is an isomorphism, so that these flags are the images under μ of orbits of the form $(\Phi\tau\varphi) \cdot N$ with $\tau \in \langle \rho_{i+1}, \ldots, \rho_{j-1} \rangle$. Nevertheless, μ is bijective, so that the orbits must actually coincide. Thus $\tau\varphi N \subset \langle \rho_{j+1}, \ldots, \rho_{k-1} \rangle \varphi N$. This proves one inclusion, and the opposite inclusion is trivial. □

Note that, if N is normal in $A(\mathscr{P})$, Proposition (12) just gives the intersection property for $A(\mathscr{P}/N) = A(\mathscr{P})/N$, the group of the regular polytope $\mathscr{P}/N$. For a general N, the condition of Proposition (12) is really a condition for orbits of flags of $\mathscr{P}$, and thus cannot be expected to characterize polytopality of $\mathscr{P}/N$ completely. In contrast, the condition of our next proposition can be seen as a condition for flags of $\mathscr{P}/N$, and does indeed characterize polytopality.

(13) Proposition. *Let $\mathscr{P}$ be a regular n-polytope with group*

$$A(\mathscr{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle,$$

and let N be a subgroup of $A(\mathscr{P})$. Then $\mathscr{P}/N$ is an n-polytope if and only if the following two conditions hold:

(a) *For each i, j, k with $-1 \le i < k < j \le n$, and each $\varphi \in A(\mathscr{P})$,*

$$\langle \rho_l | l \ne k \rangle \langle \rho_l | l > i \rangle \varphi N \cap \langle \rho_l | l \ne k \rangle \langle \rho_l | l < j \rangle \varphi N = \langle \rho_l | l \ne k \rangle \langle \rho_l | i < l < j \rangle \varphi N.$$

(b) *For each $k = 0, \ldots, n-1$ and each $\varphi \in A(\mathscr{P})$,*

$$\langle \rho_l | l \ne k \rangle \rho_k \cap \varphi^{-1} N \varphi = \varnothing.$$

Proof. We use Proposition (7). First we prove that (a) of Proposition (7) and (a) of Proposition (13) are equivalent. Let $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ be the base flag of $\mathscr{P}$. Then general chains of $\mathscr{P}$ are equivalent under $A(\mathscr{P})$ to subsets of Φ, so that for Proposition (7) we may assume that $F = F_i\varphi$, $G = F_j\varphi$, and $\Gamma = \{F_l\varphi | l \ne i+1, \ldots, j-1\}$ for some $\varphi \in A(\mathscr{P})$, and thus we have the mapping $\gamma\colon (F_j\varphi/F_i\varphi)/N_\Gamma \mapsto F_j\varphi \cdot N/F_i\varphi \cdot N$. Let $i < k < j$. The k-faces of $\mathscr{P}/N$ incident with $F_j\varphi \cdot N$ are just those of the form $F_k\alpha_1\varphi \cdot N$ with $\alpha_1 \in \langle \rho_0, \ldots, \rho_{j-1} \rangle$, while those incident with $F_i\varphi \cdot N$ are of the form $F_k\alpha_2\varphi \cdot N$ with $\alpha_2 \in \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle$. In particular, the k-faces in $F_j\varphi \cdot N/F_i\varphi \cdot N$ can be expressed both ways, and if $F_k\alpha_1\varphi \cdot N = F_k\alpha_2\varphi \cdot N$, then we have $\alpha_1\varphi \in \langle \rho_l | l \ne k \rangle \alpha_2\varphi N$.

Hence, if the condition in (a) holds, then $\alpha_1\varphi \in \langle \rho_l | l \neq k\rangle \alpha_3 \varphi N$ for some $\alpha_3 \in \langle \rho_l | i < l < j\rangle$, and thus $F_k\alpha_1\varphi \cdot N = F_k\alpha_3\varphi \cdot N$; but $F_k\alpha_3\varphi \in F_j\varphi/F_i\varphi$, so that γ is surjective. Conversely, if γ is surjective, then a k-face $F_k\alpha_1\varphi \cdot N = F_k\alpha_2\varphi \cdot N$ in $F_j\varphi \cdot N/F_i\varphi \cdot N$ is of the form $F_k\alpha_3\varphi \cdot N$ with $\alpha_3 \in \langle \rho_l | i < l < j\rangle$. Hence, if $\beta_1, \beta_2 \in \langle \rho_l | l \neq k\rangle$, $\alpha_1 \in \langle \rho_0, \ldots, \rho_{j-1}\rangle$, and $\alpha_2 \in \langle \rho_{i+1}, \ldots, \rho_{n-1}\rangle$ are such that $\beta_1\alpha_1\varphi N = \beta_2\alpha_2\varphi N$, then $F_k\alpha_1\varphi \cdot N = F_k\alpha_2\varphi \cdot N = F_k\alpha_3\varphi \cdot N$ for some $\alpha_3 \in \langle \rho_l | i < l < j\rangle$, and thus $\beta_1\alpha_1\varphi N \subset \langle \rho_l | l \neq k\rangle\langle \rho_l | i < l < j\rangle\varphi N$. This proves one inclusion of (a), and the other is trivial.

Next we need to show that parts (b) of Propositions (7) and (13) correspond to each other. First note that N does not contain a conjugate of a ρ_i (this is a weaker requirement than (b)) if and only if orbits under N of adjacent flags of $\mathscr{P}$ are distinct; in fact, if $\varphi \in A(\mathscr{P})$, then $(\Phi\varphi)^i \cdot N = (\Phi\varphi)\varphi^{-1}\rho_i\varphi \cdot N$. Hence, if (b) of Proposition (13) holds, then the orbits of adjacent flags of $\mathscr{P}$ must be distinct, so that β, γ of (b) in Proposition (7) must be injective. Conversely, assume that (b) of Proposition (13) does not hold for some k and φ, that is, $\rho_k\varphi^{-1} \in \langle \rho_l | l \neq k\rangle \varphi^{-1}N$. However, then $F_k\rho_k\varphi^{-1} \cdot N = F_k\varphi^{-1} \cdot N$, so that the two faces $F_k\rho_k\varphi^{-1}$ and $F_k\varphi^{-1}$ of the section $F_{k+1}\varphi^{-1}/F_{k-1}\varphi^{-1}$ are identified under $\pi = \beta\gamma$. Hence β, γ cannot both be injective. This completes the proof. □

4. C-Groups

In verifying that a given group is a string C-group the following *quotient lemma* is sometimes useful.

(14) Lemma. *Let $A = \langle \rho_0, \ldots, \rho_{n-1}\rangle$ be a group with property* (2), *and let $\hat{A} = \langle \sigma_0, \ldots, \sigma_{n-1}\rangle$ be a string* C*-group (with respect to the distinguished generators σ_i). If the mapping $\rho_j \mapsto \sigma_j$ $(j = 0, \ldots, n-1)$ induces a homomorphism $\alpha: A \mapsto \hat{A}$, which is one-to-one on $A_{n-1} := \langle \rho_0, \ldots, \rho_{n-2}\rangle$ or on $A_0 := \langle \rho_1, \ldots, \rho_{n-1}\rangle$, then A is also a string* C*-group, and α induces a covering $\mathscr{P}(A) \mapsto \mathscr{P}(\hat{A})$ of the corresponding polytopes.*

Proof. Assume that α is one-to-one on A_{n-1} (say). Since A_{n-1} is a string C-group, it suffices to check

$$A_{n-1} \cap \langle \rho_j, \ldots, \rho_{n-1}\rangle = \langle \rho_j, \ldots, \rho_{n-2}\rangle$$

for each $j = 1, \ldots, n-1$. Let $\varphi \in A_{n-1} \cap \langle \rho_j, \ldots, \rho_{n-1}\rangle$. Then

$$\varphi\alpha \in \langle \sigma_0, \ldots, \sigma_{n-2}\rangle \cap \langle \sigma_j, \ldots, \sigma_{n-1}\rangle = \langle \sigma_j, \ldots, \sigma_{n-2}\rangle,$$

and hence $\varphi\alpha$ has a preimage in $\langle \rho_j, \ldots, \rho_{n-2}\rangle$. However, α is one-to-one on A_{n-1}, so that φ is the only preimage of $\varphi\alpha$ in A_{n-1}. It follows that $\varphi \in \langle \rho_j, \ldots, \rho_{n-2}\rangle$.

This proves one inclusion, and the other is trivial. Therefore A is a string C-group. Finally, it is immediate that

$$\langle \rho_j | j \neq i \rangle \varphi \mapsto \langle \sigma_j | j \neq i \rangle (\varphi\alpha) \qquad (i = -1, 0, \ldots, n; \varphi \in A)$$

gives a covering $\mathscr{P}(A) \mapsto \mathscr{P}(\hat{A})$. □

Our next two propositions relate quotients of string C-groups to quotients of polytopes. In particular, in Theorem (15) it is explained why in our discussion in Section 3 it was usually assumed that the maps γ of (6) are isomorphisms. Finally, Theorem (16) is an equivalent version of Theorem (11).

(15) Theorem. *Let $\mathscr{P}$ be a regular n-polytope with group $A(\mathscr{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and let N be a normal subgroup of $A(\mathscr{P})$. If $A(\mathscr{P})/N$ is a string C-group (with distinguished generators $N\rho_0, \ldots, N\rho_{n-1}$), then the two polytopes $\mathscr{P}/N$ and $\mathscr{P}(A(\mathscr{P})/N)$ are isomorphic, with group $A(\mathscr{P}/N) = A(\mathscr{P})/N$, and all maps γ of* (6) *are isomorphisms.*

Proof. Let $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ be the base flag of $\mathscr{P}$. Write $\hat{A} := A(\mathscr{P})/N$ and let $\alpha: A(\mathscr{P}) \mapsto \hat{A}$ be the canonical projection. For $i = -1, 0, \ldots, n$ define $A_i := \langle \rho_j | j \neq i \rangle$ and $\hat{A}_i := \langle N\rho_j | j \neq i \rangle$. Note that $\hat{A}_i \alpha^{-1} = A_i N$ for each i. By assumption $\hat{A}$ is a string C-group, so that $\mathscr{P}(\hat{A})$ is defined, with base flag $\{\hat{A}_{-1}, \hat{A}_0, \ldots, \hat{A}_n\}$. Consider the mapping

$$\begin{aligned} \kappa: \quad & \mathscr{P}/N \mapsto \mathscr{P}(\hat{A}), \\ & F_i\varphi \cdot N \mapsto \hat{A}_i(\varphi\alpha) \qquad (i = -1, 0, \ldots, n; \varphi \in A(\mathscr{P})). \end{aligned}$$

For $\varphi, \psi \in A(\mathscr{P})$, we have $F_i\varphi \cdot N = F_i\psi \cdot N$ in $\mathscr{P}/N$ if and only if $A_i N\varphi = A_i N\psi$; that is, if and only if $\hat{A}_i(\varphi\alpha) = \hat{A}_i(\psi\alpha)$ in $\mathscr{P}(\hat{A})$. It follows that κ is a bijection. However, κ also preserves incidence in both directions. In fact, in $\mathscr{P}/N$ a pair of incident faces is of the form $F_i\varphi \cdot N$ and $F_j\tau\varphi \cdot N$ with $i \leq j$, $\varphi \in A(\mathscr{P})$, and $\tau \in \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle$; by κ they are mapped onto the faces $\hat{A}_i(\varphi\alpha)$ and $\hat{A}_j(\tau\alpha)(\varphi\alpha)$ of $\mathscr{P}(\hat{A})$, which are again incident because $\tau\alpha \in \langle N\rho_{i+1}, \ldots, N\rho_{n-1} \rangle$. Conversely, if $\hat{A}_i(\varphi\alpha)$ and $\hat{A}_j(\tau\alpha)(\varphi\alpha)$ are incident in $\mathscr{P}(\hat{A})$, with $\tau\alpha \in \langle N\rho_{i+1}, \ldots, N\rho_{n-1} \rangle$, then $\tau \in \langle \rho_{i+1}, \ldots, \rho_{n-1} \rangle N$ and thus $F_i\varphi \cdot N$ and $F_j\tau\varphi \cdot N$ are incident in $\mathscr{P}/N$.

It follows that $\mathscr{P}/N$ and $\mathscr{P}(\hat{A})$ are isomorphic polytopes and thus have the same group $\hat{A}$. By Theorem (9), all maps γ are isomorphisms if the mapping $\mu: \mathscr{F}(\mathscr{P})/N \mapsto \mathscr{F}(\mathscr{P}/N)$ of (8) is a bijection. To prove this, consider the chain of maps

$$\begin{aligned} & \mathscr{F}(\mathscr{P}(\hat{A})) \overset{\mu_1}{\mapsto} \hat{A} \overset{\mu_2}{\mapsto} \mathscr{F}(\mathscr{P})/N \overset{\mu}{\mapsto} \mathscr{F}(\mathscr{P}/N), \\ & \{\hat{A}_i(\varphi\alpha)\}_i \mapsto \varphi\alpha \mapsto \Phi\varphi \cdot N \mapsto \{F_i\varphi \cdot N\}_i \; (= (\Phi\varphi)\pi). \end{aligned}$$

Here, μ_1 is a bijection because $\mathscr{P}(\hat{A})$ is a regular polytope with (the simply flag-transitive) group $\hat{A}$; and μ_2 is a bijection because $\Phi\varphi \cdot N = \Phi\psi \cdot N$ in $\mathscr{F}(\mathscr{P})/N$ if and only if $\varphi\alpha = N\varphi = N\psi = \psi\alpha$. Nevertheless, $\mu_1\mu_2\mu$ is also a bijection because it is the map induced by the isomorphism $\kappa^{-1}: \mathscr{P}(\hat{A}) \mapsto \mathscr{P}/N$. Hence μ must be a bijection, and the proof is complete. □

(16) Theorem. *Let $\mathscr{P}$ be a regular n-polytope with group $A(\mathscr{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$, and let N be a normal subgroup of $A(\mathscr{P})$ such that*

$$N \cap \langle \rho_1, \ldots, \rho_{n-1} \rangle \langle \rho_0, \ldots, \rho_{n-2} \rangle = \{\varepsilon\}.$$

Then

(a) *$A(\mathscr{P})/N$ is a string* C-*group, and $\mathscr{P}/N$ and $\mathscr{P}(A(\mathscr{P})/N)$ are isomorphic regular polytopes with group $A(\mathscr{P}/N) = A(\mathscr{P})/N$.*

(b) *The facets and vertex-figures of $\mathscr{P}/N$ are isomorphic to the facets and vertex-figures of $\mathscr{P}$, respectively; that is, $\mathscr{P}$ is an $(n-1)$-covering of $\mathscr{P}/N$.*

Proof. We use the same notation as in the proof of Theorem (15). First note that $N \cap A_{n-1} = \{\varepsilon\} = N \cap A_0$, so that the restrictions of α to the subgroups A_{n-1} and A_0 are isomorphisms. In particular, $A_{n-1}\alpha = \langle N\rho_0, \ldots, N\rho_{n-2} \rangle$ and $A_0\alpha = \langle N\rho_1, \ldots, N\rho_{n-1} \rangle$ are string C-groups.

To prove that $\hat{A}$ is a string C-group observe that (2) holds trivially. For (1) it suffices to check

$$A_{n-1}\alpha \cap A_0\alpha = \langle \rho_1, \ldots, \rho_{n-2} \rangle \alpha,$$

because both groups on the left are string C-groups. Let $\varphi \in A_{n-1}$, $\psi \in A_0$, and $\varphi\alpha = \psi\alpha$. Then $\psi\varphi^{-1} \in \ker(\alpha) \cap A_0 A_{n-1} = N \cap A_0 A_{n-1} = \{\varepsilon\}$ and thus $\varphi = \psi \in A_{n-1} \cap A_0 = \langle \rho_1, \ldots, \rho_{n-2} \rangle$. However, this proves (1). Now we can apply Theorem (15) to complete the proof of part (a). Part (b) follows from $\hat{A}_{n-1} \cong A_{n-1}$ and $\hat{A}_0 \cong A_0$. □

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