

Iterated full information secretary problem

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Abstract We consider a full information best-choice problem where an administrator who has only one on-line choice in m consecutive searches has to choose the best candidate in one of them.

Keywords Monotone case · Secretary problem · Full information · Best choice · Optimal stopping time

Mathematics Subject Classification 60G40

1 Introduction and notation

In the full information best-choice problem [Gilbert and Mosteller \(1966\)](#) we deal with a discrete time stochastic process (X_1, \dots, X_n) where X_1, \dots, X_n are i.i.d. random variables with known continuous cumulative distribution function F . We observe elements of (X_1, \dots, X_n) one by one and our goal is to choose on-line the largest element of X_1, \dots, X_n which is not a priori known. Stopping the process at a given moment means choosing the object we have observed at this moment according to the knowledge obtained in the hitherto observations. The best-choice problem consists of finding a strategy of stopping the process that maximizes the probability $\mathbf{P}[X_\tau = \max\{X_1, \dots, X_n\}]$ over all stopping times $\tau \leq n$. [see [Gnedin \(1996\)](#)]

Let us recall basic results for this case.

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Numbers d_k , called *decision numbers*, are implicitly defined as satisfying the following equalities: $d_0 = 0$ and for $k = 1, 2, \dots$,

$$\sum_{j=1}^k \binom{k}{j} (d_k^{-1} - 1)^j j^{-1} = 1,$$

or, equivalently,

$$\sum_{j=1}^k (d_k^{-j} - 1) j^{-1} = 1.$$

The optimal stopping time is given by the following formula:

$$\tau_n^* = \min \{t : X_t = \max \{X_1, \dots, X_t\}, 1 \leq t \leq n \text{ and } F(X_t) \geq d_{n-t}\},$$

if the set under minimum is nonempty, otherwise $\tau_n^* = n$.

It is known [see [Samuels \(1991\)](#)], that the sequence (d_k) is increasing in k , $\lim_{k \rightarrow \infty} d_k = 1$ and $\lim_{k \rightarrow \infty} k(1 - d_k) = c$, where $c = 0.804352 \dots$ is the solution of the following equation

$$\sum_{j=1}^{\infty} \frac{c^j}{j!j} = \int_0^c x^{-1}(e^x - 1)dx = 1.$$

The maximal probability (using the optimal stopping time)

$$v_n = \mathbf{P} [X_{\tau_n^*} = \max \{X_1, \dots, X_n\}]$$

does not depend on F , is strictly decreasing in n and

$$v_{\infty} := \lim_{n \rightarrow \infty} v_n = e^{-c} + (e^c - c - 1) \int_1^{\infty} x^{-1} e^{-cx} dx \tag{1}$$

($v_{\infty} = 0.580164 \dots$) [see [Samuels \(1991\)](#)].

In this paper we consider a modification to the classical full information best-choice problem. Namely, we consider m consecutive classical full information searches. Our aim is to choose the largest element in one of them if we have only one choice. Our goal is to find a strategy that maximizes the probability of achieving this aim.

The problem considered here is related to real life situations where contests may be repeated several times but once in one of them the choice is made the procedure ends. Usually selectors know how many times they can repeat the contest and intuitively, the more contests are ahead the more selective they can be.

The solution of the no information version for the repeated contest problem was presented in [Kuchta and Morayne \(2014a\)](#).

Here is a formal description of the problem considered.

Let $n_1, \dots, n_m \in \mathbb{N}$ and $(X^{(m)}, X^{(m-1)}, \dots, X^{(1)})$ be a sequence of m consecutive searches: for $i = 0, \dots, m - 1$,

$$X^{(m-i)} = \left(X_{1+\sum_{k=0}^{i-1} n_{m-k}}, X_{2+\sum_{k=0}^{i-1} n_{m-k}}, \dots, X_{\sum_{k=0}^i n_{m-k}} \right),$$

where $X_{1+\sum_{k=0}^{i-1} n_{m-k}}, X_{2+\sum_{k=0}^{i-1} n_{m-k}}, \dots, X_{\sum_{k=0}^i n_{m-k}}$ are independent random variables with known continuous cumulative distribution function F_{m-i} . (The inverse numbering simplifies some technicalities in the proof and is adjusted to the recursion we will use.) The continuous distribution F_{m-i} is known and since the largest measurement in a sample remains the largest under all monotonic transformations of its variable, we lose no generality by assuming that F_{m-i} is the standard uniform for all searches: $F(x) = x$ on $0 \leq x \leq 1$.

Let for $1 \leq i \leq m$

$$Y^{(i)} \equiv \left(X^{(i)}, X^{(i-1)}, \dots, X^{(1)} \right)$$

and

$$\text{Max}(Y^{(i)}) = \left\{ \max \left(X^{(i)} \right), \dots, \max \left(X^{(1)} \right) \right\},$$

where $\max(X^{(i)})$ is the largest element of the search $X^{(i)}$, for $i = 1, \dots, m$.

Let t be an integer, $1 \leq t \leq \sum_{i=1}^m n_i$. For the time $t = \sum_{k=0}^{i-1} n_{m-k} + j$, where $0 \leq i \leq m - 1$, $1 \leq j \leq n_{m-i}$, the selector sees the whole sequences $X^{(m)}, X^{(m-1)}, \dots, X^{(m-i+1)}$ and the first j values of the search $X^{(m-i)}$. The goal of the selector is to stop the search at a time t maximizing the probability that $X_t \in \text{Max}(Y^{(m)})$. Formally, for $t \geq 1$, let \mathcal{F}_t be the σ -algebra generated by X_1, \dots, X_t , $\mathcal{F}_t = \sigma(X_1, \dots, X_t)$. Our aim is to find a stopping time τ_m with respect to the filtration (\mathcal{F}_t) maximizing the probability $\mathbf{P}[X_{\tau_m} \in \text{Max}(Y^{(m)})]$.

2 Optimal stopping time

Let us recall the Monotone Case Theorem [see [Chow et al. \(1971\)](#)], which is often a very useful tool when looking for an optimal stopping time.

Theorem 2.1 *If $\{(Z_i, \mathcal{F}_i) : i \leq n\}$ is a stochastic process such that the inequality $Z_i \geq E(Z_{i+1}|\mathcal{F}_i)$ implies the inequality $Z_{i+1} \geq E(Z_{i+2}|\mathcal{F}_{i+1})$ for each $i \leq n - 2$, then the stopping time*

$$\bar{\rho} = \min \{i : Z_i \geq E(Z_{i+1}|\mathcal{F}_i)\}$$

is optimal for maximizing $E(Z_\tau)$ over all (\mathcal{F}_i) -stopping times τ .

We apply this theorem to determine an optimal stopping time for m searches, i.e. for $Y^{(m)}$.

Let γ_{m-1} be the probability of success using an optimal stopping time for $Y^{(m-1)}$.

For $1 \leq k \leq n_m$ we define the sequence of *multiple search decision numbers* \hat{d}_k in the following way:

$$\hat{d}_0 = 0,$$

and if $1 \leq k \leq n_m - 1$,

$$\sum_{j=1}^k \frac{1}{j} (\hat{d}_k^{-j} - 1) = 1 - \gamma_{m-1},$$

or, equivalently,

$$\sum_{j=1}^k \binom{k}{j} \frac{1}{j} (\hat{d}_k^{-1} - 1)^j = 1 - \gamma_{m-1}. \tag{2}$$

Notice that the numbers \hat{d}_k are to be used only in the first search $X^{(m)}$. For $m = 1$, $\hat{d}_k = d_k$.

Now let us define the following stopping times τ_m for $Y^{(m)}$.

For $m = 1$:

$$\tau_1 = \min \{t : X_t = \max \{X_1, \dots, X_t\}, 1 \leq t \leq n_1 \text{ and } X_t \geq d_{n_1-t}\}$$

if the set under minimum is nonempty, otherwise $\tau_1 = n_1$,

and, for $m > 1$:

$$\tau_m = \begin{cases} \min\{t : X_t = \max\{X_1, \dots, X_t\}, 1 \leq t \leq n_m, X_t \geq \hat{d}_{n_m-t}\} & \text{if the set} \\ & \text{under minimum is nonempty,} \\ n_m + \tau_{m-1} & \text{otherwise.} \end{cases}$$

In the first case of the definition above we choose from the first search $X^{(m)} = (X_1, \dots, X_{n_m})$. In the second case we choose from among the remaining $m - 1$ searches: $Y^{(m-1)}$, so the recursion is used.

Theorem 2.2 *The stopping time τ_m is optimal for $Y^{(m)}$. When using τ_m the probability of success equals*

$$\begin{aligned} \gamma_m &= \frac{1}{n_m} \left(1 - (\hat{d}_{n_m-1})^{n_m}\right) \\ &+ \sum_{t=1}^{n_m-1} \left(\frac{1}{n_m - t} \sum_{i=1}^t \left(\frac{(\hat{d}_{n_m-i})^t}{t} - \frac{(\hat{d}_{n_m-i})^{n_m}}{n_m} \right) - \frac{(\hat{d}_{n_m-t-1})^{n_m}}{n_m} \right) \\ &+ \frac{1}{n_m} \sum_{t=1}^{n_m-1} (\hat{d}_t)^{n_m} \cdot \gamma_{m-1}, \end{aligned} \tag{3}$$

where we set $\gamma_0 = 0$.

Proof In the proof we use recursion with respect to m . If τ_{m-1} is an optimal stopping time for the case $Y^{(m-1)}$, then when looking for an optimal stopping time τ_m for m searches, i.e. for $Y^{(m)}$, the only stopping times that should be considered are the times of relative records for $X^{(m)}$ and the optimal stopping time τ_{m-1} in the remaining $m - 1$ searches $Y^{(m-1)}$. Namely,

let $\rho_1 = 1$ and, for $2 \leq i \leq \sum_{j=1}^m n_j$ if $\rho_{i-1} \leq n_m$, then

$$\rho_i = \begin{cases} \min\{j > \rho_{i-1} : X_j = \max\{X_1, \dots, X_j\}, j \leq n_m\} & \text{if the set under} \\ & \text{minimum is nonempty,} \\ n_m + \tau_{m-1} & \text{otherwise.} \end{cases}$$

(Note that if $\rho_i \leq n_m$ then ρ_i is the time of i -th relative record. If $\rho_{i-1} > n_m$ then $\rho_i = \rho_{i-1}$.)

Let

$$W_i = \begin{cases} 1 & \text{if } X_{\rho_i} \in \text{Max}(Y^{(m)}), \\ 0 & \text{if } X_{\rho_i} \notin \text{Max}(Y^{(m)}), \end{cases}$$

and

$$Z_i = E(W_i | \mathcal{F}_{\rho_i}).$$

Let us notice that in our notation the probability of stopping on a maximal element is equal to

$$\mathbf{P}[X_{\rho_i} \in \text{Max}(Y^{(m)})] = \mathbf{P}[W_i = 1] = E(Z_i).$$

Thus, our aim is maximizing $E(Z_\tau)$ over all $(\mathcal{F}_{\rho_i})_i$ -stopping times τ . By Theorems 2.3, 4.1 and Proposition 5.2 of [Kuchta and Morayne \(2014b\)](#) the process Z satisfies the hypothesis of the Monotone Case Theorem.

Suppose we have seen the t -th element x from the first search and this element is maximal so far. Thus $t = \rho_j$ for some j . There are still $n_m - t$ elements to come in $X^{(m)}$. The probability that the next $n_m - t$ elements are not bigger than x is equal to x^{n_m-t} and this is the probability of winning if we stop now. The probability of winning in the time of the next relative record in $X^{(m)}$ is equal to

$$\sum_{i=1}^{n_m-t} \binom{n_m-t}{i} \frac{1}{i} x^{n_m-t-i} (1-x)^i,$$

where the i -th summand is the probability that exactly i elements from the remaining $n_m - t$ ones are larger than x , and the maximum of those i elements appears first. Choosing the times when they come corresponds to the factor $\binom{n_m-t}{i}$, the probability that exactly these elements are bigger than x is equal to $x^{n_m-t-i} (1-x)^i$ and the probability that the largest element from this group comes before the other ones is equal to $\frac{1}{i}$.

If there is no relative record after x till the time n_m , i.e., within the first search $X^{(m)}$, we use the optimal strategy for the remaining part which consists of $m - 1$ searches $X^{(m-1)}, \dots, X^{(1)}$. The probability that this happens and that we succeed is equal to $x^{n_m-t} \gamma_{m-1}$. Thus, by the Monotone Case Theorem, we decide to stop at the t -th moment if it is the first moment of the relative record when

$$Z_j = x^{n_m-t} \geq E(Z_{j+1} | \mathcal{F}_{\rho_j}) = \sum_{i=1}^{n_m-t} \binom{n_m-t}{i} \frac{1}{i} x^{n_m-t-i} (1-x)^i + x^{n_m-t} \gamma_{m-1}. \tag{4}$$

Since

$$\sum_{i=1}^{n_m-t} \binom{n_m-t}{i} \frac{1}{i} (x^{-1} - 1)^i = \sum_{i=1}^{n_m-t} \frac{1}{i} (x^{-i} - 1),$$

(4) is equivalent to

$$1 - \gamma_{m-1} \geq \sum_{i=1}^{n_m-t} \frac{1}{i} (x^{-i} - 1). \tag{5}$$

The function $f(x) = \sum_{i=1}^{n_m-t} \frac{1}{i} (x^{-i} - 1)$ is decreasing in x , where $0 < x < 1$ and $1 \leq t \leq n_m - 1$. Thus the smallest x satisfying (5) is equal to the solution of the equation

$$1 - \gamma_{m-1} = \sum_{i=1}^{n_m-t} \frac{1}{i} (x^{-i} - 1).$$

Hence τ_m is an optimal stopping time.

The probability that we choose from $X^{(m)}$ and we are successful is equal to

$$\mathbf{P} \left[X_{\tau_m} = \max(X^{(m)}) \right] = \sum_{t=1}^{n_m} p_t, \tag{6}$$

where p_t is the probability that we stop at time t and it is successful, i.e. $X_t = \max \{ X_1, \dots, X_{n_m} \}$.

Thus

$$p_1 = \int_{\hat{d}_{n_m-1}}^1 x^{n_m-1} dx = \frac{1}{n_m} \left(1 - (\hat{d}_{n_m-1})^{n_m} \right). \tag{7}$$

Let $1 \leq t \leq n_m - 1$. The probability that no element among the first t is chosen and that the absolute largest is X_{t+1} is equal to (see the explanation below)

$$\begin{aligned} & \frac{1}{n_m - t} \sum_{i=1}^t \left(\int_0^{\hat{d}_{n_m-i}} x^{t-1} dx - \int_0^{\hat{d}_{n_m-i}} x^{n_m-1} dx \right) \\ &= \frac{1}{n_m - t} \sum_{i=1}^t \left(\frac{(\hat{d}_{n_m-i})^t}{t} - \frac{(\hat{d}_{n_m-i})^{n_m}}{n_m} \right), \end{aligned} \tag{8}$$

where the first integral is the probability that the i -th element is below \hat{d}_{n_m-i} and it is the biggest among the first t elements, the second integral is the probability that the i -th element is below \hat{d}_{n_m-i} and it is the absolute maximum, and the factor $\frac{1}{n_m-t}$ is the probability that the best element among the remaining $n_m - t$ elements is exactly the $(t + 1)$ -th one.

The probability that X_{t+1} is the largest in $X^{(m)}$ but does not pass the threshold, is equal to

$$\int_0^{\hat{d}_{n_m-t-1}} x^{n_m-1} dx = \frac{(\hat{d}_{n_m-t-1})^{n_m}}{n_m}. \tag{9}$$

Note that if the last event (whose probability is given by (9)) happens then also the previous one (whose probability is given by (8)) does, because the thresholds \hat{d}_{n_m-i} are decreasing with i . Thus by (8) and (9), for $1 \leq t \leq n_m - 1$,

$$p_{t+1} = \frac{1}{n_m - t} \sum_{i=1}^t \left(\frac{(\hat{d}_{n_m-i})^t}{t} - \frac{(\hat{d}_{n_m-i})^{n_m}}{n_m} \right) - \frac{(\hat{d}_{n_m-t-1})^{n_m}}{n_m}. \tag{10}$$

We do not stop at the first search $X^{(m)}$ if and only if, for every $1 \leq t \leq n_m - 1$, $X_t < \hat{d}_{n_m-t}$ when $X_t = \max\{X_1, \dots, X_t\}$. Thus the probability that using τ_m we do not stop at the first search $X^{(m)}$ is equal to

$$P[\tau_m > n_m] = \sum_{t=1}^{n_m-1} \int_0^{\hat{d}_{n_m-t}} x^{n_m-1} dx = \sum_{t=1}^{n_m-1} \frac{(\hat{d}_t)^{n_m}}{n_m}.$$

The above equality, (7), (10) and (6) yield (3). □

3 Asymptotics

In this section we examine the asymptotic behavior of the probability of success γ_m and the multiple search decision numbers \hat{d}_k as $n_i \rightarrow \infty$ for every $i \in \{1, \dots, m\}$.

Let us define recursively the following sequence: $r_0 = 0$, and for $i \geq 1$:

$$r_i = e^{-c_i} + (e^{c_i} - c_i - 1) \int_1^\infty x^{-1} e^{-c_i x} dx, \tag{11}$$

where c_i satisfies the following equation ($i \geq 1$):

$$\int_0^{c_i} x^{-1} (e^x - 1) dx = 1 - r_{i-1}, \tag{12}$$

or, equivalently,

$$\sum_{j=1}^{\infty} \frac{c_i^j}{j!j} = 1 - r_{i-1}.$$

Let $n_i^* = \min\{n_1, \dots, n_i\}$ for $i = 1, \dots, m$ and let, for $1 \leq k \leq n_m - 1$,

$$\alpha_k = k \left(\hat{d}_k^{-1} - 1 \right). \tag{13}$$

Theorem 3.1 $\gamma_m \longrightarrow r_m$ as $n_m^* \longrightarrow \infty$.

Proof We prove this theorem by induction with respect to m .

For $m = 1$ we have only one search and

$$\lim_{n_1^* \rightarrow \infty} \gamma_1 = v_{\infty} = r_1.$$

Of course, this is the asymptotic solution (1) of the classical full information best-choice problem.

Let $m \geq 2$ and assume that $\lim_{n_{m-1}^* \rightarrow \infty} \gamma_{m-1} = r_{m-1}$.

Note that α_k is, in fact, a function of m variables: k, n_{m-1}, \dots, n_1 ; $\alpha_k = \alpha_k(n_{m-1}, \dots, n_1)$.

Claim 1 $\alpha_k \longrightarrow c_m$ as $(k, n_m^*) \longrightarrow (\infty, \infty)$ and $k \leq n_m - 1$.

Proof of Claim 1 For $1 \leq k \leq n_m - 1$, (2) and (13) yield

$$1 - \gamma_{m-1} = \sum_{j=1}^k \binom{k}{j} \frac{1}{j} \left(\frac{\alpha_k}{k} \right)^j.$$

Thus

$$\begin{aligned} 1 - \gamma_{m-1} &= \sum_{j=1}^k \binom{k}{j} \frac{1}{k^j} \int_0^{\alpha_k} x^{j-1} dx = \int_0^{\alpha_k} \frac{1}{x} \left(\left(1 + \frac{x}{k}\right)^k - 1 \right) dx \\ &= \int_{\mathbb{R}} \mathbf{1}_{[0, \alpha_k]} \frac{1}{x} \left(\left(1 + \frac{x}{k}\right)^k - 1 \right) dx. \end{aligned}$$

Since the function $\frac{1}{x} \left(\left(1 + \frac{x}{k}\right)^k - 1 \right)$ is increasing in k for $x > 0$ and, for $k = 1$, $\int_0^{\alpha_1} 1 dx = \alpha_1 = 1 - \gamma_{m-1} \leq 1$, we always have $0 \leq \alpha_k < 1$ for $1 < k \leq n_m - 1$.

Let $U = \limsup_{(k, n_m^*) \rightarrow (\infty, \infty)} \alpha_k$ and $L = \liminf_{(k, n_m^*) \rightarrow (\infty, \infty)} \alpha_k$.

By Lebesgue’s bounded convergence theorem

$$1 - r_{m-1} = \int_{\mathbb{R}} \mathbf{1}_{[0, U]} x^{-1} (e^x - 1) dx = \int_0^U x^{-1} (e^x - 1) dx$$

and

$$1 - r_{m-1} = \int_{\mathbb{R}} \mathbf{1}_{[0,L]} x^{-1} (e^x - 1) dx = \int_0^L x^{-1} (e^x - 1) dx,$$

which implies $L = U$. Hence the limit from the statement of the claim exists. In view of the definition of c_m we also obtain $\lim_{(k,n_m^*) \rightarrow (\infty, \infty)} \alpha_k = c_m$. \square

Further we follow the method used by Samuels (1982), [see also Samuels (1991)].

Consider the first search $X^{(m)}$. Let $M_t = \max \{X_1, \dots, X_t\}$, $1 \leq t \leq n_m$, and let $M_0 = 0$. Let σ_{n_m} be the arrival time of the largest element in $X^{(m)}$ and $\hat{\sigma}_{n_m}$ be the arrival time of the largest element in $X^{(m)}$ before the time σ_{n_m} , i.e.

$$X_{\sigma_{n_m}} = M_{n_m} \quad \text{and} \quad X_{\hat{\sigma}_{n_m}} = M_{\sigma_{n_m}-1}.$$

Because \hat{d}_{n_m-t} is decreasing and M_t is increasing in t for $1 \leq t \leq n_m$, the probability that we choose from $X^{(m)}$ and we are successful is equal to

$$\mathbf{P}[X_{\tau_m} = M_{n_m}] = \mathbf{P}\left[M_{n_m} \geq \hat{d}_{n_m-\sigma_{n_m}} \ \& \ M_{\sigma_{n_m}-1} < \hat{d}_{n_m-\hat{\sigma}_{n_m}}\right]$$

and the probability that we do not choose from $X^{(m)}$ is equal to

$$\mathbf{P}[\tau_m > n_m] = \mathbf{P}\left[M_{n_m} < \hat{d}_{n_m-\sigma_{n_m}}\right].$$

Thus,

$$\lim_{n_m^* \rightarrow \infty} \gamma_m = \lim_{n_m^* \rightarrow \infty} \mathbf{P}[X_{\tau_m} = M_{n_m}] + \lim_{n_m^* \rightarrow \infty} \mathbf{P}\left[M_{n_m} < \hat{d}_{n_m-\sigma_{n_m}}\right] \cdot \lim_{n_m^* \rightarrow \infty} \gamma_{m-1}. \tag{14}$$

Claim 2

$$(i) \quad \lim_{n_m^* \rightarrow \infty} \mathbf{P}[X_{\tau_m} = M_{n_m}] = e^{-c_m} + (e^{c_m} - c_m - 1) \int_1^\infty x^{-1} e^{-c_m x} dx - r_{m-1} \left(e^{-c_m} - c_m \int_1^\infty x^{-1} e^{-c_m x} dx \right) \tag{15}$$

and

$$(ii) \quad \lim_{n_m^* \rightarrow \infty} \mathbf{P}\left[M_{n_m} < \hat{d}_{n_m-\sigma_{n_m}}\right] = e^{-c_m} - c_m \int_1^\infty x^{-1} e^{-c_m x} dx. \tag{16}$$

Proof of Claim 2 We change variables:

$$\begin{aligned} S_{n_m} &= n_m (1 - M_{n_m}); & T_{n_m} &= \frac{\sigma_{n_m}}{n_m}; \\ \hat{S}_{n_m} &= (\sigma_{n_m} - 1) \left(1 - \frac{M_{\sigma_{n_m}-1}}{M_{n_m}} \right); & \hat{T}_{n_m} &= \frac{\hat{\sigma}_{n_m}}{\sigma_{n_m} - 1}. \end{aligned}$$

Then $n_m - \sigma_{n_m} = n_m (1 - T_{n_m})$ and $n_m - \hat{\sigma}_{n_m} = n_m (1 - T_{n_m} \hat{T}_{n_m}) + \hat{T}_{n_m}$. Thus, applying (13),

$$M_{n_m} \geq \hat{d}_{n_m - \sigma_{n_m}} \Leftrightarrow S_{n_m} \left(1 - T_{n_m} + \frac{\alpha_{n_m}(1 - T_{n_m})}{n_m} \right) \leq \alpha_{n_m(1 - T_{n_m})}$$

and

$$M_{\sigma_{n_m} - 1} < \hat{d}_{n_m - \hat{\sigma}_{n_m}} \Leftrightarrow \alpha_{n_m(1 - T_{n_m} \hat{T}_{n_m}) + \hat{T}_{n_m}} < (S_{n_m} + L_{n_m}) \left[1 - T_{n_m} \hat{T}_{n_m} + \frac{1}{n_m} \left(\alpha_{n_m(1 - T_{n_m} \hat{T}_{n_m}) + \hat{T}_{n_m}} + \hat{T}_{n_m} \right) \right],$$

where

$$L_{n_m} = \left(T_{n_m} - \frac{1}{n_m} \right)^{-1} \left(1 - \frac{S_{n_m}}{n_m} \right) \hat{S}_{n_m}.$$

Let us define the following events (depending on n_m):

$$A = \left[S_{n_m} \left(1 - T_{n_m} + \frac{\alpha_{n_m}(1 - T_{n_m})}{n_m} \right) \leq \alpha_{n_m(1 - T_{n_m})} \right],$$

$$B = \left[\alpha_{K_{n_m}} < (S_{n_m} + L_{n_m}) \left(1 - T_{n_m} \hat{T}_{n_m} + \frac{1}{n_m} \left(\alpha_{K_{n_m}} + \hat{T}_{n_m} \right) \right) \right],$$

where

$$K = n_m (1 - T_{n_m} \hat{T}_{n_m}) + \hat{T}_{n_m}.$$

Then

$$\mathbf{P} [X_{\tau_m} = M_{n_m}] = \mathbf{P} [M_{n_m} \geq \hat{d}_{n_m - \sigma_{n_m}} \ \& \ M_{\sigma_{n_m} - 1} < \hat{d}_{n_m - \hat{\sigma}_{n_m}}] = \mathbf{P} [A \cap B]$$

and

$$\mathbf{P} [M_{n_m} < \hat{d}_{n_m - \sigma_{n_m}}] = 1 - \mathbf{P} [A].$$

By the properties of the uniform distribution,

$$S_{n_m} \longrightarrow S, \ \hat{S}_{n_m} \longrightarrow \hat{S}, \ T_{n_m} \longrightarrow T, \ \hat{T}_{n_m} \longrightarrow \hat{T},$$

in the weak convergence as $n_m^* \longrightarrow \infty$, where S, \hat{S}, T, \hat{T} are mutually independent variables, S, \hat{S} have the exponential distribution with parameter 1, and T, \hat{T} have the uniform distribution on $[0, 1]$.

We have

$$\begin{aligned} \lim_{n_m^* \rightarrow \infty} \mathbf{P}[X_{\tau_m} = M_{n_m}] &= \lim_{n_m^* \rightarrow \infty} \mathbf{P}[A \cap B] \\ &= \mathbf{P}\left[S(1 - T) \leq c_m \leq \left(S + \frac{\hat{S}}{T}\right)(1 - T\hat{T})\right] \end{aligned}$$

and

$$\lim_{n_m^* \rightarrow \infty} \mathbf{P}\left[M_{n_m} < \hat{d}_{n_m - \sigma_{n_m}}\right] = \lim_{n_m^* \rightarrow \infty} (1 - \mathbf{P}[A]) = \mathbf{P}[S(1 - T) > c_m].$$

The conditional probability

$$\mathbf{P}\left[S(1 - T) \leq c_m \leq \left(S + \frac{\hat{S}}{T}\right)(1 - T\hat{T}) \mid S = s, T = t, \hat{T} = \hat{t}\right]$$

is equal to

$$\begin{cases} 1 & \text{if } c_m/(1 - t\hat{t}) - s < 0 \text{ and } s \leq c_m/(1 - t), \\ 0 & \text{if } c_m/(1 - t\hat{t}) - s < 0 \text{ and } s > c_m/(1 - t), \\ e^{-t(c_m/(1-t\hat{t})-s)} & \text{otherwise.} \end{cases}$$

Now we integrate this probability multiplied by the exponential density of S and we obtain the conditional probability for given $T = t$ and $\hat{T} = \hat{t}$:

$$\begin{aligned} &\mathbf{P}\left[S(1 - T) \leq c_m \leq \left(S + \frac{\hat{S}}{T}\right)(1 - T\hat{T}) \mid T = t, \hat{T} = \hat{t}\right] \\ &= \int_0^{c_m/(1-t\hat{t})} e^{-tc_m/(1-t\hat{t})} e^{-s(1-t)} ds + \int_{c_m/(1-t\hat{t})}^{c_m/(1-t)} e^{-s} ds \\ &= (1 - t)^{-1} e^{-tc_m/(1-t\hat{t})} \left(1 - e^{-c_m(1-t)/(1-t\hat{t})}\right) + e^{-c_m/(1-t\hat{t})} - e^{-c_m/(1-t)}. \end{aligned}$$

In the next step we integrate this expression over the unit square:

$$\begin{aligned} &\int_0^1 \int_0^1 \mathbf{P}\left[S(1 - T) \leq c_m \leq \left(S + \frac{\hat{S}}{T}\right)(1 - T\hat{T}) \mid T = t, \hat{T} = \hat{t}\right] dt d\hat{t} \\ &= \int_0^1 \int_0^1 \left((1 - t)^{-1} e^{-tc_m/(1-t\hat{t})} \left(1 - e^{-c_m(1-t)/(1-t\hat{t})}\right) + e^{-c_m/(1-t\hat{t})}\right) dt d\hat{t} \\ &\quad - \int_0^1 \int_0^1 e^{-c_m/(1-t)} dt d\hat{t}. \end{aligned} \tag{17}$$

It is easy to see that

$$\int_0^1 \int_0^1 e^{-c_m/(1-t)} dt d\hat{t} = e^{-c_m} - c_m \int_1^\infty x^{-1} e^{-c_m x} dx.$$

Making the following change of variables in the first integral of (17)

$$u = \frac{1-t}{1-t\hat{t}}, \quad v = \frac{1}{1-t\hat{t}},$$

we obtain

$$\begin{aligned} & \int_0^1 \int_0^1 \left((1-t)^{-1} e^{-c_m t/(1-t\hat{t})} \left(1 - e^{-c_m(1-t)/(1-t\hat{t})} \right) + e^{-c_m/(1-t\hat{t})} \right) dt d\hat{t} \\ &= \int_1^\infty \int_0^1 v^{-2} (v-u)^{-1} e^{-c_m(v-u)} du dv \\ & \quad + \left(e^{-c_m} - c_m \int_1^\infty x^{-1} e^{-c_m x} dx \right) \int_0^{c_m} x^{-1} (e^x - 1) dx. \end{aligned}$$

We set $w = v - u$. Interchanging the order of integration we obtain

$$\int_1^\infty \int_0^1 v^{-2} (v-u)^{-1} e^{-c_m(v-u)} du dv = e^{-c_m} + (e^{c_m} - c_m - 1) \int_1^\infty x^{-1} e^{-c_m x} dx.$$

By (12) we obtain (15).

The conditional probability of $\{S \geq c_m / (1-t)\}$ for $T = t$ and $\hat{T} = \hat{t}$ is equal to

$$\mathbf{P} \left[S \geq c_m / (1-t) \mid T = t, \hat{T} = \hat{t} \right] = \int_{c_m/(1-t)}^\infty e^{-s} ds = e^{-c_m/(1-t)}.$$

Analogously,

$$\lim_{n_m^* \rightarrow \infty} \mathbf{P} \left[M_{n_m} < \hat{d}_{n_m - \sigma_{n_m}} \right] = \int_0^1 \int_0^1 e^{-c_m/(1-t)} dt d\hat{t} = e^{-c_m} - c_m \int_1^\infty x^{-1} e^{-c_m x} dx.$$

This completes the proof of the claim. □

By (14), (15) and (16) and the induction hypothesis, we get

$$\lim_{n_m^* \rightarrow \infty} \gamma_m = e^{-c_m} + (e^{c_m} - c_m - 1) \int_1^\infty x^{-1} e^{-c_m x} dx = r_m. \tag{18}$$

This completes the proof. □

The following proposition describes the asymptotic behavior of the sequences (c_m) and (r_m) when $m \rightarrow \infty$.

Proposition 3.2 *The sequence (c_m) is decreasing and $\lim_{m \rightarrow \infty} c_m = 0$. The sequence (r_m) is increasing and $\lim_{m \rightarrow \infty} r_m = 1$.*

Proof The function

$$g(y) = e^{-y} + (e^y - y - 1) \int_y^\infty x^{-1} e^{-x} dx, \quad 0 < y < 1$$

is decreasing in y , (the derivative of this function is negative i.e. $\frac{dg}{dy} = (e^y - 1) \left(\int_y^\infty e^{-x} x^{-1} dx - e^{-y} y^{-1} \right) < 0$). By (11), (12) and $r_0 = 0, c_1 \approx 0.804$. It is easy to see that the sequence (r_m) is increasing and c_m is decreasing with $m \rightarrow \infty$ and both sequences are bounded by 0 and 1. Thus, both sequences are convergent. Let $\beta = \lim_{m \rightarrow \infty} c_m$. By (11) and (12) the sequence (c_m) satisfies the following recurrence

$$\int_0^{c_{m+1}} x^{-1} (e^x - 1) dx = 1 - e^{-c_m} - (e^{c_m} - c_m - 1) \int_1^\infty x^{-1} e^{-c_m x} dx.$$

Thus, β is the solution of the following equation

$$\int_0^\beta x^{-1} (e^x - 1) dx = 1 - e^{-\beta} - (e^\beta - \beta - 1) \int_1^\infty x^{-1} e^{-\beta x} dx.$$

It is easy to check that the only β satisfying this equation is $\beta = 0$.

By (18), it is now easy to check that $\lim_{m \rightarrow \infty} r_m = 1$.

This completes the proof. □

Approximations of the first ten elements of the sequences (r_m) and (c_m) are given in Table 1, see also Fig. 1. For comparison, the first column gives the corresponding probability of success (a_m) for the iterated no information version (the classical secretary problem) [see also Kuchta and Morayne (2014a)].

Table 1 The values of r_m, c_m and a_m for $m = 1, \dots, 10$

m	$a_m \sim$	$r_m \sim$	$c_m \sim$
1	$1/e$	0.5801	0.8043
2	0.5315	0.7443	0.3803
3	0.6259	0.8200	0.2404
4	0.6879	0.8629	0.1722
5	0.7319	0.8903	0.1325
6	0.7649	0.9091	0.1067
7	0.7905	0.9228	0.0888
8	0.8110	0.9332	0.0756
9	0.8277	0.9412	0.0656
10	0.8418	0.9477	0.0578

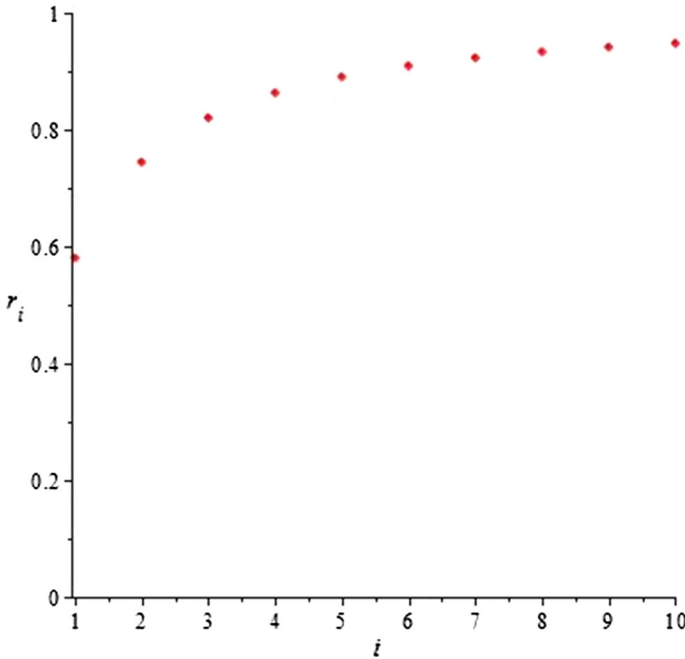


Fig. 1 The graph of $r_i, i = 1, \dots, 10$

Table 2 Approximations of probabilities of success for two searches of the same lengths: $n = 2, 3, \dots, 12$ and $n = 20$

n	$\gamma_2 \approx$
2	0.8956636342
3	0.8423563181
4	0.8159421134
5	0.8006581275
6	0.7907759043
7	0.7838725263
8	0.7787751482
9	0.7748534206
10	0.7717399044
11	0.7692062973
12	0.7671032840
20	0.7579203475

The following proposition describes the asymptotic behavior of the decision numbers \hat{d}_k when $k \rightarrow \infty$ and $n_m^* \rightarrow \infty$.

Proposition 3.3 $\hat{d}_k \rightarrow 1$ as $(k, n_m^*) \rightarrow (\infty, \infty)$ and $k \leq n_m - 1$.

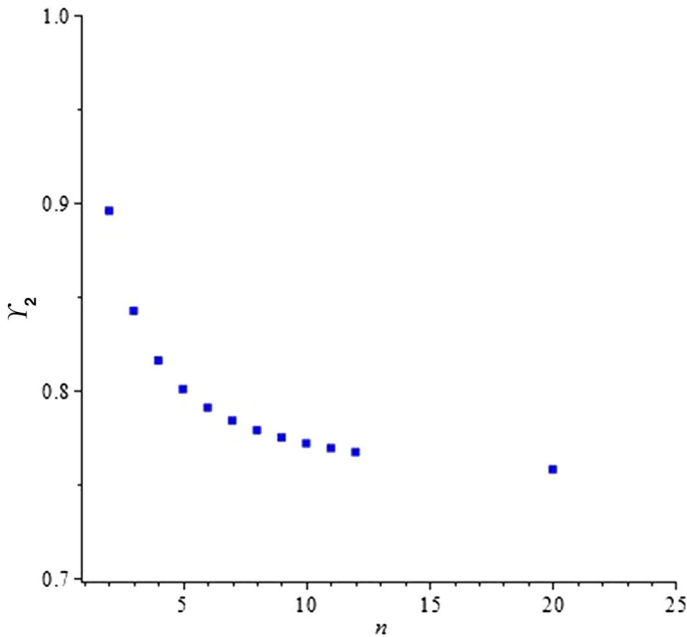


Fig. 2 The graph of γ_2 for two searches of the same lengths: $n = 2, 3, \dots, 12$ and $n = 20$

Proof By (13) we have $\hat{d}_k = \left(\frac{\alpha_k}{k} + 1\right)^{-1}$. By Claim 1 $\lim \alpha_k = c_m$ as $(k, n_m^*) \rightarrow (\infty, \infty)$ and $k \leq n_m - 1$. Since $0 < c_m < 1$, $\hat{d}(k) \rightarrow 1$ as $(k, n_m^*) \rightarrow (\infty, \infty)$ and $k \leq n_m - 1$. \square

Example Let us consider the case of two searches of the same length n . According to the optimal strategy using (13) for multiple search decision numbers $\hat{d}(k)$ for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 20$, we obtain the approximations of γ_2 from Table 2 (see Fig. 2). ($\gamma_2 \rightarrow 0.7443\dots$ for $n \rightarrow \infty$.)

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