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Computing Bend-Minimum Orthogonal Drawings of Plane Series–Parallel Graphs in Linear Time

Walter Didimo[1] · Michael Kaufmann[2] · Giuseppe Liotta[1] · Giacomo Ortali[1]

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Abstract

A *planar orthogonal drawing* of a planar 4-graph G (i.e., a planar graph with vertex-degree at most four) is a crossing-free drawing that maps each vertex of G to a distinct point of the plane and each edge of G to a polygonal chain consisting of horizontal and vertical segments. A longstanding open question in Graph Drawing, dating back over 30 years, is whether there exists a linear-time algorithm to compute an orthogonal drawing of a *plane* 4-graph with the minimum number of bends. The term "plane" indicates that the input graph comes together with a planar embedding, which must be preserved by the drawing (i.e., the drawing must have the same set of faces as the input graph). In this paper we positively answer the question above for the widely-studied class of series–parallel graphs. Our linear-time algorithm is based on a characterization of the planar series–parallel graphs that admit an orthogonal drawing without bends. This characterization is given in terms of the orthogonal spirality that each type of triconnected component of the graph can take; the orthogonal spirality of a component measures how much that component is "rolled-up" in an orthogonal drawing of the graph.

Walter Didimo, Michael Kaufmann, Giuseppe Liotta and Giacomo Ortali have contributed equally to this work.

A preliminary version of this research, restricted to rectilinear planarity testing, appears in the proceedings of the International Symposium on Graph Drawing and Network Visalization, GD 2020 [8]. The current version significantly extends the GD version, by providing complete technical details and full proofs for all lemmas/theorems, new results about the concept of spirality (Sect. 3), a new linear-time bend-minimization algorithm (Sects. 5 to 7), and a revised set of open problems (Sect. 8). Research partially supported by: (i) MIUR Project "AHeAD" under PRIN 20174LF3T8; (ii) Projects RICBA21LG "Algoritmi, modelli e sistemi per la rappresentazione visuale di reti" and RICBA22CB "Modelli, algoritmi e sistemi per la visualizzazione e l'analisi di grafi e reti", Ricerca di Base 2021 and 2022, University of Perugia.

✉ Walter Didimo
walter.didimo@unipg.it

Extended author information available on the last page of the article

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1 Introduction

Given a planar 4-graph G (i.e., a planar graph with vertex-degree at most four), a *planar orthogonal drawing* of G is a crossing-free drawing that maps each vertex of G to a distinct point of the plane and each edge of G to a polygonal chain consisting of horizontal and vertical segments [4, 12, 15, 17]. A *bend* is a point where a vertical and a horizontal segment of the same edge meet; see Fig. 1a, b, where bends are depicted as '$\times$'. Computing planar orthogonal drawings of planar graphs with the minimum number bends is one of the most studied problems in Graph Drawing. Garg and Tamassia [14] proved that this problem is NP-hard for general planar 4-graphs if the algorithm can freely choose the planar embedding; an $O(n)$-time algorithm exists for n-vertex planar 3-graphs [10] and an $O(n^3 \log^2 n)$-time algorithm exists for n-vertex series–parallel 4-graphs [6].

If G is a *plane* 4-graph, i.e., it has a planar embedding that the drawing algorithm must preserve, the problem is polynomial-time solvable. A seminal paper by Tamassia [21] describes an $O(n^2 \log n)$-time algorithm based on an elegant min-cost flow-network model. Cornelsen and Karrenbauer [2] reduced the time complexity to $O(n^{1.5})$, through a more efficient min-cost flow-network technique. Deciding whether there exists an (optimal) $O(n)$-time algorithm is a longstanding question that is still unanswered (see, e.g., [1, 4, 7]).

Contribution. In this paper we positively answer the question above for series–parallel graphs, which are a classical subject of investigation in Graph Drawing and Graph Algorithms (see, e.g., [4, 20, 22, 23]). Namely, we give an $O(n)$-time algorithm that receives as input an n-vertex plane series–parallel 4-graph G and that computes an embedding-preserving orthogonal drawing of G with the minimum number of bends. While $O(n)$-time algorithms that compute bend-minimum orthogonal drawings of plane graphs with maximum vertex-degree three are known (see, e.g., [17–19]), our result is the first linear-time algorithm for a graph family with vertices of degree four. Indeed, even for series–parallel plane graphs with degree-4 vertices, the most efficient solution known to date is the $O(n^{1.5})$-time algorithm by Cornelsen and Karrenbauer [2].

Different from the approach of Cornelsen and Karrenbauer we do not use network-flow techniques to minimize the number of bends. Instead, we rely on the observation that the bend-minimization problem for an orthogonal drawing of a plane graph G is equivalent to inserting in G the minimum number of subdivision vertices that make it rectilinear planar, i.e., drawable without bends. Following this idea, we first characterize those series–parallel graphs that are rectilinear planar. The characterization is expressed in terms of the "orthogonal spirality" for the triconnected components of G. Informally speaking, the orthogonal spirality of a component in an orthogonal drawing of G measures how much the component is "rolled-up" (see, e.g., [5, 9–11, 13]). We then consider the problem of efficiently adding the minimum number of subdivision vertices along the edges of those series–parallel graphs that are not rectilinear planar.

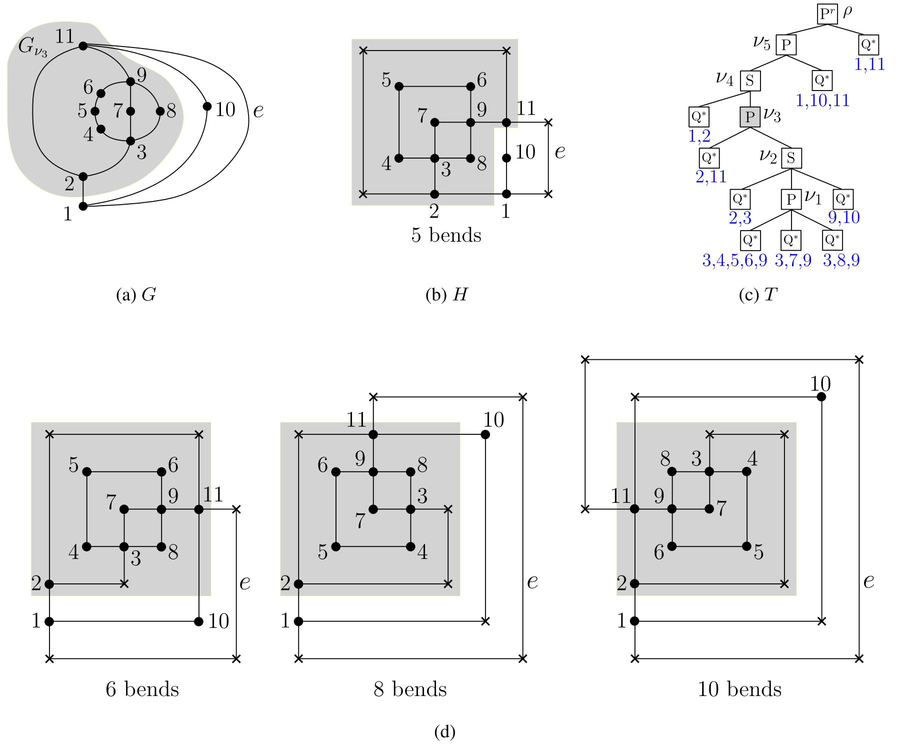

Fig. 1 **a** A series–parallel graph G with a given planar embedding. **b** A bend-minimum orthogonal drawing H of G with 5 bends, represented by cross vertices. **c** The SPQ*-tree T of G with reference edge $e = (1, 11)$; series and parallel compositions are represented by S- and P-nodes; a Q*-node represents an edge or a series of edges; the root P^r-node is a parallel composition between e and the rest of the graph. **d** Three suboptimal orthogonal drawings obtained by distributing the same number of bends of the highlighted subgraph G_{ν_3} in a different way

In a nutshell, our bend-minimization algorithm executes a post-order visit of a series–parallel *decomposition tree* T of the input graph G, also called *SPQ*-tree*. Tree T represents the parallel and the series compositions that form G (see Fig. 1a, c, and refer to Sect. 2 for a formal definition of SPQ*-trees). Suppose that ν is a node of T and G_ν its corresponding subgraph in G. When the algorithm visits ν, it must efficiently determine whether G_ν is rectilinear planar and, if not, it must compute the minimum number of subdivision vertices needed to make it rectilinear planar; how to efficiently compute such a number is a first key ingredient of our approach.

A second key ingredient is proving that, when the algorithm processes a node ν in the bottom-up visit, the addition of the minimum number of subdivision vertices that make G_ν rectilinear planar leads to the optimum in terms of total number of bends.

As a third key ingredient, the algorithm needs to concisely describe the set of rectilinear drawings of G_ν that can be obtained by distributing these subdivision vertices in all possible ways along the edges of G_ν, which gives rise to a combinatorial explosion of different possibilities. Indeed, different distributions of the same set of subdivision vertices along the edges of G_ν can lead to orthogonal drawings of G that have different number of bends. For example, consider the highlighted subgraph G_{ν_3} (associated with node ν_3 of T) in the graph of Fig. 1a. Any orthogonal drawing of G_{ν_3} requires at least three bends (subdivision vertices); placing all of them on edge (2, 11) yields the optimal solution of Fig. 1b, which additionally requires two bends on edge (1, 11). Conversely, placing the three bends on a different subset of edges of G_{ν_3} leads to (suboptimal) solutions with more bends; see Fig. 1d. To efficiently handle the combinatorially many distributions of the subdivision vertices along the edges of a subgraph G_ν, we succinctly encode in $O(1)$ space the "orthogonal shapes" that G_ν can have in a bend-minimum planar orthogonal drawing of G. This is done by looking at the set of possible orthogonal spirality values that G_ν can take in such a drawing.

The remainder of the paper is organized as follows. Section 2 recalls basic definitions used throughout the paper. Section 3 strengthens a result given in [5] about the interchangeability of orthogonal representations with the same spirality. Section 4 characterizes those plane series–parallel graphs that are rectilinear planar. Section 5 gives an overview of our bend-minimization algorithm. Section 6 provides details about the bottom-up visit performed by the algorithm. Section 7 summarizes our main result. Concluding remarks and open problems are in Sect. 8.

Some of our proofs are based on case analyses; in some proofs, we moved part of the case analysis to the paper appendix, when the analysis of a case resulted similar to the analysis of a previous case; also, the appendix reports a glossary of the main symbols used throughout the paper.

2 Preliminaries

We assume familiarity with basic concepts of graph planarity and graph drawing [4, 15–17]. We only deal with connected graphs and we focus on *orthogonal representations* rather than orthogonal drawings. An orthogonal representation H of a planar graph G describes a class of equivalent planar orthogonal drawings of G in terms of planar embedding, ordered sequence of bends along the edges (i.e., sequence of left/right turns going from an end-vertex to the other) and clockwise sequence of geometric angles at each vertex, each angle formed by two (possibly coincident) consecutive edges around the vertex and expressed as a value in the set $\{90°, 180°, 270°, 360°\}$ (angles of 360° occur only at degree-1 vertices). An orthogonal representation H of G can be described by a planar embedding of G plus an *angle labeling* specifying: (i) for each vertex v of G, the geometric angles at v; (ii) for each edge $e = (u, v)$ of G, the ordered sequence of bends along e as a sequence of angles in the left face (and hence in the right face) of e while moving along e from u to v (each bend determines an angle of 90° in one of the two faces incident to e and an angle of 270° in the other face). It is well known (see, e.g., [3]) that an angle labeling of G describes a valid orthogonal representation if and only if the following properties hold: (H1) for each vertex v, the sum of the angles at v equals 360°; (H2) for each face f, if $N_a(f)$ is the number of $a°$ angles in f, we have $N_{90}(f) - N_{270}(f) - 2N_{360}(f) = 4$ (resp. $N_{90}(f) - N_{270}(f) - 2N_{360(f)} = -4$) if f is an internal face (resp. the external face).

Note that, since the set of faces of a plane graph G is a base for the set of simple cycles of G, Properties (H1) and (H2) together are equivalent to say that for every simple cycle C of G, the number of right turns minus the number of left turns, when walking clockwise around the boundary of C, is equal to four. Namely, if v is a vertex of C we count: a right turn at v if there is an angle of 90° at v inside C; a left turn at v if the sum of the angles at v inside C equals 270°; two left turns at v if v has degree one. Also, a bend on an edge of C corresponds to a right (resp. left) turn if it determines an angle of 90° (resp., 270°) in C.

Given an orthogonal representation H of a plane graph G, a drawing of H (which corresponds to an orthogonal drawing of G) can be computed in linear time [21]. If H has no bend, H is a *rectilinear representation*.

Series–parallel graphs and decomposition trees. A *two-terminal series–parallel* graph G, also called *series–parallel graph*, has two distinct vertices s and t, called the *source* and the *sink* of G, respectively. A series–parallel graph can be inductively defined by, and naturally associated with, a *decomposition tree* T: (i) a single edge (s, t) is a series–parallel graph with source s and sink t, in which case T consists of a single Q-node, whose *poles* are s and t; (ii) given $p \geq 2$ series–parallel graphs $G_1, \ldots, G_p$, each G_i with source s_i and sink t_i ($i = 1, \ldots, p$), a new series–parallel graph G can be obtained with any of these two operations:

- *Series composition*, which identifies t_i with s_{i+1} ($i = 1, \ldots, p-1$); G has source $s = s_1$ and sink $t = t_p$. The composition is represented in T by an S-node, with *poles* s_1 and t_p, whose children are the roots of the decomposition trees T_i of G_i ($i = 1, \ldots, p$).
- *Parallel composition*, which identifies all sources s_i (resp. all sinks t_i) together ($i = 1, \ldots, p$); G has source $s = s_i$ and sink $t = t_i$. The composition is represented in T by a P-node, with *poles* are s and t, whose children are the roots of the decomposition trees T_i of G_i ($i = 1, \ldots, p$).

In our algorithm we do not distinguish between Q-nodes and S-nodes whose children are all Q-nodes. We just call any of these nodes a Q*-node. In other words, a Q*-node represents a series of edges. For a node ν of T, the *pertinent graph* G_ν of ν is the subgraph of G formed by all edges associated with the Q*-nodes in the subtree rooted at ν. We also call G_ν a *component* of G.

Let G be a plane (two-terminal) series–parallel graph with vertex-degree at most four. Note that G is either biconnected or it can be made biconnected with the addition of a single dummy edge; in this latter case we assume that the planar embedding of G is such that the dummy edge can be added on the external face of G. For any edge $e = (s, t)$ (possibly a dummy edge) on the external face, we can associate with G a decomposition tree T where the root is a P-node representing the parallel composition between e and the rest of the graph. Thus, the root of T is always a P-node with two children, one of which is a Q*-node corresponding to e. It will be called the (unique) P^r-node of T, to distinguish it by the other P-nodes. Edge e is the *reference edge* of T, and T is the *SPQ*-tree* of G *with respect to* e. Without loss of generality we assume that the external face of G is to the right of e while moving from s to t. Also, it is always possible to make T such that each (non-root) P-node has no P-node child and each S-node has no S-node child. Since G has vertex-degree at most four, a P-node has either

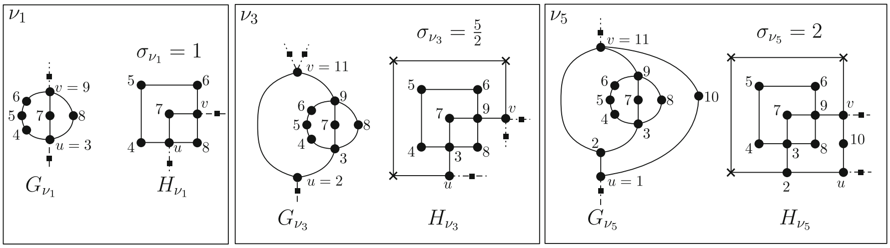

Fig. 2 The components associated with the P-nodes ν_1, ν_3, and ν_5, of the graph in Fig. 1. The alias vertices are the little squares along dashed edges. For each node ν_i, $i \in \{1, 3, 5\}$, we report G_{ν_i}, H_{ν_i}, and the spirality σ_{ν_i} of H_{ν_i} in H. In particular, the P-component of ν_3, with poles $\{u = 2, v = 11\}$, has spirality $\frac{5}{2}$; the P-component of ν_5, with poles $\{u = 1, v = 11\}$, has spirality 2

two or three children. Finally, we assume that the left-to-right order of the children of a P-node reflects the left-to-right order that their corresponding components have in the planar embedding of G. See Fig. 1a, c. From now on we assume that T satisfies the properties above for an n-vertex biconnected series–parallel graph. Observe that the number of nodes of T is $O(n)$.

Spirality of series–parallel graphs. Let G be a biconnected plane series–parallel graph and let T be an SPQ*-tree with respect to a reference edge $e = (s, t)$. Let H be an (embedding-preserving) orthogonal representation of G. Also, let ν be a node of T with poles $\{u, v\}$ and let H_ν be the restriction of H to G_ν. We also say that H_ν is a *component* of H. For each pole $w \in \{u, v\}$, let $\mathrm{indeg}_\nu(w)$ and $\mathrm{outdeg}_\nu(w)$ be the degree of w inside and outside H_ν, respectively. Define two (possibly coincident) *alias vertices* of w, denoted by w' and w'', as follows: (i) if $\mathrm{indeg}_\nu(w) = 1$, then $w' = w'' = w$; (ii) if $\mathrm{indeg}_\nu(w) = \mathrm{outdeg}_\nu(w) = 2$, then w' and w'' are dummy vertices, each splitting one of the two distinct edge segments incident to w outside H_ν; (iii) if $\mathrm{indeg}_\nu(w) > 1$ and $\mathrm{outdeg}_\nu(w) = 1$, then $w' = w''$ is a dummy vertex that splits the edge segment incident to w outside H_ν.

Let A^w be the set of distinct alias vertices of a pole w. Let P^{uv} be any simple path from u to v inside H_ν and let u' and v' be an alias vertex of u and an alias vertex of v, respectively. The path $S^{u'v'}$ obtained concatenating (u', u), P^{uv}, and (v, v') is called a *spine* of H_ν. Denote by $n(S^{u'v'})$ the number of right turns minus the number of left turns encountered along $S^{u'v'}$ while moving from u' to v'. The *spirality* $\sigma(H_\nu)$ of H_ν is introduced by Di Battista et al. [5] and it is defined based on the following cases (see also Fig. 2 for the spirality values of some P-components in the representation H of Fig. 1b):

- $A^u = \{u'\}$ and $A^v = \{v'\}$; then $\sigma(H_\nu) = n(S^{u'v'})$.
- $A^u = \{u'\}$ and $A^v = \{v', v''\}$; then $\sigma(H_\nu) = \frac{n(S^{u'v'})+n(S^{u'v''})}{2}$.
- $A^u = \{u', u''\}$ and $A^v = \{v'\}$; then $\sigma(H_\nu) = \frac{n(S^{u'v'})+n(S^{u''v'})}{2}$.
- $A^u = \{u', u''\}$ and $A^v = \{v', v''\}$; without loss of generality, assume that (u, u') precedes (u, u'') counterclockwise around u and that (v, v') precedes (v, v'') clockwise around v; then $\sigma(H_\nu) = \frac{n(S^{u'v'})+n(S^{u''v''})}{2}$.

Di Battista et al. [5] show that the spirality of H_ν does not vary with the choice of the path P^{uv}. For brevity, in the following we often denote by σ_ν the spirality of an orthogonal representation H_ν of G_ν. If ν is a Q*-node or a P-node with three children, σ_ν is always an integer. If ν is an S-node or a P-node with two children, σ_ν is either integer or semi-integer depending on whether the total number of alias vertices for the poles of ν is even or odd. When we say that the spirality σ_ν can take *all* values in an interval $[a, b]$, we mean that such values are either all the integer numbers or all the semi-integer numbers in $[a, b]$, depending on the cases described above for ν.

3 Substituting Orthogonal Components with the Same Spirality

Let G be a biconnected plane series–parallel graph and let T be an SPQ*-tree of G with respect to a given reference edge. Di Battista et al. [5] prove that two distinct orthogonal representations of the same component G_ν that have the same spirality are "interchangeable", under some additional hypotheses. Roughly speaking, they prove that if a component H_ν has a certain spirality in a given orthogonal representation H, it can be substituted with another representation H'_ν having the same spirality, under the assumption that the angles at the poles of ν that are outside G_ν do not change. In this subsection we formalize the concept of substituting an orthogonal component with another one and give a stronger version of the result in [5], which proves the interchangeability of two orthogonal components that have the same spirality, regardless of their angles at the poles.

Let H and H' be two different orthogonal representations of G with the reference edge on the external face, and let H_ν and H'_ν be the restrictions of H and H' to G_ν, respectively. If $\sigma(H_\nu) = \sigma(H'_\nu)$, the operation of *substituting* H_ν with H'_ν in H, denoted by $\mathrm{Sub}(H_\nu, H'_\nu)$, defines a new plane graph H'' with an angle labeling, such that: (a) H'' corresponds to a valid orthogonal representation of G; (b) the restriction of H'' to G_ν coincides with H'_ν; (c) the restriction of H'' to $G \setminus G_\nu$ stays as in H.

More formally, let u and v be the two poles of ν. The external boundary of H_ν contains a *left path* p_l and a *right path* p_r, such that p_l goes from u to v while traversing the external boundary of H_ν clockwise and p_r goes from u to v while traversing the external boundary of H_ν counterclockwise. Denote by f_l the face of H outside H_ν and incident to p_l, and denote by f_r the face of H outside H_ν and incident to p_r. Also, for each pole $w \in \{u, v\}$ of ν, denote by $a_{w,l}$ (resp. $a_{w,r}$) the angle at w in face f_l (resp. f_r) of H. Similarly, with respect to H'_ν and H', define p'_l, p'_r, f'_l, f'_r, and $a'_{w,l}$, $a'_{w,r}$ for each pole $w \in \{u, v\}$. The operation $\mathrm{Sub}(H_\nu, H'_\nu)$ defines H'' as follows (schematic illustrations are given in Figs. 3, 4, 5):

- The set of vertices and the set of edges of H'' are the same as in G.
- The planar embedding of H'' is such that: all faces of H outside H_ν and distinct from f_l and f_r, as well as all faces of H'_ν, are also faces of H''. Also, H'' has two faces f''_l and f''_r obtained by replacing p_l with p'_l and p_r with p'_r in the boundary of f_l and f_r, respectively.
- The angle labeling of H'' is such that: (i) all the angles at the vertices and along the edges of G not belonging to G_ν are those in H; (ii) all the angles at the vertices

of G_ν distinct from u and v are those in H'_ν; (iii) all the angles along the edges of G_ν are those in H'_ν; (iv) for each pole $w \in \{u, v\}$ of ν, the angles at w that are outside G_ν and that are neither in f''_l nor in f''_r are those in H; the angles at w that are inside G_ν are those in H'_ν; the angle $a''_{w,l}$ at w in f''_l and the angle $a''_{w,r}$ at w in f''_r are such that $a''_{w,l} = a_{w,l}$ and $a''_{w,r} = a_{w,r}$ if $\text{indeg}_\nu(w) = 1$, while $a''_{w,l} = a'_{w,l}$ and $a''_{w,r} = a'_{w,r}$ if $\text{indeg}_\nu(w) > 1$.

The next theorem proves that H'' is a valid orthogonal representation.

Theorem 1 *Let G be a biconnected series–parallel 4-graph, T be an SPQ*-tree of G with respect to a reference edge e, and ν be a non-root node of T. Let H and H' be two different orthogonal representations of G with e on the external face, and let H_ν and H'_ν be the restrictions of H and H' to G_ν, respectively. If $\sigma(H_\nu) = \sigma(H'_\nu)$ then the graph H'' defined by* $\text{Sub}(H_\nu, H'_\nu)$ *is an orthogonal representation of G.*

Proof We have to show that the embedded labeled graph H'' defined by $\text{Sub}(H_\nu, H'_\nu)$ satisfies Properties (H1) and (H2) of an orthogonal representation. Clearly, since H and H' are orthogonal representations, (H1) holds for all vertices of H'' distinct from the poles $\{u, v\}$ of G_ν; indeed, each vertex distinct from u and v inherits all its angles either from H or from H'. Analogously, each face of H'' distinct from f''_l and f''_r is either a face of H or a face of H', thus its angles satisfy Property (H2).

It remains to show that (H1) holds for u and v, and that (H2) holds for f''_l and f''_r. To this aim, we analyze different cases based on the indegree of the two poles $\{u, v\}$ of G_ν.

Case 1: $\text{indeg}_\nu(u) = 1$ **and** $\text{indeg}_\nu(v) = 1$. Refer to Fig. 3. In this case, the alias vertices u' and v', associated with u and v respectively, coincide with the poles, i.e., $u = u'$ and $v = v'$. Let $\overline{ux}$ and $\overline{yv}$ be the two edge segments of H_ν incident to u and to v, respectively. Analogously, let $\overline{ux'}$ and $\overline{y'v}$ be the two edge segments of H'_ν incident to u and to v, respectively. Without loss of generality, assume that H and H' are oriented in such a way that both $\overline{ux}$ and $\overline{ux'}$ are vertical segments and that u is below x in any drawing of H and u is below x' in any drawing of H'. By definition, since $u = u'$ and $v = v'$, the spirality $\sigma(H_\nu)$ (resp. $\sigma(H'_\nu)$) equals the number of right turns minus the number of left turns while moving from u to v along any simple path of H_ν (resp. of H'_ν). Hence, since by hypothesis $\sigma(H_\nu) = \sigma(H'_\nu)$, the edge segments $\overline{yv}$ and $\overline{y'v}$ are either both horizontal or both vertical, and more precisely they are incident to v in H and to v in H' from the same side (south, north, west, or east). In this case, $\text{Sub}(H_\nu, H'_\nu)$ defines $a''_{u,l} = a_{u,l}$, $a''_{u,r} = a_{u,r}$, $a''_{v,l} = a_{v,l}$, and $a''_{v,r} = a_{v,r}$, which implies that all the angles around u and v in H'' coincide with the angles around u and v in H. Hence, Property (H1) holds for both u and v in H''. Also, let $n(p_l)$ (resp. $n(p_r)$) be the number of right turns minus the number of left turns along p_l (resp. p_r) while moving from u to v in H. Similarly, let $n(p'_l)$ (resp. $n(p'_r)$) be the number of right turns minus the number of left turns along p'_l (resp. p'_r) while moving from u to v in H'. Since $\sigma(H_\nu) = \sigma(H'_\nu)$, and since $u = u'$ and $v = v'$, we have $n(p_l) = n(p'_l)$ and $n(p_r) = n(p'_r)$. It follows that $N_{90}(f''_l) - N_{270}(f''_l) = N_{90}(f_l) - N_{270}(f_l)$ and $N_{90}(f''_r) - N_{270}(f''_r) = N_{90}(f_r) - N_{270}(f_r)$, which imply Property (H2) for f''_l and f''_r (note that, since G is biconnected, $N_{360}(f) = 0$ for every face f of H and of H').

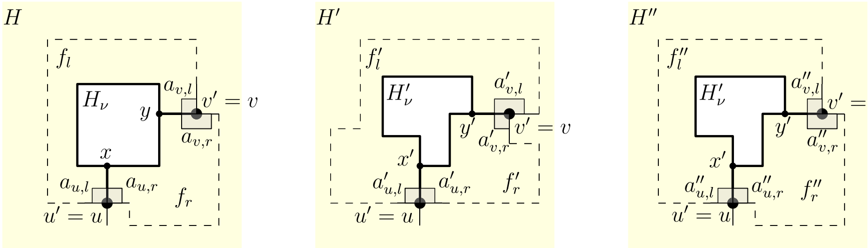

Fig. 3 Case 1 of Theorem 1: schematic illustration of the graph H'' defined by $\mathrm{Sub}(H_\nu, H'_\nu)$

Case 2: $\mathrm{indeg}_\nu(u) = 1$ **and** $\mathrm{indeg}_\nu(v) > 1$. We distinguish two subcases: $\mathrm{outdeg}_\nu(v) = 1$ or $\mathrm{outdeg}_\nu(v) = 2$. Assume first that $\mathrm{outdeg}_\nu(v) = 1$ (see Fig. 4a). In this case, u and its alias vertex u' coincide and the only alias vertex v' associated with v subdivides the edge segment incident to v in H and in H'. As in the analysis of Case 1, assume that H and H' are oriented so that each of the two edge segments $\overline{ux}$ and $\overline{ux'}$ in H and in H', respectively, is incident to u from north. By definition, the spirality $\sigma(H_\nu)$ (resp. $\sigma(H'_\nu)$) in this case equals the number of right turns minus the number of left turns along any simple path of H_ν (resp. of H'_ν) from u to v'. Since $\sigma(H_\nu) = \sigma(H'_\nu)$, this implies that the segments $\overline{vv'}$ in H and H' are incident to v from the same side. In this case, $\mathrm{Sub}(H_\nu, H'_\nu)$ defines $a''_{u,l} = a_{u,l}$, $a''_{u,r} = a_{u,r}$, $a''_{v,l} = a'_{v,l}$, and $a''_{v,r} = a'_{v,r}$, which implies that all the angles around u in H'' coincide with the angles around u and all the angles around v in H'' coincide with those around v in H'. Thus, Property (H1) holds for u and v in H''. It remains to prove (H2) for f''_l and f''_r. Denote by P_l (resp. P_r) the path of H obtained by concatenating p_l (resp. p_r) with the edge segment $\overline{vv'}$. Analogously, denote by P'_l (resp. P'_r) the path of H' obtained by concatenating p'_l (resp. p'_r) with the edge segment $\overline{vv'}$. Since $\sigma(H_\nu) = \sigma(H'_\nu)$, with the usual notation we have $n(P_l) = n(P'_l)$ and $n(P_r) = n(P'_r)$. Since, as observed above, the segments $\overline{vv'}$ in H and H' are incident to v from the same side, and since the angles at u are the same in H'' and in H, we have $N_{90}(f''_l) - N_{270}(f''_l) = N_{90}(f_l) - N_{270}(f_l)$ and $N_{90}(f''_r) - N_{270}(f''_r) = N_{90}(f_r) - N_{270}(f_r)$, which imply Property (H2) for f''_l and f''_r.

Suppose now that $\mathrm{outdeg}_\nu(v) = 2$ (see Fig. 4b). In this case, u and its alias vertex u' coincide while v has two alias vertices v' and v'', which subdivides the two edge segments incident to v in H and in H'. Since $\deg(v) = 4$, the angles at v are all $90°$ degree angles, both in H and in H'. It follows that, the angles at u and v in H'' are the same as in H, i.e., Property (H1) holds. Denote by P_l (resp. P_r) the path of H obtained by concatenating p_l (resp. p_r) with the edge segment $\overline{vv'}$ (resp. $\overline{vv''}$). Analogously, denote by P'_l (resp. P'_r) the path of H' obtained by concatenating p'_l (resp. p'_r) with the edge segment $\overline{vv'}$ (resp. $\overline{vv''}$). Since $\sigma(H_\nu) = \sigma(H'_\nu)$, we have $\frac{n(P_l)+n(P_r)}{2} = \frac{n(P'_l)+n(P'_r)}{2}$. On the other hand, since all angles at v are right angles in H and in H', we have $n(P_r) = n(P_l) + 1$ and $n(P'_r) = n(P'_l) + 1$. This implies that $n(P_l) = n(P'_l)$ and $n(P_r) = n(P'_r)$, which, together with the fact that the angles at u are the same in H and H'', implies that $N_{90}(f''_l) - N_{270}(f''_l) = N_{90}(f_l) - N_{270}(f_l)$

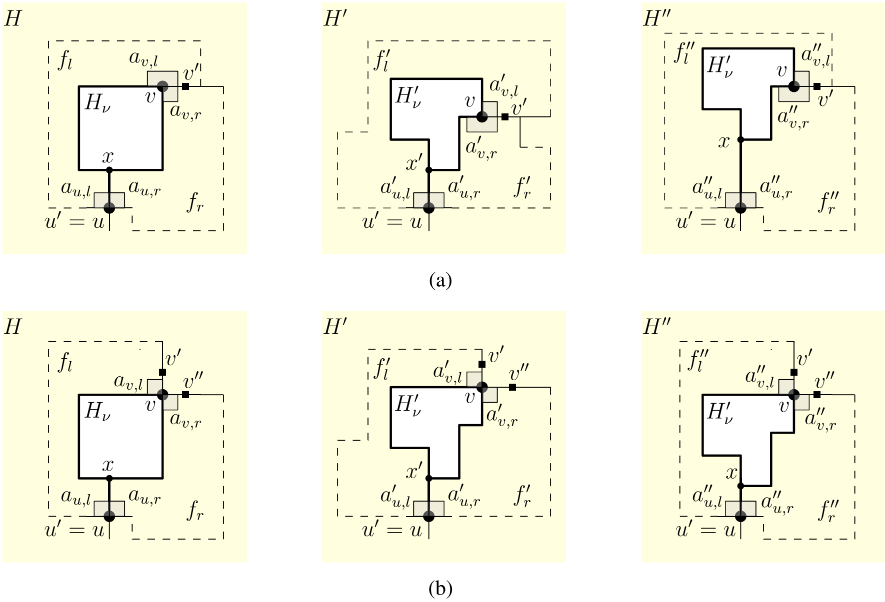

Fig. 4 Case 2 of Theorem 1: schematic illustration of the graph H'' defined by $\mathrm{Sub}(H_\nu, H'_\nu)$ when **a** $\mathrm{outdeg}(v) = 1$ and **b** $\mathrm{outdeg}(v) = 2$

and $N_{90}(f''_r) - N_{270}(f''_r) = N_{90}(f_r) - N_{270}(f_r)$. Hence, Property (H2) holds for f''_l and f''_r.
Case 3: $\mathrm{indeg}_\nu(u) > 1$ **and** $\mathrm{indeg}_\nu(v) = 1$. This case is symmetric to Case 2.
Case 4: $\mathrm{indeg}_\nu(u) > 1$ **and** $\mathrm{indeg}_\nu(v) > 1$. In this case, there are three non-symmetric subcases to analyze, depending on the outdegree of u and of v, i.e., $\mathrm{outdeg}_\nu(u) = \mathrm{outdeg}_\nu(v) = 1$, or $\mathrm{outdeg}_\nu(u) = 1$ and $\mathrm{outdeg}_\nu(v) = 2$ (symmetrically $\mathrm{outdeg}_\nu(u) = 2$ and $\mathrm{outdeg}_\nu(v) = 1$), or $\mathrm{outdeg}_\nu(u) = \mathrm{outdeg}_\nu(v) = 2$. See "Appendix A" for details and refer to Fig. 5 for a schematic illustration. □

Based on Theorem 1, in the following we can assume that two orthogonal components with the same spirality are equivalent, and we can describe the set of possible orthogonal representations for a component in terms of their spirality values.

4 Rectilinear Plane Series–Parallel Graphs

This section characterizes rectilinear plane series–parallel graphs. Let G be a plane series–parallel 4-graph. If G is biconnected let e be any edge on the external face of G; otherwise, we add a dummy edge e that makes it biconnected (recall that, if G is not biconnected we are assuming that the dummy edge e can always be added in the external face). Let T be the SPQ*-tree of G with respect to e and let ν be a node of T. We say that a component G_ν *admits spirality* σ_ν or, equivalently, that ν admits spirality σ_ν, if there exists a rectilinear planar representation H_ν of G_ν with spirality σ_ν in some rectilinear planar representation H of G. The following lemmas immediately

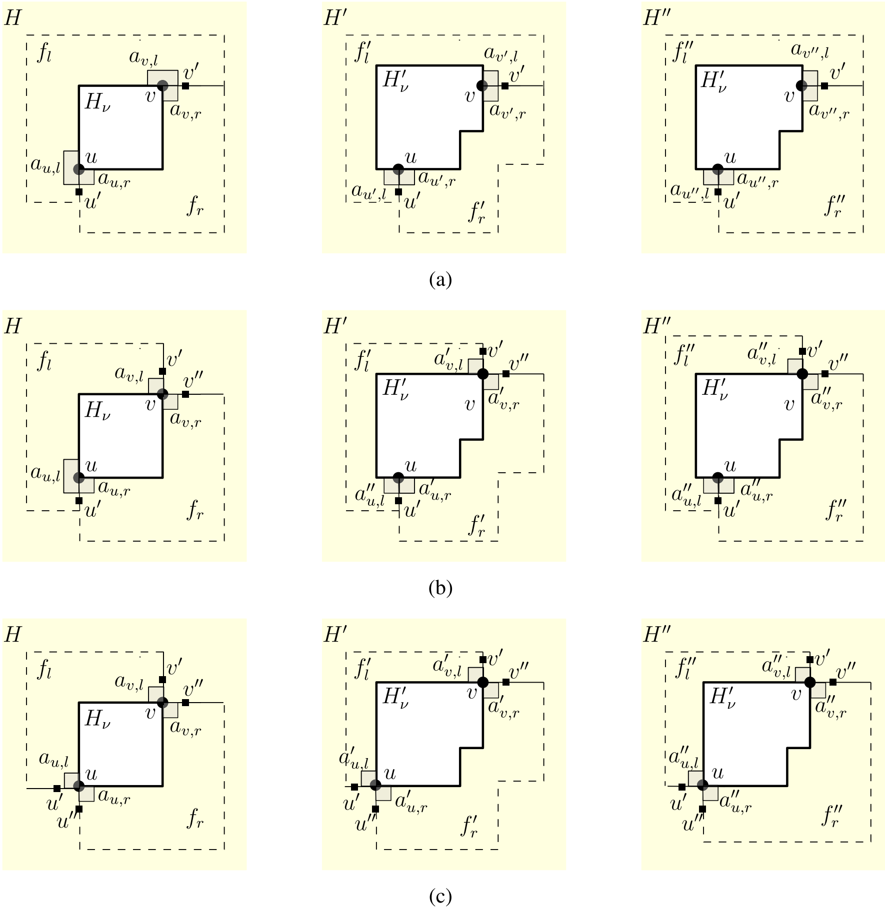

Fig. 5 Case 4 of Theorem 1: schematic illustration of the graph H'' defined by $\mathrm{Sub}(H_\nu, H'_\nu)$ when **a** $\mathrm{outdeg}(u) = \mathrm{outdeg}(v) = 1$, **b** $\mathrm{outdeg}(u) = 1$ and $\mathrm{outdeg}(v) = 2$, and **c** $\mathrm{outdeg}(u) = \mathrm{outdeg}(v) = 2$

derive from the results of Di Battista et al. [5], which for any S-node or P-node ν, relate the values of spirality for an orthogonal representation of G_ν to the values of spirality of the orthogonal representations of the child components of G_ν (i.e., the components corresponding to the children of ν). These types of relationships will be crucial to characterize those components that are rectilinear planar in a bottom-up traversal of T. Namely, Lemma 1 concentrates on S-nodes, Lemma 2 on P-nodes with three children, and Lemma 3 on P-nodes with two children. See also Fig. 6 for an illustration.

Lemma 1 [5] *Let ν be an S-node of T with children $\mu_1, \ldots, \mu_h$ ($h \geq 2$). The component G_ν admits spirality σ_ν if and only if $\sigma_\nu = \sum_{i=1}^{h} \sigma_{\mu_i}$, where σ_{μ_i} is a spirality value admitted by G_{μ_i} ($1 \leq i \leq h$).*

Lemma 2 [5] *Let ν be a P-node of T with three children μ_l, μ_c, and μ_r. G_ν admits spirality σ_ν with G_{μ_l}, G_{μ_c}, G_{μ_r} in this left-to-right order if and only if there exist*

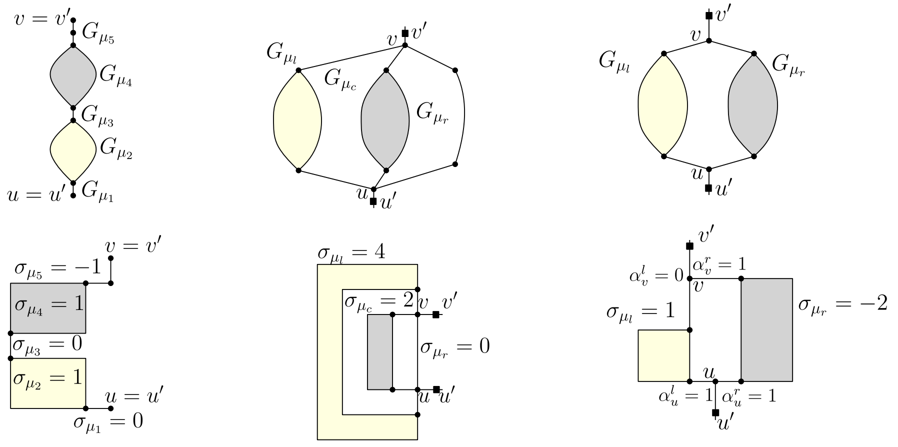

Fig. 6 Illustration of the relationships in: **a** Lemma 1 for S-nodes, **b** Lemma 2 for P-nodes with three children, and **c** Lemma 3 for P-nodes with two children

three values σ_{μ_l}, σ_{μ_c}, and σ_{μ_r} such that: (i) G_{μ_l}, G_{μ_c}, G_{μ_r} admit spirality σ_{μ_l}, σ_{μ_c}, σ_{μ_r}, respectively; and (ii) $\sigma_\nu = \sigma_{\mu_l} - 2 = \sigma_{\mu_c} = \sigma_{\mu_r} + 2$.

If ν is a P-node with two children, denote by μ_l and μ_r its left and right child in T, respectively. If ν is a P-node with three children, denote by μ_l, μ_c, and μ_r, the three children of ν from left to right. Also, for each pole $w \in \{u, v\}$ of ν, the *leftmost angle* at w in H is the angle formed by the leftmost external edge and the leftmost internal edge of H_ν incident to w. The *rightmost angle* at w in H is defined symmetrically. We define two binary variables α_w^l and α_w^r as follows: $\alpha_w^l = 0$ ($\alpha_w^r = 0$) if the leftmost (rightmost) angle at w in H is 180°, while $\alpha_w^l = 1$ ($\alpha_w^r = 1$) if this angle is 90°. Observe that if $\deg(w) = 4$ or if ν has three children, $\alpha_w^l = \alpha_w^r = 1$. Also, if ν has two children, define two additional variables k_w^l and k_w^r as follows: $k_w^d = 1$ if $\text{indeg}_{\mu_d}(w) = \text{outdeg}_\nu(w) = 1$, while $k_w^d = 1/2$ otherwise, for $d \in \{l, r\}$. For example, in Fig. 2 the component of ν_3 is such that $k_u^l = k_u^r = 1$, $k_v^l = k_v^r = \frac{1}{2}$, $\alpha_u^l = 0$, and $\alpha_u^r = \alpha_v^l = \alpha_v^r = 1$; the component of ν_5 is such that $k_u^l = k_u^r = 1$, $k_v^l = \frac{1}{2}$, $k_v^r = 1$, $\alpha_u^l = 0$, and $\alpha_u^r = \alpha_v^l = \alpha_v^r = 1$.

Lemma 3 [5] *Let ν be a P-node of T with two children μ_l and μ_r, and with poles u and v. G_ν admits spirality σ_ν with G_{μ_l} and G_{μ_r} in this left-to-right order if and only if there exist six values σ_{μ_l}, σ_{μ_r}, α_u^l, α_u^r, α_v^l, and α_v^r such that: (i) G_{μ_l} and G_{μ_r} admit spirality σ_{μ_l} and σ_{μ_r}, respectively; (ii) $\alpha_w^l \in \{0, 1\}$, $\alpha_w^r \in \{0, 1\}$, and $1 \le \alpha_w^l + \alpha_w^r \le 2$ for any $w \in \{u, v\}$; and (iii) $\sigma_\nu = \sigma_{\mu_l} - k_u^l\alpha_u^l - k_v^l\alpha_v^l = \sigma_{\mu_r} + k_u^r\alpha_u^r + k_v^r\alpha_v^r$.*

In the following we prove a condition under which the plane graph G_ν is rectilinear planar, assuming that its child components (if ν is not a leaf of T) are rectilinear planar. This condition depends on the type of node ν and is referred to as *representability condition* of ν (or, equivalently, of G_ν). Also, if the representability condition holds for ν, we denote by I_ν the set of values of spirality for which G_ν is rectilinear planar,

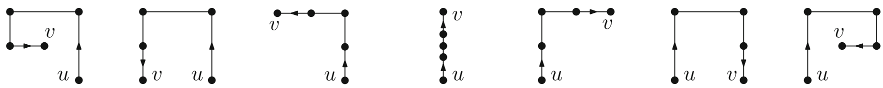

Fig. 7 Illustration of Lemma 4. For a Q^* node ν of length 4 we have $I_\nu = [-3, 3]$

i.e., G_ν admits spirality σ_ν if and only if $\sigma_\nu \in I_\nu$. We prove that I_ν is always an interval (of all integer or all semi-integer values) and call it the *representability interval* of ν (or, equivalently, of G_ν).

4.1 Representability Condition for Q*-Nodes and S-Nodes

For a Q*-node ν representing a chain of ℓ edges, we say that ℓ is the *length* of ν. As the next lemmas prove, the components of Q*- and S-nodes are always rectilinear planar, i.e., the representability condition is always true.

Lemma 4 *Let ν be a Q^*-node of length ℓ. Graph G_ν is always rectilinear planar (i.e., its representability condition is always true) and its representability interval is $I_\nu = [-\ell + 1, \ell - 1]$.*

Proof G_ν is a path with $\ell - 1$ degree-2 vertices. For any integer $k \in [-\ell + 1, 0]$, a rectilinear planar representation H_ν of G_ν with spirality k is obtained by making a left turn at k degree-2 vertices of G_ν (going from the source to the sink pole), and no turn at any remaining vertex of G_ν. Symmetrically, for any $k \in (0, \ell - 1]$, we realize H_ν with spirality k by making a right turn at exactly k degree-2 vertices of G_ν. It is clear that no values of spirality out of I_ν can be achieved. □

Figure 7 illustrates Lemma 4 for a Q*-node ν of length 4, for which $I_\nu = [-3, 3]$. The figure depicts a rectilinear planar representation of G_ν with spirality σ_ν for every $\sigma_\nu \in I_\nu$.

Lemma 5 *Let ν be an S-node with $h \geq 2$ children $\mu_1, \dots, \mu_h$. Suppose that, for every $i \in [1, h]$, the representability interval of G_{μ_i} is $I_{\mu_i} = [m_i, M_i]$. Graph G_ν is always rectilinear planar (i.e., its representability condition is always true) and its representability interval is $I_\nu = [\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} M_i]$.*

Proof We use induction on the number of children of ν. In the base case $h = 2$. By hypothesis $I_{\mu_1} = [m_1, M_1]$ and $I_{\mu_2} = [m_2, M_2]$. By Lemma 1, a series composition of a rectilinear representation of G_{μ_1} with spirality σ_{μ_1} and of a rectilinear representation of G_{μ_2} with spirality σ_{μ_2} results in a rectilinear representation of G_ν with spirality $\sigma_\nu = \sigma_{\mu_1} + \sigma_{\mu_2}$. Hence, if $M_1 = m_1 + r_1$ and $M_2 = m_2 + r_2$, for two non-negative integers r_1 and r_2, then the possible values for σ_ν are exactly $m_1 + m_2, m_1 + 1 + m_2, \dots, m_1 + r_1 + m_2, \dots, m_1 + r_1 + m_2 + 1, \dots, m_1 + r_1 + m_2 + r_2$, i.e., all values in the interval $[m_1 + m_2, M_1 + M_2]$. In the inductive case $h \geq 3$; consider the series composition G'_1 of $G_{\mu_1}, \dots, G_{\mu_{h-1}}$. Graph G_ν is the series composition of G'_1 and G_{μ_h}. By the inductive hypothesis the representability interval of G'_1 is $[\sum_{i=1}^{h-1} m_i, \sum_{i=1}^{h-1} M_i]$ and

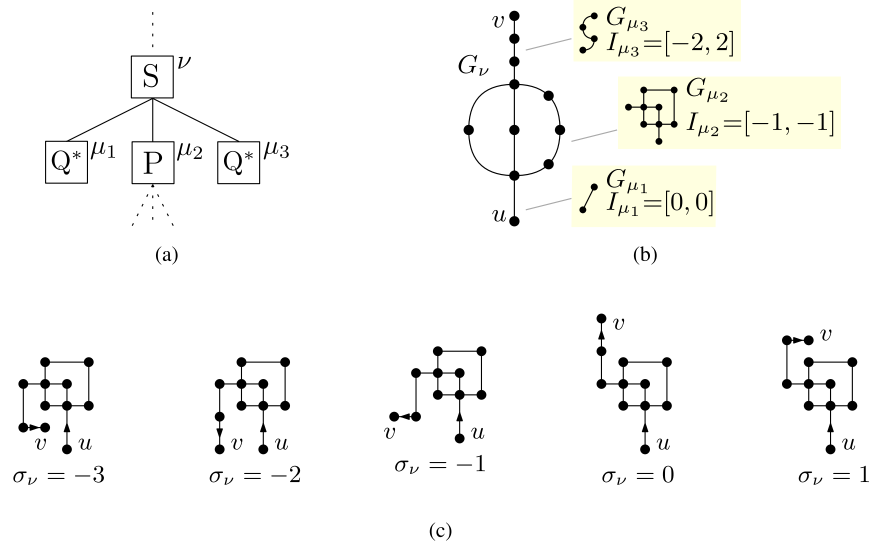

Fig. 8 Illustration of Lemma 5. **a** An S-node ν with children μ_1, μ_2, and μ_3. **b** The components G_ν, G_{μ_1}, G_{μ_2}, and G_{μ_3}. Since I_{μ_1}=[0, 0], I_{μ_2}=[−1, −1], and I_{μ_3}=[−3, 3], we have I_ν=$[\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} M_i]$=[−3, 1]. (c) A rectilinear planar representation of G_ν with spirality σ_ν for every $\sigma_\nu \in I_\nu$

by Lemma 1 applied to G'_1 and G_{μ_h} we have $I_\nu = [\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} M_i]$, using the same reasoning as for the base case. □

Figure 8 illustrates Lemma 5. Figure 8a shows an S-node ν and its three children μ_1, μ_2, and μ_3, where μ_1 and μ_3 are Q*-nodes and μ_2 is a P-node. Figure 8b shows the components G_ν, G_{μ_1}, G_{μ_2}, and G_{μ_3}, where: $I_{\mu_1} = [0, 0]$ and $I_{\mu_3} = [-2, 2]$ by Lemma 4; $I_{\mu_2} = [-1, -1]$, as G_{μ_2} only admits a rectilinear planar representation of spirality -1. By Lemma 5, $I_\nu = [\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} M_i] = [0 - 2 - 1, 0 + 2 - 1] = [-3, 1]$. Figure 8c depicts a rectilinear planar representation of G_ν with spirality σ_ν for every $\sigma_\nu \in I_\nu$.

4.2 Representability Condition for P-Nodes with Three Children

Different from S-nodes, if ν is a P-node and the pertinent graphs of the children of ν are rectilinear planar, G_ν may not be rectilinear planar. In this subsection we consider the case when ν has three children.

Lemma 6 *Let ν be a P-node with three children μ_l, μ_c, and μ_r, ordered from left to right. Suppose that G_{μ_l}, G_{μ_c}, and G_{μ_r} are rectilinear planar and that their representability intervals are $I_{\mu_l} = [m_l, M_l]$, $I_{\mu_c} = [m_c, M_c]$, and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] \neq \emptyset$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}]$.*

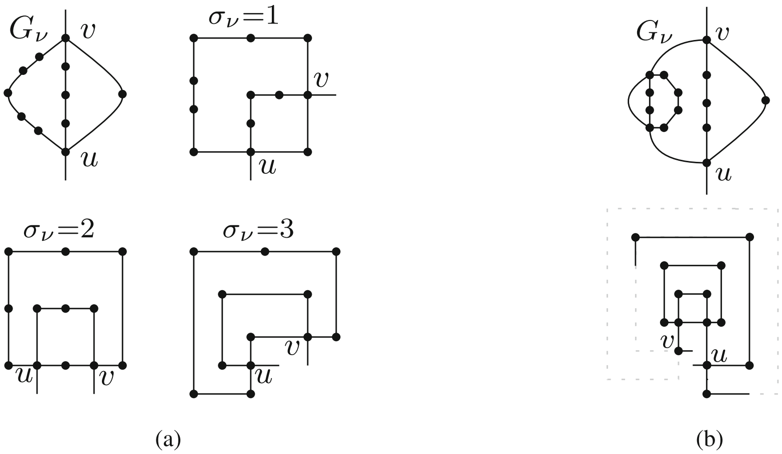

Fig. 9 Illustration of Lemma 6. **a** G_ν is rectilinear planar and $I_\nu = [1, 3]$, **b** G_ν is not rectilinear planar

Proof Representability condition. Suppose first that G_ν is rectilinear planar and let H_ν be a rectilinear planar representation of G_ν with spirality σ_ν. By Lemma 2, the spirality values σ_{μ_l}, σ_{μ_c}, and σ_{μ_r} for the representations of G_{μ_l}, G_{μ_c}, and G_{μ_r} in H_ν are such that $\sigma_{\mu_l} = \sigma_\nu + 2$, $\sigma_{\mu_c} = \sigma_\nu$, and $\sigma_{\mu_r} = \sigma_\nu - 2$. Since $\sigma_{\mu_l} \in [m_l, M_l]$, $\sigma_{\mu_c} \in [m_c, M_c]$, and $\sigma_{\mu_r} \in [m_r, M_r]$, we have $\sigma_\nu \in [m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2]$. Suppose vice versa that $[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] \neq \emptyset$, and let k be any value in such intersection. Setting $\sigma_{\mu_l} = k + 2$, $\sigma_{\mu_c} = k$, and $\sigma_{\mu_r} = k - 2$ we have $\sigma_{\mu_l} \in [m_l, M_l]$, $\sigma_{\mu_c} \in [m_c, M_c]$, and $\sigma_{\mu_r} \in [m_r, M_r]$. By Lemma 2, G_ν is rectilinear planar for a value of spirality $\sigma_\nu = k$.
Representability interval. Assume that G_ν is rectilinear planar. Clearly $[\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}] = [m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2]$, and by the truth of the feasiblity condition we have $[\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}] \neq \emptyset$. Similarly to the first part of the proof of the representability condition, any rectilinear planar representation of G_ν has a value of spirality in the interval $[\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}]$. On the other hand, let $k \in [\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}]$. Analogously to the second part of the proof of the representability condition, we can construct a rectilinear planar representation of G_ν with spirality $\sigma_\nu = k$, by combining in parallel rectilinear planar representations of G_{μ_l}, G_{μ_c}, and G_{μ_r} with spirality values $\sigma_{\mu_l} = \sigma_\nu + 2$, $\sigma_{\mu_c} = \sigma_\nu$, and $\sigma_{\mu_r} = \sigma_\nu - 2$. □

Figure 9 illustrates Lemma 6. In Fig. 9a, ν has three children that are Q*-nodes. By Lemma 4, $I_{\mu_l} = [-5, 5]$, $I_{\mu_c} = [-3, 3]$, and $I_{\mu_c} = [-1, 1]$. We have $[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] = [-7, 3] \cap [-3, 3] \cap [1, 3] = \{1, 2, 3\} \neq \emptyset$ and, consequently, G_ν is rectilinear planar. Also, $I_\nu = [\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}] = [1, 3]$. In Fig. 9b, the left child of ν is an S-node such that $I_{\mu_l} = [-2, -2]$. We have $[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] = [-4, -4] \cap [-3, 3] \cap [1, 3] = \emptyset$ and, consequently, G_ν is not rectilinear planar.

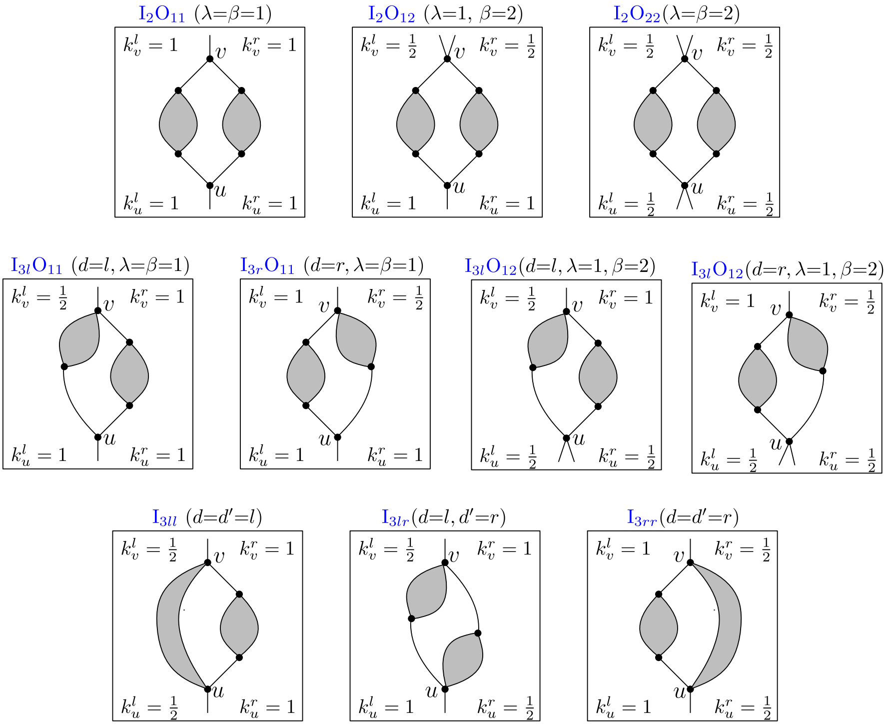

Fig. 10 Schematic illustration of the different types of P-nodes with two children

4.3 Representability Condition for P-Nodes with Two Children

For a P-node ν with two children μ_l and μ_r, the representability condition and interval depend on the different configurations of indegree and outdegree of the poles of ν in G_ν, G_{μ_l}, and G_{μ_r}. To distinguish the different configurations, we define the *type* of ν and of G_ν, using a self-contained notation; refer also to Fig. 10 for a schematic illustration. Each type is denoted using one or two symbols, I (indegree) and O (outdegree), along with suitable indices. More precisely, the possible types are defined as follows:

- $I_2O_{\lambda\beta}$: both poles of ν have indegree two in G_ν; also one pole has outdegree λ in G_ν and the other pole has outdegree β in G_ν, for $1 \le \lambda \le \beta \le 2$. This gives rise to the specific types I_2O_{11}, I_2O_{12}, and I_2O_{22}.
- $I_{3d}O_{\lambda\beta}$: one pole of ν has indegree two in G_ν, while the other pole has indegree three in G_ν and indegree two in G_{μ_d} for $d \in \{l, r\}$; also one pole has outdegree λ in G_ν and the other has outdegree β in G_ν, for $1 \le \lambda \le \beta \le 2$, where $\lambda = \beta = 2$ is not possible. This gives rise to the specific types $I_{3l}O_{11}$, $I_{3r}O_{11}$, $I_{3l}O_{12}$, $I_{3r}O_{12}$.
- $I_{3dd'}$: both poles of ν have indegree three in G_ν; one of the two poles has indegree two in G_{μ_d} and the other has indegree two in $G_{\mu_{d'}}$, for $dd' \in \{ll, lr, rr\}$ (both poles have outdegree one in G_ν). Hence, the specific types are I_{3ll}, I_{3lr}, I_{3rr}.

In the next three subsections, we analyze separately the properties of the different types of P-nodes with two children.

4.3.1 Nodes of Type $I_2O_{\lambda\beta}$

Lemma 8 states the representability condition and interval for P-nodes of type $I_2O_{\lambda\beta}$. Its proof is based on the preliminary property stated by Lemma 7.

Lemma 7 *Let G_ν be a P-node of type $I_2O_{\lambda\beta}$ with children μ_l and μ_r. G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4-\gamma]$, where $\gamma = \lambda + \beta - 2$.*

Proof We distinguish three cases, based on the values of λ and β, namely the cases I_2O_{11}, I_2O_{12}, and I_2O_{22}.

Case 1: $\lambda = \beta = 1$, i.e., G_ν is of type I_2O_{11}. We have to prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4]$. For a I_2O_{11} component we have $k_u^l = k_v^l = k_u^r = k_v^r = 1$.

If G_ν is rectilinear planar then $1 \le \alpha_u^l + \alpha_u^r \le 2$ and $1 \le \alpha_v^l + \alpha_v^r \le 2$ in any rectilinear planar representation of G_ν. Hence, by Lemma 3, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \alpha_v^l + \alpha_u^r + \alpha_v^r \in [2, 4]$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4]$. We define a rectilinear planar representation H_ν of G_ν by combining in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and by suitably assigning the values of α_u^d and α_v^d ($d \in \{l, r\}$), depending on the value of $\sigma_{\mu_l} - \sigma_{\mu_r}$. This assignment is such that for any cycle C of G_ν through u and v, the number of 90° angles minus the number of 270° angles in the interior of C is equal to four. Poles u and v split C into two paths π_l and π_r. The spirality σ_{μ_l} equals the number of right minus left turns along π_l while going from u to v, which corresponds to the number of 90° minus 270° angles in the interior of C at the vertices of π_l. Similarly, $-\sigma_{\mu_r}$ equals the number of right minus left turns along π_r while going from v to u, which corresponds to the number of 90° minus 270° angles in the interior of C at the vertices of π_r. By also considering the angles at u and v inside C, the number of 90° angles minus the number of 270° angles inside C can be expressed as $a_c = \sigma_{\mu_l} - \sigma_{\mu_r} + 4 - \alpha_u^l - \alpha_u^r - \alpha_v^l - \alpha_v^r$, and three cases are possible:

(i) if $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$, for every pole $w \in \{u, v\}$ we set α_w^l and α_w^r such that $\alpha_w^l + \alpha_w^r = 1$; (ii) if $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, for one pole $w \in \{u, v\}$ we set α_w^l and α_w^r such that $\alpha_w^l + \alpha_w^r = 1$, and for the other pole $w' \in \{u, v\}$ we set $\alpha_{w'}^l = \alpha_{w'}^r = 1$; (iii) if $\sigma_{\mu_l} - \sigma_{\mu_r} = 4$, for every pole $w \in \{u, v\}$ we set $\alpha_w^l = \alpha_w^r = 1$.

In all the cases above, we have that $a_c = 4$. Also, any other cycle not passing through u and v is an orthogonal polygon because it belongs to a rectilinear planar representation of either G_{μ_l} (with spirality σ_{μ_l}) or G_{μ_r} (with spirality σ_{μ_r}).

Case 2: $\lambda = 1$ and $\beta = 2$, i.e., G_ν is of type I_2O_{12}. We have to prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$. Suppose, w.l.o.g., that $\text{outdeg}_\nu(u) = 1$ and $\text{outdeg}_\nu(v) = 2$. We have $k_u^l = k_u^r = 1$ and $k_v^l = k_v^r = \frac{1}{2}$.

If G_ν is rectilinear planar then $\alpha_v^l + \alpha_v^r = 2$ and $\alpha_u^l + \alpha_u^r \in [1, 2]$. By Lemma 3, $\sigma_{\mu_l} - \sigma_{\mu_r} = k_u^l\alpha_u^l + k_v^r\alpha_v^l + k_u^l\alpha_u^r + k_v^r\alpha_v^r$, and hence $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \frac{1}{2}\alpha_v^l + \alpha_u^r + \frac{1}{2}\alpha_v^r \in [2, 3]$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar with spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$. As in the previous case, we define a rectilinear planar representation H_ν of G_ν, by combining in parallel the two representations of G_{μ_l} and G_{μ_r} and by suitably setting α_u^d and α_v^d ($d \in \{l, r\}$). Namely, we set $\alpha_v^l = \alpha_v^r = 1$, and the values of α_u^l and α_u^r as follows: (i) if $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$, we set α_u^l and α_u^r such that $\alpha_u^l + \alpha_u^r = 1$; (ii) if $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, we set $\alpha_u^l = \alpha_u^r = 1$. With an argument similar to the previous case, for any cycle C through u and v, the number of 90° angles minus the number of 270° angles in C can be expressed in this case by $a_c = \sigma_{\mu_l} - \sigma_{\mu_r} + 4 - \alpha_u^l - \alpha_u^r - 1$ (the angle at v in C is always of 90°). In case (i) we have $a_c = 2 + 4 - 1 - 1 = 4$; in case (ii) we have $a_c = 3 + 4 - 2 - 1 = 4$. Any other cycle not passing through u and v remains the same as in the representations of G_{μ_l} and G_{μ_r}.

Case 3: $\lambda = \beta = 2$, i.e., G_ν is of type I_2O_{22}. We prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. We have $k_u^l = k_u^r = \frac{1}{2}$.

If G_ν is rectilinear planar then $\alpha_u^l + \alpha_u^r = \alpha_v^l + \alpha_v^r = 2$. By Lemma 3, $\sigma_{\mu_l} = \sigma_\nu + 1$ and $\sigma_{\mu_r} = \sigma_\nu - 1$; hence $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$.

Suppose vice versa that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. Again, we obtain a rectilinear planar representation H_ν of G_ν by combining in parallel the representations of G_{μ_l} and G_{μ_r} and by suitably setting α_u^d and α_v^d ($d \in \{l, r\}$). In this case, for any cycle C through u and v, the number of 90° angles minus the number of 270° angles in C can be expressed by $a_c = \sigma_{\mu_l} - \sigma_{\mu_r} + 1 + 1$ (both the angles at u and v inside C is always of 90° degrees). We then set $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, which guarantees $a_c = 4$. Any other cycle not passing through u and v remains the same as in the representations of G_{μ_l} and G_{μ_r}. □

Lemma 8 *Let ν be a P-node of type $I_2O_{\lambda\beta}$ with children μ_l and μ_r. Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$, where $\gamma = \lambda + \beta - 2$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 2, m_r\} + \frac{\gamma}{2}, \min\{M_l, M_r + 2\} - \frac{\gamma}{2}]$.*

Proof We prove the correctness of the representability condition and the validity of the representability interval.

Representability condition. Suppose that G_ν is rectilinear planar. By Lemma 7, G_{μ_l} and G_{μ_r} admit spirality values σ_{μ_l} and σ_{μ_r}, respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 4 - \gamma]$. Hence, $m_l - M_r \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq 4 - \gamma$ and $M_l - m_r \geq \sigma_{\mu_l} - \sigma_{\mu_r} \geq 2$, i.e., $[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$.

Suppose, vice versa that $[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$. By hypothesis G_{μ_l} (resp. G_{μ_r}) is rectilinear planar for every integer value of spirality in the interval $[m_l, M_l]$ (resp. $[m_r, M_r]$). This implies that for every integer value k in the interval $[m_l - M_r, M_l - m_r]$, there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k$. Since by hypothesis there exists a value $k \in [m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma]$, there must be two spirality values σ_{μ_l} and σ_{μ_r} for the representations of G_{μ_l} and G_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k \in [2, 4 - \gamma]$. Hence, by Lemma 7, G_ν is rectilinear planar.

Representability interval.

As in the proof of Lemma 7, we distinguish three cases, based on the values of λ and β.

Case 1: $\lambda = \beta = 1$, i.e., G_ν is of type I_2O_{11}. We prove that $I_\nu = [\max\{m_l - 2, m_r\}, \min\{M_l, M_r + 2\}]$.

Assume first that σ_ν is the spirality of a rectilinear representation of G_ν. By Lemma 3, $\sigma_\nu \in [m_l - 2, M_r + 2]$. Also, since for a I_2O_{11} component $k_u^l = k_v^l = k_u^r = k_v^r = 1$, we have $\sigma_\nu = \sigma_{\mu_r} + \alpha_u^r + \alpha_v^r$, which implies $\sigma_\nu \geq m_r$. Analogously, $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \alpha_v^l \leq M_l$. Hence, $\sigma_\nu \in I_\nu = [\max\{m_l - 2, m_r\}, \min\{M_l, M_r + 2\}]$.

Assume vice versa that k is any integer in the interval $I_\nu = [\max\{m_l - 2, m_r\}, M = \min\{M_l, M_r + 2\}]$. We show that G_ν admits a rectilinear planar representation with spirality $\sigma_\nu = k$. By hypothesis $k \leq \min\{M_l, M_r + 2\} \leq M_l$; also, $k \geq \max\{m_l - 2, m_r\} \geq m_l - 2$, i.e., $k + 2 \geq m_l$. Hence $[k, k + 2] \cap [m_l, M_l] \neq \emptyset$. Analogously, $k \leq \min\{M_l, M_r + 2\} \leq M_r + 2$, i.e., $k - 2 \leq M_r$; also, $k \geq \max\{m_l - 2, m_r\} \geq m_r$. Hence $[k - 2, k] \cap [m_r, M_r] \neq \emptyset$. We now distinguish the following subcases:

- **Case 1.1**: $k \leq M_l - 2$. Consider any two rectilinear planar representations H_{μ_l} of G_{μ_l} and H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_l} = k+2$ and $\sigma_{\mu_r} \in [k-2, k] \cap [m_r, M_r] \neq \emptyset$, respectively. As already observed, $k + 2 \geq m_l$ and by hypothesis $k + 2 \leq M_l$; hence $\sigma_{\mu_l} \in [m_l, M_l]$. With this choice we have $2 \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq 4$, and we can combine H_{μ_l} and H_{μ_r} in parallel as in the proof of Lemma 7 to obtain a rectilinear planar representation H_ν of G_ν. By Lemma 3 the spirality of H_ν equals $\sigma_{\mu_l} - \alpha_l^u - \alpha_l^v = k + 2 - \alpha_l^u - \alpha_l^v$ and it suffices to set $\alpha_l^u = \alpha_l^v = 1$ (which is always possible, as these two values correspond to 90° angles) to get $\sigma_\nu = k$.
- **Case 1.2**: $k = M_l - 1$. Consider any rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = k + 1 = M_l$. To suitably choose the spirality of a rectilinear planar representation H_{μ_r} of G_{μ_r}, observe that by the representability condition $M_l - 2 \geq m_r$ and, as already proved, $M_r \geq k - 2$, i.e., $M_r \geq M_l - 3$. It follows that $[M_l - 3, M_l - 2] \cap [m_r, M_r] \neq \emptyset$. Hence, either $M_l - 3 \in [m_r, M_r]$ (possibly $m_r = M_r = M_l - 3$) or $M_l - 2 \in [m_r, M_r]$ (possibly $m_r = M_r = M_l - 2$). In the first case, choose any representation H_{μ_r} with spirality $\sigma_r = M_l - 3$, which implies $\sigma_{\mu_l} - \sigma_{\mu_r} = 3 \in [2, 4]$. In the second case, choose H_{μ_r} with spirality $\sigma_r = M_l - 2$, which implies $\sigma_{\mu_l} - \sigma_{\mu_r} = 2 \in [2, 4]$. H_{μ_l} and H_{μ_r} can be combined in parallel to get a representation of G_ν with spirality $\sigma_\nu = k$. Namely, by Lemma 3 we can set $\alpha_u^l = 0$ and $\alpha_v^l = 1$ (or vice versa); also, if $\sigma_{\mu_r} = M_l - 2$ we set $\alpha_u^r = 0$ and $\alpha_v^l = 1$ (or vice versa), while if $\sigma_{\mu_r} = M_l - 3$ we set $\alpha_u^r = \alpha_v^l = 1$.
- **Case 1.3**: $k = M_l$. We can combine in parallel a representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = k = M_l$ and a representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = k - 2 = M_l - 2$, which implies that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. By the representability condition we have $M_l - 2 \geq m_r$, i.e., $\sigma_{\mu_r} \geq m_r$; also, $k \leq \min\{M_l, M_r + 2\} \leq M_r + 2$, i.e., $\sigma_{\mu_r} \leq M_r$. Hence, $\sigma_{\mu_r} \in [m_r, M_r]$. By Lemma 3 we can set $\alpha_u^l = \alpha_v^l = 0$ and $\alpha_u^r = \alpha_v^l = 1$ to get a representation of G_ν with spirality $\sigma_\nu = k$.

Case 2: $\lambda = 1$ and $\beta = 2$, i.e., G_ν is of type I_2O_{12}. The arguments are similar to the previous case; see "Appendix B" for details.

Case 3: $\lambda = \beta = 2$, i.e., G_ν is of type I_2O_{22}. We prove that $I_\nu = [\max\{m_l - 2, m_r\} + 1, \min\{M_l, M_r + 2\} - 1]$.

Assume first that G_ν is rectilinear planar and let H_ν be a rectilinear planar representation of G_ν with spirality σ_ν. Let H_{μ_l} and H_{μ_r} be the rectilinear planar representations of G_{μ_l} and G_{μ_r} contained in H_ν, and let σ_{μ_l} and σ_{μ_r} be their spirality values. Since both u and v have outdegree two in G_ν we have that $\alpha_u^l + \alpha_u^r = \alpha_v^l + \alpha_v^r = 2$. By Lemma 3, $\sigma_{\mu_l} = \sigma_\nu + 1$ and $\sigma_{\mu_r} = \sigma_\nu - 1$. By the representability condition, $\sigma_{\mu_r} = \sigma_{\mu_l} - 2$. Hence $\sigma_{\mu_r} \geq m_l - 2$ and $\sigma_{\mu_r} \geq \max\{m_l - 2, m_r\}$. Also by $\sigma_\nu = \sigma_{\mu_r} + 1$, $\sigma_\nu \geq \max\{m_l - 2, m_r\} + 1$. Similarly, by the representability condition, $\sigma_{\mu_l} = \sigma_{\mu_r} + 2$. Hence $\sigma_{\mu_l} \leq M_r + 2$ and $\sigma_{\mu_l} \leq \max\{M_l, M_r + 2\}$. Since $\sigma_{\mu_l} = \sigma_\nu + 1$ we have $\sigma_\nu \leq \max\{M_l, M_r + 2\} - 1$.

Assume vice versa that k is an integer in the interval $I_\nu = [\max\{m_l - 2, m_r\} + 1, \min\{M_l, M_r + 2\} - 1]$. We show that there exists a rectilinear planar representation of G_ν with spirality $\sigma_\nu = k$. We have $k + 1 \in [\max\{m_l, m_r + 2\}, \min\{M_l, M_r + 2\}]$ and therefore $k + 1 \in [m_l, M_l]$. Hence there exists a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = k + 1$. Similarly, we have $k - 1 \in [\max\{m_l - 2, m_r\}, \min\{M_l - 2, M_r\}]$ and therefore $k - 1 \in [m_r, M_r]$. Hence there exists a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = k - 1$. By the representability condition G_ν has a rectilinear planar representation H_ν; also, following the same construction as in the proof of Lemma 7, the spirality of H_ν is $\sigma_\nu = k$. □

4.3.2 Nodes of Type $I_{3d}O_{\lambda\beta}$

Lemma 10 states the representability condition and interval for P-nodes of type $I_2O_{\lambda\beta}$. Its proof is based on the preliminary property stated by Lemma 9.

Lemma 9 *Let G_ν be a P-node of type $I_{3d}O_{\lambda\beta}$ and let μ_l and μ_r be its two children. G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r}, respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2} - \gamma]$, where $\gamma = \lambda + \beta - 2$.*

Proof We distinguish four cases, based on the values of λ, β, and d.
Case 1: $\lambda = \beta = 1$ and $d = l$, i.e., G_ν is of type $I_{3l}O_{11}$. We prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2}]$. For an $I_{3l}O_{11}$ component we have $k_u^l = k_u^r = k_v^r = 1$ and $k_v^l = \frac{1}{2}$.

If G_ν is rectilinear planar then $1 \leq \alpha_u^l + \alpha_u^r \leq 2$ and $\alpha_v^l = \alpha_v^r = 1$ in any rectilinear planar representation of G_ν. Hence, by Lemma 3, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \frac{1}{2}\alpha_v^l + \alpha_u^r + \alpha_v^r \in [\frac{5}{2}, \frac{7}{2}]$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2}]$. We define a rectilinear planar representation H_ν of G_ν, by combining in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and by suitably assigning the values of α_u^l and α_u^r, depending on the value of $\sigma_{\mu_l} - \sigma_{\mu_r}$.

Let v' be the alias vertex of G_{μ_l} that is in G_ν. Any cycle C that goes through u and v also passes through v'. We show that the number of 90° angles minus the number of 270° angles inside C is equal to four.

Vertices u and v' split C into two paths π_l and π_r. Suppose to visit C clockwise. The number of right minus left turns along π_l while going from u to v' equals $\sigma_{\mu_l} + \frac{1}{2}$. The number of right minus left turns along π_r while going from v' to u equals $-\sigma_{\mu_r}$. Hence, the sum $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 2 - \alpha_u^r - \alpha_u^l$ corresponds to the number of 90° angles minus the number of 270° angles inside C at the vertices of π_l. Notice that $\alpha_u^r + \alpha_u^l \in \{1, 2\}$ since u is a vertex of degree 3. If $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$ we set $\alpha_u^r + \alpha_u^l = 1$ and we have $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 2 - \alpha_u^r - \alpha_u^l = \frac{5}{2} + \frac{1}{2} + 2 - 1 = 4$. Else, if $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{7}{2}$ we set $\alpha_u^r + \alpha_u^l = 2$ and we have $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 2 - \alpha_u^r - \alpha_u^l = \frac{7}{2} + \frac{1}{2} + 2 - 2 = 4$.

Any other cycle not passing through u and v remains the same as in the representations of either G_{μ_l} (with spirality σ_{μ_l}) and G_{μ_r} (with spirality σ_{μ_r}).

Case 2: $\lambda = 1$, $\beta = 2$, and $d = l$, i.e., G_ν is of type $\mathrm{I}_{3l}\mathrm{O}_{12}$. We prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$ (note that this corresponds to the interval $[\frac{5}{2}, \frac{7}{2} - \gamma]$ claimed in the lemma). For an $\mathrm{I}_{3l}\mathrm{O}_{12}$ component, $k_u^l = k_u^r = k_v^l = \frac{1}{2}$ and $k_v^r = 1$.

If G_ν is rectilinear planar then $\alpha_u^l = \alpha_u^r = \alpha_v^l = \alpha_v^r = 1$ in any rectilinear planar representation of G_ν. Hence, by Lemma 3, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{1}{2}\alpha_u^l + \frac{1}{2}\alpha_v^l + \frac{1}{2}\alpha_u^r + \alpha_v^r = \frac{5}{2}$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$. As usual, we define H_ν by combining in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} and we assign values $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$.

Let v' be the alias vertex of G_{μ_l} that is in G_ν. Any cycle C that goes through u and v also passes through v'.

Vertices u and v' split C into two paths π_l and π_r. Suppose to visit C clockwise. The number of right turns minus left turns along π_l while going from u to v' equals $\sigma_{\mu_l} + \frac{1}{2}$. The number of right turns minus left turns along π_r while going from v' to u equals $-\sigma_{\mu_r}$. Also, pole u forms a 90° angle in C. Hence, the sum $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 1$ corresponds to the number of 90° angles minus the number of 270° angles in C at the vertices of π_l. Since $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$ we have $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + 1 = \frac{5}{2} + \frac{1}{2} + 1 = 4$.

Any other cycle not passing through u and v remains the same as in the rectilinear representations of G_{μ_l} (with spirality σ_{μ_l}) and G_{μ_r} (with spirality σ_{μ_r}).

Case 3: $\lambda = \beta = 1$ and $d = r$. Symmetric to Case 1.

Case 4: $\lambda = 1$, $\beta = 2$, and $d = r$. Symmetric to Case 2. □

Lemma 10 *Let ν be a P-node of type $\mathrm{I}_{3d}\mathrm{O}_{\lambda\beta}$ with children μ_l and μ_r. Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$, where $\gamma = \lambda + \beta - 2$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma - \rho(d)}{2}, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma + \rho(d)}{2}]$, where $\phi(\cdot)$ is a function such that $\phi(r) = 1$ and $\phi(l) = 0$.*

Proof We prove the correctness of the representability condition and the validity of the representability interval.

Representability condition. Suppose that G_ν is rectilinear planar. By Lemma 9, G_{μ_l} and G_{μ_r} admit spirality values σ_{μ_l} and σ_{μ_r}, respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2} - \gamma]$, where $\gamma = \lambda + \beta - 2$. Hence, $m_l - M_r \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq \frac{7}{2} - \gamma$ and $M_l - m_r \geq \sigma_{\mu_l} - \sigma_{\mu_r} \geq \frac{5}{2}$, i.e., $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$.

Suppose, vice versa that $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$. By hypothesis G_{μ_l} (resp. G_{μ_r}) is rectilinear planar for every value of spirality in the interval $[m_l, M_l]$ (resp. $[m_r, M_r]$). This implies that for every semi-integer value k in the interval $[m_l - M_r, M_l - m_r]$, there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k$. Since by hypothesis there exists a value $k \in [m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma]$, there must be two values of spirality values σ_{μ_l} and σ_{μ_r} for the representations of G_{μ_l} and G_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = k \in [\frac{5}{2}, \frac{7}{2} - \gamma]$. Hence, by Lemma 9, G_ν is rectilinear planar.

Representability interval. As for Lemma 9, the case analysis is based on the values of λ, β, and d; w.l.o.g. we assume that v is the pole of degree four.

Case 1: $\lambda = \beta = 1$, and $d = l$, i.e., G_ν is of type $\mathrm{I3}_l\mathrm{O11}$. We prove that $I_\nu = [\max\{m_l - \frac{3}{2}, m_r + 1\}, \min\{M_l - \frac{1}{2}, M_r + 2\}]$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν. Since for an $\mathrm{I3}_l\mathrm{O11}$ component we have $k_u^l = k_u^r = k_v^r = 1$ and $k_v^l = \frac{1}{2}$, by Lemma 3 we have $\sigma_\nu = \sigma_{\mu_r} + \alpha_u^r + \alpha_v^r$ and $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $\alpha_u^l + \alpha_u^r \in \{1, 2\}$ and $\alpha_v^l = \alpha_v^r = 1$, we have $\sigma_\nu \geq m_r + 1$, $\sigma_\nu \leq M_r + 2$, $\sigma_\nu \geq m_l - \frac{3}{2}$, and $\sigma_\nu \leq M_l - \frac{1}{2}$.

Conversely, we show that if $\sigma_\nu \in [\max\{m_l - \frac{3}{2}, m_r + 1\}, \min\{M_l - \frac{1}{2}, M_r + 2\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν. We have $\sigma_\nu \in [m_l - \frac{3}{2}, M_l - \frac{1}{2}]$. Hence, $\sigma_\nu + \frac{1}{2} \leq M_l$ and $\sigma_\nu + \frac{3}{2} \geq m_l$, i.e., $[\sigma_\nu + \frac{1}{2}, \sigma_\nu + \frac{3}{2}] \cap [m_l, M_l] \neq \emptyset$. Also, since m_l and M_l are both semi-integer numbers while σ_ν is integer, it is impossible to have $\sigma_\nu + 1 = m_l = M_l$. It follows that $\sigma_\nu + \frac{1}{2} \in [m_l, M_l]$ or $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$. With the same reasoning, we have $\sigma_\nu \in [m_r + 1, M_r + 2]$ and $[\sigma_\nu - 2, \sigma_\nu - 1] \cap [m_r, M_r] \neq \emptyset$. Hence, $\sigma_\nu - 2 \in [m_r, M_r]$ or $\sigma_\nu - 1 \in [m_r, M_r]$. We now observe that either $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ or $\sigma_\nu - 2 \in [m_r, M_r]$. Indeed, if it were $\sigma_\nu + \frac{3}{2} \notin [m_l, M_l]$ and $\sigma_\nu - 2 \notin [m_r, M_r]$, then $\sigma_\nu + \frac{1}{2} \in [m_l, M_l]$ and $\sigma_\nu - 1 \in [m_r, M_r]$ and consequently, $\sigma_\nu + \frac{1}{2} = M_l$ and $\sigma_\nu - 1 = m_r$; hence, it would be $M_l - m_r = \frac{3}{2}$ and, by the representability condition, G_ν would not be rectilinear planar.

Based on this observation, we can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality σ_{μ_l} and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality σ_{μ_r}, by distinguishing the following cases, one of which must be verified:

- **Case (a):** $\sigma_\nu + \frac{3}{2} \notin [m_l, M_l]$. This implies that $\sigma_\nu + \frac{1}{2} \in [m_l, M_l]$ and $\sigma_\nu - 2 \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = \sigma_\nu + \frac{1}{2}$ and $\sigma_{\mu_r} = \sigma_\nu - 2$.
- **Case (b):** $\sigma_\nu - 2 \notin [m_r, M_r]$. This implies that $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ and $\sigma_\nu - 1 \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = \sigma_\nu + \frac{3}{2}$ and $\sigma_{\mu_r} = \sigma_\nu - 1$.
- **Case (c):** $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ and $\sigma_\nu - 2 \in [m_r, M_r]$. We set $\sigma_{\mu_l} = \sigma_\nu + \frac{3}{2}$ and $\sigma_{\mu_r} = \sigma_\nu - 2$.

In all the three cases we have $\sigma_{\mu_l} - \sigma_{\mu_r} \in [\frac{5}{2}, \frac{7}{2}]$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r}. We have to prove that

in the three cases the spirality of H_ν is σ_ν. By Lemma 3 we have $\sigma'_\nu = \sigma_{\mu_l} - \alpha^l_u - \frac{1}{2}\alpha^l_v$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l}, α^l_v, and α^r_v. In Case (a) we have $\sigma'_\nu = \sigma_\nu + \frac{1}{2} - \alpha^l_u - \frac{1}{2}\alpha^l_v$; choosing $\alpha^l_u = 0$ and $\alpha^l_v = 1$ we have $\sigma'_\nu = \sigma_\nu$. In Cases (b) and (c) we have $\sigma'_\nu = \sigma_\nu + \frac{3}{2} - \alpha^l_u - \frac{1}{2}\alpha^l_v$; choosing $\alpha^l_u = 1$ and $\alpha^l_v = 1$ we have $\sigma'_\nu = \sigma_\nu$.
Case 2: $\lambda = 1$, $\beta = 2$, and $d = l$, i.e., G_ν is of type $I_{3l}O_{12}$. We prove that $I_\nu = [\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{1}{2}, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{1}{2}] = [\max\{m_l - 1, m_r + \frac{3}{2}\}, \min\{M_l - 1, M_r + \frac{3}{2}\}$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν. Since for an $I_{3l}O_{12}$ component we have $k^l_u = k^r_u = k^l_v = \frac{1}{2}$ and $k^r_v = 1$, by Lemma 3 we have $\sigma_\nu = \sigma_{\mu_r} + \frac{1}{2}\alpha^r_u + \alpha^r_v$ and $\sigma_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha^l_u - \frac{1}{2}\alpha^l_v$. Since $\alpha^l_u = \alpha^l_v = \alpha^r_u = \alpha^r_v = 1$, we have: $\sigma_\nu \geq m_r + \frac{3}{2}$, $\sigma_\nu \leq M_r + \frac{3}{2}$, $\sigma_\nu \geq m_l - 1$, and $\sigma_\nu \leq M_l - 1$.

Conversely, we show that if $\sigma_\nu \in [\max\{m_l - 1, m_r + \frac{3}{2}\}, \min\{M_l - 1, M_r + \frac{3}{2}\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν. We have $\sigma_\nu \in [m_l - 1, M_l - 1]$ and $\sigma_\nu \in [m_r + \frac{3}{2}, M_r + \frac{3}{2}]$. Hence, $\sigma_\nu + 1 \in [m_l, M_l]$ and $\sigma_\nu - \frac{3}{2} \in [m_r, M_r]$. We can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = \sigma_\nu + 1$ and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = \sigma_\nu - \frac{3}{2}$. Notice that, for this choice, we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{5}{2}$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r}. We have to prove that the spirality of H_ν is σ_ν. By Lemma 3 we have $\sigma'_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha^l_u - \frac{1}{2}\alpha^l_v$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l}, α^l_v, and α^r_v. Since $\sigma_\nu = \sigma_{\mu_l} - 1$, $\alpha^l_v = 1$, and $\alpha^r_v = 1$, we have $\sigma'_\nu = \sigma_\nu + 1 - 1 = \sigma_\nu$.
Case 3: $\lambda = \beta = 1$ and $d = r$. Symmetric to Case 1.
Case 4: $\lambda = 1$, $\beta = 2$, and $d = r$. Symmetric to Case 2. □

4.3.3 Nodes of Type $I_{3dd'}$

Lemma 12 states the representability condition and interval for P-nodes of type $I_2O_{\lambda\beta}$. Its proof is based on the preliminary property stated by Lemma 11.

Lemma 11 *Let G_ν be a P-node of type $I_{3dd'}$ and let μ_l and μ_r be its two children. G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r}, respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$.*

Proof We distinguish three cases, based on the values of d and d'. The proof for type I_{3rl} is symmetric to the one for type I_{3lr}.
Case 1: $d = d' = l$, i.e., G_ν is of type I_{3ll}. We prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. For an I_{3ll} component we have $k^l_u = k^l_v = \frac{1}{2}$ and $k^r_u = k^r_v = 1$.

If G_ν is rectilinear planar, we have $\alpha^l_u = \alpha^r_u = \alpha^l_v = \alpha^r_v = 1$ in any rectilinear planar representation of G_ν. Hence, by Lemma 3, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \frac{1}{2}\alpha^l_u + \frac{1}{2}\alpha^l_v + \alpha^r_u + \alpha^r_v = 3$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. We define a rectilinear planar representation H_ν

of G_ν, by combining in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r} with values $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$.

Let u' and v' be the alias vertices of G_{μ_l} that subdivide edges of G_ν. Any cycle C through u and v also passes through u' and v'.

Vertices u' and v' split C into two paths π_l and π_r. Suppose to visit C clockwise. The number of right turns minus left turns along π_l while going from u' to v' equals $\sigma_{\mu_l} + 1$. The number of right turns minus left turns along π_r while going from v' to u' equals $-\sigma_{\mu_r}$. The sum of these two values corresponds to the number of 90° angles minus the number of 270° angles in the interior of C at the vertices of π_l; we have $\sigma_{\mu_l} + 1 - \sigma_{\mu_r} = 3 + 1 = 4$. All other cycles not passing through u and v are orthogonal polygons as they remain the same as in G_{μ_l} (with spirality σ_{μ_l}) and G_{μ_r} (with spirality σ_{μ_r}).

Case 2: $d = d' = r$. Symmetric to Case 1, observing that $k_u^r = k_v^r = \frac{1}{2}$ and $k_u^l = k_v^l = 1$.

Case 3: $d = l$ and $d' = r$, i.e., G_ν is of type I_{3lr}. We prove that G_ν is rectilinear planar if and only if G_{μ_l} and G_{μ_r} are rectilinear planar for spirality values σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. For an I_{3lr} component we have $k_u^r = k_v^l = \frac{1}{2}$ and $k_u^l = k_v^r = 1$.

If G_ν is rectilinear planar, we have $\alpha_u^l = \alpha_u^r = \alpha_v^l = \alpha_v^r = 1$ in any rectilinear planar representation of G_ν. Hence, by Lemma 3, for any value of spirality σ_ν we have $\sigma_{\mu_l} - \sigma_{\mu_r} = \alpha_u^l + \frac{1}{2}\alpha_v^l + \frac{1}{2}\alpha_u^r + \alpha_v^r = 3$.

Suppose vice versa that G_{μ_l} and G_{μ_r} are rectilinear planar for values of spirality σ_{μ_l} and σ_{μ_r} such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. We define H_ν by combining in parallel the two rectilinear planar representations of G_{μ_l} and G_{μ_r}, with values $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$.

Let v' be the alias vertex of the pole v of G_{μ_l} such that v' subdivides an edge of G_ν. Similarly, let u' be the alias vertex of the pole u of G_{μ_r} such that u' subdivides an edge of G_ν. Any cycle C through u and v also passes through u' and v'.

Vertices u' and v' split C into two paths π_l and π_r. Visiting C clockwise, the number of right minus left turns along π_l while going from u' to v' equals $\sigma_{\mu_l} + \frac{1}{2}$. The number of right minus left turns along π_r while going from v' to u' equals $-\sigma_{\mu_r} + \frac{1}{2}$. The sum of these two values corresponds to the number of 90° angles minus the number of 270° angles in the interior of C at the vertices of π_l, and we have $\sigma_{\mu_l} + \frac{1}{2} - \sigma_{\mu_r} + \frac{1}{2} = 3 + \frac{1}{2} + \frac{1}{2} = 4$.

□

Lemma 12 *Let ν be a P-node of type $\mathrm{I}_{3dd'}$ with children μ_l and μ_r. Suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. Graph G_ν is rectilinear planar if and only if $3 \in [m_l - M_r, M_l - m_r]$. Also, if this representability condition holds then the representability interval of G_ν is $I_\nu = [\max\{m_l - 1, m_r + 2\} - \frac{\phi(d)+\phi(d')}{2}, \min\{M_l - 1, M_r + 2\} - \frac{\phi(d)+\phi(d')}{2}]$, where $\phi(\cdot)$ is a function such that $\phi(r) = 1$ and $\phi(l) = 0$.*

Proof We prove the correctness of the representability condition and the validity of the representability interval.

Representability condition. Suppose that G_ν is rectilinear planar. By Lemma 11, G_{μ_l} and G_{μ_r} admit spirality values σ_{μ_l} and σ_{μ_r}, respectively, such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$.

Hence, $m_l - M_r \leq \sigma_{\mu_l} - \sigma_{\mu_r} \leq 3$ and $M_l - m_r \geq \sigma_{\mu_l} - \sigma_{\mu_r} \geq 3$, i.e., $3 \in [m_l - M_r, M_l - m_r]$.

Suppose, vice versa that $3 \in [m_l - M_r, M_l - m_r]$. By hypothesis G_{μ_l} (resp. G_{μ_r}) is rectilinear planar for every value of spirality in the interval $[m_l, M_l]$ (resp. $[m_r, M_r]$). This implies that there exist rectilinear planar representations for G_{μ_l} and G_{μ_r} with spirality values $\sigma_{\mu_l} \in [m_l, M_l]$ and $\sigma_{\mu_r} \in [m_r, M_r]$ such that $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. Hence, by Lemma 11 G_ν is rectilinear planar.

Representability interval. We distinguish three cases, based on the values of d and d'. Note that a possible forth case for the type I_{3rl} is symmetric to the case for the type I_{3lr}.

Case 1: $d = d' = l$, i.e., G_ν is of type I_{3ll}. We prove that $I_\nu = [\max\{m_l - 1, m_r + 2\}, \min\{M_l - 1, M_r + 2\}]$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν. Since for an I_{3ll} component we have $k_u^l = k_v^l = \frac{1}{2}$ and $k_u^r = k_v^r = 1$, by Lemma 3 we have $\sigma_\nu = \sigma_{\mu_r} + \alpha_u^r + \alpha_v^r$ and $\sigma_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, we have: $\sigma_\nu \geq m_r + 2$, $\sigma_\nu \leq M_r + 2$, $\sigma_\nu \geq m_l - 1$, and $\sigma_\nu \leq M_l - 1$.

Conversely, we show that if $\sigma_\nu \in [\max\{m_l - 1, m_r + 2\}, \min\{M_l - 1, M_r + 2\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν. We have $\sigma_\nu \in [m_l - 1, M_l - 1]$ and $\sigma_\nu \in [m_r + 2, M_r + 2]$. Hence, $\sigma_\nu + 1 \in [m_l, M_l]$ and $\sigma_\nu - 2 \in [m_r, M_r]$. We can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = \sigma_\nu + 1$ and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = \sigma_\nu - 2$. Note that, for this choice, we have $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r}. We have to prove that the spirality of H_ν is σ_ν. By Lemma 3 we have $\sigma'_\nu = \sigma_{\mu_l} - \frac{1}{2}\alpha_u^l - \frac{1}{2}\alpha_v^l$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l}, α_v^l, and α_u^l. Since $\sigma_\nu = \sigma_{\mu_l} - 1$, $\alpha_u^l = 1$, and $\alpha_v^l = 1$, we have $\sigma'_\nu = \sigma_\nu + 1 - \frac{1}{2} - \frac{1}{2} = \sigma_\nu$.

Case 2: $d = d' = r$. Symmetric to Case 1.

Case 3: $d = l$ and $d' = r$, i.e., G_ν is of type I_{3lr}. We prove that $I_\nu = [\max\{m_l - 1, m_r + 2\} - \frac{1}{2}, \min\{M_l - 1, M_r + 2\} - \frac{1}{2}] = [\max\{m_l - \frac{3}{2}, m_r + \frac{3}{2}\}, \min\{M_l - \frac{3}{2}, M_r + \frac{3}{2}\}]$.

Assume first that σ_ν is the spirality of a rectilinear planar representation of G_ν. Since for an I_{3lr} component we have $k_u^r = k_v^l = \frac{1}{2}$ and $k_u^l = k_v^r = 1$, by Lemma 3 we have $\sigma_\nu = \sigma_{\mu_r} + \frac{1}{2}\alpha_u^r + \alpha_v^r$ and $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. Since $\alpha_u^l = \alpha_v^l = \alpha_u^r = \alpha_v^r = 1$, we have: $\sigma_\nu \geq m_r + \frac{3}{2}$, $\sigma_\nu \leq M_r + \frac{3}{2}$, $\sigma_\nu \geq m_l - \frac{3}{2}$ and $\sigma_\nu \leq M_l - \frac{3}{2}$. Conversely, we show that if $\sigma_\nu \in [\max\{m_l - \frac{3}{2}, m_r + \frac{3}{2}\}, \min\{M_l - \frac{3}{2}, M_r + \frac{3}{2}\}]$, there exists a rectilinear planar representation of G_ν with spirality σ_ν. We have $\sigma_\nu \in [m_l - \frac{3}{2}, M_l - \frac{3}{2}]$ and $\sigma_\nu \in [m_r + \frac{3}{2}, M_r + \frac{3}{2}]$. Hence, $\sigma_\nu + \frac{3}{2} \in [m_l, M_l]$ and $\sigma_\nu - \frac{3}{2} \in [m_r, M_r]$. We can construct a rectilinear planar representation H_{μ_l} of G_{μ_l} with spirality $\sigma_{\mu_l} = \sigma_\nu + \frac{3}{2}$ and a rectilinear planar representation H_{μ_r} of G_{μ_r} with spirality $\sigma_{\mu_r} = \sigma_\nu - \frac{3}{2}$. Notice that, for this choice, we have $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$, hence, there exists a rectilinear planar representation H_ν of G_ν given the values of σ_{μ_l} and σ_{μ_r}. We have to prove that the spirality of H_ν is σ_ν. By Lemma 3, $\sigma'_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$, where σ'_ν is the spirality of the representation H_ν of G_ν given a choice of σ_{μ_l}, α_v^l, and α_u^l. Since $\sigma_\nu = \sigma_{\mu_l} - \frac{3}{2}$, $\alpha_u^l = 1$, and $\alpha_v^l = 1$, we have $\sigma'_\nu = \sigma_\nu + \frac{3}{2} - 1 - \frac{1}{2} = \sigma_\nu$. ☐

4.4 Representability Condition for the Root

To finally achieve a characterization of rectilinear series–parallel graphs we need to consider the representability condition that must be verified at the level of the root, when the reference edge is not a dummy edge. Denote by $e = (u, v)$ the reference edge of G and let ρ be the root of T with respect to e. Let η be the child of ρ that does not correspond to e, and let u' and v' be the alias vertices associated with the poles u and v of G_η. Suppose that G_η is rectilinear planar with representability interval I_η.

We say that G satisfies the *root condition* if $I_\eta \cap \Delta_\rho \neq \emptyset$, where Δ_ρ is defined as follows: (i) $\Delta_\rho = [2, 6]$ if u' coincides with u and v' coincides with v; (ii) $\Delta_\rho = [3, 5]$ if exactly one of u' and v' coincides with u and v, respectively; (iii) $\Delta_\rho = 4$ if none of u' and v' coincides with u and v. We prove the following.

Lemma 13 *Let $e = (u, v)$ be the reference edge of G and let ρ be the root of T with respect to e. Let η be the child of ρ that does not correspond to e. Suppose that G_η is rectilinear planar with representability interval I_η. G is rectilinear planar if and only if it satisfies the root condition. Also, if G satisfies the root condition, it admits a rectilinear planar representation H for any value of spirality σ_η of H_η such that $\sigma_\eta \in I_\eta \cap \Delta_\rho$, where H_η is the restriction of H to G_η.*

Proof Let f_{int} be the internal face of G incident to e. Observe that u and v are the poles of G_η. Let u' and v' be the alias vertices associated with u and with v, respectively. H is a rectilinear planar representation of G if and only if the following two conditions hold: the restriction H_η of H to G_η is a rectilinear planar representation; the number A of right turns minus left turns of any simple cycle of G in H containing e and traversed clockwise in H is equal to 4. We have $A = \sigma_\eta + \alpha_{u'} + \alpha_{v'}$, where: σ_η is the spirality of H_η; for $w \in \{u', v'\}$, $\alpha_w = 1$, $\alpha_w = 0$, and $\alpha_w = -1$ if the angle formed by w in f_{int} is equal to 90^o, 180^o, or 270^o, respectively.

According to the definition of root condition, there are three cases to consider: (i) $\Delta_\rho = [2, 6]$, (ii) $\Delta_\rho = [3, 5]$, and (iii) $\Delta_\rho = 4$. Consider Case (i). Since in this case the alias vertices coincide with the poles, we have $\alpha_{u'} \in [-1, 1]$, $\alpha_{v'} \in [-1, 1]$, and hence $\alpha_{u'} + \alpha_{v'} \in [-2, 2]$. If G is rectilinear planar, we have that $A = \sigma_\eta + \alpha_{u'} + \alpha_{v'} = 4$ for some $\sigma_\eta \in I_\eta$ and for $\alpha_{u'} + \alpha_{v'} \in [-2, 2]$. Hence, $\sigma_\eta = 4 - \alpha_{u'} - \alpha_{v'} \in [2, 6]$, i.e., the root condition $I_\eta \cap \Delta_\rho \neq \emptyset$ holds.

Suppose vice versa that the root condition $I_\eta \cap \Delta_\rho \neq \emptyset$ holds. For any value $\sigma_\eta \in I_\eta \cap \Delta_\rho$ there exists a rectilinear planar representation of H_η of G_η with spirality σ_η. Also, since $\Delta_\rho = [2, 6]$, we have that $4 - \sigma_\eta \in [-2, 2]$, and therefore, for any possible choice of $\sigma_\eta \in I_\eta \cap \Delta_\rho$, we can suitably choose $\alpha_{u'}$ and $\alpha_{v'}$ such that $\alpha_{u'} + \alpha_{v'} = 4 - \sigma_\eta$, i.e., $A = \sigma_\eta + \alpha_{u'} + \alpha_{v'} = 4$. It follows that G is rectilinear planar and it admits a rectilinear planar representation for any value $\sigma_\eta \in I_\eta \cap \Delta_\rho$.

Cases (ii) and (iii) are proved analogously; in Case (ii) $\alpha_{u'} + \alpha_{v'} \in [-1, 1]$ and in Case (iii) $\alpha_{u'} + \alpha_{v'} = 0$. □

The next theorem summarizes the main result of this section.

Theorem 2 *Let G be a plane series–parallel 4-graph and let T be an SPQ*-tree of G. Graph G is rectilinear planar if and only if, for every node of T the corresponding representability condition of Table 1 is satisfied.*

Table 1 Representability conditions and intervals for the different types of nodes. In the formulas of the table, the term γ is such that $\gamma = \lambda + \beta - 2$, and $\phi(\cdot)$ is a function such that $\phi(r) = 1$ and $\phi(l) = 0$

Q^*-node (the pertinent graph is a path of length ℓ)	
Representability Condition	True
Representability Interval	$[-\ell + 1, \ell - 1]$
S-node with h children μ_i such that $I_{\mu_i} = [m_i, M_i]$ $(i = 1, \ldots, h)$	
Representability Condition	True
Representability Interval	$I_\nu = [m, M] = [\sum_{i=1}^{h} m_i, \sum_{i=1}^{h} M_i]$
P-node with three children μ_l, μ_c, μ_r where $I_{\mu_i} = [m_i, M_i]$ $(i \in \{l, c, r\})$	
Representability Condition	$[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] \neq \emptyset$
Representability Interval	$I_\nu = [m, M] = [\max\{m_l - 2, m_c, m_r + 2\}, \min\{M_l - 2, M_c, M_r + 2\}]$
P-node with two children μ_l and μ_r where $I_{\mu_i} = [m_i, M_i]$ $(i \in \{l, r\})$ – $\mathrm{I_2O_{\lambda\beta}}$	
Representability Condition	$[m_l - M_r, M_l - m_r] \cap [2, 4 - \gamma] \neq \emptyset$
Representability Interval	$I_\nu = [m, M] = [\max\{m_l - 2, m_r\} + \frac{\gamma}{2}, \min\{M_l, M_r + 2\} - \frac{\gamma}{2}]$
P-node with two children μ_l and μ_r where $I_{\mu_i} = [m_i, M_i]$ $(i \in \{l, r\})$ – $\mathrm{I_{3d}O_{\lambda\beta}}$	
Representability Condition	$[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] \neq \emptyset$
Representability Interval	$I_\nu = [m, M] = [\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma - \phi(d)}{2}, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma + \phi(d)}{2}]$
P-node with two children μ_l and μ_r where $I_{\mu_i} = [m_i, M_i]$ $(i \in \{l, r\})$ – $\mathrm{I_{3dd'}}$	
Representability Condition	$3 \in [m_l - M_r, M_l - m_r]$
Representability Interval	$I_\nu = [m, M] = [\max\{m_l - 1, m_r + 2\} - \frac{\phi(d)+\phi(d')}{2}, \min\{M_l - 1, M_r + 2\} - \frac{\phi(d)+\phi(d')}{2}]$
P^r-node (the root ρ)	
Root condition	$I_\eta \cap \Delta_\rho \neq \emptyset$

5 Overview of the Bend-Minimization Algorithm

Let G be a plane series–parallel 4-graph. If G is biconnected let e be any edge on the external face of G; otherwise, we add a dummy edge e that makes it biconnected. Let T be the SPQ*-tree of G with respect to e. Our bend-minimization algorithm works in two phases. It first visits T bottom-up (in post order) to determine the number of bends of a bend-minimum orthogonal representation of G. Then it visits T top-down to construct such an orthogonal representation.

When a node ν is considered in the bottom-up visit, the algorithm assigns to ν a *budget* b_ν of bends. This budget corresponds to the minimum number of extra bends that must be added to the budgets of the children of ν to realize an orthogonal representation of G_ν. In other words, b_ν can be regarded as the minimum number of extra subdivision vertices that must be inserted along the edges of G_ν (besides those already inserted for the children of ν) to make it rectilinear planar. The budget b_ν is larger than zero if and only if the representability condition of the rectilinear planarity testing for ν is not satisfied. Hence, according to Table 1, $b_\nu = 0$ if ν is a Q*- or an S-node, while it can be positive if ν is a P-node or the root of T. For instance, for the graph of Fig. 1a and the tree T of Fig. 1c, the first component that requires some bends in the bottom-up visit of T is G_{ν_3}, namely $b_{\nu_3} = 3$; two more bends are required at the root level, i.e., $b_\rho = 2$. When $b_\nu > 0$, a crucial and non-trivial aspect is how to efficiently compute b_ν. The other key aspect is how to succinctly describe the set I'_ν of spirality values that a rectilinear representation of a subdivision of G_ν can take, by considering all possible distributions of the b_ν subdivision vertices along its edges. We will show that I'_ν is still an interval, which allows us to represent it in $O(1)$ space.

Section 6 describes how to compute the budgets b_ν and the sets I'_ν in the bottom-up visit of T, and it proves the optimality of the solution. Section 7 describes the top-down visit and summarizes our main result.

6 Budgets and Optimality

In the following we denote by m and M the minimum and maximum values of the representability interval I_ν of ν when G_ν is rectilinear planar, as defined in Table 1. Also, since when we visit ν, all its children have already been visited and have received their own budget of bends (i.e., of subdivision vertices for the corresponding component), we will simply assume that each child of ν is rectilinear planar.

As observed, if ν is either a Q*-node or an S-node, $b_\nu = 0$. Hence, we assume that ν is a P-node. A child μ of a (non-root) P-node ν is either a Q*- or an S-node. To compute b_ν and I'_ν we define the concept of *exposed edge* of μ. If μ is a Q*-node, every edge of G_μ is an exposed edge of μ (and of G_μ). If μ is an S-node with at least one Q*-node child μ', every edge of G_μ that belongs to $G_{\mu'}$ is an exposed edge of μ (and of G_μ). Else, μ is an S-node that has no exposed edge. Lemma 14 states a crucial property. It implies that when we need to insert some subdivision vertices in an S-component μ that is a child of ν, these vertices can always be added along an exposed edge of μ, if one exists.

Fig. 11 Illustration of Lemma 14. **a** An orthogonal representation H_μ of an S-component with spirality 3 and having 3 bends. An exposed edge, shown as a red thick segment. **b** A different orthogonal representation H'_μ of the same component having the same spirality and number of bends as H_μ, but such that all the bends are along the red thick exposed edge (Color figure online)

Lemma 14 *Let μ be an S-node such that G_μ is rectilinear planar and μ has an exposed edge e. Let H_μ be an orthogonal representation of G_μ having $b > 0$ bends. There exists an orthogonal representation H'_μ of G_μ with b bends such that: (i) all the b bends of H'_μ lie on e; (ii) $\sigma(H'_\mu) = \sigma(H_\mu)$.*

Proof Let u and v be the poles of G_μ. Any path from u to v inside G_μ contains the exposed edge e. Consider an orthogonal representation H_μ of G_μ with $b > 0$ bends. Let $e' \neq e$ be an edge with at least one bend in H_μ. Let P^{uv} be any simple path from u to v of H_μ passing through e'. Suppose, without loss of generality, that the bend on e' corresponds to a right turn along P^{uv} while going from u to v. Since by hypothesis G_μ is rectilinear planar, we can derive from H_μ a different orthogonal representation H''_μ with b bends by simply moving the right bend from e' to e, i.e., by inserting a right bend along e and by straightening the right bend of H_μ along e'. With this transformation, the number of right and left turns along P^{uv} is the same in H''_μ and H_μ, and the angles at u and v in the two representations are also the same. This implies that $\sigma(H''_\mu) = \sigma(H_\mu)$. By repeatedly applying this transformation on H''_μ until all the b bends of H_μ are moved on e we get the desired representation H'_μ. □

An illustration of Lemma 14 is given in Fig. 11. In Fig. 11a an orthogonal representation H_μ of an S-component is shown. It has spirality 3 and 3 bends. In Fig. 11b a different orthogonal representation H'_μ of the same component is given, having the same spirality and number of bends as H_μ, where all the bends are along an exposed edge.

Observe that, if ν is a P-node with three children, each of them has an exposed edge (as the poles of ν have degree at most four). If ν has two children, it might have a child without exposed edges only if ν is of type I_{3ll} or I_{3rr} (see Fig. 10). Section 6.1 and Sect. 6.2 focus on the budget of P-nodes with three children and on the budget of P-nodes with two children, respectively. Section 6.3 concentrates on the budget of the root.

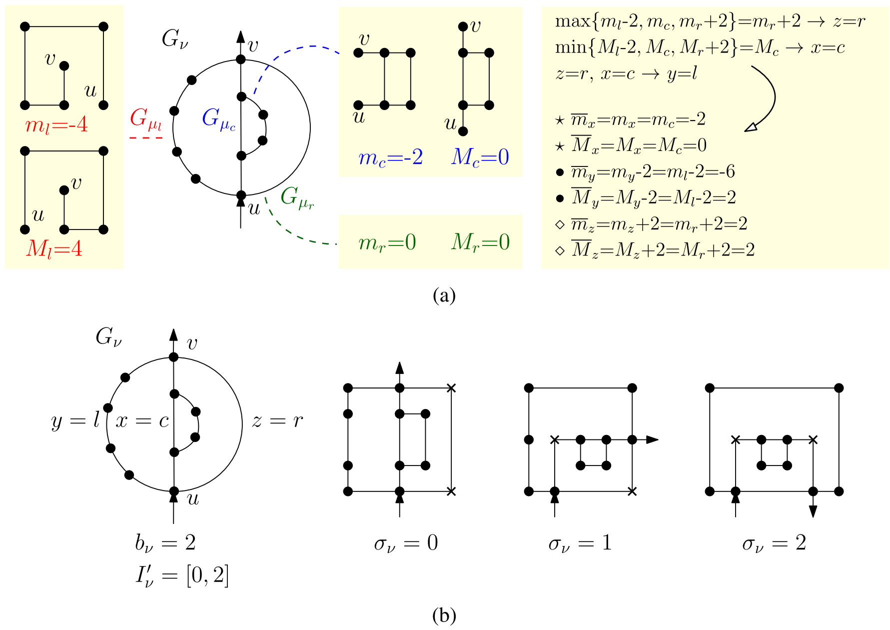

Fig. 12 **a** Illustration of the notation $\overline{m}_t$, $\overline{M}_t$ for $t \in \{x, y, z\}$. **b** Illustration of the statement of Lemma 15 for the graph G_ν. By Property (i) of the lemma $b_\nu = \overline{m}_z - \overline{M}_x = 2$ and by Property (ii) $I'_\nu = [\max\{\overline{M}_x, \overline{m}_y\}, \min\{\overline{m}_z, \overline{M}_y\}] = [0, 2]$

6.1 Budget of P-Nodes with Three Children

Lemma 15 handles the case of a P-node ν with three children μ_l, μ_c, and μ_r such that the corresponding components are rectilinear planar, while G_ν is not rectilinear planar. Denote by $I_{\mu_l} = [m_l, M_l]$, $I_{\mu_c} = [m_c, M_c]$, and $I_{\mu_r} = [m_r, M_r]$ the representability intervals of μ_l, μ_c, and μ_r, respectively. Since G_ν is not rectilinear planar, the representability condition for ν is violated, i.e., $[m_l - 2, M_l - 2] \cap [m_c, M_c] \cap [m_r + 2, M_r + 2] = \emptyset$ (see the third row of Table 1). Rename the three intervals involved in the representability condition as $[\overline{m}_x, \overline{M}_x]$, $[\overline{m}_y, \overline{M}_y]$, and $[\overline{m}_z, \overline{M}_z]$, where $x, y, z \in \{l, c, r\}$ and $x \neq y \neq z$, in such a way that $\overline{m}_z = \max\{m_l - 2, m_c, m_r + 2\}$ and $\overline{M}_x = \min\{M_l - 2, M_c, M_r + 2\}$. Namely, $z = l$ if $\overline{m}_z = m_l - 2$, $z = c$ if $\overline{m}_z = m_c$, and $z = r$ if $\overline{m}_z = m_r + 2$. Similarly, $x = l$ if $\overline{M}_x = M_l - 2$, $x = c$ if $\overline{M}_x = M_c$, and $x = r$ if $\overline{M}_x = M_r + 2$. See Fig. 12a for an example. The following simple property holds.

Proposition 1 *$[\overline{m}_x, \overline{M}_x]$ and $[\overline{m}_z, \overline{M}_z]$ are disjoint, with $\overline{m}_z > \overline{M}_x$.*

Proof Suppose for a contradiction that $\overline{m}_z \leq \overline{M}_x$. Since $\overline{M}_x$ is the minimum of the three maxima, we have that $\overline{M}_y$ and $\overline{M}_z$ are to the right of $\overline{M}_x$. Also, since $\overline{m}_z$ is the maximum of three minima, we have that $\overline{m}_y$ and $\overline{m}_x$ are to the left of $\overline{m}_z$. Hence the three intervals $[\overline{m}_x, \overline{M}_x]$, $[\overline{m}_y, \overline{M}_y]$, and $[\overline{m}_z, \overline{M}_z]$ share the interval $[\overline{m}_z, \overline{M}_x]$, which contradicts the fact that G_ν is not rectilinear planar. □

The next lemma gives the rule for computing the budget of a P-node with three children and for determining the corresponding interval of spirality values, once a number of bends equal to the budget has been added.

Lemma 15 *Let ν be a P-node with three children μ_l, μ_c, and μ_r. Let G_{μ_l}, G_{μ_c}, and G_{μ_r} be rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$, $I_{\mu_c} = [m_c, M_c]$, and $I_{\mu_r} = [m_r, M_r]$, respectively. If G_ν is not rectilinear planar then: (i) the budget for ν is $b_\nu = \overline{m}_z - \overline{M}_x$; and (ii) the interval of spirality values for an orthogonal representation of G_ν with b_ν bends is $I'_\nu = [\max\{\overline{M}_x, \overline{m}_y\}, \min\{\overline{m}_z, \overline{M}_y\}]$.*

Proof Since by hypothesis G_ν is not rectilinear planar, by the representability condition in Table 1 we have $[\overline{m}_x, \overline{M}_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z, \overline{M}_z] = \emptyset$. By Proposition 1, $[\overline{m}_x, \overline{M}_x] \cap [\overline{m}_z, \overline{M}_z] = \emptyset$. We prove Property (i) and (ii) separately. Figure 12b provides an illustration for the graph in Fig. 12a, where $x = c$, $y = l$, and $z = r$. □

Proof of Property (i). We show that $b_\nu = \overline{m}_z - \overline{M}_x$. Observe that each of the three components G_{μ_l}, G_{μ_c}, and G_{μ_r} is an S-component with an exposed edge.

We first prove that b_ν bends are necessary. Suppose for a contradiction that G_ν admits an orthogonal representation H'_ν with $b'_\nu < b_\nu$ bends. Denote by b'_x and b'_z the number of bends in the restriction of H'_ν to G_{μ_x} and to G_{μ_z}, respectively. By Lemma 14, we can assume that all the bends b'_x are along an exposed edge of G_{μ_x} and all the bends b'_z are along an exposed edge of G_{μ_z}. Consider the underlying graph G'_ν of H'_ν obtained by replacing each bend of H'_ν with a subdivision vertex. G'_ν is rectilinear planar. Denote by $[m'_x, M'_x]$ and $[m'_z, M'_z]$ the spirality intervals of G'_{μ_x} and G'_{μ_z}. Note that each subdivision vertex along an exposed edge of G'_{μ_x} allows one more turn (either to the left or to the right) in a rectilinear planar representation of this component with respect to a rectilinear planar representation of G_{μ_x}. Hence, the spirality interval of G'_{μ_x} extends the one of G_{μ_x} by b'_x units, both for the minimum value and for the maximum value. The same reasoning applies to G'_{μ_z}. It follows that $m'_x = m_x - b'_x$, $M'_x = M_x + b'_x$, $m'_z = m_z - b'_z$, and $M'_z = M_z + b'_z$. Consider the three representability intervals $[m'_l - 2, M'_l - 2]$, $[m'_c, M'_c]$, and $[m'_r + 2, M'_r + 2]$ for G'_{μ_l}, G'_{μ_c}, and G'_{μ_r}, respectively. Suppose that $x = l$, i.e., $\min\{M_l - 2, M_c, M_r + 2\} = M_l - 2$; then we define $\overline{m}'_x = m'_l - 2$ and $\overline{M}'_x = M'_l - 2$. Similarly, if $x = c$, we define $\overline{m}'_x = m'_c$ and $\overline{M}'_x = M'_c$. Finally, if $x = r$, we define $\overline{m}'_x = m'_r + 2$ and $\overline{M}'_x = M'_r + 2$. Analogously, if $z = l$, i.e. $\max\{m_l - 2, m_c, m_r + 2\} = m_l - 2$, we define $\overline{m}'_z = m_l - 2$; if $z = c$, we define $\overline{m}'_z = m'_c$ and $\overline{M}'_z = M'_c$; if $z = r$, we define $\overline{m}'_z = m'_r + 2$ and $\overline{M}'_z = M'_r + 2$. Figure 13 illustrates this notation for the graph of Fig. 12a.

We have, $\overline{m}'_x = \overline{m}_x - b'_x$, $\overline{m}'_z = \overline{m}_z - b'_z$, $\overline{M}'_x = \overline{M}_x + b'_x$, and $\overline{M}'_z = \overline{M}_z + b'_z$. Since G'_ν is rectilinear planar, we have $[\overline{m}'_x, \overline{M}'_x] \cap [\overline{m}'_z, \overline{M}'_z] = [\overline{m}_x - b'_x, \overline{M}_x + b'_x] \cap [\overline{m}_z - b'_z, \overline{M}_z + b'_z] \neq \emptyset$. Therefore, $\overline{m}_z - b'_z \leq \overline{M}_x + b'_x$, which implies $b_\nu = \overline{m}_z - \overline{M}_x \leq b'_x + b'_z \leq b'_\nu < b_\nu$, a contradiction.

We now prove that b_ν bends suffice. Let b_x and b_z be two non-negative integers such that $b_x + b_z = b_\nu$, $b_x \geq \overline{m}_y - \overline{M}_x$, and $b_z \geq \overline{m}_z - \overline{M}_y$. Note that b_x and b_z always exist because $\overline{m}_y - \overline{M}_x + \overline{m}_z - \overline{M}_y = b_\nu - \overline{M}_y + \overline{m}_y \leq b_\nu$. Insert b_x subdivision vertices on any exposed edge of G_{μ_x} and insert b_z subdivision vertices

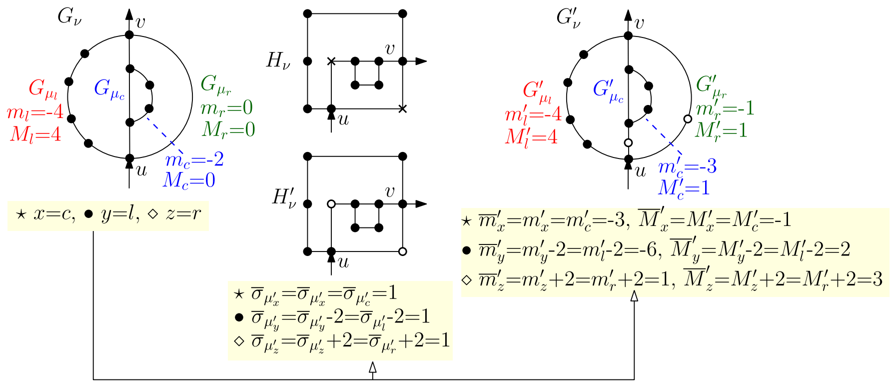

Fig. 13 Illustration for the proof of Property (i) of Lemma 15, on graph G_ν in Fig. 12a

on any exposed edge of G_{μ_z}. Clearly no subdivision vertex has been inserted on G_{μ_y} since $b_x + b_z = b_\nu$. Call G'_{μ_x}, G'_{μ_y}, and G'_{μ_z} the resulting components (note that $G'_{\mu_y} = G_{\mu_y}$). Define $\overline{m}'_x$, $\overline{m}'_z$, $\overline{M}'_x$, $\overline{M}'_z$ as in the first part of the proof. Suppose that $y = l$, i.e. $\min\{M_l - 2, M_c, M_r + 2\} \neq M_l - 2$ and $\max\{m_l - 2, m_c, m_r + 2\} \neq m_l - 2$. Then we define $\overline{m}'_y = m'_l - 2$ and $\overline{M}'_y = M'_l - 2$. Similarly, if $y = c$, define $\overline{m}'_y = m'_c$ and $\overline{M}'_y = M'_c$. Finally, if $y = r$, we define $\overline{m}'_y = m'_r + 2$ and $\overline{M}'_y = M'_r + 2$.

Consider the plane graph G'_ν obtained by the union of G'_{μ_x}, G'_{μ_y}, and G'_{μ_z}. To prove that G'_ν is rectilinear planar, by the representability condition in Table 1, it suffices to show that $[\overline{m}'_x, \overline{M}'_x] \cap [\overline{m}'_y, \overline{M}'_y] \cap [\overline{m}'_z, \overline{M}'_z] = [\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] \neq \emptyset$. We have:

- $b_x \geq \overline{m}_y - \overline{M}_x$, hence $\overline{m}_y \leq \overline{M}_x + b_x$ and $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \neq \emptyset$;
- $b_z \geq \overline{m}_z - \overline{M}_y$, hence $\overline{m}_z - b_z \leq \overline{M}_y$ and $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] \neq \emptyset$;
- $\overline{m}_z - \overline{M}_x = b_z + b_x = b_\nu$, hence $\overline{m}_z - b_z = \overline{M}_x + b_x$ and $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] \neq \emptyset$.

Hence G'_ν has a rectilinear planar representation H'_ν; replacing its subdivision vertices with bends, we get an orthogonal representation of G_ν with b_ν bends.

Proof of Property (ii). We show that set I'_ν is an interval of feasible spirality values for the orthogonal representations of G_ν with b_ν bends. Namely, we show that any orthogonal representation H_ν of G_ν with b_ν bends has spirality in the interval I'_ν and that for every value $\sigma_\nu \in I'_\nu$ there exists an orthogonal representation of G_ν with spirality σ_ν and b_ν bends.

Suppose that G_ν has an orthogonal representation H_ν with b_ν bends and let σ_ν be the spirality of H_ν. We prove that $\sigma_\nu \in I'_\nu = [\max\{\overline{M}_x, \overline{m}_y\}, \min\{\overline{m}_z, \overline{M}_y\}]$. Let b_l, b_c, and b_r be the number of bends in the restriction of H_ν to G_{μ_l}, G_{μ_C}, and G_{μ_r}, respectively, where $b_l + b_c + b_r = b_\nu$. Let H'_ν be the rectilinear planar representation obtained from H_ν by replacing each bend with a subdivision vertex and let G'_ν be the underlying graph. Clearly the spirality of H'_ν equals σ_ν. For any $t \in \{x, y, z\}$, by Lemma 14 we can assume that all the b_t bends are along an exposed edge of G_{μ_t} and, consequently, $\sigma_{\mu'_t} \in [m_t - b_t, M_t + b_t]$. By using the same notation as in the proof

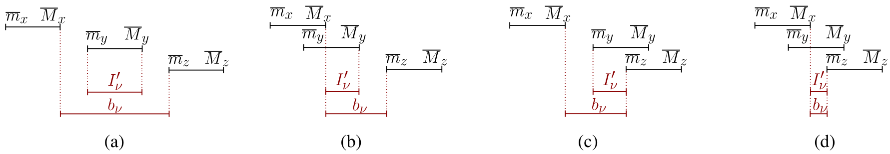

Fig. 14 The four possible cases for the proof of Property (ii) of Lemma 15

of Property (i) we define $\overline{m}'_x = m'_l - 2$ and $\overline{M}'_x = M'_l - 2$ if $x = l$, $\overline{m}'_x = m'_c$ and $\overline{M}'_x = M'_c$ if $x = c$, and $\overline{m}'_x = m'_r + 2$ and $\overline{M}'_x = M'_r + 2$ if $x = r$. The values $\overline{m}'_y, \overline{M}'_y$, $\overline{m}'_z$, and $\overline{M}'_z$ are defined analogously (see Fig. 13). Since G'_ν is rectilinear planar, by Table 1 we have $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y - b_y, \overline{M}_y + b_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] \neq \emptyset$.

The remainder of the proof exploits the following claim. For an example see Fig. 12b, where every bend is regarded as a subdivision vertex, and where $b_y = 0$, $b_x \leq \overline{M}_y - \overline{M}_x = 2 - 0 = 2$, and $b_z \leq \overline{m}_z - \overline{m}_y = 2 + 6 = 8$ in the three rectilinear representations of G'_ν.

Claim 1 *These relations hold: (1)* $b_y = 0$*; (2)* $b_x \leq \overline{M}_y - \overline{M}_x$*; (3)* $b_z \leq \overline{m}_z - \overline{m}_y$.

Claim Proof We prove the three relations by contradiction.

(1) If $b_y > 0$, then $b_x + b_z < b_\nu = \overline{m}_z - \overline{M}_x$. Hence, $\overline{M}_x + b_x < \overline{m}_z - b_z$ and $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = \emptyset$, a contradiction.
(2) If $b_x > \overline{M}_y - \overline{M}_x$, then $b_y + b_z < b_\nu - (\overline{M}_y - \overline{M}_x) = \overline{m}_z - \overline{M}_x - (\overline{M}_y - \overline{M}_x) = \overline{m}_z - \overline{M}_y$. Hence, $\overline{M}_y + b_y < \overline{m}_z - b_z$ and $[\overline{m}_y - b_y, \overline{M}_y + b_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = \emptyset$, a contradiction.
(3) If $b_z > \overline{m}_z - \overline{m}_y$, then $b_x + b_y < b_\nu - (\overline{m}_z - \overline{m}_y) = \overline{m}_z - \overline{M}_x - (\overline{m}_z - \overline{m}_y) = \overline{m}_y - \overline{M}_x$. Hence, $\overline{M}_x + b_x < \overline{m}_y - b_y$ and $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y - b_y, \overline{M}_y + b_y] = \emptyset$, a contradiction.

We now consider spirality values of the components of H'_ν and define related values $\overline{\sigma}_{\mu'_x} = \sigma_{\mu'_x} - 2$ if $x = l$, $\overline{\sigma}_{\mu'_x} = \sigma_{\mu'_x}$ if $x = c$, and $\overline{\sigma}_{\mu'_x} = \sigma_{\mu'_x} + 2$ if $x = r$. The values $\overline{\sigma}_{\mu'_y}$ and $\overline{\sigma}_{\mu'_z}$ are defined analogously. See Fig. 13 for an illustration of the notation $\overline{\sigma}_{\mu'_t}$ for $t \in \{x, y, z\}$. We have $\overline{\sigma}_{\mu'_t} \in [\overline{m}_t - b_t, \overline{M}_t + b_t]$ for any $t \in \{x, y, z\}$. Also, by Lemma 2 $\sigma_\nu = \overline{\sigma}_{\mu'_x} = \overline{\sigma}_{\mu'_y} = \overline{\sigma}_{\mu'_z}$. Hence $\sigma_\nu \in [\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y - b_y, \overline{M}_y + b_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z]$. Note that by Relation (1) of the claim $[\overline{m}_y - b_y, \overline{M}_y + b_y] = [\overline{m}_y, \overline{M}_y]$. We show that $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\max\{\overline{M}_x, \overline{m}_y\}, \min\{\overline{m}_z, \overline{M}_y\}]$. We have four cases; see Fig. 14.
Case (a): $\overline{M}_x < \overline{m}_y$ **and** $\overline{M}_y < \overline{m}_z$ (see Fig. 14a). In this case $I'_\nu = [\max\{\overline{M}_x, \overline{m}_y\}, \min\{\overline{m}_z, \overline{M}_y\}] = [\overline{m}_y, \overline{M}_y]$. Refer to Fig. 15 for an illustration of the argument that refines Fig. 14a).

First we prove that $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] = [\overline{m}_y, \overline{M}_y]$. We have $\overline{M}_x + b_x \geq \overline{m}_y$, or $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] = \emptyset$. By Relation (2) of the claim we have $b_x \leq \overline{M}_y - \overline{M}_x$. Hence, $b_x \in [\overline{m}_y - \overline{M}_x, \overline{M}_y - \overline{M}_x]$. We have $\overline{M}_x + b_x \in [\overline{M}_x + \overline{m}_y - \overline{M}_x, \overline{M}_x + \overline{M}_y - \overline{M}_x] = [\overline{m}_y, \overline{M}_y]$. It follows that $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] = [\overline{m}_y, \overline{M}_y]$.

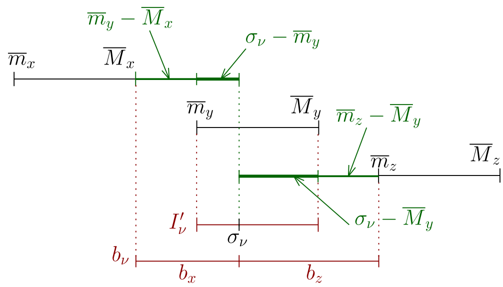

Fig. 15 A more detailed illustration for Case (a) in the proof of Lemma 15. For any value $\sigma_\nu \in I'_\nu$, we need to add $b_x = \sigma_\nu - \overline{M}_x = (\sigma_\nu - \overline{m}_y) + (\overline{m}_y - \overline{M}_x)$ bends to the representation of G_{μ_x} and $b_z = \overline{m}_z - \sigma_\nu = (\overline{m}_z - \overline{M}_y) + (\overline{M}_y - \sigma_\nu)$ bends to the representation of G_{μ_z}

Second, we prove $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{M}_y]$. We have $\overline{M}_y \geq \overline{m}_z - b_z$, or $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = \emptyset$. By Relation (3) of the claim we have $b_z \leq \overline{m}_z - \overline{m}_y$. Hence, $b_z \in [\overline{m}_z - \overline{M}_y, \overline{m}_z - \overline{m}_y]$. We have $\overline{m}_z - b_z \in [\overline{m}_z - \overline{m}_z + \overline{m}_y, \overline{m}_z - \overline{m}_z - \overline{M}_y] = [\overline{m}_y, \overline{M}_y]$.

It follows that $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{M}_y]$.

Hence, $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{M}_y] = I'_\nu$.

Case (b): $\overline{M}_x \geq \overline{m}_y$ **and** $\overline{M}_y < \overline{m}_z$ (see Fig. 14b). In this case $I'_\nu = [\overline{M}_x, \overline{M}_y]$. By the same reasoning as in Case (a) we have $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{M}_y]$.

We prove that $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] = [\overline{M}_x, \overline{M}_y]$.

By Relation (2) of the claim $b_x \in [0, \overline{M}_y - \overline{M}_x]$. We have $\overline{M}_x + b_x \in [\overline{M}_x, \overline{M}_x + \overline{M}_y - \overline{M}_x] = [\overline{M}_x, \overline{M}_y]$. It follows that $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] = [\overline{M}_x, \overline{M}_y]$.

Hence, since $\overline{M}_x \geq \overline{m}_y$, we have $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{M}_y] \cap [\overline{M}_x, \overline{M}_y] = [\overline{M}_x, \overline{M}_y] = I'_\nu$.

Case (c): $\overline{M}_x < \overline{m}_y$ **and** $\overline{M}_y \geq \overline{m}_z$ (see Fig. 14c). In this case $I'_\nu = [\overline{m}_y, \overline{m}_z]$. It is possible to prove that $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] = [\overline{m}_y, \overline{M}_y]$ as we did in Case (a).

We prove $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{m}_z]$. By Relation (3) of the claim $b_z \in [0, \overline{m}_z - \overline{m}_y]$. We have $\overline{m}_z - b_z \in [\overline{m}_z - \overline{m}_z + \overline{m}_y, \overline{m}_z] = [\overline{m}_y, \overline{m}_z]$.

It follows that $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{m}_z]$.

Hence, since $\overline{M}_y \geq \overline{m}_z$, we have $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_y, \overline{m}_z] = [\overline{m}_y, \overline{m}_z] = I'_\nu$.

Case (d): $\overline{M}_x \geq \overline{m}_y$ **and** $\overline{M}_y \geq \overline{m}_z$ (see Fig. 14d). In this case $I'_\nu = [\overline{M}_x, \overline{m}_z]$. It is possible to prove that $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] = [\overline{M}_x, \overline{M}_y]$ as we did in Case (b) and that $[\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_y, \overline{m}_z]$ as we did for Case (c).

We have $[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{M}_x, \overline{M}_y] \cap [\overline{m}_y, \overline{m}_z] = [\overline{M}_x, \overline{m}_z]$ since $\overline{M}_x \geq \overline{m}_y$ and $\overline{M}_y \geq \overline{m}_z$. Hence,

$[\overline{m}_x - b_x, \overline{M}_x + b_x] \cap [\overline{m}_y, \overline{M}_y] \cap [\overline{m}_z - b_z, \overline{M}_z + b_z] = [\overline{M}_x, \overline{m}_z] = I'_\nu$.

Suppose now that we are given $\sigma_\nu \in I'_\nu = [\max\{\overline{M}_x, \overline{m}_y\}, \min\{\overline{m}_z, \overline{M}_y\}]$. We show that there exists an orthogonal representation H_ν of G_ν with b_ν bends and with spirality σ_ν. This is equivalent to showing that there exists a plane graph G'_ν obtained by adding b_ν subdivision vertices along some edges of G_ν such that G'_ν has a rectilinear orthogonal representation with spirality σ_ν. To construct G'_ν we insert a suitable number $b_z \in [0, b_\nu]$ of subdivision vertices on an exposed edge of G_{μ_z} and $b_x = b_\nu - b_z$ subdivision vertices on an exposed edge of G_{μ_x} (as a consequence, we do not insert any subdivision vertex in G_{μ_y}). Let G'_{μ_x}, G'_{μ_y}, and G'_{μ_z} be the resulting graphs. Since by hypothesis G_{μ_x}, G_{μ_y}, and G_{μ_z} are rectilinear planar, we have that also G'_{μ_x}, G'_{μ_y}, and G'_{μ_z} are rectilinear planar. Also, with the same reasoning as in the proof of Property (i), the representability intervals of G'_{μ_x}, G'_{μ_y}, and G'_{μ_z} are $[m'_x, M'_x] = [m_x - b_x, M_x + b_x]$, $[m_y, M_y]$, and $[m_z - b_z, M_z + b_z]$, respectively. For any $t \in \{x, y, z\}$ we define $\overline{m}'_t$, $\overline{M}'_t$, and $\overline{\sigma}_{\mu'_t}$ as in the first part of the proof of Property (ii). We now describe how to compute b_x and b_z, and how to set $\overline{\sigma}_{\mu_x}$, $\overline{\sigma}_{\mu_y}$, and $\overline{\sigma}_{\mu_z}$. Let $c_1 = \max\{\overline{M}_x, \overline{m}_y\}$ and $c_2 = \min\{\overline{m}_z, \overline{M}_y\}$.

We have: $c_1 = \overline{m}_y$ and $c_2 = \overline{M}_y$ in the case of Fig. 14a; $c_1 = \overline{M}_x$ and $c_2 = \overline{M}_y$ in the case of Fig. 14b; $c_1 = \overline{m}_y$ and $c_2 = \overline{m}_z$ in the case of Fig. 14c; $c_1 = \overline{M}_x$ and $c_2 = \overline{m}_z$ in the case of Fig. 14d.

We have $\sigma_\nu \in I'_\nu = [c_1, c_2]$. We set $b_z = \overline{m}_z - \sigma_\nu$ and, consequently, $b_x = b_\nu - b_z = b_\nu - \overline{m}_z + \sigma_\nu = \overline{M}_x + \sigma_\nu$. We prove that b_z (and consequently b_x) is in the interval $[0, b_\nu]$. We have $b_z = \overline{m}_z - \sigma_\nu \in [\overline{m}_z - c_2, \overline{m}_z - c_1] \subseteq [\overline{m}_z - \overline{m}_z, \overline{m}_z - \overline{M}_x]$, since $c_2 \leq \overline{m}_z$ and $c_1 \geq \overline{M}_x$. Hence $b_z \in [\overline{m}_z - \overline{m}_z, \overline{m}_z - \overline{M}_x] = [0, b_\nu]$. We now set $\overline{\sigma}_{\mu_x} = \overline{M}_x + b_x = \sigma_\nu$ and $\overline{\sigma}_{\mu_z} = \overline{m}_z - b_z = \sigma_\nu$. Notice that $\sigma_\nu \in [c_1, c_2] \in [\overline{m}_y, \overline{M}_y]$. Hence, it is possible to set $\overline{\sigma}_{\mu_y} = \sigma_\nu$. By Lemma 2 we can get a rectilinear planar representation H'_ν of G'_ν by a parallel composition of rectilinear planar representations of G'_{μ_x}, G'_{μ_y}, and G'_{μ_z} with spirality values $\overline{\sigma}_{\mu_x}$, $\overline{\sigma}_{\mu_y}$, and $\overline{\sigma}_{\mu_z}$, respectively. By the same lemma, the spirality of H'_ν is $\sigma'_\nu = \overline{\sigma}_{\mu_x} = \overline{\sigma}_{\mu_y} = \overline{\sigma}_{\mu_z} = \sigma_\nu$. By replacing the subdivision vertices of H'_ν with bends we get an orthogonal representation of G_ν with b_ν bends and spirality σ_ν.

6.2 Budget of P-Nodes with Two Children

Let μ_l and μ_r be the two children of ν, and suppose that G_{μ_l} and G_{μ_r} are rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. The representability condition for ν given in Table 1 is expressed in terms of the intersection between the interval $[m_l - M_r, M_l - m_r]$ and another interval Δ_ν that depends on the type of P-node. Specifically: $\Delta_\nu = [2, 4 - \gamma]$ if ν is of type $\mathrm{I_2O_{\lambda\beta}}$; $\Delta_\nu = [\frac{5}{2}, \frac{7}{2} - \gamma]$ if ν is of type $\mathrm{I_{3d}O_{\lambda\beta}}$; and $\Delta_\nu = [3, 3]$ if ν is of type $\mathrm{I_{3dd'}}$. In the following, given two non-intersecting intervals of real numbers $A_1 = [m_1, M_1]$ and $A_2 = [m_2, M_2]$, the *distance* between A_1 and A_2 is defined as $\delta(A_1, A_2) = \min\{|M_1 - m_2|, |M_2 - m_1|\}$.

Section 6.2.1 handles the case of a P-node with two children both having an exposed egde. Section 6.2.2 handles the more involved cases in which either the left child or the right child of the P-node has no exposed edge. Note that, since the vertex-degree is at most four, at least one of the two children of the P-node must have an exposed edge. We start giving two simple combinatorial results (Lemmas 16 and 17), which will be used in the proof of Lemma 18. They provide some basic rules about the values that some terms of the relationships in Lemma 3 can take, depending on the type of P-nodes with two children.

Lemma 16 *Let ν be a P-node of type $\mathrm{I}_2\mathrm{O}_{\lambda\beta}$ with children μ_l and μ_r. Let G_{μ_l} and G_{μ_r} be rectilinear planar. For any rectilinear planar representation of G_{μ_l} and G_{μ_r}, and for any $d \in \{l, r\}$, the following relation holds:*

$$k_u^d \alpha_u^d + k_v^d \alpha_v^d = h \Leftrightarrow h \in [\frac{\gamma}{2}, 2 - \frac{\gamma}{2}],$$

where both h and $\frac{\gamma}{2}$ are either integer or semi-integer numbers.

Proof For the reader's convenience, we summarize in Table 2 the parameters associated with each type of P-node with two children. Based on these parameters, we analyze three cases:

- $\lambda = \beta = 1$. In this case, $\gamma = \lambda + \beta - 2 = 0$, $h \in \{0, 1, 2\}$, and $k_u^l = k_v^l = k_u^r = k_v^r = 1$ (see Table 2). Hence, $k_u^d \alpha_u^d + k_v^d \alpha_v^d = \alpha_u^d + \alpha_v^d$. Since both poles u and v have outdegree one, both α_u^d and α_v^d can take either value 0 or 1, hence $\alpha_u^d + \alpha_v^d$ can take all and only the values in the set $\{0, 1, 2\}$.
- $\lambda = 1$ and $\beta = 2$. In this case $\gamma = 1$, $h \in \{\frac{1}{2}, \frac{3}{2}\}$, $k_u^l = k_u^r = 1$, and $k_v^l = k_v^r = \frac{1}{2}$ (see Table 2). Hence, $k_u^d \alpha_u^d + k_v^d \alpha_v^d = \alpha_u^d + \frac{1}{2}\alpha_v^d$. Since we are assuming that $\mathrm{outdeg}(u) = 1$ and $\mathrm{outdeg}(v) = 2$, we have $\alpha_u^d \in \{0, 1\}$ and $\alpha_v^d = 1$, i.e., $\alpha_u^d + \frac{1}{2}\alpha_v^d$ equals either $\frac{1}{2}$ or $\frac{3}{2}$.
- $\lambda = \beta = 2$. In this case $\gamma = 2$, $c \in \{1\}$, and $k_u^l = k_u^r = k_v^l = k_v^r = \frac{1}{2}$ (see Table 2). Hence, $k_u^d \alpha_u^d + k_v^d \alpha_v^d = \frac{1}{2}\alpha_u^d + \frac{1}{2}\alpha_v^d$. Since $\mathrm{outdeg}(u) = \mathrm{outdeg}(v) = 2$, we have $\alpha_u^d = \alpha_v^d = 1$.

□

Lemma 17 *Let ν be a P-node of type $\mathrm{I}_{3l}\mathrm{O}_{\lambda\beta}$ with children μ_l and μ_r. Let G_{μ_l} and G_{μ_r} be rectilinear planar. For any rectilinear planar representation of G_{μ_l} and G_{μ_r}, the following relations hold:*

$$k_u^l \alpha_u^l + k_v^l \alpha_v^l = h_l \Leftrightarrow h_l \in [\frac{\gamma}{2} + \frac{1}{2}, \frac{3}{2} - \frac{\gamma}{2}], \tag{1}$$

where both h_l and $\frac{\gamma}{2} + \frac{1}{2}$ are either integer or semi-integer numbers.

$$k_u^r \alpha_u^r + k_v^r \alpha_v^r = h_r \Leftrightarrow h_r \in [\frac{\gamma}{2} + 1, 2 - \frac{\gamma}{2}], \tag{2}$$

where both h_r and $\frac{\gamma}{2} + 1$ are either integer or semi-integer numbers.

Proof $k_v^l = \frac{1}{2}$, $k_v^r = 1$, and $\alpha_v^d = 1$ for any $d \in \{l, r\}$. There are two subcases:

Table 2 Parameters for the P-nodes with 2 children

TYPE	k_u^l	k_u^r	k_v^l	k_v^r	λ	β	γ	d	d'
I_2O_{11}	1	1	1	1	1	1	0	–	–
I_2O_{12}	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	2	1	–	–
I_2O_{21}	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	2	1	–	–
I_2O_{22}	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	2	2	2	–	–
$I_{3l}O_{11}$	1	1	$\frac{1}{2}$	1	1	1	0	l	–
$I_{3r}O_{11}$	1	1	1	$\frac{1}{2}$	1	1	0	r	–
$I_{3l}O_{12}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	2	1	l	–
$I_{3r}O_{12}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	1	2	1	r	–
I_{3ll}	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	1	–	l	l
I_{3lr}	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	1	–	l	r
I_{3rr}	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	–	r	r

- $\beta = 1$. In this case $\gamma = 0$, $h_l \in \{\frac{1}{2}, \frac{3}{2}\}$, $h_r \in \{1, 2\}$, and $k_u^l = k_u^r = 1$. We have $k_u^d \alpha_u^d + k_v^d \alpha_v^d = \alpha_u^d + k_v^d$. Since $\alpha_u^d \in \{0, 1\}$, we have $\alpha_u^d + k_v^d \in \{k_v^d, k_v^d + 1\}$. Since $k_v^l = \frac{1}{2}$, we have $\{k_v^l, k_v^l + 1\} = \{\frac{1}{2}, \frac{3}{2}\} = \{\frac{\gamma}{2} + \frac{1}{2}, \frac{3}{2} - \frac{\gamma}{2}\}$. Hence, Relation 1 holds. Since $k_v^r = 1$, we have $\{k_v^r, k_v^r + 1\} = \{1, 2\} = \{\frac{\gamma}{2} + 1, 2 - \frac{\gamma}{2}\}$. Hence, Relation 2 holds.
- $\beta = 2$. In this case $\gamma = 1$, $h_l \in \{1\}$, $h_r \in \{\frac{3}{2}\}$, and $k_u^l = k_u^r = \frac{1}{2}$. Since $\deg(u) = 4$, $\alpha_u^d = 1$. We have $k_u^d \alpha_u^d + k_v^d \alpha_v^d = \frac{1}{2} + k_v^d$. Equivalently, $k_u^d \alpha_u^d + k_v^d \alpha_v^d \in \{\frac{1}{2} + k_v^d, \frac{1}{2} + k_v^d\}$. Since $k_v^l = \frac{1}{2}$, we have $\{\frac{1}{2} + k_v^l, \frac{1}{2} + k_v^l\} = \{\frac{\gamma}{2} + \frac{1}{2}, \frac{3}{2} - \frac{\gamma}{2}\} = \{1\}$. Hence, Relation 1 holds. Since $k_v^r = 1$, we have $\{\frac{1}{2} + k_v^r, \frac{1}{2} + k_v^r\} = \{\frac{\gamma}{2} + 1, 2 - \frac{\gamma}{2}\} = \{\frac{3}{2}\}$. Hence, Relation 2 holds.

□

6.2.1 P-Nodes with Both Children Having an Exposed Edge

As mentioned at the beginning of Sect. 6, in the following we denote by m and M the minimum and the maximum values of the interval I_ν as defined in Table 1. The next lemma gives the rule for computing the budget of a P-node with two children each having an exposed edge, and for determining the corresponding interval of spirality values.

Lemma 18 *Let ν be a P-node with two children μ_l and μ_r, each having an exposed edge. Let G_{μ_l} and G_{μ_r} be rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. If G_ν is not rectilinear planar then: (i) the budget for ν is $b_\nu = \delta([m_l - M_r, M_l - m_r], \Delta_\nu)$; and (ii) the set of spirality values for an orthogonal representation of G_ν with b_ν bends is the interval $I'_\nu = [m - b_\nu, M + b_\nu]$.*

Proof Since by hypothesis G_ν is not rectilinear planar, we have $[m_l - M_r, M_l - m_r] \cap \Delta_\nu = \emptyset$. Figure 16 illustrates the statement for a P-node ν of type I_2O_{11}, by also showing how the interval I'_ν of Property (ii) is defined.

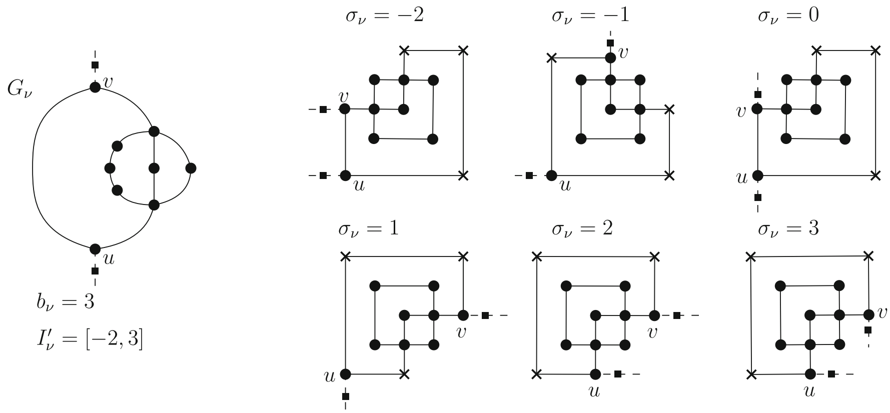

Fig. 16 Illustration of Lemma 18 for a P-node ν of type I_2O_{11}, where $m_l = M_l = 0$ and $m_r = M_r = 1$. In this case $\Delta_\nu = [2, 4]$, $m = \max\{m_l - 2, m_r\} = -1$, and $M = \min\{M_l, M_r + 2\} = 0$. Since $M_l - m_r < 2$, G_ν is not rectilinear planar. By Property (i), $b_\nu = \delta([m_l - M_r, M_l - m_r], \Delta_\nu) = 3$; by Property (ii), $I'_\nu = [m - b_\nu, M + b_\nu] = [-2, 3]$

Proof of Property (i). We show that $b_\nu = \delta([m_l - M_r, M_l - m_r], \Delta_\nu)$. We first prove that b_ν bends are necessary. Suppose for a contradiction that G_ν admits an orthogonal representation H'_ν with $b'_\nu < b_\nu$ bends. Denote by b'_l and b'_r the number of bends in the restriction of H'_ν to G_{μ_l} and to G_{μ_r}, respectively. By Lemma 14, we can assume that all the bends b'_l (resp. b'_r) are along an exposed edge of G_{μ_l} (resp. G_{μ_r}). Consider the underlying graph G'_ν of H'_ν obtained by replacing each bend of H'_ν with a subdivision vertex. G'_ν is rectilinear planar. Also, each subdivision vertex along an exposed edge of G_{μ_l} allows one more turn (either to the left or to the right) in a rectilinear representation of this component, i.e., it extends the spirality interval of G_{μ_l} by one unit, both for the minimum value and for the maximum value. The same considerations happen for G_{μ_r}. Hence G'_{μ_l} and G'_{μ_r} are rectilinear planar with representability intervals $[m_l - b'_l, M_l + b'_l]$ and $[m_r - b'_r, M_r + b'_r]$, respectively, and the representability condition for G'_ν is $[m_l - b'_l - M_r - b'_r, M_l + b'_l - m_r + b'_r] \cap \Delta_\nu \neq \emptyset$. Since $b'_l + b'_r = b'_\nu$, we have $[m_l - M_r - b'_\nu, M_l - m_r + b'_\nu] \cap \Delta_\nu \neq \emptyset$, which implies $\delta([m_l - M_r, M_l - m_r], \Delta_\nu) \leq b'_\nu < b_\nu$, a contradiction.

We now prove that b_ν bends suffice. Let b_l and b_r be two arbitrarily chosen non-negative integers such that $b_l + b_r = b_\nu$. Insert b_l (resp. b_r) subdivision vertices on any exposed edge of G_{μ_l} (resp. G_{μ_r}). Call G'_{μ_l} and G'_{μ_r} the resulting components. Since by hypothesis G_{μ_l} and G_{μ_r} are rectilinear planar, with the same argument as above, G'_{μ_l} and G'_{μ_r} are both rectilinear planar with representability intervals $[m_l - b_l, M_l + b_l]$ and $[m_r - b_r, M_r + b_r]$, respectively. Consider the plane graph G'_ν obtained by the union of G'_{μ_l} and G'_{μ_r}. Since by hypothesis $b_\nu = \delta([m_l - M_r, M_l - m_r], \Delta_\nu)$, we have that $[m_l - M_r - b_\nu, M_l - m_r + b_\nu] \cap \Delta_\nu \neq \emptyset$. Since $[m_l - b_l - M_r - b_r, M_l + b_l - m_r + b_r] = [m_l - M_r - b_\nu, M_l - m_r + b_\nu]$ we have $[m_l - b_l - M_r - b_r, M_l + b_l - m_r + b_r] \cap \Delta_\nu \neq \emptyset$. It follows that G'_ν admits a rectilinear planar representation H'_ν. By replacing the b_ν subdivision vertices of H'_ν with bends, we get an orthogonal representation of G_ν with b_ν bends.

Proof of Property (ii). Suppose without loss of generality that $\mathrm{outdeg}(u) \leq \mathrm{outdeg}(v)$. Denote by σ_{μ_l} and σ_{μ_r} the spirality values of orthogonal representations of G_{μ_l} and G_{μ_r}, respectively. We distinguish three cases, based on the types of ν, namely type $\mathrm{I_2O_{\lambda\beta}}$, $\mathrm{I_{3d}O_{\lambda\beta}}$, and type $\mathrm{I_{3dd'}}$. Since the proof line and the arguments for each case are similar, we report here only the analysis for type $\mathrm{I_2O_{\lambda\beta}}$ and move the analyses of the other two cases to "Appendix C".

Case $\mathrm{I_2O_{\lambda\beta}}$: in this case $\Delta_\nu = [2, 4-\gamma]$, $m = \max\{m_l - 2, m_r\} + \frac{\gamma}{2}$, and $M = \min\{M_l, M_r + 2\} - \frac{\gamma}{2}$. We show that the set I'_ν is an interval of feasible spirality values for the orthogonal representations of G_ν with b_ν bends. Suppose first that G_ν has an orthogonal representation H_ν with b_ν bends, and let σ_ν be the spirality of H_ν. We prove that $\sigma_\nu \in [m - b_\nu, M + b_\nu]$. Let b_l and b_r be the number of bends in the restriction of H_ν to G_{μ_l} and to G_{μ_r}, respectively, where $b_l + b_r = b_\nu$. By Lemma 14, we can assume that all the b_l bends are along an exposed edge of G_{μ_l} and all the b_r bends are along an exposed edge of G_{μ_r}. Since $\sigma_{\mu_l} \in [m_l - b_l, M_l + b_l]$ and $b_l \in [0, b_\nu]$ we have $\sigma_{\mu_l} \in [m_l - b_\nu, M_l + b_\nu]$. Also, by Lemma 3 we have $\sigma_\nu = \sigma_{\mu_l} - k_u^l \alpha_u^l - k_v^l \alpha_v^l$. By Lemma 16, we have $-k_u^l \alpha_u^l - k_v^l \alpha_v^l \in [\frac{\gamma}{2} - 2, -\frac{\gamma}{2}]$. Hence, $\sigma_\nu \in [m_l - b_\nu + \frac{\gamma}{2} - 2, M_l + b_\nu - \frac{\gamma}{2}]$. With a symmetric argument on σ_{μ_r} we have that $\sigma_\nu \in [m_r - b_\nu + \frac{\gamma}{2}, M_r + b_\nu + 2 - \frac{\gamma}{2}]$. It follows that $\sigma_\nu \in [m_l - b_\nu + \frac{\gamma}{2} - 2, M_l + b_\nu - \frac{\gamma}{2}] \cap [m_r - b_\nu + \frac{\gamma}{2}, M_r + b_\nu + 2 - \frac{\gamma}{2}] = [\max\{m_l - 2, m_r\} + \frac{\gamma}{2} - b_\nu, \min\{M_l, M_r + 2\} - \frac{\gamma}{2} + b_\nu] = [m - b_\nu, M + b_\nu]$.

To complete the proof it remains to show that for every $\sigma_\nu \in [m - b_\nu, M + b_\nu]$, there exists an orthogonal representation H_ν of G_ν with b_ν bends and with spirality σ_ν. This is equivalent to showing that there exists a plane graph G'_ν obtained by adding b_ν subdivision vertices along some edges of G_ν such that G'_ν has a rectilinear orthogonal representation with spirality σ_ν. To construct G'_ν we insert a suitable number $b_l \in [0, b_\nu]$ of subdivision vertices on an exposed edge of G_{μ_l} and $b_r = b_\nu - b_l$ subdivision vertices on an exposed edge of G_{μ_r}. Let G'_{μ_l} and G'_{μ_r} be the resulting graphs. Since by hypothesis G_{μ_l} and G_{μ_r} are rectilinear planar, also G'_{μ_l} and G'_{μ_r} are rectilinear planar. With the same reasoning as in the proof of Property (i), the representability intervals of G'_{μ_l} and G'_{μ_r} are $[m_l - b_l, M_l + b_l]$ and $[m_r - b_r, M_r + b_r]$. We now describe how to compute b_l and, consequently, b_r.

Since by hypothesis G_ν is not rectilinear planar we have $[m_l - M_r, M_l - m_r] \cap \Delta_\nu = \emptyset$, i.e., $[m_l - M_r, M_l - m_r] \cap [2, 4-\gamma] = \emptyset$. We consider two subcases:

- $M_l - m_r < 2$. In this case, by Property (i) we have $b_\nu = \delta([m_l - M_r, M_l - m_r], [2, 4-\gamma]) = 2 - M_l + m_r$. For example, in Fig. 16 we have $\gamma = 0$ and $b_\nu = 3$. We set $b_l = \sigma_\nu - M_l + h$, where h is a number (either integer or semi-integer) in the interval $[\frac{\gamma}{2}, 2 - \frac{\gamma}{2}]$ such that $b_l \in [0, b_\nu]$. We first prove that such a value h always exists.
  Suppose first that $\sigma_\nu \in [m - b_\nu, m - b_\nu + 1]$. In this case we choose $h = 2 - \frac{\gamma}{2}$. This implies that $b_l = \sigma_\nu - M_l + 2 - \frac{\gamma}{2} \in [m - b_\nu - M_l + 2 - \frac{\gamma}{2}, m - b_\nu + 1 - M_l + 2 - \frac{\gamma}{2}]$. For example, in Fig. 16 for every $\sigma_\nu \in [m - b_\nu, m - b_\nu + 1] = [-2, -1]$ we set $b_l = \sigma_\nu - M_l + 2 - \frac{\gamma}{2} = \sigma_\nu + 2$. Since $M_l - m_r < 2$ we have $m_l - 2 \leq M_l - 2 < m_r$ and hence $m_l - 2 < m_r$. Since $m = \max\{m_l - 2, m_r\} + \frac{\gamma}{2}$, we have $m = m_r + \frac{\gamma}{2}$. Also, since we have $b_\nu = 2 - M_l + m_r$, it follows $m - b_\nu - M_l + 2 - \frac{\gamma}{2} = m_r + \frac{\gamma}{2} - b_\nu - M_l + 2 - \frac{\gamma}{2} = 0$. Hence, $b_l \in [0, 1]$. Since by hypothesis G_ν is not rectilinear planar, $b_\nu \geq 1$, and therefore there exists

a value of $h \in [\frac{\gamma}{2}, 2-\frac{\gamma}{2}]$ such that $b_l \in [0, b_\nu]$.
Suppose now that $\sigma_\nu \in [m-b_\nu+2, M+b_\nu]$. In this case we choose $h = \frac{\gamma}{2}$. This implies that $b_l = \sigma_\nu - M_l + \frac{\gamma}{2} \in [m-b_\nu+2-M_l+\frac{\gamma}{2}, M+b_\nu-M_l+\frac{\gamma}{2}]$. For example, in Fig. 16 $\sigma_\nu \in [m-b_\nu+2, M+b_\nu] = [0,3]$ and we set $b_l = \sigma_\nu - M_l - \frac{\gamma}{2} = \sigma_\nu$. We have $m-b_\nu+2-M_l+\frac{\gamma}{2} = m_r + \frac{\gamma}{2} - b_\nu + 2 - M_l + \frac{\gamma}{2} = \gamma$. It follows that $b_l \in [\gamma, M+b_\nu-M_l+\frac{\gamma}{2}]$. Since $M_l < m_r+2 \leq M_r+2$ we have $M_l < M_r+2$. Also, since $M = \min\{M_l, M_r+2\} - \frac{\gamma}{2}$ it follows that $M = M_l - \frac{\gamma}{2}$. Hence, $M+b_\nu-M_l+\frac{\gamma}{2} = M_l - \frac{\gamma}{2} + b_\nu - M_l + \frac{\gamma}{2} = b_\nu$. Hence, $b_l \in [\gamma, b_\nu]$. Since $\gamma \geq 0$, also in this case there exists a value of $h \in [\frac{\gamma}{2}, 2-\frac{\gamma}{2}]$ such that $b_l \in [0, b_\nu]$. We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = M_l + b_l = M_l + \sigma_\nu - M_l + h = \sigma_\nu + c$ and G'_{μ_r} with spirality $\sigma'_{\mu_r} = m_r - b_r = m_r - (b_\nu - b_l) = m_r - b_\nu + b_l = m_r - b_\nu + \sigma_\nu - M_l + h = m_r - M_l - b_\nu + \sigma_\nu + h = b_\nu - 2 - b_\nu + \sigma_\nu + h = \sigma_\nu + h - 2$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + h - (\sigma_\nu + h - 2) = 2$, and by Lemma 7, G'_ν is rectilinear planar. It remains to show that G'_ν admits a rectilinear planar representation with spirality $\sigma'_\nu = \sigma_\nu$. Given the choice of σ'_{μ_l} and σ'_{μ_r}, by Lemma 3 every rectilinear planar representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k^l_u\alpha^l_u - k^r_u\alpha^l_v = \sigma_\nu + h - k^l_u\alpha^l_u - k^r_u\alpha^l_v$. Since $h \in [\frac{\gamma}{2}, 2-\frac{\gamma}{2}]$, by Lemma 16 there exists a value $k^l_u\alpha^l_u + k^r_u\alpha^l_v$ such that $h - k^l_u\alpha^l_u - k^r_u\alpha^l_v = 0$, and thus $\sigma'_\nu = \sigma_\nu$. For example, in Fig. 16, for every $\sigma_\nu \in I_\nu = [-2, 3]$, there is a rectilinear representation of G_ν with $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$, b_l and b_r chosen as described above, and spirality σ_ν.

- $m_l - M_r > 4 - \gamma$. In this case, by Property (i) we have $b_\nu = \delta([m_l - M_r, M_l - m_r], [2, 4-\gamma]) = m_l - M_r - 4 + \gamma$. We set $b_l = m_l - \sigma_\nu - 2 + \frac{\gamma}{2}$. We first prove that $b_l \in [0, b_\nu]$. We have $b_l = m_l - \sigma_\nu - 2 + \frac{\gamma}{2} \in [m_l - M - b_\nu - 2 + \frac{\gamma}{2}, m_l - m + b_\nu - 2 + \frac{\gamma}{2}]$. Since $M_l \geq m_l > M_r + 4 - \gamma \geq M_r + 2$, we have $M = M_r + 2 - \frac{\gamma}{2}$. Hence, $m_l - M - b_\nu - 2 + \frac{\gamma}{2} = m_l - (M_r + 2 - \frac{\gamma}{2}) - b_\nu - 2 + \frac{\gamma}{2} = m_l - M_r - 4 + \gamma - b_\nu = 0$. Also, since $m_l - 2 \geq m_l - 4 + \gamma > M_r \geq m_r$, we have $m = m_l - 2 + \frac{\gamma}{2}$. Hence, $m_l - m + b_\nu - 2 + \frac{\gamma}{2} = m_l - (m_l - 2 + \frac{\gamma}{2}) + b_\nu - 2 + \frac{\gamma}{2} = b_\nu$. It follows that $b_l \in [0, b_\nu]$.
We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = m_l - b_l = m_l - (m_l - \sigma_\nu - 2 + \frac{\gamma}{2}) = \sigma_\nu + 2 - \frac{\gamma}{2}$ and G'_{μ_r} with spirality $\sigma'_{\mu_r} = M_r + b_r = M_r + b_\nu - b_l = M_r + m_l - M_r - 4 + \gamma - m_l + \sigma_\nu + 2 - \frac{\gamma}{2} = \sigma_\nu + \frac{\gamma}{2} - 2$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + 2 - \frac{\gamma}{2} - (\sigma_\nu + \frac{\gamma}{2} - 2) = 4 - \gamma$ and, by Lemma 7, G'_ν is rectilinear planar. It remains to show that G'_ν admits a rectilinear representation with spirality $\sigma'_\nu = \sigma_\nu$. Given the choice of σ'_{μ_l} and σ'_{μ_r}, by Lemma 3 every rectilinear representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k^l_u\alpha^l_u - k^r_u\alpha^l_v = \sigma_\nu + 2 - \frac{\gamma}{2} - k^l_u\alpha^l_u - k^r_u\alpha^l_v$. Since $2 - \frac{\gamma}{2} \in [\frac{\gamma}{2}, 2-\frac{\gamma}{2}]$, by Lemma 16 we can set $k^l_u\alpha^l_u + k^r_u\alpha^l_v = 2 - \frac{\gamma}{2}$, and thus $\sigma'_\nu = \sigma_\nu$.

Case $\mathrm{I}_{3d}\mathrm{O}_{\lambda\beta}$ and **Case** $\mathrm{I}_{3dd'}$: see "Appendix C". □

6.2.2 P-Nodes Having a Child with No Exposed Edges

We now consider the case of a P-node ν having a child that does not contain an exposed edge (see Lemmas 20 and 21). Denote by μ such a child node; observe that μ is an

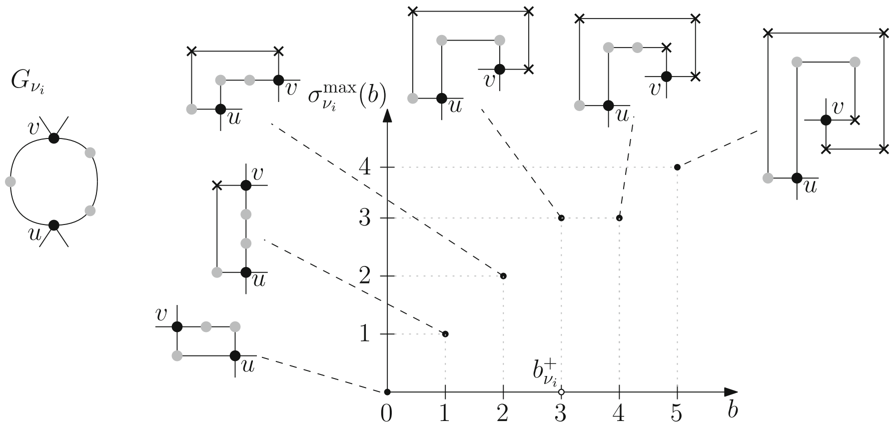

Fig. 17 A P-component G_{ν_i} of type I_2O_{22} and a plot of function $\sigma^{\max}_{\nu_i}(b)$ for $b = 0, 1, \ldots, 5$. For each value of b, the figure depicts an orthogonal representation of (maximum) spirality $\sigma^{\max}_{\nu_i}(b)$ among those with b bends (cross vertices). The positive flexibility breakpoint is $b^+_{\nu_i} = 3$

S-node whose children $\nu_1, \nu_2, \ldots, \nu_h$ ($h \geq 2$) (e.g., ordered from the bottom pole to the top pole of μ) are all P-nodes of type I_2O_{22}. This is because, if one ν_i among $\nu_2, \ldots, \nu_{h-1}$ had a pole of outdegree one, then the external edge of G_{ν_i} incident to this pole would be part of a Q*-node child of μ, thus contradicting the fact that μ has no exposed edges; also, the bottom pole of ν_1 (resp. of ν_h) coincides with the bottom pole (resp. top pole) of ν, and has outdegree two in G_{ν_1} (resp. in G_{ν_h}).

For a given integer value $b \geq 0$, denote by $\sigma^{\max}_{\nu_i}(b)$ the *maximum* value of spirality that any orthogonal representation of G_{ν_i} with at most b bends can have ($i \in \{1, \ldots, h\}$). When we consider the value $b+1$, the value $\sigma^{\max}_{\nu_i}(b+1)$ may or may not increase by one unit with respect to $\sigma^{\max}_{\nu_i}(b)$, i.e., either $\sigma^{\max}_{\nu_i}(b+1) = \sigma^{\max}_{\nu_i}(b)+1$ or $\sigma^{\max}_{\nu_i}(b+1) = \sigma^{\max}_{\nu_i}(b)$. By plotting the function $\sigma^{\max}_{\nu_i}(b)$ as b increases, we define the *positive flexibility breakpoint of* G_{ν_i} as the maximum number of bends $b^+_{\nu_i}$ such that for every non-negative integer $b < b^+_{\nu_i}$, we have $\sigma^{\max}_{\nu_i}(b+1) = \sigma^{\max}_{\nu_i}(b) + 1$. For example, Fig. 17 shows a component G_{ν_i} and a plot of the function $\sigma^{\max}_{\nu_i}(b)$ for $b = 0, 1, \ldots, 5$. In the figure, the positive flexibility breakpoint is $b^+_{\nu_i} = 3$ because passing from $b = 3$ to $b = 4$ does not allow us to have an orthogonal representation of G_{ν_i} with a larger value of spirality. The *positive flexibility breakpoint of the S-node* μ is denoted as b^+_μ and it is defined as the sum of the positive flexibility breakpoints of its children, i.e., $b^+_\mu = \sum_{i=1}^{h} b^+_{\nu_i}$.

Symmetrically, for a given integer $b \geq 0$, let $\sigma^{\min}_{\nu_i}(b)$ be the *minimum* value of spirality that an orthogonal representation of G_{ν_i} with at most b bends can have ($i \in \{1, \ldots, h\}$). The value of $\sigma^{\min}_{\nu_i}(b+1)$ may or may not decrease by one unit with respect to $\sigma^{\min}_{\nu_i}(b)$. We define the *negative flexibility breakpoint of* G_{ν_i} as the maximum number of bends $b^-_{\nu_i}$ such that for every non-negative integer $b < b^-_{\nu_i}$, we have $\sigma^{\min}_{\nu_i}(b+1) = \sigma^{\min}_{\nu_i}(b) - 1$. For example, Fig. 18 shows the same component G_{ν_i} as in Fig. 17 and a plot of the function $\sigma^{\min}_{\nu_i}(b)$ for $b = 0, 1, \ldots, 4$. In the figure,

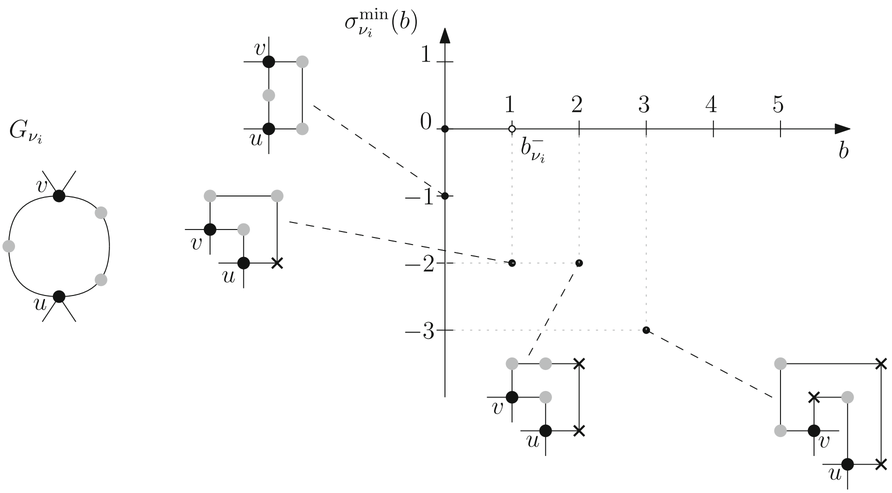

Fig. 18 A P-component G_{ν_i} of type I_2O_{22} and a plot of function $\sigma^{\min}_{\nu_i}(b)$ for $b = 0, 1, \ldots, 4$. For each value of b, the figure depicts an orthogonal representation of (minimum) spirality $\sigma^{\min}_{\nu_i}(b)$ among those with b bends (cross vertices). The negative flexibility breakpoint is $b^-_{\nu_i} = -1$

the negative flexibility breakpoint is $b^-_{\nu_i} = 1$ because passing from $b = 1$ to $b = 2$ does not allow us to have an orthogonal representation of G_{ν_i} with a smaller value of spirality. The *negative flexibility breakpoint of the S-node* μ is denoted as b^-_{μ} and it is defined as the sum of the negative flexibility breakpoints of its children, i.e., $b^-_{\mu} = \sum_{i=1}^{h} b^-_{\nu_i}$. The next lemma establishes how to compute in constant time the positive and negative flexibility breakpoints for a P-node ν_i of type I_2O_{22}, given the representability intervals of its children.

Lemma 19 *Let ν_i be a P-node of type I_2O_{22} with children μ_l and μ_r such that G_{ν_i} is rectilinear planar. Let $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$ be the representability intervals of G_{μ_l} and G_{μ_r}, respectively. We have $b^+_{\nu_i} = |M_r + 2 - M_l|$ and $b^-_{\nu_i} = |m_l - 2 - m_r|$. Also, an orthogonal representation of G_{ν_i} with spirality $\sigma^{\max}_{\nu_i}(b^+_{\nu_i})$ (resp. with $\sigma^{\min}_{\nu_i}(b^-_{\nu_i})$) can be obtained by inserting all the bends on an exposed edge of either G_{μ_l} or G_{μ_r}.*

Proof We prove that $b^+_{\nu_i} = |M_r + 2 - M_l|$. The proof that $b^-_{\nu_i} = |m_l - m_r - 2|$ is symmetric. Consider a rectilinear planar representation H_{ν_i} of G_{ν_i} with maximum value of spirality, that is spirality $\sigma^{\max}_{\nu_i}(0)$ which, by Table 1

is $\sigma^{\max}_{\nu_i}(0) = \min\{M_l - 1, M_r + 1\}$. Let σ_{μ_l} and σ_{μ_r} be the spirality values of the left and the right orthogonal components of H_{ν_i}, respectively. By Lemma 7, we know that $\sigma_{\mu_l} - \sigma_{\mu_r} = 2$. Also, by Lemma 3, $\sigma^{\max}_{\nu_i}(0) = \sigma_{\mu_l} - 1 = \sigma_{\mu_r} + 1$. By Table 1, either $\sigma_{\mu_l} = M_l$ or $\sigma_{\mu_r} = M_r$ (possibly both). Three cases are possible:

- $\sigma_{\mu_l} = M_l$ and $\sigma_{\mu_r} = M_r$. Note that, since $\sigma_{\mu_l} = \sigma_{\mu_r} + 2$, in this case we have $|M_r + 2 - M_l| = 0$. We show that if we are allowed to subdivide an edge of G_{ν_i} with exactly one degree-2 vertex, so to obtain a graph G'_{ν_i}, the maximum value of spirality that a rectilinear planar representation H'_{ν_i} of G'_{ν_i} equals $\sigma^{\max}_{\nu_i}(0)$. In other words, we show that $\sigma^{\max}_{\nu_i}(1) = \sigma^{\max}_{\nu_i}(0)$. By Lemma 14, we can assume that the degree-2 vertex subdivides an exposed edge, which belongs either to G_{μ_l} or to G_{μ_r}. Suppose that this degree-2 vertex is inserted along an exposed edge of G_{μ_l}. Denote by G'_{μ_l} and G'_{μ_r} the left component and the right component of G'_{ν_i}, respectively. The maximum spirality of a rectilinear planar representation of G'_{μ_l} is $M_l + 1$, while the maximum spirality of a rectilinear planar representation of G'_{μ_r} remains M_r, because G'_{μ_r} coincides with G_{μ_r}. Assume for a contradiction that G'_{ν_i} admits a rectilinear planar representation H'_{ν_i} with spirality $\sigma^{\max}_{\nu_i}(0) + 1$, and denote by σ'_{μ_l} and σ'_{μ_r} the spirality values of the left and right components of H'_{ν_i}. We should have that $\sigma^{\max}_{\nu_i}(0)+1 = \sigma'_{\mu_l} - 1$ and $\sigma^{\max}_{\nu_i}(0)+1 = \sigma'_{\mu_r} + 1$. Now, $\sigma^{\max}_{\nu_i}(0)+1 = M_l$ and $\sigma^{\max}_{\nu_i}(0)+1 = (M_r+1)+1$, which implies that G'_{μ_r} should have a rectilinear planar representation with spirality $M_r + 1$, a contradiction. A symmetric argument applies if we subdivide an exposed edge of G_{μ_r}. Hence, $b^+_{\nu_i} = 0 = |M_r + 2 - M_l|$.

- $\sigma_{\mu_l} = M_l$ and $\sigma_{\mu_r} < M_r$. Subdivide an exposed edge of G_{ν_l} with a degree 2-vertex, and call G'_{ν_i} the graph resulting from G_{ν_i} after this subdivision. As in the previous case, denote by G'_{μ_l} and G'_{μ_r} the left and the right components of G'_{ν_i}, where G'_{μ_r} coincides with G_{μ_r}. We show that G'_{ν_i} admits a rectilinear planar representation with spirality $\sigma'_{\nu} = \sigma^{\max}_{\nu_i}(0)+1$. Since we have added a subdivision vertex on an exposed edge of G_{μ_l}, there exists a rectilinear planar representation H'_{μ_l} of G'_{μ_l} with spirality $\sigma'_{\mu_l} = M_l + 1$. Also, since $\sigma_{\mu_r} < M_r$, there exists a rectilinear planar representation of G'_{μ_r} with spirality $\sigma'_{\mu_r} = \sigma_{\mu_r} + 1$. Hence, $\sigma'_{\mu_l} - \sigma'_{\mu_r} = 2$ and therefore, by Lemma 7, we can merge the representations H'_{μ_l} and H'_{μ_r} into a rectilinear planar representation H'_{ν_i} of G'_{ν_i}. The spirality of H'_{ν_i} is $\sigma'_{\nu_i} = \sigma^{\max}_{\nu_i}(0)+1$. By replacing the subdivision vertex of H'_{μ_l} with a bend, we get an orthogonal representation of G_{ν_i} with one bend and with spirality $\sigma^{\max}_{\nu_i}(0)+1$, i.e., $\sigma^{\max}_{\nu_i}(1) = \sigma^{\max}_{\nu_i}(0)+1$. Iterate this procedure until $\sigma'_{\mu_r} = M_r$, i.e., until $\sigma'_{\nu_i} = M_r + 1$. Denote by b the number of bends added in total. By Lemma 3, the spirality of the resulting orthogonal representation is $\sigma^{\max}_{\nu_i}(b) = M_l + b - 1 = M_r + 1$, and hence $b = M_r + 2 - M_l$. If we consider the rectilinear representation H'_{ν_i} where the b bends are replaced with degree-2 vertices, its left and right components have the maximum possible spirality in their representability intervals. Hence, the previous case applies, and inserting exactly one subdivision vertex to G'_{ν_i} does not result into a graph that admits a rectilinear planar representation with spirality greater than the one of H'_{ν_i}. It follows that $b^+_{\nu_i} = M_r + 2 - M_l$.
- $\sigma_{\mu_l} < M_l$ and $\sigma_{\mu_r} = M_r$. With a symmetric argument as in the previous case, $b^+_{\nu_i} = M_l - (M_r + 2)$.

□

The next lemma gives the rule for computing the budget of a P-node with two children such that the left one has no exposed edge, and for determining the corresponding interval of spirality values.

Lemma 20 *Let ν be a P-node with two children μ_l and μ_r, such that μ_l has no exposed edge. Let G_{μ_l} and G_{μ_r} be rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. If G_ν is not rectilinear planar* *then: (i) the budget for* ν *is* $b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3])$*; and (ii) the interval of spirality values for an orthogonal representation of* G_ν *with* b_ν *bends is* $I'_\nu = [m - b_\nu, M + \min\{b^+_{\mu_l}, b_\nu\}]$ *if* $M_l - m_r < 3$ *and* $I'_\nu = [m - \min\{b^-_{\mu_l}, b_\nu\}, M + b_\nu]$ *if* $m_l - M_r > 3$.

Proof Since by hypothesis G_ν is not rectilinear planar, by Table 1 we have $3 \notin [m_l - M_r, M_l - m_r] \cap \Delta_\nu = \emptyset$, where $\Delta_\nu = [3, 3]$. In other words, $3 \notin [m_l - M_r, M_l - m_r]$. Proof of Property (i). We first prove that $b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3])$ bends are necessary. Suppose for a contradiction that G_ν admits an orthogonal representation H'_ν with $b'_\nu < b_\nu$ bends. Let b'_l and b'_r be the number of bends in the restriction of H'_ν to G_{μ_l} and to G_{μ_r}, respectively. By Lemma 14, we can assume that all the bends b'_r are along an exposed edge of G_{μ_r}. Consider the underlying graph G'_ν of H'_ν obtained by replacing each bend of H'_ν with a subdivision vertex. G'_ν is rectilinear planar. Let G'_{μ_l} and G'_{μ_r} be the left and right component of G'_ν, respectively. Hence G'_{μ_r} is rectilinear planar with representability interval $[m_r - b'_r, M_r + b'_r]$. About G'_{μ_l}, its representability interval $I'_{\mu_l} = [p, q]$ is such that $[p, q] \subseteq [m_l - b'_l, M_l + b'_l]$. In particular $[p, q] = [m_l - b'_l, M_l + b'_l]$ when $b^-_{\mu_l} \geq b'_l$ and $b^+_{\mu_l} \geq b'_l$. By Table 1, the representability condition for G'_ν is $[p - M_r - b'_r, q - m_r + b'_r] \cap [3, 3] \neq \emptyset$. Since $p \geq m_l - b'_l$ and $q \leq M_l + b'_l$ we have $[p - M_r - b'_r, q - m_r + b'_r] \subseteq [m_l - b'_l - M_r - b'_r, M_l + b'_l - m_r + b'_r]$, and therefore $[m_l - b'_l - M_r - b'_r, M_l + b'_l - m_r + b'_r] \cap [3, 3] \neq \emptyset$. This equals to say that $[m_l - M_r - b'_\nu, M_l - m_r + b'_\nu] \cap [3, 3] \neq \emptyset$, which implies that $\delta([m_l - M_r, M_l - m_r], [3, 3]) \leq b'_\nu < b_\nu$, a contradiction.

We now prove that b_ν bends suffice. Insert b_ν subdivision vertices on any exposed edge of G_{μ_r}, and let G'_{μ_r} be the resulting component. Since G_{μ_r} is rectilinear planar by hypothesis, G'_{μ_r} is also rectilinear planar and its representability interval is $[m_r - b_\nu, M_r + b_\nu]$. Consider the plane graph G'_ν obtained by the union of G_{μ_l} and G'_{μ_r}. Since by hypothesis $b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3])$, we have that $[m_l - M_r - b_\nu, M_l - m_r + b_\nu] \cap [3, 3] \neq \emptyset$. It follows that G'_ν is rectilinear planar. Consider any rectilinear representation H'_ν of G'_ν and replace each of its subdivision vertices with a bend. Since by construction G'_ν has b_ν subdivision vertices, the resulting orthogonal representation has at most b_ν bends.

Proof of Property (ii). Since by hypothesis $3 \notin [m_l - M_r, M_l - m_r]$ we have two cases: either $M_l - m_r < 3$ or $M_r - m_l > 3$. For each of them, we must show that every orthogonal representation of G_ν with b_ν bends has spirality σ_ν in the interval I'_ν and that for every value $\sigma_\nu \in I'_\nu$ there exists an orthogonal representation of G_ν with b_ν bends and spirality σ_ν. We show the argument for the case $M_l - m_r < 3$ (the other case is treated analogously).

By Table 1 we have $M = \min\{M_l - 1, M_r + 2\}$ and $m = \max\{m_l - 1, m_r + 2\}$. Since $M_l < m_r + 3 \leq M_r + 3$, we have $M_l - 1 < M_r + 2$ and since $m_l - 3 \leq M_l - 3 < m_r$, we have $m_l - 1 < m_r + 2$. Hence, $M = M_l - 1$ and $m = m_r + 2$.

Suppose first that G_ν has an orthogonal representation H_ν with b_ν bends. We prove that $\sigma_\nu \in [m - b_\nu, M + \min\{b^+_{\mu_l}, b_\nu\}]$. Let σ_ν be the spirality of H_ν, and let σ_{μ_l} and σ_{μ_r} be the spirality values of the restrictions H_{μ_l} and H_{μ_r} of H_ν to G_{μ_l} and G_{μ_r}, respectively. By Lemma 7 we have $\sigma_{\mu_l} - \sigma_{\mu_r} = 3$. Let b_l and b_r be the number of bends in H_{μ_l} and H_{μ_r}, respectively. Clearly, $b_l + b_r = b_\nu$. We now prove the following claim. □

Claim 2 $b_l \in [0, \min\{b^+_{\mu_l}, b_\nu\}]$.

Claim Proof Suppose for a contradiction that $b_l \notin [0, \min\{b^+_{\mu_l}, b_\nu\}]$. Clearly, $b_l \in [0, b_\nu]$, and thus, under this hypothesis, it must be $\min\{b^+_{\mu_l}, b_\nu\} = b^+_{\mu_l}$ and $b_l \in [b^+_{\mu_l} + 1, b_\nu]$. We show that there exists an orthogonal representation H'_ν of G_ν with less than b_ν bends, which is impossible by definition of budget b_ν. We will denote by H'_{μ_l} and H'_{μ_r} the restrictions of H'_ν to G_{μ_l} and to G_{μ_r}, respectively. Also, b'_l and b'_r will denote the number of bends of H'_{μ_l} and H'_{μ_r}, while σ'_{μ_l} and σ'_{μ_r} will denote the spirality values of H'_{μ_l} and H'_{μ_r}, respectively. We distinguish two cases: (a) $b_l = b^+_{\mu_l} + 1$ and (b) $b_l \in [b^+_{\mu_l} + 2, b_\nu]$.

In case (a), we set $b'_l = b_l - 1 = b^+_{\mu_l}$ and $b'_r = b_r$. By definition of positive flexibility breakpoint it is possible to set $\sigma'_{\mu_l} = \sigma_{\mu_l}$. Also, we set $\sigma'_{\mu_r} = \sigma_{\mu_r}$. Since $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_{\mu_l} - \sigma_{\mu_r} = 3$, by Lemma 7, H'_ν exists and it has $b'_l + b'_r = b_\nu - 1 < b_\nu$ bends, a contradiction.

In case (b), we set $b'_l = b_l - 2$ and $b'_r = b_r + 1$. Again by definition of positive flexibility breakpoint, it is possible to set $\sigma'_{\mu_l} = \sigma_{\mu_l} - 1$. Also, since G_{μ_r} has an exposed edge, by Lemma 14 we can use such an edge to place the extra bend of the right component. It follows that, we can set $\sigma'_{\mu_r} = \sigma_{\mu_r} - 1$. Therefore, $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_{\mu_l} - \sigma_{\mu_r} = 3$ and, by Lemma 7, H'_ν exists and it has $b'_l + b'_r < b_l + b_r = b_\nu$ bends, a contradiction.

Since G_ν is of type I_{3ll}, by Lemma 3 and Table 2 we have $\sigma_\nu = \sigma_{\mu_l} - 1 = \sigma_{\mu_r} + 2$. By the claim we have $b_l \in [0, \min\{b^+_{\mu_l}, b_\nu\}]$, hence $m_l - \min\{b^+_{\mu_l}, b_\nu\} \le \sigma_{\mu_l} \le M_l + \min\{b^+_{\mu_l}, b_\nu\}$ and hence $\sigma_\nu \in [m_l - \min\{b^+_{\mu_l}, b_\nu\} - 1, M_l + \min\{b^+_{\mu_l}, b_\nu\} - 1]$. Also, $b_r \in [0, b_\nu]$ and $\sigma_\nu \in [m_r - b_\nu + 2, M_r + b_\nu + 2]$. Hence $\sigma_\nu \ge \max\{m_l - \min\{b^+_{\mu_l}, b_\nu\} - 1, m_r - b_\nu + 2\}$. By Property (i), $b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3]) = 3 - M_l + m_r$, hence, $m_r - b_\nu + 2 = m_r - 3 + M_l - m_r + 2 = M_l - 1 \ge m_l - \min\{b^+_{\mu_l}, b_\nu\} - 1$. It follows that $\sigma_\nu \ge m_r - b_\nu + 2$. Also, $\sigma_\nu \le \min\{M_l + \min\{b^+_{\mu_l}, b_\nu\} - 1, M_r + b_\nu + 2\}$. Since by hypothesis $M_l - 1 \le m_r + 2 \le M_r + 2$ and since $\min\{b^+_{\mu_l}, b_\nu\} \le b_\nu$, we have $M_l + \min\{b^+_{\mu_l}, b_\nu\} - 1 \le M_r + b_\nu + 2$ and $\sigma_\nu \le M_l + \min\{b^+_{\mu_l}, b_\nu\} - 1$. Hence, $\sigma_\nu \in [m_r - b_\nu + 2, M_l + \min\{b^+_{\mu_l}, b_\nu\} - 1] = [m - b_\nu, M + \min\{b^+_{\mu_l}, b_\nu\}] = I'_\nu$.

It remains to show that for every $\sigma_\nu \in I'_\nu = [m - b_\nu, M + \min\{b^+_{\mu_l}, b_\nu\}]$, there exists an orthogonal representation H_ν of G_ν with b_ν bends and with spirality σ_ν. With the same notation and arguments as in Lemma 18, we show how to set the number of bends b_l and b_r that can be assigned to the left and right component to guarantee the existence of H_ν. By Property (i), $b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3]) = 3 - M_l + m_r$. By setting $b_l = \sigma_\nu - M_l + 1$ and $b_r = b_\nu - b_l$, we have $b_l \in [m - b_\nu - M_l + 1, M + \min\{b^+_{\mu_l}, b_\nu\} - M_l + 1]$, and hence $b_l \in [m_r + 2 - b_\nu - M_l + 1, M_l - 1 + \min\{b^+_{\mu_l}, b_\nu\} - M_l + 1] = [0, \min\{b^+_{\mu_l}, b_\nu\}]$, i.e., $b_l \in [0, \min\{b^+_{\mu_l}, b_\nu\}]$.

Let $\nu_1, \ldots, \nu_h$ be the children of μ_l. Since G_{μ_l} has no exposed edge, each G_{ν_i} ($i = 1, \ldots, h$) is a parallel component of type $\mathrm{I}_2\mathrm{O}_{22}$ and Lemma 19 applies. We distribute the b_l bends on the components G_{ν_i} in such a way that the number of bends assigned to G_{ν_i} is at most $b^+_{\nu_i}$ ($i = 1, \ldots, h$). This is always possible, because, as showed above, $b_l \in [0, \min\{b^+_{\mu_l}, b_\nu\}]$. The bends assigned to each G_{ν_i} are inserted on an exposed edge according to the procedure in the proof of Lemma 19. The graph G'_{μ_l}

obtained from G_{μ_l} by regarding each bend as a subdivision vertex admits a rectilinear planar representation H'_{μ_l} with spirality $\sigma'_{\mu_l} = M_l + b_l = M_l + \sigma_\nu - M_l + 1 = \sigma_\nu + 1$. We now place b_r bends along an exposed edge of G_{μ_r}. The graph G'_{μ_r} obtained from G_{μ_r} by regarding each bend as a subdivision vertex admits a rectilinear planar representation H'_{μ_r} with spirality $\sigma'_{\mu_r} = m_r - b_r = m_r - (b_\nu - b_l) = m_r - b_\nu + \sigma_\nu - M_l + 1 = (m_r - M_l + 1) - b_\nu + \sigma_\nu = b_\nu - 2 - b_\nu + \sigma_\nu = \sigma_\nu - 2$. It follows that $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + 1 - (\sigma_\nu - 2) = 3$ and, by Lemma 11, G'_ν is rectilinear planar and it admits a rectilinear planar representation H'_ν obtained by combining in parallel H'_{μ_l} and H'_{μ_r}. By Lemma 3, H'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k_u^{l\,\prime}\alpha_u^{l\,\prime} - k_u^{l\,\prime}\alpha_v^{l\,\prime}$. Since u and v have degree four, we have $\alpha_u^{l\,\prime} = \alpha_v^{l\,\prime} = \alpha_u^{r\,\prime} = \alpha_v^{r\,\prime} = 1$. Also, by Table 2, $k_u^{l\,\prime} = k_u^{r\,\prime} = \frac{1}{2}$. Hence, $\sigma'_\nu = \sigma_\nu + 1 - 1 = \sigma_\nu$. Replacing the subdivision vertices of H'_ν with bends we get an orthogonal representation of G_ν with b_ν bends and spirality σ_ν.

The proof of Lemma 21 considers the case where the right child of a P-node with two children has no exposed edge. The proof is symmetric to that of Lemma 20, hence it is omitted.

Lemma 21 *Let ν be a P-node with two children μ_l and μ_r, such that μ_r has no exposed edge. Let G_{μ_l} and G_{μ_r} be rectilinear planar with representability intervals $I_{\mu_l} = [m_l, M_l]$ and $I_{\mu_r} = [m_r, M_r]$, respectively. If G_ν is not rectilinear planar, then: (i) the budget for ν is $b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3])$; and (ii) the interval of spirality values for an orthogonal representation of G_ν with b_ν bends is $I'_\nu = [m - \min\{b^-_{\mu_r}, b_\nu\}, M + b_\nu]$ if $m_l - M_r > 3$ and $I'_\nu = [m - b_\nu, M + \min\{b^+_{\mu_r}, b_\nu\}]$ if $M_l - m_r < 3$.*

6.3 Budget of the Root

We finally show how to compute the budget of the root ρ of T. Recall that ρ is the P^r-node describing the parallel composition of the reference edge $e = (u, v)$ with the rest of the graph. If e is a dummy edge, the budget of ρ is zero, because e does not need to be drawn. Thus we assume that the graph is biconnected and e is a real edge. Let η be the child of ρ that does not correspond to e, and let u' and v' be the alias vertices associated with the poles u and v of G_η. If G_η is rectilinear planar with representability interval I_η, by the root condition in Table 1 we know that G is rectilinear planar if and only if $I_\eta \cap \Delta_\rho \neq \emptyset$. Recall that Δ_ρ is defined as follows: (i) $\Delta_\rho = [2, 6]$ if u' coincides with u and v' coincides with v; (ii) $\Delta_\rho = [3, 5]$ if exactly one of u' and v' coincides with u and v, respectively; (iii) $\Delta_\rho = 4$ if none of u' and v' coincides with u and v. We prove the following.

Lemma 22 *Let $e = (u, v)$ be the reference edge of G and let ρ be the root of T with respect to e. Let η be the child of ρ that does not correspond to e. Suppose that G_η is rectilinear planar with representability interval I_η. If G is not rectilinear planar then $b_\rho = \delta(I_\eta, \Delta_\rho)$.*

Proof See Fig. 19 for an illustration of the statement.

Let f_{int} be the internal face of G incident to e. Note that H is an orthogonal representation of G if and only if the following two conditions hold: the restriction

H_η of H to G_η is an orthogonal representation; the number A of right turns minus left turns of any simple cycle of G in H containing e and traversed clockwise in H is equal to 4. We have $A = \sigma_\eta - \sigma_e + \alpha_{u'} + \alpha_{v'}$, where: σ_η is the spirality of H_η; σ_e is the spirality of e; for $w \in \{u', v'\}$, $\alpha_w = 1$, $\alpha_w = 0$, and $\alpha_w = -1$ if the angle formed by w in f_{int} equals 90^o, 180^o, or 270^o, respectively.

Since G is not rectilinear planar, H must contain some bends. Denote by b the number of bends of H. Each of these b bends placed along an edge of G_η contributes to increase or decrease σ_η by at most one unit, therefore increasing or decreasing A by at most one unit. Also placing this bend along e contributes to increase or decrease A by at most one unit. Hence, without loss of generality, we can assume that all the b bends are placed along e, which implies that H_η does not contain any bend. It follows that $\sigma_\eta \in I_\eta = [m_\eta, M_\eta]$ and $\sigma_e \in [-b, b]$. We now show that if b is the minimum value such that $A = 4$ then $b = b_\rho$. Observe that when b is minimum, we have $|\sigma_e| = b$, and therefore $|\sigma_e| > 0$.

Consider first Case (i) in the definition of Δ_ρ, i.e., $\Delta_\rho = [2, 6]$. Since in this case the alias vertices coincide with the poles, we have $\alpha_{u'} \in [-1, 1]$, $\alpha_{v'} \in [-1, 1]$, and hence $\alpha_{u'} + \alpha_{v'} \in [-2, 2]$. Since $\sigma_\eta + \alpha_{u'} + \alpha_{v'} + \sigma_e = 4$ there are two possible subcases for any possible choice of the values σ_η, $\alpha_{u'}$, and $\alpha_{v'}$: either (a) $\sigma_\eta + \alpha_{u'} + \alpha_{v'} < 4$ or (b) $\sigma_\eta + \alpha_{u'} + \alpha_{v'} > 4$.

- Case (a). The maximum value for $\sigma_\eta + \alpha_{u'} + \alpha_{v'}$ is $M_\eta + 2$. Hence $M_\eta + 2 < 4$, which implies $M_\eta < 2$. It follows that $b_\rho = \delta(I_\eta, \Delta_\rho) = 2 - M_\eta$. Also, since b is the minimum value such that $A = 4$ and $\sigma_\eta + \alpha_{u'} + \alpha_{v'} < 4$, we have that $b = \sigma_e$ and $M_\eta + 2 + b = 4$, which implies $b = 2 - M_\eta = b_\rho$. Therefore, an orthogonal representation of G with b bends is constructed by placing b bends along e and choosing $\sigma_\eta = M_\eta$.
- Case (b). The minimum value for $\sigma_\eta + \alpha_{u'} + \alpha_{v'}$ is $m_\eta - 2$. Hence $m_\eta - 2 > 4$, which implies $m_\eta > 6$. It follows that $b_\rho = \delta(I_\eta, \Delta_\eta) = m_\eta - 6$. Also, since b is the minimum value such that $A = 4$ and $\sigma_\eta + \alpha_{u'} + \alpha_{v'} > 4$, we have that $b = -\sigma_e$ and $m_\eta - 2 - b = 4$, which implies $b = m_\eta - 6 = b_\rho$. Therefore, an orthogonal representation of G with b bends is constructed by placing b bends along e and choosing $\sigma_\eta = m_\eta$.

The proofs for Case (ii) ($\Delta_\rho = [3, 5]$) and Case (iii) ($\Delta_\rho = 4$) are analogous, observing that in Case (ii) we have $\alpha_{u'} + \alpha_{v'} \in [-1, 1]$ and in Case (iii) we have $\alpha_{u'} + \alpha_{v'} = 0$. □

Table 3 summarizes how to compute b_ν and I'_ν for the different types of nodes ν.

6.4 Optimality of the Approach

Our bottom-up algorithm incrementally computes for each node ν of T the budget of bends needed to realize an orthogonal representation of G_ν. We prove that, the total budget at the level of the root of T corresponds to the number of bends of a bend-minimum orthogonal representation of G. More formally, for a node ν of T, the *cumulative budget* B_ν of ν is the sum of the budgets of all nodes in the subtree of T rooted at ν. If ν is a leaf of T, $B_\nu = b_\nu = 0$. If ν is an internal node with children

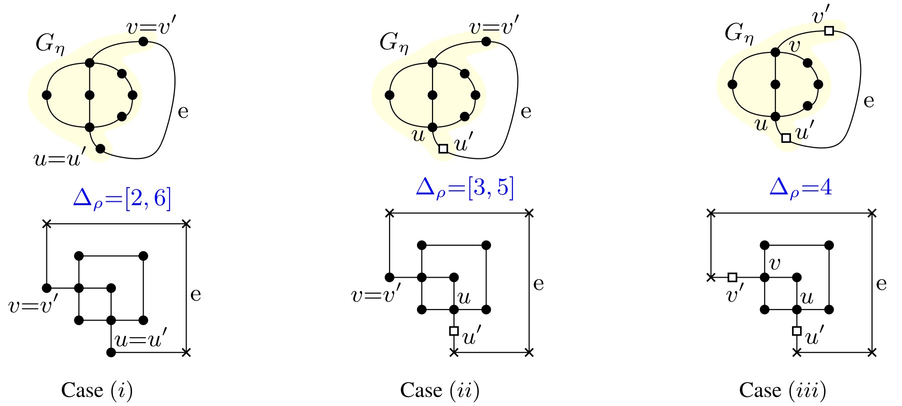

Fig. 19 Illustration of Lemma 22. We have $I_\eta = [-1, -1]$ and: in Case (i), $\Delta_\rho = [2, 6]$ and $b_\rho = \delta(I_\eta, \Delta_\rho) = 3$; in Case (ii), $\Delta_\rho = [3, 5]$ and $b_\rho = \delta(I_\eta, \Delta_\rho) = 4$; in Case (iii), $\Delta_\rho = 4$ and $b_\rho = \delta(I_\eta, \Delta_\rho) = 5$

Table 3 Summary of how to compute b_ν and I'_ν for the different types of nodes ν. In the formulas, we have $\gamma = \lambda + \beta - 2$ and $\phi(\cdot)$ is such that $\phi(r) = 1$ and $\phi(l) = 0$. Values m and M are computed as shown in Table 1

P-node with two children, each having an exposed edge—Lemma 18	
Budget	$\delta([m_l - M_r, M_l - m_r], \Delta_\nu)$
Representability Interval	$[m - b_\nu, M + b_\nu]$
P-node with three children (each child has an exposed edge)—Lemma 15	
Budget	$\overline{m}_z - M_x$
Representability Interval	$[\max\{\overline{M}_x, \overline{m}_y\}, \min\{\overline{m}_z, \overline{M}_y\}]$
P-node with two children, such that μ_d, with $d \in \{l, r\}$ has no exposed edge—Lemma 20 if $d = l$ and Lemma 21 if $d = r$	
Budget	$b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3])$
Representability Interval	If $d = l$:
	$[m - b_\nu, M + \min\{b^+_{\mu_l}, b_\nu\}]$ if $M_l - m_r < 3$
	$[m - \min\{b^-_{\mu_l}, b_\nu\}, M + b_\nu]$ if $m_l - M_r > 3$.
	If $d = r$:
	$[m - \min\{b^-_{\mu_r}, b_\nu\}, M + b_\nu]$ if $m_l - M_r > 3$
	$[m - b_\nu, M + \min\{b^+_{\mu_r}, b_\nu\}]$ if $M_l - m_r < 3$.
P^r-node (the root ρ)	
Budget	$b_\rho = \delta(I_\eta, \Delta_\rho)$

$\mu_1, \mu_2, \ldots, \mu_h$ then $B_\nu = b_\nu + \sum_{i=1,\ldots,h} B_{\mu_i}$. Hence, the cumulative budget B_ρ of the root corresponds to the total number of bends at the end of the bottom-up visit.

Theorem 3 *Let G be a plane series–parallel 4-graph, let T be an SPQ*-tree of G, and let ρ be the root of T. The cumulative budget B_ρ computed by the bottom-up visit equals the number of bends of a bend-minimum orthogonal representation of G.*

Proof We prove that once the algorithm has processed a node ν in the bottom-up visit, the cumulative budget B_ν equals the number of bends of a bend-minimum orthogonal representation of G_ν. This implies that once the algorithm has visited the root ρ of T, the cumulative budget B_ρ equals the number of bends of a bend-minimum orthogonal representation of $G_\rho = G$. The proof is by induction on the depth of the subtree of T rooted at ν.
Base Case. For a leaf ν of T (ν is a Q*-node), the statement is trivial, as $B_\nu = 0$.
Inductive Case. Let ν be a node of T that is not a leaf. Denote by $\mu_1, \mu_2, \ldots, \mu_h$ the children of ν. By the inductive hypothesis, each B_{μ_i} $(i = 1, \ldots, h)$ corresponds to the number of bends of a bend-minimum representation of G_{μ_i}. By definition, $B_\nu = b_\nu + \sum_{i=1,\ldots,h} B_{\mu_i}$, where b_ν is the minimum number of bends that must be used in addition to $\sum_{i=1,\ldots,h} B_{\mu_i}$ to realize an orthogonal representation of G_ν ($b_\nu = 0$ if ν is an S-node). Budget B_ν corresponds to the minimum number of bends of any orthogonal representation H_ν of G_ν with the property that the number of bends $b(H_{\mu_i})$ of the restriction H_{μ_i} of H_ν to G_{μ_i} is such that $b(H_{\mu_i}) \geq B_{\mu_i}$ $(i \in \{1, \ldots, h\})$. Let H'_ν be any bend-minimum orthogonal representation of G_ν, and let H'_{μ_i} be the restriction of H'_ν to G_{μ_i}. Since, by the inductive hypothesis, there is no orthogonal representation of G_{μ_i} with less bends than B_{μ_i}, we have $b(H'_{\mu_i}) \geq B_{\mu_i}$, and hence, by the previous observation, $b(H'_\nu) \geq B_\nu$. On the other hand, since H'_ν is bend-minimum for G_ν, we also have $b(H'_\nu) \leq B_\nu$, i.e., $b(H'_\nu) = B_\nu$. □

7 Bend-Minimization in Linear Time

The bottom-up visit described above, equips each node ν of T with three information: the budget b_ν, the cumulative budget B_ν, and, if $\nu \neq \rho$, an interval $I'_\nu = [m'_\nu, M'_\nu]$ of all possible spirality values that an orthogonal representation of G_ν with B_ν bends can have. Once the bottom-up visit of T has been completed, our algorithm performs a top-down visit of T to suitably add to G a number B_ρ of subdivision vertices, so that the resulting graph G' admits a rectilinear planar representation.

At the beginning of the top-down visit, the root ρ is considered. Let η be the child of ρ that does not correspond to e. If $I'_\eta \cap \Delta_\rho \neq \emptyset$, the algorithm selects an arbitrary value $\sigma_\eta \in I'_\eta \cap \Delta_\rho$ as target spirality value for a representation of G'_η within a rectilinear planar representation of G'. If, vice versa, $I'_\eta \cap \Delta_\rho = \emptyset$, according to the proof of Lemma 22, the algorithm subdivides e with $b_e = \delta(I'_\eta, \Delta_\rho)$ bends and sets the target spirality value σ_η as either $\sigma_\eta = M'_\eta$ (if M'_η is smaller than the infimum of Δ_ρ) or $\sigma_\eta = m'_\eta$ (if m'_η is larger than the supremum of Δ_ρ). In the next step of the top-down visit, the algorithm considers node η, for which the target spirality value σ_η has been previously fixed. If η is an S-node then $b_\eta = 0$, i.e., no subdivision vertices must be added in this step. If η is a P-node and $b_\eta > 0$, the algorithm suitably adds b_η subdivision vertices along some edges of G_η. To do so, it applies the procedures described in the second part of the proof of Property (ii) of Lemma 15, of Lemma 18, or of Lemmas 20 and 21, depending on whether η is a P-node with three children, a P-node with two children both having an exposed edge, or a P-node with two children one of which has no exposed edge. Then, the algorithm sets the spirality values for each child of η. Namely, we distinguish the following cases:

Case 1: η is an S-node, with children $\mu_1, \ldots, \mu_h$ ($i \in \{1, \ldots, h\}$). Let $I'_{\mu_i} = [m'_i, M'_i]$ be the representability interval of μ_i (assuming that any orthogonal representation of G_{μ_i} will have B_{ν_i} bends). We must find a value $\sigma_{\mu_i} \in [m'_i, M'_i]$ for each $i = 1, \ldots, h$ such that $\sum_{i=i}^{h} \sigma_{\mu_i} = \sigma_\eta$. To this aim, initially set $\sigma_{\mu_i} = M'_i$ for each $i = 1, \ldots, h$ and consider $s = (\sum_{i=i}^{h} \sigma_{\mu_i}) - \sigma_\eta$. By Lemma 1, $s \geq 0$. If $s = 0$ we are done. Otherwise, iterate over all $i = 1, \ldots, h$ and for each i decrease both σ_{μ_i} and s by the value $\min\{s, M'_i - m'_i\}$, until $s = 0$.
Case 2: η is a P-node with three children, μ_l, μ_c, and μ_r. By Lemma 2, it suffices to set $\sigma_{\mu_l} = \sigma_\eta + 2$, $\sigma_{\mu_c} = \sigma_\eta$, and $\sigma_{\mu_r} = \sigma_\eta - 2$.
Case 3: η is a P-node with two children, μ_l and μ_r. Let u and v be the poles of η. By Lemma 3, σ_{μ_l} and σ_{μ_r} must be determined in such a way that $\sigma_{\mu_l} = \sigma_\eta + k^l_u \alpha^l_u + k^l_v \alpha^l_v$ and $\sigma_{\mu_r} = \sigma_\eta - k^r_u \alpha^r_u - k^r_v \alpha^r_v$. The values of $k^l_u, k^l_v, k^r_u, k^r_v$ are fixed by the indegree and outdegree of u and v. Hence, it suffices to choose the values of $\alpha^l_u, \alpha^l_v, \alpha^r_u, \alpha^r_v$ such that they are consistent with the type of η and yield $\sigma_{\mu_l} \in I'_{\mu_l}$ and $\sigma_{\mu_r} \in I'_{\mu_r}$. Since each α^d_w ($w \in \{u, v\}, d \in \{l, r\}$) is either 0 or 1, there are at most four combinations of values to consider.
Case 4: η is a Q*-node. In this case the algorithm does nothing, as no further subdivision vertices must be added.
In the subsequent steps of the top-down visit, for every node ν the algorithm applies the same procedure as for η to determine a target spirality value σ_ν and to suitably distribute the b_ν subdivision vertices along the edges of G_ν.

Theorem 4 *Let G be an n-vertex plane series–parallel 4-graph. There exists an $O(n)$-time algorithm that computes a bend-minimum orthogonal representation of G.*

Proof If G is biconnected let e be any edge of G on the external face; otherwise, let e be a dummy edge added on the external face to make G biconnected. Let T be an SPQ*-tree of G with respect to e. The algorithm executes the bottom-up and the top-down visits described above. Once the top-down visit is completed and B_ρ subdivision vertices have been suitably inserted in G, a rectilinear planar representation of the subdivision of G is easily computed from the spirality values of each component and from the values of the angles at the poles of each component. From this representation we obtain a bend-minimum orthogonal representation of G by replacing the subdivision vertices with bends. Since the obtained orthogonal representation has B_ρ bends, by Theorem 3 it has the minimum number of bends.

We now analyze the time complexity of the algorithm. T can be computed in $O(n)$ time and it consists of $O(n)$ nodes [4]. For a node ν of T that is not a Q*-node, we denote by n_ν the number of children of ν.

Consider first the bottom-up visit. Let ν be a visited node of T. If ν is a Q*-node then $b_\nu = 0$ and, by Table 1, $I'_\nu = I_\nu$ is computed in $O(1)$ time (we can assume that the length ℓ of the chain of edges represented by ν is stored at ν during the construction of T). If ν is an S-node, we still have $b_\nu = 0$ and, by Table 1, $I'_\nu = I_\nu$ is computed in $O(n_\nu)$ time. If ν is a P-node with three children, by Lemma 15 b_ν and I'_ν are computed in $O(1)$ time. If ν is a P-node with two children each having an exposed edge, by Lemma 18 b_ν and I'_ν are computed in $O(1)$ time. Suppose now that ν is a P-node with an S-node child that has no exposed edge, and assume that this S-node is the left child μ_l of ν. By Lemma 20, b_ν and I'_ν can be computed in $O(1)$ time if we know the flexibility breakpoints (positive and negative) of μ_l. By Lemma 19, the flexibility breakpoints of μ_l can be computed in $O(n_{\mu_l})$ time. By Lemma 21, the same reasoning applies if the right child of ν is an S-node with no exposed edge. Finally, if ν coincides with the root ρ of T, by Lemma 22 b_ν is easily computed in $O(1)$ time. In summary, for each S-node ν of T, the bottom-up visit requires $O(n_\nu)$ time to compute b_ν, I'_ν, and the flexibility breakpoints of ν if needed. For any other type of node, the visit takes $O(1)$ time. Thus, the bottom-up visit requires $O(n)$ overall time.

In the top-down visit, for every non-leaf node ν of T, the algorithm spends $O(b_\nu)$ time to add b_ν subdivision vertices. Also, we should consider the extra time t required to decide what are the edges along which these bends must be added and what is the target spirality value for each child of ν. Namely, if ν is the root, $t = O(1)$. If ν is an S-node, $t = O(n_\nu)$ by Case 1 of the top-down visit described above. If ν is a P-node with three children, by Case 2 of the top-down visit and by Lemma 15, $t = O(1)$. If ν is a P-node with two children each having an exposed edge, by Case 3 of the top-down visit and by Lemma 18, $t = O(1)$. Finally, if ν is a P-node with two children, one of which is an S-node μ with no exposed edge, by Case 3 and by Lemmas 20 and 21, $t = O(n_\mu)$. Hence, since $B_\rho = \sum_\nu b_\nu = O(n)$ [21], the top-down visit takes $O(n)$ overall time, and a rectilinear planar representation of the subdivision of G is easily computed in $O(n)$ time from the spirality values of each component and from the values of the angles at the poles of each component.

□

8 Conclusions and Open Problems

We proved that there exists an optimal linear-time algorithm that computes a bend-minimum orthogonal drawing of a plane series–parallel 4-graph; this result solves, for a popular and widely studied family of plane graphs, a question opened for over 30 years, thus shedding new light on the complexity of computing orthogonal drawings of plane graphs with the minimum number of bends. It is also worth remarking that, despite the sophisticated analysis and key ingredients needed to prove our main theorem, the resulting bend-minimization algorithm is relatively easy to implement, as at every node of the SPQ*-tree it just requires to apply some simple formulas, as summarized in Tables 1 and 3. We conclude by suggesting two open problems:

- **Problem 1.** Our result holds for the series–parallel graphs that are also called two-terminal, which are either biconnected or which can be made biconnected with the addition of a single edge. Can we extend Theorem 4 to 1-connected plane 4-graphs whose biconnected components are two-terminal series–parallel graphs, also known as partial 2-trees? Here the main difficulty of extending our approach is to succinctly describe the possible spirality values for those components that contain cut-vertices. In fact, a cut-vertex requires the imposition of some constraints at its angles, in order to correctly compose the different biconnected components that contain it. These constraints forbid some vertex angles, and in turns some spirality values that an orthogonal representation can take.

– **Problem 2.** Is it possible to find a linear-time algorithm for the bend-minimization problem of triconnected plane 4-graphs? A positive answer to this question, together with our result, could be used to solve the problem of computing a bend-minimum orthogonal drawing of a general plane 4-graph in linear time.

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A Analysis of the Missing Cases in the Proof of Theorem 1

Case 4: $\text{indeg}_\nu(u) > 1$ **and** $\text{indeg}_\nu(v) > 1$. We analyze the three non-symmetric subcases, depending on the outdegree of u and of v, i.e., $\text{outdeg}_\nu(u) = \text{outdeg}_\nu(v) = 1$, or $\text{outdeg}_\nu(u) = 1$ and $\text{outdeg}_\nu(v) = 2$ (symmetrically $\text{outdeg}_\nu(u) = 2$ and $\text{outdeg}_\nu(v) = 1$), or $\text{outdeg}_\nu(u) = \text{outdeg}_\nu(v) = 2$. In all these cases, for a pole $w \in \{u, v\}$, the angles defined by $\text{Sub}(H_\nu, H'_\nu)$ around w in H'' are the same as in H' (note that if $\text{outdeg}_\nu(w) = 2$, the angles at w are all right angles, and they coincide both in H and H'). Hence, Property (H1) holds for u and v in H''. About Property (H2), we analyze the different subcases separately:

- $\text{outdeg}_\nu(u) = \text{outdeg}_\nu(v) = 1$ (see Fig. 5a). Each of the two poles u and v has a single alias vertex, denoted as u' and v', respectively. Let P_l (resp. P_r) be the path of H obtained by concatenating p_l (resp. p_r) with the segments $\overline{u'u}$ and $\overline{vv'}$. Analogously, let P'_l (resp. P'_r) be the path of H' obtained by concatenating p'_l (resp. p'_r) with the segments $\overline{u'u}$ and $\overline{vv'}$. Using the same notation as in the previous cases, we have $\sigma(H_\nu) = n(P_l) = n(P_r)$ and $\sigma(H'_\nu) = n(P'_l) = n(P'_r)$. Hence, since $\sigma(H_\nu) = \sigma(H'_\nu)$, we have $n(P_l) = n(P_r) = n(P'_l) = n(P'_r)$, which implies that $N_{90}(f''_l) - N_{270}(f''_l) = N_{90}(f_l) - N_{270}(f_l)$ and $N_{90}(f''_r) - N_{270}(f''_r) = N_{90}(f_r) - N_{270}(f_r)$. Hence, Property (H2) holds for f''_l and f''_r.
- $\text{outdeg}_\nu(u) = 1$ and $\text{outdeg}_\nu(v) = 2$ (see Fig. 5b). The pole u has a single alias vertex u', while v has two alias vertices v' and v''. Let P_l (resp. P_r) be the path of H obtained by concatenating p_l (resp. p_r) with the segments $\overline{u'u}$ and $\overline{vv'}$ (resp. $\overline{vv''}$). Analogously, let P'_l (resp. P'_r) be the path of H' obtained by concatenating p'_l (resp. p'_r) with the segments $\overline{u'u}$ and $\overline{vv'}$ (resp. $\overline{vv''}$). Since $\sigma(H_\nu) = \sigma(H'_\nu)$, we have $\frac{n(P_l)+n(P_r)}{2} = \frac{n(P'_l)+n(P'_r)}{2}$. Also, since all the angles at v are right angles, we have $n(P_r) = n(P_l) + 1$ and $n(P'_r) = n(P'_l) + 1$, which implies $n(P_l) = n(P'_l)$ and $n(P_r) = n(P'_r)$. Hence, $N_{90}(f''_l) - N_{270}(f''_l) = N_{90}(f_l) - N_{270}(f_l)$ and $N_{90}(f''_r) - N_{270}(f''_r) = N_{90}(f_r) - N_{270}(f_r)$, i.e., Property (H2) holds for f''_l and f''_r.

- $\text{outdeg}_\nu(u) = \text{outdeg}_\nu(v) = 2$ (see Fig. 5c). Each of the two poles u and v has two alias vertices, denoted as $\{u', u''\}$ and $\{v', v''\}$, respectively. Let P_l (resp. P_r) be the path of H obtained by concatenating p_l (resp. p_r) with the segments $\overline{u'u}$ and $\overline{vv'}$ (resp. $\overline{u''u}$ and $\overline{vv''}$). Analogously, let P'_l (resp. P'_r) be the path of H' resulting from the concatenation of p'_l (resp. p'_r) with the segments $\overline{u'u}$ and $\overline{vv'}$ (resp. $\overline{u''u}$ and $\overline{vv''}$). Since $\sigma(H_\nu) = \sigma(H'_\nu)$, we have $\frac{n(P_l)+n(P_r)}{2} = \frac{n(P'_l)+n(P'_r)}{2}$. Also, since all the angles at u and v are right angles, we have $n(P_r) = n(P_l) + 2$ and $n(P'_r) = n(P'_l) + 2$. This implies that $n(P_l) = n(P'_l)$ and $n(P_r) = n(P'_r)$, which in turns implies that $N_{90}(f''_l) - N_{270}(f''_l) = N_{90}(f_l) - N_{270}(f_l)$ and $N_{90}(f''_r) - N_{270}(f''_r) = N_{90}(f_r) - N_{270}(f_r)$. Hence, Property (H2) holds for f''_l and f''_r.

B Analysis of the Missing Cases in the Proof of Lemma 8

Case 2: $\lambda = 1$ and $\beta = 2$, i.e., G_ν is of type I_2O_{12}. We prove that $I_\nu = [\max\{m_l - 2, m_r\} + \frac{1}{2}, \min\{M_l, M_r + 2\} - \frac{1}{2}]$. Assume first that G_ν is rectilinear planar and let H_ν be a rectilinear planar representation of G_ν with spirality σ_ν. Let H_{μ_l} and H_{μ_r} be the representations of G_{μ_l} and G_{μ_r} contained in H_ν, and let σ_{μ_l} and σ_{μ_r} be their corresponding spirality values. By Lemma 7, $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$, i.e., $\sigma_{\mu_l} \in [2 + \sigma_{\mu_r}, 3 + \sigma_{\mu_r}]$. Since $\sigma_{\mu_l} \in [m_l, M_l]$ and $\sigma_{\mu_r} \in [m_r, M_r]$, we have $\sigma_{\mu_l} \geq \max\{m_l, m_r + 2\}$. Suppose, w.l.o.g, that $\text{outdeg}_\nu(v) = 2$ and $\text{outdeg}_\nu(u) = 1$. We have $k^r_u = k^l_u = 1$, $k^r_v = k^l_v = \frac{1}{2}$, $\alpha^l_u \in [0, 1]$, $\alpha^r_u \in [0, 1]$, and $\alpha^l_v = \alpha^r_v = 1$. By Lemma 3, $\sigma_\nu = \sigma_{\mu_l} - \alpha^l_u - \frac{1}{2}\alpha^l_v$. Since $-\alpha^l_u - \frac{1}{2}\alpha^l_v \geq -\frac{3}{2}$, we have $\sigma_\nu \geq \max\{m_l, m_r + 2\} - \frac{3}{2}$. It follows that $\sigma_\nu \geq \max\{m_l - 2, m_r\} + \frac{1}{2}$. Analogously, since $\sigma_{\mu_r} \in [\sigma_{\mu_l} - 3, \sigma_{\mu_l} - 2]$, we have $\sigma_{\mu_r} \leq \min\{M_l - 2, M_r\}$. By Lemma 3, $\sigma_\nu = \sigma_{\mu_r} + \alpha^r_u + \frac{1}{2}\alpha^r_v$. Since $\alpha^r_u + \frac{1}{2}\alpha^r_v \leq \frac{3}{2}$, we have $\sigma_\nu \leq \min\{M_l - 2, M_r\} + \frac{3}{2}$. It follows that $\sigma_\nu \leq \min\{M_l, M_r + 2\} - \frac{1}{2}$, hence $\sigma_\nu \in I_\nu$.

Assume vice versa that k is a semi-integer in the interval $I_\nu = [\max\{m_l - 2, m_r\} + \frac{1}{2}, \min\{M_l, M_r + 2\} - \frac{1}{2}]$. We show that G_ν has a rectilinear planar representation with spirality $\sigma_\nu = k$. Since $k \in [m_l - \frac{3}{2}, M_l - \frac{1}{2}]$ we have $k + \frac{1}{2} \leq M_l$ and $k + \frac{3}{2} \geq m_l$, i.e., $[k + \frac{1}{2}, k + \frac{3}{2}] \cap [m_l, M_l] \neq \emptyset$. Also, since m_l and M_l are both integer numbers while k is semi-integer, it is impossible to have $k + 1 = m_l = M_l$. It follows that $k + \frac{1}{2} \in [m_l, M_l]$ or $k + \frac{3}{2} \in [m_l, M_l]$. With the same reasoning, we have $k \in [m_r + \frac{1}{2}, M_r + \frac{3}{2}]$ and $[k - \frac{3}{2}, k - \frac{1}{2}] \cap [m_r, M_r] \neq \emptyset$. Hence, $k - \frac{3}{2} \in [m_r, M_r]$ or $k - \frac{1}{2} \in [m_r, M_r]$. We now prove that $k + \frac{3}{2} \in [m_l, M_l]$ or $k - \frac{3}{2} \in [m_r, M_r]$. Suppose for a contradiction that $k + \frac{3}{2} \notin [m_l, M_l]$ and $k - \frac{3}{2} \notin [m_r, M_r]$. In that case $k + \frac{1}{2} \in [m_l, M_l]$ and $k - \frac{1}{2} \in [m_r, M_r]$. Consequently, $k + \frac{1}{2} = M_l$ and $k - \frac{1}{2} = m_r$. Hence, $M_l - m_r = 1$ and, by the representability condition, G_ν is not rectilinear planar, a contradiction. As in the previous case, a rectilinear representation of G_ν with spirality k is obtained by combining in parallel a representation H_{μ_l} of G_{μ_l} with spirality σ_{μ_l} and a representation H_{μ_r} of G_{μ_r} with spirality σ_{μ_r}, for two suitable values σ_{μ_l} and σ_{μ_r}. Based on the previous considerations, we distinguish the following subcases.

- **Case 2.1:** $k + \frac{3}{2} \notin [m_l, M_l]$. This implies that $k + \frac{1}{2} \in [m_l, M_l]$ and $k - \frac{3}{2} \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = k + \frac{1}{2}$ and $\sigma_{\mu_r} = k - \frac{3}{2}$.
- **Case 2.2:** $k - \frac{3}{2} \notin [m_r, M_r]$. This implies that $k + \frac{3}{2} \in [m_l, M_l]$ and $k - \frac{1}{2} \in [m_r, M_r]$, and therefore we set $\sigma_{\mu_l} = k + \frac{3}{2}$ and $\sigma_{\mu_r} = k - \frac{1}{2}$.
- **Case 2.3:** $k + \frac{3}{2} \in [m_l, M_l]$ and $k - \frac{3}{2} \in [m_r, M_r]$. We set $\sigma_{\mu_l} = k + \frac{3}{2}$ and $\sigma_{\mu_r} = k - \frac{3}{2}$.

Note that, in all the three subcases we have $\sigma_{\mu_l} - \sigma_{\mu_r} \in [2, 3]$, hence by Lemma 7 there exists a rectilinear planar representation H_ν of G_ν that contains H_{μ_l} and H_{μ_r}. It remains to prove that the spirality σ_ν of H_ν is equal to k. Suppose, w.l.o.g, that $\text{outdeg}_\nu(u) = 1$ and $\text{outdeg}_\nu(v) = 2$. We have $k_u^r = k_u^l = 1$ and $k_v^r = k_v^l = \frac{1}{2}$. Since G_ν is rectilinear planar, $\alpha_u^l \in [0, 1]$ and $\alpha_v^l = 1$. By Lemma 3, $\sigma_\nu = \sigma_{\mu_l} - \alpha_u^l - \frac{1}{2}\alpha_v^l$. In Case 2.1 we have $\sigma_\nu = k + \frac{1}{2} - \alpha_u^l - \frac{1}{2}\alpha_v^l$; choosing $\alpha_u^l = 0$ and $\alpha_v^l = 1$ we have $\sigma_\nu = k$. In Cases 2.2 and 2.3 we have $\sigma_\nu = k + \frac{3}{2} - \alpha_u^l - \frac{1}{2}\alpha_v^l$; choosing $\alpha_u^l = 1$ and $\alpha_v^l = 1$ we have $\sigma_\nu = k$.

C Analysis of the Missing Cases in the Proof of Lemma 18

Case $\text{I}_{3d}\text{O}_{\lambda\beta}$: recall that in this case $\lambda = 1$ and $\beta \in \{1, 2\}$. Assume, without loss of generality, that $\text{indeg}(v) = 3$ and $\text{indeg}(u) = 2$. Also, assume $d = l$ (the case $d = r$ is treated symmetrically). In this case $\Delta_\nu = [\frac{5}{2}, \frac{7}{2} - \gamma]$, $m = \max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma}{2}$, and $M = \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma}{2}$. We show that set I'_ν is an interval of feasible spirality values for the orthogonal representations of G_ν with b_ν bends. Suppose first that G_ν has an orthogonal representation H_ν with b_ν bends, and let σ_ν be the spirality of H_ν. We prove that $\sigma_\nu \in [m - b_\nu, M + b_\nu]$. Let b_l and b_r be the number of bends in the restriction of H_ν to G_{μ_l} and to G_{μ_r}, respectively, where $b_l + b_r = b_\nu$. By Lemma 3, we have $\sigma_\nu = \sigma_{\mu_l} - k_u^l\alpha_u^l - k_v^l\alpha_v^l$. By Relation 1 of Lemma 17, we have $-k_u^l\alpha_u^l - k_v^l\alpha_v^l \in [-\frac{3}{2} + \frac{\gamma}{2}, -\frac{\gamma}{2} - \frac{1}{2}]$. Hence, by the same reasoning as in the case $\text{I}_2\text{O}_{\lambda\beta}$, $\sigma_\nu \in [m_l - b_\nu - \frac{3}{2} + \frac{\gamma}{2}, M_l + b_\nu - \frac{\gamma}{2} - \frac{1}{2}]$. Also, by Relation 2 of Lemma 17, we have $k_u^r\alpha_u^r + k_v^r\alpha_v^r \in [\frac{\gamma}{2} + 1, 2 - \frac{\gamma}{2}]$. Hence, $\sigma_\nu \in [m_r - b_\nu + \frac{\gamma}{2} + 1, M_r + b_\nu + 2 - \frac{\gamma}{2}]$. It follows that $\sigma_\nu \in [m_l - b_\nu - \frac{3}{2} + \frac{\gamma}{2}, M_l + b_\nu - \frac{\gamma}{2} - \frac{1}{2}] \cap [m_r - b_\nu + 1 + \frac{\gamma}{2}, M_r + b_\nu + 2 - \frac{\gamma}{2}] = [\max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma}{2} - b_\nu, \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma}{2} + b_\nu] = [m - b_\nu, M + b_\nu]$.

It remains to show that for every $\sigma_\nu \in [m - b_\nu, M + b_\nu]$, there exists an orthogonal representation H_ν of G_ν with b_ν bends and with spirality σ_ν. With the same notation as in the previous case, we show how to compute the values b_l and b_r. Also, we show how to compute the spirality for the rectilinear planar representations of G'_{μ_l} and G'_{μ_r}, within their representability intervals $[m_l - b_l, M_l + b_l]$ and $[m_r - b_r, M_r + b_r]$. Since by hypothesis G_ν is not rectilinear planar we have $[m_l - M_r, M_l - m_r] \cap \Delta_\nu = \emptyset$, i.e., $[m_l - M_r, M_l - m_r] \cap [\frac{5}{2}, \frac{7}{2} - \gamma] = \emptyset$. We analyze the two possible subcases:

- $M_l - m_r < \frac{5}{2}$. By Property (i), $b_\nu = \delta([m_l - M_r, M_l - m_r], [\frac{5}{2}, \frac{7}{2} - \gamma]) = \frac{5}{2} - M_l + m_r$. We set $b_l = \sigma_\nu - M_l + h_l$, where h_l is a number (either integer or semi-integer) in the interval $[\frac{\gamma}{2} + \frac{1}{2}, \frac{3}{2} - \frac{\gamma}{2}]$ such that $b_l \in [0, b_\nu]$. We first prove that such a value h_l always exists for any given $\sigma_\nu \in [m - b_\nu, M + b_\nu]$. If $\sigma_\nu = m - b_\nu$, we choose $h_l = \frac{3}{2} - \frac{\gamma}{2}$. This implies that $b_l = m - b_\nu - M_l + \frac{3}{2} - \frac{\gamma}{2}$. Since $M_l - m_r < \frac{5}{2}$, we have $m_l - \frac{5}{2} \le M_l - \frac{5}{2} < m_r$, and therefore $m_l - \frac{3}{2} < m_r + 1$. Since $m = \max\{m_l - \frac{3}{2}, m_r + 1\} + \frac{\gamma}{2}$, we have $m = m_r + 1 + \frac{\gamma}{2}$. Also, since $b_\nu = \frac{5}{2} - M_l + m_r$, we have $b_l = m - b_\nu - M_l + \frac{3}{2} - \frac{\gamma}{2} = m_r + 1 + \frac{\gamma}{2} - b_\nu - M_l + \frac{3}{2} - \frac{\gamma}{2} = 0$. If $\sigma_\nu \in [m - b_\nu + 1, M + b_\nu]$, we choose $h_l = \frac{\gamma}{2} + \frac{1}{2}$. This implies that $b_l = \sigma_\nu - M_l + \frac{\gamma}{2} + \frac{1}{2} \in [m - b_\nu + 1 - M_l + \frac{\gamma}{2} + \frac{1}{2}, M + b_\nu - M_l + \frac{\gamma}{2} + \frac{1}{2}]$. We have $m - b_\nu + 1 - M_l + \frac{\gamma}{2} + \frac{1}{2} = m_r + 1 + \frac{\gamma}{2} - b_\nu + 1 - M_l + \frac{\gamma}{2} + \frac{1}{2} = (\frac{5}{2} + m_r - M_l) - b_\nu + \gamma = \gamma$. Since $M_l < m_r + \frac{5}{2} \le M_r + \frac{5}{2}$ we have $M_l - \frac{1}{2} < M_r + 2$. Also, since $M = \min\{M_l - \frac{1}{2}, M_r + 2\} - \frac{\gamma}{2}$ it follows that $M = M_l - \frac{1}{2} - \frac{\gamma}{2}$. Hence, $M + b_\nu - M_l + \frac{\gamma}{2} + \frac{1}{2} = M_l - \frac{1}{2} - \frac{\gamma}{2} + b_\nu - M_l + \frac{\gamma}{2} + \frac{1}{2} = b_\nu$. It follows that $b_l \in [\gamma, b_\nu]$. Since $\gamma \ge 0$, also in this case there exists a value of $h_l \in [\frac{\gamma}{2} + \frac{1}{2}, \frac{3}{2} - \frac{\gamma}{2}]$ such that $b_l \in [0, b_\nu]$. We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = M_l + b_l = M_l + \sigma_\nu - M_l + h_l = \sigma_\nu + h_l$ and G'_{μ_r} with spirality $\sigma'_{\mu_r} = m_r - b_r = m_r - (b_\nu - b_l) = m_r - b_\nu + b_l = m_r - b_\nu + \sigma_\nu - M_l + h_l = b_\nu - \frac{5}{2} - b_\nu + \sigma_\nu + h_l = \sigma_\nu + h_l - \frac{5}{2}$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + h_l - (\sigma_\nu + h_l - \frac{5}{2}) = \frac{5}{2}$ and, by Lemma 9, G'_ν is rectilinear planar. It remains to show that G'_ν admits a rectilinear planar representation with spirality $\sigma'_\nu = \sigma_\nu$. Given the choice of σ'_{μ_l} and σ'_{μ_r}, by Lemma 3 every rectilinear planar representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k^l_u\alpha^l_u - k^l_u\alpha^l_v = \sigma_\nu + h_l - k^l_u\alpha^l_u - k^r_u\alpha^l_v$. Since $h_l \in [\frac{\gamma}{2} + \frac{1}{2}, \frac{3}{2} - \frac{\gamma}{2}]$, by Relation 1 of Lemma 17 there exists a value $k^l_u\alpha^l_u + k^r_u\alpha^l_v$ such that $h_l - k^l_u\alpha^l_u - k^r_u\alpha^l_v = 0$, and thus $\sigma'_\nu = \sigma_\nu$.

- $m_l - M_r > \frac{7}{2} - \gamma$. In this case, by Property (i) we have $b_\nu = m_l - M_r - \frac{7}{2} + \gamma$. We set $b_l = m_l - \sigma_\nu - \frac{3}{2} + \frac{\gamma}{2}$. As before, we first prove that $b_l \in [0, b_\nu]$. We have $b_l \in [m_l - M - b_\nu - \frac{3}{2} + \frac{\gamma}{2}, m_l - m + b_\nu - \frac{3}{2} + \frac{\gamma}{2}]$. Since $M_l \ge m_l > M_r + \frac{7}{2} - \gamma \ge M_r + \frac{5}{2}$, we have $M_l - \frac{1}{2} \ge M_r + 2$. It follows that $M = M_r + 2 - \frac{\gamma}{2}$. Hence, $m_l - M - b_\nu - \frac{3}{2} + \frac{\gamma}{2} = m_l - (M_r + 2 - \frac{\gamma}{2}) - b_\nu - \frac{3}{2} + \frac{\gamma}{2} = m_l - M_r - \frac{7}{2} + \gamma - b_\nu = 0$. Also, since $m_l - \frac{5}{2} \ge m_l - \frac{7}{2} + \gamma > M_r$ we have $m_l - \frac{3}{2} \ge M_r + 1 \ge m_r + 1$. It follows that $m = m_l - \frac{3}{2} + \frac{\gamma}{2}$. Hence, $m_l - m + b_\nu - \frac{3}{2} + \frac{\gamma}{2} = m_l - (m_l - \frac{3}{2} + \frac{\gamma}{2}) + b_\nu - \frac{3}{2} + \frac{\gamma}{2} = b_\nu$. It follows that $b_l \in [0, b_\nu]$. We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = m_l - b_l = m_l - (m_l - \sigma_\nu - \frac{3}{2} + \frac{\gamma}{2}) = \sigma_\nu + \frac{3}{2} - \frac{\gamma}{2}$ and G'_{μ_r} with spirality $\sigma'_{\mu_r} = M_r + b_r = M_r + b_\nu - b_l = M_r + m_l - M_r - \frac{7}{2} + \gamma - m_l + \sigma_\nu + \frac{3}{2} - \frac{\gamma}{2} = \sigma_\nu + \frac{\gamma}{2} - 2$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + \frac{3}{2} - \frac{\gamma}{2} - (\sigma_\nu + \frac{\gamma}{2} - 2) = \frac{7}{2} - \gamma$ and, by Lemma 9, G'_ν is rectilinear planar. Also, by Lemma 3 every rectilinear planar representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k^l_u\alpha^l_u - k^l_u\alpha^l_v = \sigma_\nu + \frac{3}{2} - \frac{\gamma}{2} - k^l_u\alpha^l_u - k^l_u\alpha^l_v$. By Relation 1 of Lemma 17, we can set $k^l_u\alpha^l_u + k^r_u\alpha^l_v = \frac{3}{2} - \frac{\gamma}{2}$, and thus $\sigma'_\nu = \sigma_\nu$.

Case $\mathrm{I}_{3dd'}$: assume that $d = l$ (the case $d = r$ is symmetric). In this case $k^l_v = \frac{1}{2}$, $k^r_v = 1$, $\Delta_\nu = [3, 3]$, $m = \max\{m_l - 1, m_r + 2\} - \frac{\phi(d')}{2}$, and $M = \min\{M_l - 1, M_r + 2\} - \frac{\phi(d')}{2}$. Since both u and v have degree four, in any rectilinear representation of G_{μ_l} and G_{μ_l} we have $\alpha^l_u = \alpha^r_u = \alpha^l_v = \alpha^r_v = 1$. If $d' = l$ then $k^l_u = \frac{1}{2}$ and $k^r_u = 1$; hence $k^l_u\alpha^l_u + k^l_v\alpha^l_v = 1$ and $k^r_u\alpha^r_u + k^r_v\alpha^r_v = 2$. If $d' = r$ then $k^l_u = 1$ and $k^r_u = \frac{1}{2}$;

hence $k_u^l\alpha_u^l + k_v^l\alpha_v^l = k_u^r\alpha_u^r + k_v^r\alpha_v^r = \frac{3}{2}$. We show that set I'_ν is an interval of feasible spirality values for the orthogonal representations of G_ν with b_ν bends. Suppose first that G_ν has an orthogonal representation H_ν with b_ν bends, and let σ_ν be the spirality of H_ν. We prove that $\sigma_\nu \in [m - b_\nu, M + b_\nu]$. Let b_l and b_r be the number of bends in the restriction of H_ν to G_{μ_l} and to G_{μ_r}, respectively, where $b_l + b_r = b_\nu$. By Lemma 3, we have $\sigma_\nu = \sigma_{\mu_l} - k_u^l\alpha_u^l - k_v^l\alpha_v^l = \sigma_{\mu_r} + k_u^r\alpha_u^r + k_v^r\alpha_v^r$. Suppose first $d' = l$. In this case $\sigma_\nu = \sigma_{\mu_l} - 1 = \sigma_{\mu_r} + 2$. Hence, by the same reasoning as in the case $I_2O_{\lambda\beta}$, $\sigma_\nu \in [m_l - b_\nu - 1, M_l + b_\nu - 1]$ and $\sigma_\nu \in [m_r - b_\nu + 2, M_r + b_\nu + 2]$. Therefore, $\sigma_\nu \in [m_l - b_\nu - 1, M_l + b_\nu - 1] \cap [m_r - b_\nu + 2, M_r + b_\nu + 2] = [\max\{m_l - 1, m_r + 2\} - b_\nu, \min\{M_l - 1, M_r + 2\} + b_\nu]$. Since $d' = l$, we have $\frac{\phi(d')}{2} = 0$ and $\sigma_\nu \in [\max\{m_l - 1, m_r + 2\} - b_\nu, \min\{M_l - 1, M_r + 2\} + b_\nu] = [m - b_\nu, M + b_\nu]$. Suppose now $d' = r$. In this case $\sigma_\nu = \sigma_{\mu_l} - \frac{3}{2} = \sigma_{\mu_r} + \frac{3}{2}$. Hence, $\sigma_\nu \in [m_l - b_\nu - \frac{3}{2}, M_l + b_\nu - \frac{3}{2}] \cap [m_r - b_\nu + \frac{3}{2}, M_r + b_\nu + \frac{3}{2}] = [\max\{m_l - \frac{3}{2}, m_r + \frac{3}{2}\} - b_\nu, \min\{M_l - \frac{3}{2}, M_r + \frac{3}{2}\} + b_\nu] = [\max\{m_l - 1, m_r + 2\} - \frac{1}{2} - b_\nu, \min\{M_l - 1, M_r + 2\} - \frac{1}{2} + b_\nu]$. Since $d' = r$, we have $\frac{\phi(d')}{2} = \frac{1}{2}$ and $\sigma_\nu \in [\max\{m_l - 1, m_r + 2\} - \frac{1}{2} - b_\nu, \min\{M_l - 1, M_r + 2\} - \frac{1}{2} + b_\nu] = [m - b_\nu, M + b_\nu]$.

It remains to show that for every $\sigma_\nu \in [m - b_\nu, M + b_\nu]$, there exists an orthogonal representation H_ν of G_ν with b_ν bends and with spirality σ_ν. With the same notation as in the previous cases, we show how to compute the values b_l and b_r. Also, we show how to compute the spirality for the rectilinear planar representations of G'_{μ_l} and G'_{μ_r}, within their representability intervals $[m_l - b_l, M_l + b_l]$ and $[m_r - b_r, M_r + b_r]$. Since by hypothesis G_ν is not rectilinear planar we have $[m_l - M_r, M_l - m_r] \cap \Delta_\nu = \emptyset$, i.e., $[m_l - M_r, M_l - m_r] \cap [3, 3] = \emptyset$. We analyze the two possible subcases:

- $M_l - m_r < 3$. By Property (i), $b_\nu = \delta([m_l - M_r, M_l - m_r], [3, 3]) = 3 - M_l + m_r$. Suppose first that $d' = l$. By setting $b_l = \sigma_\nu - M_l + 1$, we have $b_l \in [0, b_\nu]$. Namely, $b_l \in [m - b_\nu - M_l + 1, M + b_\nu - M_l + 1]$. Since $M_l < m_r + 3 \leq M_r + 3$ we have $M_l - 1 < M_r + 2$. Also, since $M = \min\{M_l - 1, M_r + 2\}$, we have $M = M_l - 1$. Since $m_l - 3 \leq M_l - 3 < m_r$ we have $m_l - 1 < m_r + 2$. Also, since $m = \max\{m_l - 1, m_r + 2\}$, we have $m = m_r + 2$. Hence, $b_l \in [m_r + 2 - b_\nu - M_l + 1, M_l - 1 + b_\nu - M_l + 1] = [0, b_\nu]$.
  We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = M_l + b_l = M_l + \sigma_\nu - M_l + 1 = \sigma_\nu + 1$ and G'_{μ_r} with spirality $\sigma'_{\mu_r} = m_r - b_r = m_r - (b_\nu - b_l) = m_r - b_\nu + \sigma_\nu - M_l + 1 = (m_r - M_l + 1) - b_\nu + \sigma_\nu = b_\nu - 2 - b_\nu + \sigma_\nu = \sigma_\nu - 2$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + 1 - (\sigma_\nu - 2) = 3$ and, by Lemma 11, G'_ν is rectilinear planar. Also, given the choice of σ'_{μ_l} and σ'_{μ_r}, by Lemma 3 every rectilinear planar representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k_u^l\alpha_u^l - k_u^l\alpha_v^l = \sigma_\nu + 1 - 1 = \sigma_\nu$.
  Suppose now that $d' = r$. By setting $b_l = \sigma_\nu - M_l + \frac{3}{2}$ we have $b_l \in [0, b_\nu]$. Namely, $b_l \in [m - b_\nu - M_l + \frac{3}{2}, M + b_\nu - M_l + \frac{3}{2}]$. Since $M_l < M_r + 3$ we have $M_l - \frac{3}{2} < M_r + \frac{3}{2}$. Also, since $M = \min\{M_l - \frac{3}{2}, M_r + \frac{3}{2}\}$, we have $M = M_l - \frac{3}{2}$. Since $m_l - 3 < m_r$ we have $m_l - \frac{3}{2} < m_r + \frac{3}{2}$. This implies that $m = \max\{m_l - \frac{3}{2}, m_r + \frac{3}{2}\} = m_r + \frac{3}{2}$. Hence, $b_l \in [m_r + \frac{3}{2} - b_\nu - M_l + \frac{3}{2}, M_l - \frac{3}{2} + b_\nu - M_l + \frac{3}{2}] = [0, b_\nu]$.
  We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = M_l + b_l = M_l + \sigma_\nu - M_l + \frac{3}{2} = \sigma_\nu + \frac{3}{2}$ and

G'_{μ_r} with spirality $\sigma'_{\mu_r} = m_r - b_r = m_r - (b_\nu - b_l) = m_r - b_\nu + \sigma_\nu - M_l + \frac{3}{2} = (m_r - M_l + \frac{3}{2}) - b_\nu + \sigma_\nu = b_\nu - \frac{3}{2} - b_\nu + \sigma_\nu = \sigma_\nu - \frac{3}{2}$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + \frac{3}{2} - (\sigma_\nu - \frac{3}{2}) = 3$ and, by Lemma 11, G'_ν is rectilinear planar.
Also, by Lemma 3 every rectilinear planar representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k^l_u \alpha^l_u - k^l_u \alpha^l_\nu = \sigma_\nu + \frac{3}{2} - \frac{3}{2} = \sigma_\nu$.

- $m_l - M_r > 3$. In this case, by Property (i) we have $b_\nu = m_l - M_r - 3$.
Suppose first that $d' = l$. By setting $b_l = m_l - \sigma_\nu - 1$, we have $b_l \in [0, b_\nu]$. Namely, $b_l \in [m_l - b_\nu - M - 1, m_l - m + b_\nu - 1]$. Since $M_l \geq m_l > M_r + 3$, we have $M_l - 1 \geq M_r + 2$. This implies that $M = \min\{M_l - 1, M_r + 2\} = M_r + 2$. Since $m_l > M_r + 3 \geq m_r + 2$, we have $m_l - 1 \geq m_r + 2$, which implies that $m = \max\{m_l - 1, m_r + 2\} = m_l - 1$. Hence, $b_l \in [m_l - b_\nu - M_r - 2 - 1, m_l - m_l - 1 + b_\nu + 1] = [0, b_\nu]$.
We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = m_l - b_l = m_l - m_l + \sigma_\nu + 1 = \sigma_\nu + 1$ and G'_{μ_r} with spirality $\sigma'_{\mu_r} = M_r + b_r = M_r + b_\nu - b_l = M_r + b_\nu - (m_l - \sigma_\nu - 1) = M_r + b_\nu - m_l + \sigma_\nu + 1 = \sigma_\nu + b_\nu - b_\nu - 2 = \sigma_\nu - 2$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + 1 - (\sigma_\nu - 2) = 3$ and, by Lemma 11, G'_ν is rectilinear planar.
Also, by Lemma 3 every rectilinear planar representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k^l_u \alpha^l_u - k^l_u \alpha^l_\nu = \sigma_\nu + 1 - 1 = \sigma_\nu$.
Suppose now that $d' = r$. By setting $b_l = m_l - \sigma_\nu - \frac{3}{2}$ we have $b_l \in [0, b_\nu]$. Namely, $b_l \in [m_l - b_\nu - M - \frac{3}{2}, m_l - m + b_\nu - \frac{3}{2}]$. Since $M_l > M_r + 3$, we have $M_l - \frac{3}{2} \geq M_r + \frac{3}{2}$. This implies that $M = \min\{M_l - \frac{3}{2}, M_r + \frac{3}{2}\} = M_r + \frac{3}{2}$. Since $m_l > m_r + 3$, we have $m_l - \frac{3}{2} \geq m_r + \frac{3}{2}$. This implies that $m = \max\{m_l - \frac{3}{2}, m_r + \frac{3}{2}\} = m_l - \frac{3}{2}$. Hence, $b_l \in [m_l - b_\nu - M - \frac{3}{2}, m_l - m + b_\nu - \frac{3}{2}] = [m_l - b_\nu - M_r - \frac{3}{2} - \frac{3}{2}, m_l - m_l - \frac{3}{2} + b_\nu + \frac{3}{2}] = [0, b_\nu]$.
We represent G'_{μ_l} with spirality $\sigma'_{\mu_l} = m_l - b_l = m_l - m_l + \sigma_\nu + \frac{3}{2} = \sigma_\nu + \frac{3}{2}$ and G'_{μ_r} with spirality $\sigma'_{\mu_r} = M_r + b_r = M_r + b_\nu - b_l = M_r + b_\nu - (m_l - \sigma_\nu - \frac{3}{2}) = \sigma_\nu + b_\nu - b_\nu - \frac{3}{2} = \sigma_\nu - \frac{3}{2}$. We have $\sigma'_{\mu_l} - \sigma'_{\mu_r} = \sigma_\nu + \frac{3}{2} - (\sigma_\nu - \frac{3}{2}) = 3$ and, by Lemma 11, G'_ν is rectilinear planar.
Also, by Lemma 2 every rectilinear planar representation of G'_ν has spirality $\sigma'_\nu = \sigma'_{\mu_l} - k^l_u \alpha^l_u - k^l_u \alpha^l_\nu = \sigma_\nu + \frac{3}{2} - \frac{3}{2} = \sigma_\nu$.

D Glossary of Symbols and Terminology

See Tables 4 and 5.

Table 4 Glossary of the main symbols used in the paper

NOTATION	SHORT DESCRIPTION
G	Graph (with vertex-degree at most four)
H	Orthogonal representation
$N_a(f)$	Number of $a°$ angles in face f of an orthogonal representation
T	SPQ*-tree
G_ν	Pertinent graph of the node ν in an SPQ*-tree (component of G)
H_ν	Orthogonal representation of a component G_ν
$\sigma(H_\nu)$ or σ_ν	Spirality of an orthogonal representation H_ν (component of H)
P^{uv}	Simple path from u to v in a component
$S^{u'v'}$	Spine of a component with poles u and v (u' and v' being alias vertices associated with u and v)
$n(P)$	Number of right turns minus left turns along an oriented path P
A^u	Set of the alias vertices of a pole u
$\mathrm{Sub}(H_\nu, H'_\nu)$	Orthogonal representation obtained from H by substituting H_ν with H'_ν
α_l^w, α_r^w	Binary coefficients used to denote the leftmost and the rightmost external angles at a pole w of a component
	(the value 0 denotes a 180° angle, while the value 1 denotes a 90° angle)
k_l^w, k_r^w	Coefficients that take value 1 or 1/2, based on the indegree/outdegree of a pole w in the left/right child
	of a P-node with two children
I_ν	Representability interval of a node ν, i.e., the set of spirality values that can be taken from a rectilinear
	representation of G_ν
M, m	Maximum and minimum values of the representability interval of a node
$\mathrm{I_2O_{\lambda\beta}}, \mathrm{I_{3d}O_{\lambda\beta}}, \mathrm{I_{3dd'}}$	Different types of P-nodes with two children, where $1 \leq \lambda \leq \beta \leq 2$ and $d \in \{l, r\}$ (refer to Fig. 10)
$\phi(d)$	Binary function defined on the values $d \in \{l, r\}$ and such that $\phi(l) = 0$ and $\phi(r) = 1$
Δ_ρ	Interval associated with the root ρ of an SPQ*-tree: $\Delta_\rho = [2, 6]$ if each pole of ρ coincides with its alias vertex;
	$\Delta_\rho = [3, 5]$ if only one pole of ρ coincides with its alias vertex; $\Delta_\rho = 4$ if each pole of ρ is distinct from its alias vertex

Table 4 continued

NOTATION	SHORT DESCRIPTION
$\delta(A_1, A_2)$	Distance between two non-intersecting intervals of real numbers A_1 and A_2
b_ν	Budget of a node ν, i.e., number of extra bends added to those of the children of ν to guarantee
	the existence of an orthogonal representation of G_ν
B_ν	Cumulative budget of a node ν, i.e., sum of the budgets of all nodes in the subtree rooted at ν
I'_ν	Set of spirality values that an orthogonal representation of G_ν with b_ν extra bends can take
$\sigma_\nu^{\max}(b)$, $\sigma_\nu^{\min}(b)$	Maximum and minimum spirality value for an orthogonal representation H_ν with at most b bends
b_ν^+	Positive flexibility breakpoint of G_ν, i.e., the maximum number of bends such that
	for every $b < b_\nu^+$, we have $\sigma_\nu^{\max}(b+1) = \sigma_\nu^{\max}(b) + 1$
b_ν^-	Negative flexibility breakpoint of G_ν, i.e., the maximum number of bends such that
	for every $b < b_\nu^-$, we have $\sigma_\nu^{\min}(b+1) = \sigma_\nu^{\min}(b) - 1$

Table 5 Glossary of the main terminology

TERMINOLOGY	SHORT DESCRIPTION
k-graph	A graph whose vertices have degree at most k.
Orthogonal drawing	Drawing of a graph where each vertex is represented as a point of the plane and each edge
	as a sequence of horizontal and vertical segments.
Bend	Contact point between a vertical and a horizontal segment along an edge of an orthogonal drawing
Orthogonal representation	Description of the “shape” of an orthogonal drawing, i.e., angles at the vertices
	and sequences of left/right bends along the edges; vertex and bend coordinates are not specified
Rectilinear drawing/representation	Orthogonal drawing/representation without bends
Planar orthogonal drawing/representation	Orthogonal drawing/representation without edge crossings
Rectilinear planar graph	A graph that admits a rectilinear planar drawing
Series–parallel graph	A graph obtained in an inductive way by means of series- and parallel-compositions
SPQ*-tree	Rooted tree whose nodes describe the structure of a series–parallel graph in terms of
	its series- and parallel-compositions
S-node	Node of an SPQ*-tree associated with a series-composition that is not a chain of edges

Table 5 continued

TERMINOLOGY	SHORT DESCRIPTION
P-node	Node of an SPQ*-tree associated with a parallel-composition
Q*-node	Leaf of an SPQ*-tree, associated with either an edge or a chain of edges
Reference edge	A designated edge for an SPQ*-tree; the root of the tree corresponds to the parallel-composition
	between the reference edge and the rest of the graph
Poles of a node ν	The two terminal vertices for the composition represented by the node ν of an SPQ*-tree
Pertinent graph or component of a node ν	The subgraph G_ν induced by the leaves of the subtree of an SPQ*-tree rooted at ν
Alias vertex of a pole	Dummy vertex associated with a pole of a node ν of an SPQ*-tree. This vertex may or may not
	coincide with the pole, depending on its indegree/outdegree in the component that ν represents
Orthogonal component	Restriction of an orthogonal representation to a component of the graph
Spirality of an orthogonal component	A measure of how much an orthogonal component is rolled-up
Representability condition of a node ν	A Boolean condition that is true if and only if the component G_ν is rectilinear planar
Representability interval of a node ν	Interval of the spirality values that a rectilinear representation of G_ν can take
Exposed edge	For a child μ of a P-node, if μ is a Q*-node then all the edges of μ are exposed edges;
	if μ is an S-node, the exposed edges of μ are those corresponding to leaf-children of μ, if any

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Authors and Affiliations

Walter Didimo[1] · Michael Kaufmann[2] · Giuseppe Liotta[1] · Giacomo Ortali[1]

Michael Kaufmann
mk@informatik.uni-tuebingen.de

Giuseppe Liotta
giuseppe.liotta@unipg.it

Giacomo Ortali
giacomo.ortali@unipg.it

[1] Department of Engineering, University of Perugia, Perugia, Italy

[2] University of Tübingen, Tübingen, Germany