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# Chiral Polyhedra in Ordinary Space, I*

Egon Schulte

Department of Mathematics, Northeastern University,
Boston, MA 02115, USA
schulte@neu.edu

To the memory of Donald Coxeter

**Abstract.** Chiral polyhedra in ordinary euclidean space $\mathbb{E}^3$ are nearly regular polyhedra; their geometric symmetry groups have two orbits on the flags, such that adjacent flags are in distinct orbits. This paper completely enumerates the discrete infinite chiral polyhedra in $\mathbb{E}^3$ with finite skew faces and finite skew vertex-figures. There are several families of such polyhedra of types $\{4, 6\}$, $\{6, 4\}$ and $\{6, 6\}$. Their geometry and combinatorics are discussed in detail. It is also proved that a chiral polyhedron in $\mathbb{E}^3$ cannot be finite. Part II of the paper will complete the classification of all chiral polyhedra in $\mathbb{E}^3$. All chiral polyhedra not described in Part I have infinite, helical faces and again occur in families. So, in effect, Part I enumerates all chiral polyhedra in $\mathbb{E}^3$ with finite faces.

## 1. Introduction

The study of highly symmetric polyhedra in ordinary euclidean space $\mathbb{E}^3$ has a long history. With the passage of time, the concept of a polyhedron has undergone a number of changes which have brought to light new classes of regular polyhedra. Coxeter's famous *Regular Polytopes* [6] and his various other writings treat the Platonic solids, the Kepler–Poinsot polyhedra and the Petrie–Coxeter polyhedra in great detail, and cover what might be called the classical theory. Around 1975, Grünbaum [11] generalized the notion of a polyhedron by permitting discrete polyhedral structures with finite or infinite, planar or skew, polygonal faces or vertex-figures, and discovered all, save one, generalized regular polyhedra; Dress [8], [9] discovered the final instance around 1980, and he also proved the completeness of the enumeration. We refer to Section 7E of [18] (or [17]) for a quick method of arriving at the full characterization, as well as for presentations of the symmetry groups.

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This paper deals with chiral polyhedra in ordinary space. A polyhedron is (*geometrically*) *chiral* if its geometric symmetry group has two orbits on the flags, such that adjacent flags are in distinct orbits. Recall that a polyhedron is *regular* if its geometric symmetry group is transitive on the flags. Thus chiral polyhedra are nearly regular. Most regular polyhedra in $\mathbb{E}^3$ have either skew faces or skew vertex-figures, but it is quite remarkable that none has both finite skew faces and finite skew vertex-figures. However, this phenomenon changes drastically in the context of chiral polyhedra.

This paper describes a complete classification of the discrete chiral polyhedra with finite skew faces and finite skew vertex-figures in $\mathbb{E}^3$. There are three integer-valued two-parameter families of chiral polyhedra of this kind for each type $\{4, 6\}$, $\{6, 4\}$ or $\{6, 6\}$. Each chiral polyhedron with finite skew faces and finite skew vertex-figures in $\mathbb{E}^3$ necessarily belongs to one infinite family. Some infinite families split further into several smaller subfamilies. The geometry and combinatorics of these polyhedra are discussed in detail. It is also proved that there are no chiral polyhedra in $\mathbb{E}^3$ which are finite.

The paper is organized as follows. In Section 2 we begin with some basic notions about chiral and regular polyhedra. Then in Section 3 we establish that a chiral polyhedron in $\mathbb{E}^3$ must necessarily be an *apeirohedron*, that is, a polyhedron with infinitely many faces. All remaining sections of the paper deal with infinite polyhedra. In Section 4 we introduce some general considerations, in particular those concerning the special group of the symmetry group. The actual enumeration is then carried out in Sections 5 and 6. In particular, we describe in detail the chiral polyhedra of types $\{6, 6\}$ and $\{4, 6\}$ with skew faces and vertex-figures. Finally, in Section 7 we briefly discuss relationships among them.

In Part II we then complete the enumeration of the chiral polyhedra in $\mathbb{E}^3$. All chiral polyhedra not described in Part I have infinite, helical faces and again occur in families. So, in effect, Part I enumerates all chiral polyhedra in $\mathbb{E}^3$ with finite faces.

## 2. Chiral Polyhedra

Since we discuss chiral polyhedra on the abstract as well as the geometric level, we begin with a brief introduction to the underlying general theory (see Chapter 2 of [18]). An (*abstract*) *polyhedron* (*abstract* 3-*polytope*) is a partially ordered set $\mathcal{P}$ with a strictly monotone *rank* function whose range is $\{-1, 0, \ldots, 3\}$. The elements of rank $j$ are called the $j$-*faces* of $\mathcal{P}$. For $j = 0$, 1 or 2, we also call $j$-faces *vertices*, *edges* and *facets*, respectively. When there is no possibility of confusion, we adopt standard terminology for polyhedra and use the term "face" to mean "2-face" (facet). The *flags* (maximal totally ordered subsets) of $\mathcal{P}$ each contain one vertex, one edge and one facet, as well as the unique minimal face $F_{-1}$ and unique maximal face $F_3$ of $\mathcal{P}$. Further, $\mathcal{P}$ is *strongly flag-connected*, meaning that any two flags $\Phi$ and $\Psi$ of $\mathcal{P}$ can be joined by a sequence of flags $\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi$, where $\Phi_{i-1}$ and $\Phi_i$ are *adjacent* (differ by one face), and $\Phi \cap \Psi \subseteq \Phi_i$ for each $i$. Finally, if $F$ and $G$ are a $(j-1)$-face and a $(j+1)$-face with $F < G$ and $0 \leq j \leq 2$, then there are exactly *two* $j$-faces $H$ such that $F < H < G$.

When $F$ and $G$ are two faces of a polyhedron $\mathcal{P}$ with $F \leq G$, we call $G/F := \{H \mid F \leq H \leq G\}$ a *section* of $\mathcal{P}$. We may usually safely identify a face $F$ with the section $F/F_{-1}$. For a face $F$, the section $F_3/F$ is called the *co-face of* $\mathcal{P}$ *at* $F$, or the *vertex-figure at* $F$ if $F$ is a vertex.

An abstract polyhedron $\mathcal{P}$ is *regular* if its (*combinatorial automorphism*) *group* $\Gamma(\mathcal{P})$ is transitive on its flags. Let $\Phi := \{F_0, F_1, F_2\}$ be a fixed or *base* flag of $\mathcal{P}$ (we usually omit $F_{-1}$ and $F_3$ from the notation for flags). The group $\Gamma(\mathcal{P})$ of a regular polyhedron $\mathcal{P}$ is generated by *distinguished generators* $\rho_0, \rho_1, \rho_2$ (*with respect to* $\Phi$), where $\rho_j$ is the unique automorphism which keeps all but the $j$-face of $\Phi$ fixed. These generators satisfy the standard relations

$$\rho_0^2 = \rho_1^2 = \rho_2^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = \varepsilon, \tag{2.1}$$

with $p$ and $q$ determined by the (*Schläfli*) *type* $\{p, q\}$ of $\mathcal{P}$; in general there are also other independent relations. Observe that, in a natural way, the group of the facet of $\mathcal{P}$ is $\langle\rho_0, \rho_1\rangle$, while that of the vertex-figure is $\langle\rho_1, \rho_2\rangle$.

An abstract polyhedron $\mathcal{P}$ is *chiral* if $\Gamma(\mathcal{P})$ has two orbits on the flags, such that adjacent flags are in distinct orbits. Once again let $\Phi := \{F_0, F_1, F_2\}$ be a base flag of $\mathcal{P}$, and let $F_j'$, with $j = 0, 1, 2$, denote the $j$-face of $\mathcal{P}$ with $F_{j-1} < F_j' < F_{j+1}$ and $F_j' \neq F_j$. The group $\Gamma(\mathcal{P})$ of a chiral polyhedron $\mathcal{P}$ is generated by *distinguished generators* $\sigma_1, \sigma_2$ (*with respect to* $\Phi$), where $\sigma_1$ fixes the base facet $F_2$ and cyclically permutes its vertices such that $F_1\sigma_1 = F_1'$ (and thus $F_0'\sigma_1 = F_0$),[1] and $\sigma_2$ fixes the base vertex $F_0$ and cyclically permutes the vertices in the vertex-figure at $F_0$ such that $F_2\sigma_2 = F_2'$ (and thus $F_1'\sigma_2 = F_1$). Now we have the standard relations

$$\sigma_1^p = \sigma_2^q = (\sigma_1\sigma_2)^2 = \varepsilon, \tag{2.2}$$

where again $p$ and $q$ are determined by the type $\{p, q\}$ of $\mathcal{P}$ and in general other independent relations occur. We often take $\tau := \sigma_1\sigma_2$ and $\sigma_2$ as generators of $\Gamma(\mathcal{P})$. Note that $\tau$ acts like a "half-turn" about the "midpoint" of the base edge $F_1$; it interchanges the vertices $F_0$ and $F_0'$ of $F_1$, as well as the two facets $F_2$ and $F_2'$ that meet at $F_1$.

In a chiral polyhedron, adjacent flags are not equivalent under the group. If $\Phi$ is replaced by the adjacent flag $\Phi^2 := \{F_0, F_1, F_2'\}$ (say), then the generators $\sigma_1, \sigma_2$ of $\Gamma(\mathcal{P})$ must be replaced by the new generators $\sigma_1\sigma_2^2, \sigma_2^{-1}$. Note that their product is again $\tau$. A chiral polyhedron occurs in two (*combinatorially*) *enantiomorphic forms* (see [22] and [23]); an enantiomorphic form simply is a pair consisting of the underlying abstract polyhedron and an orbit of flags (specifying a "combinatorial orientation"). These two enantiomorphic forms of $\mathcal{P}$ correspond to the two sets of generators $\sigma_1, \sigma_2$ and $\sigma_1\sigma_2^2, \sigma_2^{-1}$ of $\Gamma(\mathcal{P})$.

If $\mathcal{P}$ is a regular polyhedron, then the generators $\sigma_1 := \rho_0\rho_1$ and $\sigma_2 := \rho_1\rho_2$ of the *rotation subgroup* $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$ also satisfy the relations (2.2). Moreover, $\tau = \rho_0\rho_2$. Now the two sets of generators of $\Gamma^+(\mathcal{P})$ are conjugate in $\Gamma(\mathcal{P})$ under $\rho_2$ (so that the two enantiomorphic forms can be identified).

Let $\mathcal{P}$ be an abstract polyhedron, and let $\mathcal{P}_j$ denote its set of $j$-faces. Following Section 5A of [18], a *realization* of $\mathcal{P}$ is a mapping $\beta\colon \mathcal{P}_0 \to E$ of the vertex-set $\mathcal{P}_0$ into some euclidean space $E$. In our applications, $E = \mathbb{E}^3$. Define $\beta_0 := \beta$ and $V_0 := V := \mathcal{P}_0\beta$, and write $2^X$ for the family of subsets of the set $X$. The realization $\beta$ recursively induces surjections $\beta_j\colon \mathcal{P}_j \to V_j$, for $j = 1, 2, 3$, with $V_j \subset 2^{V_{j-1}}$ consisting

[1] Throughout, mappings act on the right.

of the elements

$$F\beta_j := \{G\beta_{j-1} \mid G \in \mathcal{P}_{j-1} \text{ and } G \leq F\}$$

for $F \in \mathcal{P}_j$; further, $\beta_{-1}$ is given by $F_{-1}\beta_{-1} := \emptyset$. Even though each $\beta_j$ is determined by $\beta$, it is helpful to think of the realization as given by all the $\beta_j$. A realization $\beta$ is *faithful* if each $\beta_j$ is a bijection; otherwise, $\beta$ is *degenerate*. See Section 5B in [18], or [19], for classes of abstract polyhedra or polytopes all of whose realizations in euclidean spaces are known.

Except in one instance, we work with discrete and faithful realizations. In this case the vertices, edges and facets of $\mathcal{P}$ are in one-to-one correspondence with certain points, line segments and simple (finite or infinite) polygons in $E$, and it is safe to identify a face of $\mathcal{P}$ and its image in $E$. The resulting family of points, line segments and polygons is a *geometric polyhedron* in $E$ and is denoted by $P$; it is understood that $P$ inherits the partial ordering of $\mathcal{P}$, and when convenient $P$ is identified with $\mathcal{P}$. For related work see also [12] and [13].

A realization $\beta$ of $\mathcal{P}$ is *symmetric* if each automorphism of $\mathcal{P}$ induces an isometric permutation of the vertex-set $V = \mathcal{P}_0\beta$; such an isometric permutation extends to an isometry of $E$, uniquely if $E$ is the affine hull of $V$. Thus associated with a realization $\beta$ of $\mathcal{P}$ is a euclidean representation of $\Gamma(\mathcal{P})$ as a group of isometries. The Grünbaum–Dress polyhedra mentioned in the Introduction are precisely the realizations $P$ of abstract regular polyhedra in ordinary space $\mathbb{E}^3$, which are discrete, faithful and symmetric. They are geometric polyhedra in $\mathbb{E}^3$ which are *geometrically regular*, meaning that they have a flag-transitive symmetry group $G(P)$.

In this paper we are mainly concerned with geometric polyhedra $P$ in $\mathbb{E}^3$. We call such a polyhedron $P$ *geometrically chiral* if its symmetry group $G(P)$ has two orbits on the flags of $P$, such that adjacent flags are in distinct orbits. Then it is immediate that the underlying abstract polyhedron $\mathcal{P}$ must be combinatorially chiral or combinatorially regular. In any case, the above general results for abstract chiral polyhedra carry over to geometrically chiral polyhedra. In particular, we now have *distinguished generators* $S_1, S_2$ for $G(P)$ corresponding to $\sigma_1, \sigma_2$, as well as their product $T := S_1S_2$ corresponding to $\tau = \sigma_1\sigma_2$. If $\mathcal{P}$ is also chiral, then $\Gamma(\mathcal{P})$ and $G(P)$ are isomorphic, and the realization is symmetric. If $\mathcal{P}$ is regular, then the geometric group $G(P)$ is isomorphic to the subgroup $\Gamma^+(\mathcal{P})$ of $\Gamma(\mathcal{P})$, and the generators $S_1, S_2$ for $G(P)$ correspond to those of $\Gamma^+(\mathcal{P})$; in this case the involutory automorphism $\rho_0$ of $\mathcal{P}$ does not correspond to a symmetry of $P$, so that only one-half of the automorphisms of $\mathcal{P}$ are realized as symmetries of $P$. In this situation we call $P$ a *chiral realization* of the regular polyhedron $\mathcal{P}$. Not much is known about chiral realizations of regular polyhedra in euclidean spaces.

Regular or chiral polyhedra $P$ (or $\mathcal{P}$) can be obtained by *Wythoff's construction*. There are two variants, one based on reflections and applying only to regular polytopes (see [6] and p. 124 of [18]), and the other based on rotations and applying to both kinds of polyhedra.

Let $\mathcal{P}$ be an abstract regular polyhedron, and let $G := \langle R_0, R_1, R_2\rangle$ be a euclidean representation of its group $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2\rangle$ in $\mathbb{E}^3$, where the (point, line or plane) reflection $R_i$ corresponds to $\rho_i$ for each $i$. Each point $v$ which is fixed by $R_1$ and $R_2$ yields a realization $P$ of $\mathcal{P}$; the base (or *initial*) vertex, base edge and base facet of $P$ are given by $v$, $v\langle R_0\rangle$ and $v\langle R_0, R_1\rangle$, respectively, and the other vertices, edges and facets of $P$ are their images under $G$.

Let $\mathcal{P}$ be abstractly chiral or abstractly regular, and let $G := \langle S_1, S_2 \rangle$ be a euclidean representation of its group $\Gamma(\mathcal{P})$ or its rotation subgroup $\Gamma^+(\mathcal{P})$, respectively. Now each point $v$ which is fixed by $S_2$ can serve as the *initial* vertex of a realization $P$, again with vertex-set $V = vG$. Its base vertex, base edge and base facet are $v$, $v\langle T \rangle$ and $v\langle S_1 \rangle$, respectively, and the other vertices, edges and facets are again their images under $G$. Note that, a priori, an abstract regular polyhedron can have a realization which is geometrically chiral.

Our main goal is to describe the geometrically chiral polyhedra $P$ whose symmetry group $G(P)$ is generated by rotatory reflections $S_1, S_2$ of finite period. In particular, this will yield a complete enumeration of the geometrically chiral polyhedra with finite skew facets and finite skew vertex-figures. Moreover, we shall see that there are no finite geometric polyhedra that are geometrically chiral.

The regular polyhedra in $\mathbb{E}^3$ were extensively studied in Section 7E of [18] (see also [8], [9], [11], [14], and [15]). It is interesting to observe that some of the basic operations that can be applied to them have analogs that also apply to abstract chiral polyhedra, and frequently to geometrically chiral polyhedra as well. Two examples are the *duality operation* $\delta$, yielding the generators for the group of the dual, and the (2*nd*) *facetting operation* $\varphi_2$ (see pp. 192 and 194 of [18]). If $\mathcal{P}$ is a chiral polyhedron with group $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$, then $\delta$ and $\varphi_2$ are given by

$$\delta\colon (\sigma_1, \sigma_2) \mapsto (\sigma_2^{-1}, \sigma_1^{-1}) \quad \text{and} \quad \varphi_2\colon (\sigma_1, \sigma_2) \mapsto (\sigma_1\sigma_2^{-1}, \sigma_2^2), \tag{2.3}$$

respectively. When applied to the rotation subgroup $\Gamma^+(\mathcal{P})$ of a regular polyhedron $\mathcal{P}$, the generators on the right become the distinguished generators for the rotation subgroups of the polyhedra $\mathcal{P}^\delta$ (the dual $\mathcal{P}^*$) and $\mathcal{P}^{\varphi_2}$ of Section 7B of [18], respectively. In Section 7 we shall meet a third example, the *halving operation* $\eta$ (see p. 197 of [18]).

Finally, recall that the *Petrial* of a polyhedron $P$ (or $\mathcal{P}$) has the same vertices and edges as $P$, and has as its facets the *Petrie polygons* of $P$, whose defining property is that two successive edges, but not three, are edges of a facet of $P$. Note that the Petrie operation on p. 192 of [18] cannot be expressed in terms of rotations only; in fact, this should not come as a surprise, because a chiral polyhedron will generally have “right-handed” and “left-handed” Petrie polygons of different lengths.

## 3. Finite Polyhedra

Let $P$ be a geometric polyhedron in $\mathbb{E}^3$ which is (geometrically) regular or chiral, and let $\Phi := \{F_0, F_1, F_2\}$ be a base flag of $P$. Then $G(P)$ contains elements $S_1, S_2$ as above, and these generate $G(P)$ if $P$ is chiral, or a subgroup of index 2 in $G(P)$ (isomorphic to $\Gamma^+(\mathcal{P})$) if $P$ is regular. As before, let $T := S_1 S_2$. If $P$ is finite, then $G(P)$ leaves a point invariant, the origin $o$ (say). Then $G(P)$ is a finite subgroup of $\mathcal{O}_3$, the orthogonal group.

We now establish that a chiral polyhedron in $\mathbb{E}^3$ cannot be finite. One possible proof appeals to the enumeration of the finite subgroups of $\mathcal{O}_3$ (see Chapter 2 of [10]) and exploits the geometry of the possible generators $S_1, S_2$ in the context of Wythoff’s construction. However, here we borrow the shorter proof, suitably adapted, of a more general non-existence result for certain classes of chiral polytopes in euclidean $n$-space $\mathbb{E}^n$, recently obtained by McMullen [16]; in fact, there are no finite chiral polytopes of rank $n$ in $\mathbb{E}^n$. The following theorem is the special case $n = 3$.

**Theorem 3.1.** *There are no chiral geometric polyhedra in* $\mathbb{E}^3$ *which are finite. In other words, in the above situation,* $P$ *must be geometrically regular.*

*Proof.* Since $P$ is finite, its vertices must lie on a sphere centered at $o$. The vertices adjacent to the base vertex $F_0$ are cyclically permuted by $S_2$, hence lie on a sphere centered at $F_0$. Thus the vertex-figures of $P$ must be planar *regular* polygons. Moreover, $T$ is an involutory symmetry of $P$ which interchanges the vertex-figures $P/F_0$ at $F_0$ and $P/F_0'$ at the adjacent vertex $F_0'$ of $F_1$. As $P/F_0$ is regular, we can also find two reflections $R_1$, $R_2$ in planes through $o$ which act on $P/F_0$ like the standard generators of its planar dihedral symmetry group. In particular, $S_2 = R_1R_2$. The generator $R_2$ fixes each point of $F_1$ (its mirror contains $F_1$) and determines the symmetry group of the (regular one-dimensional) co-face $P/F_1$ at $F_1$. Moreover, $R_2$ is also a symmetry of the vertex-figure $P/F_0'$ at the adjacent vertex $F_0'$.

We then readily see that this local symmetry $R_2$ is in fact a global symmetry of $P$. The details of the proof exploit the existence of global symmetries (such as $S_2$ and $T$) and local symmetries determined by edges (such as $R_2$). For example, the regularity of the vertex-figures already forces $R_2$ to be a symmetry of a "belt" of faces near the mirror of $R_2$; here it is useful to observe that, because they have full rotational symmetry, the polygonal faces of $P$ are actually also regular. We can further extend the belt by employing global symmetries; for example, the vertex-figures at the vertices $F_0'S_2$ and $F_0'S_2^{-1}$ are related by symmetry under $R_2$, as are their images under $T$, and so on.

Hence $R_2$ is a symmetry of $P$, and thus $R_1 = S_2R_2$ and $R_0 := S_1R_1$ are also symmetries of $P$. In particular, $P$ must be regular. (See also Lemma 2.1 of [21] for another argument why $P$ must be regular.) $\square$

The eighteen finite regular polyhedra in $\mathbb{E}^3$ split into two groups of nine, according as $T$ is a half-turn or a plane reflection (see p. 218 of [18]).

- If $T$ is a half-turn, then $P$ is either a Platonic solid $\{3, 3\}$, $\{3, 4\}$, $\{4, 3\}$, $\{3, 5\}$ or $\{5, 3\}$, or a Kepler–Poinsot polyhedron $\{3, \frac{5}{2}\}$, $\{\frac{5}{2}, 3\}$, $\{5, \frac{5}{2}\}$ or $\{\frac{5}{2}, 5\}$.
- If $T$ is a plane reflection, then $P$ is the Petrial of either a Platonic solid or a Kepler-Poinsot polyhedron.

Recall that $P$ and its Petrial have the same group but with different sets of generators, namely $R_0, R_1, R_2$ and $R_0R_2, R_1, R_2$, respectively [18, p. 192]; since $R_2$ is a plane reflection, the corresponding mappings $T$, given by $R_0R_2$ and $R_0$, respectively, have the property that one is a half-turn if and only if the other is a plane reflection.

It is quite remarkable that there are examples of finite geometric polyhedra in $\mathbb{E}^3$ which are not chiral but still have a symmetry group with only two orbits on the flags. Their flag orbits must necessarily contain pairs of adjacent flags. Wills [24] proved that, up to similarity, there are precisely five such polyhedra with planar faces which are realizations of orientable abstract regular polyhedra (see also [2]). The corresponding abstract polyhedra are given by the two dual pairs $\{4, 5\}_6$ and $\{5, 4\}_6$ of genus 4, and $\{6, 5\}_4$ and $\{5, 6\}_4$ of genus 9 (see [7]), as well as a certain self-dual map of type $\{6, 6\}$ of genus 11 (also described on p. 151 of [25]).

Observe that these examples also illustrate that we cannot omit the requirement that adjacent flags be in distinct orbits from the definition of chirality.

With a very similar proof as above we can also establish the following:

**Theorem 3.2.** *A chiral geometric polyhedron in* $\mathbb{E}^3$ *cannot be planar.*

This is the special case $n = 3$ of the more general result that there are no discrete faithful chiral realizations of polytopes of rank $n$ in $\mathbb{E}^{n-1}$ (see again [16]). An alternative proof of Theorem 3.2 can be given, which again is based on Wythoff's construction and inspection of the geometry of the possible generators $S_1$, $S_2$.

## 4. General Considerations for Infinite Polyhedra

We now investigate geometrically chiral *apeirohedra*, or infinite geometric polyhedra, in $\mathbb{E}^3$. Throughout we assume discreteness. In particular, this implies that the vertex-figures are finite polygons.

We begin with a lemma which restricts the groups that can occur as symmetry groups. Recall that an infinite discrete group $G$ of isometries of $\mathbb{E}^3$ is said to *act irreducibly* if there is no non-trivial linear subspace $L$ of $\mathbb{E}^3$ which is *invariant* in the sense that $G$ permutes the translates of $L$. If such an invariant subspace $L$ exists, then its orthogonal complement $L^\perp$ is also invariant in the same sense. In effect, the following lemma was proved on p. 220 of [18].

**Lemma 4.1.** *An irreducible infinite discrete group of isometries in* $\mathbb{E}^3$ *is a crystallographic group. In particular, it does not contain rotations of periods other than* 2, 3, 4 *or* 6.

Bieberbach's theorem now tells us that such a group $G$ contains a subgroup $T(G)$ of the group $\mathcal{T}_3$ of translations of $\mathbb{E}^3$, such that the quotient $G/T(G)$ is finite; in effect, $T(G)$ can be thought of as a lattice in $\mathbb{E}^3$ (see [1] and Section 7.4 of [20]). If $R\colon\ x \mapsto xR' + t$ is a general element of $G$, with $R' \in \mathcal{O}_3$ and $t \in \mathbb{E}^3$ a translation vector (we may thus think of $t \in \mathcal{T}_3$), then the mappings $R'$ clearly form a subgroup $G_0$ of $\mathcal{O}_3$, called the *special group* of $G$. Thus $G_0$ is the image of $G$ under the homomorphism on $\mathcal{I}_3$, the group of isometries of $\mathbb{E}^3$, whose kernel is $\mathcal{T}_3$ (the image is, of course, $\mathcal{O}_3$). In other words,

$$G_0 = G\mathcal{T}_3/\mathcal{T}_3 \cong G/(G \cap \mathcal{T}_3) = G/T(G),$$

if $T(G)$ is the full translation subgroup of $G$.

In this context the following lemma is useful.

**Lemma 4.2.** *Let* $R \in \mathcal{I}_3$, $R' \in \mathcal{O}_3$ *and* $t \in \mathbb{E}^3$, *such that* $xR = xR' + t$ *for each* $x \in \mathbb{E}^3$. *Let* $H$ *be a plane through* $o$, *with orthogonal complement* $H^\perp$, *and let* $t = t_1 + t_2$ *with* $t_1 \in H$ *and* $t_2 \in H^\perp$.

(a) *If* $R'$ *is the reflection in* $H$, *then* $R$ *is the glide reflection given by the reflection in the plane through* $\frac{1}{2}t_2$ *parallel to* $H$, *followed by the translation by* $t_1$ *parallel to* $H$.

(b) *If $R'$ is a non-trivial rotation about $H^{\perp}$, then $R$ is a twist (screw motion) given by a rotation about an axis parallel to $H^{\perp}$, followed by the translation by $t_2$ in the direction of this axis.*
(c) *If $R'$ is a rotatory reflection with reflection plane $H$ and rotation axis $H^{\perp}$, such that $R'$ is not the reflection in $H$, then $R$ is a rotatory reflection with a reflection plane parallel to $H$ and passing through $\frac{1}{2}t_2$, and with a rotation axis parallel to $H^{\perp}$.*

Now let $P$ be a geometrically chiral apeirohedron in $\mathbb{E}^3$ of type $\{p, q\}$, and let $G(P)$ act irreducibly on $\mathbb{E}^3$ (that is, $P$ is a *pure* realization in the sense of [18, p. 126]). We assume that the base vertex is at the origin $o$. Then $G(P) = \langle S_1, S_2\rangle$ must be a crystallographic group, whose special group $G_0(P)$ is a finite subgroup of $\mathcal{O}_3$ generated by the images $S_1'$ of $S_1$ and $S_2'$ of $S_2$ under the above homomorphism on $\mathcal{I}_3$. The isometries $S_1'$ and $S_2'$ are rotations or rotatory reflections of finite period at least 3. Note that $S_2' = S_2$, because $S_2$ fixes the base vertex $o$.

If the apeirohedron $P$ has finite faces, then each of the four isometries $S_1$, $S_2$, $S_1'$, $S_2'$ must be a rotation or rotatory reflection of finite period. Moreover, if $P$ has finite skew faces and skew vertex-figures, then indeed all four must be rotatory reflections. In this case the products $T = S_1S_2$ and $T' = S_1'S_2'$ are proper involutory isometries and thus are half-turns.

From now on we assume that the generators $S_1$ and $S_2$ of $G(P)$ are rotatory reflections of finite period. The next lemma limits the groups that can occur as special groups to only two possibilities. Let $C = \{4, 3\}$ be a cube centered at $o$ with edges parallel to the coordinate axes, and let $K = \{3, 3\}$ be a tetrahedron inscribed in $C$ as in Fig. 4.1. Consider the subgroup

$$[3, 3]^* := [3, 3]^+ \cup (-I)[3, 3]^+ \tag{4.1}$$

of $[3, 4] = G(C)$ isomorphic to $A_4 \times C_2$. Then $[3, 3]^*$ consists of the rotational symmetries of $C$ which map $K$ to itself, as well as of the symmetries of $C$ obtained from those by adjoining $-I$, the negative of the identity mapping $I$.

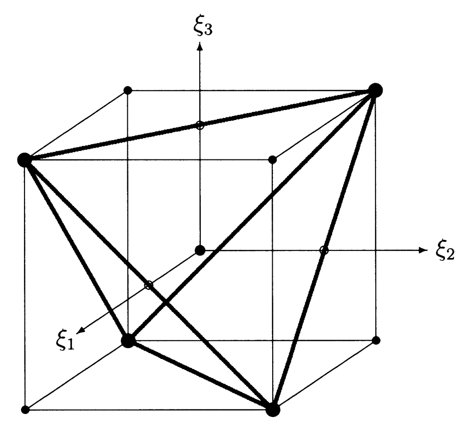


**Fig. 4.1.** A regular tetrahedron inscribed in a cube.

**Lemma 4.3.** *As before, let $P$ be a geometrically chiral apeirohedron in $\mathbb{E}^3$ with group $G(P) = \langle S_1, S_2\rangle$, let $P$ be discrete and pure and let $o$ be the base vertex of $P$. If $S_1$ and $S_2$ are rotatory reflections of finite period (this is true if $P$ has finite skew faces and finite skew vertex-figures), then $G_0(P) = \langle S_1', S_2'\rangle$ is either the group* $[3, 4]$ *or its subgroup* $[3, 3]^*$. *Moreover, $P$ is of type* $\{4, 6\}$, $\{6, 4\}$ *or* $\{6, 6\}$.

*Proof.* We know that $G_0(P)$ is a finite subgroup of $\mathcal{O}_3$ generated by two rotatory reflections of period at least 3, whose product is of period 2. Since $P$ is pure, we must have rotatory reflections in $G_0(P)$ for more than one axis. Moreover, by Lemma 4.1 we cannot have five-fold rotations in $G_0(P)$. Inspection of the finite subgroups of $\mathcal{O}_3$ (see Chapter 2 of [10]) now limits the admissible groups to only two, namely $[3, 4]$ and $[3, 3]^*$.

In $[3, 3]^*$ we have eight rotatory reflections of period 6, each given by a rotation by $\pm\pi/3$ about a diagonal of $C$, followed by a reflection in the plane through $o$ perpendicular to the diagonal (see Fig. 4.1). They correspond to Petrie polygons of $C$, or, equivalently, faces of $K$. In particular, we obtain pairs $S_1'$, $S_2'$ of generators from pairs of adjacent faces of $K$, suitably oriented to yield products of period 2. Thus $P$ must be a polyhedron of type $\{6, 6\}$, whose faces and vertex-figures are skew hexagons of type $\{6\}\#\{\,\}$ (see p. 222 of [18]). Note that $T' = S_1'S_2'$ is a half-turn about a coordinate axis (passing through the centers of antipodal faces of $C$).

In $[3, 4]$ we have six rotatory reflections of period 4 and eight of period 6, the latter being those of the subgroup $[3, 3]^*$. Each rotatory reflection of period 4 is given by a rotation by $\pm\pi/2$ about a coordinate axis, followed by a reflection in the plane through $o$ perpendicular to the axis. Since the product of the two generators $S_1'$, $S_2'$ must be of period 2, one generator must be of period 4 and the other of period 6. In fact, given a rotatory reflection of period 6 in $[3, 4]$, exactly three rotatory reflections of period 4 will yield a product with it of period 2, one for each coordinate axis. Now $P$ must be a polyhedron of type $\{4, 6\}$ or $\{6, 4\}$, whose faces and vertex-figures are skew quadrilaterals $\{4\}\#\{\,\}$ or skew hexagons $\{6\}\#\{\,\}$, respectively. Note that $T' = S_1'S_2'$ is a half-turn about the midpoint of an edge of $C$. $\square$

Next we investigate translations in $G(P)$. Once again, let $P$ be a geometrically chiral apeirohedron with base vertex $o$, and let $G(P)$ be generated by rotatory reflections $S_1$ and $S_2$. We concentrate on the types $\{4, 6\}$ and $\{6, 6\}$, and later derive the type $\{6, 4\}$ by duality. Then $S_2$ has period 6, and hence

$$S_2^3 = -I \in G(P).$$

Now let $R := S_2^3 T$, and let $R' := S_2^3 T' = -T'$ be its image in $G_0(P)$; since $T'$ is a half-turn, $R'$ is a plane reflection. The base edge $F_1$ with vertices $o$ and $v := oT$ is perpendicular to the axes of $T$ and $T'$, and thus lies in the reflection plane of $R'$. Moreover, $oR = oS_2^3 T = v$, and hence $R$ is the glide reflection given by

$$xR = xR' + v \qquad (x \in \mathbb{E}^3).$$

This gives us the translation $R^2$ in $G(P)$ by the vector $2v$. However, then the conjugates of $R^2$ by elements of $\langle S_2\rangle$ will yield all translations by vectors $2w$, with $w$ in the hexagonal vertex-figure of $P$ at $o$. In particular, if we identify a translation with its translation vector, we see that $2\Lambda_0$, with

$$\Lambda_0 := \langle w \mid w \text{ a vertex of } P \text{ adjacent to } o\rangle_{\mathbb{Z}}, \tag{4.2}$$

is a subgroup of $G(P)$. Note that $\Lambda_0$ will generally be a lattice spanned by any three independent vectors in the (generally skew) vertex-figure at $o$. However, there are more translations in $G(P)$, as the following general considerations explain.

Let $P$ be a realization of an abstract $n$-polytope in euclidean $d$-space $\mathbb{E}^d$ (see Section 5A of [18]), and let $o$ be a vertex of $P$. In our applications, $n = d = 3$. Now define the *edge-module* $\Lambda$ of $P$ by

$$\Lambda := \langle x - y \mid x, y \text{ adjacent vertices of } P\rangle_{\mathbb{Z}}. \tag{4.3}$$

Then $\Lambda$ is the $\mathbb{Z}$-module generated by the "oriented" edges of $P$; alternatively, if for a given vertex $y$ we call the set of vectors $\{x - y \mid x \text{ a vertex adjacent to } y\}$ the *vertex-star* of $P$ at $y$, then $\Lambda$ is the $\mathbb{Z}$-module generated by all the vertex-stars of $P$. If $V$ denotes the vertex-set of $P$, then

$$V \subset \Lambda,$$

because $P$ is connected. In fact, if $x$ is a vertex of $P$, then there is a chain of vertices $o = x_0, x_1, \dots, x_k = x$ of $P$ such that consecutive vertices are adjacent; then

$$x = \sum_{i=1}^{k}(x_i - x_{i-1}) \in \Lambda$$

because each summand is in $\Lambda$.

More can be said if the realization $P$ is vertex-transitive; in particular, this applies if $P$ is chiral or regular. If $G_0(P)$ is the special group of $G(P)$, then

$$\Lambda = \langle xR' \mid R' \in G_0(P),\ x \in V,\ x \text{adjacent to } o\rangle_{\mathbb{Z}},$$

and $\Lambda$ is invariant under $G_0(P)$. (In fact, under an element $R \in G(P)$ direction vectors are changed by $R'$.) In our applications, $P$ will be centrally symmetric with respect to $o$, and $-I$ will be an element of $G(P)$ contained in the group of the vertex-figure at $o$. In this context we then have

$$2\Lambda \subset V \subset \Lambda, \tag{4.4}$$

so that $V$ is the union of certain cosets of $\Lambda$ modulo $2\Lambda$. In fact, if $y$ is a vertex adjacent to $o$, and if $y = oR$ with $R \in G(P)$, then the translation by $2y$ is simply $(-I)R^{-1}(-I)R$, and hence belongs to $G(P)$. The conjugates of such translations by elements in $G(P)$ then yield all the generating translations of $2\Lambda$. Therefore,

$$2\Lambda \leq G(P). \tag{4.5}$$

Its orbit containing $o$, which we identify with $2\Lambda$, then is a subset of $V$. However, $2\Lambda$ yields translations, so $V$ must be the union of certain cosets of $\Lambda$ modulo $2\Lambda$. Observe also that, in effect, we have proved that "twice an oriented edge" (pointing from a vertex to its neighbor) will always determine a translation in $G(P)$.

We mention in passing that edge-modules are particular examples of *diagonal-modules*. The diagonals (pairs of vertices) of a realization $P$ fall into *diagonal classes*, consisting of equivalent diagonals modulo $G(P)$. With any diagonal class of $P$ is associated the $\mathbb{Z}$-module spanned by the diagonals (the vectors $x - y$) in this class. Every diagonal module of $P$ is a submodule of the edge module.

In the next sections we describe all chiral polyhedra of types $\{4, 6\}$ and $\{6, 6\}$ with a group generated by rotatory reflections. In particular, this yields the polyhedra with skew faces and skew vertex-figures. We construct these polyhedra from their vertex-sets and their edge-modules by identifying the vectors which point from a given vertex to the adjacent vertices. The pure *regular* polyhedra of types $\{4, 6\}$ and $\{6, 6\}$ with finite faces all have their generators $S_1$, $S_2$ given by rotatory reflections (see p. 225 of [18]), so they also naturally arise in this context.

In Part II of the paper we shall prove that a polyhedron $P$ with reducible group $G(P)$ cannot be chiral; that is, a chiral polyhedron in $\mathbb{E}^3$ cannot be a blend (see p. 125 of [18]). This then settles the enumeration of polyhedra with finite skew faces and vertex-figures. Moreover, we have already seen in Theorem 3.2 that a chiral polyhedron cannot be planar (see also [16]).

## 5. Type $\{6, 6\}$

In this section we derive the chiral polyhedra of type $\{6, 6\}$ and describe their geometry and combinatorics in detail.

For a polyhedron $P$ of type $\{6, 6\}$ we must begin with the group $[3, 3]^*$ and realize it as the special group of a suitable group $G$, the group of $P$. Once again we pick $o$ as the base vertex of $P$; its orbit under $G$ will then be the vertex-set $V(P)$ of $P$. The six faces which contain $o$ must correspond to six rotatory reflections of period 6; their reflection planes are parallel in pairs, with one pair for each pair of antipodal faces which contain $o$. If we move these planes into $o$ and include the mirror of the reflection component of $S_2$, we obtain four planes, each perpendicular to one of four diagonals of a cube (see Fig. 4.1).

The following lemma implies that there is essentially only one way in which the group $G$ and its generators may be taken to give a chiral polyhedron.

**Lemma 5.1.** *Let $S_2$ $(= S_2')$ be a rotatory reflection of period* 6 *in* $[3, 3]^*$. *Then there are precisely three rotatory reflections $S_1'$ of period* 6 *in* $[3, 3]^*$ *such that the product $S_1'S_2$ is of period* 2. *If $S_1'$ is one of them, then the other two are $S_2^{-1}S_1'S_2$ and $S_2^{-2}S_1'S_2^2$, and their products with $S_2$ are $S_2^{-1}(S_1'S_2)S_2$ and $S_2^{-2}(S_1'S_2)S_2^2$, respectively.*

*Proof.* Let $H := [3, 3]^* = [3, 3]^+ \cup (-I)[3, 3]^+$. Each rotatory reflection $S$ of period 6 in $H$ is of the form $S = -R$ with $R \in [3, 3]^+$ $(\cong A_4)$. If $S_1' = -R_1$ and $S_2 = -R_2$ (say), then $S_1'S_2 = R_1R_2$. It is easy to see that, given $R_2$, there are just three possible choices for $R_1$ such that $R_1R_2$ is of period 2. If $R_1$ is one of them, then the other two are $R_2^{-1}R_1R_2$ and $R_2^{-2}R_1R_2^2$. Accordingly, we obtain $S_2^{-1}S_1'S_2$ and $S_2^{-2}S_1'S_2^2$ from $S_1'$. Note that, since $S_2^3 = -I$, we have $S_2^{-j}S_1'S_2^j = S_2^{-(j+3)}S_1'S_2^{j+3}$ for $j = 0, 1, 2$. $\square$

Since the four diagonals of the cube are equivalent under the group $[3,3]^*$, and there are just three equivalent ways to pick the first generator for $[3,3]^*$ once the second is chosen, we may confine ourselves to some very specific choices for $S_2$ and $T = S_1 S_2$. There is of course the further possibility of reversing the orientation of the generator $S_2$ and replacing it by its inverse $S_2^{-1}$. However, as we shall see, this just replaces the polyhedron $P$ by its enantiomorphic image (with an adjacent base flag).

Thus we take the group $G = G(a,b)$ generated by

$$\begin{array}{rcll} S_2\colon & x & \mapsto & -(\xi_3, \xi_1, \xi_2), \\ T\colon & x & \mapsto & (-\xi_1, \xi_2, -\xi_3) + (a, 0, b), \end{array} \tag{5.1}$$

described in terms of $x = (\xi_1, \xi_2, \xi_3)$, and with real parameters $a$ and $b$, not both zero. Since $S_1 = T S_2^{-1}$, we then have

$$S_1\colon \quad x \quad \mapsto \quad (-\xi_2, \xi_3, \xi_1) + (0, -b, -a). \tag{5.2}$$

When necessary we indicate the parameters $a$ and $b$ more explicitly and write $T = T(a,b)$, $S_1 = S_1(a,b)$, and so on; note that $S_2$ does not depend on $a$ and $b$. The polyhedron $P = P(a,b)$ is now obtained by Wythoff's construction applied to $G$. In particular, $v := oT = (a, 0, b)$ is the vertex in the base edge $F_1$ distinct from $o$. Note that $F_1$ lies in the $\xi_1\xi_3$-plane and is perpendicular to the rotation axes of $T$, which is parallel to the $\xi_2$-axis and passes through $\frac{1}{2}v$.

Before we move on, let us note that

$$P(-a,-b) = -P(a,b) = P(a,b) \tag{5.3}$$

for all $a$ and $b$. In fact, $-I = S_2^3 \in G$, and conjugation by $-I$ maps the generators $T$, $S_2$ of $G = G(a,b)$ to those of $G(-a,-b)$.

The orbit $V_0$ of $v$ under $\langle S_2 \rangle$ is given by

$$V_0 := \{(a,0,b), (-b,-a,0), (0,b,a), (-a,0,-b), (b,a,0), (0,-b,-a)\}; \tag{5.4}$$

its elements are the vertices in the vertex-figure at $o$ and are listed in cyclic order. We think of them as a set of six vectors, called the *vertex-star* at the vertex $o$. Similarly, $V_2 := V_0 T' = V_0 T - v$ is the vertex-star at the vertex $v$, and is given by

$$V_2 = \{(-a,0,-b), (b,-a,0), (0,b,-a), (a,0,b), (-b,a,0), (0,-b,a)\}. \tag{5.5}$$

The vertex-stars at the other vertices adjacent to $o$ can be computed using the half-turns

$$\begin{array}{rcll} T_1 := S_2^{-2} T S_2^2\colon & x & \mapsto & (\xi_1, -\xi_2, -\xi_3) + (0, b, a), \\ T_3 := S_2^{-4} T S_2^4\colon & x & \mapsto & (-\xi_1, -\xi_2, \xi_3) + (b, a, 0). \end{array} \tag{5.6}$$

In particular, the vertex-stars $V_1 := V_0 T_1 - (0,b,a)$ at $(0,b,a)$ and $V_3 := V_0 T_3 - (b,a,0)$ at $(b,a,0)$ are given by

$$\begin{array}{l} V_1 = \{(a,0,-b), (-b,a,0), (0,-b,-a), (-a,0,b), (b,-a,0), (0,b,a)\}, \\ V_3 = \{(-a,0,b), (b,a,0), (0,-b,a), (a,0,-b), (-b,-a,0), (0,b,-a)\}, \end{array} \tag{5.7}$$

respectively. It is also convenient to set $T_2 := T$, so that $V_2 = V_0 T_2 - (a,0,b)$.

Alternatively we can obtain the vertex-star at a vertex $w := vS_2^j$ as $V_2S_2^j$, for $j = 0, \dots, 5$. Since $V_i = -V_i$ for each $i$, this also shows that the vertex-stars at $\pm w$ are the same. In fact, the latter should not come as a surprise, because the translation by $2w$ is an element in $G$, namely $S_2^{-j}(S_2^3T)^2S_2^j$ with $j$ as above (see Section 4). Note also that $|V_i \cap V_j| = 2$ for all $i, j$ with $i \neq j$, provided that $a, b \neq 0$.

Before we move on, let us recall that the base face $F_2$ of $P$ is determined by the orbit of $o$ under $S_1$. If we identify a face with its vertex-set, then $F_2$ is given by

$$\begin{aligned} F_2 \;=\; &\{(0,0,0), (0,-b,-a), (b,-a-b,-a), \\ &(a+b,-a-b,-a+b), (a+b,-a,b), (a,0,b)\}, \end{aligned} \tag{5.8}$$

where the vertices are listed in cyclic order. The other faces with vertex $o$ are the images of $F_2$ under the non-trivial elements of $\langle S_2 \rangle$. Note that the faces are skew hexagons of type {6}#{ } (see p. 222 of [18]); their vertices are among those of a hexagonal prism.

Next consider the set of (generally 12) vectors

$$V := V_0 \cup V_1 \cup V_2 \cup V_3 = \{(\pm a, 0, \pm b), (\pm b, \pm a, 0), (0, \pm b, \pm a)\}. \tag{5.9}$$

The corresponding points are the vertices of a convex 3-polytope, which is combinatorially equivalent to an icosahedron if $a, b \neq 0$ and $a \neq \pm b$, or is a cuboctahedron if $a = \pm b \neq 0$, or an octahedron if $a = 0$ or $b = 0$. Figure 5.1 shows the 12 points of $V$ for $a = 1$ and $b = 3$. The fat lines indicate the five triangles meeting at a vertex of the convex polytope with vertex set $V$. The fine grid lines are drawn in for reference.

Note that $V$ is invariant under $S_2$ and $T'$, and thus under the special group $[3, 3]^*$ of $G$. In fact,

$$V_0S_2 = V_0, \quad V_1S_2 = V_2, \quad V_2S_2 = V_3, \quad V_3S_2 = V_1, \quad V_0T' = V_2, \quad V_1T' = V_3. \tag{5.10}$$

In particular, since the vertex-stars at the vertices of $P$ are the images of the vertex-star at $o$ under the special group, (5.10) implies that $V_0, V_1, V_2, V_3$ are the only vertex-stars which can occur in $P$. Moreover, (5.10) gives a permutation representation on the indices

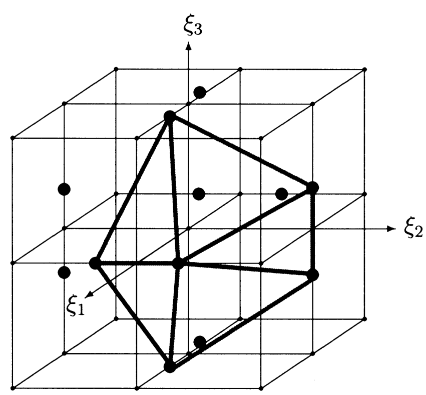


**Fig. 5.1.** The points of $V$ for $a = 1$ and $b = 3$.

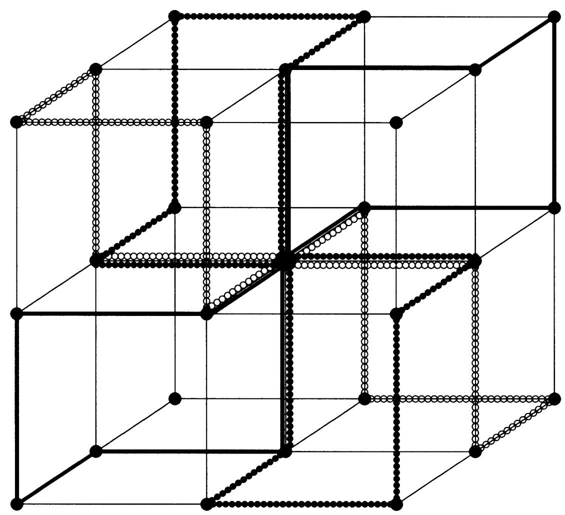

**Fig. 5.2.** The six faces of the polyhedron $P(1,0)$ containing $o$.

0, 1, 2, 3 of the vertex-stars; in particular,

$$S_2 = (1\,2\,3), \qquad T' = (0\,2)\,(1\,3), \qquad S_1' = (0\,1\,2), \tag{5.11}$$

and these permutations generate the alternating group $A_4$ $(= [3,3]^*/\{\pm I\} = [3,3]^+)$.

Observe that if $a = 0$ or $b = 0$, then the vertex-stars $V_i$ all coincide as sets, but not as sets equipped with the cyclic ordering of their vectors as in (5.4), (5.5) and (5.7); these orderings correspond to the Petrie polygons of the octahedron.

Figure 5.2 shows the polyhedron $P(1,0)$ obtained for the parameters $a = 1$ and $b = 0$. Its skew hexagonal faces are Petrie polygons of cubes in the cubical tessellation $\{4,3,4\}$. The six faces which contain $o$ are represented by fat lines, dotted lines or circled lines, such that opposite faces are indicated in the same way. The vertex-figure of $P(1,0)$ at $o$ is also a skew hexagon given by a Petrie polygon of the vertex-figure $\{3,4\}$ of $\{4,3,4\}$. Note that the edges of $P(1,0)$ are edges of $\{4,3,4\}$, but this is generally not the case for the other polyhedra.

Let $\Lambda := \mathbb{Z}[V]$ denote the $\mathbb{Z}$-module spanned by the vectors in $V$; this is the edge-module of $P$. Then it follows from the above considerations that each vector in $2\Lambda$ determines a translation in $G$. More precisely, the generating translations by vectors in $2V$ are either conjugates of $(S_2^3T)^2$ by elements in $\langle S_2\rangle$, or their conjugates by $T_1$, $T_2$ or $T_3$; the former correspond to the vectors in $2V_0$, the latter to those in $2V_1$, $2V_2$ or $2V_3$, respectively. In general, $2\Lambda$ will not be the full translation subgroup of $G$ (see Lemma 5.5).

We now discuss discreteness. Since $2\Lambda$ is a subgroup of $G$, it maps vertices to vertices; it follows that $2\Lambda$ itself, being the orbit which contains $o$, is a subset of the vertex-set (see (4.4)).

**Lemma 5.2.** *$G$ is discrete if and only if $a$ or $b$ is zero or $a$ and $b$ are rational multiples of each other.*

*Proof.* If $G$ is discrete, then $\Lambda$ must also be discrete. We have

$$2(a, 0, 0) = (a, 0, b) + (a, 0, -b) \in \Lambda,$$

and, similarly, $2(b, 0, 0) \in \Lambda$; hence $2(na + mb, 0, 0) \in \Lambda$ for all integers $n$ and $m$. However, the subset $\{na + mb \mid n, m \in \mathbb{Z}\}$ of $\mathbb{R}$ is dense in $\mathbb{R}$, unless $a$ or $b$ is zero or $a$ and $b$ are rational multiples of each other. This proves one direction.

Conversely, if $a = (k/l)\,b$ with integers $k, l$ such that $(k, l) = 1$, then $(a, 0, b) = (b/l)\,(k, 0, l)$, and we can rescale to obtain an equivalent group with parameters $k$ and $l$. This group is a subgroup of $[4, 3, 4]$, the symmetry group of the cubical tessellation with vertex-set $\mathbb{Z}^3$; hence it is discrete. Similarly, if $a$ or $b$ is zero, we can rescale to a group with parameters 0 and 1, so once again we have a subgroup of $[4, 3, 4]$. □

In the present context we are mainly interested in discrete groups and discrete polyhedra. However, the reader should be aware that there are interesting chiral polyhedra which are non-discrete. For example, if $a = 1$ and $b = \tau$, the golden ratio, then the polyhedron in Fig. 5.1 is a regular icosahedron and $\Lambda$ is the $\mathbb{Z}$-module spanned by its vertices. We remark that similar $\mathbb{Z}$-modules have occurred in the context of icosahedral quasicrystals (see [3]).

Thus, up to similarity, we can take $a$ and $b$ to be integers with $(a, b) = 1$. If nothing is said to the contrary, we explicitly allow $a = 0$ or $b = 0$; in this case, $b = \pm 1$ or $a = \pm 1$, respectively. Now $G$ is a subgroup of $[4, 3, 4]$ and each vertex is in $\mathbb{Z}^3$, the vertex-set of $\{4, 3, 4\}$. In fact, we have

$$2\mathbb{Z}^3 \subset \Lambda \subset \mathbb{Z}^3. \tag{5.12}$$

If $a = 0$ or $b = 0$, then $\Lambda = \mathbb{Z}^3$. To prove the first inclusion in general, choose integers $n$ and $m$ with $na + mb = 1$ and argue as in the proof of Lemma 5.2; this yields $(2, 0, 0) \in \Lambda$, and then by symmetry also $(0, 2, 0), (0, 0, 2) \in \Lambda$.

The further properties of the edge-module $\Lambda$ depend on the parity of $a$ and $b$. Let $s$ be a positive integer, let $k = 1, 2$ or 3, and let $\mathbf{s} := (s^k, 0^{3-k})$, the vector with $k$ components $s$ and $3 - k$ components 0. Following [18, p. 166], we write $\Lambda_{\mathbf{s}}$ for the sublattice of $\mathbb{Z}^3$ generated by $\mathbf{s}$ and its images under permutation and changes of sign of coordinates. Observe that

$$\Lambda_{\mathbf{s}} = s\Lambda_{(1^k, 0^{3-k})},$$

when $\mathbf{s} = (s^k, 0^{3-k})$. Of course, $\Lambda_{(1,0,0)}$ is just the *cubic lattice* $\mathbb{Z}^3$. The lattice $\Lambda_{(1,1,0)}$ is the *face-centered cubic lattice* (the root lattice $D_3$) and consists of all integral vectors whose coordinate sum is even (see [4]); a basis (with determinant 2) is given by $(1, 1, 0), (-1, 1, 0), (0, -1, 1)$. The lattice $\Lambda_{(1,1,1)}$ is the *body-centered cubic* lattice, with a basis (with determinant 4) given by $(2, 0, 0), (0, 2, 0), (1, 1, 1)$.

**Lemma 5.3.** *As before, let $\Lambda$ be the lattice spanned by $V$. Then*

(a) $\Lambda = \Lambda_{(1,0,0)} = \mathbb{Z}^3$ *if $a$ or $b$ is even*;
(b) $\Lambda = \Lambda_{(1,1,0)}$ *if $a$ and $b$ are odd.*

*Proof.* We know that $2\mathbb{Z}^3 \subset \Lambda$. Since $(a, b) = 1$, the parameters $a$ and $b$ cannot both be even. If $a$ is odd and $b$ is even (say), the generator $(a, 0, b)$ of $\Lambda$ is equivalent to $(1, 0, 0)$ modulo $2\mathbb{Z}^3$; hence $(1, 0, 0) \in \Lambda$, and then also $(0, 1, 0), (0, 0, 1) \in \Lambda$. This proves the first part. If both $a$ and $b$ are odd, we similarly obtain $(1, 0, 1) \in \Lambda$, and hence also $(1, 1, 0), (0, 1, 1) \in \Lambda$; but the coordinate sum of each generator, and thus of each element, of $\Lambda$ is even, so this also proves the second part. □

Next we determine the vertex-set $V(P)$ of the polyhedron $P$, that is, the orbit of $o$ under $G$. From (4.4) we know that

$$2\Lambda \subset V(P) \subset \Lambda. \tag{5.13}$$

More precisely, we have

**Lemma 5.4.** *The vertex-set $V(P)$ of the polyhedron $P$ is given by*

(a) $V(P) = \mathbb{Z}^3\ (= \Lambda)$ *if $a$ or $b$ is even*;
(b) $V(P) = \Lambda_{(1,1,0)}\ (= \Lambda)$ *if $a$ and $b$ are odd with $a + b \equiv 0 \pmod 4$*;
(c) $V(P) = \bigcup_{i=0}^{3} (x_i + 2\Lambda_{(1,1,0)})$, *with* $x_0 := (0, 0, 0)$, $x_1 := (0, 1, 1)$, $x_2 := (1, 0, 1)$ *and* $x_3 := (1, 1, 0)$, *if $a$ and $b$ are odd with $a + b \not\equiv 0 \pmod 4$.*

*Proof.* We generate new vertices as images of $o$ under $G$ and use the fact that $2\Lambda$ is both a subgroup of $G$ and a subset of $V(P)$.

If $a$ is odd and $b$ is even (say), the vertices $(a, 0, b)$, $(a+b, -a, b)$ and $(a+b, -a-b, -a+b)$ of $F_2$ are equivalent to $(1, 0, 0)$, $(1, 1, 0)$ or $(1, 1, 1)$ modulo $2\Lambda = 2\mathbb{Z}^3$, respectively, and hence the latter are vertices. From those we obtain $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$ and $(0, 1, 1)$ as vertices by applying $S_2$ and once again reducing modulo $2\Lambda$. It follows that each coset of $\Lambda$ modulo $2\Lambda$ is represented by a vertex, and hence $V(P) = \Lambda$.

If $a$ and $b$ are odd, then $4\mathbb{Z}^3 \subset 2\Lambda_{(1,1,0)} = 2\Lambda$. If $a + b \equiv 0 \pmod 4$, then $(a, 0, b)$ and $(a + b, -a, b)$ are equivalent to $(1, 0, -1)$ or $(0, 1, 1)$ modulo $2\Lambda$, respectively; applying $S_2$ to them now yields all points $(\pm1, \pm1, 0)$, $(\pm1, 0, \pm1)$ and $(0, \pm1, \pm1)$ as vertices. Moreover, we also obtain $(2, 0, 0)$ as vertex from $(a + b, -a - b, -a + b)$ in $F_2$ modulo $2\Lambda$. However, then each coset of $\Lambda$ modulo $2\Lambda$ is represented by a vertex, and hence $\Lambda$ itself is the vertex-set.

Finally, let $a$ and $b$ be odd with $a + b \not\equiv 0 \pmod 4$. Now $(a, 0, b)$ is equivalent to $x_2 = (1, 0, 1)$, and under $S_2$ we also obtain $x_1 = (0, 1, 1)$ and $x_3 = (1, 1, 0)$ as vertices. This proves that each coset $x_i + 2\Lambda$, with $i = 0, 1, 2, 3$ (and $x_0 = o$), is a subset of $V(P)$. However, these cosets are permuted by the generators $S_2$ and $T$, so $G$ must map a vertex contained in their union $U$ (say) to a vertex which is again contained in $U$. Since the base vertex $o$ is also in $U$, and $G$ is transitive on the vertices, this proves that $V(P) = U$. □

We now determine the full translation subgroup $T(G)$ of $G$. We already know that it must contain $2\Lambda$.

**Lemma 5.5.** *The subgroup $T(G)$ of all translations in G is given by*

(a) $\Lambda_{(1,1,1)}$ *if $a$ or $b$ is even*;

(b) $2\mathbb{Z}^3$ *if a and b are odd with* $a + b \equiv 0 \pmod 4$;
(c) $2\Lambda_{(1,1,0)} = 2\Lambda$ *if a and b are odd with* $a + b \not\equiv 0 \pmod 4$.

*In particular*, $T(G)$ *contains* $2\Lambda$ *as a subgroup of index* 2, 2 *or* 1, *respectively.*

*Proof.* In general we have additional translations arising from products like $T_2T_1T_3$, with $T_2 = T$ as in (5.1) and $T_1, T_3$ as in (5.6). In fact, $T_2T_1T_3$ is the translation by $(-a + b, a - b, a - b)$, since the sign changes in the three linear parts cancel out.

If $a$ is odd and $b$ is even (say), then this vector is equivalent to $(1, 1, 1)$ modulo $2\Lambda = 2\mathbb{Z}^3$, hence the translation by $(1, 1, 1)$ is also in $G$. However, the lattice $\Lambda_{(1,1,1)}$ is generated by $2\mathbb{Z}^3$ and $(1, 1, 1)$ and thus yields translations in $G$.

If $a$ and $b$ are odd with $a + b \equiv 0 \pmod 4$, the vector $(-a + b, a - b, a - b)$ is equivalent to $(2, 0, 0)$ modulo $2\Lambda = 2\Lambda_{(1,1,0)}$. However, $2\Lambda_{(1,1,0)}$ and $(2, 0, 0)$ generate $2\mathbb{Z}^3$, so now the latter yields translations in $G$.

If $a$ and $b$ are odd with $a + b \not\equiv 0 \pmod 4$, there are no translations in $G$ other than those in $2\Lambda = 2\Lambda_{(1,1,0)}$. The proof is rather tedious, so we only sketch it here. The arguments also extend to the previous two cases and will show that the full translation subgroup of $G$ cannot be larger than $\Lambda_{(1,1,1)}$ or $2\mathbb{Z}^3$, respectively.

Since $G = \langle T, S_2\rangle$, we have $G = N \cdot \langle S_2\rangle$ (as a product of subgroups) with $N := \langle S_2^{-j} T S_2^j \mid j = 0, \dots, 5\rangle$. Define $\widehat{T}_j := S_2^{-j} T S_2^j$; then $\widehat{T}_0 = T$, $\widehat{T}_2 = T_1$, $\widehat{T}_4 = T_3$, and $\widehat{T}_{j+3} = (-I)\widehat{T}_j(-I)$ for each $j$. The translation part of the generator $\widehat{T}_j$ of $N$ belongs to $V_0$, and the linear part of $\widehat{T}_j$ involves two sign changes but no permutation of coordinates. Then a translation in $G$ must necessarily belong to $N$; in fact, in the special group, the images of a translation in $G$ or an element of $N$ must involve an even number of sign changes but no permutation of coordinates, whereas an element in $\langle S_2\rangle$ either involves an odd number of sign changes or a permutation of coordinates.

Now let $R$ be any element of $N$, and let $R = \widehat{T}_{j_1} \cdots \widehat{T}_{j_n}$ for some $j_1, \dots, j_n$. If $j_2 \equiv j_1 + 3 \pmod 6$, then $\widehat{T}_{j_1}\widehat{T}_{j_2}$ is a translation by a vector in $2V_0$ and thus belongs to $2\Lambda$. If the first three generators $\widehat{T}_{j_1}, \widehat{T}_{j_2}, \widehat{T}_{j_3}$ involve the three sign changes $(-, -, +), (-, +, -), (+, -, -)$ (in some order), then their product is also a translation. If $j_2 \not\equiv j_1 + 3 \pmod 6$, then this certainly can be achieved by inserting a trivial subproduct $\widehat{T}_j\widehat{T}_j$ between the second and third term, and this would only increase $n$ by 2. Therefore, by applying a combination of these two operations, we can rewrite $R$ as a product involving only translations and at most two additional generators $\widehat{T}_j$. Clearly, no additional factor $\widehat{T}_j$ can occur if $R$ itself is a translation.

It remains to identify the translations obtained as products of three generators involving all three sign changes. In a product of this kind, if we replace one of the factors $\widehat{T}_j$ by $\widehat{T}_{j+3}$, then the new translation vector is equivalent to the old modulo $2\Lambda$. The same remains true if we cyclically permute the factors in the product (by conjugation). This, then, reduces the consideration to the two products $\widehat{T}_0\widehat{T}_1\widehat{T}_2$ and $\widehat{T}_2\widehat{T}_1\widehat{T}_0$, the latter being equivalent to the product $T_2T_1T_3$ modulo $2\Lambda$ discussed earlier, and the former being its inverse.

It follows that the structure of the full translation subgroup of $G$ is entirely determined by the translation vector $(-a+b, a-b, a-b)$ of $T_2T_1T_3$ modulo $2\Lambda$. In particular, this completes the proofs for the first and second parts. Finally, for the third part observe that $(-a + b, a - b, a - b)$ itself belongs to $2\Lambda$ if $a$ and $b$ are odd with $a + b \not\equiv 0 \pmod 4$, so there are no translations in addition to those of $2\Lambda$.

Note that the proof gives more. In fact, the translation subgroup of $G$ is a subgroup of $N$ of index 4, and $I$, $T_1$, $T_2$, $T_3$ is a system of coset representatives in $N$ (and the quotient group is $C_2 \times C_2$). Moreover, $G = N \cdot \langle S_2 \rangle$ is a semidirect product. □

Now that we know the vertex-set and the translation group for given parameters $a$ and $b$, we can give a direct description of the vertex-figure of the polyhedron $P$ at any given vertex $x$. This is done in Theorem 5.8. First consider the canonical mapping

$$\begin{array}{rccc} \pi: & V(P) & \to & V(P)/T(G), \\ & x & \mapsto & x + T(G). \end{array}$$

In the case when $V(P)$ is a lattice (that is, when $a$ or $b$ is even, or $a$ or $b$ is odd and $a+b \equiv 0 \pmod 4$), then $T(G)$ is a sublattice of $V(P)$ of index 4, $\pi$ is a homomorphism, and $V(P)/T(G)$ is isomorphic to $C_2 \times C_2$. However, when $V(P)$ itself is not a lattice (that is, when $a$ and $b$ are odd with $a + b \not\equiv 0 \pmod 4$), then $V(P)$ is the union of four cosets of $\Lambda_{(1,1,0)}$ modulo $T(G)$ (see Lemmas 5.4 and 5.5); these cosets are the elements of a set, again denoted by $V(P)/T(G)$.

We now *color* each vertex $x$ of $P$ with one of four elements of

$$C := \{0, 1, 2, 3\},$$

the *set of colors*; the color of $x$ will be the suffix of the vertex-star at $x$. We take the mapping

$$c'\colon\ V(P)/T(G) \to C$$

which associates with a coset $x + T(G)$ a label $i \in C$ as specified in Table 1 for its coset representative, and then consider the induced *coloring mapping*

$$c\colon\ V(P) \to C$$

defined by $(x)c := (x + T(G))c' = (x\pi)c'$. We call $(x)c$ the *color* of the vertex $x$. Thus the color of a vertex $x$ is obtained by reducing $x$ modulo $T(G)$, and then assigning to $x$ the color which is associated with its coset modulo $T(G)$ according to Table 1. Note that the vectors in the columns of Table 1 give a complete set of coset representatives for the cosets in $V(P)/T(G)$; they represent the vertices

$$o, \qquad oT_1 = (0, b, a), \qquad oT_2 = (a, 0, b), \qquad oT_3 = (b, a, 0) \tag{5.14}$$

of $P$, in this order (see (5.1) and (5.6)). If $a$ or $b$ is even, then, in effect, we are coloring the vertices of a cube with colors 0, 1, 2, 3, such that antipodal vertices receive the

**Table 1.** The colors $i$ assigned to the cosets in $V(P)/T(G)$.

| $i$ | $a$ odd, $b$ even | $a$ even, $b$ odd | $a, b$ odd, $a + b \equiv 0 \pmod 4$ | $a, b$ odd, $a + b \not\equiv 0 \pmod 4$ |
|---|---|---|---|---|
| 0 | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) |
| 1 | (0, 0, 1) | (0, 1, 0) | (0, 1, 1) | (0, 1, 1) |
| 2 | (1, 0, 0) | (0, 0, 1) | (1, 0, 1) | (1, 0, 1) |
| 3 | (0, 1, 0) | (1, 0, 0) | (1, 1, 0) | (1, 1, 0) |

same color, and are extending this coloring to a coloring of the vertices of the cubical tessellation, such that antipodal vertices of a cube always receive the same color.

When $V(P)$ is a lattice, it is convenient to make $C$ into a group such that $c'$ becomes an isomorphism and $c$ a homomorphism between groups. This is done by defining addition by

$$\begin{aligned} 0 \oplus i &:= i \qquad \text{if} \quad i = 0, 1, 2, 3, \\ i \oplus j &:= k \qquad \text{if} \quad \{i, j, k\} = \{1, 2, 3\}. \end{aligned}$$

Then $C$ is isomorphic to $C_2 \times C_2$.

Note that $G$ acts on the four cosets in $V(P)/T(G)$ as an alternating group $A_4$. In fact, since $S_2$ and $T'$ (the linear part of $T$) map $T(G)$ onto itself, the generators $S_2$ and $T$ of $G$ map cosets of $V(P)$ modulo $T(G)$ to cosets of $V(P)$ modulo $T(G)$. It follows that $G$ also acts as a permutation group on $C$. In particular, $S_2 = (1\,2\,3)$ and $T = (0\,2)(1\,3)$, as permutations on $C$.

For the proof of Theorem 5.8 we require the following two lemmas about the vectors in the intersections $V_i \cap V_j$ of two vertex-stars. The first deals with the case when $V(P)$ is a lattice, and the second with the case when $V(P)$ is not a lattice. The proof of one lemma does not extend to the other.

**Lemma 5.6.** *Let $a$ or $b$ be even and non-zero, or let $a$ and $b$ be odd with $a + b \equiv 0$ (mod 4).*

(a) *If $y \in V_i \cap V_j$ for some $i, j = 0, 1, 2, 3$ with $i \neq j$, then $(y)c = i \oplus j$.*
(b) *If $y \in V_i$ and $k := (y)c$ $(\neq 0)$, then $y \in V_j$ with $j = i \oplus k$.*

*Proof.* Every set $V_i \cap V_j$ consists of one pair of antipodal vectors for each $i \neq j$ (we have excluded the case that $a$ or $b$ is zero). Reducing a vector $y \in V_i \cap V_j$ modulo $T(G)$, and then applying $c'$, yields the color $(y)c$ of $y$. In the cases we are considering, $y$ is a vertex of $P$, so $(y)c$ is defined. A simple case-by-case inspection then shows that indeed $(y)c = i \oplus j$ in each case. This proves the first part.

None of the vectors in a vertex-star is equivalent to $o$ modulo $T(G)$. So in the second part we must have $k \neq 0$. However, $y \in V_j$ for some $j$, and $k = (y)c = i \oplus j$ by the first part. Hence $j = i \oplus k$, as required. □

The coset representatives $x_0 = (0, 0, 0)$, $x_1 = (0, 1, 1)$, $x_2 = (1, 0, 1)$ and $x_3 = (1, 1, 0)$ occurring in Lemma 5.4(c) allow a simple description of the vertex-stars $V_l$ modulo $2\Lambda_{(1,1,0)}$, for $l = 0, 1, 2, 3$. First note that the sums $x_i + x_j$ of two representatives will generally not represent elements in $V(P)$.

**Lemma 5.7.** *Let $a$ and $b$ be odd with $a + b \not\equiv 0$ (mod 4), and let $l = 0, 1, 2$ or 3.*

(a) *Then $\widehat{V}_l := \{x_l + x_i, x_l + x_j, x_l + x_k\}$, with $\{i, j, k, l\} = \{0, 1, 2, 3\}$, is a system of representatives for the vectors in $V_l$ modulo $2\Lambda_{(1,1,0)}$.*
(b) *If $y \in V_l$ is equivalent to $x_l + x_m$ modulo $2\Lambda_{(1,1,0)}$ for some $m \neq l$, then also $y \in V_m$.*

*Proof.* A simple computation shows that, modulo $2\Lambda_{(1,1,0)}$, the set $x_l + V_l$ is represented by $\{x_i, x_j, x_k\}$, with $i, j, k$ as above. Hence $V_l$ itself is represented by $\widehat{V_l}$. This proves the first part.

We know that $V_l \cap V_n$ consists of one pair of antipodal vectors for each $n \neq l$. If $y \in V_l \cap V_n$, then $y$ must be equivalent to $x_l + x_r$ and $x_n + x_s$ for some $r$ and $s$ with $r \neq l$ and $s \neq n$. An inspection of the possible sums now shows that indeed $\{l, r\} = \{n, s\}$, and thus $r = n$ and $s = l$. Therefore, if $y \in V_l$ and $y$ is equivalent to $x_l + x_m$ for some $m \neq l$, then necessarily $y \in V_m$ (and $y \notin V_p$ for $p \neq l, m$). □

We now have the following alternative description of the edge-graph of $P$.

**Theorem 5.8.** *Let $a$ and $b$ be integers with $(a, b) = 1$. Let $P$ be the polyhedron associated with $G$, and let $V(P)$ and $T(G)$ be as in Lemmas* 5.4 *and* 5.5, *respectively. If $x$ is a vertex of $P$, then $x + V_{(x)c}$ is the set of vertices of $P$ adjacent to $x$; that is, the vertex-star at $x$ is given by $V_{(x)c}$. Moreover, $P$ has no multiple vertices; that is, if $x \in V(P)$ and $y \in V_{(x)c}$, then $-y \in V_{(x+y)c}$.*

*Proof.* There are only four orbits of vertices of $P$ under $T(G)$, namely the four cosets in $V(P)/T(G)$, with coset representatives as in Table 1. Any two vertices of $P$ belonging to the same orbit have the same vertex-stars, since they are equivalent under a translation; in particular, a vertex $x$ of $P$ has the same vertex-star as its coset representative modulo $T(G)$. The coset representatives of Table 1 are representing the four vertices in (5.14) modulo $T(G)$, and hence their vertex-stars are $V_0$, $V_1$, $V_2$ and $V_3$, respectively.

It remains to show that the assignment of vertex-stars to vertices is indeed consistent and produces only simple vertices each of valency 6; that is, if $x$ is a vertex and $y \in V_{(x)c}$, then $x + y$ is again a vertex and $-y \in V_{(x+y)c}$ (so that we have $x = (x + y) + (-y) \in (x + y) + V_{(x+y)c}$). Now Lemmas 5.6 and 5.7 come in.

First, let $a$ or $b$ be even and non-zero, or let $a$ and $b$ be odd with $a+b \equiv 0 \pmod 4$. Let $x \in V(P)$ with $(x)c = i$, and let $y \in V_i$ with $(y)c = k$. Then $x + y$ is a vertex because $V(P) = \Lambda$, and $y \in V_j$ with $j = i \oplus k$ by Lemma 5.6(b). Now $c$ is a homomorphism between groups and thus $(x+y)c = (x)c \oplus (y)c = i \oplus k = j$. It follows that $y \in V_{(x+y)c}$, and hence also that $-y \in V_{(x+y)c}$.

If $a = 0$ or $b = 0$, then $V(P) = \mathbb{Z}^3 = \Lambda$ and $V_0 = V_1 = V_2 = V_3$ (as sets). Now the statement holds trivially because $V_{(x+y)c} = V_{(x)c}$.

Finally, let $a$ and $b$ be odd with $a + b \not\equiv 0 \pmod 4$. Let $x \in V(P)$ with $(x)c = i$. Then Lemma 5.7(a) shows that $x + y$ is equivalent to one vector in $x_i + \widehat{V_i} = \{x_j \mid j \neq i\}$ modulo $T(G)$, so must belong to $V(P)$. If $x + y$ is equivalent to $x_j$ (say) with $j \neq i$, then $(x + y)c = j$ and $y$ is equivalent to $x_i + x_j$. Now Lemma 5.7(b) applies and proves that $y \in V_j = V_{(x+y)c}$. Therefore, $-y \in V_{(x+y)c}$, as required. □

We can rephrase the theorem (and its proof) to obtain a new definition of the edge-graph of $P$ which is independent of $G$. Let $a$ and $b$ be integers with $(a, b) = 1$, and let $V_{a,b}$ and $T_{a,b}$ denote the vertex-set and the translation group associated with $a$ and $b$ (that is, the sets $V(P)$ and $T(G)$ of Lemmas 5.4 and 5.5), respectively. Take $V_{a,b}$ as the vertex-set of the polyhedron. Then color each vertex $x \in V_{a,b}$ by $(c)x$ as above, and assign to $x$ the set $V_{(x)c}$ as its vertex-star. This can be done in a consistent way. The resulting adjacency relationships between vertices now yield the edge-graph of the polyhedron.

Next we investigate the faces of the polyhedron. In particular, we are interested in cyclic sequences of six vectors in $V = V_0 \cup \cdots \cup V_3$. These sequences describe how we can move around the faces of $P$, going from one vertex to the next. We begin with a lemma about the vertex-stars.

**Lemma 5.9.** *Let $a, b \neq 0$, let $i = 0, 1, 2$ or $3$ and let $V_i = \{x, y, z, -x, -y, -z\}$ (say). Let $\widehat{z}$ be obtained from $z$ by changing one sign of a non-zero coordinate of $z$ (it does not matter which). If $u := \widehat{z}$, then*

(i) $\{x, y, u\} \not\subset V_l$ *for* $l = 0, 1, 2, 3$;
(ii) $x, y \in V_i$, $y, u \in V_j$ *and* $u, x \in V_k$, *for some* $j, k$ *with* $i, j, k$ *mutually distinct*;
(iii) $x \in V_i \cap V_k$, $y \in V_i \cap V_j$ *and* $u \in V_j \cap V_k$, *for some* $j, k$ *with* $i, j, k$ *mutually distinct*.

*Moreover, if a vector $u$ satisfies the first two properties for some $j$ and $k$, then necessarily $u = \widehat{z}$.*

*Proof.* This is easily verified by inspection. Note that any two vertex-stars intersect in a pair of vectors, and that each such pair determines the two vertex-stars which contain it. Clearly, either one of the second or third property of the lemma follows from the other. For the proof of the last statement observe that if the two pairs $\pm x$ and $\pm y$ of vectors in $V_i$ are also known to belong to $V_k$ or $V_j$, respectively, then the third pair in $V_i$ is obtained from the unique pair of vectors in $V_j \cap V_k$ by changing one sign of a non-zero component. □

Let $a, b \neq 0$. A sequence of vectors $\zeta := (z_1, \ldots, z_6)$ in $V$ is said to be *admissible* if the following properties hold, with indices considered modulo 6:

$$\begin{array}{ll} \text{(i)} & z_{i+3} = -z_i \text{ and } z_i \cdot z_{i+1} = ab, \text{ for } i = 1, \ldots 6 \text{ (here } \cdot \text{ denotes} \\ & \text{the scalar product);} \\ \text{(ii)} & z_1, z_2 \in V_i,\ z_2, z_3 \in V_j \text{ and } z_3, -z_1 \in V_k, \text{ for some } i, j, k, \\ & \text{mutually distinct.} \end{array} \tag{5.15}$$

Clearly, any cyclic permutation of the vectors in an admissible sequence gives again an admissible sequence. The same remains true if their order is reversed and each vector is replaced by its negative. Note that the second condition implies that the vectors $z_1, z_2, z_3$ in $\zeta$ cannot all be contained in one vertex-star.

With any face $F$ of $P$ we can associate a sequence $\zeta(F) = (z_1, \ldots, z_6)$ in $V$, defined up to a cyclic permutation and the reversal of order and signs. We say that $\zeta(F)$ is *associated* with $F$. More precisely, if $F = \{y_1, \ldots, y_6\}$ (say), with vertices $y_1, \ldots, y_6$ in cyclic order, we set $z_i := y_{i+1} - y_i$ for $i = 1, \ldots, 6$. Then we can recover $F$ from $\zeta(F)$ and $y := y_1$, that is,

$$F = \{y, y+z_1, y+z_1+z_2, \ldots, y+z_1+\cdots+z_5\} =: y+\{o, z_1, z_1+z_2, \ldots, z_1+\cdots+z_5\}.$$

Notice that $\{o, z_1, z_1 + z_2, \dots, z_1 + \cdots + z_5\}$ is the vertex-set of a (generally skew) regular hexagon. In particular, $z_{i+3} = -z_i$ for each $i$, and $\sum_{i=1}^{6} z_i = o$. The *center* $c(F)$ of $F$ is the centroid of the vertex-set of $F$ and is given by

$$c(F) := y + \tfrac{1}{2}(z_1 + z_2 + z_3).$$

If $F$ is the base face $F_2$ given by (5.8), then $\zeta(F)$ is easily seen to be admissible. In fact, we have

**Theorem 5.10.** *Let $a$ and $b$ be non-zero integers with $(a, b) = 1$. The admissible sequences in $V$ are precisely the sequences $\zeta(F)$ associated with the faces $F$ of the polyhedron $P$. In particular, if $\zeta = (z_1, \dots, z_6)$ is an admissible sequence with indices $i, j, k$ as in* (5.15), *and if $y$ is a vertex of $P$ colored $k$, then $F := y + \{o, z_1, z_1 + z_2, \dots, z_1 + \cdots + z_5\}$ is the vertex-set of a face of $P$, and $\zeta = \zeta(F)$.*

*Proof.* Each element of the special group of $G$ permutes the four vertex-stars $V_i$ as well as the four cosets (colors) of vertices of $P$, so takes admissible sequences to admissible sequences. We already know that $\zeta(F_2)$ is admissible; if the vertices of $F_2$ are taken in the order in which they occur in (5.8), then the corresponding indices are given by $(i, j, k) = (1, 2, 0)$. Now each face $F$ of $P$ is an image of $F_2$ under some element $R$ in $G$. Hence, if $R'$ denotes the image of $R$ in the special group, then $R'$ takes $\zeta(F_2)$ to $\zeta(F)$, and thus $\zeta(F)$ must also be admissible. This proves that the sequences associated with faces of $P$ are admissible.

Conversely, let $\zeta = (z_1, \dots, z_6)$ be an admissible sequence with indices $i, j, k$ as in (5.15). Then $z_6, z_1 \in V_k$, $z_1, z_2 \in V_i$ and $z_2, z_6 \in V_j$. From Lemma 5.9 applied to the vertex-star $V_k$ we obtain $V_k = \{z_6, z_1, \widehat{z_2}, -z_6, -z_1, -\widehat{z_2}\}$, where $\widehat{z_2}$ is defined as in Lemma 5.9. Hence $z_2$ is determined up to sign, and then in fact uniquely, since $z_i \cdot z_{i+1} = ab$ for each $i$. Thus $\zeta$ is uniquely determined by $z_1$ and $z_6$. Now, if $y$ is a vertex of $P$ colored $k$, then $V_k$ is the vertex-star at $y$, and $z_1$ and $z_6$ determine a face $F$ of $P$ which has $y$ as a vertex. Then $\zeta(F)$ is an admissible sequence by the first part of the proof, and $\zeta(F)$ contains $z_1$ and $z_6$. However, since there is only one admissible sequence which contains $z_1$ and $z_6$, we now have $\zeta = \zeta(F)$ and therefore also $F = y + \{o, z_1, z_1 + z_2, \dots, z_1 + \cdots + z_5\}$. □

We can say more about the faces of $P$. Each face $F$ which contains a given vertex $y$ of $P$ is of the form $F := y + \{o, z_1, z_1 + z_2, \dots, z_1 + \cdots + z_5\}$ for some admissible sequence $\zeta = (z_1, \dots, z_6)$. The same face $F$ also occurs at the vertex $y + z_1 + z_2 + z_3$ opposite to $y$ in $F$. However, $z_1 + z_2 + z_3$ yields a translation in $G$, so $F$ is a translate of the face opposite to $F$ in the vertex-figure at $y$. For the proof that $z_1 + z_2 + z_3$ indeed yields a translation we observe that the vertex opposite to $o$ in the base face $F_2$ is obtained from $o$ by a translation in $G$ (it has the same color), and that under $G$ this property must continue to hold for any pair of opposite vertices of a face. Thus, modulo $T(G)$, there are only three faces at a given vertex. This also remains true if $a = 0$ or $b = 0$, because once again opposite vertices in a face are related by a translation in $T(G)$.

Since the four classes of vertices modulo $T(G)$ are represented by the base vertex $o$ and the adjacent vertices $oT_2 = oT$, $oT_1$ and $oT_3$, every face of $P$ must be equivalent modulo $T(G)$ to a face which contains such a vertex, and every center of a face of $P$ must be equivalent to the center of such a face. The following lemma describes the full set of face centers that occur. The details of the proof are omitted.

**Lemma 5.11.** *Let $P$ be the polyhedron associated with $G$. Then the set of centers of faces of $P$ is given by*

(a) $\frac{1}{2}(1,1,1)+\mathbb{Z}^3$ *if $a$ or $b$ is even*;
(b) $(1,1,1)+\Lambda_{(1,1,0)}$ *if $a$ and $b$ are odd and $a+b\equiv 0 \pmod 4$*;
(c) $z+V(P)$ *if $a$ and $b$ are odd and $a+b\not\equiv 0 \pmod 4$, with $V(P)$ as in Lemma* 5.4(*c*), *where $z=o$ if $a\equiv b \pmod 8$, or $z=(2,0,0)$ if $a\not\equiv b \pmod 8$.*

Note that the set of face centers in Lemma 5.11(c) consists of four cosets of $\Lambda_{(1,1,0)}$ modulo $T(G)=2\Lambda_{(1,1,0)}$. If $a\equiv b \pmod 8$, then every vertex of $P$ is the center of a face of $P$, and vice versa; their cosets are represented by the vectors $x_0=(0,0,0)$, $x_1=(0,1,1)$, $x_2=(1,0,1)$ and $x_3=(1,1,0)$ as in Lemma 5.4(c). However, if $a\not\equiv b$ $(\mathrm{mod}\, 8)$, the four cosets modulo $T(G)$ are just those which do not yield vertices of $P$; now they are represented by $(2,0,0)$, $(0,1,-1)$, $(1,0,-1)$ and $(1,-1,0)$.

We can also decide when the polyhedron has planar faces or vertex-figures.

**Lemma 5.12.** *Let $a$ and $b$ be integers with $(a,b)=1$. Let $P$ be the polyhedron associated with $G$. Then*

(a) *$P$ has planar vertex-figures if and only if $a=-b=\pm 1$*;
(b) *$P$ has planar faces if and only if $a=b=\pm 1$.*

*Proof.* The points in $V_0$ are the vertices of $P$ adjacent to $o$, so $P$ has planar vertex-figures if and only if these points lie in a plane. The latter holds if and only if the determinant of any three mutually non-collinear vectors in $V_0$ is 0. We have

$$\begin{vmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{vmatrix} = a^3+b^3\,;$$

hence the points lie in a plane if and only $a=-b$. Since $a$ and $b$ must be coprime, this settles the first part. Note that $P(-1,1)=P(1,-1)$ (see (5.3)).

For the second part consider an admissible sequence $(z_1,\ldots,z_6)$ for the base face $F_2$. Then $P$ has planar faces if and only if $\det(z_1,z_2,z_3)=0$. However, $\det(z_1,z_2,z_3)=\pm(a^3-b^3)$, so $P$ has planar faces if and only if $a=b$. Once again, since $(a,b)=1$, the latter means that $a=b=\pm 1$. Note that $P(-1,-1)=P(1,1)$. $\square$

Next we discuss duality. In the present context the generators $S_1$ and $S_2$ of $G$ are rotatory reflections and thus have a unique fixed point. When taken as initial vertices for Wythoff's construction, these fixed points yield a pair of "geometrically" dual polyhedra. The original polyhedron $P$ is obtained from the point fixed by $S_2$, namely $o$. Its dual $P^*$ is derived from

$$w:=\tfrac{1}{2}(a+b,-a-b,-a+b),$$

the fixed point of $S_1$. Actually, as we remarked near the end of Section 2, it is more natural to take $S_2^{-1}$, $S_1^{-1}$ as distinguished generators when $G$ is considered as the symmetry group of $P^*$. Note that $S_2^{-1}S_1^{-1} = (S_1S_2)^{-1} = T$, so the distinguished element $T$ will remain the same.

Since the base vertex $w$ of $P^*$ is the center of the base face $F_2$ of $P$, the vertex-set $V(P^*)$ of $P^*$ is just the set of face centers of $P$ and thus is given by Lemma 5.11. Moreover, $wT$ is the vertex in the base edge of $P^*$ distinct from $w$, and the orbit of $wT$ under $\langle S_1\rangle$ (or, rather, $\langle S_1^{-1}\rangle$) consists of the vertices in the vertex-figure at $w$. In particular, we obtain the vertex-star at $w$ from $V_3$ by interchanging $a$ and $b$ in each vector. The action of the special group $G_0$ of $G$ on the collection of vertex-stars $V_i = V_i(a, b)$ of $P$ (see (5.10)) induces a corresponding action on the collection of sets $V_i(b, a)$ obtained by interchanging $a$ and $b$ in each vector. The vertex-star at a general vertex $wR$ of $P^*$ is given by $V_3(b, a)R'$, where $R'$ is the image of the element $R$ of $G$ in $G_0$. It follows that $P^*$ is the polyhedron associated with the vertex-stars $V_i(b, a)$. Hence we have established the following:

**Theorem 5.13.** *The dual $P(a, b)^*$ of $P(a, b)$ is congruent to $P(b, a)$, or, less formally, $P(a, b)^* \cong P(b, a)$.*

By Lemma 5.12 it is trivially true that the vertex-figures of $P$ are planar if and only if those of $P^*$ are planar. However, it is not true that $P$ has planar vertex-figures if and only if $P^*$ has planar faces.

Next we shall prove that the two polyhedra $P(a, b)$ and $P(b, a)$ are congruent for any $a$ and $b$. Indeed, if $R$ denotes the reflection in the plane $\xi_1 - \xi_3 = 0$, so that

$$R\colon (\xi_1, \xi_2, \xi_3) \mapsto (\xi_3, \xi_2, \xi_1), \tag{5.16}$$

then

$$P(a, b)R = P(b, a). \tag{5.17}$$

For the proof, let $S_2$ and $T = T(a, b)$ be the generators of $G = G(a, b)$ as in (5.1). Then $R^{-1}S_2R = S_2^{-1}$ and $R^{-1}T(a, b)R = T(b, a)$, and thus $R^{-1}G(a, b)R = G(b, a)$. However, $oR = o$, so $R$ maps $V(P(a, b)) = oG(a, b)$ to $V(P(b, a)) = oG(b, a)$. In fact, $R$ also maps $T(G(a, b))$ to $T(G(b, a))$ (see Lemma 5.5) and takes each vertex of $P(a, b)$ colored 0, 1, 2 or 3 to a vertex of $P(b, a)$ colored 0, 3, 2 or 1, respectively (see Table 1 and note that the first two columns are interchanged under $R$ if $a$ or $b$ is even). Finally, since also $V_1(a, b)R = V_3(b, a)$, $V_3(a, b)R = V_1(b, a)$ and $V_i(a, b)R = V_i(b, a)$ for $i = 0$ or 2, we must in fact have $P(a, b)R = P(b, a)$.

Now the following corollary is immediate.

**Corollary 5.14.** *The polyhedron $P(a, b)$ is geometrically self-dual, that is, $P(a, b)$ is congruent to its dual $P(a, b)^*$.*

Before we move on, let us further comment on the case $b = \pm a$. If $b = \pm a$, then the above mapping $R$ takes $P(a, b)$ to itself and induces an automorphism (see (5.3)); in fact, since $oR = o$ and $V_0R = V_0$, it induces an automorphism of the vertex-figure at $o$.

On the other hand, $R$ acts like a transposition on the colors of the vertices, so it certainly does not belong to $G_0$, which acts on the colors like a group $A_4$ (see (5.11)). However, then $R$ also cannot belong to $G$ (it already fixes $o$), so even combinatorially it must act on the vertex-figure at $o$ like a reflection. Hence $P(a, b)$ is geometrically regular if $b = \pm a$, and its full symmetry group is generated by $G$ and $R$. We shall see later that $R$ maps the base flag of $P(a, b)$ to either an adjacent flag if $b = a$, or the image of an adjacent flag under the central symmetry $-I$ $(= S_2^3)$ if $b = -a$.

Next we briefly discuss the general question when two polyhedra $P(a, b)$ and $P(c, d)$ are affine images of each other. Here we do not need to assume that the parameters are integers.

**Lemma 5.15.** *Let $a, b, c$ and $d$ be real numbers, and let $(a, b) \neq (0, 0) \neq (c, d)$. Let $P(a, b)$ and $P(c, d)$ be the polyhedra associated with $a, b$ and $c, d$, respectively. Then $P(a, b)$ and $P(c, d)$ are affinely equivalent if and only if $(c, d) = s(a, b)$ or $t(b, a)$ for some real numbers $s$ or $t$. Moreover, $P(a, b)$ and $P(c, d)$ are congruent if and only if this holds with $s, t = \pm 1$.*

*Proof.* We only sketch the proof. Suppose that we have $P(a, b)R = P(c, d)$ for some affine mapping $R$. Since the group $G(c, d)$ acts transitively on the vertices, and the stabilizer of the vertex $o$ in $G(c, d)$ acts transitively on the vertices adjacent to $o$, we can further assume that $oR = o$ and that $(a, 0, b)R = (c, 0, d)$. The affine transformation $R^{-1}T(a, b)R$ acts on the polyhedron $P(c, d)$ locally in exactly the same way as the symmetry $T(c, d)$ of $P(c, d)$, so it must indeed be the same transformation (because an affine transformation of $\mathbb{E}^3$ is uniquely determined by its effect on four independent points). Moreover, the cyclic order of the vertices in the vertex-figure at $o$ is either preserved or reversed by $R$ (see (5.4)); that is, $R^{-1}S_2R = S_2$ or $S_2^{-1}$ (and hence $R^{-1}G(a, b)R = G(c, d)$). These two properties translate directly into conditions for the matrix entries of $R$ and prove that there are essentially only two possibilities. If the order of the vertices in the vertex-figure is preserved, then $R$ is the identity mapping, up to scaling by $s := c/a$ (or $d/b$ if $a = 0$). If the order is reversed, then $R$ is the mapping defined in (5.16), up to scaling by $t := c/b$ (or $d/a$ if $b = 0$). In particular, $(c, d) = s(a, b)$ or $t(b, a)$. Furthermore, $R$ can only be an isometry if $s = \pm 1$ or $t = \pm 1$, respectively. The other direction is obvious. □

The same proof actually establishes that the polyhedra are generally chiral. In fact, we have

**Theorem 5.16.** *The polyhedron $P(a, b)$ is geometrically chiral if $b \neq \pm a$, or geometrically regular if $b = \pm a$.*

*Proof.* It is clear that every polyhedron $P(a, b)$ is either geometrically chiral or geometrically regular; in fact, $G$ acts transitively on the vertices, and $S_2$ permutes cyclically the vertices adjacent to $o$. Now suppose that $P(a, b)$ is geometrically regular. Then its symmetry group must contain an element $R$ which maps the base flag of $P(a, b)$ to the adjacent flag differing in the 2-face. In particular, $R$ must fix the two vertices $o$ and

$(a, 0, b)$ of the base edge. Now apply the arguments of the proof of Lemma 5.15 with $c = a$ and $d = b$. Since $R^{-1}S_2R = S_2^{-1}$, the order of the vertices in the vertex-figure is reversed, so $R$ must be the mapping as defined in (5.16), or its negative. In particular, we must have $(a, b) = t(b, a)$ with $t = \pm 1$. Hence we must have $b = \pm a$. On the other hand, we already proved above that $P(a, b)$ is geometrically regular if $b = \pm a$. □

We now address enantiomorphism. As explained in Section 2, the two enantiomorphic forms of a chiral polyhedron are represented by different pairs of generators of its group. If $S_1, S_2$ is the pair associated with the base flag $\{F_0, F_1, F_2\}$ (say) of $P(a, b)$, then $S_1S_2^2, S_2^{-1}$ is the pair associated with the adjacent flag $\{F_0, F_1, F_2'\}$ of $P(a, b)$. For each pair, the product of the generators is $T$. If Wythoff's construction is applied to $G$ with the new generators $S_1S_2^2, S_2^{-1}$ and with the same initial vertex $o$, which again is fixed by $S_2^{-1}$, then we indeed obtain the same underlying polyhedron, namely $P(a, b)$, but now with a new base flag adjacent to the original base flag. To see this, we once again employ the mapping $R$ defined in (5.16).

In fact, we know that $R^{-1}G(a, b)R = G(b, a)$, and that

$$\begin{aligned} R^{-1}S_2R &= S_2^{-1}, \qquad R^{-1}T(a, b)R = T(b, a), \\ R^{-1}S_1(a, b)R &= R^{-1}T(a, b)S_2^{-1}R = T(b, a)S_2 = S_1(b, a)S_2^2. \end{aligned} \tag{5.18}$$

In other words, conjugation by $R$ transforms the pair of generators $S_1(a, b), S_2$ of $G(a, b)$ into the pair $S_1(b, a)S_2^2, S_2^{-1}$ of $G(b, a)$, and vice versa; the same also remains true with $a$ and $b$ interchanged. It follows that the polyhedron $P$ (say), obtained by Wythoff's construction from $G(a, b)$ and its generators $S_1(a, b)S_2^2, S_2^{-1}$, is mapped by $R$ to the polyhedron obtained from $G(b, a)$ and its generators $S_1(b, a), S_2$. However, the latter is just $P(b, a)$, so its preimage under $R$ is $P(a, b)$ itself. Hence, $P = P(a, b)$. Note that $R$ takes the base flag $\{F_0(a, b), F_1(a, b), F_2(a, b)\}$ of $P(a, b)$ to the flag $\{F_0(b, a), F_1(b, a), F_2'(b, a)\}$ adjacent to the base flag $\{F_0(b, a), F_1(b, a), F_2(b, a)\}$ of $P(b, a)$ (and differing from it in the 2-face). If $b = \pm a$, this once again proves that $P(a, b)$ is geometrically regular. In fact, then $R$ maps $P(a, b)$ to itself, and takes the base flag to either an adjacent flag if $b = a$, or the image of an adjacent flag under the central symmetry $-I$ $(= S_2^3)$ if $b = -a$.

These considerations also justify our initial hypothesis, pointed out at the beginning of this section, that it is enough to concentrate on only one orientation for the generator $S_2$. Indeed, the opposite orientation (given by $S_2^{-1}$) is implied by enantiomorphism.

We now address the question of abstract isomorphism between the polyhedra constructed in this section. The two cases $a = b$ and $a = -b$ are special because the corresponding polyhedra $P(1, 1)$ and $P(1, -1)$ are regular. Initially we might conjecture that each polyhedron $P(a, b)$ is combinatorially isomorphic to $P(1, 1)$ or $P(1, -1)$, but this turns out to be false.

First we identify the two regular polyhedra. We know from Theorem 7E15 of [18] that there are only two pure regular polyhedra of type $\{6, 6\}$ in $\mathbb{E}^3$, namely the Petrie–Coxeter polyhedron $\{6, 6\,|\,3\}$ (with planar faces and skew vertex-figures), and $\{6, 6\}_4$ (with skew faces and planar vertex-figures). The polyhedron $\{6, 6\}_4$ has its vertices at alternating vertices of the Petrie–Coxeter polyhedron $\{4, 6\,|\,4\}$; its faces are the vertex-figures at the remaining vertices of $\{4, 6\,|\,4\}$. We know from Lemma 5.12 that $P(1, 1)$ has planar faces and that $P(1, -1)$ has planar vertex-figures. So we have

**Theorem 5.17.** *$P(1,1) = \{6,6\,|\,3\}$ and $P(1,-1) = \{6,6\}_4$ are the only polyhedra $P(a,b)$ which are regular.*

It is instructive to see how the general polyhedron $P(a,b)$ is different from the two special polyhedra $P(1,1)$ and $P(1,-1)$. We know that $P(1,1) = \{6,6\,|\,3\}$ has 2-holes of length 3 (see also [5]), and that $P(1,-1) = \{6,6\}_4$ has Petrie polygons of length 4 (see pp. 193 and 196 of [18]); in other words, if their full symmetry group is $\langle R_0, R_1, R_2\rangle$ (say), then their elements

$$S_1 S_2^{-1} = R_0 R_1 R_2 R_1, \qquad S_1^2 S_2^2 = (R_0 R_1 R_2)^2$$

have periods 3 and 2, respectively. Now consider the two elements $S_1 S_2^{-1}$ and $S_1^2 S_2^2$ in $G$ for general $a$ and $b$. Once again they are associated with a right-handed 2-hole or Petrie polygon of the polyhedron $P(a,b)$. Now we have

$$\begin{array}{rccl} (S_1 S_2^{-1})^3: & x & \mapsto & x + s(-1,1,1), \\ (S_1^2 S_2^2)^2: & x & \mapsto & x + t(1,0,0), \end{array} \tag{5.19}$$

with $s = a - b$ and $t = -2a - 2b$. Hence $(S_1 S_2^{-1})^3$ and $(S_1^2 S_2^2)^2$ are genuine translations unless $a = b$ or $a = -b$, respectively. The two cases $a = b$ and $a = -b$ yield $P(1,1)$ and $P(1,-1)$ as $a$ and $b$ are relatively prime, and for them we already know that the periods of $S_1 S_2^{-1}$ and $S_1^2 S_2^2$ are 3 and 2, respectively. Note that the elements $(S_1 S_2^2)(S_2^{-1})^{-1} = S_1(-I)$ and $(S_1 S_2^2)^2 (S_2^{-1})^2 = S_1 S_2^2 S_1$ of $G$ are associated with a left-handed 2-hole or Petrie polygon of $P(a,b)$, respectively; their cube or square is again a translation unless $a = b = \pm 1$ or $a = -b = \pm 1$.

In particular, these considerations imply that none of the polyhedra $P(a,b)$ with $a \neq \pm b$ is combinatorially isomorphic to $P(1,1)$ or $P(1,-1)$. More generally, the following theorem holds.

**Theorem 5.18.** *Let $a, b$ and $c, d$ be pairs of relatively prime integers. Then the polyhedra $P(a,b)$ and $P(c,d)$ are combinatorially isomorphic if and only if $(c,d) = \pm(a,b)$ or $(c,d) = \pm(b,a)$.*

*Proof.* Suppose that $\kappa$ is a combinatorial isomorphism between (the face lattices of) $P(a,b)$ and $P(c,d)$. Consider the image of the base flag $\{F_0(a,b), F_1(a,b), F_2(a,b)\}$ of $P(a,b)$ under $\kappa$. Since $G(c,d)$ has at most two orbits on the flags of $P(c,d)$, we can compose $\kappa$ with an element of $G(c,d)$ (if need be) and obtain an isomorphism $\kappa'$ between $P(a,b)$ and $P(c,d)$ which maps the base flag of $P(a,b)$ to either the base flag $\{F_0(c,d), F_1(c,d), F_2(c,d)\}$ or the adjacent flag $\{F_0(c,d), F_1(c,d), F_2'(c,d)\}$ of $P(c,d)$. Moreover, in the latter case we can further compose $\kappa'$ with the isomorphism between $P(c,d)$ and $P(d,c)$ determined by the reflection $R$ of (5.16) (see also (5.18)); the resulting isomorphism $\kappa''$ between $P(a,b)$ and $P(d,c)$ then maps the base flag of $P(a,b)$ to the base flag of $P(d,c)$. It follows that, up to interchanging the parameters $c$ and $d$, we may assume that we have an isomorphism between $P(a,b)$ and $P(c,d)$ which maps the base flag of $P(a,b)$ to the base flag of $P(c,d)$.

Then this isomorphism of polyhedra induces an isomorphism of groups

$$\mu\colon\ G(a,b) \to G(c,d)$$

which maps the generators $S_1(a,b), S_2$ of $G(a,b)$ to the generators $S_1(c,d), S_2$ of $G(c,d)$. (By composing with $R$ as above, we have already accounted for the possibility that $\mu$ would take the pair of generators $S_1(a,b), S_2$ into the other distinguished pair of generators $S_1(c,d)S_2^2, S_2^{-1}$ of $G(c,d)$). Then $\mu$ also maps $T(a,b)$ to $T(c,d)$.

First we show that $\mu$ maps translations in $G(a,b)$ to translations in $G(c,d)$. Let $T(z)$ denote the translation of $\mathbb{E}^3$ by the vector $z \in \mathbb{E}^3$. Now, if $x$ is a vertex of $P(a,b)$ adjacent to $o$, then $x = oL$ with $L := T(a,b)S_2^j$ for some $j$, and

$$T(2x) = (-I)L^{-1}(-I)L = S_2^3 L^{-1} S_2^3 L$$

(see the proof of (4.5)). The conjugates of such translations by elements in $G(a,b)$ then yield all the generating translations of the subgroup $2\Lambda = 2\Lambda(a,b)$ of $T(G(a,b))$. Then under $\mu$ we obtain

$$T(2x)\mu = (S_2^3 L^{-1} S_2^3 L)\mu = S_2^3 (L\mu)^{-1} S_2^3 (L\mu) = (-I)(L\mu)^{-1}(-I)(L\mu) = T(2y),$$

with $y := o(L\mu)$ a vertex in $P(c,d)$; but $y$ is adjacent to $o$ because $L\mu = (T(a,b)S_2^j)\mu = T(c,d)S_2^j$. However, $\mu$ takes conjugates to conjugates, so $\mu$ must map $2\Lambda(a,b)$ to $2\Lambda(c,d)$. Finally, to account for all translations in $G(a,b)$ we must also consider the translation

$$T_2(a,b)T_1(a,b)T_3(a,b)$$

in $G(a,b)$ (see the proof of Lemma 5.5). This is mapped by $\mu$ onto the corresponding product for the parameters $c, d$, which again is a translation. Hence $\mu$ takes translations to translations.

We now employ the translations in (5.19). Let $e_1, e_2, e_3$ denote the canonical basis of $\mathbb{E}^3$. Then the two translations are given by

$$(S_1^2(a,b)S_2^2)^2 = T(-2(a+b)e_1), \qquad (S_1(a,b)S_2^{-1})^3 = T((a-b)(-e_1+e_2+e_3)), \tag{5.20}$$

and similarly for the parameters $c, d$. However, $\mu$ takes $S_1(a,b), S_2$ to $S_1(c,d), S_2$, so $\mu$ must also take the two translations for $a, b$ to those for $c, d$. By Lemmas 5.3 and 5.5 we have $2e_1 \in \Lambda(a,b)$ and $T(4e_1) \in G(a,b)$ for every choice of parameters $a, b$. We also know that $T(4e_1)\mu$ is a translation. Clearly, $(T(z)\mu)^k = T(kz)\mu$ for any integer $k$ and any translation $T(z)$ in $G(a,b)$. In particular, setting $k := a+b$ we obtain

$$\begin{array}{rcl}(T(4e_1)\mu)^{a+b} & = & T(4(a+b)e_1)\mu \\ & = & (T(-2(a+b)e_1))^{-2}\mu \\ & = & (T(-2(c+d)e_1))^{-2} \\ & = & T(4(c+d)e_1).\end{array}$$

Therefore, if $a+b \neq 0$, then $T(4e_1)\mu$ itself must be translation in the direction of $e_1$. Moreover, if $a = -b = \pm 1$, then also $c = -d = \pm 1$, as required.

Now let $a+b \neq 0$ (and hence $c+d \neq 0$), and let $T(4e_1)\mu = T(re_1)$ for some integer $r$. Then

$$T(r(a+b)e_1) = (T(re_1))^{a+b} = T(4(c+d)e_1),$$

and thus we have the linear equation $r(a+b) = 4(c+d)$ for $a, b, c, d$. To derive a second linear equation we first observe that $T(4e_i)\mu = T(re_i)$ for each $i = 1, 2, 3$. This is proved by conjugation with $S_2^2$ or $S_2^4$; for example, we have

$$T(4e_3)\mu = (S_2^{-2}T(4e_1)S_2^2)\mu = S_2^{-2}T(re_1)S_2^2 = T(re_3).$$

We now use the fourth power of the second translation in (5.20). In fact, we have

$$\begin{aligned} T(4(c-d)(-e_1+e_2+e_3)) &= T(4(a-b)(-e_1+e_2+e_3))\mu \\ &= (T(4e_1)^{-1}T(4e_2)T(4e_3))^{a-b}\mu \\ &= (T(re_1)^{-1}T(re_2)T(re_3))^{a-b} \\ &= T(r(a-b)(-e_1+e_2+e_3)). \end{aligned}$$

This leads to the second linear equation, $r(a-b) = 4(c-d)$. Together with the first equation this implies $ra = 4c$ and $rb = 4d$. Hence, bearing in mind that $a, b$ and $c, d$ are relatively prime, we arrive at $r = \pm 4$; that is, $(c, d) = \pm(a, b)$, as required. This concludes the proof. □

In general we do not know if any polyhedron $P(a, b)$ other than $P(1, 1)$ and $P(1, -1)$ is combinatorially regular, but we do know that none is geometrically regular. If indeed any such $P(a, b)$ is combinatorially regular, then $P(a, b)$ would be a chiral realization of itself.

## 6. Types {4, 6} and {6, 4}

In this section we describe the geometrically chiral polyhedra of type {4, 6}. Their duals are the chiral polyhedra of type {6, 4}.

Once again we consider only polyhedra whose symmetry group is generated by rotatory reflections of finite periods. We now prefer to denote such a polyhedron by $Q$. We know from Lemma 4.3 that there are only two groups, namely $[3, 3]^*$ and $[3, 4]$, which can occur as the special group for a geometrically chiral polyhedron $Q$. In this section we discuss the polyhedra associated with $[3, 4]$. They must necessarily be of type {4, 6} or {6, 4}. We begin with the polyhedra of type {4, 6} and obtain those of type {6, 4} by duality.

As in the previous section we begin by realizing $[3, 4]$ as the special group of a suitable group $H$ (we now denote the group by $H$), and then obtain the polyhedron $Q$ by Wythoff's construction. In particular, $o$ is the initial vertex of $Q$, and the orbit of $o$ under $H$ is the vertex-set $V(Q)$ of $Q$. Geometrically we take $[3, 4]$ in the form $[4, 3]$, that is, as the symmetry group of the cube. The following lemma shows that there is essentially only one way in which $H$ and its generators may be taken.

**Lemma 6.1.** *Let $S_2$ $(= S_2')$ be a rotatory reflection of period* 6 *in* $[3, 4]$. *Then there are precisely three rotatory reflections $S_1'$ of period* 4 *in* $[3, 4]$ *such that the product $S_1'S_2$ is of period* 2. *If $S_1'$ is one of them, then the other two are $S_2^{-1}S_1'S_2$ and $S_2^{-2}S_1'S_2^2$, and their products with $S_2$ are $S_2^{-1}(S_1'S_2)S_2$ and $S_2^{-2}(S_1'S_2)S_2^2$, respectively.*

*Proof.* Let $C$ be a cube centered at $o$ with edges parallel to the coordinate axes. Each rotatory reflection of period 4 in its symmetry group $[3,4]$ is given by a rotation by $\pm\pi/2$ about a coordinate axis, followed by a reflection in the plane through $o$ perpendicular to the axis. Each rotatory reflection of period 6 in $[3,4]$ is given by a rotation about a vertex of $C$, followed by a reflection in the plane through $o$ perpendicular to the rotation axis. Any two rotatory reflections of period 4 or 6, respectively, are conjugate in $[3,4]$. (Note that the analogous statement was not true for $[3,3]^*$.) Once $S_2$ is chosen, then exactly three rotatory reflections of period 4 will yield a product with $S_2$ of period 2, one for each coordinate axis; in particular, any two of them are conjugate by an element of $\langle S_2^2\rangle$. □

It follows from Lemma 6.1 that any two pairs of admissible generators $S_1', S_2$ of $[3,4]$ are conjugate in $[3,4]$. In each case, $S_1'S_2$ is necessarily the half-turn about the midpoint of an edge of $C$ which contains a vertex of $C$ invariant under $S_2$. (The half-turns about face centers of $C$ yield the group $[3,3]^*$, and the half-turns about the midpoints of those edges which do not contain a vertex invariant under $S_2$ correspond to elements $S_1'$ of period 2.)

Thus, as in the previous section, we may confine ourselves to some very specific choices for $S_2$ and $T := S_1S_2$. We take the group $H = H(c,d)$ generated by

$$\begin{array}{rccl} S_2\colon & x & \mapsto & -(\xi_3,\xi_1,\xi_2), \\ T\colon & x & \mapsto & (\xi_2,\xi_1,-\xi_3)+(c,-c,d), \end{array} \tag{6.1}$$

with real parameters $c$ and $d$, not both zero. Then $S_1 := TS_2^{-1}$ is given by

$$S_1\colon \quad x \quad \mapsto \quad (-\xi_1,\xi_3,-\xi_2)+(c,-d,-c). \tag{6.2}$$

The base vertex of the corresponding polyhedron $Q = Q(c,d)$ is $w := oT = (c,-c,d)$, and the base edge $F_1$ with vertices $o$ and $w$ lies in the plane $\xi_1+\xi_2 = 0$. In particular, $F_1$ is perpendicular to the rotation axes of $T$, which is the line through $\frac{1}{2}w$ with direction vector $(1,1,0)$. Observe that the same argument as in the previous section shows that

$$Q(-c,-d) = Q(c,d). \tag{6.3}$$

Now the orbit $W_0$ of $w$ under $\langle S_2\rangle$ is given by

$$W_0 := \{(c,-c,d),(c,-d,-c),(d,c,-c),(-c,c,-d),(-c,d,c),(-d,-c,c)\}, \tag{6.4}$$

where again the points are listed in cyclic order; we simply write

$$W_0 = \{\pm(c,-c,d),\pm(c,-d,-c),\pm(d,c,-c)\}, \tag{6.5}$$

with the understanding that, up to cyclic permutation, plussigns precede minussigns. Then $W_0$ is the vertex-star of $Q$ at its vertex $o$, and the vertex-stars at the vertices adjacent to $o$ are the images of $W_0$ under the conjugates of $T$ by elements of $\langle S_2\rangle$ (more exactly, under their images in the special group $H_0$ of $H$). In particular, the vertex-star $W_2 := W_0T' = W_0T - w$ at $(c,-c,d) = w$ is given by

$$W_2 = \{\pm(c,-c,d),\pm(c,d,c),\pm(-d,c,c)\}. \tag{6.6}$$

As before, define

$$T_1 := S_2^{-2} T S_2^2, \qquad T_2 := T, \qquad T_3 := S_2^{-4} T S_2^4.$$

Then the vertex-stars $W_1 := W_0 T_1 - (-c, d, c)$ at $(-c, d, c)$ and $W_3 := W_0 T_3 - (d, c, -c)$ at $(d, c, -c)$ are given by

$$\begin{aligned} W_1 &= \{\pm(c, c, -d), \pm(c, -d, -c), \pm(-d, -c, -c)\}, \\ W_3 &= \{\pm(c, c, d), \pm(c, -d, c), \pm(-d, -c, c)\}, \end{aligned} \tag{6.7}$$

respectively. Moreover, since $-I = S_2^3$ belongs to $H$, the vertex-stars at pairs of opposite vertices $\pm z$ in the vertex-figure at $o$ are necessarily the same. This also follows from the observation that the translation by $2z$, which maps $-z$ to $z$, is an element of $H$, namely $S_2^{-j}(S_2^3 T)^2 S_2^j$, with $j$ such that $z = w S_2^j$.

So far we have identified four vertex-stars for $Q$. Their vectors comprise certain triples of the four vertices on each square face of the "truncated octahedron" shown in Fig. 6.3 for the case $0 < |c| < |d|$. We shall see that there are four more vertex-stars which occur in $Q$; together with the first set of four, these yield all the vertices of the truncated octahedron. The missing vertex in each square face is the image of the opposite vertex in the square face under the reflection in the mirror half-way between the two vertices. This reflection is indeed contained in [3, 4], but it remains to prove that it is also the image of an element of $H$ in $H_0$. The latter can be accomplished as follows.

The image $T_1' S_2^3$ of $T_1 S_2^3$ in $H_0$ is the reflection in the plane perpendicular to the rotation axis of $T_1$. We now conjugate to obtain the desired reflection. In fact, the image of

$$R := T_3 T_1 S_2^3 T_3\colon \quad x \quad \mapsto \quad (\xi_2, \xi_1, \xi_3) + (-2c + d, -d, -d) \tag{6.8}$$

in $H_0$ is the reflection $R' = T_3' T_1' S_2^3 T_3'$ in the plane $\xi_1 = \xi_2$, and the vertex-star $W_4 := W_0 R'$ at the vertex $oR = (-2c + d, -d, -d)$ of $Q$ is given by

$$W_4 = \{\pm(-c, c, d), \pm(-c, -d, c), \pm(d, -c, c)\}. \tag{6.9}$$

More generally, we obtain the vertex-star $W_i := W_{i-4} R'$ for $i = 4, 5, 6, 7$ at a vertex which is the image of a vertex with vertex-star $W_i$ under $R$. In particular,

$$\begin{aligned} W_5 &= \{\pm(c, c, -d), \pm(c, d, c), \pm(d, -c, c)\}, \\ W_6 &= \{\pm(-c, c, d), \pm(-c, d, -c), \pm(-d, -c, -c)\}, \\ W_7 &= \{\pm(c, c, d), \pm(c, d, -c), \pm(d, -c, -c)\}. \end{aligned} \tag{6.10}$$

We now have a full set of vertex-stars for $Q$. In fact, since $H$ acts transitively on the vertices of $Q$, the vertex-stars of $Q$ are images of $W_0$ under $H_0$, so $H_0$ must act transitively on them. However, since $H_0$ contains $S_2$, the stabilizer of $W_0$ must be at least of order 6, and hence the number of vertex-stars cannot exceed 8.

Observe that the elements $S_1'$, $S_2$ and $T'$ of $H_0$ act on the labels of the vertex-stars in the following way:

$$S_1' = (0\,1\,5\,2)\,(3\,6\,4\,7), \qquad S_2 = (1\,2\,3)\,(5\,7\,6), \qquad T' = (0\,2)\,(1\,7)\,(3\,5)\,(4\,6). \tag{6.11}$$

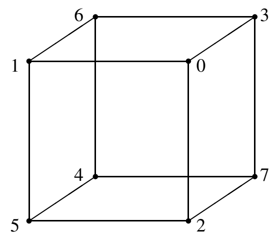


**Fig. 6.1.** The cube representing the action of $H_0$.

These permutations generate the rotation subgroup $[3,4]^+$ ($= [3,4]/\{\pm I\} \cong S_4$) for the cube with vertices $0,\dots,7$ shown in Fig. 6.1. The figure represents the action of $H_0$ on the vertex-stars; bear in mind that $-I$ acts trivially on the labels, so the above permutations really correspond to the geometric symmetries $-S_1'$, $S_2$ and $T'$, respectively.

The base face $F_2$ of $Q$ is given by

$$F_2 = \{(0,0,0), (c,-d,-c), (0,-c-d,-c+d), (c,-c,d)\}, \tag{6.12}$$

where the vertices are listed in cyclic order. Thus the faces are generally skew quadrangles.

Figure 6.2 shows the six faces of the polyhedron $Q(1,1)$ which contain $o$. They are represented by circled lines or by dotted lines with small or large dots, respectively, such that opposite faces are indicated in the same way. The vertex-figure of $Q(1,1)$ at $o$ is a skew hexagon given by a Petrie polygon of the cube.

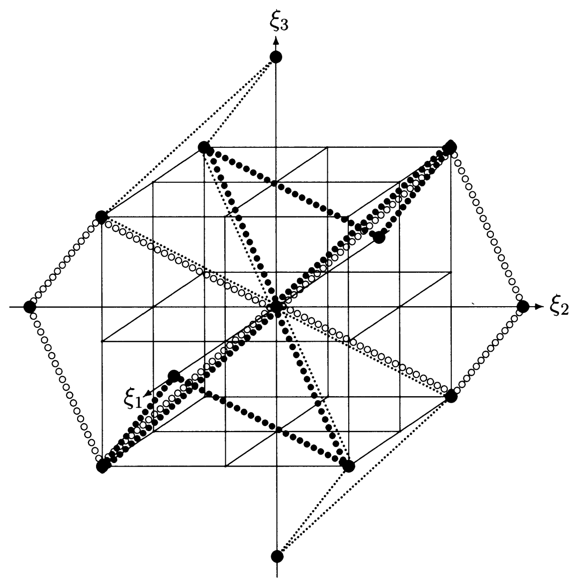


**Fig. 6.2.** The six faces of the polyhedron $Q(1,1)$ containing $o$.

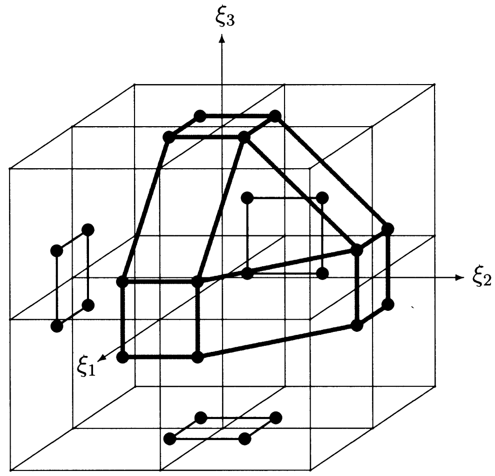


**Fig. 6.3.** The points of $W$ for $c = 1$ and $d = 4$.

Now consider the set of (generally 24) vectors

$$W := \bigcup_{i=0}^{7} W_i = \{(\pm d, \pm c, \pm c), (\pm c, \pm d, \pm c), (\pm c, \pm c, \pm d)\}. \tag{6.13}$$

This is the vertex-set of a convex 3-polytope which is one of the following: a suitably truncated octahedron or cube, with 24 vertices, if $0 < |c| < |d|$ or $0 < |d| < |c|$, respectively; a cube if $c = \pm d$; an octahedron if $c = 0$; or a cuboctahedron if $d = 0$. Figure 6.3 illustrates the truncated octahedron obtained for $c = 1$ and $d = 4$; the fat lines indicate the squares or rectangles surrounding a triangular face, and the fine grid lines are drawn in for reference.

If $c, d \neq 0$ and $c \neq \pm d$, then each vector in $W$ belongs to exactly two vertex-stars; moreover, if two vertex-stars have a vector in common, then they intersect in precisely two vectors, one being the negative of the other. In particular, the eight vertex-stars are distinct as sets. In all other cases the vertex-stars coincide (as cyclically ordered sets) in pairs, namely we have

$$W_0 = W_4, \qquad W_1 = W_7, \qquad W_2 = W_6, \qquad W_3 = W_5. \tag{6.14}$$

If $c = \pm d$ or $c = 0$, then they correspond to the Petrie polygons of the cube or octahedron, respectively; if $c = 0$, they all coincide as sets of vectors. Finally, if $d = 0$, they are planar and correspond to the four equatorial hexagons in the cuboctahedron.

Let $\Lambda := \mathbb{Z}[W]$ denote the $\mathbb{Z}$-module spanned by the vectors in $W$. Then each vector in $2\Lambda$ determines again a translation in $H$; in fact, we shall see that $2\Lambda$ is the full translation subgroup $T(H)$ of $H$. Moreover, by (4.4), $2\Lambda$ is a subset of the vertex-set $V(Q)$ of the polyhedron $Q$. In particular, we again have the following criterion for discreteness.

**Lemma 6.2.** *$H = H(c, d)$ is discrete if and only if c or d is zero or c and d are rational multiples of each other.*

*Proof.* The proof of Lemma 5.2 carries over with appropriate changes. Now

$$2(d,0,0) = (d,c,c) + (d,-c,-c) \in \Lambda,$$

and, similarly, $2(c,0,0) \in \Lambda$. In the discrete case, after rescaling (if need be), $H$ is again a subgroup of $[4,3,4]$. □

Therefore, up to similarity, we may assume that $c$ and $d$ are integers with $(c,d) = 1$. Again we explicitly allow $c = 0$ or $d = 0$. Then $H$ is a subgroup of $[4,3,4]$ and each vertex of $Q$ is in $\mathbb{Z}^3$. Moreover,

$$2\mathbb{Z}^3 \subset \Lambda \subset \mathbb{Z}^3,$$

for the same reason as in the previous section. Now we have

**Lemma 6.3.** *Let $\Lambda$ be the lattice spanned by $W$. Then*

(a) $\Lambda = \Lambda_{(1,1,0)}$ *if $c$ is odd and $d$ is even*;
(b) $\Lambda = \Lambda_{(1,0,0)} = \mathbb{Z}^3$ *if $c$ is even and $d$ is odd*;
(c) $\Lambda = \Lambda_{(1,1,1)}$ *if $c$ and $d$ are odd.*

*Proof.* Reduce certain vectors modulo $\Lambda$ and use the fact that $2\mathbb{Z}^3 \subset \Lambda$. For example, in the three cases for $c$ and $d$, the vector $(c,c,d)$ of $W$ is equivalent to $(1,1,0)$, $(0,0,1)$ and $(1,1,1)$, respectively. Now appeal to symmetry. □

Next we determine the vertex-set $V(Q)$ of $Q$. We know from (4.4) that $2\Lambda \subset V(Q) \subset \Lambda$, such that $V(Q)$ is a union of cosets of $\Lambda$ modulo $2\Lambda$. Leaving aside the special cases when $c = 0, d = 0$ or $c = \pm d$ (that is, $c, d = \pm 1$) for the moment, we now are tempted to proceed as follows, but unfortunately the argument is flawed. There are at most eight such cosets, and in each coset the vertex-stars at its vertices are necessarily the same (because any two vertices are related by a translation). Since $Q$ has altogether eight vertex-stars, all eight cosets must actually occur and thus $V(Q) = \Lambda$. The trouble is that the argument assumes vertex-faithfulness, that is, no vertex occurs with multiplicity. However, as we shall see, in one parameter family the polyhedra have vertices of multiplicity 2, so only four of the eight cosets occur. In fact, we have

**Lemma 6.4.** *The vertex-set $V(Q)$ of the polyhedron $Q$ is given by*

(a) $V(Q) = \bigcup_{i=0}^{3} (x_i + 2\Lambda_{(1,1,0)})$, *with* $x_0 := (0,0,0)$, $x_1 := (1,0,1)$, $x_2 := (1,1,0)$ *and* $x_3 := (0,1,1)$, *if $c$ is odd and* $d \equiv 2 \pmod 4$;
(b) $V(Q) = \Lambda$ *otherwise.*

*Proof.* Once again we generate new vertices as images of $o$ under $H$ and employ the translations in $2\Lambda$. In particular, once we find one representative vertex for a coset of $\Lambda$ modulo $2\Lambda$, then the entire coset is a subset of $V(Q)$. Obviously, $o$ yields $2\Lambda$ itself.

If $c$ is even and $d$ is odd, the vertices $(c,-d,-c)$ and $(0,-c-d,-c+d)$ of $F_2$ (see (6.12)) are equivalent to $(0,1,0)$ and $(0,1,1)$ modulo $2\Lambda = 2\mathbb{Z}^3$, respectively; their images under $\langle S_2 \rangle$ also yield $(1, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$. Finally we obtain $(1, 1, 1)$ from $oR = (-2c + d, -d, -d)$, with $R$ as in (6.8). Hence, $V(Q) = \Lambda$.

If $c$ is odd and $d$ is even, then the vertex $(0, -c - d, -c + d)$ of $F_2$, as well as its images under $\langle S_2 \rangle$, give the coset representatives $x_1 = (1, 0, 1)$, $x_2 = (1, 1, 0)$ and $x_3 = (0, 1, 1)$ modulo $2\Lambda = 2\Lambda_{(1,1,0)}$. From the vertices in $W_0$ we also obtain $(-1, d, 1)$, $(1, -1, d)$ and $(d, 1, -1)$ modulo $2\Lambda$; these three coincide with the first three if $d \equiv 2 \pmod 4$, but yield $(-1, 0, 1)$, $(1, -1, 0)$ and $(0, 1, -1)$ if $d \equiv 0 \pmod 4$. The vertex $oR = (-2c + d, -d, -d)$, with $R$ as in (6.8), also contributes $(2, 0, 0)$ if $d \equiv 0 \pmod 4$, so that all eight cosets are present in this case, proving that $V(Q) = \Lambda$. Finally, if $d \equiv 2 \pmod 4$, the union of the four cosets $x_i + 2\Lambda_{(1,1,0)}$, with $i = 0, 1, 2, 3$, is seen to be invariant under the two generators $S_2$ and $T$ of $H$, and hence must be the full vertex-set $V(Q)$, as it contains the base vertex. In this case each vertex of $Q$ is of multiplicity 2; we discuss this again below.

If both $c$ and $d$ are odd, then the vertices in $W_0$ contribute the representatives $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$ modulo $2\Lambda = 2\Lambda_{(1,1,1)}$, and the vertex $(0, -c - d, -c + d)$ and its images under $\langle S_2 \rangle$ yield $(2, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 2)$. The eighth coset representative $(1, 1, 1)$ is derived from $(-2c + d, -d, -d)$. Hence, $V(Q) = \Lambda$. □

In the exceptional case when $c$ is odd and $d \equiv 2 \pmod 4$, each point $x \in \mathbb{E}^3$ taken by a vertex is in fact occupied by exactly two vertices. Each vertex is assigned a vertex-star, so there are always two vertex-stars positioned at $x$. Once we know the two vertex-stars at one point, then we can determine them at any point by appealing to vertex-transitivity. Consider the point $x = (-2c + d, -d, -d)$ occupied by the vertex $oR$, with $R$ as in (6.8). By construction, $x$ receives the vertex-star $W_4 = W_0 R'$. On the other hand, since $-2c + d \equiv 0 \pmod 4$, the point $x$ is equivalent to $o$ modulo the translations in $2\Lambda = 2\Lambda_{(1,1,0)}$, so $x$ must also receive the vertex-star $W_0$. Hence both $W_0$ and $W_4$ occur at $(-2c + d, -d, -d)$. Now, appealing to $R$ once again we see that $W_0$ and $W_4$ must also be the vertex-stars at $o$, the point occupied by the base vertex. Moreover, observe that the two vertex-stars at a point taken by a vertex are always disjoint, so that there are twelve edges of $Q$ emanating from it.

We should point out here that a polyhedron $Q(c, d)$ with $c$ odd and $d \equiv 2 \pmod 4$ is *not* a faithful realization of its underlying abstract polyhedron, so in particular is not a (faithful) geometric polyhedron or apeirohedron as defined in Sections 2 and 4. Each such $Q(c, d)$ is an infinite discrete realization and is geometrically chiral (as we shall see).

Next we determine the translations which map the polyhedron $Q$ onto itself.

**Lemma 6.5.** *Let $c, d$ be integers with $(c, d) = 1$. The subgroup $T(H)$ of all translations in $H$ is given by $T(H) = 2\Lambda$.*

*Proof.* The translates of the base vertex are again vertices of $Q$, so we certainly have

$$2\Lambda \subset T(H) = oT(H) \subset V(Q) \subset \Lambda.$$

Moreover, $V(Q)$ is the union of either four or eight cosets of $\Lambda$ modulo $2\Lambda$; in particular, $V(Q) = \Lambda$ unless $c$ is odd and $d \equiv 2 \pmod 4$ (see Lemma 6.4(a)). Clearly, if two vertices of $Q$ are equivalent under a translation, then their vertex-stars or pairs of vertex-stars, respectively, must be the same.

First consider the case $c, d \neq 0$ and $c \neq \pm d$, when there are eight distinct vertex-stars. If $V(Q)$ consists of eight cosets of $\Lambda$ modulo $2\Lambda$, then each coset must uniquely determine the vertex-star at its vertices, and no two cosets can be associated with the same vertex-star. If $V(Q)$ consists of only four cosets, then the same remains true for the pairs of vertex-stars at the vertices. In any case, we must have $T(H) = 2\Lambda$; otherwise $Q$ could have at most four distinct vertex-stars.

If $c = \pm d = \pm 1$, then $2\Lambda_{(1,1,1)} \subset T(H) \subset \Lambda_{(1,1,1)} = \Lambda$. Bearing in mind that $T(H)$ is invariant under $H_0$, particularly under $S_2$ and $T'$, we find that a translation group strictly larger than $2\Lambda_{(1,1,1)}$ would necessarily lead to a polyhedron with at most two distinct vertex-stars; hence $T(H) = 2\Lambda$ also in this case. For the remaining cases $c = 0$ or $d = 0$ we can refer to Theorem 6.12 and directly use the geometry of $\{4, 6\,|\,4\}$ and $\{4, 6\}_6$ (the Petrial of $\{6, 6\}_4$). $\square$

We should mention that there is also a direct proof of Lemma 6.5. This considers the factorization $H = N \cdot \langle S_2 \rangle$, with $N$ the normal closure of the generator $T$ in $H$, and then describes the elements of $N$. In particular, $N$ is generated by the half-turns $R_0, \ldots, R_5$, where $R_i$ is given by $R_i := S_2^{-i} T S_2^{i}$. However, while instructive, the details are rather tedious, so we omit them here.

We now discuss an analog of Theorem 5.8. Once again we color the vertices $x$ of $Q$, now with *colors* from the set $C := \{0, \ldots, 7\}$. We consider the mapping

$$c'\colon\ V(Q)/T(H) \to C$$

which associates with a coset $x + T(H)$ a label $i \in C$ as specified in Table 2 for its coset representative, and then take the induced *coloring mapping*

$$c\colon\ V(Q) \to C$$

defined by $(x)c := (x + T(H))c'$. (If $V(Q)$ is not a lattice, then $V(Q)/T(H)$ simply denotes the set of cosets of $V(Q)$ modulo $T(H)$; it is not a group in this case.) In effect, $c$ assigns to every vertex $x$ the index $i = (x)c$ (say) of its vertex-star $W_i$ as its color. In the exceptional case when points are doubly occupied by vertices of $Q$ (that is, $c$ is odd and $d \equiv 2 \pmod 4$), this associates with such a point two colors, one for each vertex; in fact, in the penultimate column of Table 2 each representative occurs twice. Moreover, when the eight vertex-stars of $Q$ coincide in pairs (see (6.14)), then the eight assignments of vertex-stars to cosets modulo $T(H)$ also coincide in pairs.

Note that the vectors in a column of Table 2 represent the vertices

$$\begin{gathered}
(0,0,0) = o, \qquad (-c,d,c) = oT_1, \qquad (c,-c,d) = oT_2, \qquad (d,c,-c) = oT_3,\\
(-2c+d,-d,-d) = oR, \qquad (0,-c-d,-c+d) = oS_1^2,\\
(-c-d,-c+d,0) = oS_1^2S_2^2, \qquad (-c+d,0,-c-d) = oS_1^2S_2^4,
\end{gathered} \tag{6.15}$$

in this order, with $R$ as in (6.8).

**Table 2.** The colors $i$ assigned to the cosets in $V(Q)/T(H)$.

| | $c, d$ odd, $c \equiv d \pmod 4$ | $c, d$ odd, $c \not\equiv d \pmod 4$ | $c$ odd, $d \equiv 0 \pmod 4$ | $c$ odd, $d \equiv 2 \pmod 4$ | $c$ even, $d$ odd |
|---|---|---|---|---|---|
| 0 | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) | (0, 0, 0) |
| 1 | (−1, 1, 1) | (1, 1, −1) | (−1, 0, 1) | (1, 0, 1) | (0, 1, 0) |
| 2 | (1, −1, 1) | (−1, 1, 1) | (1, −1, 0) | (1, 1, 0) | (0, 0, 1) |
| 3 | (1, 1, −1) | (1, −1, 1) | (0, 1, −1) | (0, 1, 1) | (1, 0, 0) |
| 4 | (1, 1, 1) | (1, 1, 1) | (2, 0, 0) | (0, 0, 0) | (1, 1, 1) |
| 5 | (0, 2, 0) | (0, 0, 2) | (0, 1, 1) | (0, 1, 1) | (0, 1, 1) |
| 6 | (2, 0, 0) | (0, 2, 0) | (1, 1, 0) | (1, 1, 0) | (1, 1, 0) |
| 7 | (0, 0, 2) | (2, 0, 0) | (1, 0, 1) | (1, 0, 1) | (1, 0, 1) |

We summarize the above discussion in the following:

**Theorem 6.6.** *Let $c$ and $d$ be integers with $(c, d) = 1$. Let $Q$ be the polyhedron associated with $H$, and let $V(Q)$ and $T(H)$ be as in Lemmas* 6.4 *and* 6.5, *respectively. If $x$ is a vertex of $Q$, then $x + W_{(x)c}$ is the set of vertices of $Q$ adjacent to $x$; that is, the vertex-star at $x$ is given by $W_{(x)c}$. If $c$ is odd and $d \equiv 2 \pmod 4$, then every vertex is a double vertex. In all other cases, every vertex is single.*

Next we discuss when the polyhedron has planar vertex-figures or faces.

**Lemma 6.7.** *Let $Q$ be the polyhedron associated with $H$. Then*

(a) *$Q$ has planar vertex-figures if and only if $d = 0$;*
(b) *$Q$ has planar faces if and only if $c = 0$.*

*Proof.* The polyhedron $Q$ has planar vertex-figures if and only if the points in $W_0$ lie in a plane. Now the corresponding determinant is

$$\begin{vmatrix} c & -c & d \\ c & -d & -c \\ d & c & -c \end{vmatrix} = d\,(3c^2 + d^2).$$

Hence the points lie in a plane if and only if $d = 0$.

The vertices of the base face $F_2$ of $Q$ are $(0, 0, 0)$, $(c, -d, -c)$, $(0, -c-d, -c+d)$ and $(c, -c, d)$, in this order (see (6.12)). Hence $F_2$ is planar if and only if $(0, -c-d, -c+d)$ is the sum of $(c, -d, -c)$ and $(c, -c, d)$; this gives the condition $c = 0$. □

We now characterize the faces of the polyhedron. We only consider the case when $Q$ has eight distinct vertex-stars. Once again we associate with each face $F$ of $Q$ a sequence $\zeta(F) := (z_1, \ldots, z_4)$ in $W$, defined up to cyclic permutation and reversal of order. If $F = \{y_1, \ldots, y_4\}$ (say), with vertices $y_1, \ldots, y_4$ in cyclic order, we set $z_i := y_{i+1} - y_i$ for $i = 1, \ldots, 4$ (all indices are considered modulo 4); then $\sum_{i=1}^{4} z_i = o$ and

$$F = y + \{o, z_1, z_1 + z_2, z_1 + z_2 + z_3\},$$

with $y := y_1$. If $F = F_2$, with the vertices as in (6.12), then

$$z_1 = (c, -d, -c), \quad z_2 = (-c, -c, d), \quad z_3 = (c, d, c), \quad z_4 = (-c, c, -d),$$

and $z_1, z_2 \in W_1$, $z_2, z_3 \in W_5$, $z_3, z_4 \in W_2$ and $z_4, z_1 \in W_0$; note that the labels of the vertex-stars occurring here are just those of the cycle of 0 in the cycle representation for $S_1'$ in (6.11). Moreover, $z_i \cdot z_{i+1} = -c^2$ (scalar product) for each $i$. Observe that if $F = F_2 S$ for some $S \in H$, then the vertex-stars occurring for $\zeta(F)$ will be just those of the cycle of $i$, with $W_i = W_0 S'$, in the cycle representation for $(S')^{-1} S_1' S'$.

Accordingly we now define a sequence of vectors $\zeta := (z_1, \ldots, z_4)$ in $W$ to be *admissible* if

(i) $z_i \cdot z_{i+1} = -c^2$ for $i = 1, \ldots, 4$;
(ii) $z_1, z_2 \in W_i,\ z_2, z_3 \in W_j,\ z_3, z_4 \in W_k$ and $z_4, z_1 \in W_l$, where $i, j, k, l$ are such that the cycle $(i\ j\ k\ l)$ represents a face, in cyclic order, of the cube in Fig. 6.1.

(6.16)

Any cyclic permutation or reversal of order of the vectors in an admissible sequence gives again an admissible sequence. Although the appearance is quite different, this is the exact analog of the definition in (5.15). In the previous section the special group acts on a tetrahedron instead of a cube (see (5.11)), so faces are necessarily triples of indices. In the present context we must allow all faces of the cube of Fig. 6.1, because their vertex cycles are precisely the cycles which occur in the cycle representation of the conjugates of $S_1'$ in $H_0$.

**Theorem 6.8.** *Let $c$ and $d$ be non-zero integers with $c \neq \pm d$ and $(c, d) = 1$. The admissible sequences in $W$ are precisely the sequences $\zeta(F)$ associated with the faces $F$ of the polyhedron $Q$. In particular, if $\zeta = (z_1, \ldots, z_4)$ is an admissible sequence with indices $i, j, k, l$ as in* (6.16), *and if $y$ is a vertex of $Q$ colored $l$, then $F = y + \{o, z_1, z_1 + z_2, z_1 + z_2 + z_3\}$ is the vertex-set of a face of $Q$, and $\zeta = \zeta(F)$.*

*Proof.* We already know that the sequence associated with a face of $Q$ is admissible. If $\zeta(F_2) = (z_1, \ldots, z_4)$ is as above, then $z_1 \in W_0 \cap W_1$, $z_2 \in W_1 \cap W_5$, $z_3 \in W_5 \cap W_2$ and $z_4 \in W_2 \cap W_0$, so the corresponding cycle is given by (0 1 5 2). Using (6.11) we see that the cycles for $\zeta(F_2 S_2)$ and $\zeta(F_2 S_2^2)$ are (0 2 7 3) and (0 3 6 1), respectively. These three cycles represent the three faces of the cube in Fig. 6.1 which contain the vertex labeled 0. Now recall that the images of $F_2$ under $\langle S_2 \rangle$ give all the faces of $Q$ containing $o$, and that 0 is the color of $o$ as a vertex of $Q$. However, then we can appeal to transitivity and conclude that the sequences $\zeta$ which are associated with the faces containing a given vertex of $Q$ colored $l$ (say), must correspond to cycles which represent faces of the cube in Fig. 6.1 containing the vertex labeled $l$.

Now let $\zeta = (z_1, \ldots, z_4)$ be an admissible sequence with cycle $(i\ j\ k\ l)$ as in (6.16); then $z_1 \in W_l \cap W_i$, $z_2 \in W_i \cap W_j$, $z_3 \in W_j \cap W_k$ and $z_4 \in W_k \cap W_l$. However, each such intersection of vertex-stars consists of a single vector and its negative, so it determines its vectors up to sign. Once a vector in one intersection is chosen, then the others are determined by the scalar product condition of (6.16). Hence $\zeta$ is determined by $(i\ j\ k\ l)$ and a single vector, for example, $z_1$.

Now, if $y$ is a vertex of $Q$ colored $l$, then $W_l$ is the vertex-star at $y$ and the vectors $z_1, z_4$ of $W_l$ determine a face $F'$ (say) of $Q$ which has $y$ as a vertex. Then $\zeta(F')$ is an admissible sequence which contains $z_1, z_4$, and it has a cycle of indices which represents a face of the cube in Fig. 6.1 with vertex $l$. Since the sequences for the six faces of $Q$ with vertex $y$ lead to the three cycles which contain $l$, the sequence associated with one such face $F$ (say) must have $(i\ j\ k\ l)$ as its cycle, and $F$ must be either $F'$ itself or adjacent to $F'$. The two vectors of $W_l$ which determine $F$ thus include $z_1$ or $z_4$, or both. Hence $\zeta(F)$ is an admissible sequence with cycle $(i\ j\ k\ l)$, and $\zeta(F)$ includes $z_1$ or $z_4$. However, then $\zeta = \zeta(F)$, because an admissible sequence is determined by its cycle and a single vector. Now the theorem follows. $\square$

We now compute the face centers of $Q$. Every face of $Q$ must be equivalent modulo $T(H) = 2\Lambda$ to a face which contains a vertex from the list in (6.15) of vertex representatives modulo $T(H)$, and every center of a face must be equivalent to the center of such a face. The face center of the base face $F_2$ (see (6.12)) is given by

$$c(F_2) = \tfrac{1}{2}(c, -c-d, -c+d),$$

so every face center of $Q$ must be a point in $\frac{1}{2}\mathbb{Z}^3$. The following lemma describes the full set of face centers that occur. The details of the computation are omitted.

**Lemma 6.9.** *Let $Q$ be the polyhedron associated with $H$, and let $\Lambda$ be as in Lemma* 6.3. *Then the set of face centers of $Q$ is the union of cosets of $\frac{1}{2}\mathbb{Z}^3$ modulo $2\Lambda$. The cosets which occur can be represented by the following vectors*:

(a) $\pm\frac{1}{2}(2,-1,0), \pm\frac{1}{2}(2,3,0)$, *all cyclic permutations of coordinates*, *if $c$ and $d$ are odd with $c+d \equiv 0 \pmod 4$*;
(b) $\pm\frac{1}{2}(1,-2,0), \pm\frac{1}{2}(3,2,0)$, *all cyclic permutations of coordinates*, *if $c$ and $d$ are odd with $c+d \equiv 2 \pmod 4$*;
(c) $\pm\frac{1}{2}(0,1,-1), \pm\frac{1}{2}(2,1,-1)$, *all cyclic permutations of coordinates*, *if $c$ is even and $d$ is odd*;
(d) $\pm\frac{1}{2}(1,1,-1), \pm\frac{1}{2}(1,1,3)$, *all cyclic permutations of coordinates*, *if $c$ is odd and $d \equiv 0 \pmod 4$*;
(e) $\pm\frac{1}{2}(1,1,-3)$, *if $c \equiv \pm 1 \pmod 8$ and $d \equiv 2 \pmod 4$*;
(f) $\pm\frac{1}{2}(1,1,1)$, *if $c \equiv \pm 3 \pmod 8$ and $d \equiv 2 \pmod 4$*.

Note that the number of cosets occurring in Lemma 6.9 is twelve in each of the first four cases, but is only two in the last two cases.

The face centers of $Q = Q(c,d)$ are just the vertices of the dual polyhedron of type $\{6,4\}$ denoted by $Q^* = Q(c,d)^*$. Its face centers are the vertices of $Q$. This polyhedron $Q^*$ can also be obtained by Wythoff's construction applied to the same group $H$, but with new generators $S_2^{-1}, S_1^{-1}$ and initial (base) vertex

$$w := \tfrac{1}{2}(c, -c-d, -c+d) = c(F_2),$$

the fixed point of $S_1^{-1}$. Alternatively we could describe $Q^*$ in terms of the vertex-stars at its vertices, but we do not discuss this here in detail. Suffice it to say that the vertex in the base edge of $Q^*$ distinct from $w$ is given by $wT$, and that the orbit of $wT$ under $\langle S_1^{-1}\rangle$ yields the vertices in the vertex-figure at $w$. This determines the vertex-star at $w$, and then the remaining vertex-stars are just its images under $H_0$.

We can also decide when two polyhedra $Q(c, d)$ and $Q(e, f)$ are affinely equivalent. Once again we need not assume that the parameters are integers.

**Lemma 6.10.** *Let $c, d, e$ and $f$ be real numbers, and let $Q(c, d)$ and $Q(e, f)$ be the polyhedra associated with $c, d$ and $e, f$, respectively. Then $Q(c, d)$ and $Q(e, f)$ are affinely equivalent if and only if $(e, f) = s(c, d)$ or $t(-c, d)$ for some real numbers $s$ and $t$. Moreover, $Q(c, d)$ and $Q(e, f)$ are congruent if and only if this holds with $s, t = \pm 1$.*

*Proof.* We can argue as in the proof of Lemma 5.15. Suppose that we have $Q(c, d)R = Q(e, f)$ for some affine mapping $R$, and that $oR = o$ and $(c, -c, d)R = (e, -e, f)$. We can prove as before that there are essentially only two choices for $R$. If the cyclic order of the vertices in the vertex-figure at $o$ is preserved by $R$ (see (6.4)), then $R$ is the identity mapping, up to scaling (by $s := e/c$ if $c \neq 0$). On the other hand, if the order is reversed by $R$, then $R$ is the reflection in the plane $\xi_1 = \xi_2$ of $\mathbb{E}^3$, up to scaling (by $t := e/c$ if $c \neq 0$). In particular, $(e, f) = s(c, d)$ or $t(-c, d)$, respectively, with $s, t = \pm 1$ in the case of congruence. □

Using Lemma 6.10 we now settle the question when a polyhedron $Q(c, d)$ is chiral or regular. We have

**Theorem 6.11.** *The polyhedron $Q(c, d)$ is geometrically chiral if $c, d \neq 0$, or geometrically regular if $c = 0$ or $d = 0$.*

*Proof.* By construction, each polyhedron $Q(c, d)$ is either geometrically chiral or geometrically regular. Suppose that $Q(c, d)$ is geometrically regular. Then its symmetry group must contain an element $R$ which fixes the two vertices $F_0 := o$ and $w := (c, -c, d)$ of the base edge $F_1$ and interchanges the two faces $F_2, F_2'$ containing $F_1$. Now appeal to the proof of Lemma 6.10 with $e = c$ and $f = d$. Since $R^{-1}S_2R = S_2^{-1}$, the order of the vertices in the vertex-figure is reversed, so $R$ or $-R$ must be the reflection in the plane $\xi_1 = \xi_2$ given by

$$R\colon \quad (\xi_1, \xi_2, \xi_3) \quad \mapsto \quad (\xi_2, \xi_1, \xi_3). \tag{6.17}$$

In particular, $(c, d) = t(-c, d)$ with $t = \pm 1$. Hence either $c = 0$ or $d = 0$.

Conversely, let $c = 0$ or $d = 0$; then $d = 1$ or $c = 1$, respectively. Although we could appeal to Theorem 6.12 below, we give a direct proof. Let $R$ be the reflection in (6.17). We show that $R$ or $-R$, respectively, belongs to the symmetry group of $Q(c, d)$ and maps the base flag $\{F_0, F_1, F_2\}$ onto the adjacent flag $\{F_0, F_1, F_2'\}$. In fact, if $c = 0$, we have $oR = o$, $wR = w$, $R^{-1}TR = T$ and $R^{-1}S_2R = S_2^{-1}$, and hence also $R^{-1}S_1R = S_1S_2^2$ and $R^{-1}HR = H$. However, since

$$F_2'(S_1S_2^2) = (F_2'T)S_2 = F_2S_2 = F_2',$$

we must have

$$F_2R = (o\langle S_1\rangle)R = (oR)\langle S_1S_2^2\rangle = o\langle S_1S_2^2\rangle = F_2',$$

as required. Moreover, if $S \in H$ and $F_i S$ is any vertex, edge or face of $Q(c, d)$, then $(F_i S)R = (F_i R)(R^{-1}SR)$ is again a vertex, edge or face, so $R$ maps $Q(c, d)$ onto itself. This settles the case $c = 0$. If $d = 0$, we can replace $R$ by $-R$ and argue in exactly the same way. □

For non-zero parameters $c$ and $d$, the reflection $R$ of (6.17) maps the polyhedron $Q(c, d)$ onto the polyhedron $Q(-c, d)$, or rather the enantiomorphic image of $Q(-c, d)$. In fact, conjugation by $R$ transforms the group $H(c, d)$ with generators $S_1(c, d)$, $S_2$ into the new group $H(-c, d)$ with generators $S_1(-c, d)S_2^2$, $S_2^{-1}$. However, since $oR = o$, $(c, -c, d)R = (-c, c, d)$ and $(o\langle S_1\rangle)R = o\langle S_1 S_2^2\rangle$, the reflection $R$ maps $Q(c, d)$ onto $Q(-c, d)$ and takes the base flag $\{F_0(c, d), F_1(c, d), F_2(c, d)\}$ (say) of $Q(c, d)$ to $\{F_0(-c, d), F_1(-c, d), F_2'(-c, d)\}$, the flag adjacent to the base flag of $Q(-c, d)$. In fact, the image $Q(c, d)R$ is just the polyhedron obtained from $H(-c, d)$ by Wythoff's construction with generators $S_1(-c, d)S_2^2$, $S_2$ and initial vertex $(-c, c, d)$. Observe that the mapping $-R$ takes $Q(c, d)$ to $Q(c, -d)$, but the latter is simply $Q(-c, d)$ again.

Next we identify the two polyhedra $Q(c, d)$ which are regular. We know from Theorem 7E15 of [18] that there are only two regular polyhedra of type $\{4, 6\}$ in $\mathbb{E}^3$, namely the Petrie–Coxeter polyhedron $\{4, 6\,|\,4\}$ (with planar faces and skew vertex-figures), and $\{4, 6\}_6$ (with skew faces and planar vertex-figures). The latter is the Petrial of the polyhedron $P(1, -1) = \{6, 6\}_4$ occurring in Theorem 5.17; its skew "square" faces are inscribed in three fourths of the cubes of the cubical tessellation $\{4, 3, 4\}$ of $\mathbb{E}^3$, such that the vertex-figures are planar hexagons. We know from Lemma 6.7 that $Q(0, 1)$ has planar faces and that $Q(1, 0)$ has planar vertex-figures. So we have

**Theorem 6.12.** *$Q(0, 1) = \{4, 6\,|\,4\}$ and $Q(1, 0) = \{4, 6\}_6$ are the only polyhedra $Q(c, d)$ which are regular. Their duals are $Q(0, 1)^* = \{6, 4\,|\,4\}$ and $Q(1, 0)^* = \{6, 4\}_6$.*

The two elements $(S_1 S_2^{-1})^4$ and $(S_1^2 S_2^2)^3$ of $H(c, d)$ are given by

$$\begin{array}{rccl} (S_1S_2^{-1})^4: & x & \mapsto & x + 4c(0, 1, 0), \\ (S_1^2S_2^2)^3: & x & \mapsto & x + 2d(-1, 1, -1). \end{array} \tag{6.18}$$

In particular, $(S_1 S_2^{-1})^4 = I$ if $c = 0$, and $(S_1^2 S_2^2)^3 = I$ if $d = 0$. These relations confirm that $Q(0, 1)$ has 2-holes of length 4, and that $Q(1, 0)$ has Petrie polygons of length 6 (see pp. 193 and 196 of [18]). For non-zero parameters $c$ and $d$, these elements are genuine translations. Note that it is generally not true that $Q(c, d)$ is the Petrial of some polyhedron $P(a, b)$ (the Petrial generally has infinite faces, but $Q(c, d)$ does not); the only exception is $Q(1, 0)$.

We conclude with an analog of Theorem 5.18 about combinatorial isomorphism.

**Theorem 6.13.** *Let $c, d$ and $e, f$ be pairs of relatively prime integers. Then the polyhedra $Q(c, d)$ and $Q(e, f)$ are combinatorially isomorphic if and only if $(e, f) = \pm(c, d)$ or $(e, f) = \pm(-c, d)$.*

*Proof.* We adapt the proof of Theorem 5.18. The same general argument (but with $R$ as in (6.17)) shows that, up to replacing $e$ by $-e$, we may assume that there is an isomorphism between $Q(c,d)$ and $Q(e,f)$ which maps the base flag of $Q(c,d)$ to the base flag of $Q(e,f)$. Then there also is a group isomorphism $\mu\colon H(c,d) \to H(e,f)$ which maps $S_1(c,d)$, $S_2$ to $S_1(e,f)$, $S_2$. In particular, $\mu$ maps translations to translations.

Now the two translations in (6.18) take the form

$$(S_1(c,d)S_2^{-1})^4 = T(4ce_2), \qquad (S_1(c,d)^2S_2^2)^3 = T(2d(-e_1+e_2-e_3)), \tag{6.19}$$

and similarly for the parameters $e, f$. Then

$$(T(4e_2)\mu)^c = T(4ce_2)\mu = T(4ee_2) = T(4e_2)^e.$$

Hence, if $c = 0$, then also $e = 0$ and we are done. Let $c \neq 0$. Then $T(4e_2)\mu = T(re_2)$ for some integer $r$, and thus $T(cre_2) = T(re_2)^c = T(4ee_2)$. This yields the equation $cr = 4e$. Moreover, we have $T(4e_i)\mu = T(re_i)$ for each $i$ and therefore obtain the equation $dr = 4f$ from

$$\begin{aligned} T(4f(-e_1+e_2-e_3)) &= T(4d(-e_1+e_2-e_3))\mu \\ &= (T(4e_1)^{-1}T(4e_2)T(4e_3)^{-1})^d\mu \\ &= (T(re_1)^{-1}T(re_2)T(re_3)^{-1})^d \\ &= T(dr(-e_1+e_2-e_3)). \end{aligned}$$

Finally, bearing in mind that $c, d$ and $e, f$ are relatively prime, we must have $r = \pm 4$, that is, $(e, f) = \pm(c, d)$. $\square$

We do not know if any polyhedron $Q(c,d)$ other than $Q(0,1)$ and $Q(1,0)$ is combinatorially regular, but we do know that none is geometrically regular.

## 7. Relationships Among Chiral Polyhedra

In this final section we briefly discuss relationships among the chiral polyhedra of Sections 5 and 6 (see Theorem 7.1) which are based on an analog of the *halving operation* for regular polyhedra (see p. 197 of [18]). They were observed by Peter McMullen and are reproduced here with his permission.

The halving operation $\eta$ of [18] applies to an (abstract) regular polyhedron $\mathcal{Q}$ of type $\{4, q\}$ for some $q \geq 3$, and turns it into a self-dual regular polyhedron $\mathcal{P} := \mathcal{Q}^\eta$ of type $\{q, q\}$. If $\Gamma(\mathcal{Q}) = \langle \alpha_0, \alpha_1, \alpha_2 \rangle$ (say), then $\Gamma(\mathcal{P}) = \langle \rho_0, \rho_1, \rho_2 \rangle$ is the subgroup of $\Gamma(\mathcal{Q})$ of index at most 2 determined by

$$\eta\colon (\alpha_0, \alpha_1, \alpha_2) \mapsto (\alpha_0\alpha_1\alpha_0, \alpha_2, \alpha_1) =: (\rho_0, \rho_1, \rho_2).$$

The index is 2 if and only if the edge-graph of $\mathcal{Q}$ is bipartite. Alternatively, in terms of the generators $\beta_1 := \alpha_0\alpha_1$, $\beta_2 := \alpha_1\alpha_2$ of $\Gamma^+(\mathcal{Q})$ and $\sigma_1 := \rho_0\rho_1$, $\sigma_2 := \rho_1\rho_2$ of $\Gamma^+(\mathcal{P})$, the same operation is given by

$$\eta\colon (\beta_1, \beta_2) \mapsto (\beta_1^2\beta_2, \beta_2^{-1}) =: (\sigma_1, \sigma_2). \tag{7.1}$$

Now, for a chiral polyhedron $\mathcal{Q}$ of type $\{4, q\}$ with group $\Gamma(\mathcal{Q}) = \langle \beta_1, \beta_2 \rangle$, the halving operation $\eta$ is directly defined by (7.1). This will generally yield (except perhaps in a few degenerate cases) a self-dual polyhedron $\mathcal{P} := \mathcal{Q}^\eta$ of type $\{q, q\}$ with group $\Gamma(\mathcal{P}) = \langle \sigma_1, \sigma_2 \rangle$, which is either chiral or regular. Note that $\sigma_1\sigma_2 = \beta_1^2$ is indeed an involution because $\mathcal{Q}$ is of type $\{4, q\}$. Moreover, $\Gamma(\mathcal{P})$ is again a subgroup of $\Gamma(\mathcal{Q})$ of index at most 2, and $\beta_1 \notin \Gamma(\mathcal{P})$ if and only if the index is 2. The self-duality of $\mathcal{P}$ can be verified as follows.

We first conjugate by $\beta_1$ to replace the generators $\sigma_1, \sigma_2$ for $\Gamma(\mathcal{P})$ by the generators $\sigma_2, (\sigma_1\sigma_2^2)^{-1}$ for the conjugate subgroup $\beta_1^{-1}\Gamma(\mathcal{P})\beta_1$ in $\Gamma(\mathcal{Q})$ (in effect, this replaces $\mathcal{P}$ by an isomorphic copy), and then obtain their images $\sigma_1\sigma_2^2, \sigma_2^{-1}$ under the duality operation $\delta$ of (2.3); but the latter are just the basic generators for $\Gamma(\mathcal{P})$ associated with a flag adjacent to the base flag (and determining the other enantiomorphic form of $\mathcal{P}$), so $\mathcal{P}$ must indeed be isomorphic to its dual. In fact, we have

$$\begin{aligned}
\beta_1^{-1}\sigma_1\beta_1 &= \beta_1^{-1}\beta_1^2\beta_2\beta_1 = \beta_1\beta_2\beta_1 = \beta_2^{-1} = \sigma_2,\\
\beta_1^{-1}\sigma_2\beta_1 &= \beta_1^2(\beta_1\sigma_2\beta_1^{-1})\beta_1^{-2} = \beta_1^2\sigma_1\beta_1^{-2} = \beta_2\beta_1^{-2} = \sigma_2^{-2}\sigma_1^{-1} = (\sigma_1\sigma_2^2)^{-1}.
\end{aligned}$$

Here the first equation also elucidates why $\Gamma(\mathcal{P})$ will generally satisfy the intersection property with respect to its generators $\sigma_1, \sigma_2$ (see [22]); in fact, since $\sigma_1 = \beta_1\sigma_2\beta_1^{-1}$, any element in $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle$ must necessarily fix both the base vertex $F_0$ and the adjacent vertex $F_0\beta_1^{-1}$ of the original polyhedron $\mathcal{Q}$, and hence must be trivial, except perhaps in a few degenerate cases (when adjacent vertices are joined by more than one edge).

If $\Gamma(\mathcal{P})$ is of index 2 in $\Gamma(\mathcal{Q})$, we can recover the original polyhedron $\mathcal{Q}$ from its image $\mathcal{P} = \mathcal{Q}^\eta$ by twisting with $\beta_1$ (in the sense of [18, p. 245]). In fact, $\beta_1$ acts on $\Gamma(\mathcal{P})$ by conjugation and we can adjoin it to $\Gamma(\mathcal{P})$ to recover $\Gamma(\mathcal{Q})$; the corresponding twisting operation is

$$\kappa\colon\ (\sigma_1, \sigma_2; \beta_1) \mapsto (\beta_1, \sigma_2^{-1}) = (\beta_1, \beta_2). \tag{7.2}$$

We now apply these considerations to the polyhedra $Q(c, d)$ and $P(a, b)$ of the previous sections. Let $H(c, d) = \langle B_1(c, d), B_2\rangle$ and $G(a, b) = \langle S_1(a, b), S_2\rangle$ be their groups with generators as in (6.1), (6.2), (5.1) and (5.2), respectively; here we have renamed the generators of $H(c, d)$ to distinguish them better from those of $G(a, b)$. Note that $B_2 = S_2$. In the geometric context it is more convenient to modify $\eta$ by conjugating the new generators of $\eta$ by $C := B_1(c, d)B_2^4$. The resulting operation $\eta'$ then takes the form

$$\eta'\colon\ (B_1(c, d), B_2) \mapsto (C^{-1}B_1(c, d)^2B_2C,\ C^{-1}B_2^{-1}C) = (S_2^{-1}, S_1(a, b)^{-1}), \tag{7.3}$$

with $a := -c + d$ and $b := -c - d$. However, the pair of generators on the right is just the image of the distinguished pair of generators $S_1(a, b), S_2$ of $G(a, b)$ under the duality operation $\delta$ of (2.3). Using Corollary 5.14 and (5.3) we therefore have

$$Q(c, d)^{\eta'} = P(-c + d, -c - d)^* \cong P(-c + d, -c - d) = P(c - d, c + d), \tag{7.4}$$

where $\cong$ means congruence. Note that the parameters $c - d, c + d$ are relatively prime unless both $c$ and $d$ are odd; in the latter case they should be halved for classification purposes (and then congruence replaced by similarity). Now the first part of the following theorem follows.

**Theorem 7.1.** *Let $c$ and $d$ be integers with $(c, d) = 1$. Then we have*

(a) $Q(c, d)^\eta \cong P(c - d, c + d)$,
(b) $(Q(c, d)^\eta)^\kappa = Q(c, d)$.

The first part of Theorem 7.1 implies that each polyhedron $P(a, b)$ is the image under $\eta$ of a suitable polyhedron $Q(c, d)$, once again up to rescaling. Moreover, as expected, $\eta$ pairs up the regular polyhedra among the $P(a, b)$ and $Q(c, d)$; in particular, we have

$$\begin{aligned} \{4, 6\,|\,4\}^\eta = Q(0, 1)^\eta &\cong P(1, -1) = \{6, 6\}_4, \\ \{4, 6\}_6^\eta = Q(1, 0)^\eta &\cong P(1, 1) = \{6, 6\,|\,3\} \end{aligned}$$

(see also p. 224 of [18]).

Under the original operation $\eta$, the two generators $B_1(c, d)$, $B_2$ of $H(c, d)$ are changed to new generators $B_1(c, d)^2 B_2$, $B_2^{-1}$ whose product is $B_1(c, d)^2$. However,

$$B_1(c, d)^2 = T_1(-c + d, -c - d),$$

with $T_1$ as in (5.6), so the parameters for $Q(c, d)^\eta$ can be read off directly from $\eta$. This also shows that the opposite vertex of $o$ in the base face of $Q(c, d)$ becomes the vertex of the base edge of $Q(c, d)^\eta$ distinct from the base vertex $o$. The six edges of $Q(c, d)^\eta$ at $o$ are thus the diagonals containing $o$, of the six faces of $Q(c, d)$ at $o$.

Finally, we know from the above that we can recover $Q(c, d)$ from $Q(c, d)^\eta$ by twisting with the generator $B_1(c, d)$ of $H(c, d)$, that is, by applying $\kappa$. Note that $B_1(c, d)$ is not contained in the group of $Q(c, d)^\eta$, so the latter is indeed of index 2 in $H(c, d)$. In fact, the image $B_1(c, d)'$ of $B_1(c, d)$ in the special group of $H(c, d)$ is given by

$$B_1(c, d)' \colon\ (\xi_1, \xi_2, \xi_3) \ \mapsto\ (-\xi_1, \xi_3, -\xi_2)$$

(see (6.2)), so in particular does not belong to $[3, 3]^*$, with $[3, 3]^*$ as in (4.1); but the latter is just the special group of $Q(c, d)^\eta$.

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