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Finite-Time Synchronization of Coupled Markovian Discontinuous Neural Networks with Mixed Delays

Xinsong Yang[1] · Jinde Cao[2] · Qiang Song[3] · Chen Xu[4] · Jianwen Feng[4]

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Abstract This paper is concerned with finite-time synchronization in an array of coupled neural networks with discontinuous activation functions, Markovian jumping parameters, as well as discrete and infinite-time distributed delays (mixed delays) under the framework of Filippov solution. Based on novel Lyapunov–Krasovskii functionals and analytical techniques and M-matrix method, the difficulties caused by the uncertainties of Filippov solutions, time delays, as well as Markov chain are overcome. Several sufficient conditions are obtained to guarantee the synchronization in finite time. Different from existing results on finite-time synchronization of non-delayed systems, the settling time for time-delay systems is dependent not only on the values

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✉ Xinsong Yang
xinsongyang@163.com

Jinde Cao
jdcao@seu.edu.cn

Qiang Song
songqhdu@163.com

Chen Xu
xuchen_szu@szu.edu.cn

Jianwen Feng
fengjw@szu.edu.cn

1 Department of Mathematics, Chongqing Normal University, Chongqing 401331, China

2 Department of Mathematics, Southeast University, Nanjing 210096, China

3 College of Electrical Engineering, Henan University of Technology, Zhengzhou 450001, China

4 College of Mathematics and Statistics, Shenzhen University, Shenzhen 518060, China

of the error state at time zero, but also on the histories of the error state, the time delays, and the initial value of Markov chain. Moreover, finite-time synchronization of the coupled neural networks with nonidentical uncertain perturbations is also considered. The obtained results are also applicable to continuous nonlinear systems, which essentially extend existing results which can only finite-timely synchronize or stabilize non-delayed systems. Finally, numerical examples are given demonstrate the effectiveness of the theoretical results.

Keywords Discontinuous neural networks · Filippov solutions · Markov Chain · Finite-time synchronization · Mixed delays

1 Introduction

In the past decades, synchronization of chaotic systems has been extensively studied thanks to the pioneering work [25]. Although different kinds of synchronization, such as generalized synchronization, projective synchronization, lag synchronization, have been proposed and extensively studied [5,17,20,23,38,40,48,49], all these synchronization can be classified as the following two kinds: synchronization as time goes to infinity (or asymptotic synchronization) and synchronization in finite time. Compared with the former, finite-time synchronization is optimal [41,42]. It is well known that the range of time during which the chaotic oscillators are not synchronized corresponds to the range of time during which the encoded message can unfortunately not be recovered [41]. Therefore, finite-time technique enable us to recover the transmitted signals in a setting time, while the transmitted signals can only be obtained as time goes to infinity if the other synchronization technique is utilized. Obviously, compared with asymptotic synchronization, finite-time synchronization improves the efficiency and confidentiality greatly when it is applied to secure communication. Moreover, the finite-time control techniques possess better robustness and disturbance rejection properties [35], which are also considered in the present paper. Due to these advantages, many researchers have devoted themselves to finite-time synchronization of coupled chaotic systems. For example, authors in [36] addressed finite-time synchronization of multi-agent systems, and authors in [1,33,37,41,43] considered finite-time synchronization of some class of coupled chaotic systems.

Neural networks, as a class of important chaotic systems, have been extensively applied pattern recognition, image processing, secure communication, automatic control, and associative memory [15,50]. Since the combination of a set of neural networks could achieve higher level information processing [4], synchronization in an array of coupled neural networks, as a typical collective behavior, has been extensively studied in various fields [3,14,44]. However, the activation functions of the neural network in the above-mentioned references are continuous. When the activation functions are not continuous, most of existing results on synchronization of neural networks including those in the former mentioned papers are not applicable any more. It is reported that neural networks with discontinuous (or non-Lipschitz) neuron activations are an ideal model when the gain of the neuron amplifiers is very high, which is frequently encountered in applications [13]. For instance, in the classical model of Hopfield neural networks with graded response neurons, when the activations are assumed to be in a high-gain limit, they actually closely approach discontinuous and comparator functions [22]. Neural networks with discontinuous activations are a special kind of differential equations with discontinuous state on the right-hand side. Since their solution cannot be guaranteed to be unique [10] (i.e., its solution is uncertain but belongs to a collection), it has analysis difficulty in studying their synchronization.

In [39,43,45,46], synchronization of discontinuous dynamical systems has been considered by using discontinuous controllers. Specially, finite-time synchronization in an array of coupled nonidentical systems of differential equations with discontinuous state on the right-hand side was investigated in [43], and [46] studied finite-time synchronization of coupled discontinuous neural networks with mixed delays. One phenomenon for neural networks which should not be ignored is the interconnections among the neurons, which are often affected by some random factors such as such environmental changes, random failures, and repairs [32]. Such change in the interconnection among the neurons usually takes place according to the rule of Markov chain with finite state space [7]. In recent years, increasing attention has been attracted to stability and stabilization of Markovian systems [18,19]. Although there were many results on asymptotic or exponential synchronization of Markovian neural networks with continuous activations and various time delays [21,30,47], seldom result on synchronization of Markovian neural networks with discontinuous activations and delays is reported till now, not to mention finite-time synchronization of such kind of neural networks, so the present paper solves this problem. Note that it is not an easy work to deal with the Markov chain in studying finite-time synchronization. It is well known that, for a given generator matrix, the Markov chain generated by the generator can be completely different [47], which leads to uncertainty of the Markov chain. The usual method to surmount the effect of the uncertainty is to introduce some free parameters or matrices in studying dynamics of systems with Markovian jumping parameters [30]. However, the free parameters shall make it difficult to ascertain the setting time. Hence, we have to establish new analytical techniques to cope with this difficulty, which is challenging.

Motivated by the above discussions, this paper aims to investigate finite-time synchronization in an array of coupled neural networks with discontinuous activation functions, Markovian jumping parameters, as well as mixed delays. Novel methods are proposed to study the finite-time synchronization of the coupled neural networks. Under a class of simple controller, difficulties caused by the uncertainties of Filippov solutions, time delays, and Markov chain are well coped with by using the new method, M-matrix method, and designing some new Lyapunov–Krasovskii functionals. Our synchronization criteria do not have any free parameters and can be easily verified. Numerical examples demonstrate the effectiveness of the theoretical results.

The rest of this paper is organized as follows. Section 2 presents the model of linearly coupled neural networks with discontinuous activations and mixed delays. Some necessary preliminaries are provided in this section. Several finite-time synchronization criteria are obtained in Sect. 3. Section 4 investigates the finite-time synchronization of the coupled neural networks with nonidentical uncertain perturbations. Then, Sect. 5 gives numerical simulations to show the effectiveness of our results. Finally, Sect. 6 gives conclusions and future work.

Notations In the sequel, if not explicitly stated, matrices are assumed to have compatible dimensions; $\mathbb{R}^n$ denotes the set of $n \times 1$ real vectors, and $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ matrices; I_n denotes the identity matrix of n-dimension; $\mathbf{1}_n$ is a column vector with all n elements being 1; $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are the 1-norm and ∞-norm of a vector or a matrix, respectively. $|\Psi|$ is a vector (or a matrix) derived by taking absolute values of all the elements of the vector (or matrix) Ψ; $\overline{co}[E]$ is the closure of the convex hull of the set $E \subset \mathbb{R}^n$. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., the filtration contains all P-null sets and is right continuous). Denote by $L_{\mathcal{F}_0}((-\infty, 0]; \mathbb{R}^n)$ the family of all $\mathcal{F}_0$-measurable $C((-\infty, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(s) : s \leq 0\}$ such that $\sup_{s\leq 0} \mathbf{E}\{\|\xi(s)\|\} < \infty$, where $\mathbf{E}\{\cdot\}$ stands for mathematical expectation operator with respect to the given probability measure P.

2 Model Formulation and Preliminaries

Generally, neural networks with discrete and infinite-time distributed delays can be described as follows:

$$\begin{aligned} \dot{x}(t) = &-Cx(t) + Af(x(t)) + Bf(x(t-\tau(t))) \\ &+D\int_{-\infty}^{t} K(t-s)f(x(s))\mathrm{d}s + J, \end{aligned} \tag{1}$$

where $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}} \in \mathbb{R}^n$ represents the state vector of the neural network at time t; n corresponds to the number of neurons; $f(x(t)) = (f_1(x_1(t)), \ldots, f_n(x_n(t)))^{\mathrm{T}}$ is the neuron activation function; $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ is a diagonal matrix with $c_i > 0$, which represents the rate with which the ith neuron will reset its potential to the resting state; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $D = (d_{ij})_{n\times n}$ are the connection weight matrix, time-delayed weight matrix, and the distributively time-delayed weight matrix, respectively; $J = (J_1, J_2, \ldots, J_n)^{\mathrm{T}} \in \mathbb{R}^n$ is an external input vector; $\tau(t)$ is the time-varying delay; $K(t)$ is a nonnegative bounded scalar function defined on $[0, +\infty)$ describing the delay kernel of the infinite-time distributed delay.

The trajectory of the solution $x(t)$ to neural network (1) can be any desired state: equilibrium point, a non-trivial periodic or almost periodic orbit, or even a chaotic orbit.

In this paper, we suppose that the activation function $f(x(t))$ is not continuous on $\mathbb{R}^n$. Hence, system (1) becomes a differential equation with discontinuous right-hand side. In this case, the existence and uniqueness of the solution to (1) might be lost, and at the worst case, one cannot define a solution in the conventional sense.

In order to study the dynamics of a system of differential equation with discontinuous right-hand side, we first transform it into a differential inclusion [10] by using Filippov regularization; then by the measurable selection theorem in [2], we reach an uncertain differential equation. Thus, studying the dynamics of the system of differential equation with discontinuous right-hand side has at last been transformed into studying the corresponding problem of the uncertain differential equation. The Filippov regularization is defined as follows:

Definition 1 [10] The Filippov set-valued map of $f(x)$ at $x \in \mathbb{R}^n$ is defined as follows:

$$F(x) = \bigcap_{\varsigma>0} \bigcap_{\mu(\Omega)=0} \overline{co}[f(B(x,\varsigma) \setminus \Omega)],$$

where $B(x,\varsigma) = \{y : \|y - x\| \leq \varsigma\}$, and $\mu(\Omega)$ is the Lebesgue measure of set Ω.

By Definition 1, the Filippov set-valued map gives the convex hull of $f(\cdot)$ at the discontinuity points (ignoring sets of measure zero) when applied to the discontinuity points, but is otherwise the same as $f(\cdot)$ at continuous points. A vector-value function $x(t)$ defined on the interval $[0, T]$ is called a Filippov solution of $\dot{x}(t) = f(x(t))$ if it is absolutely continuous on $[0, T]$ and satisfies the differential inclusion $\dot{x}(t) \in F(x(t))$ for $t \in [0, T]$. By the measurable selection theorem in [2], we can find a measurable function $\gamma : [0, T] \to \mathbb{R}^n$ such that $\gamma(t) \in F(x(t))$ for almost all (a.a.) $t \in [0, T]$ and $\dot{x}(t) = \gamma(t)$ for a.a. $t \in [0, T]$.

As for the neural network (1), we assume that

- (H_1) For every $i = 1, 2, \ldots, n$, $f_i : \mathbb{R} \to \mathbb{R}$ is continuous except on a countable set of isolate points $\{\rho_k^i\}$, where there exist finite right and left limits $f_i^+(\rho_k^i)$ and $f_i^-(\rho_k^i)$, respectively. Moreover, f_i has at most a finite number of jump discontinuities in every compact interval of $\mathbb{R}$.
- (H_2) For each $i = 1, 2, \ldots, n$, there exist nonnegative constants z_i and p_i such that $\sup|\xi_i - \eta_i| \leq z_i|u - v| + p_i$ for $\forall u, v \in \mathbb{R}$, where $\xi_i \in F_i(u)$, $\eta_i \in F_i(v)$, $F_i(u) = [\min\{f_i^-(u), f_i^+(u)\}, \max\{f_i^-(u), f_i^+(u)\}]$.
- (H_3) There exist two constants $\mu < 1$ and τ such that $\dot{\tau}(t) \leq \mu$ and $0 < \tau(t) \leq \tau$.
- (H_4) There is a positive constant q such that $\int_0^{+\infty} K(u)\mathrm{d}u = q$.

Note that, when $p_i = 0$ in (H_2), the function $f_i(u)$ is continuous on $\mathbb{R}$. Hence, the assumption (H_2) includes continuous activation function $f(x)$, $x \in \mathbb{R}^n$ as a special case. In the following, we denote $F(x(t)) = (F_1(x_1(t)), F_2(x_2(t)), \ldots, F_n(x_n(t)))^{\mathrm{T}}$.

The following Definition 2 specifies what a Filippov solution of system (1) is.

Definition 2 [12] A function $x : (-\infty, T] \to \mathbb{R}^n$, $T \in (0, +\infty]$, is a Filippov solution of the discontinuous system (1) on $(-\infty, T]$ if:

- (i) x is continuous on $(-\infty, T]$ and absolutely continuous on $[0, T]$;
- (ii) There exists a measurable function $\gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))^{\mathrm{T}} : (-\infty, T] \to \mathbb{R}^n$, such that $\gamma(t) \in F(x(t))$ for almost all (a.a.) $t \in (-\infty, T]$ and

$$\begin{aligned}\dot{x}(t) = &-Cx(t) + A\gamma(t) + B\gamma(t - \tau(t)) + D\int_{-\infty}^{t} K(t-s)\gamma(s)\mathrm{d}s \\ &+J, \quad \text{for a.a. } t \in [0, T]. \end{aligned} \tag{2}$$

Definition 3 (IVP) [13] For any continuous function $\varphi : (-\infty, 0] \to \mathbb{R}^n$ and measurable selection $\psi : (-\infty, 0] \to \mathbb{R}^n$ such that $\psi(s) \in F(\varphi(s))$ for a.a. $s \in (-\infty, 0]$

by an initial value problem associated to (2) with initial condition (φ, ψ), we mean the following problem: find a couple of functions $[x(t), \gamma(t)] : (-\infty, T] \to \mathbb{R}^n \times \mathbb{R}^n$, such that $x(t)$ is a solution of (2) on $(-\infty, T]$ for some $T > 0$, $\gamma(t)$ is an output associated with $x(t)$, and

$$\begin{cases} \dot{x}(t) = -Cx(t) + A\gamma(t) + B\gamma(t-\tau(t)) + D\int_{-\infty}^{t} K(t-s)\gamma(s)\mathrm{d}s \\ \qquad +J, \quad \text{for a.a. } t \in [0, T], \\ \gamma(t) \in F(x(t)), \text{ for a.a. } t \in [0, T], \\ x(s) = \varphi(s), \qquad \forall s \in (-\infty, 0], \\ \gamma(s) = \psi(s), \qquad \text{for a.a. } s \in (-\infty, 0]. \end{cases} \tag{3}$$

Lemma 1 [46] *Suppose that* (H_1)–(H_4) *are satisfied. Then, there exists at least one solution* $x(t)$ *of discontinuous neural network* (1) *on* $[0, +\infty)$ *in the sense of equation* (3).

Let $\{r_t, t \geq 0\}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ taking values in a finite state space $\varpi = \{1, 2, \ldots, w\}$ with generator $\Pi = (\pi_{ij})_{w\times w}$ given by:

$$P\{r_{t+\Delta t} = j : r_t = i\} = \begin{cases} \pi_{ij}\Delta t + O(\Delta t), & \text{if } \ i \neq j, \\ 1 + \pi_{ii}\Delta t + O(\Delta t), & \text{if } \ i = j, \end{cases}$$

where w is a positive integer, $\Delta t > 0$ and $\lim\limits_{\Delta t\to 0} \frac{O(\Delta t)}{\Delta t} = 0$. Here, $\pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\pi_{ii} = -\sum\limits_{j=1, j\neq i}^{w} \pi_{ij}$.

As a standing hypothesis, we assume that Π is irreducible. This is equivalent to the condition that, for any $i, j \in \varpi$, we can find $i_1, i_2, \ldots, i_k \in \varpi$ such that $\pi_{ii_1}\pi_{i_1 i_2} \ldots, \pi_{i_k j} > 0$.

Usually, a coupled system is related to a digraph. Let $\mathcal{G} = \{\mathcal{N}, \varepsilon, \mathcal{V}\}$ be a digraph with a node set $\mathcal{N} = \{1, 2, \ldots, N\}$, an edge set $\mathcal{E} \subseteq \mathcal{N}\times\mathcal{N}$, and a weighted adjacency matrix $\mathcal{V} = (v_{ij})_{N\times N}$ with nonnegative elements. A directed edge denoted by (j, i) means that node i has access to node j, i.e., node i can receive information from node j. The elements of the adjacency matrix $\mathcal{V} = (v_{ij})_{N\times N}$ are defined as follows: if there is a directed link from node j to i $(j \neq i)$, then $v_{ij} > 0$; otherwise, $v_{ij} = 0$. We assume that $v_{ii} = 0$ for all $i \in \mathcal{N}$. The Laplacian matrix with respect to the digraph $\mathcal{G}$ is $L = (l_{ij})_{N\times N}$ with $l_{ij} = -v_{ij}$ $i \neq j$, and $l_{ii} = \sum_{j=1, j\neq i} v_{ij}$.

Considering linear state coupling, an array of coupled neural network (1) with Markovian parameters under the Markov chain defined above can be described as

$$\begin{aligned} \dot{x}_i(t) = &-C(r_t)x_i(t) + A(r_t)f(x_i(t)) + B(r_t)f(x_i(t-\tau(t))) + J(r_t) \\ &+ D(r_t)\int_{-\infty}^{t} K(t-s)f(x_i(s))\mathrm{d}s - \sum_{j=1}^{N} l_{ij}(r_t)\Phi(x_j(t) - x_i(t)) \\ &+ R_i(r_t, t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{4}$$

where $\{r_t, t \geq 0\}$ is the continuous-time Markov process describing the evolution of the mode at time t, $x_i(t) = (x_{i1}(t), \ldots, x_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n$ represents the state vector of the ith node of the network at time t; $L(r_t) = (l_{ij}(r_t))_{N\times N}$ is the Laplacian matrix of the coupled network, $\Phi = \mathrm{diag}(\phi_1, \ldots, \phi_n)$ with $\phi_l > 0, l = 1, \ldots, n$; $R_i(r_t, t)$ are controllers to be designed. The other parameters have the same physical meanings as those in the neural network (1).

The initial condition associated with system (4) is given by $x_i(t) = \phi_i(t) \in C((-\infty, 0]; \mathbb{R}^n), i = 1, 2, \ldots, N$.

Definition 4 The coupled neural network (4) is said to be finite-timely synchronized if, by adding suitable designed controllers, there exists a constant $t_1 > 0$, such that $\lim\limits_{t\to t_1} \mathbf{E}\{\|x_i(t) - x_1(t)\|_1\} = 0$ and $\mathbf{E}\{\|x_i(t) - x_1(t)\|_1\} \equiv 0$ for $t > t_1, i = 2, \ldots, N$, where t_1 is called the settling time.

For the convenience of study, we denote $\Theta(r_t) = \Theta_k$ when $t \geq 0$ and $r_t = k$. Moreover, in the remaining section of this paper, we need the following matrix notations: $C_k = \mathrm{diag}(c_{1k}, c_{2k}, \ldots, c_{nk})$, $A_k = (a_{ijk})_{n\times n}$, $B_k = (b_{ijk})_{n\times n}$, $D_k = (d_{ijk})_{n\times n}$. The coupled neural network (4) with Filippov solution in the sense of Definition 2 are presented as follows:

$$\begin{aligned}\dot{x}_i(t) &= -C_k x_i(t) + A_k\alpha_i(t) + B_k\alpha_i(t-\tau(t)) + D_k\int_{-\infty}^{t} K(t-s)\alpha_i(s)\mathrm{d}s \\ &\quad + J_k - \sum_{j=1}^{N} l_{ijk}\Phi(x_j(t) - x_i(t)) + R_{ik}(t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{5}$$

where $\alpha_i(t) \in F(x_i(t))$.

According to Definitions 2 and 3 , investigating synchronization of the coupled neural networks with discontinuous activations (4) is equivalent to studying the same problem for eq. (5). Let $e_i(t) = x_i(t) - x_1(t)$, $\beta_i(t) = \alpha_i(t) - \alpha_1(t), i = 1, 2, \ldots, N$. Considering $e_1(t) \equiv 0$, we derive error dynamical system from system (5) as follows:

$$\begin{aligned}\dot{e}_i(t) &= -C_k e_i(t) + A_k\beta_i(t) + B_k\beta_i(t-\tau(t)) + D_k\int_{-\infty}^{t} K(t-s)\beta_i(s)\mathrm{d}s \\ &\quad - \sum_{j=2}^{N} \tilde{l}_{ijk}\Phi e_j(t) + R_{ik}(t), \quad i = 2, \ldots, N, \end{aligned} \tag{6}$$

where $\tilde{l}_{ijk} = l_{ijk} - l_{1jk}, i, j = 2, 3, \ldots, N$.

The initial condition of (6) is $\varphi_i(t) = \phi_i(t) - \phi_1(t) \in C((-\infty, 0]; \mathbb{R}^n), i = 2, 3, \ldots, N$.

Remark 1 Generally, the Markov chain state of the underlying coupled systems is assumed to be available to the isolate neural network at any time. However, this assumption may sometimes be impossible to be satisfied for systems without time stamp information. Thus, the ideal requirement inevitably limits the applications of the obtained results. Therefore, we do not synchronize the coupled neural network (4) onto an isolate neural network with identical Markov chain generator matrix.

Function $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular [6], if $V(x)$ is:

(i) regular in $\mathbb{R}^n$;
(ii) positive definite, i.e., $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$;
(iii) radially unbounded, i.e., $V(x) \to +\infty$ as $\|x\| \to +\infty$.

Note that a C-regular Lyapunov function $V(x)$ is not necessarily differentiable.

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. The Clarke's generalized gradient of V at $x \in \mathbb{R}^n$ [6] is defined by $\partial V(x) = \overline{co}[\lim \nabla V(x_i) : x_i \to x, x_i \notin \Omega \cup \mathcal{N}]$, where $\Omega \subset \mathbb{R}^n$ is the set of Lebesgue measure zero where ∇V does not exist, and $\mathcal{N} \subset \mathbb{R}^n$ is an arbitrary set with measure zero.

The next lemma will be useful to compute the time derivative along solutions (4) of the Lyapunov function designed in the later sections.

Lemma 2 (Chain rule) [6] *If $V(x) : \mathbb{R}^n \to \mathbb{R}$ is C-regular, and $x(t)$ is absolutely continuous on any compact subinterval of $[0, +\infty)$. Then, $x(t)$ and $V(x(t)) : [0, +\infty) \to \mathbb{R}$ are differentiable for a.a. $t \in [0, +\infty)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \gamma(t)\dot{x}(t), \quad \forall \gamma(t) \in \partial V(x(t)),$$

where $\partial V(x(t))$ is the Clarke generalized gradient of V at $x(t)$.

Lemma 3 [16] *If $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ with $a_{ij} \leq 0$ $(i \neq j)$, then the following statements are equivalent:*

(i) *A is an M-matrix.*
(ii) *A^{-1} exists and all the elements of A^{-1} are nonnegative.*
(iii) *All the eigenvalues of A have positive real parts.*

Lemma 4 [42] *Let $A = (a_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$ with $a_{ij} \leq 0$ $(i \neq j)$, $\sum_{j=1}^{n} a_{ij} = 0$, $i, j = 1, 2, \ldots, n$. If A is irreducible, then, for any $\zeta > 0$, $A + \zeta I_n$ is a non-singular M-matrix.*

3 Finite-Time Synchronization of the Coupled Neural Networks

It is well known that classical results on synchronization or stability of systems with Markov jumping parameters need to introduce some free parameters or matrices to manage the uncertainty of the Markov chain [21–32]. If the free-parameters-based result is on asymptotic synchronization or stability, it cannot cause much trouble in real applications since its achieving time is infinity. But in the case of finite-time synchronization or stability, free parameters will make it difficult to ascertain the settling time for a given system. In order that our results are optimal and easy to ascertain the settling time in practical applications, special analysis techniques are established in this section. On the other hand, the uncertainties induced by Filippov solutions and the delays (including finite-time discrete and infinite-time distributed delays) shall also be dealt with at the same time. Specifically, a set of simple discontinuous controllers are employed to deal with the uncertainties of the Filippov solutions, a special M-matrix technique is developed to overcome the uncertain effects of Markov chain. Moreover, some new analytical methods are established to cope with the uncertain factors of the mixed delays. Through this series of control and analytical methods, several synchronization criteria for the coupled discontinuous neural network (4) are derived. Furthermore, the upper bounds of the synchronization time are estimated when the distributed delays are bounded or there is no delay in (4).

Inspired by [43], we use the following simple controller:

$$R_{ik}(t) = -\xi_i^k e_i(t) - \eta^k \mathrm{sgn}(e_i(t)), \tag{7}$$

where $\mathrm{sgn}(e_i(t)) = (\mathrm{sgn}(e_{i1}(t)), \mathrm{sgn}(e_{i2}(t)), \ldots, \mathrm{sgn}(e_{in}(t)))^{\mathrm{T}}$, $\xi_i^k > 0$ and $\eta^k > 0$ are constants to be determined, $i = 2, 3, \ldots, N$.

Note that controller (7) is discontinuous, and its discontinuous points are a special case of the condition (H_1). By using the same analysis method in [46], it is easy to get that, under the assumptions (H_1)–(H_4), the Filippov solution to system (4) exists on $[0, +\infty)$. In order to avoid unnecessary repetition, we do not prove them here.

Denote $z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}$, $p = (p_1, p_2, \ldots, p_n)^{\mathrm{T}}$, $c_k = \min\{c_{lk}, l = 1, 2, \ldots, n\}$, $\underline{\phi} = \min\{\phi_l, l = 1, 2, \ldots, n\}$, $\overline{\phi} = \max\{\phi_l, l = 1, 2, \ldots, n\}$, $\overline{\|Bz\|} = \max\{\||B_k|z\|_\infty, k \in \varpi\}$, $\overline{\|Dz\|} = \max\{\||D_k|z\|_\infty, k \in \varpi\}$, $e(t) = (e_2^{\mathrm{T}}(t), e_3^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t))$. The following Theorem 1 is one of our main results.

Theorem 1 *Assume that the assumptions* (H_1)–(H_4) *are satisfied. If the control gains* ξ_i^k, η^k, $i = 2, 3, \ldots, N$, $k \in \varpi$, *are chosen such that* $\eta^k > \||A_k|p\|_\infty + \||B_k|p\|_\infty + q\||D_k|p\|_\infty$ *and* $\xi_i^k > -c_k + \||A_k|z\|_\infty - \tilde{l}_{iik}\underline{\phi} + \sum_{j=2, j\neq i}^{N} |\tilde{l}_{jik}|\overline{\phi} + q\overline{\|Dz\|} + \frac{\overline{\|Bz\|}}{1-\mu} \triangleq \chi_i^k$, $k = 1, 2, \ldots, w$, $i = 2, 3, \ldots, N$, *then the coupled neural network* (4) *is finite-timely synchronized under controller* (7).

Proof Since $\{e(t), r_t, t \geq 0\}$ is not a Markov process, in order to cast our model into the framework for a Markov system, let us define a new Markov process $\{e_t, r_t, t \geq 0\}$ by $e_t(s) = e(t+s)$, $s \leq 0$. Then, $\{e_t, r_t, t \geq 0\}$ is a Markov process $\{e_t, r_t, t \geq 0\}$ [7].

From $\xi_i^k > \chi_i^k$, $k = 1, 2, \ldots, w$, $i = 2, 3, \ldots, N$, we get that $\zeta = \max\{\chi_i^k - \xi_i^k, k = 1, 2, \ldots, w, i = 2, 3, \ldots, N\} < 0$. On the other hand, since Π is irreducible, $-\Pi$ is also irreducible. From Lemma 4, $-\Pi - \zeta I_w$ is a non-singular M-matrix. According to Lemma 3, $(-\Pi - \zeta I_w)^{-1}$ exists and all the elements of $(-\Pi - \zeta I_w)^{-1}$ are nonnegative. Since $(-\Pi - \zeta I_w)^{-1}$ is also invertible, there exists at least one positive element in each row of $(-\Pi - \zeta I_w)^{-1}$. Let δ be the maximum of the row sums of $(-\Pi - \zeta I_w)^{-1}$. Then, all the elements of $(\rho_1, \rho_2, \ldots, \rho_w)^{\mathrm{T}} = \frac{1}{\delta}(-\Pi - \zeta I_w)^{-1}\mathbf{1}_w$ are positive and $\max\{\rho_k, k \in \varpi\} = 1$ and

$$\sum_{l\in\varpi}\pi_{kl}\rho_l+\rho_k\zeta=-\frac{1}{\delta}<0. \tag{8}$$

When $r_t = k \in \varpi$, consider the following Markovian switching Lyapunov–Krasovskii functional:

$$V(e_t,k,t)=\sum_{l=1}^{3}V_l(e_t,k,t), \tag{9}$$

where

$$\begin{aligned}
V_1(e_t,k,t)&=\rho_k\sum_{i=2}^{N}\|e_i(t)\|_1,\\
V_2(e_t,k,t)&=\frac{\overline{\|Bz\|}}{1-\mu}\sum_{i=2}^{N}\int_{t-\tau(t)}^{t}\|e_i(s)\|_1\mathrm{d}s,\\
V_3(e_t,k,t)&=\overline{\|Dz\|}\sum_{i=2}^{N}\int_{-\infty}^{0}\int_{t+s}^{t}K(-s)\|e_i(u)\|_1\mathrm{d}u\mathrm{d}s.
\end{aligned}$$

Let $\mathcal{L}$ be the weak infinitesimal generator of the random process (e_t,k,t), then based on Lemma 2, differentiating $V_1(e_t,k,t)$ along the solutions of (6) and considering controller (7) produce that

$$\begin{aligned}
\mathcal{L}V_1(e_t,k,t)=\rho_k\sum_{i=2}^{N}\mathbf{1}_n^{\mathrm{T}}\mathrm{diag}(\mathrm{sgn}(e_i(t)))\Big[&-C_ke_i(t)+A_k\beta_i(t)+B_k\beta_i(t-\tau(t))\\
&-\sum_{j=2}^{N}\tilde{l}_{ijk}\Phi e_j(t)+D_k\int_{-\infty}^{t}K(t-s)\beta_i(s)\mathrm{d}s\\
&-\xi_i^ke_i(t)-\eta^k\mathrm{sgn}(e_i(t))\Big]+\sum_{l\in\varpi}\pi_{kl}\rho_l\sum_{i=2}^{N}\|e_i(t)\|_1.
\end{aligned} \tag{10}$$

It is obvious that, if $e_{il}(t)=0$, then $\mathrm{sgn}(e_{il}(t))\sum_{j=1}^{n}a_{ljk}\beta_{ij}(t)=0$, otherwise,

$$\mathrm{sgn}(e_{il}(t))\sum_{j=1}^{n}a_{ljk}\beta_{ij}(t)\le\sum_{j=1}^{n}|a_{ljk}||\beta_{ij}(t)|\le\sum_{j=1}^{n}z_j|a_{ljk}||e_{ij}(t)|+\sum_{j=1}^{n}|a_{ljk}|p_j,$$

$l=1,2,\ldots,n$, $i=2,3,\ldots,N$. In any case, we have

$$\mathrm{sgn}(e_{il}(t))\sum_{j=1}^{n}a_{ljk}\beta_{ij}(t)\le\sum_{j=1}^{n}z_j|a_{ljk}||e_{ij}(t)|+\sum_{j=1}^{n}|a_{ljk}|p_j\lambda_{il},$$

where $\lambda_{il} = 1$ if $e_{il}(t) = 0$, otherwise $\lambda_{il} = 0$. Therefore,

$$\begin{aligned}\mathbf{1}_n^{\mathrm{T}}\mathrm{diag}\{\mathrm{sgn}(e_i(t)\}A_k\beta_i(t) &\leq \sum_{l=1}^{n}\sum_{j=1}^{n} z_j|a_{ljk}||e_{ij}(t)| + \sum_{l=1}^{n}\sum_{j=1}^{n}|a_{ljk}|p_j\lambda_{il} \\ &\leq \||A_k|z\|_\infty\|e_i(t)\|_1 + \||A_k|p\|_\infty\sum_{l=1}^{n}\lambda_{il}. \end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned}&\mathbf{1}_n^{\mathrm{T}}\mathrm{diag}\{\mathrm{sgn}(e_i(t)\}B_k\beta_i(t-\tau(t)) \\ &\leq \||B_k|z\|_\infty\|e_i(t-\tau(t))\|_1 + \||B_k|p\|_\infty\sum_{l=1}^{n}\lambda_{il}, \end{aligned} \tag{12}$$

and

$$\begin{aligned}&\mathbf{1}_n^{\mathrm{T}}\mathrm{diag}\{\mathrm{sgn}(e_i(t)\}D_k\int_{-\infty}^{t}K(t-s)\beta_i(s)\mathrm{d}s \\ &\leq \||D_k|z\|_\infty\int_{-\infty}^{t}K(t-s)\|e_i(s)\|_1\mathrm{d}s + q\||D_k|p\|_\infty\sum_{l=1}^{n}\lambda_{il}, \end{aligned} \tag{13}$$

and

$$\mathbf{1}_n^{\mathrm{T}}\mathrm{diag}\{\mathrm{sgn}(e_i(t)\}\eta^k\mathrm{sgn}(e_i(t)) = \eta^k\sum_{l=1}^{n}\lambda_{il}. \tag{14}$$

Substituting (11)–(14) into (10) yields

$$\begin{aligned}\mathcal{L}V_1(e_t,k,t) \leq \sum_{i=2}^{N}\Bigg[\Bigg(&\sum_{l\in\varpi}\pi_{kl}\rho_l + \rho_k\Bigg(-c_k - \xi_i^k + \||A_k|z\|_\infty - \tilde{l}_{iik}\underline{\phi} \\ &+ \sum_{j=2,j\neq i}^{N}|\tilde{l}_{jik}|\overline{\phi}\Bigg)\Bigg)\|e_i(t)\|_1 + \rho_k\||B_k|z\|_\infty\|e_i(t-\tau(t))\|_1 \\ &+\rho_k\||D_k|z\|_\infty\int_{-\infty}^{t}K(t-s)\|e_i(s)\|_1\mathrm{d}s - \rho_k\Bigg[\eta^k \\ &-\Bigg(\||A_k|p\|_\infty + \||B_k|p\|_\infty + q\||D_k|p\|_\infty\Bigg)\Bigg]\sum_{l=1}^{n}\lambda_{il}\Bigg] \\ \leq \sum_{i=2}^{N}\Bigg[\Bigg(&\sum_{l\in\varpi}\pi_{kl}\rho_l + \rho_k(-c_k - \xi_i^k + \||A_k|z\|_\infty - \tilde{l}_{iik}\underline{\phi}\end{aligned}$$

$$+\sum_{j=2, j\neq i}^{N} |\tilde{l}_{jik}|\overline{\phi})\Bigg)\|e_i(t)\|_1 + \overline{\|Bz\|}\|e_i(t-\tau(t))\|_1$$

$$+\overline{\|Dz\|}\int_{-\infty}^{t} K(t-s)\|e_i(s)\|_1 \mathrm{d}s - \underline{\rho}\varepsilon\sum_{l=1}^{n}\lambda_{il}\Bigg], \tag{15}$$

where $\max\{\rho_k, k \in \varpi\} = 1$ has been used, $\underline{\rho} = \min\{\rho_k, k \in \varpi\}$, $\varepsilon = \min\{\eta^k - (\|A_k\|_\infty + \||B_k|p\|_\infty + q\||D_k|p\|_\infty), k \in \varpi\} > 0$.

Differentiating $V_2(e_t, k, t)$ and $V_3(e_t, k, t)$ derives that

$$\mathcal{L}V_2(e_t, k, t) \leq \frac{\overline{\|Bz\|}}{1-\mu}\sum_{i=2}^{N}[\|e_i(t)\|_1 - (1-\mu)\|e_i(t-\tau(t))\|_1], \tag{16}$$

and

$$\begin{aligned}\mathcal{L}V_3(e_t, k, t) &= \overline{\|Dz\|}\sum_{i=2}^{N}\int_{-\infty}^{0} K(-s)\|e_i(t)\|_1 \mathrm{d}s \\ &\quad -\overline{\|Dz\|}\sum_{i=2}^{N}\int_{-\infty}^{0} K(-s)\|e_i(t+s)\|_1 \mathrm{d}s \\ &= q\overline{\|Dz\|}\sum_{i=2}^{N}\|e_i(t)\|_1 \\ &\quad -\overline{\|Dz\|}\sum_{i=2}^{N}\int_{-\infty}^{t} K(t-s)\|e_i(s)\|_1 \mathrm{d}s. \end{aligned} \tag{17}$$

It is followed from (9) and (15)–(17) that

$$\mathcal{L}V(e_t, k, t) \leq \sum_{i=2}^{N}\Bigg[\Bigg(\sum_{l\in\varpi}\pi_{kl}\rho_l + \rho_k(\chi_i^k - \xi_i^k)\Bigg)\|e_i(t)\|_1 - \underline{\rho}\varepsilon\sum_{l=1}^{n}\lambda_{il}\Bigg]. \tag{18}$$

Inequality (8) means that $\sum_{l\in\varpi}\pi_{kl}\rho_l + \rho_k(\chi_i^k - \xi_i^k) \leq -\frac{1}{\delta} < 0$ for $k = 1, 2, \ldots, w$, $i = 2, 3, \ldots, N$. From this inequality and (18), we get that

$$\mathcal{L}V(e_t, k, t) \leq -\underline{\rho}\varepsilon\sum_{i=2}^{N}\sum_{l=1}^{n}\lambda_{il}. \tag{19}$$

Inequality (19) and the arbitrariness of $k \in \varpi$ imply that

$$\mathcal{L}V(e_t, r_t, t) \leq -\underline{\rho}\varepsilon\sum_{i=2}^{N}\sum_{l=1}^{n}\lambda_{il}.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\{V(e_t, r_t, t)\} \leq -\underline{\rho}\varepsilon \sum_{i=2}^{N}\sum_{l=1}^{n}\lambda_{il}. \tag{20}$$

Because $\mathbf{E}\{V(e_t, r_t, t)\}$ is positive definite and non-increasing, known from (20), there exists nonnegative constant V^* such that

$$\lim_{t\to+\infty}\mathbf{E}\{V(e_t, r_t, t)\} = V^* \text{ and } \mathbf{E}\{V(e_t, r_t, t)\} \geq V^*, \ \forall t \geq 0. \tag{21}$$

On the other hand, integrating both sides of inequality (20) from 0 to t gets the following inequality:

$$\mathbf{E}\{V(e_t, r_t, t)\} - V(e_0, r_0, 0) \leq -\underline{\rho}\varepsilon \sum_{i=2}^{N}\sum_{l=1}^{n}\lambda_{il}t. \tag{22}$$

Now we prove that there exists $t_1 \in (0, +\infty)$ such that

$$\lim_{t\to t_1}\mathbf{E}\{\|e(t)\|_1\} = 0. \tag{23}$$

On the contrary, suppose that $\mathbf{E}\{\|e(t)\|_1\} > 0$ for all $t > 0$. Then, at any instant $t \in [0, +\infty)$, there exists at least one pair (i, l), $i \in \{1, 2, \ldots, N\}$, $l \in \{1, 2, \ldots, n\}$ such that $\mathbf{E}\{|e_{il}(t)|\} > 0$, and so $\sum_{i=2}^{N}\sum_{l=1}^{n}\lambda_{il} \geq 1$, which further leads to $\mathbf{E}\{V(e_t, r_t, t)\} - V(e_0, r_0, 0) \leq -\underline{\rho}\varepsilon t$ for all $t \in [0, +\infty)$, known from (22). Therefore, $\lim_{t\to+\infty}\mathbf{E}\{V(e_t, r_t, t)\} = -\infty$, which contradicts (21). So there exists $t_1 \in (0, +\infty)$ such that the conditions in (23) hold true.

Next, we prove that $\mathbf{E}\{\|e(t)\|_1\} \equiv 0$ for $\forall t \geq t_1$. On the contrary, there exists $t_2 > t_1$ such that $\mathbf{E}\{\|e(t_2)\|_1\} > 0$. Let $t_s = \sup\{t \in [t_1, t_2] : \mathbf{E}\{\|e(t)\|_1\} = 0\}$. We have $t_s < t_2$, $\mathbf{E}\{\|e(t_s)\|_1\} = 0$ and $\mathbf{E}\{\|e(t)\|_1\} > 0$ for all $t \in (t_s, t_2]$. By $\mathbf{E}\{\|e(t)\|_1\} > 0$ for all $t \in (t_s, t_2]$, there exists at least one pair (i_0, l_0), $i_0 \in \{1, 2, \ldots, N\}$, $l_0 \in \{1, 2, \ldots, n\}$ such that $\mathbf{E}\{|e_{i_0l_0}(t)|\} > 0$ at any instant $t \in (t_s, t_3]$, where $t_s < t_3 \leq t_2$. By the same argument as above, we get that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}\{V(e_t, r_t, t)\} \leq -\underline{\rho}\varepsilon t < 0$ in the time interval $(t_s, t_3]$. Hence, $0 < \mathbf{E}\{V(e_{t_3}, r_{t_3}, t_3)\} = V(e_s, r_s, s) + \int_{t_s}^{t_3}\mathrm{d}\mathbf{E}\{V(e_\mu, r_\mu, \mu)\} < -\underline{\rho}\varepsilon(t_3 - t_s) < 0$, which is a contradiction.

To sum up, there exists a constant $t_1 > 0$ such that $\lim_{t\to t_1}\mathbf{E}\{\|e(t)\|_1\} = 0$ and $\mathbf{E}\{\|e(t)\|_1\} \equiv 0$ for $t > t_1$, which implies that $\lim_{t\to t_1}\mathbf{E}\{\|e_i(t)\|_1\} = 0$ and $\mathbf{E}\{\|e_i(t)\|_1\} \equiv 0$ for $t > t_1$, $i = 2, \ldots, N$. According to Definition 4, the coupled neural network (4) is synchronized in a finite-time under controller (7). The proof is completed. □

Remark 2 From the above deduction, we can see that there exists a $t_2 > t_1$ such that $\lim_{t\to t_2}\mathbf{E}\{V_2(e_t, r_t, t)\} = 0$, and $\lim_{t\to t_2}\mathbf{E}\{V_3(e_t, r_t, t)\} = V^*$. Moreover, the constant V^* is zero only in the case that all the initial conditions of (6) are zero, i.e., $\varphi_i(t) = 0$ for $t \leq 0$. Actually, V^* is positive as long as there exists one $i_0 \in \{2, 3, \ldots, N\}$ such that $\|\varphi_{i_0}(t)\|_1 \neq 0$ for all $t \leq 0$ due to the infinite interval of the double integral. Note that the exact value of V^* is related to the initial value of error system (6) and the error of the measurable selection $\alpha_i(t)$, $i = 1, 2, \ldots, N$. Since the measurable selections $\alpha_i(t)$, $i = 1, 2, \ldots, N$ are uncertain, it is difficult to get the exact value of V^*, and hence, it is not easy to estimate the settling time t_1 though it exists. However, it can be seen from (20) that increasing the values of $\underline{\rho}$ and ε can accelerate the synchronization. Therefore, larger values of ξ_i^k and η^k $(k = 1, 2, \ldots, w, i = 2, 3, \ldots, N)$ can decrease the settling time.

Remark 3 Since controller (7) is very simple and has no time delay, it is easy to be implemented in practice if the number of coupled nodes is not large. Numerical simulations in Sect. 5 demonstrate that the designed controllers are very effective. However, $N - 1$ controllers are needed for a network with N nodes. When N is very large, it is difficult to control all nodes in practice though computing the inequalities in Theorem 1 is not difficult.

When the delay kernel satisfies the following condition:

$$K(t) = \begin{cases} 0, & t > \theta, \\ 1, & 0 \leq t \leq \theta, \end{cases} \tag{24}$$

where $\theta > 0$ is a constant, then the coupled neural network (4) becomes the following system with finite-time discrete and distributed delays:

$$\begin{aligned} \dot{x}_i(t) = &-C(r_t)x_i(t) + A(r_t)f(x_i(t)) + B(r_t)f(x_i(t - \tau(t))) \\ &+D(r_t)\int_{t-\theta}^{t} f(x_i(s))\mathrm{d}s + J(r_t) - \sum_{j=1}^{N} l_{ij}(r_t)\Phi(x_j(t) - x_i(t)) \\ &+R_i(r_t)(t), \quad i = 1, 2, \ldots, N. \end{aligned} \tag{25}$$

Correspondingly, error dynamical system (6) turns out to the following form:

$$\begin{aligned} \dot{e}_i(t) = &-C_k e_i(t) + A_k\beta_i(t) + B_k\beta_i(t - \tau(t)) + D_k\int_{t-\theta}^{t} \beta_i(s)\mathrm{d}s \\ &-\sum_{j=2}^{N} \tilde{l}_{ijk}\Phi e_j(t) + R_{ik}(t), \quad i = 2, \ldots, N, \end{aligned} \tag{26}$$

It is obvious that Theorem 1 is applicable to system (25). The following Theorem 2 shows that not only the network (25) can be finite-timely synchronized, but also the settling time can be explicitly given.

Theorem 2 *Assume that the assumptions* (H_1)–(H_3) *are satisfied. If the control gains* ξ_i^k, η^k, $i = 2, 3, \ldots, N$, $k \in \varpi$, *are chosen such that* $\eta^k > \||A_k|p\|_\infty + \||B_k|p\|_\infty + \theta\||D_k|p\|_\infty$ *and* $\xi_i^k > -c_k + \||A_k|z\|_\infty - \tilde{l}_{iik}\underline{\phi} + \sum_{j=2, j\neq i}^{N} |\tilde{l}_{jik}|\overline{\phi} +$

$\theta\overline{\|Dz\|} + \frac{\overline{\|Bz\|}}{1-\mu} = \hat{\chi}_i^k$, $k = 1, 2, \ldots, w, i = 2, 3, \ldots, N$, *then the coupled neural network* (25) *is finite-timely synchronized under controller* (7)*. Moreover, the settling time satisfies* $t_1 \leq \frac{1}{\hat{\rho}_m \epsilon}\big[\sum_{i=2}^N \|\varphi_i(0)\|_1 + \frac{\overline{\|Bz\|}}{1-\mu}\sum_{i=2}^N \int_{-\tau_0}^0 \|\varphi_i(s)\|_1 \mathrm{d}s + \overline{\|Dz\|}\sum_{i=2}^N \int_{-\theta}^0 \int_s^0 \|\varphi_i(u)\|_1 \mathrm{d}u\mathrm{d}s\big] - \max\{\tau, \theta\}$*, where* $(\hat{\rho}_1, \hat{\rho}_2, \ldots, \hat{\rho}_w)^{\mathrm{T}} = \frac{1}{\hat{\delta}}(-\varPi - \hat{\zeta} I_w)^{-1}\boldsymbol{1}_w$, $\hat{\zeta} = \max\{\hat{\chi}_i^k - \xi_i^k, k = 1, 2, \ldots, w, i = 2, 3, \ldots, N\}$, $\hat{\delta}$ *is the maximum of the sow sums of* $(-\varPi - \hat{\zeta} I_w)^{-1}$, $\hat{\rho}_m = \min\{\hat{\rho}_k, k \in \varpi\}$, $\epsilon = \min\{\eta^k - (\||A_k|p\|_\infty + \||B_k|p\|_\infty + \theta\||D_k|p\|_\infty), k \in \varpi\}$, $\tau_0 = \tau(0)$.

Proof By the similar discussions as that given in the proof of Theorem 1, we get that

$$\sum_{l\in\varpi} \pi_{kl}\rho_l + \rho_k\hat{\zeta} = -\frac{1}{\hat{\delta}} < 0. \tag{27}$$

So,

$$\sum_{l\in\varpi} \pi_{kl}\rho_l + \rho_k(\hat{\chi}_i^k - \xi_i^k) < 0. \tag{28}$$

When $r_t = k \in \varpi$, modify the Lyapunov–Krasovskii functional candidate in (9) as:

$$\hat{V}(e_t, k, t) = \sum_{l=1}^{3} \hat{V}_l(e_t, k, t), \tag{29}$$

where

$$\hat{V}_1(e_t, k, t) = \hat{\rho}_k \sum_{i=2}^{N} \|e_i(t)\|_1,$$
$$\hat{V}_2(e_t, k, t) = \frac{\overline{\|Bz\|}}{1-\mu} \sum_{i=2}^{N} \int_{t-\tau(t)}^{t} \|e_i(s)\|_1 \mathrm{d}s,$$
$$\hat{V}_3(e_t, k, t) = \overline{\|Dz\|} \sum_{i=2}^{N} \int_{-\theta}^{0} \int_{t+s}^{t} K(-s)\|e_i(u)\|_1 \mathrm{d}u\mathrm{d}s.$$

Using the same proof procedure as that given in the proof of Theorem 1 and the conditions in Theorem 2 yields that

$$\mathrm{d}\mathbf{E}\{\hat{V}(e_t, r_t, t) \leq -\hat{\rho}_m\epsilon \sum_{i=2}^{N}\sum_{l=1}^{n} \lambda_{il}\mathrm{d}t. \tag{30}$$

Moreover, there exists $t_1 \in (0, +\infty)$ such that

$$\lim_{t\to t_1} \mathbf{E}\{\|e_i(t)\|_1\} = 0 \text{ and } \mathbf{E}\{\|e_i(t)\|_1\} \equiv 0 \quad \text{for all } t > t_1, i = 2, \ldots, N. \tag{31}$$

It is obvious that $\hat{V}(e_{t_2}, r_{t_2}, t_2) = 0$ and $\hat{V}(e_t, r_t, t) \equiv 0$, $\forall t \geq t_2$, where $t_2 = t_1 + \max\{\tau, \theta\}$.

By the same discussions as those given in the proof of Theorem 1, we get $\sum_{i=1}^{N}\sum_{l=1}^{n}\lambda_{il} \geq 1$ before t_2. Therefore, it is followed from (30) that

$$\mathrm{d}\mathbf{E}\{\hat{V}(e_t, r_t, t)\} \leq -\hat{\rho}_m \epsilon \mathrm{d}t,\ \ t \in [0, t_2). \tag{32}$$

Integrating both sides of inequality (32) from 0 to t_2 obtains that:

$$t_2 \leq \frac{\hat{V}(e_0, r_0, 0)}{\hat{\rho}_m \epsilon}.$$

Hence, $t_1 \leq \frac{\hat{V}(e_0, r_0, 0)}{\hat{\rho}_m \epsilon} - \max\{\tau, \theta\}$. The proof is completed. □

Remark 4 It can be seen from Theorem 2 that larger values of $\hat{\rho}_m$ and ϵ lead to smaller settling time. Hence, increasing the values of ξ_i^k and η^k ($k = 1, 2, \ldots, w, i = 2, 3, \ldots, N$) can decrease the synchronization time in practice. Since $0 < \hat{\rho}_m \leq 1$, the effect of large value of ξ_i^k ($k = 1, 2, \ldots, w, i = 2, 3, \ldots, N$) on the settling time is limit. Therefore, it is better to increase the value of η^k ($k = 1, 2, \ldots, w$) in order to effectively decrease the settling time.

When there is no delay in (1), i.e., $B_k = D_k = 0$ (zero matrix), the following Corollary 1 can be easily obtained from Theorem 2.

Corollary 1 *Assume that the assumptions* (H_1)–(H_2) *are satisfied. If the control gains* $\xi_i^k, \eta^k, i = 2, 3, \ldots, N, k \in \varpi$, *are chosen such that* $\eta^k > \||A_k|p\|_\infty$ *and* $\xi_i^k > -c_k + \||A_k|z\|_\infty - \tilde{l}_{iik}\underline{\phi} + \sum_{j=2, j\neq i}^{N}|\tilde{l}_{jik}|\overline{\phi} = \tilde{\chi}_i^k$, $k = 1, 2, \ldots, w, i = 2, 3, \ldots, N$, *then the coupled neural network* (1) *with* $B_k = D_k = 0$ *is finite-timely synchronized under controller* (7). *Moreover, the settling time is estimated as* $t_1 \leq \frac{1}{\tilde{\rho}_m \underline{\epsilon}}\sum_{i=2}^{N}\|\varphi_i(0)\|_1$, *where* $\underline{\epsilon} = \min\{\eta^k - \||A_k|p\|_\infty, k \in \varpi\}$, $(\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_w)^{\mathrm{T}} = \frac{1}{\tilde{\delta}}(-\Pi - \tilde{\zeta} I_w)^{-1}\boldsymbol{1}_w$, $\tilde{\zeta} = \max\{\tilde{\chi}_i^k - \xi_i^k, k = 1, 2, \ldots, w, i = 2, 3, \ldots, N\}$, $\tilde{\rho}_m = \min\{\tilde{\rho}_k, k \in \varpi\}$, $\tilde{\delta}$ *is the maximum of the sow sums of* $(-\Pi - \tilde{\zeta} I_w)^{-1}$.

Remark 5 The analysis methods developed in this paper are applicable to finite-time synchronization of differential equations with or without delay and are completely different from those utilized in [33,37,41,43], and [1], which used the finite-time stability theorems developed in [34] and [11]. Notice that it is crucial to select the variable λ_{il} in the proofs of Theorems 1 and 2 according to the value of $e_{il}(t)$ which makes our proposed method work for this finite-time problem. It should be emphasized that studying finite-time stability and synchronization of general nonlinear time-delay systems is extremely difficult in the literature. For instance, the authors of [24] tried to study the finite-time stability and stabilization of retarded-type functional differential equations by using Lyapunov functionals. Unfortunately, the theoretical result in [24] cannot be applied in practice for studying the finite-time stabilization problem because it is extremely difficult to find a Lyapunov functional satisfying the assumptions in [24]. According to [9], the finite-time stability theorem in [34] and [11] is not applicable to time-delay systems because its framework is based on the inequality $\dot{V}(x) \leq -\alpha V^{\eta}(x)$, where $\alpha > 0$ and $0 < \eta < 1$ are constants, $\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|)$ with class-$\mathcal{K}$ functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$. Note that our analytical technique is based on 1-norm; hence, our Lyapunov functionals are not the positive definite quadratic form as those in [33,37,41,43], and [1]. It is proved theoretically that the 1-norm-based analytical technique is effective for finite-time synchronization of delayed systems though the conditions and t_1 in Theorem 2 seem to be complex.

Remark 6 Based on the new M-matrix approach, the settling time does not involve any free parameters, which means that, for a given system with known initial value and generator matrix of the Markov chain, the settling time is fixed when the control gains have been suitably chosen. Different from existing results on finite-time synchronization, the settling time in Theorem 2 is determined not only by the values of the error state at $t = 0$, but also by the history of the error state, time delays, and the generator matrix of the Markov chain.

The above synchronization criteria show that the control gains η^k have the two roles: (a) to tune the synchronization time; (b) to overcome the uncertainties of the Filippov solutions. Specially, when the activation functions f are continuous, the η^k can be any positive constant. On the other hand, it can be seen from the proofs above that the role of ξ_i^k is to keep the error system stable. Due to the uncertainty of Markov chain, the above synchronization criteria are some what conservative, known from the definition of $\overline{\|Bz\|}$, $\overline{\|Dz\|}$, ε, ϵ, and $\underline{\epsilon}$, *etc*. This problem is inevitable because Markov chain is really difficult to be dealt with in determining control gains and synchronization time with the designed Lyapunov–Krasovskii functionals. For the time being, we do not find a suitable Lyapunov–Krasovskii functionals in quadric form to get a better result. However, if there is only one mode in (4), the conservativeness of the corresponding synchronization condition can be reduced. Consider the following coupled neural networks without Markov jumping parameters:

$$\begin{aligned} \dot{x}_i(t) &= -Cx_i(t) + Af(x_i(t)) + Bf(x_i(t-\tau(t))) + D\int_{-\infty}^{t} K(t-s)f(x_i(s))\mathrm{d}s \\ &\quad + J - \sum_{j=1}^{N} l_{ij}\Phi(x_j(t) - x_i(t)) + R_i(t),\ i = 1, 2, \ldots, N, \end{aligned} \tag{33}$$

with the controller:

$$R_i(t) = -\xi_i e_i(t) - \eta \mathrm{sgn}(e_i(t)), \tag{34}$$

where $\xi_i > 0$ and $\eta > 0$ are constants to be determined, $\mathrm{sgn}(e_i(t))$ is the same as that above, $i = 2, 3, \ldots, N$.

The following corollaries show that the conservativeness of the control gains and the settling time is further reduced for (4) without Markovian jumping parameters. They can be easily obtained from Theorems 1, 2 and Corollary 1, respectively, and hence, their proof are omitted here. Let $\tilde{l}_{ij} = l_{ij} - l_{1j}$, $i, j = 2, 3, \ldots, N$, $c = \min\{c_l, l = 1, 2, \ldots, n\}$.

Corollary 2 *Assume that the assumptions* (H_1)–(H_4) *are satisfied. If the control gains* $\xi_i, \eta, i = 2, 3, \ldots, N$, *are chosen such that* $\eta > \||A|p\|_\infty + \||B|p\|_\infty + q\||D|p\|_\infty$ *and* $\xi_i = -c + \||A|z\|_\infty - \tilde{l}_{ii}\underline{\phi} + \sum_{j=2, j\neq i}^{N} |\tilde{l}_{ji}|\overline{\phi} + q\||D|z\|_\infty + \frac{\||B|z\|_\infty}{1-\mu}$, *then the coupled neural network* (33) *is finite-timely synchronized under controller* (34).

Corollary 3 *Assume that the assumptions* (H_1)–(H_3) *and* (24) *are satisfied. If the control gains* $\xi_i, \eta, i = 2, 3, \ldots, N$, *are chosen such that* $\eta > \||A|p\|_\infty + \||B|p\|_\infty + \theta\||D|p\|_\infty$ *and* $\xi_i = -c + \||A|z\|_\infty - \tilde{l}_{ii}\underline{\phi} + \sum_{j=2, j\neq i}^{N} |\tilde{l}_{ji}|\overline{\phi} + \theta\||D|z\|_\infty + \frac{\||B|z\|_\infty}{1-\mu}$, *then the coupled neural network* (33) *is finite-timely synchronized under controller* (34). *Moreover, the settling time is* $t_1 \leq \frac{1}{\epsilon}\big[\sum_{i=2}^{N} \|\varphi_i(0)\|_1 + \frac{\||B|z\|_\infty}{1-\mu}\sum_{i=2}^{N}\int_{-\tau(0)}^{0} \|\varphi_i(s)\|_1 \mathrm{d}s + \||D|z\|_\infty \sum_{i=2}^{N}\int_{-\theta}^{0}\int_{s}^{0} \|\varphi_i(u)\|_1 \mathrm{d}u\mathrm{d}s\big] - \max\{\tau, \theta\}$, *where* $\epsilon = \min\{\eta - \||A|p\|_\infty + \||B|p\|_\infty + \theta\||D|p\|_\infty\}$.

Corollary 4 *Assume that the assumptions* (H_1)–(H_2) *are satisfied. If the control gains* $\xi_i, \eta, i = 2, 3, \ldots, N$, *are chosen such that* $\eta > \||A|p\|_\infty$ *and* $\xi_i = -c + \||A|z\|_\infty - \tilde{l}_{ii}\underline{\phi} + \sum_{j=2, j\neq i}^{N} |\tilde{l}_{ji}|\overline{\phi}$, $i = 2, 3, \ldots, N$, *then the coupled neural network* (33) *with* $B_k = D_k = 0$ *is finite-timely synchronized under controller* (34). *Moreover, the settling time is* $t_1 \leq \frac{1}{\underline{\epsilon}}\sum_{i=2}^{N} \|\varphi_i(0)\|_1$, *where* $\underline{\epsilon} = \min\{\eta - \||A|p\|_\infty, k \in \varpi\}$.

4 Finite-Time Synchronization with Perturbations

In real-world applications, the coupled neural network (4) might represent a nominal model that is valid only under ideal conditions, while a more accurate description of the system might be provided by a perturbed model. In this section, we investigate the finite-time synchronization of the coupled neural network (4), which is presented as follows:

$$\begin{aligned}\dot{x}_i(t) = & -C(r_t)x_i(t) + A(r_t)f(x_i(t)) + B(r_t)f(x_i(t-\tau(t))) \\ & + D(r_t)\int_{-\infty}^{t} K(t-s)f(x_i(s))\mathrm{d}s \\ & + J(r_t) - \sum_{j=1}^{N} l_{ij}(r_t)\Phi(x_j(t) - x_i(t)) \\ & + g_i(x_i(t), t, r_t) + \tilde{R}_i(r_t, t), \quad i = 1, 2, \ldots, N, \end{aligned} \tag{35}$$

where the perturbation term $g_i(x_i(t), t, r_t) = (g_{i1}(x_i(t), t, r_t), g_{i2}(x_i(t), t, r_t), \ldots, g_{in}(x_i(t), t, r_t))^{\mathrm{T}}$, $i = 1, 2, \ldots, N$ result from disturbances, uncertainties, parameters variations, or modeling errors, $\tilde{R}_i(r_t, t)$ is the controller to be designed, which is similar to controller (7).

For simplicity, we consider only continuous norm-bounded perturbations so that the Filippov solution of each node system of the perturbed network (35) is guaranteed. Precisely, the following assumption (H_5) is used.

(H$_5$) $g_i(x_i(t), t, r_t), i = 1, 2, \ldots, N$, are continuous, and there are constants $h_i(r_t) \geq 0$ such that $\|g_i(x_i(t), t, r_t)\|_\infty \leq h_i(r_t), i = 1, 2, \ldots, N$.

Remark 7 The $g_i(x_i(t), t, r_t), i = 1, 2, \ldots, N$, are not required to be differentiable or satisfy Lipchitz condition. Moreover, each node may subject to different perturbation. Since the states of chaotic systems are bounded, it is reasonable to assume that the perturbations are bounded. Hence, the condition (H$_5$) is general.

When $r_t = k$, it can be obtained from (35) that

$$\begin{aligned}\dot{e}_i(t) = &-C_k e_i(t) + A_k\beta_i(t) + B_k\beta_i(t-\tau(t)) + D_k\int_{-\infty}^{t} K(t-s)\beta_i(s)\mathrm{d}s\\ &-\sum_{j=2}^{N}\tilde{l}_{ijk}\Phi e_j(t) + g_{ik}(x_i(t), t) - g_{1k}(x_1(t), t) + \tilde{R}_{ik}(t),\\ &i = 2, \ldots, N,\end{aligned} \tag{36}$$

where $g_{ik}(x_i(t), t) = (g_{i1,k}(x_i(t), t), g_{i2,k}(x_i(t), t), \ldots, g_{in,k}(x_i(t), t))^{\mathrm{T}}$, $i = 1, 2, \ldots, N$.

The controller $\tilde{R}_{ik}(t)$ is designed as follows, which can be derived by a small modification of controller (7).

$$\tilde{R}_{ik}(t) = -\xi_i^k e_i(t) - \eta_i^k \mathrm{sgn}(e_i(t)), \tag{37}$$

where $\mathrm{sgn}(e_i(t)) = (\mathrm{sgn}(e_{i1}(t)), \mathrm{sgn}(e_{i2}(t)), \ldots, \mathrm{sgn}(e_{in}(t)))^{\mathrm{T}}$, $\xi_i^k > 0$ and $\eta_i^k > 0$ are constants to be determined, $i = 2, 3, \ldots, N$.

Theorem 3 *Assume that the assumptions* (H$_1$)–(H$_5$) *are satisfied. If the control gains* $\xi_i^k, \eta_i^k, i = 2, 3, \ldots, N, k \in \varpi$, *are chosen such that* $\eta_i^k > \||A_k|p\|_\infty + \||B_k|p\|_\infty + q\||D_k|p\|_\infty + h_{ik} + h_{1k}$ *and* $\xi_i^k > \chi_i^k$, $k = 1, 2, \ldots, w, i = 2, 3, \ldots, N$, *then the coupled neural network* (35) *is finite-timely synchronized under controller* (37)*, where* χ_i^k *are the same as those in Theorem* 1.

Proof Denote $\tilde{g}_{ik}(t) = g_{ik}(x_i(t), t) - g_{1k}(x_1(t), t)$. It follows from (H$_5$) that

$$\begin{aligned}\mathbf{1}_n^{\mathrm{T}}\mathrm{diag}\{\mathrm{sgn}(e_i(t))\}\tilde{g}_{ik}(t) &= \sum_{l=1}^{n}\mathrm{sgn}(e_{il}(t))\tilde{g}_{il,k}(t) \leq \sum_{l=1}^{n}|\tilde{g}_{il,k}(t)|\lambda_{il}\\ &\leq \|\tilde{g}_{ik}(t)\|_\infty\sum_{l=1}^{n}\lambda_{il} \leq (h_{ik} + h_{1k})\sum_{l=1}^{n}\lambda_{il},\end{aligned} \tag{38}$$

where λ_{il} is defined in the proof of Theorem 1.

The other part of the proof is same as that given in the proof of Theorem 1. The proof is completed. □

We remark that the corresponding results on finite-time synchronization can also be easily obtained for Theorem 2, Corollaries 1–4 when the perturbations satisfying

(H_5) are considered. The only change is to raise the control gains η^k by $h_{ik} + h_{1k}$. This means that controller (7) possesses good perturbation rejection and robustness properties.

Remark 8 Controllers (7) and (37) are discontinuous, and the phenomenon of chattering will appear [8]. In order to eliminate the chattering, controllers (7) and (37) can be modified as

$$R_{ik}(t) = -\xi_i^k e_i(t) - \eta^k \frac{e_i(t)}{\|e_i(t)\|_1 + \nu}, \tag{39}$$

and

$$\tilde{R}_{ik}(t) = -\xi_i^k e_i(t) - \eta_i^k \frac{e_i(t)}{\|e_i(t)\|_1 + \nu}, \tag{40}$$

respectively, where ν is a sufficiently small positive constant.

5 Numerical Examples

In this section, three numerical examples are given to verify the above theoretical analysis. Specifically, Example 1 is to verify the Theorem 1, Example 2 is to verify Theorem 3, and Example 3 aims to illustrate the Theorem 2. Moreover, the accuracy of the estimation of settling time is also discussed. The synchronization errors are $\|e_i(t)\|_1 = |x_{i1}(t) - x_{11}(t)| + |x_{i2}(t) - x_{12}(t)|, i = 2, 3, 4, 5$.

Example 1 Consider the following two modes:

$$\begin{aligned} \dot{x}(t) = & -C_k x(t) + A_k f(x(t)) + B_k f(x(t-\tau(t))) + D_k \int_{-\infty}^{t} K(t-s) f(x(s))\mathrm{d}s \\ & + J_k + u_k(x(t), t), \quad k = 1, 2, \end{aligned} \tag{41}$$

where $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$, $u_k(x(t), t)$ is external perturbation, $\tau(t) = 1$, $K(t) = e^{-0.5t}$, $J_1 = J_2 = (1, 1.2)^{\mathrm{T}}$,

$$C_1 = \begin{pmatrix} 1.2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 3 & -0.3 \\ 4 & 4.5 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1.4 & 0.1 \\ 0.3 & -8 \end{pmatrix}, \quad D_1 = \begin{pmatrix} -1.2 & 0.1 \\ -2.8 & -1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1.2 & 0 \\ 0 & 1.1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -0.3 \\ 4 & 5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1 & 0.12 \\ 0.35 & -9.5 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -1 & 0.09 \\ -2.6 & -1.2 \end{pmatrix},$$

the activation function is $f(x) = (f_1(x_1), f_2(x_2))$ with

$$f_i(x_i) = \begin{cases} \tanh(x_i) + 0.35, & x_i > 0, i = 1, 2, \\ \tanh(x_i) - 0.35, & x_i < 0, i = 1, 2. \end{cases}$$

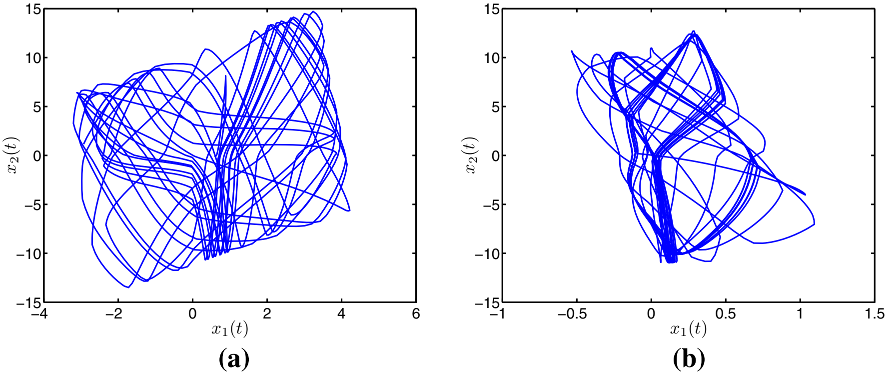

Fig. 1 Chaotic-like trajectories of models 1 (**a**) and 2 (**b**) in system (41) without perturbation

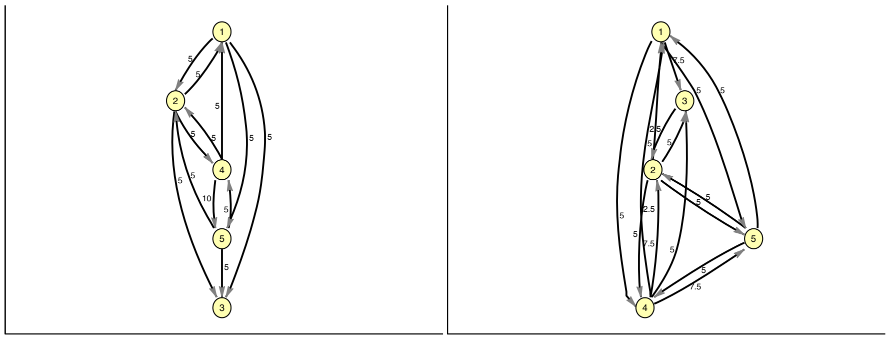

Fig. 2 Network topologies $\mathcal{G}_1$ (*left*) and $\mathcal{G}_2$ (*right*) in the example

Figure 1 shows the chaotic-like trajectories of models 1 and 2 in system (41) without perturbation, where the initial condition of model 1 is $x(t) = (0.1, 0.2)^{\mathrm{T}}, \forall t \in [-1, 0]$, and $x(t) = 0$ for $t < -1$, the initial condition of model 2 is $x(t) = (-0.1, 0.2)^{\mathrm{T}}$, $\forall t \in [-1, 0]$, and $x(t) = 0$ for $t < -1$.

It is easy to check that the discontinuous activation function f satisfies (H_1). Moreover, by simple computation, we get $z_1 = z_2 = 1$, $p_1 = p_2 = 0.7$, $\mu = 0$, $\tau = 1$, $q = 2$. So (H_1)–(H_4) are all satisfied.

When there is no perturbation, consider a coupled neural networks consisting of five identical models (41) with Markovian parameters, where the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ of the coupled network are shown in Fig. 2.

By simple computation, we get that $\|A_1 p\|_\infty + \|B_1 p\|_\infty + q\|D_1 p\|_\infty = 17.08$, $\|A_2 p\|_\infty + \|B_2 p\|_\infty + q\|D_2 p\|_\infty = 18.515$, $\chi_2^1 = 21.95$, $\chi_3^1 = 21.95$, $\chi_4^1 = 23.95$, $\chi_5^1 = 24.95$, $\chi_2^2 = 21.35$, $\chi_3^2 = 22.35$, $\chi_4^2 = 24.35$, $\chi_5^2 = 21.85$. By Theorem 1, if we take $\eta^1 = 18$, $\eta^2 = 19$, $\xi_2^1 = \xi_3^1 = 22$, $\xi_4^1 = 24$, $\xi_5^1 = 25$, $\xi_2^2 = 21.5$, $\xi_3^2 = 22.5$, $\xi_4^2 = 24.5$, $\xi_5^2 = 22$, then, with any irreducible Π, the coupled neural networks with the Markovian jumping parameters in (41) and the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ can be finite-timely synchronized by controller (7).

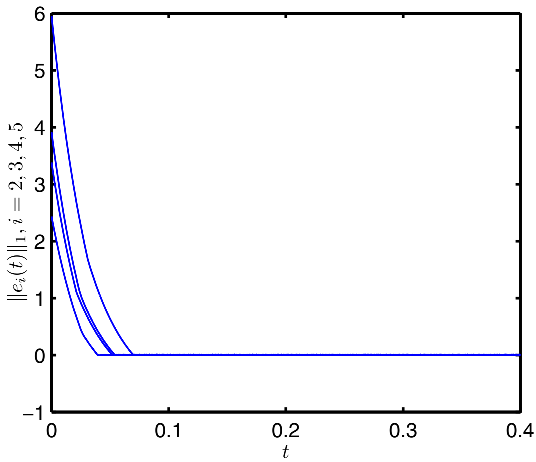

Fig. 3 Time response of synchronization errors under controller (39)

According to Remark 8, we use controller (39) with $\nu = 0.001$ instead of controller (7) in the simulations. The initial values of the coupled network are chosen randomly in the real number interval $[-3, 3]$ for $t \in [-1, 0]$, and all the states of the coupled neural networks are zero for $t < -1$, the time step size is $\delta = 0.0002$, $\Pi = \begin{pmatrix} -3000 & 3000 \\ 4000 & -4000 \end{pmatrix}$, $r_0 = 1$. Figure 3 describes the time evolutions of the synchronization errors, from which one can see that the synchronization is realized before 0.1, which matches Theorem 1.

Example 2 Now consider the coupled neural network (35). The parameters are the same as those in Example 1 except the perturbations. When $u_1(x(t), t) = (0.5|\cos(t)|, -0.3\sin(t))^{\mathrm{T}} = w_1(x(t), t)$, $u_1(x(t), t) = (-0.5\cos(t), 0)^{\mathrm{T}} = w_2(x(t), t)$, $u_1(x(t), t) = (0.65\sin(x_2(t)), -0.2|\sin(x_1(t))|)^{\mathrm{T}} = w_3(x(t), t)$, the trajectories of the mode 1 with the same initial condition as that in Fig. 1a are shown in Fig. 4, respectively.

When $u_2(x(t), t) = (0.5|\sin(x_1(t))|, -0.3\sin(t))^{\mathrm{T}} = \bar{w}_1(x(t), t)$, $u_2(x(t), t) = (0.5|\cos(x_1(t))|, -0.3\cos(t))^{\mathrm{T}} = \bar{w}_2(x(t), t)$, $u_2(x(t), t) = (0.13\sin(x_1(t)), -0.4|\cos(x_2(t))|)^{\mathrm{T}} = \bar{w}_3(x(t), t)$, the trajectories of the mode 2 with the same initial condition as that in Fig. 1b are shown in Fig. 5, respectively.

Suppose the perturbations to each node in coupled mode 1 are $g_1(x_1(t), t, 1) = w_1(x_1(t), t)$, $g_2(x_2(t), t, 1) = w_2(x_2(t), t)$, $g_3(x_3(t), t, 1) = w_2(x_3(t), t)$, $g_4(x_4(t), t, 1) = w_3(x_3(t), t)$, $g_5(x_5(t), t, 1) = w_3(x_4(t), t)$, the perturbations to each node in coupled mode 2 are $g_1(x_1(t), t, 2) = \bar{w}_1(x_1(t), t)$, $g_2(x_2(t), t, 2) = \bar{w}_2(x_3(t), t)$, $g_3(x_3(t), t, 2) = \bar{w}_2(x_3(t), t)$, $g_4(x_4(t), t, 2) = \bar{w}_3(x_4(t), t)$, $g_5(x_5(t), t, 2) = \bar{w}_3(x_5(t), t)$. It follows that $h_{11} = h_{21} = h_{31} = h_{12} = h_{22} = h_{32} = 0.5$, $h_{41} = h_{51} = 0.65$, $h_{42} = h_{52} = 0.4$.

By Theorem 3, the coupled neural networks with Markov jumping parameters and nonidentical perturbations can be finite-timely synchronized under controller (37) if the control gains satisfy the conditions: $\eta_i^1 > 18.08$ for $i = 2, 3$, $\eta_i^1 > 18.23$ for

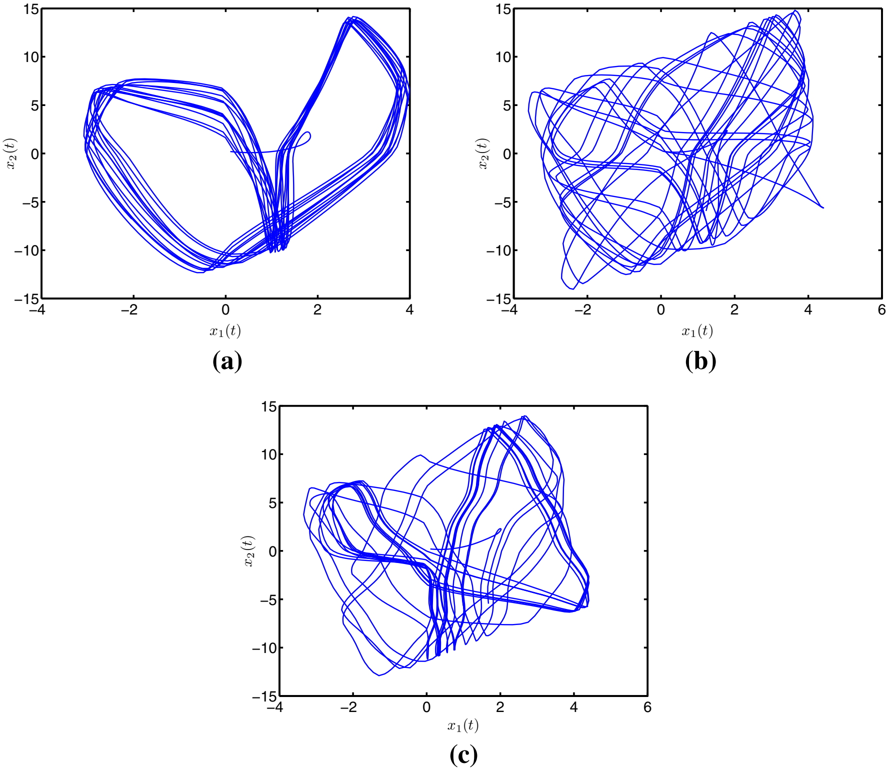

Fig. 4 Trajectories of mode 1 in (41) with the same initial condition as that in Fig. 1a under different perturbations: **a** $u_1(x(t), t) = w_1(x(t), t)$; **b** $u_1(x(t), t) = w_2(x(t), t)$; **c** $u_1(x(t), t) = w_3(x(t), t)$

$i = 4, 5$, $\eta_i^2 > 19.515$ for $i = 2, 3$, $\eta_i^2 > 19.415$ for $i = 4, 5$, $\xi_i^k > \chi_i^k$, $i = 2, 3, 4, 5$, $k = 1, 2$, are the same as those in Example 1.

Take controller (40) with $\nu = 0.001$ instead of controller (37) in the simulations. The initial values of the coupled network are chosen randomly in the real number interval $[-3, 3]$ for $t \in [-1, 0]$ and all the states of the coupled neural networks are zero for $t < -1$, the time step size is $\delta = 0.0002$. Figure 6 describes the time evolutions of the synchronization errors, from which one can see that the synchronization of the Markovian coupled neural networks with nonidentical perturbations is achieved quickly, and the Theorem 3 is verified.

Example 3 Consider the following discontinuous Markovian jumping neural network:

$$\begin{aligned}\dot{x}(t) &= -C_k x(t) + A_k f(x(t)) + B_k f(x(t-\tau(t))) \\ &\quad + D_k \int_{t-\theta}^{t} f(x(s))\mathrm{d}s + J_k, k = 1, 2, \end{aligned} \tag{42}$$

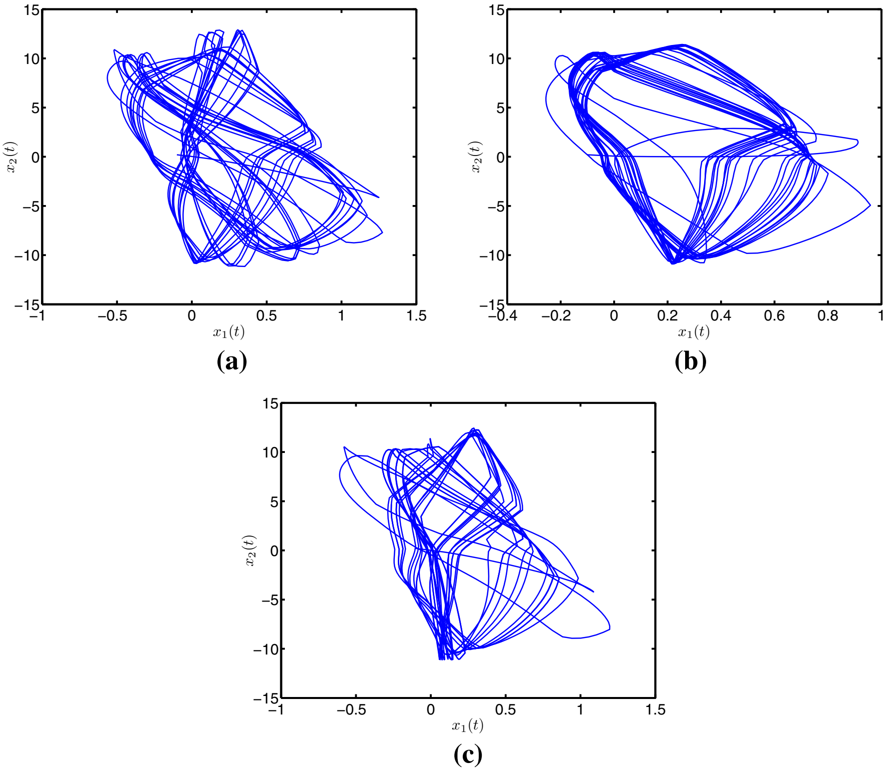

Fig. 5 Trajectories of mode 2 in (41) with the same initial condition as that in Fig. 1b under different perturbations: **a** $u_2(x(t), t) = \bar{w}_1(x(t), t)$; **b** $u_2(x(t), t) = \bar{w}_2(x(t), t)$; **c** $u_2(x(t), t) = \bar{w}_3(x(t), t)$

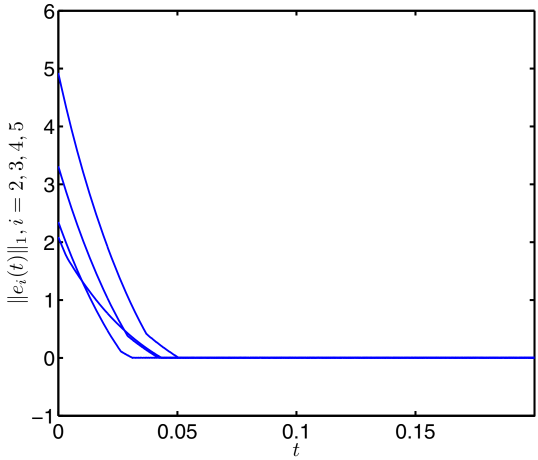

Fig. 6 Time response of synchronization errors under controller (40)

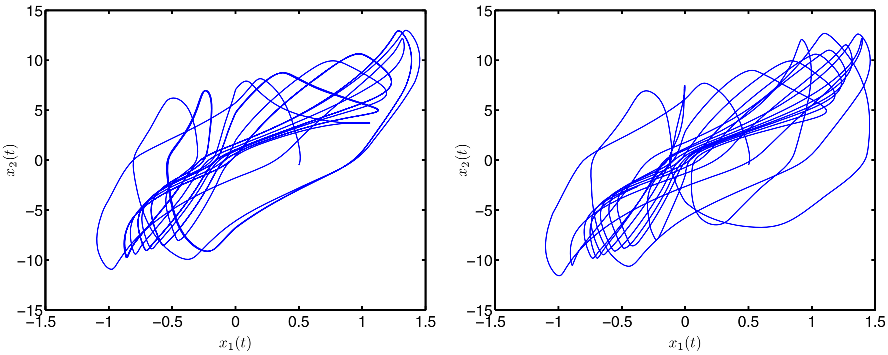

Fig. 7 Chaotic-like trajectories of models 1 (*left*) and 2 (*right*) in system (42)

where $x(t) = (x_1(t), x_2(t))^{\mathrm{T}}$, $\tau(t) = 1$, $\theta = 0.3$, $J_1 = (0.1, 1.2)^{\mathrm{T}}$, $J_2 = (0.1, 0.45)^{\mathrm{T}}$,

$$C_1 = \begin{pmatrix} 1.2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 3 & -0.3 \\ 8 & 5 \end{pmatrix}, \quad B_1 = \begin{pmatrix} -1.4 & 0.1 \\ 0.3 & -8 \end{pmatrix}, \quad D_1 = \begin{pmatrix} -1.2 & 0.1 \\ -2.8 & -1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 2.7 & -0.3 \\ 7.8 & 5.3 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -1.42 & 0.1 \\ 0.28 & -8.1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} -0.98 & 0.12 \\ -2.7 & -1.2 \end{pmatrix},$$

the activation function is $f(x) = (f_1(x_1), f_2(x_2))$ with

$$f_i(x_i) = \begin{cases} \tanh(x_i) + 0.05, & x_i > 0, i = 1, 2, \\ \tanh(x_i) - 0.05, & x_i < 0, i = 1, 2. \end{cases}$$

Figure 7 shows the Chaotic-like trajectories of models 1 and 2 in system (42), where the initial condition of models 1 and 2 is the same as $x(t) = (0.5, -0.5)^{\mathrm{T}}$, $\forall t \in [-1, 0]$.

It is not difficult to verified that (H_1)–(H_4) are satisfied and $z_1 = z_2 = 1$, $p_1 = p_2 = 0.1$, $\mu = 0$, $\tau = 1$, $\theta = 0.3$.

Consider a coupled neural networks consisting of five identical models (42) with Markovian parameters and the digraphs $\mathcal{G}_1, \mathcal{G}_2$ shown in Fig. 2.

It is followed that $\|A_1 p\|_\infty + \|B_1 p\|_\infty + \theta\|D_1 p\|_\infty = 2.244$, $\|A_2 p\|_\infty + \|B_2 p\|_\infty + \theta\|D_2 p\|_\infty = 2.265$, $\hat{\chi}_2^1 = \hat{\chi}_3^1 = 18.55$, $\hat{\chi}_4^1 = 20.55$, $\hat{\chi}_5^1 = 21.55$, $\hat{\chi}_2^2 = 17.65$, $\hat{\chi}_3^2 = 18.65$, $\hat{\chi}_4^2 = 20.65$, $\hat{\chi}_5^2 = 18.15$. According to Theorem 2, the coupled neural networks with any irreducible Π and the Markovian jumping parameters in (42) and the digraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ can be finite-timely synchronized by controller (7) with the control gains $\eta^1 = 2.25$, $\eta^2 = 2.27$ (i.e., $\epsilon = 0.05$), $\xi_2^1 = \xi_3^1 = 18.6$, $\xi_4^1 = 20.6$, $\xi_5^1 = 21.6$, $\xi_2^2 = 17.7$, $\xi_3^2 = 18.7$, $\xi_4^2 = 20.7$, $\xi_5^2 = 18.2$.

In the simulations, we use controller (39) with $\nu = 0.001$ instead of controller (7). The initial values of the coupled network are randomly chosen as $x_1(t) = (0.8801, -0.7524)^{\mathrm{T}}$, $x_2(t) = (-0.0535, 0.3450)^{\mathrm{T}}$, $x_3(t) = (0.2666, -0.1450)^{\mathrm{T}}$, $x_4(t) = (0.0998, 0.9211)^{\mathrm{T}}$, $x_5(t) = (-0.6171, -0.0849)^{\mathrm{T}}$ for $t \in [-1, 0]$, the time

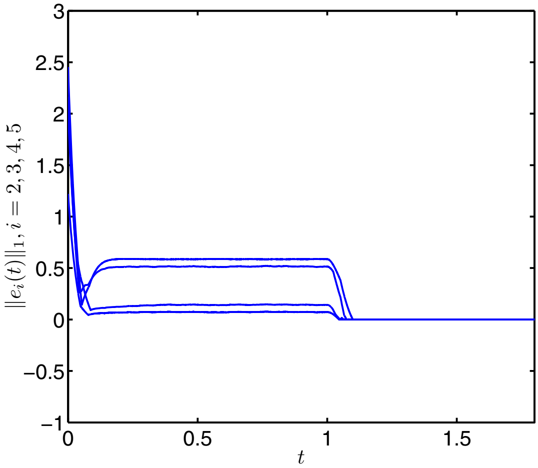

Fig. 8 Time response of synchronization errors under controller (39)

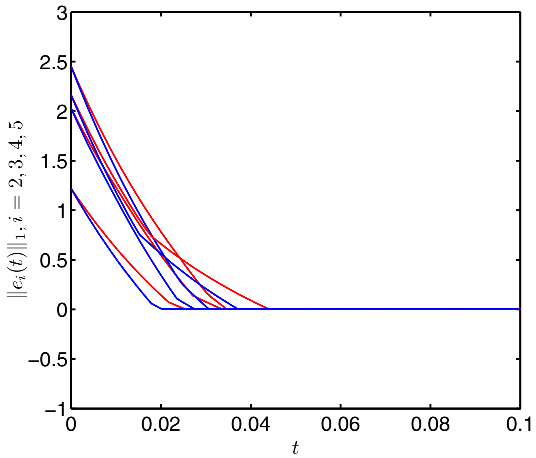

Fig. 9 Time response of synchronization errors under controller (39) with $\eta^1 = \eta^2 = 17$ (*red*) and $\eta^1 = \eta^2 = 23$ (*blue*)

step size is $\delta = 0.0005$, $\Pi = \begin{pmatrix} -700 & 700 \\ 800 & -800 \end{pmatrix}$, $r_0 = 1$. Based on Theorem 2, the coupled network is synchronized in the time $t_1 = 1476.9$s. Figure 8 describes the time evolutions of the synchronization errors. One can see from Fig. 8 that all the synchronization errors become zero before 1.2s.

Now we verify the accuracy of the settling time t_1 by changing the control gains η^1 and η^2 and keeping the other parameters same as those in Fig. 8. By Theorem 2, we get $t_1 \leq 4.0217$ when $\eta^1 = \eta^2 = 17$ ($\epsilon = 14.735$), and $t_2 \leq 2.5686$ when $\eta^1 = \eta^2 = 23$ ($\epsilon = 20.735$). Figure 9 shows that the synchronization is realized before 0.05s for the case of $\eta^1 = \eta^2 = 17$ and before 0.04s in the case of $\eta^1 = \eta^2 = 23$. Figure 9 not only demonstrates the correctness of Theorem 2, but also shows that the control gains η^k in controller (7) can be used to tune the synchronization time. In the three simulations, the errors between the synchronization time in the simulation and the settling time

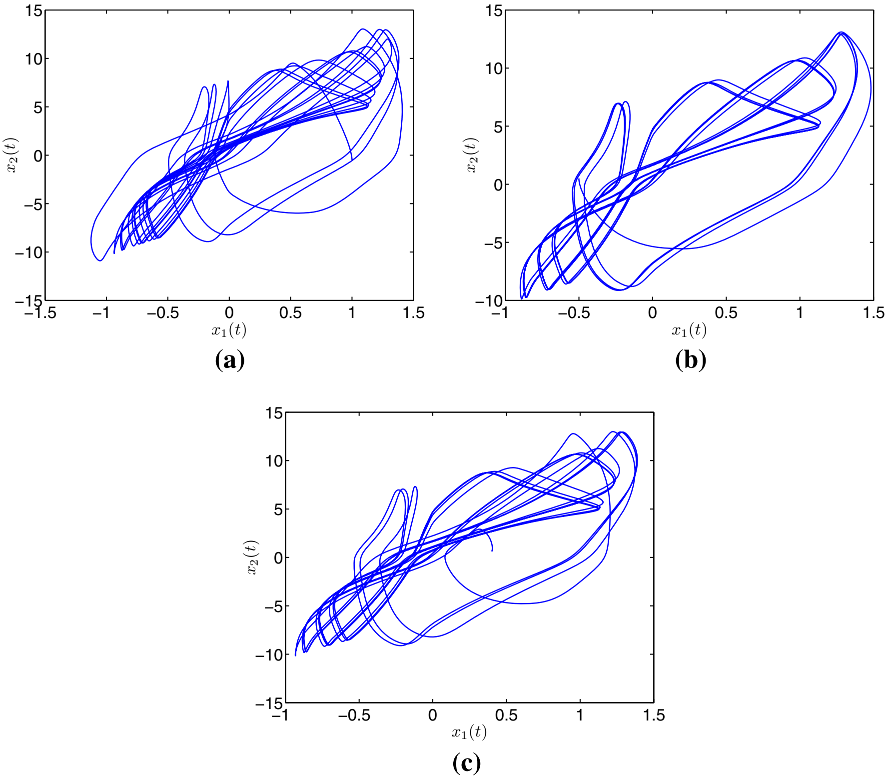

Fig. 10 Different trajectories of models 1 in system (42) with different initial values: $x(t) = (1, -0.5)^{\mathrm{T}}$ (**a**), $x(t) = (-0.5, 0.5)^{\mathrm{T}}$ (**b**), $x(t) = (0.4, 0.6)^{\mathrm{T}}$ (**c**), where $\forall t \in [-1, 0]$

t_1 are roughly estimated as $\Delta t = 1475.7$ ($\epsilon = 0.05$), $\Delta t = 3.9717$ ($\epsilon = 14.735$), $\Delta t = 2.5286$ ($\epsilon = 20.735$), respectively. These data suggest that the accuracy of the estimation of the settling time improves as the increasing of control gains η^k (or the number ϵ).

Remark 9 The trajectory of the delayed neural networks with discontinuous activation functions is heavily dependent on the initial condition. We take the model 1 in Example 2 as an example. Its trajectories are completely different with initial different values: $x(t) = (1, -0.5)^{\mathrm{T}}$, $x(t) = (-0.5, 0.5)^{\mathrm{T}}$, $x(t) = (0.4, 0.6)^{\mathrm{T}}$, $\forall t \in [-1, 0]$, which are shown in Fig. 10. Although the node systems with different initial conditions exhibit different trajectories, the finite-time synchronization is still achieved and verifies the theoretical analysis in the examples. So the controller has desired robust property.

6 Conclusions

In this paper, finite-time synchronization in an array of coupled delayed neural networks with discontinuous activation functions, Markovian jumping parameters as well as nonidentical uncertain perturbations has been studied. The time delay includes discrete delay, finite-time and infinite-time distributed delay. By using a class of simple controller, novel analytical techniques, M-matrix method and newly designed Lyapunov–Krasovskii functionals, sufficient conditions guaranteeing the finite-time synchronization of the considered system are derived. The uncertainties caused by the mixed delays, Filippov solutions and Markov chain and the uncertainties of external perturbations have been properly handled by using a class of simple controllers. Moreover, the settling time is also given for coupled neural networks with finite-time mixed delays and without delay. Results of this paper essentially extend existing ones concerning finite-time synchronization of systems of differential equations without delay. Several useful corollaries have also given for neural networks without Markovian jumping parameters. The accuracy of the estimation of settling time is also discussed with numerical simulations.

Recently, dynamical behaviors of general nonlinear systems based on fuzzy dynamic models have been extensively investigated [26–29]. However, to the best of our knowledge, few results are published on finite-time control of fuzzy dynamic models with time delays, packet dropouts, as well as quantization. Our future research is to solve this problem, which is also challenging.

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