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Regular Polytopes in Ordinary Space*

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Abstract. The three aims of this paper are to obtain the proof by Dress of the completeness of the enumeration of the Grünbaum–Dress polyhedra (that is, the regular apeirohedra, or apeirotopes of rank 3) in ordinary space $\mathbb{E}^3$ in a quicker and more perspicuous way, to give presentations of those of their symmetry groups which are affinely irreducible, and to describe all the discrete regular apeirotopes of rank 4 in $\mathbb{E}^3$. The paper gives a complete classification of the discrete regular polytopes in ordinary space.

1. Introduction

The theory of regular polytopes has undergone a number of changes since its origins in the classification by the Greeks of the five regular ("Platonic") solids, but most particularly during this century. At the forefront of this development is Coxeter, whose book *Regular Polytopes* [3] covers what might be called the classical theory. However, already in Coxeter's work a more abstract approach begins to be manifested. One generalization of regular polyhedron is that of a *regular map* (see, for example, [4]).

The starting point of the present paper is 1926, when Petrie found the two dual regular apeirohedra (infinite polyhedra) $\{4, 6|4\}$ and $\{6, 4|4\}$ in $\mathbb{E}^3$, which have planar faces but skew vertex-figures (these technical terms are defined in Section 2). Immediately afterwards, Coxeter discovered a third such example: $\{6, 6|3\}$ (see [2]). Around 1975 Grünbaum (see [8]) restored the symmetry by allowing skew faces as well (although implicitly these were permitted by Coxeter also in using the Petrie operation); he found

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20 more regular apeirohedra in $\mathbb{E}^3$. A final instance was discovered by Dress around 1980 (see [5] and [6]), who also proved the completeness of the enumeration.

Our aims here are threefold. First, we describe a far quicker method of arriving at Dress's characterization result. The key to this method is a "trick" employed in the proofs of Theorems 4.5 and 6.2, the essence of which is to replace a reflexion whose mirror is a line by one whose mirror is a perpendicular plane. This idea also leads to new pairings between the finite regular polyhedra. Second, we give presentations of the symmetry groups of those discrete regular apeirohedra in $\mathbb{E}^3$ whose groups are affinely irreducible; an important ingredient here is Theorem 2.5, which says that such a presentation arises solely from the vertex-figure and the edge-circuits. Third, we describe seven new discrete regular apeirotopes of rank 4 in $\mathbb{E}^3$, to add to the familiar honeycomb $\{4, 3, 4\}$ of cubes, and prove that there are no others. Thus the paper contains a complete classification of the discrete regular polytopes in ordinary space.

2. Abstract Regular Polytopes

Since we discuss regular polytopes on the abstract as well as the geometric level, we begin with a brief introduction to the underlying general theory (see, for example, [12] and [13]). An (*abstract*) *polytope of rank* n, or simply an n-*polytope*, satisfies the following properties. It is a partially ordered set $\mathcal{P}$ with a strictly monotone *rank* function whose range is $\{-1, 0, \dots, n\}$. The elements of rank j are called the j-*faces* of $\mathcal{P}$, and the family of such j-faces is denoted $\mathcal{P}_j$. For $j = 0, 1, n-2$, or $n-1$, we also call j-faces *vertices*, *edges*, *ridges*, and *facets*, respectively. The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ each contain exactly $n+2$ faces, including the unique minimal face F_{-1} and unique maximal face F_n of $\mathcal{P}$. Further, $\mathcal{P}$ is *strongly flag-connected*, meaning that any two flags Φ and Ψ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \dots, \Phi_k = \Psi$, where Φ_{i-1} and Φ_i are *adjacent* (differ by one face), and $\Phi \cap \Psi \subseteq \Phi_i$ for each i. Finally, if F and G are a $(j-1)$-face and a $(j+1)$-face with $F < G$, then there are exactly *two* j-faces H such that $F < H < G$.

When F and G are two faces of a polytope $\mathcal{P}$ with $F \leq G$, we call $G/F := \{H \mid F \leq H \leq G\}$ a *section* of $\mathcal{P}$. We may usually safely identify a face F with the section F/F_{-1}. For a face F the section F_n/F is called the *co-face of* $\mathcal{P}$ *at* F, or the *vertex-figure at* F if F is a vertex.

An n-polytope $\mathcal{P}$ is *regular* if its (*automorphism*) *group* $\Gamma(\mathcal{P})$ is transitive on its flags. Let $\Phi := \{F_{-1}, F_0, \dots, F_{n-1}, F_n\}$ be a fixed or *base* flag of $\mathcal{P}$. The group $\Gamma(\mathcal{P})$ of a regular n-polytope $\mathcal{P}$ is generated by *distinguished generators* $\rho_0, \dots, \rho_{n-1}$ (*with respect to* Φ), where ρ_j is the unique automorphism which keeps all but the j-face of Φ fixed. These generators satisfy relations

(2.1) $$(\rho_i \rho_j)^{p_{ij}} = \varepsilon \qquad (i, j = 0, \dots, n-1),$$

with

(2.2) $$p_{ii} = 1, \qquad p_{ij} = p_{ji} \geq 2 \qquad (i \neq j),$$

and

(2.3) $$p_{ij} = 2 \quad \text{if} \quad |i - j| \geq 2.$$

The numbers $p_j := p_{j-1,j}$ $(j = 1, \dots, n-1)$ determine the (*Schläfli*) *type* $\{p_1, \dots, p_{n-1}\}$ of $\mathcal{P}$. Further, $\Gamma(\mathcal{P})$ has the *intersection property* (with respect to the distinguished generators), namely,

(2.4) $\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \qquad \text{for all} \quad I, J \subset \{0, \dots, n-1\}.$

Observe that, in a natural way, the group of the facet of $\mathcal{P}$ is $\langle \rho_0, \dots, \rho_{n-2} \rangle$, while that of the vertex-figure is $\langle \rho_1, \dots, \rho_{n-1} \rangle$.

By a *C-group*, we mean a group which is generated by involutions such that (2.1), (2.2), and (2.4) hold. If, in addition, (2.3) holds, then the group is called a *string C-group*. The group of a regular polytope is a string C-group. Conversely, given a string C-group, there is an associated regular polytope of which it is the automorphism group [12]. In verifying that a given group is a C-group, it is usually only the intersection property which causes difficulty. Note that Coxeter groups are examples of C-groups (see [12] and [21]).

Given regular n-polytopes $\mathcal{P}_1$ and $\mathcal{P}_2$ such that the vertex-figures of $\mathcal{P}_1$ are isomorphic to the facets of $\mathcal{P}_2$, we denote by $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$ the *class* of all regular $(n+1)$-polytopes $\mathcal{P}$ with facets isomorphic to $\mathcal{P}_1$ and vertex-figures isomorphic to $\mathcal{P}_2$. If $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle \neq \emptyset$, then any such $\mathcal{P}$ is a quotient of a universal member of $\langle \mathcal{P}_1, \mathcal{P}_2 \rangle$; this *universal polytope* is denoted by $\{\mathcal{P}_1, \mathcal{P}_2\}$ (see [12], [17], and [20]).

We end the general discussion of regular polytopes and their groups with a useful remark. Let $\Gamma = \langle \rho_0, \dots, \rho_{n-1} \rangle$ be the group of a regular n-polytope $\mathcal{P}$, and suppose that $\gamma \in \Gamma$. Then we can express γ in the form

$$\gamma = \alpha_0 \rho_0 \alpha_1 \rho_0 \cdots \alpha_{k-1} \rho_0 \alpha_k,$$

with $\alpha_i \in \Gamma_0 := \langle \rho_1, \dots, \rho_{n-1} \rangle$, the group of the vertex-figure of $\mathcal{P}$ at its base vertex $v := F_0$, for $i = 0, \dots, k$. With γ, we can associate a path in $\mathcal{P}$ with k edges leading from v to $v\gamma$. If $k = 0$, the path consists of v $(= v\alpha_0)$ alone. For $k > 0$, let $(E'_1, \dots, E'_{k-1})$ be an edge-path associated with $\alpha_0 \rho_0 \alpha_1 \rho_0 \cdots \alpha_{k-1}$. With γ is then associated the path $(E_1, \dots, E_k)$, given by

$$\begin{aligned} E_1 &:= E\alpha_k \qquad (= E\rho_0\alpha_k), \\ E_i &:= E'_{i-1}\rho_0\alpha_k \qquad \text{for} \quad i = 2, \dots, k, \end{aligned}$$

where $E := F_1$ is the base edge of $\mathcal{P}$. Of course, this path will not generally be unique, since it depends on the particular expression for γ.

Conversely, an edge-path $(E_1, \dots, E_k)$ from v corresponds to such an element $\gamma \in \Gamma$, in which ρ_0 occurs k times. If $k > 0$, then there is an $\alpha_k \in \Gamma_0$ such that $E_1 = E\alpha_k$. The shorter path $(E'_1, \dots, E'_{k-1})$, given by

$$E'_i := E_{i+1}\alpha_k^{-1}\rho_0$$

for $i = 1, \dots, k-1$, also starts at v, and we can repeat to obtain γ as above, with a free choice of α_0.

In the context of group presentations, we deduce:

(2.5) Theorem. *Let P be a regular polytope. Then the group $\Gamma = \Gamma(\mathcal{P})$ of $\mathcal{P}$ is determined by the group of its vertex-figure, and the relations on the distinguished generators of Γ induced by the edge-circuits of $\mathcal{P}$ which contain the initial vertex.*

Proof. A relation on Γ can be written in the form

$$\alpha_0\rho_0\alpha_1\rho_0\cdots\alpha_{k-1}\rho_0 = \varepsilon,$$

with $\alpha_i \in \Gamma_0$ for $i = 0, \ldots, k-1$, which corresponds to an edge-circuit starting and ending at v. Conversely, such an edge-circuit is equivalent under Γ_0 to one beginning with E, and this gives rise to a relation as above (now the element α_0 will be determined by the circuit). This is the result. ☐

We now come to the geometric aspects of the theory. Following [10] and [11], a *realization* of a regular polytope $\mathcal{P}$ is a mapping β: $\mathcal{P}_0 \to E$ of the vertex-set $\mathcal{P}_0$ into some euclidean space E, such that each automorphism of $\mathcal{P}$ induces an isometry of $V := \mathcal{P}_0\beta$; such an isometry extends to one of all of E, uniquely if we make the natural assumption that $E = \operatorname{aff} V$, the affine hull of V. In this latter case, we call $\dim E$ the *dimension* of β also. Thus associated with a realization β of $\mathcal{P}$ is a representation of $\Gamma := \Gamma(\mathcal{P})$ as a group of isometries, which we may also denote by β; we write $G := \Gamma\beta$.

Let β be a realization of $\mathcal{P}$. For $j = 0, \ldots, n-1$, we define

$$R_j := \rho_j\beta \in G.$$

If R_j is not the identity mapping, then it is an involutory isometry or *reflexion* of E, which whenever convenient we identify with its *mirror* of fixed points

$$\{x \in E \mid xR_j = x\}.$$

Of course, we have $G = \langle R_0, \ldots, R_{n-1}\rangle$. We then obtain the points in V, which we also refer to as *vertices*, by means of *Wythoff's construction*: if $v := F_0\beta$ is the image of the vertex of $\mathcal{P}$ in the base flag, then $V = vG$. Now certainly $v \in R_1 \cap \cdots \cap R_{n-1}$; if $v \in R_0$ as well, then β is *trivial*, in that V reduces to a single point. Thus for β to be non-trivial, we must have $v \in (R_1 \cap \cdots \cap R_{n-1})\backslash R_0$.

The realization β induces one of each section of $\mathcal{P}$ as follows. The facet $\mathcal{P}_{n-1}$ in the base flag is realized by

$$v\langle R_0, \ldots, R_{n-2}\rangle,$$

with v as before the initial vertex, while if $w := vR_0$ is the other vertex of the initial edge, then the vertex-figure is realized by

$$w\langle R_1, \ldots, R_{n-1}\rangle.$$

Any other section is obtained by iteration of taking facets or vertex-figures, and so the general case is clear.

If β is a realization of $\mathcal{P}$, then we often write $P := \mathcal{P}\beta$ for a *geometric* regular polytope, with the understanding that P inherits the implied partial ordering induced by the basic faces and their images under G. If P is isomorphic to $\mathcal{P}$ under this partial ordering, then we call β *faithful*. Observe that a faithful realization of $\mathcal{P}$ yields a faithful realization of each of its sections.

The realization β is *blended* if there are proper orthogonal complementary subspaces L and M of E, such that the representation $\Gamma\beta$ permutes the family of translates of L (and hence of M also). In this case we refer to $\Gamma\beta$ as being *affinely reducible*, or just *reducible* if the context prevents confusion with the usual (linear) reducibility. A vertex x of P is thus expressed as $x = (y, z)$, with $y \in L$ and $z \in M$, and there are then induced realizations of $\mathcal{P}$ in L and M. Conversely, such realizations P_1 in L and P_2 in M may be *blended* by pairing up corresponding vertices as $x = (y, z) \in E = L \times M$; we then write $P := P_1 \# P_2$ for the *blend* of P_1 and P_2. In a blend, a mirror of a generating reflexion in E decomposes as $R_j = S_j \times T_j$, with $S_j \subseteq L$ and $T_j \subseteq M$; it may happen that $S_j = L$ or $T_j = M$. If P is not blended in a non-trivial way, then we call P *pure*.

A 0-polytope can only be realized as a point, and a 1-polytope only non-trivially as a (line) segment { }.

In $\mathbb{E}^3$, a (faithfully realized) finite regular polygon P can only be either planar, and thus a pure polygon $\{p\}$, or skew, being the blend $\{p\} \# \{\,\}$ of a planar polygon and a segment; the latter are three-dimensional. Our notation is rather sloppy in this latter case (and similarly elsewhere); strictly speaking, what we mean is that the projections of P on the two orthogonal subspaces in the decomposition of the blend are coverings of $\{p\}$ and { } by P, where in general $p > 2$ is a fraction. If $p \geq 3$ is an integer, then the vertices of a skew polygon $\{p\} \# \{\,\}$ are among those of a p-gonal prism; they will form all the vertices if p is odd, in which case the blend is a $2p$-gon, and half of them if p is even; in each case, they will lie alternately on the two p-gonal faces of the prism.

Similarly, an apeirogon (infinite regular polygon) in $\mathbb{E}^3$ is a linear one $\{\infty\}$, a (planar) skew one, which is the blend $\{\infty\} \# \{\,\}$ with a segment, or a *helix*, which is the blend of $\{\infty\}$ with a bounded regular polygon. Note that the bounded regular polygon in the last type need not itself be finite, although it will always be so in this paper.

We end the section with two useful remarks.

(2.6) Theorem. *A faithful realization of a finite regular n-polytope $\mathcal{P}$ has dimension at least n.*

Proof. The result clearly holds if $n \leq 1$, so suppose that $n \geq 2$, and make the obvious inductive assumption. The vertex-figure of $\mathcal{P}$ is also realized faithfully, and so has dimension at least $n - 1$. The vertices of the realization lie on a sphere, and hence no vertex v can lie in the affine hull of the vertex-figure at v. Thus the realization must have dimension at least n, completing the inductive step. □

(2.7) Corollary. *A discrete faithful realization of a regular n-apeirotope has dimension at least $n - 1$.*

Proof. The vertex-figure of such a realization is a faithfully realized finite regular $(n - 1)$-polytope, and so the corollary follows at once from Theorem 2.6. □

3. Mixing Operations

The idea of a mixing operation is very general. Let Δ be a group generated by involutions, say $\Delta = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle$; usually, but not necessarily, Δ will be a C-group. A *mixing operation* then derives a new group Γ from Δ by taking as generators $\rho_0, \ldots, \rho_{m-1}$ for Γ certain suitably chosen (involutory) products of the σ_i. Then Γ is a subgroup of Δ, and the mixing operation is denoted by

$$(\sigma_0, \ldots, \sigma_{n-1}) \mapsto (\rho_0, \ldots, \rho_{m-1}).$$

In this section we largely follow [12], to which we refer for further details.

Mixing operations are particularly powerful when applied to a polyhedron (3-polytope) $\mathcal{Q}$, which we may also think of as a (finite or infinite) regular map. The underlying surface for such a map is, of course, the order complex $\mathcal{C}(\mathcal{Q})$ of $\mathcal{Q}$, or, more exactly, its underlying (topological) polyhedron. Recall that a triangle $T(\Psi)$ of $\mathcal{C}(\mathcal{Q})$ is associated with each flag Ψ of $\mathcal{Q}$, with each vertex of $T(\Psi)$ associated with a face of Ψ, and two triangles share an edge precisely when the corresponding flags are adjacent. Many of the operations have direct geometric interpretations in this context.

The effect of a mixing operation on a polyhedron $\mathcal{Q}$ can often be pictured geometrically by applying Wythoff's construction (or, rather, its abstract analogue) in the underlying surface $\mathcal{C}(\mathcal{Q})$. However, observe that the new faces (that is, 2-faces) which are obtained, regarded as circuits of vertices and edges, will not usually bound discs in $\mathcal{C}(\mathcal{Q})$.

We thus take our regular polyhedron $\mathcal{Q}$ to have $\Gamma(\mathcal{Q}) = \Delta = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$, and we suppose that $\mathcal{Q}$ is of Schläfli type $\{p, q\}$. Each operation μ will lead to a new group Γ, and a new polyhedron $\mathcal{P} := \mathcal{Q}^\mu$ with $\Gamma(\mathcal{P}) = \Gamma$.

Duality

Our first mixing operation is not commonly thought of as such. This is *duality*, denoted here by δ, and given by

(3.1) $$\delta\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_2, \sigma_1, \sigma_0) =: (\rho_0, \rho_1, \rho_2).$$

The dual of $\mathcal{Q}$ is thus denoted $\mathcal{Q}^\delta$, rather than $\mathcal{Q}^*$ as is more common in other contexts.

The Petrie Operation

Next, we have the *Petrie operation* π, defined by

(3.2) $$\pi\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0\sigma_2, \sigma_1, \sigma_2) =: (\rho_0, \rho_1, \rho_2).$$

The resulting polyhedron $\mathcal{Q}^\pi$ is often called the *Petrie dual* or, more briefly, the *Petrial* of $\mathcal{Q}$. It has the same vertices and edges as $\mathcal{Q}$; however, its faces are the *Petrie polygons* of $\mathcal{Q}$, whose defining property is that two successive edges, but not three, are edges of a face of $\mathcal{Q}$. Thus the faces of $\mathcal{Q}^\pi$ are *zigzags*, leaving a face of $\mathcal{Q}$ after traversing two of its edges.

It is clear that the Petrie operation π is involutory, so that $\pi^{-1} = \pi$, and $(\mathcal{Q}^\pi)^\pi = \mathcal{Q}$. If $\mathcal{Q}^\pi$ is isomorphic to $\mathcal{Q}$, then we call $\mathcal{Q}$ *self-Petrie*; it should, however, be observed that a self-Petrie polyhedron and its Petrial do not coincide, since while they share the same vertices and edges, their faces are different. An example of a self-Petrie polyhedron is the *hemi-dodecahedron* $\{5, 3\}_5 = \{5, 3\}/2$, obtained from the dodecahedron $\{5, 3\}$ by identifying its faces of each dimension under the central involutory symmetry. We should recall that, in this context, a regular polyhedron of type $\{p, q\}$ is denoted $\{p, q\}_r$ if the length r of its Petrie polygons determines its combinatorial type; observe that $(\{p, q\}_r)^\pi = \{r, q\}_p$. We shall meet further examples below. The rare instances in which the Petrial of a polyhedron is not polytopal will not concern us here; in all cases under discussion, the intersection property for the corresponding group is easy to verify directly, since everything will be firmly geometric.

In general, then, the Petrial of a regular polyhedron will also be a regular polyhedron (that is, it will also be polytopal). The polyhedra obtained from a given one by iterating the Petrie operation and duality then form a family of six; that is, we have

(3.3) Lemma. $(\pi\delta)^3 = \varepsilon$, *the identity operation on classes of polyhedra*.

Proof. Indeed, considering the groups, we have

$$\begin{aligned}(\sigma_0, \sigma_1, \sigma_2) &\xrightarrow{\pi} (\sigma_0\sigma_2, \sigma_1, \sigma_2)\\ &\xrightarrow{\delta} (\sigma_2, \sigma_1, \sigma_0\sigma_2)\\ &\xrightarrow{\pi} (\sigma_0, \sigma_1, \sigma_0\sigma_2)\\ &\xrightarrow{\delta} (\sigma_0\sigma_2, \sigma_1, \sigma_0)\\ &\xrightarrow{\pi} (\sigma_2, \sigma_1, \sigma_0)\\ &\xrightarrow{\delta} (\sigma_0, \sigma_1, \sigma_2),\end{aligned}$$

as claimed. □

In (5.5) and (5.6) below, the Petrie operation relates blended apeirohedra. This exhibits a general phenomenon, whose proof is an easy consequence of the definition of a blend in Section 2.

(3.4) Lemma. *The Petrial of the blend of two polyhedra is the blend of their Petrials.*

In the application in Section 5, the second component of the blend will be a segment or apeirogon; the Petrie operation will not affect this, as the corresponding reflexion T_2 (in the notation of Section 2) is absent.

Facetting

We now have an operation which replaces σ_1 by some other reflexion (conjugate of σ_1 or σ_2) in $\langle\sigma_1, \sigma_2\rangle$. More specifically, the kth *facetting operation* φ_k is given by the operation

(3.5) $$\varphi_k\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0, \sigma_1(\sigma_2\sigma_1)^{k-1}, \sigma_2) =: (\rho_0, \rho_1, \rho_2).$$

We suppose that $2 \le k < \frac{1}{2}q$, since φ_{q-k} has the same effect as φ_k up to isomorphism (actually, conjugation of the whole group by σ_2), and the case $k = \frac{1}{2}q$ can only yield a polyhedron $\{r, 2\}$ for some r. (In this last case, the number r may be of independent interest, but it is not in the present context. In some special circumstances, we also find it convenient to allow the case $k = 1$ as well, where φ_1 is just the identity mixing operation ε.) When the highest common factor $(k, q) = 1$, then $\rho_1\rho_2$ has the same period q as $\sigma_1\sigma_2$; indeed, the groups are the same, and φ_k is inverted by $\varphi_{k'}$, where $kk' \equiv \pm 1 \pmod{q}$. In fact, we have

(3.6) Lemma. *$\varphi_k\varphi_m = \varphi_{km}$, where the suffix is to be read as that number between* 0 *and $\frac{1}{2}q$ which is congruent to $\pm km$ modulo q.*

Proof. We apply φ_k and φ_m in succession to the group. Noting that only σ_1 changes, and writing $\rho_1 = \sigma_1(\sigma_2\sigma_1)^{k-1} = (\sigma_1\sigma_2)^k\sigma_2$, it becomes

$$((\sigma_1\sigma_2)^k\sigma_2^2)^m\sigma_2 = (\sigma_1\sigma_2)^{km}\sigma_2,$$

as required. □

This lemma covers all possible k and m. Generally speaking, we are rather less interested in the case $(k, q) > 1$, although it will occasionally be useful. In particular, we employ φ_2 with q even (actually, $q = 6$) in Section 6 below.

Geometrically, φ_k has the following effect when $(k, q) = 1$. The new polyhedron $\mathcal{P} := \mathcal{Q}^{\varphi_k}$ has the same vertices and edges as $\mathcal{Q}$. However, a typical face of $\mathcal{P}$ is a *k-hole* of $\mathcal{Q}$, which is formed by the edge-path which leaves a vertex by the kth edge from which it entered, in the same sense (that is, keeping always to the left, say, in some local orientation of $\mathcal{C}(\mathcal{Q})$). The faces of $\mathcal{P}$ then comprise all the k-holes of $\mathcal{Q}$. Hence, if such a k-hole is an r-gon, so that r is the period of

$$\rho_0\rho_1 = \sigma_0 \cdot (\sigma_1\sigma_2)^{k-1}\sigma_1,$$

then $\mathcal{Q}^{\varphi_k}$ is of type $\{r, q\}$. If $\mathcal{Q}$ is infinite, then it is possible that $r = \infty$, even if p is finite. Of course, the 1-holes of $\mathcal{P}$ are just its faces.

Naturally, we must not forget to verify the intersection property, but in this case it is much easier to do this “geometrically”, thinking of $\mathcal{P} = \mathcal{Q}^{\varphi_k}$ as embedded in a surface. Generally, this will not be the same as the original surface $\mathcal{C}(\mathcal{Q})$ underlying $\mathcal{Q}$, although in practice we are able to work with $\mathcal{C}(\mathcal{Q})$ instead of the new surface $\mathcal{C}(\mathcal{P})$, and employ Wythoff’s construction, as we said earlier. In any event, we may use even more directly geometric arguments in our context.

When $(k, q) > 1$, the situation is similar, except that now in general a compound of several polyhedra of type $\{r, q/(k, q)\}$ will be formed. However, if $\mathcal{Q}^{\varphi_k}$ remains connected, it will fail to be polytopal, since the vertex-figure at a vertex will no longer be connected.

The Petrie operation and facetting are related as follows.

(3.7) Lemma. *The Petrie operation π and the facetting operation φ_k commute.*

Proof. This is easily verified algebraically, but it is even more instructive to look at the geometry. Whether we apply π or φ_k first, the result will be (assuming that it exists) a polyhedron $\mathcal{P}$ whose vertices and edges are those of $\mathcal{Q}$, but whose typical face is a *k-zigzag*, given by an edge-path which (as for φ_k) leaves a vertex at the kth edge from the one by which it entered, but in the oppositely oriented sense at alternate vertices. The Petrie polygons themselves are thus 1-zigzags. □

At this point, it is appropriate to introduce some general notation. We denote by $\{p, q|h\}$ a regular polyhedron of type $\{p, q\}$, whose combinatorial type is determined by the fact that its (2-)holes are h-gons. Analogously, if a regular polyhedron $\mathcal{P}$ of type $\{p, q\}$ is determined by the lengths h_j of its j-holes for certain j in the range $2 \leq j \leq k := \lfloor \frac{1}{2}q \rfloor$, then we denote it by

(3.8) $$\mathcal{P} := \{p, q|h_2, \ldots, h_k\};$$

any unnecessary h_j (that is, one which is not needed for the specification) is replaced by "·", with those at the end of the sequence omitted. An example where all the h_j are required for the specification is provided by Coxeter's polyhedron $\{4, n|4^{\lfloor n/2-1 \rfloor}\}$ (see [2]; in [14]–[16] embeddings in $\mathbb{E}^3$ are found of this polyhedron, but of course without full symmetry).

Similarly, the notation $\{p, q\}_r$ for a regular polyhedron $\mathcal{P}$ of type $\{p, q\}$ determined by the length r of its Petrie polygons is generalized to

(3.9) $$\mathcal{P} := \{p, q\}_{r_1,\ldots,r_k},$$

with $\mathcal{P}$ now determined by the lengths r_j of its j-zigzags for $j = 1, \ldots, k$ with k as before. The same conventions for unnecessary r_j apply.

These notations can be combined, to give regular polyhedra

(3.10) $$\{p, q|h_2, \ldots, h_k\}_{r_1,\ldots,r_k}$$

of type $\{p, q\}$, determined by certain of its holes and zigzags. The notation is not symmetric between holes and zigzags; the 1-holes are, of course, just the faces $\{p\}$.

The corresponding defining relations for the groups of such regular polyhedra are easily obtained from the discussion above. Thus $\mathcal{P}$ is forced to have j-holes of length h_j by imposing the relation

(3.11) $$(\rho_0\rho_1(\rho_2\rho_1)^{j-1})^{h_j} = \varepsilon$$

on the group $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 \rangle$, while $\mathcal{P}$ is forced to have j-zigzags of length r_j by imposing the relation

(3.12) $$(\rho_0(\rho_1\rho_2)^j)^{r_j} = \varepsilon.$$

Of course, it is a consequence of Lemma 3.7 that the Petrie operation interchanges j-holes and j-zigzags.

Halving

The *halving operation* η applies only to a regular polyhedron $\mathcal{Q}$ of type $\{4, q\}$ for some $q \geq 3$, and turns it into a self-dual polyhedron $\mathcal{P} := \mathcal{Q}^\eta$ of type $\{q, q\}$. We define η by

(3.13) $$\eta\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0\sigma_1\sigma_0, \sigma_2, \sigma_1) =: (\rho_0, \rho_1, \rho_2).$$

The intersection property is easily checked for $\Gamma := \langle \rho_0, \rho_1, \rho_2 \rangle$; it will become clear from the discussion below. When we think of $\Delta = \Gamma(\mathcal{Q})$ acting on the surface $\mathcal{C}(\mathcal{Q})$, the triangle $T = T(\Phi)$ associated with the base flag Φ of $\mathcal{Q}$ is a fundamental region for Δ, and σ_0, σ_1, and σ_2 act as reflexions in the sides of T. Now let $T' := T \cup T\sigma_0$. Then T' is the fundamental region for Γ, and Γ is similarly generated by the reflexions in the sides of T'.

If we now apply Wythoff's construction (or, rather, its abstract analogue, in the underlying surface $\mathcal{C}(\mathcal{Q})$), then we see that there are two possibilities.

First, suppose that the (edge-)graph of $\mathcal{Q}$ is bipartite, so that all the edge-circuits of $\mathcal{Q}$ have even length. Then $\mathcal{P}$ will be a map on the same surface $\mathcal{C}(\mathcal{Q})$. It will have half as many vertices as $\mathcal{Q}$, namely, those in the same partition of the vertex-set $\mathcal{Q}_0$ of $\mathcal{Q}$ as the initial vertex in the base flag Φ. Further, Γ will have index 2 in Δ. As we asserted above, $\mathcal{P}$ will be self-dual, since $\sigma_0 \in \Delta$ acts as an automorphism of Γ, which interchanges ρ_0 and ρ_2 and leaves ρ_1 fixed. The vertices of the dual $\mathcal{P}^\delta$ will then be those in the other partition of $\mathcal{Q}_0$.

In the other case, the graph of $\mathcal{Q}$ is not bipartite. Unless $q = 4$ also, $\mathcal{P}$ will be a map on a different surface from $\mathcal{C}(\mathcal{Q})$. In actual fact, $\mathcal{C}(\mathcal{P})$ will be a double cover of $\mathcal{C}(\mathcal{Q})$ in every case; we must be careful to note that $\mathcal{Q}$ itself does not cover $\mathcal{P}$ in general. We now have $\Gamma = \Delta$, and $\mathcal{P}$ will have the same vertex-set $\mathcal{Q}_0$ as $\mathcal{Q}$. Finally, $\mathcal{P}$ will still be self-dual, although now the conjugating element σ_0 is in Γ.

In either case, we have

$$\rho_0\rho_1\rho_2\rho_1 = \sigma_0\sigma_1\sigma_0\sigma_2\sigma_1\sigma_2 = (\sigma_0\sigma_1\sigma_2)^2.$$

This shows that if the original polyhedron $\mathcal{Q}$ has Petrie polygons of length h, then the new polyhedron $\mathcal{P}$ will have 2-holes of length h or $h/2$ according to whether h is odd or even. Note that the latter will be the case when the graph of $\mathcal{Q}$ is bipartite (but possibly in other cases also); then $[\Delta : \Gamma] = 2$.

In this spirit, in certain cases the combinatorial type of a polyhedron $\mathcal{P} = \mathcal{Q}^\eta$ is easily determined from that of $\mathcal{Q}$.

(3.14) Theorem. *Let $q \geq 4$ and $s \geq 2$. Then*

(a) $\{4, q\,|\,2s\}^\eta = \{q, q\}_{2s}$;
(b) $(\{4, q\}_{2s})^\eta = \{q, q\,|\,s\}$.

Proof. The graphs of the two polyhedra $\{4, q\,|\,2s\}$ and $\{4, q\}_{2s}$ are bipartite, since their defining circuits are faces and holes or zigzags, of lengths 4 and $2s$, respectively. The operation $\mathcal{Q} \mapsto \mathcal{Q}^\eta =: \mathcal{P}$ is given by

$$\eta\colon\ (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0\sigma_1\sigma_0, \sigma_2, \sigma_1) =: (\rho_0, \rho_1, \rho_2).$$

In case (a), we therefore have

$$\begin{aligned} \varepsilon &= (\sigma_0\sigma_1\sigma_2\sigma_1)^{2s} \\ &= (\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_1)^s \\ &= (\sigma_0\sigma_1\sigma_2 \cdot \sigma_0\sigma_1\sigma_0\sigma_1\sigma_0 \cdot \sigma_2\sigma_1)^s \end{aligned}$$

$$\begin{aligned}
&\sim (\sigma_1\sigma_2 \cdot \sigma_0\sigma_1\sigma_0 \cdot \sigma_1\sigma_2 \cdot \sigma_0\sigma_1\sigma_0)^s \\
&= (\sigma_1\sigma_2 \cdot \sigma_0\sigma_1\sigma_0)^{2s} \\
&= (\rho_2\rho_1\rho_0)^{2s} \\
&\sim (\rho_0\rho_1\rho_2)^{2s}.
\end{aligned}$$

Similarly, in case (b), we have

$$\begin{aligned}
\varepsilon &= (\sigma_0\sigma_1\sigma_2)^{2s} \\
&= (\sigma_0\sigma_1\sigma_2\sigma_0\sigma_1\sigma_2)^s \\
&= (\sigma_0\sigma_1\sigma_0 \cdot \sigma_2\sigma_1\sigma_2)^s \\
&= (\rho_0\rho_1\rho_2\rho_1)^s.
\end{aligned}$$

In each case, the defining relation of the original polyhedron $\mathcal{Q}$ is equivalent to the corresponding defining relation for the new polyhedron $\mathcal{P}$. □

Skewing

Finally we have the *skewing* operation σ, or, as it would perhaps be better named, *skew halving*. It applies to a regular polyhedron $\mathcal{Q}$ of type $\{p, 4\}$, and is defined by

(3.15) $$\sigma\colon (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_1, \sigma_0\sigma_2, (\sigma_1\sigma_2)^2) =: (\rho_0, \rho_1, \rho_2).$$

It is remotely related to halving; in fact

$$\sigma = \pi\delta\eta\pi\delta,$$

since, because $(\sigma_1\sigma_2)^4 = \varepsilon$, we have

$$\begin{aligned}
(\sigma_0, \sigma_1, \sigma_2) &\overset{\pi}{\longmapsto} (\sigma_0\sigma_2, \sigma_1, \sigma_2) \\
&\overset{\delta}{\longmapsto} (\sigma_2, \sigma_1, \sigma_0\sigma_2) \\
&\overset{\eta}{\longmapsto} (\sigma_2\sigma_1\sigma_2, \sigma_0\sigma_2, \sigma_1) \\
&\overset{\pi}{\longmapsto} ((\sigma_1\sigma_2)^2, \sigma_0\sigma_2, \sigma_1) \\
&\overset{\delta}{\longmapsto} (\sigma_1, \sigma_0\sigma_2, (\sigma_1\sigma_2)^2),
\end{aligned}$$

as claimed. This also indicates that σ halves the order of the group just when η does, modulo the double application of $\pi\delta$, that is, when the graph of $\mathcal{Q}^{\pi\delta}$ is bipartite. If this is the case, then $\sigma_2 \notin \Gamma := \Gamma(\mathcal{Q}^\sigma) = \langle\rho_0, \rho_1, \rho_2\rangle$, but acts on it as an automorphism. In any event, $\mathcal{P} := \mathcal{Q}^\sigma$ is self-Petrie; the isomorphism between $\mathcal{P}$ and $\mathcal{P}^\pi$ is given by conjugation of $\Gamma(\mathcal{P})$ by σ_2, since

$$\sigma_2\sigma_1\sigma_2 = \sigma_1 \cdot (\sigma_1\sigma_2)^2.$$

The type $\{s, t\}$ of $\mathcal{P} = \mathcal{Q}^\sigma$ is determined by the periods of

$$\rho_0\rho_1 = \sigma_1 \cdot \sigma_0\sigma_2 \sim \sigma_0\sigma_1\sigma_2$$

and

$$\rho_1\rho_2 = \sigma_0\sigma_2 \cdot (\sigma_1\sigma_2)^2 = \sigma_0\sigma_1\sigma_2\sigma_1.$$

Hence s is the length of the Petrie polygons of $\mathcal{Q}$, while t is the length of its 2-holes.

We illustrate these techniques with a few examples. In particular, we see how they apply to the Platonic polyhedra. In this section we treat them combinatorially; in later sections, where we give further examples and complete classifications in $\mathbb{E}^3$, we look at them more geometrically.

With the regular tetrahedron $\{3, 3\}$, only one of the above operations yields anything new, namely, the Petrie operation π. The new polyhedron $\{3, 3\}^\pi$ is $\{4, 3\}_3$; it can also be obtained from the cube $\{4, 3\}$ by identifying its antipodal faces of each rank, so that an alternative notation for it is $\{4, 3\}/2$. Observe that the facets of $\{4, 3\}_3$ are skew (non-planar) polygons $\{4\}\#\{\,\}$. For this reason, while the dual $\{3, 4\}_3$ of this polyhedron exists in a combinatorial sense, it is not faithfully realizable in $\mathbb{E}^3$ (or, indeed, in any space).

With the cube, again only the Petrie operation leads to a new polyhedron (we leave aside duality for the moment); we have $\{4, 3\}^\pi = \{6, 3\}_4 = \{6, 3\}_{(2,0)}$, a toroidal polyhedron. Similarly, for the octahedron, $\{3, 4\}^\pi = \{6, 4\}_3$. Duality completes the family; the cube and octahedron are dual, and the duals of the two Petrials, namely, $\{3, 6\}_4$ and $\{4, 6\}_3$, are each the Petrial of the other, but are again not realizable faithfully. Note that the operation φ_2 can be applied to the octahedron, but will only yield the degenerate polyhedron $\{4, 2\}$.

The icosahedron $\{3, 5\}$ and dual dodecahedron $\{5, 3\}$ give rise to a rich family. Since its realizable members are listed in (4.3), we confine ourselves here to some combinatorial remarks. The Petrial of $\{3, 5\}$ is $\{10, 5\}_3$, and that of $\{5, 3\}$ is $\{10, 3\}_5$, both with skew decagonal faces. Applying the facetting operation φ_2 to $\{3, 5\}$ yields $\{3, 5\}^{\varphi_2} = \{5, \frac{5}{2}\} \cong \{5, 5|3\}$ (see [2]). Its Petrial is $\{3, 5\}^{\pi\varphi_2} \cong \{6, 5\}_{5,3}$; in fact, abstractly this polyhedron is actually $\{6, 5\}_{\cdot,3}$, and geometrically it is $\{6, \frac{5}{2}\}_{\cdot,3}$, with vertex-figures which are pentagrams. Combinatorially, of course, $\{5, \frac{5}{2}\}$ is self-dual, but geometrically (in $\mathbb{E}^3$) its dual $\{\frac{5}{2}, 5\}$ is distinct; therefore reversing these mixing operations yields the other half of the symmetric table (4.3), with the interchange of 5 and $\frac{5}{2}$, and 10 and $\frac{10}{3}$. No mixing operations other than those in (4.3) yield polyhedra faithfully realizable in $\mathbb{E}^3$, but we observe that

$$(\{10, 3\}_5)^{\delta\varphi_3\delta} = \{\tfrac{10}{3}, 3\}_{5/2}.$$

4. Finite Regular Polyhedra

We now employ the techniques above to describe the possible three-dimensional realizations of abstract regular polytopes, which are both discrete and faithful. These restrictions are the natural geometric ones; they are assumed to hold henceforth, and will not be repeated. We then prove by direct methods that the enumeration is complete. In this section we treat the finite regular polyhedra; subsequent ones are devoted to the (infinite) apeirohedra, and to the apeirotopes of rank 4.

Before we embark on the classification problem, we briefly describe the key idea behind our approach. This is the following simple “trick”: *each generating reflexion* *whose mirror is a line is replaced by one whose mirror is a suitable perpendicular plane*. The problem thus reduces to one involving groups generated by plane reflexions, and these are well known.

We begin by classifying the finite cases. We do this to illustrate the various mixing operations we have described in Section 3 on examples which should be more familiar. Theorem 2.6 says that a faithfully realized finite regular polytope in $\mathbb{E}^3$ can have rank at most 3, and then its vertex-figure must be planar. So we begin by listing the finite regular 3-polytopes in $\mathbb{E}^3$, with the relationships between them. These polytopes will have the same symmetry groups as the tetrahedron, octahedron, or icosahedron. In our lists we do not repeat self-dual polytopes, for example. We should emphasize again that we are really considering realizations of polytopes; thus a polytope may be realizable, while its dual is not (at least, not faithfully).

The three groupings are then:

Tetrahedral Symmetry

(4.1) $$\{3,3\} \stackrel{\pi}{\longleftrightarrow} \{4,3\}_3.$$

Octahedral Symmetry

(4.2) $$\{6,4\}_3 \stackrel{\pi}{\longleftrightarrow} \{3,4\} \stackrel{\delta}{\longleftrightarrow} \{4,3\} \stackrel{\pi}{\longleftrightarrow} \{6,3\}_4.$$

Icosahedral Symmetry

(4.3) $$\begin{array}{ccccccc}
\{10,5\} & \stackrel{\pi}{\longleftrightarrow} & \{3,5\} & \stackrel{\delta}{\longleftrightarrow} & \{5,3\} & \stackrel{\pi}{\longleftrightarrow} & \{10,3\} \\
\updownarrow{\scriptstyle\varphi_2} & & \updownarrow{\scriptstyle\varphi_2} & & & & \\
\{6,\frac{5}{2}\} & \stackrel{\pi}{\longleftrightarrow} & \{5,\frac{5}{2}\} & \stackrel{\delta}{\longleftrightarrow} & \{\frac{5}{2},5\} & \stackrel{\pi}{\longleftrightarrow} & \{6,5\} \\
 & & & & \updownarrow{\scriptstyle\varphi_2} & & \updownarrow{\scriptstyle\varphi_2} \\
\{\frac{10}{3},3\} & \stackrel{\pi}{\longleftrightarrow} & \{\frac{5}{2},3\} & \stackrel{\delta}{\longleftrightarrow} & \{3,\frac{5}{2}\} & \stackrel{\pi}{\longleftrightarrow} & \{\frac{10}{3},\frac{5}{2}\}.
\end{array}$$

We should say a few things about the last display. First, we have suppressed the exact description of most of the polyhedra. Instead, we have given symbols more akin to those used by Coxeter in [3]. In fact, the polyhedra occur in isomorphic pairs, given by symmetry of the display about its centre, or by interchanging 5 with $\frac{5}{2}$ and 10 with $\frac{10}{3}$. The remaining details are:

(4.4) $$\begin{aligned}
\{10,5\} &\cong \{10,5\}_3, \\
\{10,3\} &\cong \{10,3\}_5, \\
\{\tfrac{5}{2},5\} &\cong \{5,5|3\}, \\
\{6,5\} &\cong \{6,5\}_{\cdot,3}.
\end{aligned}$$

For the last of these, recall from Section 3 that the subscripts denote the lengths of the 1- and 2-zigzags of the polyhedron.

(4.5) Theorem. *The list of* 18 *finite regular polyhedra in* (4.1), (4.2), *and* (4.3) *is complete*.

Proof. The enumeration is performed by a systematic investigation of the various possibilities. A main reason for treating this familiar problem is to introduce the techniques employed. Let P be a regular polyhedron P in $\mathbb{E}^3$, and let $G(P) = \langle R_0, R_1, R_2 \rangle$ be its symmetry group. We may suppose that $G(P)$ is an orthogonal group, so that P has centre o. Further, we adopt our usual convention of identifying a reflexion with its mirror.

Our first observation is that R_1 and R_2 must be planes. The simplest way to see this is that the initial vertex v of P satisfies $v \in (R_1 \cap R_2) \setminus \{o\}$. Thus $R_1 \cap R_2$ contains a line. However, R_1 and R_2 must also be distinct and non-commuting, and the desired conclusion follows. Next R_0 is also either a line or a plane. Indeed, it clearly cannot be the point-set $\{o\}$, which gives the only other kind of involution in an orthogonal group in $\mathbb{E}^3$, because then R_0 would be central, and so would commute with R_1.

We now employ our trick. If R_0 is a line, we replace it by the orthogonal plane $S_0 := R_0^{\perp}$. As an orthogonal mapping, $S_0 = -R_0$, that is, the product of R_0 with the central reflexion. We note that S_0 still commutes with R_2, but does not with R_1. If R_0 is a plane, we set $S_0 := R_0$. Further, we set $S_j := R_j$ for $j = 1, 2$ in each case. Define $H := \langle S_0, S_1, S_2 \rangle$. Then H is a finite (plane) reflexion group in $\mathbb{E}^3$, which is, as is well known (see [3]), the symmetry group of some classical regular polyhedron Q.

We may clearly reverse the argument. If we take the group $H = \langle S_0, S_1, S_2 \rangle$ of one of the nine classical regular polyhedra Q (again, see [3], or for a different approach to the classification, [9]), we may replace S_0 by $-S_0$, to obtain a new finite orthogonal group in $\mathbb{E}^3$ generated by involutions. This then adds another nine regular polyhedra to the nine classical ones, and thus we arrive at the 18 polyhedra listed above. □

We make a remark about this pairing of polyhedra. When the group is $[3, 5]$, a line which is the axis of a twofold rotation is perpendicular to a reflexion plane; thus the two groups $G(P)$ and H are the same. But when the group of P is $[3, 3]$ or $[3, 4]$, something strange can happen, in that the two groups may be interchanged by the procedure of replacing S_0 by $-S_0$. However, it is clear that applying this procedure to Petrie duals will yield Petrie duals. The resulting pairings among the 18 polyhedra are

$$\begin{aligned} \{3,3\} &\longleftrightarrow \{6,3\}_4, \\ \{3,4\} &\longleftrightarrow \{6,4\}_3, \\ \{4,3\} &\longleftrightarrow \{4,3\}_3, \end{aligned}$$

(4.6)

$$\begin{aligned} \{5,3\} &\longleftrightarrow \{\tfrac{10}{3},3\}, \\ \{3,5\} &\longleftrightarrow \{6,5\}, \\ \{5,\tfrac{5}{2}\} &\longleftrightarrow \{\tfrac{10}{3},\tfrac{5}{2}\}, \\ \{\tfrac{5}{2},5\} &\longleftrightarrow \{10,5\}, \\ \{3,\tfrac{5}{2}\} &\longleftrightarrow \{6,\tfrac{5}{2}\}, \\ \{\tfrac{5}{2},3\} &\longleftrightarrow \{10,3\}. \end{aligned}$$

For the polyhedra with symmetry group $[3, 5]$, we have adopted the same abbreviated notation as above.

5. Blended Regular Apeirohedra

We now move on to the regular apeirohedra, or infinite polyhedra, in $\mathbb{E}^3$. We repeat our blanket assumptions of discreteness and faithfulness.

A key tool in our investigations is a refinement of Bieberbach's theorem [1], [7]. We say that a subgroup G of the whole group $\mathcal{I}_n$ of isometries of $\mathbb{E}^n$ *disperses* if, whenever $v \in \mathbb{E}^n$ is any point, then $\mathbb{E}^n = \text{conv}(vG)$, the convex hull of the images of v under G. When G is also discrete, this implies that the topological quotient space $\mathbb{E}^n/G$ is compact.

(5.1) Lemma. *A discrete infinite group G of isometries which disperses on $\mathbb{E}^2$ or $\mathbb{E}^3$ does not contain rotations of periods other than* 2, 3, 4, *or* 6.

Proof. Let G be such a subgroup of the whole group $\mathcal{I}_n$ of isometries of $\mathbb{E}^n$. Then G cannot have any non-trivial invariant (linear or affine) subspaces. Bieberbach's theorem then tells us that G contains a full subgroup T of the group $\mathcal{T}_n$ of translations of $\mathbb{E}^n$, and that the quotient G/T is finite; in effect, T can be thought of as a lattice of rank n in $\mathbb{E}^n$. If $x \mapsto x\varphi + t$ is a general element of G, with $\varphi \in \mathcal{O}_n$, the orthogonal group, and $t \in \mathbb{E}^n$ a translation vector (we may thus think of $t \in \mathcal{T}_n$), then the mappings φ clearly form a subgroup G_0 of $\mathcal{O}_n$, called the *special group* of G. Thus G_0 is the image of G under the homomorphism on $\mathcal{I}_n$, whose kernel is $\mathcal{T}_n$ (the image of $\mathcal{I}_n$ is, of course, $\mathcal{O}_n$). In other words,

$$G_0 = G\mathcal{T}_n/\mathcal{T}_n \cong G/(G \cap \mathcal{T}_n) = G/T.$$

There is no loss of generality in assuming that G_0 contains the central inversion $-I$; if it does not, then we adjoin it. Suppose now that $n = 2$ or 3, and that G_0 does contain a rotation with period $k \neq 2, 3, 4, 6$; then it contains such a rotation φ through an angle $2\pi/k$. We first consider the planar case $n = 2$. Since $-I \in G_0$, we may suppose that $k \geq 8$ is even. There is a minimal length δ among the translations of T. If $t \in T$ has $\|t\| = \delta$, then the distance between t and $t\varphi$ is $2\delta \sin(\pi/k) < \delta$, an obvious contradiction. In the case $n = 3$ of ordinary space, we argue similarly, except that we consider the minimal distance between parallel axes of k-fold symmetry. □

When we consider discrete regular polytopes in $\mathbb{E}^3$ of higher rank, we note that the three regular tessellations $\{3, 6\}$, $\{4, 4\}$, and $\{6, 3\}$ are planar, as are their Petrials. However, we can treat these cases with the others, all of which are genuinely three-dimensional. We thus obtain the two families of planar apeirohedra

(5.2) $$\{4, 4\} \overset{\pi}{\longleftrightarrow} \{\infty, 4\}_4$$

and

(5.3) $$\{\infty, 6\}_3 \overset{\pi}{\longleftrightarrow} \{3, 6\} \overset{\delta}{\longleftrightarrow} \{6, 3\} \overset{\pi}{\longleftrightarrow} \{\infty, 3\}_6.$$

The other operations we have described in Section 3 lead to no new polyhedra, even when they are applicable. In fact, we have $\{4, 4\}^{\eta} = \{4, 4\}$ (actually, another copy, whose edges are diagonals of the original squares) and $\{3, 6\}^{\varphi_2} = \{6, 3\}$ (a different copy from the geometric dual).

(5.4) Theorem. *The list of six planar regular apeirohedra in* (5.2) *and* (5.3) *is complete.*

Proof. The argument is similar to that for the finite regular polyhedra, but a little easier. Let P be a planar regular apeirohedron, with group $G(P) = \langle R_0, R_1, R_2\rangle$. The initial vertex v satisfies $v \in R_1 \cap R_2$, and R_1 and R_2 are non-commuting involutions in $\mathbb{E}^2$; hence R_1 and R_2 must be intersecting lines. In view of Lemma 5.1, the angle between R_1 and R_2 can only be $\pi/3$, $\pi/4$ or $\pi/6$. R_0 may be a line or a point. In the former case $G(P)$ must be one of [4, 4] or [3, 6], yielding the three ordinary planar tessellations. In the latter case, since R_0 commutes with R_2, the point must lie in R_2, and we obtain the three Petrials of the planar tessellations (observe that $R_0 R_2$ will be the reflexion in a line perpendicular to R_2). □

We now consider the three-dimensional discrete regular apeirohedra in $\mathbb{E}^3$. We shall see that they fall into two families of 12 each. The first comprises those which are blends in a non-trivial way, while the second consists of the pure apeirohedra; we deal with the latter in Section 6.

The non-pure three-dimensional apeirohedra are derived from the six planar regular apeirohedra by blending with either a segment { } or the linear apeirogon $\{\infty\}$. The apeirohedra in each of these two families form one-dimensional classes under similarity; the parameter is the ratio between the edge-length of the planar apeirohedron and either the length of the segment or the edge-length of the apeirogon. These apeirohedra are thus listed as follows. First, we have the blends with segments:

(5.5)
$$\begin{aligned} \{4,4\}\,\#\,\{\ \} &\overset{\pi}{\longleftrightarrow} \{\infty,4\}_4\,\#\,\{\ \},\\ \{3,6\}\,\#\,\{\ \} &\overset{\pi}{\longleftrightarrow} \{\infty,6\}_3\,\#\,\{\ \},\\ \{6,3\}\,\#\,\{\ \} &\overset{\pi}{\longleftrightarrow} \{\infty,3\}_6\,\#\,\{\ \}.\end{aligned}$$

Then we have the blends with apeirogons:

(5.6)
$$\begin{aligned} \{4,4\}\,\#\,\{\infty\} &\overset{\pi}{\longleftrightarrow} \{\infty,4\}_4\,\#\,\{\infty\},\\ \{3,6\}\,\#\,\{\infty\} &\overset{\pi}{\longleftrightarrow} \{\infty,6\}_3\,\#\,\{\infty\},\\ \{6,3\}\,\#\,\{\infty\} &\overset{\pi}{\longleftrightarrow} \{\infty,3\}_6\,\#\,\{\infty\}.\end{aligned}$$

The Petrie relationships are given by Lemma 3.4.

If p is finite, the facets of a blend $\{p, q\}\,\#\,\{\ \}$ are skew polygons $\{p\}\,\#\,\{\ \}$, and their vertex-figures are flat polygons $\{q\}$, parallel to the plane of the tessellation $\{p, q\}$. Note as in Section 2 that $\{3\}\,\#\,\{\ \}$ is actually a skew hexagon. For a blend $\{\infty, q\}_p\,\#\,\{\ \}$, the facets are (planar) zigzag apeirogons $\{\infty\}\,\#\,\{\ \}$, since we have

$$(\{\infty\}\,\#\,\{\ \})\,\#\,\{\ \} = \{\infty\}\,\#\,\{\ \},$$

if we ignore the implicit parameter giving the relative sizes of the components of the blend. The vertex-figures are again flat polygons $\{q\}$, since the Petrie operation preserves vertex-figures.

Similarly, the facets of a blend $\{p, q\}\,\#\,\{\infty\}$ with p finite are helical apeirogons $\{p\}\,\#\,\{\infty\}$ (spiralling around a cylinder with a p-gonal base), and the vertex-figures are skew polygons $\{q\}\#\{\,\}$. The facets of a blend $\{\infty, q\}_p \# \{\infty\}$ are now zigzag apeirogons, since

$$(\{\infty\}\#\{\,\})\#\{\infty\} = \{\infty\}\#\{\,\},$$

again ignoring the implicit parameter. The vertex-figures are still skew polygons $\{q\}\#\{\,\}$.

(5.7) Theorem. *The list of* 12 *blended apeirohedra in* (5.5) *and* (5.6) *is complete.*

Proof. Again, there is little to say here. A regular apeirohedron P in $\mathbb{E}^3$ which is a non-trivial blend must have components of dimensions 2 and 1. Moreover, the two-dimensional component must be one of the six planar apeirohedra listed above. Lemma 5.1 eliminates any other possibilities; in particular, the projection of P on the corresponding plane must be discrete. The required classification is then immediate. □

6. Pure Regular Apeirohedra

We finally come to the pure three-dimensional apeirohedra. We begin by listing them, with the various relationships between them (these are elucidated in Section 7). We see that, in a sense, they fall into a single family, derived from the regular honeycomb $\{4, 3, 4\}$.

(6.1)

$$\begin{array}{ccccccc}
\{\infty,4\}_{6,4} & \overset{\pi}{\longleftrightarrow} & \{6,4|4\} & \overset{\delta}{\longleftrightarrow} & \{4,6|4\} & \overset{\pi}{\longleftrightarrow} & \{\infty,6\}_{4,4} \\
 & & \downarrow\sigma & & \downarrow\eta & & \\
 & & \{\infty,4\}_{\cdot,*3} & & \{6,6\}_4 & \overset{\varphi_2}{\longrightarrow} & \{\infty,3\}^{(a)} \\
 & & & & \updownarrow\pi & & \updownarrow\pi \\
 & & \{6,4\}_6 & \overset{\delta}{\longleftrightarrow} & \{4,6\}_6 & \overset{\varphi_2}{\longrightarrow} & \{\infty,3\}^{(b)} \\
 & & \downarrow\sigma\delta & & \downarrow\eta & & \\
 & & \{\infty,6\}_{6,3} & \overset{\pi}{\longleftrightarrow} & \{6,6|3\}. & &
\end{array}$$

The notation for one of the second suffixes has the following meaning; we prefix a $*$ to a number denoting a k-hole for $k \geq 3$, or a k-zigzag for $k \geq 2$, to mean that this is the size of the corresponding hole or zigzag of the dual polyhedron. Thus the $*$ prefix here indicates the 2-zigzag of the dual.

The two apeirohedra of type $\{\infty, 3\}$ are described in terms of their groups in Section 7. In addition to the relationships given in the diagram, the two apeirohedra of type $\{6, 6\}$ are of course self-dual, since they are obtained from other apeirohedra by the halving operation η. Further, $\{6, 4\}_6$ is self-Petrie as the notation indicates, as is $\{\infty, 4\}_{\cdot,*3}$ since it is obtained by means of the skewing operation σ from another apeirohedron. Finally, chasing through the above diagram and using $\sigma = \pi\delta\eta\pi\delta$, we also have

$$(\{\infty, 4\}_{6,4})^\sigma = \{6, 4\}_6.$$

There are no other new apeirohedra which can be derived from any of these. For example, further applications of φ_2 give

$$\{4,6|4\}^{\varphi_2} = \{4,3\},$$
$$\{6,6|3\}^{\varphi_2} = \{3,3\},$$

both finite polyhedra.

(6.2) Theorem. *The list of* 12 *discrete pure three-dimensional apeirohedra in* (6.1) *is complete.*

Proof. Let P be a pure three-dimensional apeirohedron in $\mathbb{E}^3$, with symmetry group $G(P) = \langle R_0, R_1, R_2 \rangle$. Thus R_0, R_1, and R_2 are involutory isometries of $\mathbb{E}^3$ such that R_0 and R_2 commute, while R_1 does not commute with R_0 or R_2. As usual, we identify a reflexion with its mirror.

We first show that each of R_0, R_1, and R_2 must be a line or a plane; in other words, generating reflexions in points are excluded. Without loss of generality, we may take the initial vertex of P to be o. We then have $o \in R_1 \cap R_2$, and so this latter intersection must be non-empty and strictly contained in both. Hence $\dim R_j \geq 1$ for $j = 1, 2$. For these j, we write $S_j = R_j$ or $-R_j$ as R_j is a plane or line (as before, $-R_j$ is the orthogonal complement of R_j), and L for the plane through o perpendicular to S_1 and S_2. Further, $o \notin R_0$. If $\dim R_0 = 0$, then it easily follows that $G(P)$ is reducible, since each R_j permutes the planes parallel to L, which is contrary to the assumption that P is pure. Indeed, we would have $R_0 \subset R_2$, since these reflexions commute. Thus $\dim R_0 \geq 1$ also.

We next exclude the possibility that $\dim R_j = 2$ for each $j = 1, 2$. In this case R_1 and R_2 would be planes through o, with some acute angle between them. Then R_0 is a line or plane, whose reflexion commutes with R_2, but not with R_1. If R_0 is a plane, then since $G(P)$ is irreducible it will follow that $R_0 \cap R_1 \cap R_2 \neq \emptyset$; hence $G(P)$ will be a discrete orthogonal group, and so finite. Similarly, if R_0 is a line, there are two possibilities. First, R_0 may lie in R_2, giving $R_0 \cap R_1 \cap R_2 \neq \emptyset$ since $G(P)$ is irreducible, and as before $G(P)$ is finite. Second, R_0 may be perpendicular to R_2; this makes the group $G(P)$ reducible, which again is not permitted.

There is now a further case to be excluded; we cannot have $\dim R_0 = 2$ and $\dim R_2 = 1$. If this were so, the line R_2 would have to be perpendicular to the plane R_0 (since $o \notin R_0$, the possibility $R_2 \subset R_0$ is forbidden). As in the previous case, the group $G(P)$ would then be reducible, which we do not allow.

In conclusion, then, the *dimension vector* $(\dim R_0, \dim R_1, \dim R_2)$ for the mirrors can take only four values, namely, $(2, 1, 2)$, $(1, 1, 2)$, $(1, 2, 1)$, and $(1, 1, 1)$.

We have already introduced the planar reflexions S_1 and S_2. We now define a third reflexion S_0, whose mirror is also a plane, as follows. We let R_0' be the translate of R_0 which contains the origin o, and then set $S_0 := R_0'$ or $-R_0'$ as R_0 is a plane or a line. In other words, we are employing the same trick as in the proof of Theorem 4.5. We write $G' := \langle R_0', R_1, R_2 \rangle$, which is the special group of $G(P)$ (see the proof of Lemma 5.1), and set $H := \langle S_0, S_1, S_2 \rangle$. Then H is a finite irreducible (plane) reflexion group, namely, one of $[3, 3]$, $[3, 4]$, or $[3, 5]$, and G' is either again one of these reflexion groups, or its rotation subgroup (this can happen only when $\dim R_j = 1$ for each j). Since $G(P)$ has to be discrete, Lemma 5.1 excludes fivefold rotations, and hence G' cannot be [3, 5] or its rotation subgroup. In other words, H must be [3, 3] or [3, 4].

With four possibilities for the vector ($\dim R_0$, $\dim R_1$, $\dim R_2$), and three for the group H (which can also be taken as [4, 3], of course), we see that we have just 12 possibilities. These 12 all occur; we may reverse the method of the proof, and observe that different positions of R_0 not containing o, but meeting R_2, lead to similar apeirohedra. □

We may now list these 12 apeirohedra, according to the different scheme given by the proof of Theorem 6.2.

	$\{3, 3\}$	$\{3, 4\}$	$\{4, 3\}$
(2, 1, 2)	$\{6, 6\vert 3\}$	$\{6, 4\vert 4\}$	$\{4, 6\vert 4\}$
(1, 1, 2)	$\{\infty, 6\}_{4,4}$	$\{\infty, 4\}_{6,4}$	$\{\infty, 6\}_{6,3}$
(1, 2, 1)	$\{6, 6\}_4$	$\{6, 4\}_6$	$\{4, 6\}_6$
(1, 1, 1)	$\{\infty, 3\}^{(a)}$	$\{\infty, 4\}_{\cdot,*3}$	$\{\infty, 3\}^{(b)}$

In this table the entries on the left are the dimension vectors ($\dim R_0$, $\dim R_1$, $\dim R_2$). The columns are indexed by the finite regular polyhedra to which the respective apeirohedra correspond.

It is appropriate to make one further comment here. The symmetry groups of the three apeirohedra associated with the dimension vector (1, 1, 1) are generated by rotations (half-turns) in $\mathbb{E}^3$; thus the whole groups contain only direct isometries. This implies that the three apeirohedra occur in enantiomorphic (mirror-image) pairs, with their facets consisting of either all left-hand helices or all right-hand helices. The other six apeirohedra with helical facets (three blended and three pure) contain both left- and right-handed helices, since there is a plane or point reflexion among the generators of each of their symmetry groups.

7. Group Presentations

It remains for us to prove that ten of the apeirohedra do have the automorphism groups that their notations signify, and to determine the groups of the remaining two. In doing this, we also verify the relationships of (6.1). (We do not treat the blended apeirohedra here, since they are described by their geometric structures.)

For the moment, we leave aside $\{4, 6|4\}$ and its dual; we thus take their groups as given. In Corollary 8.7 below, we demonstrate that their groups are as indicated by the notation. (The same assumption could apply to $\{6, 6|3\}$, but we actually obtain its group here, as it fits into our general scheme.) Since the Petrie operation interchanges k-holes and k-zigzags, we see that

$$\{4, 6|4\} \overset{\pi}{\longmapsto} \{\infty, 6\}_{4,4},$$
$$\{6, 4|4\} \overset{\pi}{\longmapsto} \{\infty, 4\}_{6,4},$$

as claimed. (Strictly speaking, perhaps we ought to replace "∞" here by "$\cdot$"!)

We next appeal to Theorem 3.14, to obtain

$$\{4, 6|4\} \stackrel{\eta}{\longmapsto} \{6, 6\}_4.$$

The Petrie operation and duality then yield $\{4, 6\}_6$ and $\{6, 4\}_6$. Another appeal to Theorem 3.14 then yields

$$\{4, 6\}_6 \stackrel{\eta}{\longmapsto} \{6, 6|3\}.$$

From this last, just as above we obtain

$$\{6, 6|3\} \stackrel{\pi}{\longmapsto} \{\infty, 6\}_{6,3}.$$

We have three apeirohedra remaining, one obtained by an application of σ $(= \pi\delta\eta\pi\delta)$, and the other two by applications of φ_2. The first we can do directly:

$$\begin{aligned} \{6, 4|4\} &\stackrel{\pi}{\longmapsto} \{\infty, 4\}_{6,4} \\ &\stackrel{\delta}{\longmapsto} \{4, \infty\}_{6,*4} \\ &\stackrel{\eta}{\longmapsto} \{\infty, \infty|3\}_4 \\ &\stackrel{\pi}{\longmapsto} \{4, \infty\}_{\infty,3} \\ &\stackrel{\delta}{\longmapsto} \{\infty, 4\}_{\cdot,*3}. \end{aligned}$$

The $*$ prefix to a suffix was explained in Section 6. We have replaced the ∞ in the final suffix by $\cdot$, to indicate that the corresponding value is unspecified. The only step left unexplained is the third, namely, the application of η to $\{4, \infty\}_{6,*4}$. The operation is

$$\eta\colon\ (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0\sigma_1\sigma_0, \sigma_2, \sigma_1) =: (\rho_0, \rho_1, \rho_2).$$

The first suffix 6 is dealt with by Theorem 3.14; observe that the graph of $\{4, \infty\}_{6,*4}$ is indeed bipartite. For the second suffix $*4$, the relation gives the period of

$$\sigma_2(\sigma_1\sigma_0)^2 = \sigma_2\sigma_1\sigma_0\sigma_1\sigma_0 = \rho_1\rho_2\rho_0 \sim \rho_0\rho_1\rho_2,$$

namely, that of the Petrie polygon of the second apeirohedron.

The last two apeirohedra, those of type $\{\infty, 3\}$, must be characterized by direct methods. We work with $P^{(b)} := \{\infty, 3\}^{(b)}$ rather than with $P^{(a)} := \{\infty, 3\}^{(a)}$, because its structure is a little easier to describe.

The symmetry group of $\{4, 6|4\}$ is generated by the three involutions

$$\begin{aligned} S_0\colon x &\mapsto (1 - \xi_1, \xi_2, \xi_3), \\ S_1\colon x &\mapsto (\xi_2, \xi_1, -\xi_3), \\ S_2\colon x &\mapsto (\xi_1, \xi_3, \xi_2), \end{aligned}$$

in terms of $x = (\xi_1, \xi_2, \xi_3)$, and the initial vertex is o. These are all symmetries of the honeycomb $\{4, 3, 4\}$ of unit cubes in $\mathbb{E}^3$, whose vertices are the points with integer cartesian coordinates; hence all the groups which occur in the discussion are subgroups of its symmetry group $[4, 3, 4]$. Indeed, we shall see in Section 8 that if $[4, 3, 4] =$

$\langle T_0, \ldots, T_3\rangle$ in the natural way, then $S_0 = T_0$, $S_1 = T_1 T_3$, and $S_2 = T_2$. These "standard" generators T_j are

$$\begin{aligned} T_0\colon x &\mapsto (1-\xi_1, \xi_2, \xi_3),\\ T_1\colon x &\mapsto (\xi_2, \xi_1, \xi_3),\\ T_2\colon x &\mapsto (\xi_1, \xi_3, \xi_2),\\ T_3\colon x &\mapsto (\xi_1, \xi_2, -\xi_3). \end{aligned}$$

If we start from $\{4, 6|4\}$ and trace through the three mixing operations which lead to $P^{(b)}$ (namely, η, π, and φ_2), we find that the symmetry group of the latter has generators (which are reflexions in lines)

$$\begin{aligned} R_0 = (S_0 S_1)^2\colon x &\mapsto (1-\xi_1, 1-\xi_2, \xi_3),\\ R_1 = S_2 S_1 S_2\colon x &\mapsto (\xi_3, -\xi_2, \xi_1),\\ R_2 = S_1\colon x &\mapsto (\xi_2, \xi_1, -\xi_3). \end{aligned}$$

We may picture $P^{(b)}$ in the following way. As we have already remarked, its facets are all helices with the same sense. The initial vertex is still o, and hence all the vertices are points of $\mathbb{E}^3$ with integer cartesian coordinates. We easily see from the generators that, in fact, the sum of the coordinates of each vertex is even.

The initial edge has vertices $o = (0, 0, 0)$ and $oR_0 = (1, 1, 0)$, which is a diagonal of a 2-face of $\{4, 3, 4\}$; hence all edges are such diagonals. Next, $R_1 R_0$, which preserves the initial facet and takes o into $(1, 1, 0)$, is

$$R_1 R_0\colon\ x \mapsto (1-\xi_3, 1+\xi_2, \xi_1),$$

which is a translation by $(0, 1, 0)$ together with a right-hand (or negative) twist of $\pi/2$ about the axis through $(\frac{1}{2}, 0, \frac{1}{2})$ in direction $(0, 1, 0)$. Hence the facets are helices of type $\{\infty\} \# \{4\}$. Finally, R_1 takes $(1, 1, 0)$ into $(1, 1, 0)R_1 = (0, -1, 1)$, and R_2 takes $(0, -1, 1)$ into $(0, -1, 1)R_2 = (-1, 0, -1)$; indeed, $R_2 R_1\colon\ x \mapsto (-\xi_3, -\xi_1, \xi_2)$ is a cyclic permutation of the signed basis vectors $e_1, -e_2, -e_3$.

It follows from this that $(R_1 R_0)^4$ is a translation, by $(0, 4, 0)$. However, such translations in the directions of the three coordinate axes do not generate the whole translation group. Instead, we observe that

$$x R_2 R_1 R_0 = (1+\xi_3, 1+\xi_1, \xi_2),$$

so that $(R_2 R_1 R_0)^3$ is the translation by $(2, 2, 2)$. Since the images of $(2, 2, 2)$ under $R_1 R_2$ and its inverse are $(-2, 2, -2)$ and $(-2, -2, 2)$, we see that the translation subgroup is actually the lattice $\Lambda := \Lambda_{(2,2,2)}$, which is generated by $(2, 2, 2)$ and its transforms under changes of signs of the coordinates. Incidentally, since $R_2 R_1 R_0$ is conjugate to the "translation" $R_0 R_2 \cdot R_1$ which takes one vertex of a facet of the Petrial $P^{(a)}$ of $P^{(b)}$ into the next vertex, we see that these facets are of type $\{\infty\} \# \{3\}$; this time, they are helices with a left-hand (positive) twist.

The axes of the helical facets of $P^{(b)}$ are parallel to the three coordinate axes; as we have seen, these three axes are permuted by $R_2 R_1$. To visualize the way in which the facets fit together, it is more convenient to concentrate on the vertical ones. The cubes in $\{4, 3, 4\}$ fall into vertical stacks or (infinite) towers. Just an eighth of these towers are

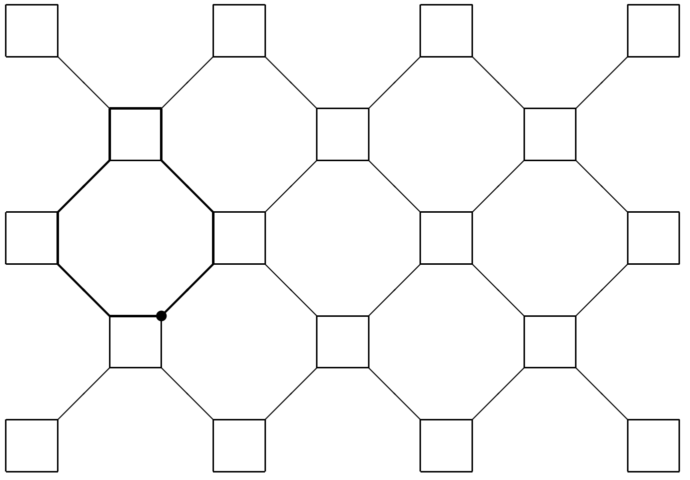

Fig. 1. The apeirohedron $\{\infty, 3\}^{(b)}$.

associated with facets; they are all the images of one fixed tower under the translation lattice Λ. A typical facet winds upwards (or downwards) in a right-hand spiral around the tower, crossing its square faces diagonally; we may envisage it as a staircase. (In Fig. 1 we are looking at the vertical towers from above. As we go around a tower in the clockwise direction, we rise by a floor each time we traverse an edge.)

The origin o is a vertex of a vertical tower; we think of it as lying at ground level. Ascending four flights of stairs brings us to $(0, 0, 4)$ on the fourth floor, immediately above our starting point. At each floor is a single horizontal bridge, leading away from one tower to an "adjacent" tower, across the diagonal of a horizontal square of $\{4, 3, 4\}$. If we ascend one flight to the first floor, cross the bridge, descend one flight in the adjacent tower to the ground floor, and then cross the next bridge, we shall similarly have gone four edges along a facet of $P^{(b)}$ with a horizontal axis. Each bridge belongs to two such horizontal facets, of course, according to whether it was reached by an ascending or descending flight.

Theorem 2.5 shows that we can find a presentation of the automorphism group of $P^{(b)} = \{\infty, 3\}^{(b)}$ by considering its edge-circuits (the vertex-figure is, of course, known). A minimal or *basic* circuit is constructed as follows. Ascend four floors of a tower by the staircase, cross the bridge to the adjacent tower, descend four floors by its staircase, and then cross back over the bridge to the starting point. The other basic circuits are then the images of this one under the symmetry group $G^{(b)} := G(P^{(b)})$ of $P^{(b)}$. A typical basic circuit using horizontal towers is formed similarly, although the description superficially appears different. From a starting vertex, cross a bridge and ascend one floor of a staircase. Repeat this twice, then cross a final bridge and descend three floors. Of course, we may interchange "ascend" and "descend" in this description. Such a circuit uses four towers in a square formation; the circuit goes round the inside of three towers, and the outside of the fourth. (In Fig. 1 such a horizontal basic circuit is indicated by heavy lines.)

If C and D are two edge-circuits, then we *concatenate* them by taking their symmetric difference $C \triangle D$. (In taking the symmetric difference, we are of course only considering the edges, not their vertices; isolated vertices obviously disappear.) Observe that $(C \triangle D) \triangle D = C$, so that concatenating twice with a fixed circuit has no effect. The key result for categorizing $G^{(b)}$ is

(7.1) Lemma. *An arbitrary edge-circuit in* $\{\infty, 3\}^{(b)}$ *is a concatenation of basic circuits.*

Proof. It is clear that, at any stage, we may confine our attention to a single connected circuit C; if, after any concatenation, a circuit becomes disconnected, then we simply consider the resulting components.

We now reduce the circuit C to a vertex by means of two kinds of operation. First, if C uses two or more bridges between the same towers, we may concatenate with vertical basic circuits to eliminate these bridges in pairs. Thus we may assume that C contains no more than one bridge between any two towers.

We now look down on C from a vertical direction, as in Fig. 1. Since the plane is simply connected, we may contract the projection of C to a single vertex. For this purpose, we can safely identify the vertices of C in any one tower, since there is now no more than one bridge between any two towers. A contraction over a diamond formed by four towers is achieved by concatenating with a horizontal basic circuit (like one of those indicated in Fig. 1) which uses these four towers and shares one of the bridges of C. Of course, further reductions of the first kind will then also generally be needed, since horizontal bridges along the other three sides of the diamond may be introduced (and some may disappear). It is clear that systematic application of these two kinds of operation will eventually reduce C to a single vertex of $P^{(b)}$. Hence, if we reverse the successive concatenations, we shall recover the original circuit, as was claimed. □

By Theorem 2.5, a basic edge-circuit in $P^{(b)}$ corresponds to a relation in $G^{(b)}$ between its distinguished generators R_0, R_1, and R_2. We consider the following horizontal basic circuit. It starts from the initial vertex o, and contains the first four successive edges of the initial facet F_2. This sequence of four edges is continued at each end by the two edges (corresponding to bridges) joining F_2 to the facet in an adjacent (horizontal) stack of cubes, and is completed by the four intermediate edges of that facet. The symmetry group of this basic circuit has two generators. The first is the conjugate $U_1 := (R_0R_1)^3R_0$ of R_1 by $(R_1R_0)^2$, which fixes F_2 and interchanges the two bridging edges. The second is the conjugate $U_2 := R_2R_1R_0R_1R_2$ of R_0 by R_1R_2, which interchanges F_2 and the "adjacent" facet, and fixes the two bridging edges. The relation which imposes this basic circuit is then $U_1U_2 = U_2U_1$, or $(U_1U_2)^2 = I_3$, the identity in $\mathbb{E}^3$. When expressed in terms of the generators R_0, R_1, and R_2, the relation $(U_1U_2)^2 = I_3$ involves R_0 ten times, in keeping with the fact that a basic circuit in $P^{(b)}$ has ten edges.

In order to state the main theorem, we provide an alternative interpretation of the group relation given by this basic circuit. Define

$$S := (R_0R_1)^4,$$
$$T := (R_0R_1R_2)^3.$$

Thus S and T are the translations given by the facet of $P^{(b)}$ and by its Petrie polygon (the facet of $P^{(a)}$). Then we have

$$\begin{aligned}(U_1U_2)^2 &= (R_0R_1)^3R_0 \cdot R_2R_1R_0R_1R_2 \cdot (R_0R_1)^3R_0 \cdot R_2R_1R_0R_1R_2 \\ &= (R_0R_1)^4 \cdot R_1R_2R_1R_0R_1R_2R_0R_1R_0 \cdot (R_1R_0)^4 \cdot R_0R_1R_0R_1R_2R_1R_0R_1R_2 \\ &= (R_0R_1)^4 \cdot R_2R_1R_2R_0R_1R_0R_2R_1R_0 \cdot (R_1R_0)^4 \cdot R_0R_1R_0R_2R_1R_2R_0R_1R_2 \\ &= ST^{-1}S^{-1}T,\end{aligned}$$

where we have freely used $R_0R_2 = R_2R_0$ and $R_1R_2R_1 = R_2R_1R_2$, but not the fact that S and T are actually translations, and hence commute. Rearranging, we see that $(U_1U_2)^2 = I_3$ and $ST = TS$ are equivalent relations; it is the latter which we employ.

(7.2) Theorem. *The automorphism groups of the two non-planar pure apeirohedra of type* $\{\infty, 3\}$ *in* $\mathbb{E}^3$ *are the Coxeter group* $[\infty, 3] = \langle \rho_0, \rho_1, \rho_2 \rangle$, *with the imposition of the single extra relation*

$$\sigma\tau = \tau\sigma,$$

where $\sigma := (\rho_0\rho_1)^3$ *and* $\tau := (\rho_0\rho_1\rho_2)^4$ *for* $\{\infty, 3\}^{(a)}$, *or* $\sigma := (\rho_0\rho_1)^4$ *and* $\tau := (\rho_0\rho_1\rho_2)^3$ *for* $\{\infty, 3\}^{(b)}$.

Proof. The given relation for $P^{(b)}$ is equivalent to the one given above, since σ corresponds to S and τ corresponds to T. By Theorem 2.5, any relation on the automorphism group $\Gamma^{(b)}$ of $P^{(b)}$ corresponds to an edge-circuit, and we have shown in Lemma 7.1 that these are formed by concatenating basic circuits, each of which is obtained by conjugating the extra relation. Thus the automorphism group of $P^{(b)}$ is as claimed.

The corresponding relation for $P^{(a)}$ is obtained from that for $P^{(b)}$ by means of the Petrie operation substitution of $\rho_0\rho_2$ for ρ_0. Indeed, in terms of the generators of $\Gamma^{(b)}$, and with σ and τ retaining (for the moment) their original definitions, we have

$$\begin{aligned}(\rho_0\rho_2\rho_1)^3 &= \rho_2\tau\rho_2 \\ (\rho_0\rho_2\rho_1\rho_2)^4 &= \rho_2\sigma\rho_2.\end{aligned}$$

Thus the relations between the new σ and τ are just the old ones (conjugated by ρ_2) with σ and τ interchanged, again as asserted. □

We make a further comment on this group. By definition, $P^{(b)} = (\{4, 6\}_6)^{\varphi_2}$, by means of the operation

$$(\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0, \sigma_1\sigma_2\sigma_1, \sigma_2) =: (\rho_0, \rho_1, \rho_2)$$

on the (larger) automorphism group $[4, 6]_6 := \Gamma(\{4, 6\}_6) = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$. If we substitute for ρ_0, ρ_1, and ρ_2 in this way, it is easy to check that (as we must) we do obtain a valid relation in $[4, 6]_6$.

We end with a remark on the groups of the pure apeirohedra with finite faces. We put them in a table, together with the pairs of finite regular polyhedra to which they correspond; as before, the entry in the first column of the table is the dimension vector (dim R_0, dim R_1, dim R_2).

(2, 2, 2)	$\{3, 3\}$	$\{3, 4\}$	$\{4, 3\}$
(1, 2, 2)	$\{6, 3\}_4$	$\{6, 4\}_3$	$\{4, 3\}_3$
(2, 1, 2)	$\{6, 6\|3\}$	$\{6, 4\|4\}$	$\{4, 6\|4\}$
(1, 2, 1)	$\{6, 6\}_4$	$\{6, 4\}_6$	$\{4, 6\}_6$

A convex regular polyhedron of type $\{3, q\}$ (or $\{q, 3\}$) has holes $\{h\}$ with $h = q$, while its Petrie polygon is an r-gon with

$$r = \frac{2q + 10}{7 - q}$$

(this is derived from 4.91 of [3], and is just one of many possible expressions; we write r here for Coxeter's h, for obvious reasons). For $p > 2$, we define p' by

$$\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}.$$

The corresponding polyhedra in each column are then

$$\{p, q\}, \quad \{p', q\}_{r'}, \quad \{p', q'|h\}, \quad \{p', q'\}_r.$$

Of course, the fact that the holes or Petrie polygons of the derived polyhedra are those given follows from the relationship between the generating reflexions R_0, R_1, and R_2 of their groups, and the corresponding plane reflexions S_0, S_1, and S_2 which generate the group of the convex polyhedron.

8. Regular 4-Apeirotopes

In view of Corollary 2.7, we may confine our attention to 4-apeirotopes in any discussion of possible faithfully realized regular polytopes of rank at least 4 in $\mathbb{E}^3$. We treat these largely geometrically.

So, let P be a discrete faithfully realized regular 4-apeirotope in $\mathbb{E}^3$, with symmetry group $\langle R_0, R_1, R_2, R_3\rangle$. Its facets are (finite or infinite) regular polyhedra. Furthermore, its ridges (2-faces) must be planar regular polygons, again finite or infinite; they cannot be three-dimensional, because this would force the stabilizing element R_3 of the base ridge to be the identity, and no regular apeirohedron has linear apeirogons as facets, because then the stabilizing element R_2 of the base edge would fix the whole line containing the base facet, and hence all the vertices would lie on this line.

It follows that R_3 must be the reflexion in the plane of the base ridge. Moreover, the facets cannot be planar, because then all the vertices would lie in the plane of this base ridge.

The vertex-figure P/v of P at its initial vertex v must be a finite regular polyhedron, and hence one of the 18 in the list of (4.1), (4.2), and (4.3); see Theorem 4.5. However, Bieberbach's theorem and discreteness again exclude fivefold rotational symmetries; see Lemma 5.1. The vertex-figures must therefore belong to the crystallographic family in

(4.1) and (4.2), namely

$$\{3,3\},\ \{4,3\}_3; \quad \{3,4\},\ \{6,4\}_3; \quad \{4,3\},\ \{6,3\}_4.$$

We have listed Petrie duals together.

The only polyhedra which can be vertex-figures of a regular 4-apeirotope P with finite planar ridges are $\{3, 4\}$ and $\{6, 4\}_3$; the 2-faces are then squares $\{4\}$. (Note, incidentally, that blended regular apeirohedra cannot have finite planar facets.) To see this, we may use the same argument as that on p. 69 of [3]. Recall that the ratio of the edge-length of the vertex-figure of a planar regular polygon $\{p\}$ (joining the two vertices adjacent to a given one) to the edge-length of $\{p\}$ itself is $2\cos(\pi/p)$. Hence, if P has 2-faces $\{p\}$ for some rational number p, its vertex-figure must be a finite regular polyhedron whose ratio of edge-length to circumradius is of the form $2\cos(\pi/p)$; the only instances are the octahedron $\{3, 4\}$ and its Petrial $\{6, 4\}_3$, where $p = 4$. We thus obtain the two apeirohedra

(8.1) $$\{4,3,4\} = \{\{4,3\},\{3,4\}\}, \quad \{\{4,6|4\},\{6,4\}_3\}.$$

We justify the notation for the second apeirotope below; it is indeed the universal regular polytope of its kind.

In our listing of the regular apeirohedra, we found that the only ones with planar (zigzag) apeirogons as facets are

$$\{\infty,6\}_3 \# \{\ \}, \quad \{\infty,4\}_4 \# \{\ \}, \quad \{\infty,3\}_6 \# \{\ \};$$
$$\{\infty,6\}_3 \# \{\infty\}, \quad \{\infty,4\}_4 \# \{\infty\}, \quad \{\infty,3\}_6 \# \{\infty\}.$$

In each case, the reflexion R_0 is that in a point (obtained as the product of two such reflexions, one for each component of the blend). It follows that the only other possibilities for discrete regular 4-apeirotopes are obtained by taking R_0 to be a point of $R_2 \cap R_3$, since R_0 must commute with R_2 and R_3. (Observe that the six possible vertex-figures do have $R_2 \cap R_3$ as a line.) Since R_0 and R_1 must not commute, we have $R_0 \notin R_1$. In effect, this amounts to choosing R_0 to be a vertex w of the vertex-figure P/v at v, or, more strictly perhaps, half-way between v and w (this ensures that w is the image of v under R_0). Each possible choice will yield an apeirotope.

The resulting six apeirotopes are of type (in a general sense)

(8.2) $$\begin{gathered} \{\{\infty,3\}_6 \# \{\ \},\{3,3\}\}, \quad \{\{\infty,4\}_4 \# \{\infty\},\{4,3\}_3\}; \\ \{\{\infty,3\}_6 \# \{\ \},\{3,4\}\}, \quad \{\{\infty,6\}_3 \# \{\infty\},\{6,4\}_3\}; \\ \{\{\infty,4\}_4 \# \{\ \},\{4,3\}\}, \quad \{\{\infty,6\}_3 \# \{\infty\},\{6,3\}_4\}. \end{gathered}$$

This notation suppresses the exact definitions of the apeirotopes, and should not be taken to imply universality.

The identification of the apeirotopes in the list (8.2) is facilitated by the observation that the vertex-figures of an apeirohedron $\{\infty, q\}_s \# \{\ \}$ are planar polygons $\{q\}$, while those of $\{\infty, q\}_s \# \{\infty\}$ are skew polygons $\{q\} \# \{\ \}$. (The 2-faces of $\{4, 3\}_3$ are skew polygons $\{4\} \# \{\ \}$, while those of $\{6, 4\}_3$ and $\{6, 3\}_4$ are skew polygons $\{6\} \# \{\ \}$.)

To summarize, we thus have

(8.3) Theorem. *The list of eight discrete regular* 4*-apeirotopes in* $\mathbb{E}^3$ *in* (8.1) *and* (8.2) *is complete.*

Before we go on to describe the groups of these eight apeirotopes (although we only find presentations for the first two), we make some remarks. The two lists (8.1) and (8.2) group the apeirotopes in pairs; their vertex-figures are Petrie duals, and so their automorphism (or symmetry) groups are related by the involutory mixing operation

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_0, \sigma_1\sigma_3, \sigma_2, \sigma_3) =: (\rho_0, \rho_1, \rho_2, \rho_3).$$

This may also be seen to induce the appropriate changes of kind in the facets in (8.2), namely, that between blends with the segment $\{\,\}$ and the apeirogon $\{\infty\}$. Geometrically, when the vertex-figure is a convex regular polyhedron $\{q, r\}$, with symmetry group $\langle R_1, R_2, R_3\rangle$ generated by plane reflexions, then R_0 is a point of the line $R_2 \cap R_3$, showing that a 2-face, with group $\langle R_0, R_1\rangle$, is a zigzag (planar) apeirogon. If we replace R_1 by $R_1R_3 = R_1 \cap R_3$, the reflexion in a line, we see that the new 2-face is the same zigzag apeirogon, since that is fixed by R_3. However, under this change of generators, the vertex-figure of the facet $\{\infty, q\}_s \# \{\,\}$ is changed from a planar polygon $\{q\}$ to a skew polygon $\{t\} \# \{\,\}$, namely, the Petrie polygon of the original vertex-figure $\{q, r\}$. The new facet must then be an apeirohedron of the form $\{\infty, t\} \# \{\infty\}$. This explains the pairing of the apeirotopes in (8.2). Further, if we have a presentation for the automorphism group of one of the pair, then we have it for the other, just by making the substitution of $\rho_1\rho_3$ for ρ_1 wherever it occurs.

Now we already know that $\{4, 3, 4\} = \{\{4, 3\}, \{3, 4\}\}$ is the universal regular polytope of its Schläfli type. When we replace its vertex-figure $\{3, 4\}$ by its Petrial $\{6, 4\}_3$, we obtain the following presentation for the automorphism group of $P := \{\{4, 6|4\}, \{6, 4\}_3\}$:

$$\begin{aligned}\rho_0^2 &= (\rho_1\rho_3)^2 = \rho_2^2 = \rho_3^2\\ &= (\rho_0\rho_1\rho_3)^4 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2\\ &= (\rho_1\rho_3\rho_2)^3 = (\rho_1\rho_3^2)^2 = (\rho_2\rho_3)^4 = \varepsilon.\end{aligned}$$

Simplifying these relations and reordering them, we obtain

(8.4)
$$\begin{aligned}\rho_0^2 &= \rho_1^2 = \rho_2^2 = \rho_3^2\\ &= (\rho_0\rho_1)^4 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = (\rho_1\rho_3)^2 = (\rho_2\rho_3)^4\\ &= (\rho_1\rho_2\rho_3)^3 = \varepsilon.\end{aligned}$$

The relations involving ρ_1, ρ_2, and ρ_3 certainly specify the group of $\{6, 4\}_3$, which must therefore be the vertex-figure. The relation $(\rho_1\rho_2)^6 = \varepsilon$ is implied by the other relations, and it is only conventional to insert the number "6" in $\{\cdot, 4\}_3$, as its omission looks a little strange.

The relations involving ρ_0, ρ_1, and ρ_2, on the other hand, are clearly inadequate as they stand to specify the facet $\{4, 6|4\}$. It is curious, therefore, that the relations of (8.4) must serve to specify the group of P itself. In this context, we show in Corollary 8.7 that the mixing operation

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_0, \sigma_1\sigma_3, \sigma_2) =: (\rho_0, \rho_1, \rho_2)$$

applied to $[4, 3, 4]$ indeed yields the group of $\{4, 6|4\}$. Here, it is appropriate to demonstrate:

(8.5) Theorem. *An abstract regular* 4*-polytope of type* $\{4, 6, 4\}$ *with vertex-figure of type* $\{6, 4\}_3$ *is a quotient of* $\{\{4, 6|4\}, \{6, 4\}_3\}$.

Proof. In fact, we could really describe the type of the polytope as $\{4, \cdot, 4\}$. Under the given conditions, the group of such a polytope Q, say, satisfies the relations (8.4); we have observed that the vertex-figure then must be of type $\{6, 4\}_3$. We now reverse the Petrie operation on the vertex-figure, whereby we recover the relations for the Coxeter group $[4, 3, 4]$.

It remains to show that we come to the same conclusion, even if we impose the extra relations which specify the facet $\{4, 6|4\}$. We may certainly set $(\rho_1\rho_2)^6 = \varepsilon$, if this is not already given to us. Under the (reverse) mixing operation,

$$\begin{aligned} \rho_0\rho_1\rho_2\rho_1 &= \sigma_0\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3 \\ &\sim \sigma_0\sigma_1\sigma_2\sigma_1 \\ &= \sigma_0\sigma_2\sigma_1\sigma_2 \\ &\sim \sigma_0\sigma_1, \end{aligned}$$

so that $(\rho_0\rho_1\rho_2\rho_1)^4 = \varepsilon$ is compatible with the presentation of $[4, 3, 4]$. □

There are two immediate consequences of this argument.

(8.6) Corollary. *The apeirotope of type* $\{\{4, 6|4\}, \{6, 4\}_3\}$ *in* $\mathbb{E}^3$ *is universal*.

(8.7) Corollary. *The mixing operation*

$$(\sigma_0, \sigma_1, \sigma_2, \sigma_3) \mapsto (\sigma_0, \sigma_1\sigma_3, \sigma_2) =: (\rho_1, \rho_1, \rho_2)$$

applied to $[4, 3, 4]$ *yields the group of* $\{4, 6|4\}$.

For the other six apeirotopes, on the face of it the procedure appears very simple. To the group $\langle\rho_1, \rho_2, \rho_3\rangle$ of the vertex-figure, we adjoin a new generator ρ_0 such that

$$\rho_0^2 = (\rho_0\rho_2)^2 = (\rho_0\rho_3)^2 = \varepsilon.$$

This is just the dual procedure to the construction of the free extension in [18] and [19]. However, in general we do not actually obtain the free extension, because the translation subgroup of the apeirotope imposes extra relations on the group.

In fact, we confine ourselves to some brief remarks. When the vertex-figure is $\{3, 3\}$ or its Petrie dual $\{4, 3\}_3$, the translation subgroup T is generated by products of pairs of conjugates of ρ_0; that is,

$$T = \langle \alpha\rho_0\beta\rho_0\gamma \mid \alpha, \beta, \gamma \in \langle\rho_1, \rho_2, \rho_3\rangle \text{ and } \alpha\beta\gamma = \varepsilon\rangle.$$

(General products of pairs of conjugates of ρ_0 can be expressed as products of these.) The imposed extra conditions on the group have to say that T is abelian.

For the remaining cases, the product of a conjugate of ρ_0 by an element of $\langle\rho_1, \rho_2, \rho_3\rangle$

with the central reflexion in the vertex-figure, namely, $(\rho_1\rho_2\rho_3)^3$ for {3, 4} and {4, 3} or $(\rho_1\rho_2)^3$ for their Petrie duals, will also belong to T. Again, conditions must be imposed which force T to be abelian. In fact, these are precisely the conditions which arise from applying Theorem 2.5, though we do not give any details. However, it is worth noting one curiosity. We observe that

$$\{\infty, q\}_s \# \{\ \} \cong \{\infty, q\}_s$$

for the pairs $(q, s) = (3, 6)$ or $(4, 4)$, since all edge-circuits of the planar apeirohedra have even lengths. It turns out that two of the three apeirotopes with facets of type $\{\infty, q\}_s \# \{\ \}$ are universal:

(8.8) $$\{\{\infty, 3\}_6 \# \{\ \}, \{3, 3\}\} \cong \{\{\infty, 3\}_6, \{3, 3\}\}$$

and

(8.9) $$\{\{\infty, 4\}_4 \# \{\ \}, \{3, 3\}\} \cong \{\{\infty, 4\}_4, \{4, 3\}\};$$

that is, they have automorphism groups which the latter notation signifies.

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