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On the Pentomino Exclusion Problem

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Abstract. In this paper we are interested in the *Pentomino Exclusion Problem* due to Golomb: *Find the minimum number of unit squares to be placed on a $k \times n$ chessboard so as to exclude all pentominoes*. Using an appropriate definition of density of a tiling, we obtain the asymptotic value of this number, and we establish this number for the $k \times n$ chessboard when $k \leq 4$.

1. Introduction

A *polyomino* is a pattern formed by the connection of a specified number of equal-sized squares along common edges (see [2]). A *pentomino* is a polyomino composed of five squares. The *interior boundary* $\delta_{\text{int}}(P)$ of a polyomino P is the set of squares of P having a common edge with the "exterior" of P. The *exterior boundary* $\delta_{\text{ext}}(P)$ of a polyomino P is the interior boundary of the complement of P. The *perimeter* of a polyomino P is $|\delta_{\text{int}}(P)|$. For a given polyomino P, $\Delta(P)$ denotes the area of P.

Consider the adjacency relations α and β, which defines what is usually called respectively 8-connectivity and 4-connectivity in discrete geometry, between squares in $\mathbb{Z}^2$: write $C\alpha C'$ (resp. $C\beta C'$) iff C and C' have a common vertex (resp. edge).

For a given polyomino P, we can build a graph $G(P) = (V, E)$ defined by $V = \{p | p$ is the center of a unit square in $P\}$ and $E = \{UV | U\alpha V\}$. A vertex v of $G(P)$ can be seen as a square of $\mathbb{R}^2$, so for brevity sometimes v should be seen as the unique corresponding square (see Fig. 1). Moreover, $G(\mathbb{Z}^2)$ is in graph-theoretical language usually called the *total infinite complete grid graph* which can be defined as a total product of two infinite paths.

Golomb [2] proposed the following *Pentomino Exclusion Problem*, denoted ($PEP_{k\times n}$): *Find the minimum number of unit squares to be placed on a $k \times n$ chessboard so as to exclude all pentominoes*.

In a previous paper [3] we introduced a notion of density of a tiling with (for example)

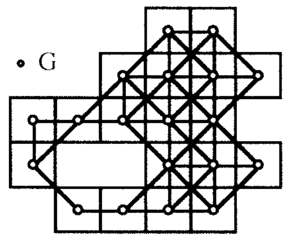

Fig. 1. A polyomino P, the graph $G(P)$.

polyominoes. Using this definition in the next section, we determine the asymptotic value (i.e., the density of a tiling where the tiles are polyominoes composed of less than five squares) of the Pentomino Exclusion Problem and other related problems.

Finally, in Section 3, we investigate ($\mathrm{PEP}_{k\times n}$) problems when $k = 4$.

2. Assymptotic Results

In order to state our results we need some preliminary definitions.

We denote by (G_Δ) the problem:

Find the minimum density of unit squares to be placed on the plane so as to exclude all polyominoes of area $> \Delta$.

For instance, the Pentomino Exclusion Problem in the plane is equivalent to the problem (G_4).

An *admissible* solution of (G_Δ) is a set $\mathcal{S}$ of squares centered on $\mathbb{Z}^2$ such that any connected component in the β adjacency of $\mathbb{R}^2 - \mathcal{S}$ has area less than or equal to Δ. The squares belonging to an admissible solution $\mathcal{S}$ are filled in (i.e. black) and the others are left white.

We now need a measure, called "density," defined on an admissible solution of (G_Δ) in order to compare two admissible solutions. If T is a finite subset of $\mathbb{Z}^2$, a natural way to define the density of $\mathcal{S}$ relative to T is $|\mathcal{S} \cap T|/|T - \mathcal{S}|$. We now show a way to extend this definition to the infinite case:

For an admissible solution $\mathcal{S}$ of (G_Δ), observe that if we remove one "crossing edge" of each K_4 (complete graph on four vertices) in $G(\mathcal{S})$, then the resulting **plane** graph $G'(\mathcal{S})$ defines a tiling of $\mathbb{R}^2$ (see Fig. 2) where the tiles are the faces of $G'(\mathcal{S})$. For a face (or a tile) $\langle C\rangle$ of $G'(\mathcal{S})$ there corresponds a unique polyomino C where $\delta_{\mathrm{ext}}(C) \subset \mathcal{S}$. Some of these tiles correspond to some connected components (in the β connectivity) of $\mathbb{Z}^2 - \mathcal{S}$. Other tiles are triangles corresponding to three mutually adjacent elements of $\mathcal{S}$ (in this case $C = \emptyset$).

Let D be a finite subset of $\mathbb{R}^2$. The density of an admissible solution $\mathcal{S}$ of (G_Δ) relative to D is

$$d(\mathcal{S}, D) = \frac{\text{black area of } \bar{D}}{\text{white area of } \bar{D}},$$

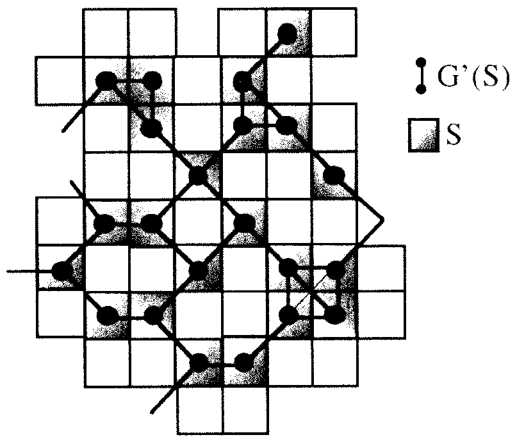

Fig. 2. $\mathcal{S}$ and $G'(\mathcal{S})$.

where $\bar{D}$ is the union of all faces of $G'(\mathcal{S})$ which intersect D. Notice that $\bar{D}$ defines a polyomino P with unit squares from $\mathcal{S}$, and each square in the interior boundary of P belongs to $\mathcal{S}$. Moreover, observe that $d(\mathcal{S}, D)$ is well-defined since each face of $G'(\mathcal{S})$ define a polyomino with bounded area.

Let B_r be a ball of $\mathbb{R}^2$ of radius r. Then

$$\underline{d}(\mathcal{S}) = \liminf_{r\to\infty} d(\mathcal{S}, B_r) \quad \text{and} \quad \bar{d}(\mathcal{S}) = \limsup_{r\to\infty} d(\mathcal{S}, B_r),$$

are called the *lower* and *upper* density, respectively. If these two values coincide, their common value is called *density* $d(\mathcal{S}, D)$. This kind of definition of density is more or less standard (see, for example, [4]). In [3] we proved:

Theorem 1. *Let Δ_n be the maximum number of squares belonging to a polyomino of perimeter n with $n = 4q + r > 0$ and $0 \le r \le 3$; Δ_n is given by the following function*:

$$\Delta_n = \begin{cases} 2q^2 + 2q + 1 & \text{if} \quad r = 0, \\ 2q^2 + 3q + 1 & \text{if} \quad r = 1, \\ 2q^2 + 4q + 2 & \text{if} \quad r = 2, \\ 2q^2 + 5q + 3 & \text{if} \quad r = 3. \end{cases}$$

Theorem 2. *Let $n = 4q + r \ge 5$, with $0 \le r \le 3$, be an integer such that $\Delta \ge \Delta_n$. If $q > 1$ and $\Delta - \Delta_n \le \lceil q/2 \rceil$, then an optimal solution $\mathcal{S}$ of (G_Δ) satisfies*

$$\underline{d}(\mathcal{S}) \ge \frac{(n+4)/2 - 1}{\Delta_n}.$$

For any $\Delta \ge \Delta_n$, we have

$$\bar{d}(\mathcal{S}) \le \begin{cases} \dfrac{(n+4)/2 - 1}{\Delta_n} & \text{if} \quad r \in \{0, 2\}, \\ \dfrac{(n+5)/2 - 1}{\Delta_n} & \text{if} \quad r \in \{1, 3\}. \end{cases}$$

As noticed in [3], a direct consequence of Theorem 2 is that when $r \in \{0, 2\}$, $q > 1$ and when $\Delta - \Delta_n \leq \lceil q/2 \rceil$ the density of an optimal solution of (G_Δ) exists and is equal to

$$\frac{(n+4)/2 - 1}{\Delta_n.}$$

Moreover, this density is independent from the position of the ball B_r.

In this section we complete Theorem 2 for $\Delta \leq 7$ since the first values given by Theorem 2 deal with $\Delta \geq 8$.

Theorem 3. *The only admissible solution of* (G_0) *is* $\mathcal{S} = \mathbb{Z}^2$. *An optimal solution of* (G_Δ) *satisfies*

$$\begin{array}{lll} d(\mathcal{S}) = 1 & \textit{when} & \Delta \leq 2, \\ \left.\begin{array}{l}\bar{d}(\mathcal{S}) \leq 1 \\ \underline{d}(\mathcal{S}) \geq \frac{5}{6}\end{array}\right\} & \textit{when} & \Delta = 3, \\ d(\mathcal{S}) = \frac{3}{4} & \textit{when} & \Delta = 4, \\ d(\mathcal{S}) = \frac{3}{5} & \textit{when} & \Delta = 5, \\ \left.\begin{array}{l}\bar{d}(\mathcal{S}) \leq \frac{3}{5} \\ \underline{d}(\mathcal{S}) \geq \frac{7}{12}\end{array}\right\} & \textit{when} & \Delta = 6, \\ d(\mathcal{S}) = \frac{4}{7} & \textit{when} & \Delta = 7. \end{array}$$

Proof. Let $\mathcal{S}$ be an optimal solution of (G_Δ). Let $D \subset \mathbb{R}^2$ and let

$$\bar{D} = \bigcup_{\{\langle C\rangle \in G'(\mathcal{S}) \,|\, \langle C\rangle \cap D \neq \emptyset\}} \langle C\rangle.$$

First we claim that:

$$\text{It may be assumed that every } \langle C\rangle \text{ has no hole.} \tag{1}$$

If $\langle C\rangle$ has a hole, then move it closer to the exterior boundary of $\langle C\rangle$ in order to obtain a new face $\langle C'\rangle$ with no hole. If we repeat this operation for any $\langle C\rangle$ having a hole, then we obtain a new admissible solution of (G_Δ) with the same density.

Now we assume that any face of $G'(\mathcal{S})$ has no hole. Using the structure of $\mathbb{Z}^2$, we claim that

$$d(\mathcal{S}, D) = \frac{\sum_{\langle C\rangle \in \bar{D}} (|\delta_{\text{ext}}(C)|/2 - 1)}{\sum_{\langle C\rangle \in \bar{D}} |C|}. \tag{2}$$

If P is the polyomino defined by $\bar{D}$, then, by Pick's theorem, we obtain that the area of $\bar{D}$ is given by $|P - \delta_{\text{int}}(P)| + |\delta_{\text{int}}(P)|/2 - 1$ and the area of each $\langle C\rangle$ is given by $|C| + |\delta_{\text{ext}}(C)|/2 - 1$, since by assumption no $\langle C\rangle$ has a hole. Now by additivity of the area and since $\{\langle C\rangle | \langle C\rangle \in \bar{D}\}$ is a tiling of $\bar{D}$, we have that $\sum_{\langle C\rangle \in \bar{D}} (|\delta_{\text{ext}}(C)|/2 - 1)$ is equal to the number of squares in $\mathcal{S} \cap P$ not in the interior boundary of P, plus half the number of squares in the interior boundary of P which corresponds to the black area of $\bar{D}$. Also, $\sum_{\langle C\rangle \in \bar{D}} |C|$ is the number of squares in P not in $\mathcal{S}$, which corresponds to the white area of $\bar{D}$.

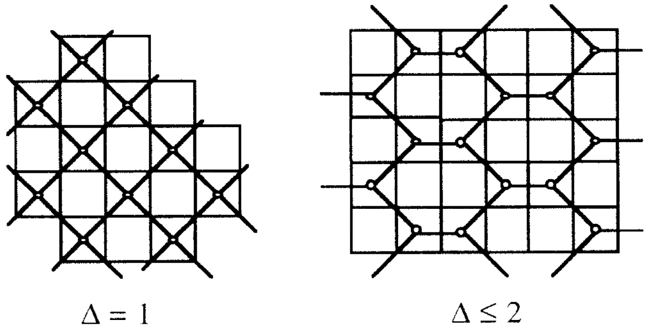

Fig. 3. $1 \leq \Delta \leq 2$.

From (2), we have

$$d(\mathcal{S}, D) \geq \min_{\langle C\rangle \in \bar{D}} \frac{|\delta_{\text{ext}}(C)|/2 - 1}{|C|}. \tag{3}$$

Hence, to obtain the lower bounds on $\underline{d}(\mathcal{S})$, it is sufficient to check that

$$|\delta_{\text{ext}}(C)| \geq \begin{cases} 4 & \text{if} \quad |C| = 1, \\ 6 & \text{if} \quad |C| = 2, \\ 7 & \text{if} \quad |C| = 3, \\ 8 & \text{if} \quad |C| = 4, \\ 8 & \text{if} \quad |C| = 5, \\ 9 & \text{if} \quad |C| = 6, \\ 10 & \text{if} \quad |C| = 7. \end{cases}$$

To prove the upper bounds on $\bar{d}(\mathcal{S})$ it is sufficient to exhibit tilings having the appropriate density (see Figs. 3–6). To determine the density of the tilings described in Figs. 3–6, it is sufficient to observe that, by (2), we have

$$d(\mathcal{S}, D) \leq \max_{\langle C\rangle \in \bar{D}} \frac{|\delta_{\text{ext}}(C)|/2 - 1}{|C|}. \quad \square \tag{4}$$

3. Finite Cases

In this section we investigate the problem ($\text{PEP}_{k\times n}$) for some values of k and n. We denote by $G_{k,n}$ an instance of ($\text{PEP}_{k\times n}$). For given k and n, $C_1, \ldots, C_n$ (resp. $R_1, \ldots, R_k$) denote the *columns* (resp. the *rows*) of $G_{k,n}$. The squares of $G_{k,n}$ are denoted by $s_{i,j}$ where $\{s_{i,j}\} = R_i \cap C_j$. A *free* polyomino of a solution $\mathcal{S}$ of ($\text{PEP}_{k\times n}$) is a polyomino which does not intersect $\mathcal{S}$.

First we give some upper bounds on the cardinality of a solution of ($\text{PEP}_{k\times n}$) for small values of k and n. It is easy to see that:

Lemma 1. *Every solution $\mathcal{S}$ to* ($\text{PEP}_{2\times 3}$) *satisfies* $|\mathcal{S}| \geq 2$.

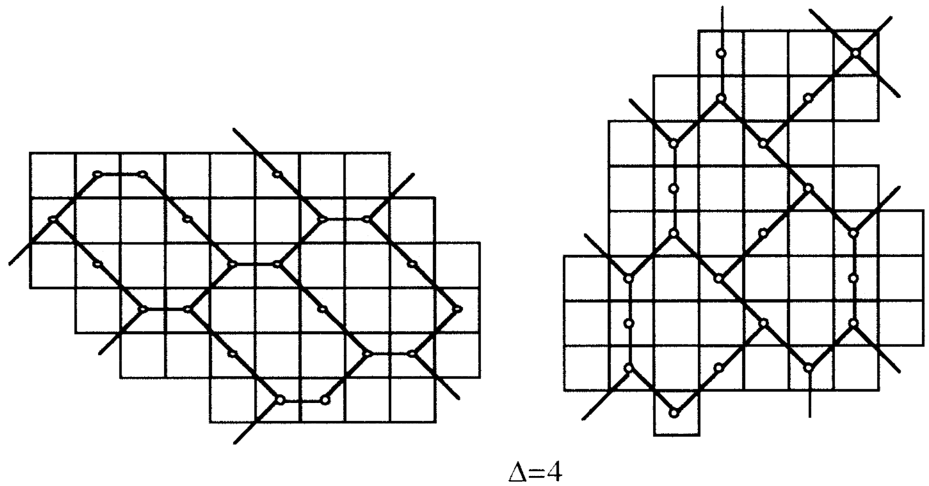

Fig. 4. $\Delta = 4$.

Theorem 4. *Let $\mathcal{S}_{k,n}$ be an optimal solution of* ($\mathrm{PEP}_{k\times n}$). *Then*

$$|\mathcal{S}_{k,n}| = \begin{cases} \lfloor n/5 \rfloor & \text{if} \quad k = 1, \\ 2\lfloor n/3 \rfloor & \text{if} \quad k = 2, \\ n & \text{if} \quad k = 3 \quad \text{and} \quad n \geq 2. \end{cases}$$

Proof. When $k = 1$, the theorem is obvious. When $k = 2$, then Theorem 4 is a direct consequence of Lemma 1. If $k = 3$, then let $\mathcal{S}$ be a solution of ($\mathrm{PEP}_{3\times n}$). We prove the lower bound by induction on n. It is easy to see that if $n \leq 3$, then Theorem 4 holds. Assume that $n > 3$. By Lemma 1, we have $|\mathcal{S} \cap (C_1 \cup C_2)| \geq 2$. Now, Theorem 4 holds by induction hypothesis on $C_3 \cup \cdots \cup C_n$.

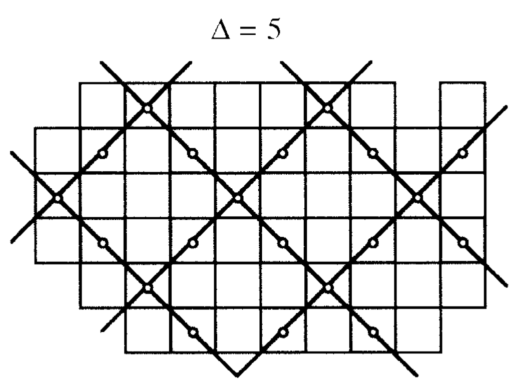

Fig. 5. $5 \leq \Delta \leq 6$.

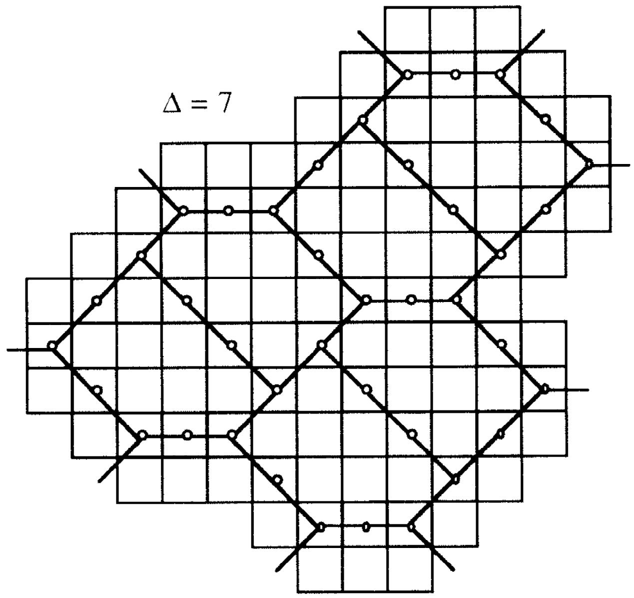

Fig. 6. $\Delta = 7$.

To achieve the proof of Theorem 4, we exhibit a solution:

$$\mathcal{S} = \left\{ s_{i,j} \;\middle|\; \begin{array}{c} i = 2 \text{ if } j \text{ is even} \\ i + j \equiv 0[4], \text{ otherwise} \end{array} \right\}$$

satisfying $|\mathcal{S}| = n$. □

Lemma 2. *Every solution* $\mathcal{S}$ *to* (PEP$_{4\times4}$) *satisfies* $|\mathcal{S}| \geq 5$. *Moreover, there exists a unique, up to rotation, solution* F (*see Fig.* 7) *with only five squares in* (PEP$_{4\times4}$).

Proof. Let $\mathcal{S}$ be a solution of (PEP$_{4\times4}$) with less than six squares. We claim that

$$\mathcal{S} \cap C_i \neq \emptyset \quad \text{and} \quad \mathcal{S} \cap R_i \neq \emptyset \qquad \text{for all} \quad i = 1, \ldots, 4. \tag{5}$$

Indeed, assume, in the opposite case, that $\mathcal{S} \cap R_1 = \emptyset$ or $\mathcal{S} \cap R_2 = \emptyset$. If $\mathcal{S} \cap R_2 = \emptyset$,

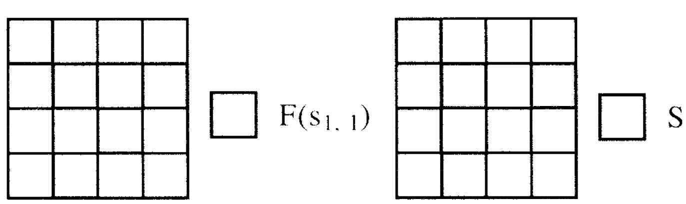

Fig. 7. The solution F of (PEP$_{4\times4}$) and a solution S.

then $R_2 \cup R_3 \subseteq \mathcal{S}$. If $\mathcal{S} \cap R_1 = \emptyset$, then $R_1 \subseteq \mathcal{S}$. Now, by (1), $|\mathcal{S} \cap (R_3 \cup R_4)| \geq 2$. In any case, by symmetry, we obtain that $|\mathcal{S}| > 5$, a contradiction.

Case 1: *none of the squares* $s_{1,1}$, $s_{1,4}$, $s_{4,1}$ *and* $s_{4,4}$ *belongs to* $\mathcal{S}$. In this case, by (5), we may assume that $s_{2,1} \in \mathcal{S}$.

By Lemma 1 applied on $(C_1 \cup C_2) \cap (R_2 \cup R_3 \cup R_4)$ and on $(C_3 \cup C_4) \cap (R_2 \cup R_3 \cup R_4)$, we have $|\mathcal{S} \cap (R_2 \cup R_3 \cup R_4)| \geq 4$. Hence, since $|\mathcal{S}| \leq 5$, we have $s_{1,3} \notin \mathcal{S}$.

By Lemma 1 applied on $(C_1 \cup C_2) \cap (R_2 \cup R_3 \cup R_4)$ and on $(C_3 \cup C_4) \cap (R_1 \cup R_2 \cup R_3)$, we have $|\mathcal{S} - \{s_{1,1}, s_{1,2}, s_{3,4}, s_{4,4}\}| \geq 4$. Hence, since $|\mathcal{S}| \leq 5$, we have $s_{4,3} \notin \mathcal{S}$.

By (5), we have $s_{4,2} \in \mathcal{S}$. So, by symmetry and by (5), we may assume that $s_{2,1} \in \mathcal{S}$.

Since $|\mathcal{S}| \leq 5$ and since, by Lemma 1, $|(\mathcal{S} \cap (R_1 \cup R_2 \cup R_3)) \cap (C_3 \cup C_4)| \geq 2$, we have $\mathcal{S} \cap (R_1 \cup R_2) = \{s_{1,2}, s_{2,1}, s_{4,2}\}$. However, now observe that $\{s_{3,1}, s_{4,1}, s_{2,2}, s_{3,2}, s\}$ induces a free pentomino for any $s \in \{s_{2,3}, s_{3,3}\}$. Thus $s_{2,3}, s_{3,3} \in \mathcal{S}$.

Since $|\mathcal{S}| \leq 5$, we have that $(C_3 \cup C_4) - \mathcal{S}$ contains a free pentomino, which contradicts the fact that $\mathcal{S}$ is a solution of $(\text{PEP}_{4\times 4})$.

Case 2: $s_{1,1}$ *belongs to* $\mathcal{S}$. Assume that $s_{1,1} \in \mathcal{S}$.

By Lemma 1, we have $|(\mathcal{S} \cap (R_1 \cup R_2)) \cap (C_2 \cup C_3 \cup C_4)| \geq 2$ and $|(\mathcal{S} \cap (R_3 \cup R_4)) \cap (C_2 \cup C_3 \cup C_4)| \geq 2$. Hence, since $|\mathcal{S}| \leq 5$, we obtain that $s_{1,2}$, $s_{1,3}$ and $s_{1,4}$ do not belong to $\mathcal{S}$. Moreover, by symmetry, we obtain that $s_{2,1}$, $s_{3,1}$ and $s_{4,1}$ do not belong to $\mathcal{S}$.

By Lemma 1, we have $|(\mathcal{S} \cap (R_1 \cup R_2)) \cap (C_2 \cup C_3 \cup C_4)| \geq 2$ and $|(\mathcal{S} \cap (R_3 \cup R_4)) \cap (C_1 \cup C_2 \cup C_3)| \geq 2$. Hence, since $|\mathcal{S}| \leq 5$, we obtain that $s_{4,3}$ and $s_{4,4}$ do not belong to $\mathcal{S}$. Moreover, by symmetry, we obtain that $s_{3,4} \notin \mathcal{S}$.

We must have $\{s_{2,2}, s_{2,4}, s_{4,2}\} \subset \mathcal{S}$. Finally, since $|\mathcal{S}| \leq 5$ and by (5) we must have $s_{3,3} \in \mathcal{S}$, which completes the proof of lemma.

The unique solution, up to rotation, is $\mathcal{S} = \{s_{1,1}, s_{2,2}, s_{2,4}, s_{3,3}, s_{4,2}\}$. □

We denote by $F(s)$ a solution $\mathcal{S}$ of $(\text{PEP}_{4\times 4})$ of cardinality five where $s \in \mathcal{S}$ is a "corner" (see Fig. 7).

In order to study the structure of a solution of $(\text{PEP}_{k\times n})$, we now need some additional definitions. A $P(3, 4)$ *configuration* of a solution $\mathcal{S}$ of a $(\text{PEP}_{4\times n})$ (for some n) is a column C_i such that $s_{2,i} \in \mathcal{S}$ and $s_{1,i}$ belong to a white polyomino in $G_{4,i}$ of size 4 and $s_{3,i}$, $s_{4,i}$ belong to a white polyomino in $G_{4,i}$ of size 3. A $P(4)$ *configuration* of a solution $\mathcal{S}$ of a $(\text{PEP}_{4\times n})$ (for some n) is a column C_i such that three squares of C_i belong to a white polyomino in $G_{4,i}$ of size 4. Note that the fourth column of an $F(s)$ is either a $P(3, 4)$ or a $P(4)$ configuration.

Lemma 3. *Let* $i \leq n - 5$. *If* C_i *is a* $P(4)$ *configuration of* $\mathcal{S}$, *then* $|\mathcal{S} \cap (C_{i+1} \cup C_{i+2} \cup C_{i+3} \cup C_{i+4})| \geq 7$. *If* C_i *is a* $P(3, 4)$ *configuration of* $\mathcal{S}$, *then* $|\mathcal{S} \cap (C_{i+1} \cup C_{i+2} \cup C_{i+3} \cup C_{i+4})| \geq 6$. *Moreover, if we have* $|\mathcal{S} \cap (C_{i+1} \cup C_{i+2} \cup C_{i+3} \cup C_{i+4})| = 6$, *then* C_{i+4} *is a* $P(3, 4)$ *or a* $P(4)$ *configuration* (*see Fig.* 7).

Proof. First, suppose that C_i is a $P(4)$ configuration. Then at least three squares must belong to $\mathcal{S} \cap C_{i+1}$. Moreover, by Lemma 1, $|\mathcal{S} \cap (C_{i+2} \cup C_{i+3} \cup C_{i+4} \cap (R_1 \cup R_2))| \geq 2$ and $|\mathcal{S} \cap (C_{i+2} \cup C_{i+3} \cup C_{i+4} \cap (R_3 \cup R_4))| \geq 2$, so we obtain that $|\mathcal{S} \cap (C_{i+1} \cup C_{i+2} \cup C_{i+3} \cup C_{i+4})| \geq 7$.

Now suppose that C_i is a $P(3,4)$ configuration. Then $s_{1,i+1} \in \mathcal{S}$. We claim that:

$$\text{We may assume that } |\mathcal{S} \cap C_{i+1}| \leq 2. \tag{6}$$

Indeed, in the opposite case by Lemma 1 applied on $C_{i+2} \cup C_{i+3} \cup C_{i+4}$, we obtain $\mathcal{S} \cap (C_{i+1} \cup \cdots \cup C_{i+4})| \geq 7$.

If $s_{3,i+1} \notin \mathcal{S}$, then $s_{2,i+1}$ and $s_{4,i+1}$ belong to $\mathcal{S}$, which contradicts claim (6). Assume now that $s_{3,i+1} \in \mathcal{S}$. This implies that $s_{4,i+2} \in \mathcal{S}$. We may assume that $s_{3,i+2} \notin \mathcal{S}$. Since by Lemma 1 applied on $(R_2 \cup R_3 \cup R_4) \cap (C_{i+3} \cup C_{i+4})$ and since $\mathcal{S} \cap \{s_{2,i+1}, s_{1,i+2}, s_{2,i+2}, s_{1,i+3}, s_{1,i+4}\} \neq \emptyset$, we obtain that $|\mathcal{S} \cap (C_{i+1} \cup \cdots \cup C_{i+4})| \geq 7$. We claim that:

$$\text{We may assume that } |\mathcal{S} \cap (C_{i+2} - s_{4,i+2})| = 1,\ |\mathcal{S} \cap C_{i+3}| = 1 \text{ and } |\mathcal{S} \cap C_{i+4})| = 1. \tag{7}$$

If $\mathcal{S} \cap (C_{i+2} - s_{4,i+2}) = \emptyset$, then $\mathcal{S} \cap C_{i+3} \supseteq \{s_{1,i+3}, s_{2,i+3}, s_{3,i+3}\}$. Moreover, $\mathcal{S} \cap (C_{i+4} \cup \{s_{4,i+3}\}) \neq \emptyset$; and so $|\mathcal{S} \cap (C_{i+1} \cup \cdots \cup C_{i+4})| \geq 7$. If $|\mathcal{S} \cap C_{i+2}| \geq 3$, then, by Lemma 1 applied on $C_{i+3} \cup C_{i+4}$, we obtain that $|\mathcal{S} \cap (C_{i+1} \cup \cdots \cup C_{i+4})| \geq 7$.

To prove that we may assume $|\mathcal{S} \cap C_{i+3}| = 1$ and $|\mathcal{S} \cap C_{i+4})| = 1$, it is sufficient to observe that $\mathcal{S}$ intersects columns C_{i+3}, C_{i+4} and that $|\mathcal{S} \cap (C_{i+1} \cup C_{i+2})| = 4$.

By Lemma 1 applied on $(C_{i+3} \cup C_{i+4}) - \{s, s'\}$, we can see that we may assume that $s = s_{4,i+3}$ and $s' = s_{4,i+4}$ (resp. $s = s_{1,i+3}$ and $s' = s_{1,i+4}$) do not belong to $\mathcal{S}$.

This last remark implies that $\mathcal{S}$ intersects both $\{s_{2,i+3}, s_{2,i+4}\}$ and $\{s_{3,i+3}, s_{3,i+4}\}$. Combining (7), and the previous remarks, we obtain that the only solutions when $|\mathcal{S} \cap (C_{i+1} \cup \cdots \cup C_{i+4})| < 7$ are $\mathcal{S} = \{s_{1,i+1}, s_{3,i+1}, s_{2,i+2}, s_{4,i+2}, s_{2,i+3}, s_{3,i+4}\}$ and $\mathcal{S} = \{s_{1,i+1}, s_{3,i+1}, s_{2,i+2}, s_{4,i+2}, s_{3,i+3}, s_{2,i+4}\}$. In any case, C_{i+4} (up to rotation) is respectively a $P(4)$ or a $P(3,4)$ configuration. □

Lemma 4. *Let* $2 \leq i \leq n-4$. *If* C_{i-1} *and* C_{i+4} *are a* $P(4)$ *or a* $P(3,4)$ *configuration of* $\mathcal{S}$, *then* $|\mathcal{S} \cap (C_i \cup C_{i+1} \cup C_{i+2} \cup C_{i+3})| \geq 8$.

Proof. First assume that C_i and C_{i+4} are $P(3,4)$ configurations.

If $s_{2,i-1} \in C_{i-1}$ and $s_{2,i+4} \in C_{i+4}$, then, similarly as in the proof of Lemma 3, we have that $s_{1,i}$, $s_{1,i+3}$, $s_{3,i}$, $s_{4,i+1}$, $s_{4,i+2}$ and $s_{3,i+3}$ are in $\mathcal{S}$. We conclude by applying Lemma 1 on $(C_{i+1} \cup C_{i+2}) \cap (R_1 \cup R_2 \cup R_3)$.

If $s_{2,i-1} \in C_{i-1}$ and $s_{3,i+4} \in C_{i+4}$, then, similarly as in the proof of Lemma 3, we have that $s_{1,i}$, $s_{4,i+3}$, $s_{3,i}$, $s_{4,i+1}$, $s_{1,i+2}$ and $s_{2,i+3}$ are in $\mathcal{S}$. It is now straightforward to check that we need three more squares to exclude all pentominoes in $C_{i-1} \cup C_i \cup \cdots \cup C_{i+4}$.

If C_{i-1} is a $P(3,4)$ configuration and C_{i+4} is a $P(4)$ configuration, then, similarly as in the proof of Lemma 3, we have that $s_{1,i}$, $s_{3,i}$ and $s_{4,i+1}$ are in $\mathcal{S}$. Moreover, since C_{i+4} is a $P(4)$ configuration, $|\mathcal{S} \cap C_{i+3}| \geq 3$, we conclude by applying Lemma 1 on $(C_{i+1} \cup C_{i+2}) \cap (R_1 \cup R_2 \cup R_3)$.

By symmetry, the lemma holds again if C_{i+4} is a $P(3,4)$ configuration and C_{i-1} is a $P(4)$ configuration.

If C_{i-1} and C_{i+4} are both $P(4)$ configurations, then $|\mathcal{S} \cap (C_i \cup C_{i+3})| \geq 6$. Hence, we complete the proof of Lemma 4 by applying Lemma 1 on $(C_{i+1} \cup C_{i+2}) \cap (R_1 \cup R_2 \cup R_3)$. □

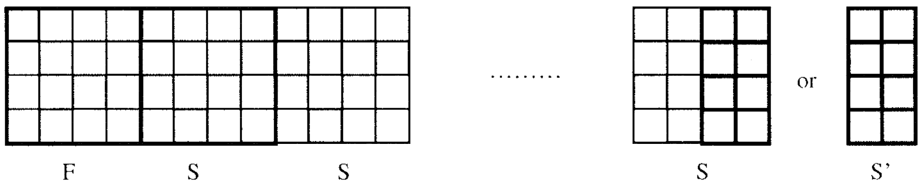

Fig. 8. The solutions of ($\mathrm{PEP}_{4\times n}$).

From Lemmas 3 and 4, we obtain:

Theorem 5. *An optimal solution of* ($\mathrm{PEP}_{4\times n}$) *with* $n = 4q \geq 4$ *has cardinality* $6q - 1$. *Moreover, up to rotation, any optimal solution of* ($\mathrm{PEP}_{4\times 4q}$) *can be described as shown in Fig.* 8.

Proof. We consider an intermediate problem, denoted by ($\mathrm{PEP}'_{4\times 4q}$): *If the column* C_0 *is a* $P(3,4)$ *configuration, then what is the smallest cardinality of a set which excludes all pentominoes in* $C_1 \cup C_2 \cup \cdots \cup C_n$?

We claim that:

> Any solution $\mathcal{S}$ of ($\mathrm{PEP}_{4\times 4q}$) has cardinality at least $6q - 1$. Any solution $\mathcal{S}'$ of ($\mathrm{PEP}'_{4\times 4q}$) has cardinality at least $6q$. Moreover, if $|\mathcal{S}| = 6q - 1$, then, up to rotation, C_{4q} is a $P(3,4)$ or a $P(4)$ configuration and C_1 is a $P(4)$ configuration. If $|\mathcal{S}'| = 6q$, then, up to rotation, C_{4q} is a $P(4)$ or a $P(3,4)$ configuration. (8)

The proof works by induction on q. If $q = 1$, then Theorem 5 follows from Lemmas 2 and 3. Assume now that $q > 1$.

Case A: $\mathcal{S}$ *is a solution to* ($\mathrm{PEP}_{4\times 4q}$). Let $A_1, \ldots, A_q$ be a partition of $G_{4,n}$ where the A_i's are blocks of four consecutive columns. Observe that, by the induction hypothesis, we have $|\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)| \geq 6(q-1) - 1$, $|\mathcal{S} \cap A_1| \geq 5$.

Subcase A.1: $|\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)| > 6(q-1) - 1$. If $|\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)| > 6(q-1)$, then, by Lemma 2, we obtain $|\mathcal{S}| \geq 6q$, and so the claim holds. Assume that $|\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)| = 6(q-1)$. If $|\mathcal{S} \cap A_1| > 5$, then the claim holds similarly as in the previous case. So, we may assume that $|\mathcal{S} \cap A_1| = 5$. Hence, $|\mathcal{S}| = 6q - 1$.

If $|\mathcal{S} \cap A_2| \geq 7$, then, by the induction hypothesis applied on columns $A_3, \ldots, A_q$ and since $|\mathcal{S}| = 6q - 1$, we obtain $|\mathcal{S} \cap A_2| = 7$ and $|\mathcal{S} \cap (A_3 \cup \cdots \cup A_q)| = 6(q-2) - 1$. However, now by the induction hypothesis applied on columns $A_3, \ldots, A_q$, we have that $\mathcal{S} \cap A_3$ is either a $P(4)$ or a $P(3,4)$ configuration. So, in any case, this contradicts Lemma 4.

Thus, by Lemma 3, we have $|\mathcal{S} \cap A_2| = 6$ and so $\mathcal{S} \cap C_4$ is a $P(3,4)$ configuration. Observe that $\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)$ is a solution $\mathcal{S}'$ of ($\mathrm{PEP}'_{4\times 4q}$) of cardinality $6(q-1)$. Thus, by the induction hypothesis, claim (8) holds.

Subcase A.2: $|\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)| = 6(q-1) - 1$. Without loss of generality, we may assume that $|\mathcal{S} \cap A_1| = 6$. Moreover, by the induction hypothesis, we have either $|\mathcal{S} \cap A_2| = 5$ or $|\mathcal{S} \cap A_q| = 5$.

If $|\mathcal{S} \cap A_2| = 5$, then we may assume $q > 2$ for otherwise we conclude as in Subcase A.1. Hence, $\mathcal{S} \cap A_2$ is a $P(4)$ configuration. Thus, $|\mathcal{S} \cap A_1| = 6$ contradicts Lemma 3.

If $|\mathcal{S} \cap A_2| = 6$, then, by Lemma 3, $\mathcal{S} \cap A_2$ is a $P(3, 4)$ configuration. So claim (8) holds by Lemma 3.

Case B: $\mathcal{S}$ *is a solution to* $(\mathrm{PEP}'_{4\times 4q})$. Let $A_1, \ldots, A_q$ be a partition of $G_{4,n}$ where the A_i's are blocks of four consecutive columns. Observe that by the induction hypothesis we have $|\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)| \geq 6(q-1) - 1$, $|\mathcal{S} \cap A_1| \geq 6$.

If $|\mathcal{S} \cap A_2 \cap \cdots \cap A_q| > 6(q-1)-1$ and if $|\mathcal{S} \cap A_1| > 6$, then clearly the claim holds. If $|\mathcal{S} \cap A_1| = 6$, then, by Lemma 3, $\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)$ is a solution of $(\mathrm{PEP}'_{4\times 4(q-1)})$. So claim (8) holds, by the induction hypothesis.

Suppose that $|\mathcal{S} \cap (A_2 \cup \cdots \cup A_q)| = 6(q-1) - 1$. By the induction hypothesis, we have either $|\mathcal{S} \cap A_2| = 5$ or $|\mathcal{S} \cap A_q| = 5$.

If $|\mathcal{S} \cap A_2| = 5$, then C_5 is a $P(4)$ configuration and so, by Lemma 4, $|\mathcal{S} \cap A_1| \geq 8$. Hence, claim (8) holds.

If $|\mathcal{S} \cap A_q| = 5$, then C_5 is a $P(4)$ or a $P(3, 4)$ configuration and so, by Lemma 4 and since C_0 is a $P(3, 4)$ configuration, we obtain again that $|\mathcal{S} \cap A_1| \geq 8$, which completes the proof of claim (8).

Now, as in the proof of claim (8), we have that any optimal solution $\mathcal{S}$ of $(\mathrm{PEP}_{4\times 4q})$ has cardinatity $6q - 1$ and is obtained as shown in Fig. 8. □

Using the same technique employed in the proofs of Lemmas 2–4, we can prove that any solution $\mathcal{S}$ and $\mathcal{S}'$ of (PEP_{4+r}) and respectively (PEP'_{4+r}) satisfies

$$|\mathcal{S} \cap (C_1 \cup \cdots \cup C_{4+r})| \geq \begin{cases} 7 & \text{if} \quad r = 1, \\ 8 & \text{if} \quad r = 2, \\ 10 & \text{if} \quad r = 3, \end{cases}$$

and $|\mathcal{S}' \cap (C_1 \cup \cdots \cup C_{4+r})| \geq |\mathcal{S} \cap (C_1 \cup \cdots \cup C_{4+r})| + 1$.

Finally, by a simple induction and using Theorem 5, we can prove that

$$\mathcal{S} \cap G_{4,4q+r} = \begin{cases} 6q - 1 & \text{if} \quad r = 0, \\ 6q + 1 & \text{if} \quad r = 1, \\ 6q + 2 & \text{if} \quad r = 2, \\ 6q + 4 & \text{if} \quad r = 3. \end{cases}$$

Unfortunately, for $r \neq 0$, the optimal solutions are not “unique.”

4. Concluding Remarks

It is straightforward from Lemmas 2 and 3 to prove that any solution of $(\mathrm{PEP}_{8\times 8})$ has at least 24 squares. We do not give the details of the proof here since Bosch [1] solved it using a computer and an integer linear programming approach. Our approach should be helpful to solve $(\mathrm{PEP}_{k\times n})$-type problems for any fixed k. Another family of problems should be to consider other lattices instead of the chessboard (grid).

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