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Compatibly involutive residuated lattices and the Nelson identity

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Abstract

Nelson's constructive logic with strong negation **N3** can be presented (to within definitional equivalence) as the axiomatic extension $\mathbf{NInFL}_{ew}$ of the involutive full Lambek calculus with exchange and weakening by the *Nelson axiom*

$$\vdash \big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x))\big) \Rightarrow (x \Rightarrow y). \qquad (\text{Nelson}_{\vdash})$$

The algebraic counterpart of $\mathbf{NInFL}_{ew}$ is the recently introduced class of *Nelson residuated lattices*. These are commutative integral bounded residuated lattices $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ that: (i) are *compatibly involutive* in the sense that ${\sim}{\sim} a = a$ for all $a \in A$, where ${\sim} a := a \Rightarrow 0$, and (ii) satisfy the *Nelson identity*, namely the algebraic analogue of ($\text{Nelson}_{\vdash}$), *viz.*

$$\big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)\big) \approx x \Rightarrow y. \qquad (\text{Nelson})$$

The present paper focuses on the role played by the Nelson identity in the context of compatibly involutive commutative integral bounded residuated lattices. We present several characterisations of the identity (Nelson) in this setting, which variously permit us to comprehend its model-theoretic content from order-theoretic, syntactic, and congruence-theoretic perspectives. Notably, we show that a compatibly involutive commutative integral bounded residuated lattice $\mathbf{A}$ is a Nelson residuated lattice iff for all $a, b \in A$, the congruence condition

$$\Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(0, b) \quad \text{and} \quad \Theta^{\mathbf{A}}(1, a) = \Theta^{\mathbf{A}}(1, b) \quad \text{implies} \quad a = b$$

holds. This observation, together with others of the main results, opens the door to studying the characteristic property of Nelson residuated lattices (and hence Nelson's constructive logic with strong negation) from a purely abstract perspective.

Keywords Nelson algebra · Nelson logic · Compatibly involutive residuated lattices · Congruence orderable · Fregean

Dedicated to Lluís Godo on the occasion of his 60th birthday.

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1 Introduction

The starting point for the present paper is our previous work (Nascimento et al. 2018a, b) devoted to the logic **S** of strong negation introduced by Nelson (1959). There we established that **S** is the axiomatic extension of the full Lambek calculus with exchange and weakening by the axioms of *double negation* and (3, 2)*-contraction*, *viz.*

$$\vdash {\sim}{\sim} x \Rightarrow x$$
$$\vdash \big(x \Rightarrow (x \Rightarrow (x \Rightarrow y))\big) \Rightarrow \big(x \Rightarrow (x \Rightarrow y)\big).$$

In view of results due to Spinks and Veroff (2008a, b) and Busaniche and Cignoli (2010), Nelson's well-known constructive logic with strong negation **N3** (Nelson 1949; Rasiowa 1974; Sendlewski 1984; Vakarelov 1977) is in turn precisely the axiomatic extension of **S** (to within definitional equivalence) by the *Nelson axiom*

$$\vdash \big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x))\big) \Rightarrow (x \Rightarrow y). \quad (\text{Nelson}_{\vdash})$$

In Nascimento et al. (2018a, b), we further showed that **S** is algebraisable and characterised its algebraic counterpart as the variety of *compatibly involutive 3-potent commutative integral residuated lattices* (dubbed for short as **S**-*algebras* in Nascimento et al. 2018a, b). In consequence, the algebraic counterpart of Nelson's logic **N3** is, up to term equivalence, precisely the subvariety of **S**-algebras that additionally satisfies the algebraic analogue of the axiom (Nelson$_{\vdash}$). By algebraisability, this is the identity

$$\big((x \Rightarrow (x \Rightarrow y)) \wedge ({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x))\big) \Rightarrow (x \Rightarrow y) \approx \mathbf{1}. \quad (1.1)$$

Busaniche and Cignoli (2010, Remark 2.1) observed that (1.1) is equivalent (over compatibly involutive commutative integral residuated lattices) to the following

$$\big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)\big) \approx x \Rightarrow y \quad (\text{Nelson})$$

which we shall take as our official version of the *Nelson identity*.

The present paper is the outgrowth of our interest in understanding the essential difference between the logics **S** and **N3** in a (universal) algebraic context; our main focus shall thus be on the meaning and role of the Nelson identity in the context of (3-potent) compatibly involutive commutative integral residuated lattices. In this endeavour, we have naturally been led to formulate more abstract order-theoretic/algebraic properties which go hand in hand, in our context, with the Nelson identity. We have thus, for instance, introduced generalisations of the *congruence orderable algebras* and the *Fregean varieties* of Idziak et al. (2009). The main interest in the approach we take is, in our opinion, the fact that it may open the way to further universal algebraic investigation beyond the context of Nelson's logics and even beyond residuated lattices.

The remainder of the paper is organised as follows. Having fixed terminology and notation in Preliminaries, in Sect. 2 we review the main results about Nelson algebras and (Nelson) residuated lattices that will be needed throughout the paper. In Sect. 3, we show how a (variant of a) well-known dualising construction for residuated structures turns out to be especially insightful when applied to compatibly involutive commutative integral residuated lattices. In Sect. 4, we turn our attention to Nelson algebras and Nelson residuated lattices, showing (on the one hand) that 3-potent compatibly involutive commutative integral residuated lattices are 'almost' Nelson residuated lattices, and hence, that (on the other hand) the distinctive model-theoretic properties of Nelson algebras arise solely from satisfaction of the Nelson axiom. In Sect. 5, we isolate a congruence property, generalising the well-known notion of congruence orderability, that is equivalent, over compatibly involutive commutative integral residuated lattices, to satisfaction of the Nelson identity. We prove this equivalence in Sect. 7 making use of a syntactic characterisation of Nelson residuated lattices introduced in Sect. 6. As mentioned earlier, our main results open the door to studying the characteristic property of Nelson algebras at a more general, purely algebraic level: these directions for further research are mentioned, in the form of open problems, in final Sect. 8.

Preliminaries. The set of natural numbers is denoted ω. Given a set A, the diagonal relation $\{\langle a, a\rangle : a \in A\}$ is denoted Δ_A and the universal relation A^2 is denoted ∇_A. For an equivalence relation θ on A, the equivalence class of $a \in A$ is denoted a/θ. We assume familiarity with the rudiments of general algebra and model theory, especially that part of first-order logic known as equational logic. For general algebraic background, see Burris and Sankappanavar (1981), Grätzer (2008), McKenzie et al. (1987). Algebras are denoted **A**, **B**, etc. Given an algebra **A**, the set of all congruences on **A** is denoted Con **A** and the principal congruence on **A** generated by $\{a, b\} \subseteq A$ is denoted $\Theta^{\mathbf{A}}(a, b)$. The set of all compact (i.e. finitely generated) congruences on **A** is denoted Cp **A**. Classes of algebras are denoted K, V, etc. Standard use is made in the paper of the class operators H, S, and P (see e.g. Burris and Sankappanavar 1981).

Throughout the paper, we work with algebraic languages possessing a binary operation symbol $\wedge$ (which will be interpreted, as usual, as the lattice meet operation on the algebras in question). For the sake of brevity, given terms s, t, we write $s \leqslant t$ as shorthand for the identity $s \approx s \wedge t$. Throughout the paper, we overload a number of symbols, most notably $\preceq, \rightarrow$, and $\Rightarrow$, repeatedly and without warning. Additionally, we sometimes make use of the symbol $\Rightarrow$ in a meta-theoretic context for convenience.

Let K be a class of similar algebras and let $\mathbf{A} \in \mathsf{K}$. A *constant term* of **A** or K is any nullary or constant unary term function, or, less precisely, the element of **A** (or of each member of K) that constitutes the range of such a function (Blok and Pigozzi 1994a, p. 551). A *constant* is a nullary fundamental operation. A class K of similar algebras is *pointed* if it has at least one constant term. Given an algebra **A**, elements $a, b \in A$ are said to be *residually distinct* if they have distinct images in every non-trivial homomorphic image of **A**; or, equivalently, if $\Theta^{\mathbf{A}}(a, b) = \nabla_A$ (Blok and Pigozzi 1994a, p. 551). A class K of similar algebras is *double-pointed* if it has at least two constant terms **c**, **d** that realise residually distinct elements c, d on every member **A** of K.

We assume familiarity with the fundamentals of abstract algebraic logic, especially the part concerning logics that are (strongly) algebraisable in the sense of Blok and Pigozzi (1989). For background on abstract algebraic logic, see Blok and Pigozzi (1989), Czelakowski (2001), Font et al. (2003); for particulars on Blok–Pigozzi algebraisable logics, see Blok and Pigozzi (1989, 2001) or Czelakowski (2001, Chapter 4, §6). Following Blok and Pigozzi (1989, Chapter 1), a *deductive system* is a pair $\langle \Lambda, \vdash \rangle$ where Λ is a language type and $\vdash$ is a finitary and substitution-invariant consequence relation over Λ; we identify *logics* with deductive systems. Logics are denoted by **L**, etc., with **S** reserved for the logic of strong negation of Nelson (1959). Unless otherwise specified, all deductive systems considered in the sequel are Blok–Pigozzi (finitary and finitely) algebraisable.

2 Nelson algebras and Nelson residuated lattices

In this section, we recall some fundamental results about the classes of algebras that correspond (via algebraisability) to the various logics introduced by David Nelson, as well as about related residuated structures that are well known in the world of substructural logics (the main reference for these is Galatos et al. 2007). For our purposes, the most important result is the term equivalence between Nelson algebras and Nelson residuated lattices, first established in Spinks and Veroff (2008a, b); an alternative, more algebraically oriented proof can be found in Busaniche and Cignoli (2010). This equivalence ensures that we can view the algebraic counterpart of Nelson's constructive logic with strong negation **N3** as either a class of De Morgan algebras structurally enriched with a certain "weak" implication ($\to$), i.e. *Nelson algebras*, or as a class of lattices having a residuated implication ($\Rightarrow$) and an involutive negation ($\sim$) satisfying the Nelson identity (i.e. *Nelson residuated lattices*). We shall work mainly with the latter, as this will allow us to take advantage of several results and techniques that have been introduced within the study of residuated structures.

Over the course of some four decades of investigations into the notion of constructible falsity, David Nelson introduced a number of deductive systems of non-classical logic that have aroused considerable interest in the (algebraic) logic community. The oldest (1949) and most well known among them is his *constructive logic with strong negation* **N3** (Nelson 1949; Rasiowa 1974; Sendlewski 1984; Vakarelov 1977), often referred to as just *Nelson's logic*. In 1984, Nelson and Almukdad introduced (though other authors had independently considered it earlier) a paraconsistent weakening of **N3**, called *(Nelson's) paraconsistent constructive logic with strong negation* **N4** (Almukdad and Nelson 1984; Odintsov 2003, 2004, 2008). Both **N3** and **N4** are conservative axiomatic expansions of the negation-free fragment of the intuitionistic propositional calculus (Rasiowa 1974, Chapter X) by a unary logical connective $\sim$ of strong negation.

The logic **S** mentioned at the beginning of Sect. 1 also originates with Nelson's investigations, though it received almost no attention in the literature until our recent papers (Nascimento et al. 2018a, b). As said earlier, we proved there that **N3** can be viewed as the extension of **S** by the axiom (Nelson$_\vdash$); it was already well known that **N3** is the axiomatic extension of **N4** by the *ex contradictione quodlibet* law $\vdash x \to (\sim x \to y)$. To complete the picture, we have also shown that **S** and **N4** are incomparable and that the least logic extending both is precisely **N3**. Since all three logics are algebraisable, these considerations entail that we can introduce the algebraic counterpart of **N3** (the class of *Nelson algebras*) as a subclass of either **N4**-*lattices* or **S**-*algebras* (the algebraic counterparts of, respectively, **N4** and **S**): this is precisely the above-mentioned alternative of viewing Nelson algebras as De Morgan algebras structurally enriched with a weak implication or as a class of lattices having a residuated implication and an involutive negation. In what follows, we shall first take the former perspective.

Nelson algebras. Recall from lattice theory (Balbes and Dwinger 1974; Kalman 1958; Pynko 1999) that a *De Morgan lattice* is an algebra $\langle A; \wedge, \vee, \sim \rangle$ where $\langle A; \wedge, \vee \rangle$ is a distributive lattice and $\sim$ is a unary operation such that the following identities are satisfied:

$$\sim\sim x \approx x, \qquad \sim(x \wedge y) \approx \sim x \vee \sim y, \qquad \text{and} \qquad \sim(x \vee y) \approx \sim x \wedge \sim y.$$

A *De Morgan algebra* is an algebra $\langle A; \wedge, \vee, \sim, 0, 1 \rangle$ of type $\langle 2, 2, 1, 0, 0 \rangle$ where $\langle A; \wedge, \vee, \sim \rangle$ is a De Morgan lattice and $\langle A; \wedge, \vee, 0, 1 \rangle$ is a bounded lattice. It is well known that the class of all De Morgan lattices, hence De Morgan algebras, is equationally definable.

An algebra $\mathbf{A} = \langle A; \wedge, \vee, \to, \sim \rangle$ of type $\langle 2, 2, 2, 1 \rangle$ is an **N4**-*lattice* (Odintsov 2003, Definition 5.1) if:

(N1) The reduct $\langle A; \wedge, \vee, \sim \rangle$ is a De Morgan lattice with lattice ordering $\leq$.
(N2) The relation $\preceq$ on A defined for all $a, b \in A$ by $a \preceq b$ iff $a \to b = (a \to b) \to (a \to b)$ is a quasiorder (i.e. is reflexive and transitive) on A.

(N3) The relation $\Xi := \preceq \cap (\preceq)^{-1}$ is a congruence on the reduct $\langle A; \wedge, \vee, \rightarrow\rangle$, and the quotient algebra $\langle A; \wedge, \vee, \rightarrow\rangle/\Xi$ is an implicative lattice.[1]
(N4) For all $a, b \in A$, it holds that $\sim(a \rightarrow b) \equiv a \wedge \sim b \pmod{\Xi}$.
(N5) For all $a, b \in A$, it holds that $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$.

Although it is not quite obvious from this definition, the class of **N4**-lattices forms a variety (Odintsov 2003, Theorem 6.3).

An **N4**-lattice $\mathbf{A}$ is a *Nelson algebra* if, in addition to (N1)–(N5) above, it satisfies the following condition:

(N6) For all $a, b \in A$, it holds that $a \wedge \sim a \preceq b$.

The lattice reduct of an **N4**-lattice $\mathbf{A}$ need not be bounded. However, if $\mathbf{A}$ is a Nelson algebra, then $\mathbf{1} := x \rightarrow x$ defines a constant term in $\mathbf{A}$ that realises the top element 1 of its lattice reduct (consequently, $\mathbf{0} := \sim(x \rightarrow x)$ defines a constant term that induces the bottom element 0 of the lattice). Thus, every Nelson algebra has a term-definable De Morgan algebra reduct. The converse also holds: if $x \rightarrow x$ defines a constant term, then $\mathbf{A}$ is Nelson algebra; therefore, (N6) above could be equivalently replaced by the requirement that $a \rightarrow a = b \rightarrow b$ for all $a, b \in A$. For proofs of these and related statements, see (Spinks and Veroff, Chapter 1).

We now turn our attention to residuated structures that are, as mentioned earlier, the other main perspective from which one can view and study (the algebraic semantics of) logics in the Nelson family. The principal insight is that Nelson algebras can be presented (to within term equivalence) as a class of residuated lattices (the standard algebraic counterpart of the so-called substructural logics) that satisfies certain additional properties (commutativity, integrality, etc.) that we proceed to define below.

Commutative integral residuated lattices. Let $\langle A; \leq\rangle$ be a poset. A binary operation $*$ on A is *compatible* with $\leq$ if, for all $a, b, c \in A$, it holds that

$$\text{if } a \leq b, \text{ then } a * c \leq b * c \text{ and } c * a \leq c * b. \qquad \text{(Compat)}$$

A structure $\langle A; *, \Rightarrow, 1; \leq\rangle$ of type $\langle 2, 2, 1; 2\rangle$ is called a *partially ordered commutative residuated integral monoid* (briefly, pocrim) (Blok and Raftery 1997) if: (i) $\langle A, \leq\rangle$ is a poset with greatest element $1 \in A$; (ii) $\langle A; *, 1\rangle$ is a monoid whose product $*$ is compatible with $\leq$; and (iii) for all $a, b, c \in A$, it holds that

$$a * b \leq c \text{ iff } a \leq b \Rightarrow c. \qquad \text{(Res)}$$

[1] Implicative lattices, also called *Brouwerian lattices* or *generalised Heyting algebras* in the literature, are precisely the 0-free subreducts of Heyting algebras $\langle A; \wedge, \vee, \rightarrow, 0, 1\rangle$.

Observing that for every pocrim $\mathbf{A}$ and all $a, b, c \in A$, we have:

$$a \leq b \text{ iff } a \Rightarrow b = 1.$$
$$\text{If } a \leq b, \text{ then } b \Rightarrow c \leq a \Rightarrow c \text{ and } c \Rightarrow a \leq c \Rightarrow b. \qquad (2.1)$$

(We make implicit use of all unnumbered displayed expressions in the sequel.) Observe also that every pocrim satisfies the identities

$$\mathbf{1} \Rightarrow x \approx x \text{ and } x \Rightarrow \mathbf{1} \approx \mathbf{1}$$
$$x \Rightarrow (y \Rightarrow z) \approx (x * y) \Rightarrow z. \qquad (2.2)$$

A *commutative integral residuated lattice* is a structure $\langle A; \wedge, \vee, *, \Rightarrow, 1; \leq\rangle$ of type $\langle 2, 2, 2, 2, 0; 2\rangle$ such that: (i) $\langle A; \wedge, \vee\rangle$ is a lattice with lattice order $\leq$, and (ii) $\langle A; *, \Rightarrow, 1; \leq\rangle$ is a pocrim. Because $\leq$ is a lattice order, it is equationally definable; thus, up to first-order definitional equivalence, the relation $\leq$ may be elided in the signature and commutative integral residuated lattices may be treated as pure *algebras* of the form $\langle A; \wedge, \vee, *, \Rightarrow, 1\rangle$. We adopt this perspective in the sequel and abbreviate the term "commutative integral residuated lattice" by CIRL. It is well known that CIRLs satisfy the identity

$$(x \vee y) * z \approx (x * z) \vee (y * z). \qquad (2.3)$$

Filters and congruences. A non-empty subset F of a CIRL $\mathbf{A}$ is said to be a *filter* if for all $a, b \in A$ it holds that: (i) $a \leq b$ and $a \in F$ implies $b \in F$, and (ii) $a, b \in F$ implies $a * b \in F$. Every filter of a CIRL is a lattice filter (in the usual sense) of the underlying lattice, but not every lattice filter is a filter in the above sense because condition (ii) may fail to hold. On the other hand, filters correspond exactly to the implicative or "deductive" filters usually studied by algebraic logicians; these are the subsets $F \subseteq A$ satisfying: (i) $1 \in F$, and (ii) if $a, a \Rightarrow b \in F$, then $b \in F$.

Proposition 2.1 (Ono 2010, Proposition 3.6) *Let $\mathbf{A}$ be a CIRL, $F \subseteq A$ a filter, and θ a congruence on $\mathbf{A}$. The following statements hold:*

1. *The relation θ_F defined for all $a, b \in A$ by $(a, b) \in \theta_F$ iff $a \Rightarrow b, b \Rightarrow a \in F$ is a congruence on $\mathbf{A}$.*
2. *The subset F_θ defined for all $a \in A$ by $a \in F_\theta$ iff $(a, 1) \in \theta$ is a filter of $\mathbf{A}$.*
3. *The natural maps induced by (1) and (2) above are mutually inverse and establish an isomorphism between the lattice of all filters of $\mathbf{A}$ and the lattice of all congruences on $\mathbf{A}$.* □

Let S be a non-empty subset of a CIRL $\mathbf{A}$. Then, the set $\{b \in A : a_1 * \cdots * a_k \leq b$, for some $a_1, \ldots, a_k \in S\}$ is a filter, namely the filter generated by S; we write $[S)$ for the filter generated by S. In particular, the filter $[a)$ generated by a singleton set $\{a\} \subseteq A$ is the set
$\{b \in A : \underbrace{a * \cdots * a}_{k \text{ times}} \leq b, \text{ for some positive integer } k\}$.

Corollary 2.2 (Kowalski 2004, Proposition 1.2; Ono 2010, Lemma 4.1) *A CIRL $\mathbf{A}$ is subdirectly irreducible iff there exists an element $a < 1$ such that for every $b < 1$ there exists a positive integer m for which $\underbrace{b * \cdots * b}_{m \text{ times}} \leq a$ holds.* □

$k + 1$-potency and EDPC. As Proposition 2.1 and Corollary 2.2 suggest, expressions of the form $\underbrace{a * \cdots * a}_{k \text{ times}}$ play an important role in the theory of commutative (integral) residuated lattices. This leads to the following definition. For each integer $k \geq 0$, consider the unary $\{*\}$-terms x^k defined recursively by $x^0 := 1$ and $x^{n+1} := x^n * x$ when $0 \leq n \in \omega$. The following technical lemma is very useful in practice.

Lemma 2.3 (n-fold construction) *Let $\langle A; \leq\rangle$ be a partially ordered set and let $*$ be a binary operation on A such that (Compat) holds for all $a, b, c \in A$. Then, for all $a, b \in A$ and every $n \geq 1$, if $a \leq b$, then $a^n \leq b^n$.*

Proof By induction on n. Suppose $a \leq b$. For the basis case, we have $a^1 = a \overset{\text{(Hyp.)}}{\leq} b = b^1$. Assume now that the induction hypothesis holds for some $m \geq 1$. By the induction hypothesis, $a^m \leq b^m$, so by (Compat), $a * a^m \leq a * b^m$. Also, $a \leq b$ by assumption, so by (Compat) again, $a * b^m \leq b * b^m$. By transitivity, $a * a^m \leq b * b^m$, which is to say $a^{m+1} \leq b^{m+1}$. By induction, we conclude that $a^{n+1} \leq b^{n+1}$ for every $n \geq 1$. □

Going forward, by the phrase 'n-fold construction with $n = k$', we shall mean the application of Lemma 2.3 to deduce $a^k \leq b^k$ given $a \leq b$, for some $k \geq 1$. When $n = 2$, we simply refer to the 'doubling construction' rather than 'the n-fold construction with $n = 2$'.

Given $k \in \omega$, an element a of a CIRL $\mathbf{A}$ is said to be $k+1$-*potent* if $a^{k+1} = a^k$. The algebra $\mathbf{A}$ is said to be $k+1$-*potent* if it satisfies the identity

$$x^{k+1} \approx x^k. \qquad (k + 1\text{-potency})$$

Clearly, the class E_k^* of all CIRLs satisfying ($k + 1$-potency) is equationally definable. The following identities are easily seen to hold over $k+1$-potent CIRLs using integrality, $k+1$-potence, and induction:

$$\mathbf{1}^k \approx \mathbf{1} \qquad (x * y)^k \approx x^k * y^k$$
$$x^k \leqslant x \qquad x^j \approx x^k \qquad \text{for every } j \geq k.$$

For each integer $k \geq 0$, consider the binary $\{\Rightarrow\}$-terms $x \overset{k}{\Rightarrow} y$ defined recursively by $x \overset{0}{\Rightarrow} y := y$ and $x \overset{n+1}{\Rightarrow} y := x \Rightarrow (x \overset{n}{\Rightarrow} y)$ when $0 \leq n \in \omega$. An easy proof by induction shows that for every $k \in \omega$, the identity

$$x \overset{k}{\Rightarrow} y \approx x^k \Rightarrow y \qquad (2.4)$$

holds over CIRLs (Ferreirim 1992, Lemma 1.21(i)). For $k \in \omega$, it is well known (Ferreirim 1992, Lemma 1.21(ii)) that a CIRL $\mathbf{A}$ is $k+1$-potent iff it satisfies $x \overset{k+1}{\Rightarrow} y \approx x \overset{k}{\Rightarrow} y$; the variety of all CIRLs satisfying ($k + 1$-potency) can thus be alternatively presented as the class of all CIRLs satisfying this last identity.

It turns out that $k + 1$-potency has profound consequences for CIRLs. In more detail, recall from Baldwin and Berman (1975) that a class K of similar algebras has *definable principal congruences* (DPC) iff there exists a formula $\varphi(x, y, z, w)$ in the first-order language of K (whose only free variables are x, y, z, w) such that for every $\mathbf{A} \in \mathsf{K}$ and all $a, b, c, d \in A$, $c \equiv d \pmod{\Theta^{\mathbf{A}}(a, b)}$ iff $\mathbf{A} \models \varphi[a, b, c, d]$. When $\varphi(x, y, z, w)$ can be taken as a conjunction (*viz.*, finite set) of equations, then K is said to have *equationally definable principal congruences* (EDPC) (Köhler and Pigozzi 1980).

Theorem 2.4 (Kowalski 2004, Theorem 2.1) *For a variety V of CIRLs t. f. a. e.:*

1. *V has DPC.*
2. *V has EDPC.*
3. *$\mathsf{V} \models$ ($k + 1$-potency) for some $k \in \omega$.*
4. *$\mathsf{V} \subseteq \mathsf{E}_k^*$ for some $k \in \omega$.* □

Of the several intrinsic characterisations of EDPC that have been given in the literature, we shall be most interested in the following. Let $\mathbf{A} = \langle A; \vee, 0\rangle$ be a join semilattice with least element 0. If $a, b \in A$, the *dual relative pseudocomplement* of b with respect to a is the smallest element c, if it exists, having the property that $a \leq b \vee c$; the element c is denoted $b * a$. The semilattice $\mathbf{A}$ is said to be *dually relatively pseudocomplemented* (Köhler and Pigozzi 1980) if $b * a$ exists for all $a, b \in A$. A *dual Brouwerian semilattice* is a dually relatively pseudocomplemented semilattice in which the operation $*$ is distinguished.

Let V be a variety with EDPC. By Köhler and Pigozzi (1980, Theorem 5), for every $\mathbf{A} \in \mathsf{V}$ the join semilattice $\langle \mathrm{Cp}\,\mathbf{A}; \vee, \Delta_A\rangle$ of compact congruences on $\mathbf{A}$ is dually relatively pseudocomplemented. Conversely, if V is a variety such that $\langle \mathrm{Cp}\,\mathbf{A}; \vee, \Delta_A\rangle$ is dually relatively pseudocomplemented for every $\mathbf{A} \in \mathsf{V}$, then V has EDPC (Köhler and Pigozzi 1980, Theorem 8).

We conclude this section with the following observation. In a $k+1$-potent CIRL $\mathbf{A}$, for every $a \in A$, we can assume w.l.o.g. that the filter generated by a is given by:

$$[a) = \{b \in A : a^k \le b\}.$$

Indeed, suppose $c \in [a)$. Then, $a^j \le c$ for some positive integer j. If $j < k$, then $a^k \le a^j \le c$ by integrality, whence $a^k \le c$. On the other hand, if $j \ge k$, then $a^j = a^k$ by $k+1$-potency, so again $a^k \le c$. It follows that $[a) = \{b \in A : a^k \le b\}$.

WBSO varieties. Recall from general algebra that a variety V with a constant term $\mathbf{c}$ is said to be $\mathbf{c}$-*regular* if, for every $\mathbf{A} \in \mathsf{V}$ and all $\theta_1, \theta_2 \in \operatorname{Con}\mathbf{A}$ with $c/\theta_1 = c/\theta_2$, it holds that $\theta_1 = \theta_2$. A point regular variety is a variety that is $\mathbf{c}$-regular for some constant term $\mathbf{c}$. It is well known that $\mathbf{c}$-regularity can be expressed as a Mal'cev-type condition: by a classic result of Fichtner (1970), V is $\mathbf{c}$-regular iff there exist binary terms $d_0(x,y), \ldots, d_{n-1}(x,y)$ in the language of V such that: (i) $\mathsf{V} \models d_i(x,x) \approx \mathbf{c}$ for all $i < n$, and (ii) $\mathsf{V} \models \text{and}_{i<n}\, d_i(x,y) \approx \mathbf{c}$ implies $x \approx y$.[2] See Gumm and Ursini (1984, Corollary 1.6).

A *variety of weak Brouwerian semilattices with* $\mathbf{c}$-*filter preserving operations* (briefly, a $\mathbf{c}$-*WBSO variety*) is a variety V with a constant term $\mathbf{c}$ and binary terms $\to_{\mathbf{c}}$, $\cdot_{\mathbf{c}}$, and $\Delta_{\mathbf{c}}$ in the language of V such that for every algebra $\mathbf{A} \in \mathsf{V}$, the following statements hold (Blok et al. 1984, Theorem 2.6):

(WBSO1) $\mathsf{HSP}(\mathbf{A})$ is $\mathbf{c}$-regular with witness term $\Delta_{\mathbf{c}}(x,y)$.
(WBSO2) The relation $\preceq_{\mathbf{c}}$ defined for all $a, b \in A$ by $a \to_{\mathbf{c}} b = c$ is a quasiorder on A.
(WBSO3) The relation $\Xi_{\mathbf{c}} := \preceq \cap (\preceq)^{-1}$ is a congruence on the reduct $\langle A; \cdot_{\mathbf{c}}, \to_{\mathbf{c}}, \mathbf{c}^{\mathbf{A}}\rangle$ and the quotient algebra $\langle A; \cdot_{\mathbf{c}}, \to_{\mathbf{c}}, \mathbf{c}^{\mathbf{A}}\rangle/\Xi_{\mathbf{c}}$ is an implicative semilattice.[3]
(WBSO4) The c-equivalence classes of the congruences on $\mathbf{A}$ are precisely the subsets of the form $\bigcup F$, where the set F is an implicative semilattice filter of $\langle A; \cdot_{\mathbf{c}}, \to_{\mathbf{c}}, \mathbf{c}^{\mathbf{A}}\rangle/\Xi_{\mathbf{c}}$.

We typically drop all instances of the subscript from $\to$, $\cdot$, Δ, $\preceq$, and Ξ when $\mathbf{c}$ is clear from context. The terms $\to$, $\cdot$, and Δ are called a *weak relative pseudocomplementation*, *weak meet*, and *Gödel equivalence* for V, respectively. In general, none of these terms need be unique; see Blok et al. (1984, p. 357).

The defining conditions for Nelson algebras immediately suggest that they form a $\mathbf{1}$-WBSO variety (compare (N2)–(N3) with (WBSO2)–(WBSO3)) with weak meet $x \wedge y$ and weak relative pseudocomplementation $x \to y$. This is indeed the case; see the discussion of Blok et al. (1984, pp. 357–358). Modulo the presentation of Nelson algebras as residuated structures alluded to in Introduction (and discussed in more detail below), this observation is an instance of the following more general lemma, which is part of the folklore.

Lemma 2.5 (cf. Nascimento et al. 2018b, Theorem 4.5) *A variety* V *of CIRLs is a* $\mathbf{1}$-*WBSO variety iff it is* $k+1$-*potent for some* $k \in \omega$. *If* V *is a* $\mathbf{1}$-*WBSO variety, then weak meet, weak relative pseudocomplementation, and Gödel equivalence terms for* V *are given, respectively, by:*

$$x \cdot y := x \wedge y, \qquad x \to y := x \overset{k}{\Rightarrow} y, \quad \textit{and}$$
$$x \Delta y := (x \Rightarrow y) \wedge (y \Rightarrow x).$$

□

In the context of $k+1$-potent CIRLs, $k \in \omega$, we shall be especially interested in the quasiordering $\preceq$ induced by the derived operation $\to$, and the interaction of $\preceq$ with $\le$, $\Rightarrow$, and $\to$. To this end, we gather together some rules for calculating with $\to$ and $\preceq$. But first, the following auxiliary lemma, which generalises (Busaniche and Cignoli 2010, Lemma 3.3) to the setting of $k+1$-potent CIRLs.

Lemma 2.6 *Let* $\mathbf{A}$ *be a* $k+1$-*potent CIRL,* $k \in \omega$. *Then,* $\mathbf{A}$ *satisfies the identity*

$$(x^k \Rightarrow y^k)^k \approx (x^k \Rightarrow y)^k. \tag{2.5}$$

Proof Let $a, b \in A$. From $a^k \le a$ and (2.1), we have $b^k \Rightarrow a^k \le b^k \Rightarrow a$, so by the n-fold construction with $n = k$, $(b^k \Rightarrow a^k)^k \le (b^k \Rightarrow a)^k$. Conversely, $(b^k \Rightarrow a)^k \le b^k \Rightarrow a$ by integrality, whence $(b^k \Rightarrow a)^k * b^k \le a$ by (Res). That is to say, $\big((b^k \Rightarrow a) * b\big)^k \le a$, so by the n-fold construction with $n = k$, $\big(((b^k \Rightarrow a) * b)^k\big)^k \le a^k$. By $k+1$-potence, $\big((b^k \Rightarrow a) * b\big)^k \le a^k$, which is to say $(b^k \Rightarrow a)^k * b^k \le a^k$. By (Res), $(b^k \Rightarrow a)^k \le b^k \Rightarrow a^k$, so by the n-fold construction with $n = k$ again, $\big((b^k \Rightarrow a)^k\big)^k \le (b^k \Rightarrow a^k)^k$. By $k+1$-potence once more, $(b^k \Rightarrow a)^k \le (b^k \Rightarrow a^k)^k$ as desired. Hence, $\mathbf{A} \models$ (2.5). □

Lemma 2.7 *Let* $\mathbf{A}$ *be a* $k+1$-*potent CIRL,* $k \in \omega$. *The relation* $\preceq$ *defined for all* $a, b \in A$ *by* $a \preceq b$ *iff* $a \to b = 1$ *is a quasiorder on* A. *For all* $a, b, c \in A$, *the following statements hold:*

1. $a \preceq b$ *iff* $a^k \Rightarrow b = 1$ *iff* $a^k \le b$ *iff* $a^k \le b^k$.

[2] We avoid using $\wedge, \vee, \to, \Rightarrow$, etc., as logical symbols of the first-order language with equality $\Lambda[\text{FOL}, \approx]$, as determined by the algebraic language type Λ, since these symbols are employed extensively to denote connective symbols of the algebraic language types considered throughout the paper.

[3] Implicative semilattices, also called *Brouwerian semilattices* in the literature, are precisely the $\langle\to,\wedge\rangle$-subreducts of Heyting algebras $\langle A; \wedge, \vee, \to, 0, 1\rangle$.

2. $a \preceq 1$; *moreover,* $a/\Xi = \{1\}$.
3. *If* $a \leq b$*, then* $a \preceq b$.
4. *If* $a \leq b$*, then* $b \to c \leq a \to c$ *and* $c \to a \leq c \to b$.
5. *If* $a \preceq b$*, then* $b \to c \preceq a \to c$ *and* $c \to a \preceq c \to b$.

Proof The first assertion is clear in view of the preceding discussion. For the remaining statements,

1. We have $a \preceq b$ iff $a^k \Rightarrow b = 1$ (by (2.4)) iff $a^k \leq b$. Suppose $a^k \leq b$. By the n-fold construction with $n = k$, $(a^k)^k \leq b^k$, whence $a^k \leq b^k$ by $k+1$-potence. Conversely, suppose $a^k \leq b^k$. By integrality, $b^k \leq b$ and so $a^k \leq b$.
2. Observe $a \to 1 = a^k \Rightarrow 1 = 1$, so $a \preceq 1$. Suppose $1 \preceq a$. Then, $a = 1 \Rightarrow a = 1^k \Rightarrow a = 1 \to a = 1$. Hence $a/\Xi = \{1\}$.
3. Suppose $a \leq b$. By the n-fold construction with $n = k$, $a^k \leq b^k$. Hence, $a \preceq b$.
4. Suppose $a \leq b$. By the n-fold construction with $n = k$, we get $a^k \leq b^k$, whence $b^k \Rightarrow c \leq a^k \Rightarrow c$ by (2.1). By (2.4), $b \overset{k}{\Rightarrow} c \leq a \overset{k}{\Rightarrow} c$, which is to say $b \to c \leq a \to c$. Suppose $a \leq b$ again. By (2.1), $c^k \Rightarrow a \leq c^k \Rightarrow b$, so by (2.4), $c \overset{k}{\Rightarrow} a \leq c \overset{k}{\Rightarrow} b$. Hence, $c \to a \leq c \to b$.
5. Suppose $a \preceq b$. Then, $a^k \leq b^k$. By (2.1), $b^k \Rightarrow c \leq a^k \Rightarrow c$, so by the n-fold construction with $n = k$, $(b^k \Rightarrow c)^k \leq (a^k \Rightarrow c)^k$. By (2.4), $(b \overset{k}{\Rightarrow} c)^k \leq (a \overset{k}{\Rightarrow} c)^k$, which is to say $b \to c \preceq a \to c$. Suppose $a \preceq b$ again. Then, $a^k \leq b^k$. By (2.1), $c^k \Rightarrow a^k \leq c^k \Rightarrow b^k$, so by the n-fold construction with $n = k$, we have $(c^k \Rightarrow a^k)^k \leq (c^k \Rightarrow b^k)^k$. By Lemma 2.6, $(c^k \Rightarrow a)^k \leq (c^k \Rightarrow b)^k$, whence $(c \overset{k}{\Rightarrow} a)^k \leq (c \overset{k}{\Rightarrow} b)^k$ by (2.4). Thus, $c \to a \preceq c \to b$. □

Remark 2.8 In the deductive systems naturally associated with the $k+1$-potent varieties of (compatibly involutive) CIRLs, $k \geq 1$, the formula $\varphi^k \Rightarrow \psi$ witnesses the deduction theorem (Galatos et al. 2007, Theorem 11.3). Thus, $x \to y$ is truly a conditional. □

Compatible involutions. A *commutative integral bounded residuated lattice* (CIBRL) is an algebra $\langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that: (i) $\langle A; \wedge, \vee, *, \Rightarrow, 1\rangle$ is a CIRL, and (ii) $0 \leq a$ for all $a \in A$. Without proof, we note the following elementary properties of CIBRLs. For each CIBRL $\mathbf{A}$ and for all $a, b \in A$,

$$0 = a * {\sim} a, \quad 1 = {\sim} 0 \text{ and } 0 = {\sim} 1, \quad \text{and } 0 \leq a.$$

The following lemma will also be useful for our congruence-theoretic study of the Nelson identity (see Sect. 5).

Lemma 2.9 *Let* $\mathbf{A}$ *be a CIBRL and* $a \in A$*. Then,* $\Theta^{\mathbf{A}}({\sim} a, 1) = \Theta^{\mathbf{A}}(a, 0)$.

Proof Let θ be a congruence on $\mathbf{A}$, and suppose that $({\sim} a, 1) \in \theta$. Then, $({\sim}{\sim} a, {\sim} 1) = ({\sim}{\sim} a, 0) \in \theta$, from which we have $({\sim}{\sim} a \vee a, 0 \vee a) = ({\sim}{\sim} a, a) \in \theta$ using that $a \leq {\sim}{\sim} a$. But then $(a, 0) \in \theta$ by transitivity of θ. Conversely, if $(a, 0) \in \theta$, then we immediately have $({\sim} a, {\sim} 0) = ({\sim} a, 1) \in \theta$. □

Let $\mathbf{A}$ be a CIBRL and consider the derived unary operation $\sim$ defined for all $a \in A$ by ${\sim} a := a \Rightarrow 0$. We say $\mathbf{A}$ is *compatibly involutive*[4] if $\sim$ is *self-inverting* in the sense that ${\sim}{\sim} a = a$ for all $a \in A$. On the other hand, a *compatibly involutive CIRL* is an algebra $\langle A; \wedge, \vee, *, \Rightarrow, {\sim}, 1\rangle$ of type $\langle 2, 2, 2, 2, 1, 0\rangle$ such that: (i) $\langle A; \wedge, \vee, *, \Rightarrow, 1\rangle$ is a CIRL, and (ii) the following identities are satisfied:

$${\sim}{\sim} x \approx x \quad \text{and} \quad x \Rightarrow {\sim} y \approx y \Rightarrow {\sim} x.$$

The identity ${\sim}{\sim} x \approx x$ is called the *law of double negation*. As mentioned early on in Sect. 1, the class of $\mathbf{S}$-algebras of Nascimento et al. (2018a, b) is precisely the variety of compatibly involutive 3-potent CIRLs. It is well known and easy to see that compatibly involutive CIRLs satisfy the following useful collection of identities and quasi-identities:

$$x \leqslant y \text{ implies } {\sim} y \leqslant {\sim} x \qquad (\sim\text{-Contra})$$

$$x \Rightarrow y \approx {\sim} y \Rightarrow {\sim} x \qquad (\Rightarrow\text{-Contra})$$

$$x \Rightarrow y \approx {\sim}(x * {\sim} y) \qquad (\Rightarrow, *\text{-Equiv})$$

$$x * y \approx {\sim}(x \Rightarrow {\sim} y). \qquad (*, \Rightarrow\text{-Equiv})$$

Every compatibly involutive CIRL $\mathbf{A}$ gives rise to a compatibly involutive CIBRL $\mathbf{A}^{\mathbf{0}}$ upon setting $0 := {\sim} 1$. Conversely, every compatibly involutive CIBRL induces a compatibly involutive CIRL $\mathbf{A}^{\sim}$ upon setting ${\sim} a := a \Rightarrow 0$. Moreover, the algebras $\mathbf{A}^{\mathbf{0}}$ and $\mathbf{A}^{\sim}$ are term equivalent (Galatos and Raftery 2004, Section 5). Modulo applications sensitive to changes in signature, it is therefore a matter of taste and convenience whether one works with compatibly involutive CIRLs or compatibly involutive CIBRLs. We employ both perspectives in the sequel, often simultaneously, with the particular choice of formulation driven by the immediate need at hand.

[4] In many texts, compatibly involutive commutative (integral) residuated lattices are simply referred to as involutive commutative (integral) residuated lattices. Here we follow the terminological conventions of Hsieh and Raftery (2007); note that in our previous work (Nascimento et al. 2018a, b) (and likewise in the earlier paper Galatos and Raftery 2004 of Galatos and Raftery) compatibly involutive commutative (integral) residuated lattices are called involutive commutative (integral) residuated lattices.

Lemma 2.10 *Let* $\mathbf{A}$ *be a compatibly involutive CIRL and let* $\mathbf{A}^-$ *be its CIRL reduct. Then,* $\mathrm{Con}\,\mathbf{A} = \mathrm{Con}\,\mathbf{A^0} = \mathrm{Con}\,\mathbf{A}^-$.

Proof By term equivalence, $\mathbf{A}$ and $\mathbf{A^0}$ have the same congruences. Also, $\mathbf{A^0}$ and $\mathbf{A}^-$ have the same congruences, since $\mathbf{A}^-$ is the algebra obtained from $\mathbf{A^0}$ upon deleting the nullary operation symbol $\mathbf{0}$ from the latter's type, and deleting a nullary operation symbol from the type of an algebra does not disturb the algebra's congruences. □

Owing to Lemma 2.10, characterisations of congruences on CIRLs (such as order- or filter-theoretic descriptions) extend to compatibly involutive CIRLs. In particular, for a $k+1$-potent compatibly involutive CIRL $\mathbf{A}$, it holds that $[a)_{\mathbf{A}} = [a)_{\mathbf{A}^-} = \{b \in A : a^k \leq b\}$. We will use this and similar observations repeatedly in the sequel without further warning.

We are now finally ready to introduce Nelson residuated lattices, which are the algebraic counterpart of Nelson's logic **N3**, viewed as the axiomatic extension $\mathbf{NInFL}_{ew}$ of the involutive full Lambek calculus with exchange and weakening by the Nelson axiom.

Nelson residuated lattices. A compatibly involutive CIRL $\langle A; \wedge, \vee, \Rightarrow, *, \sim\rangle$ is a *Nelson residuated lattice* (NRL) if it satisfies the Nelson identity:

$$\big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big(\sim y \Rightarrow (\sim y \Rightarrow \sim x)\big) \approx x \Rightarrow y. \tag{Nelson}$$

Clearly, the class of all Nelson residuated lattices forms a variety. The Nelson identity, which is one of the main objects of our interest in the present work, encodes (or, one might say, hides) a great deal of information about a compatibly involutive CIRL. In particular, we may observe that, as a consequence of Theorem 2.11 and Proposition 4.4, we know that the Nelson identity entails that the lattice reduct of a CIRL must be distributive (and hence a De Morgan lattice; *cf.* item (N1) from the definition of Nelson algebras) and 3-potent; see Corollary 4.3 and Proposition 4.10 in Sect. 4 for direct proofs of this. On the other hand, a compatibly involutive 3-potent CIRL need not satisfy the Nelson identity: in other words, an **S**-algebra in the sense of Nascimento et al. (2018a, b) need not be a NRL.

The connection between Nelson algebras and NRLs is established by the following result.

Theorem 2.11 (Spinks and Veroff 2008a, Theorem 1.1) *The variety of Nelson algebras and the variety of Nelson residuated lattices are term equivalent.* □

Given a Nelson algebra, the residuated pair $(*, \Rightarrow)$ is recovered by defining:

$$x * y := \sim(x \to \sim y) \vee \sim(y \to \sim x)$$

$$x \Rightarrow y := (x \to y) \wedge (\sim y \to \sim x),$$

while given a Nelson residuated lattice, the weak implication $\to$ is recovered via:

$$x \to y := x \Rightarrow (x \Rightarrow y).$$

Nelson algebras *qua* Nelson residuated lattices have been investigated in Busaniche and Cignoli (2010).

3 The dual of a compatibly involutive CIBRL

In this section, we introduce and exploit a construction that allows one to view a compatibly involutive CIBRL as what we shall call a compatibly involutive *dual* CIBRL. This can be seen as just an equivalent presentation of the "same" algebra or class of algebras in a different algebraic language, analogous to the presentation of Boolean algebras as Boolean rings or to that of MV algebras as certain lattice-ordered groups, and just like these constructions, it will afford additional insight into the structure of compatibly involutive CIRLs (and hence of Nelson algebras).

The 'horizontal' dual of a compatibly involutive CIBRL. The construction that we introduce is a variant of a well-known dualising construction that can be applied to MV algebras or, more generally, residuated lattices. Let $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a CIBRL. According to Galatos et al. (2007, Section 3.4.17) , the dual of $\mathbf{A}$ (with respect to $0 \in A$) is the algebra $\mathbf{A}^\partial_\nu = \langle A; \vee, \wedge, +, \div, 1, 0\rangle$ where for all $a, b \in A$,

$$a \wedge^\partial_\nu b := a \vee b, \qquad a +^\partial_\nu b := \sim a \Rightarrow b, \qquad 0^\partial_\nu := 1$$
$$a \vee^\partial_\nu b := a \wedge b, \qquad a \div^\partial_\nu b := \sim a * b, \qquad 1^\partial_\nu := 0.$$

The dual $\mathbf{A}^\partial_\nu$ acts 'vertically' on $\mathbf{A}$ in the sense that the underlying order $\leq^\partial_\nu$ of the lattice reduct of $\mathbf{A}^\partial_\nu$ is the dual of the order $\leq$ of the lattice reduct of $\mathbf{A}$. That is, $\langle A; \leq^\partial_\nu\rangle = \langle A; \geq\rangle$. From Galatos et al. (2007, Proposition 3.43), it follows readily that the dual $\mathbf{A}^\partial_\nu$ of a compatibly involutive CIBRL $\mathbf{A}$ is itself a compatibly involutive CIBRL with neutral element 0. The residuated pair $(+, \div)$ of $\mathbf{A}^\partial_\nu$ thus satisfies, for all $a, b, c \in A$,

$$a + b \leq^\partial_\nu c \text{ iff } a \leq^\partial_\nu b \div c \qquad \text{iff}$$
$$c \leq a + b \text{ iff } b \div c \leq a.$$

Here we shall be interested in a variant dualising construction closely related to the 'vertical' dual. We define the *dual* of a CIBRL $\mathbf{A}$ (with respect to $0 \in A$) to be the algebra

$\mathbf{A}_h^\partial = \langle A; \wedge, \vee, +, \dot{-}, 0, 1\rangle$ where for all $a, b \in A$,

$$a \wedge_h^\partial b := a \wedge b, \qquad a +_h^\partial b := {\sim} a \Rightarrow b, \qquad 0_h^\partial := 0$$
$$a \vee_h^\partial b := a \vee b, \qquad a \dot{-}_h^\partial b := a * {\sim} b, \qquad 1_h^\partial := 1.$$

This notion seems to be new, even as several closely related constructions have been considered in the literature; see especially (Paoli 2005, Section 2.2). Our dual $\mathbf{A}_h^\partial$ acts 'horizontally' on $\mathbf{A}$ inasmuch as the underlying order $\leq_h^\partial$ of the lattice reduct of $\mathbf{A}_h^\partial$ *coincides* with the underlying order $\leq$ of the lattice reduct of $\mathbf{A}$, while the operation $\dot{-}_h^\partial$ *reflects* the residual $\dot{-}_v^\partial$ in the sense that $a \dot{-}_h^\partial b = b \dot{-}_v^\partial a$ for all $a, b \in A$.

Let $\mathbf{A}$ be a CIBRL. In general, the operations $+$ and $\dot{-}$ on $\mathbf{A}_h^\partial$ are not related. However, if $\mathbf{A}$ is compatibly involutive, the pair $(+, \dot{-})$ of $\mathbf{A}_h^\partial$ is *dually* residuated in that it satisfies, for all $a, b, c \in A$,

$$c \leq_h^\partial a + b \text{ iff } c \dot{-} b \leq_h^\partial a \quad \text{iff}$$
$$c \leq a + b \text{ iff } c \dot{-} b \leq a.$$

In this case, $\mathbf{A}_h^\partial$ identically satisfies $\neg\neg a = a$, where for all $b \in A$, $\neg b := 1 \dot{-} b$.

A structure $\langle A; +, \dot{-}, 0; \leq\rangle$ of type $\langle 2, 2, 0; 2\rangle$ is called a *partially ordered commutative dually residuated integral monoid* (dual pocrim) (Blok and Raftery 1997; Higgs 1984) if: (i) $\langle A, \leq\rangle$ is a poset with least element $0 \in A$; (ii) $\langle A; +, 1\rangle$ is a monoid whose product $+$ is compatible with $\leq$; and (iii) for all $a, b, c \in A$, it holds that $c \leq a + b$ iff $c \dot{-} b \leq a$. A *commutative integral dually residuated lattice* (dual CIRL) is a structure $\langle A; \wedge, \vee, +, \dot{-}, 0; \leq\rangle$ of type $\langle 2, 2, 2, 2, 0; 2\rangle$ such that: (i) $\langle A; \wedge, \vee\rangle$ is a lattice with lattice order $\leq$, and (ii) $\langle A; +, \dot{-}, 0; \leq\rangle$ is a dual pocrim. As with CIBRLs, such structures are first-order definitionally equivalent to pure *algebras* $\langle A; \wedge, \vee, +, \dot{-}, 0\rangle$ and will be so treated.

A *commutative integral bounded dually residuated lattice* (dual CIBRL) is an algebra $\langle A; \wedge, \vee, +, \dot{-}, 0, 1\rangle$ of type $\langle 2, 2, 2, 2, 0, 0\rangle$ such that: (i) $\langle A; \wedge, \vee, +, \dot{-}, 0\rangle$ is a dual CIRL, and (ii) $a \leq 1$ for all $a \in A$. A dual CIBRL is *compatibly involutive* if $\neg\neg a = a$ for all $a \in A$. The next lemma, which is evident given the preceding definitions and discussion, put our presentation of the horizontal dual into context.

Lemma 3.1 *The dual $\mathbf{A}_h^\partial$ (with respect to $0 \in A$) of a compatibly involutive CIBRL $\mathbf{A}$ is a compatibly involutive dual CIBRL.* □

Lemma 3.1 ensures that, for a suitably modified notion of horizontal dual, we can also take the horizontal dual $\mathbf{A}_{h'}^\partial$ of a compatibly involutive dual CIBRL $\mathbf{A}$; in the compatibly involutive CIBRL $\mathbf{A}_{h'}^\partial$ that results, the non-trivial operations $*'$ and $\Rightarrow'$ yielding the residuated pair $(*', \Rightarrow')$ are obtained by setting $a *' b := a \dot{-} \neg b$ and $a \Rightarrow' b := \neg a + b$ for all $a, b \in A$.

Let $\mathbf{A}$ be a compatibly involutive CIBRL and $\mathbf{B}$ be a compatibly involutive dual CIBRL. Direct computation establishes $(\mathbf{A}_h^\partial)_{h'}^\partial = \mathbf{A}$ and $(\mathbf{B}_{h'}^\partial)_h^\partial = \mathbf{B}$. Indeed, we have:

Proposition 3.2 *Every compatibly involutive CIBRL $\mathbf{A}$ is term equivalent to its dual $\mathbf{A}_h^\partial$.*

Proof Evidently, $\mathbf{A}$ and $\mathbf{A}_h^\partial$ have the same n-ary term operations for every $n > 0$. Therefore, $\mathbf{A}$ and $\mathbf{A}_h^\partial$ are term equivalent. □

By exploiting the proof of Galatos et al. (2007, Proposition 3.43), it can similarly be shown that every compatibly involutive CIBRL $\mathbf{A}$ is term equivalent to $\mathbf{A}_v^\partial$. Since term equivalence is an equivalence relation on varieties (McKenzie et al. 1987, Section 4.12), the dualising algebras $\mathbf{A}_h^\partial$ and $\mathbf{A}_v^\partial$ are also term equivalent.

Given the preceding remarks, when dealing with compatibly involutive CIBRLs, it is a matter of taste and convenience whether one works with a compatibly involutive CIBRL $\mathbf{A}$ *simpliciter*, or with its dual $\mathbf{A}_h^\partial$, or with $\mathbf{A}_v^\partial$. It is traditional to work with compatibly involutive CIBRLs directly.[5] However, in this paper we shall profit by working with a compatibly involutive CIBRL $\mathbf{A}$ and its dual $\mathbf{A}_h^\partial$ *simultaneously*. In particular, we shall work primarily in the setting of compatibly involutive CI(B)RLs, while at the same time making free use of the terminology and notation of the 'horizontal' dual. Our use of the dual $\mathbf{A}_h^\partial$ rather than the more traditional $\mathbf{A}_v^\partial$ is not essential, but is driven by applications; the 'horizontal' dual helps to clarify the order- and congruence-theoretic descriptions of Nelson residuated lattices presented in the sequel. Going forward, by the dual of a compatibly involutive CIBRL we will always mean the 'horizontal' dual $\mathbf{A}_h^\partial$, unless otherwise specified.

Ideals and congruences. Since we are dualising, we now shift our attention to the dual of the notion of residuated lattice filter considered in the preceding section, i.e. to ideals. A non-empty subset I of a compatibly involutive CIRL $\mathbf{A}$ is an *ideal* if for all $a, b \in A$ it satisfies: (i) $a \leq b$ and $b \in I$ implies $a \in I$, and (ii) $a, b \in I$ implies $a+b \in I$. As with filters, each ideal of $\mathbf{A}$ is a lattice ideal of $\mathbf{A}$ (as well as of $\mathbf{A}_h^\partial$), but not all lattice ideals need to satisfy condition (ii) above. Let S be a non-empty subset of a compatibly involutive CIRL $\mathbf{A}$. Then, the set $\{b \in A : b \leq a_1 + \cdots + a_k$, for some $a_1, \ldots, a_k \in S\}$ is an ideal, namely the ideal generated by S; we write $(S]$ for the ideal generated by S. In particular, the ideal $(a]$ generated by a singleton set $\{a\} \subseteq A$ is the set $\{b \in A : b \leq \underbrace{a + \cdots + a}_{k \text{ times}}$, for some positive integer $k\}$. The next result is dual to Proposition 2.1.

[5] Probably, this is because CIBRLs are **1**-regular but not **0**-regular.

Proposition 3.3 (cf. Ono 2010, Proposition 3.6) *Let* $\mathbf{A}$ *be a compatibly involutive CIRL,* $I \subseteq A$ *an ideal, and* θ *a congruence on* $\mathbf{A}$*. The following statements hold:*

1. *The relation* θ_I *defined for all* $a, b \in A$ *by* $(a, b) \in \theta_I$ *iff* $a \dot{-} b, b \dot{-} a \in I$ *is a congruence on* $\mathbf{A}$*.*
2. *The subset* I_θ *defined for all* $a \in A$ *by* $a \in I_\theta$ *iff* $(a, 0) \in \theta$ *is an ideal of* $\mathbf{A}$*.*
3. *The natural maps induced by (1) and (2) above are mutually inverse and establish an isomorphism between the lattice of all ideals of* $\mathbf{A}$ *and the lattice of all congruences on* $\mathbf{A}$*.* □

Similarly, the next result is dual to that of Corollary 2.2.

Corollary 3.4 (Kowalski 2004, Proposition 1.2; Ono 2010, Lemma 4.1) *A compatibly involutive CIRL* $\mathbf{A}$ *is subdirectly irreducible iff there exists an element* $a > 0$ *such that for every* $b > 0$ *there exists a positive integer* m *for which* $a \leq \underbrace{b + \cdots + b}_{m \text{ times}}$ *holds.* □

Proposition 3.3 evidently entails that the variety of compatibly involutive CIRLs is $\mathbf{0}$-regular. Given a compatibly involutive CIRL $\mathbf{A}$ and $a, b \in A$, therefore, to see $\Theta^{\mathbf{A}}(0, a) \subseteq \Theta^{\mathbf{A}}(0, b)$ it suffices to show that $(a] \subseteq (b]$; or, using Lemma 2.9, that $[\sim b) \subseteq [\sim a)$.

We note in passing that Proposition 3.3 can be slightly sharpened. Indeed, recall from Gumm and Ursini (1984) that a variety V with a constant term $\mathbf{c}$ is $\mathbf{c}$-*subtractive* if there exists a binary term $s(x, y)$ in the language of V such that $\mathsf{V} \models s(x, x) \approx \mathbf{c}$ and $\mathsf{V} \models s(x, \mathbf{c}) \approx x$. An algebra $\mathbf{A}$ with a constant term $\mathbf{c}$ is said to be $\mathbf{c}$-*ideal determined* if every ideal I of $\mathbf{A}$ is the c-class of a unique congruence relation θ_I on $\mathbf{A}$, for a suitably defined syntactic notion of ideal (Gumm and Ursini 1984; Ursini 1972). By Gumm and Ursini (1984, Corollary 1.9), a variety V with a constant term $\mathbf{c}$ is $\mathbf{c}$-ideal determined iff it is $\mathbf{c}$-regular and $\mathbf{c}$-subtractive; here, V is $\mathbf{c}$-*ideal determined* if every $\mathbf{A} \in \mathsf{V}$ is $\mathbf{c}$-ideal determined.

Theorem 3.5 *The variety of compatibly involutive CIRLs is* $\mathbf{0}$*-ideal determined.*

Proof In view of previous remarks, it suffices to observe that the terms $\{x \dot{-} y, y \dot{-} x\}$ witness $\mathbf{0}$-regularity for the variety of compatibly involutive CIRLs, while the term $x \dot{-} y$ witnesses $\mathbf{0}$-subtractivity. □

Of course, the variety of (compatibly involutive) CIRLs is also $\mathbf{1}$-ideal determined; the terms $\{x \Rightarrow y, y \Rightarrow x\}$ and $\{x \Rightarrow y\}$ are witnesses to $\mathbf{1}$-regularity and $\mathbf{1}$-subtractivity, respectively. Given a CIRL $\mathbf{A}$ and $a, b \in A$, therefore, to see $\Theta^{\mathbf{A}}(1, a) \subseteq \Theta^{\mathbf{A}}(1, b)$ it suffices to show $[a) \subseteq [b)$; again, we use this and similar observations in the sequel without further comment.

$\overline{k+1}$-*potency.* For each integer $k \geq 0$, consider the (derived) unary $\{+\}$-terms $x^{\bar{k}}$ defined inductively by $x^{\bar{0}} := 0$ and $x^{\overline{n+1}} := x^{\bar{n}} + x$ when $0 \leq n \in \omega$. Recall also that a^k abbreviates $\underbrace{a * \cdots * a}_{k \text{ times}}$. In the context of compatibly involutive CIRLs, terms of the form $x^{\bar{k}}$ have been considered in Cignoli and Torrens (2012) and other places; here we recall only that for each compatibly involutive CIRL $\mathbf{A}$, all $a \in A$, and every $k \geq 0$, it holds that (Cignoli and Torrens 2012, Lemma 1.3)

$$a^{\bar{k}} = \sim((\sim a)^k). \tag{3.1}$$

Given $k \in \omega$, an element a of a compatibly involutive CIRL $\mathbf{A}$ is said to be $\overline{k+1}$-*potent* if $a^{\overline{k+1}} = a^{\bar{k}}$. The algebra $\mathbf{A}$ is said to be $\overline{k+1}$-*potent* if it satisfies the identity

$$x^{\overline{k+1}} \approx x^{\bar{k}}. \qquad (\overline{k+1}\text{-potency})$$

For a $\overline{k+1}$-potent compatibly involutive CIRL $\mathbf{A}$, the analogue of the term $x \to y$ is the term $x \doteq y := x \dot{-} y^{\bar{k}}$; the induced operation $\doteq$ plays a role in $\mathbf{A}_h^\partial$ similar to that played by $\to$ in $\mathbf{A}$. Thus, we can also consider the counterpart of $\preceq$ on $\mathbf{A}_h^\partial$, namely the relation $\preceq_h^\partial$ defined for all $a, b \in A$ by $a \preceq_h^\partial b$ iff $a \doteq b = 0$. The relation $\preceq_h^\partial$ is a quasiorder on $\mathbf{A}_h^\partial$ (hence $\mathbf{A}$) that enjoys properties similar to its counterpart $\preceq$; in particular, for all $a, b \in A$, it holds that $a \preceq_h^\partial b$ iff $b \leq a^{\bar{k}}$ iff $b^{\bar{k}} \leq a^{\bar{k}}$.

The next lemma clarifies the relationship between $k+1$-potency, $\overline{k+1}$-potency, the partial orders $\leq$ and $\leq_h^\partial$, and the quasiorders $\preceq$ and $\preceq_h^\partial$.

Lemma 3.6 *Let* $\mathbf{A}$ *be a compatibly involutive CIRL with dual* $\mathbf{A}_h^\partial$*. Then,* $\mathbf{A}$ *is* $k+1$*-potent iff* $\mathbf{A}_h^\partial$ *is* $\overline{k+1}$*-potent. Moreover, the following statements hold for all* $a, b \in A$*:*

1. $a \leq_h^\partial b$ *iff* $\sim b \leq \sim a$ *(iff* $a \leq b$*).*
2. $a \preceq_h^\partial b$ *iff* $\sim b \preceq \sim a$*.*

Proof Let $c \in A$. For the first assertion, suppose $\mathbf{A} \models x^{k+1} \approx x^k$. By hypothesis, $(\sim c)^{k+1} = (\sim c)^k$, so by two applications of ($\sim$-Contra), $\sim((\sim c)^{k+1}) = \sim((\sim c)^k)$. By (3.1), $c^{\overline{k+1}} = c^{\bar{k}}$. Hence $\mathbf{A}_h^\partial \models x^{\overline{k+1}} \approx x^{\bar{k}}$. The converse is similar.

(1) Obvious, by ($\sim$-Contra).
(2) It suffices to observe that $a \preceq_h^\partial b$ iff $a \leq b^{\bar{k}}$ iff $a \leq \sim((\sim b)^k)$ (by (3.1)) iff $(\sim b)^k \leq \sim a$ (by ($\sim$-Contra), as $\sim$ is self-inverting) iff $\sim b \preceq \sim a$.

□

Remark 3.7 Let V be a variety of $\overline{k+1}$-potent compatibly involutive CIRLs. Modulo our discussion of duals, it is easy to see from first principles that V is a **0**-WBSO variety in addition to being a **1**-WBSO variety. This may also be inferred from Blok et al. (1984, Theorem 3.6). Weak meet, weak relative pseudocomplementation, and Gödel equivalence terms for V are given by

$$x \cdot_{\mathbf{0}} y := x \vee y, \qquad x \rightarrow_{\mathbf{0}} y := y \dot{-} x^{\bar{k}}, \qquad \text{and}$$
$$x \Delta_{\mathbf{0}} y := (x \dot{-} y) \vee (y \dot{-} x).$$

It follows that every algebra $\mathbf{A} \in \mathsf{V}$ has the global outline or 'weak structure' of an implicative semilattice 'at 0'. To dispel any confusion, we explicitly point out here that the canonical quasiorder $\preceq_{\mathbf{0}}$ induced by $\rightarrow_{\mathbf{0}}$ is the *converse* of the relation $\preceq_h^{\partial}$; that is, $\langle A; \preceq_h^{\partial}\rangle = \langle A; \succeq_{\mathbf{0}}\rangle$. (*En passant*, this vindicates the claim that $\preceq_h^{\partial}$ is a quasiorder.) □

We conclude this section with the following observation. In a $k+1$-potent compatibly involutive CIRL **A**, for every $a \in A$, we can assume w.l.o.g. that:

$$(a] = \{b \in A : b \leq a^{\bar{k}}\}.$$

This follows by an argument analogous to the case for filters, since **A** is $\overline{k+1}$-potent iff **A** is $k+1$-potent.

4 An order-theoretic characterisation of NRLs

As mentioned towards the end of Sect. 2, a remarkable consequence of the Nelson identity is that Nelson residuated lattices are not only distributive but also 3-potent. This result, which has profound (perhaps not yet fully understood) consequences for the structure of such lattices, was first established in Busaniche and Cignoli (2010, Theorem 2.2). The proof given in Busaniche and Cignoli (2010), however, is not quite immediate; so here our first order of business shall be to show that 3-potency follows immediately from the Nelson identity.

Proposition 4.1 *Let* **A** *be a compatibly involutive CIRL. The following statements hold:*

1. $\mathbf{A} \models \big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)\big) \leqslant x \Rightarrow y$ *iff* $\mathbf{A} \models x * y \leqslant (x^2 * y) \vee (y^2 * x)$.
2. $\mathbf{A} \models x \Rightarrow y \leqslant \big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)\big)$ *and* $\mathbf{A} \models (x^2 * y) \vee (y^2 * x) \leqslant x * y$.
3. $\mathbf{A} \models$ *(Nelson) iff it satisfies the identity*

$$x * y \approx (x^2 * y) \vee (y^2 * x). \tag{4.1}$$

Proof We show only (1); both identities of (2) follow by integrality, while (3) follows from (1) and (2).

Suppose $\mathbf{A} \models \big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)\big) \leqslant x \Rightarrow y$ and let $a, b \in A$. By hypothesis,

$$\big(a \Rightarrow (a \Rightarrow {\sim} b)\big) \wedge \big({\sim}{\sim} b \Rightarrow ({\sim}{\sim} b \Rightarrow {\sim} a)\big) \leq a \Rightarrow {\sim} b,$$

so by (∼-Contra),

$$\begin{aligned} {\sim}(a \Rightarrow {\sim} b) \leq {\sim}\big(&(a \Rightarrow (a \Rightarrow {\sim} b)) \\ &\wedge (b \Rightarrow (b \Rightarrow {\sim} a))\big), \end{aligned} \tag{4.2}$$

as ∼ is self-inverting. But then,

$$\begin{aligned}
a * b &= {\sim}(a \Rightarrow {\sim} b) && \text{by } (*, \Rightarrow\text{-Equiv}) \\
&\leq {\sim}\big((a \Rightarrow (a \Rightarrow {\sim} b)) \wedge (b \Rightarrow (b \Rightarrow {\sim} a))\big) && \text{by } (4.2) \\
&= {\sim}\big(a \Rightarrow (a \Rightarrow {\sim} b)\big) \vee {\sim}\big(b \Rightarrow (b \Rightarrow {\sim} a)\big) \\
&= {\sim}{\sim}\big(a * {\sim}{\sim}(a * {\sim}{\sim} b)\big) \vee \\
&\quad {\sim}{\sim}\big(b * ({\sim}{\sim} b * {\sim}{\sim} a)\big) && \text{by } (*, \Rightarrow\text{-Equiv}) \\
&= \big(a * (a * b)\big) \vee \big(b * (b * a)\big) && \text{as } {\sim} \text{ is self-inverting.}
\end{aligned}$$

Hence $\mathbf{A} \models x * y \leqslant (x^2 * y) \vee (y^2 * x)$.

Conversely, suppose $\mathbf{A} \models x * y \leqslant (x^2 * y) \vee (y^2 * x)$. By hypothesis,

$$a * {\sim} b \leq (a^2 * {\sim} b) \vee \big(({\sim} b)^2 * {\sim} a\big),$$

so by (∼-Contra),

$${\sim}\big((a^2 * {\sim} b) \vee (({\sim} b)^2 * {\sim} a)\big) \leq {\sim}(a * {\sim} b), \tag{4.3}$$

whence

$$\begin{aligned}
&\big(a \Rightarrow (a \Rightarrow b)\big) \wedge \big({\sim} b \Rightarrow ({\sim} b \Rightarrow {\sim} a)\big) \\
&\quad = {\sim}\big(a * {\sim}{\sim}(a * {\sim} b)\big) \wedge \\
&\qquad {\sim}\big({\sim} b * {\sim}{\sim}({\sim} b * {\sim}{\sim} a)\big) && \text{by } (\Rightarrow, *\text{-Equiv}) \\
&\quad = {\sim}\big(a * (a * {\sim} b)\big) \wedge {\sim}\big({\sim} b * ({\sim} b * a)\big) && \text{as } {\sim} \text{ is self-inverting} \\
&\quad = {\sim}\big((a^2 * {\sim} b) \vee (({\sim} b)^2 * a)\big) \\
&\quad \leq {\sim}(a * {\sim} b) && \text{by } (4.3) \\
&\quad = a \Rightarrow b && \text{by } (\Rightarrow, *\text{-Equiv}).
\end{aligned}$$

Hence $\mathbf{A} \models \big(x \Rightarrow (x \Rightarrow y)\big) \wedge \big({\sim} y \Rightarrow ({\sim} y \Rightarrow {\sim} x)\big) \leqslant x \Rightarrow y$. □

Remark 4.2 The form of identity (4.1) suggests that a (cut-free) sequent calculus for constructive logic with strong negation *qua* a substructural logic may be obtained upon adjoining the structural rule

$$\frac{\Gamma, \Gamma, \Pi \vdash \Sigma, \Delta, \Delta \qquad \Gamma, \Pi, \Pi \vdash \Sigma, \Sigma, \Delta}{\Pi, \Gamma \vdash \Sigma, \Delta} \ (\mathrm{N})$$

to a sequent calculus for $\mathbf{FL}_{ew}$, the full Lambek calculus with exchange and weakening. This has been verified in Metcalfe (2009);[6] for additional remarks on the special '(3, 2)-contraction rule' (N), see Slaney et al. (1989, Section II, p. 9), Slaney (2004, Section 4, p. 289), and Spinks and Veroff (2010, Section 3). □

The next result is Busaniche and Cignoli (2010, Theorem 2.2); another proof is given in Kozak (2014, Fact 2.20).

Corollary 4.3 *Let* $\mathbf{A}$ *be a Nelson residuated lattice. Then,* $\mathbf{A}$ *is 3-potent.*

Proof By hypothesis, $\mathbf{A} \models$(4.1). On identifying the variables x and y, we get that $\mathbf{A} \models x^3 \approx x^2$. Hence $\mathbf{A}$ is 3-potent. □

The following characterisation of Nelson residuated lattices, which turns out to be very useful in practice, is a slight sharpening of a result due to Busaniche and Cignoli.

Proposition 4.4 (cf. Busaniche and Cignoli 2010, Corollary 3.8) *For a compatibly involutive CIRL* $\mathbf{A}$*, t.f.a.e.:*

1. $\mathbf{A}$ *is 3-potent and satisfies the quasi-identity:*

$$x^2 \approx y^2 \text{ and } (\sim x)^2 \approx (\sim y)^2 \text{ implies } x \approx y. \tag{4.4}$$

2. $\mathbf{A}$ *is a Nelson residuated lattice.*

Proof Suppose $\mathbf{A}$ is 3-potent and $\mathbf{A} \models$ (4.4). By 3-potency and Busaniche and Cignoli (2010, Corollary 3.8), $\mathbf{A}$ is a Nelson residuated lattice. Conversely, suppose $\mathbf{A}$ is a Nelson residuated lattice. By Corollary 4.3, $\mathbf{A}$ is 3-potent, whence $\mathbf{A} \models$ (4.4) by Busaniche and Cignoli (2010, Corollary 3.8) again. □

Let V be a $\mathbf{c}$-WBSO variety and let $\mathbf{A} \in \mathsf{V}$. It follows from (WBSO1)–(WBSO4) that the join semilattice $\langle \mathrm{Cp}\,\mathbf{A}; \vee, \Delta_A \rangle$ of compact congruences on $\mathbf{A}$ is dually relatively pseudocomplemented, and hence that the map $a \mapsto \Theta^{\mathbf{A}}(c, a)$ from A to $\mathrm{Con}\,\mathbf{A}$ is a homomorphism from $\langle A; \cdot, \rightarrow, \mathbf{c}^{\mathbf{A}} \rangle$ onto $\langle \mathrm{Cp}\,\mathbf{A}; \vee, *, \Delta_A \rangle$ with Ξ as its relation kernel (Blok et al. 1984, p. 354). The binary terms $\cdot$ and $\rightarrow$ thus realise, for every algebra $\mathbf{A} \in \mathsf{V}$, term operations that faithfully reflect, within the clone of the variety, the conjunction and implication operations of the dual Brouwerian semilattice of compact congruences.

[6] Numerous sequent calculi for constructive logic with strong negation have been proposed in the literature, including the systems of Almukdad and Nelson (1984), Gurevich (1977), Kutschera (1969), López-Escobar (1972), Thomason (1969), and Zaslavsky (1978). The sequent calculus of Metcalfe (2009) is distinguished among these systems in that—unlike any of the other calculi cited above—it does not contain rules acting on more than one connective at a time. See Kozak (2014, Footnote 3).

An algebra $\mathbf{A}$ in a $\mathbf{c}$-WBSO variety V thus possesses a global outline or 'weak structure' ('at c') closely resembling that of implicative semilattices. For the case of Nelson algebras and (more generally) $k+1$-potent (compatibly involutive) CIRLs, even more is true. As it turns out, both classes have a rather special character that is not shared by WBSO varieties in general. Let $x \wedge^k y$ and $x \xrightarrow{k} y$ abbreviate $(x \wedge y)^k$ and $(x \rightarrow y)^k$, respectively. Since for every $k+1$-potent CIRL $\mathbf{A}$ and all $a, b \in A$,

$$\Theta^{\mathbf{A}}(a \wedge^k b, 1) = \Theta^{\mathbf{A}}(a \wedge b, 1) \quad \text{and}$$
$$\Theta^{\mathbf{A}}(a \xrightarrow{k} b, 1) = \Theta^{\mathbf{A}}(a \rightarrow b, 1),$$

the terms $x \wedge^k y$ and $x \xrightarrow{k} y$ are a weak meet and weak relative pseudocomplementation for E^*_k, respectively (recall these terms need not be unique). The set $B = \{b^k : b \in A\}$ of 'open' elements then forms the universe of a subalgebra $\mathbf{B} = \langle B; \wedge^k, \xrightarrow{k}, 1 \rangle$ of the weak Brouwerian semilattice $\langle A; \wedge^k, \xrightarrow{k}, 1 \rangle$, and the mapping $a \mapsto a^k$ $(a \in A)$ is a retraction of $\langle A; \wedge^k, \xrightarrow{k}, 1 \rangle$ onto $\mathbf{B}$. Moreover, under the map $b \mapsto \Theta^{\mathbf{A}}(b, 1)$ $(b \in B)$ the algebra $\mathbf{B}$ is (dually) isomorphic to the dual Brouwerian semilattice $\langle \mathrm{Cp}\,\mathbf{A}; \vee, *, \Delta_A \rangle$. These observations generalise (Blok et al. 1984, p. 358); for proofs, see Spinks and Veroff (Chapter 4). See also Blok and Pigozzi (1994b, Theorem 2.23, Corollary 2.24) and the surrounding discussion.

The upshot of the preceding discussion is that a Nelson residuated lattice $\mathbf{A}$ actually includes, upon identifying b^2 with $\Theta^{\mathbf{A}}(b, 1)$, the (dual of) $\langle \mathrm{Cp}\,\mathbf{A}; \vee, *, 1 \rangle$ as a subalgebra of the term reduct $\langle A; \wedge^2, \xrightarrow{2}, 1 \rangle$. The 'intuitionistic' flavour of the weak structure $\langle A; \wedge, \vee, \rightarrow, 1 \rangle$ of a Nelson algebra is thus accounted for by the observation that Nelson residuated lattices are 3-potent, rather than by any properties peculiar to Nelson algebras.

We now reify the preceding discussion by showing that a compatibly involutive 3-potent CIRL (i.e. an $\mathbf{S}$-algebra in the sense of Nascimento et al. 2018a, b) is *almost* a Nelson algebra in that it satisfies (N1)–(N4) and (N6) from the definition of Nelson algebras given at the beginning of Sect. 2. But first, a lemma that is very useful for calculations.

Lemma 4.5 (Centripetal lemma) *Each 3-potent CIRL* $\mathbf{A}$ *satisfies the identities:*

$$(x \wedge y)^2 \approx (x^2 \wedge y^2)^2 \tag{$\wedge$-Centrip.}$$
$$(x \vee y)^2 \approx x^2 \vee (x * y) \vee y^2. \tag{$\vee$-Centrip.}$$

Proof Let $a, b \in A$.

($\wedge$-Centrip.): From $a^2 \leq a$ and $b^2 \leq b$ we get $a^2 \wedge b^2 \leq a \wedge b$. By the doubling construction, $(a^2 \wedge b^2)^2 \leq (a \wedge b)^2$. Conversely, from $a \wedge b \leq a, b$, we have $(a \wedge b)^2 \leq$

a^2, b^2 by two applications of the doubling construction, whence $(a \wedge b)^2 \leq a^2 \wedge b^2$. By the doubling construction, $\big((a \wedge b)^2\big)^2 \leq (a^2 \wedge b^2)^2$, whence $(a \wedge b)^2 \leq (a^2 \wedge b^2)^2$ by 3-potence. Hence $\mathbf{A} \models$ ($\wedge$-Centrip.).

($\vee$-Centrip.): Observe that $(a \vee b)^2 = (a \vee b) * (a \vee b) \overset{(2.3)}{=} \big((a \vee b) * a\big) \vee \big((a \vee b) * b\big) \overset{(2.3)}{=} (a * a) \vee (b * a) \vee (a * b) \vee (b * b) = a^2 \vee (a * b) \vee b^2$. Hence $\mathbf{A} \models$ ($\vee$-Centrip.). □

Lemma 4.6 *Every 3-potent compatibly involutive CIRL* $\mathbf{A}$ *satisfies the identities:*

$$x * y \leqslant (x \vee y)^2 \tag{4.5}$$
$$(x \vee y)^2 \approx x^2 \vee y^2 \tag{4.6}$$
$$(x \wedge {\sim} x)^2 \approx \mathbf{0}. \tag{4.7}$$

Proof Let $a, b \in A$.

For (4.5), it suffices to observe $a * b \leq a^2 \vee (a * b) \vee b^2 \overset{(\vee\text{-Centrip.})}{=} (a \vee b)^2$.

Identity (4.6) is (Nascimento et al. 2018b, Lemma 3.8).

For (4.7), from $a \wedge {\sim} a \leq a \vee {\sim} a$ we have $a \wedge {\sim} a \leq {\sim}(a \wedge {\sim} a)$ (by De Morgan's laws, as $\sim$ is self-inverting) $= (a \wedge {\sim} a) \Rightarrow 0$, so by (Res), $(a \wedge {\sim} a) * (a \wedge {\sim} a) \leq 0$. Hence $(a \wedge {\sim} a)^2 = 0$. □

A *Kleene algebra* is a De Morgan algebra that satisfies the identity

$$x \wedge {\sim} x \leqslant y \vee {\sim} y. \tag{4.8}$$

It is well known (Brignole and Monteiro 1967) that the De Morgan algebra reduct of a Nelson algebra is a Kleene algebra; this result generalises to 3-potent compatibly involutive CIRLs as follows.

Proposition 4.7 *Every 3-potent compatibly involutive CIRL* $\mathbf{A}$ *satisfies identity* (4.8). *Hence, if* $\mathbf{A}$ *is distributive, the De Morgan algebra term reduct* $\langle A; \wedge, \vee, {\sim}, 0, 1\rangle$ *is a Kleene algebra.*

Proof Let $a, b \in A$. Put $\alpha := a \wedge {\sim} a$ and $\beta := b \wedge {\sim} b$. Observe that

$$\begin{aligned}(a \wedge {\sim} a) * (b \wedge {\sim} b) &= \alpha * \beta \\ &\overset{(4.5)}{\leq} (\alpha \vee \beta)^2 \overset{(4.6)}{=} \alpha^2 \vee \beta^2 = (a \wedge {\sim} a)^2 \\ &\quad \vee (b \wedge {\sim} b)^2 \overset{((4.7))}{=} 0 \vee 0 = 0.\end{aligned}$$

From $(a \wedge {\sim} a) * (b \wedge {\sim} b) \leq 0$, we get $a \wedge {\sim} a \leq (b \wedge {\sim} b) \Rightarrow 0$ by (Res), whence $a \wedge {\sim} a \leq {\sim}(b \wedge {\sim} b) = b \vee {\sim} b$ as desired. The remaining statement is clear. □

Proposition 4.8 *Let* $\mathbf{A} = \langle A; \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be a compatibly involutive CIBRL. The following statements hold:*

1. *If* $\mathbf{A}$ *is distributive, then* $\langle A; \wedge, \vee, {\sim}, 0, 1\rangle$ *is a De Morgan algebra.*
2. *If* $\mathbf{A}$ *is 3-potent, then:*
   (a) *If* $\mathbf{A}$ *is distributive, then* $\langle A; \wedge, \vee, {\sim}, 0, 1\rangle$ *is a Kleene algebra.*
   (b) *The relation* $\preceq$ *defined for all* $a, b \in A$ *by* $a \preceq b$ *iff* $a \to b = 1$, *where* $a \to b$ *abbreviates* $a \Rightarrow (a \Rightarrow b)$, *is a quasiorder on* A.
   (c) *The relation* $\varXi$ *defined for all* $a, b \in A$ *by* $a \ \varXi \ b$ *iff* $a \preceq b$ *and* $b \preceq a$ *is a congruence on the term reduct* $\langle A; \wedge, \vee, \to, 0, 1\rangle$. *Moreover, the quotient algebra* $\langle A; \wedge, \vee, \to, 0, 1\rangle/\varXi$ *is a Heyting algebra.*
   (d) *For all* $a, b \in A$, *it holds that* ${\sim}(a \to b) \equiv a \wedge {\sim} b \pmod{\varXi}$.
   (e) *For all* $a, b \in A$, *it holds that* $a \wedge {\sim} a \preceq b$.

Proof (1) This is clear.

(2a) This follows from Proposition 4.7.

(2b) Observe that for all $c, d \in A$, $c \preceq d$ iff $c \to d = 1$ iff $c^2 \leq d$ iff $c^2 \leq d^2$, where the last equivalence follows by integrality and 3-potence. It follows vacuously that $\preceq$ is a quasiorder on A.

(2c) This is Busaniche and Cignoli (2010, Theorem 3.4).

(2d) Observe ${\sim}(a \to b) \overset{(2.4)}{=} {\sim}(a^2 \Rightarrow b) \overset{(\Rightarrow,\, *\text{-Equiv})}{=} {\sim}{\sim}(a^2 * {\sim} b) = a^2 * {\sim} b$, so it suffices to show: (i) $a^2 * {\sim} b \preceq a \wedge {\sim} b$, and (ii) $a \wedge {\sim} b \preceq a^2 * {\sim} b$. For (i), we have $a^2 * {\sim} b \leq a^2 \wedge {\sim} b$ (by integrality) $\leq a \wedge {\sim} b$ (by integrality again). The result now follows, because $\leq \subseteq \preceq$ by Lemma 2.7. For (ii), from $a^2 \wedge ({\sim} b)^2 \leq a^2, ({\sim} b)^2$ we get $(a^2 \wedge {\sim} b^2)^2 \leq a^2 * ({\sim} b)^2$ (by (Compat)) $\leq a^2 * {\sim} b$ (by integrality), which is to say $(a \wedge {\sim} b)^2 \leq a^2 * {\sim} b$ by ($\wedge$-Centrip.). Thus, $a \wedge {\sim} b \preceq a^2 * {\sim} b$.

(2e) By (4.7), $(a \wedge {\sim} a)^2 \leq b$. Thus, $(a \wedge {\sim} a)^2 \preceq b$ as $\leq \subseteq \preceq$ by Lemma 2.7. □

Remark 4.9 Although it will not be needed here, we note that Items (2b)–(2e) of Proposition 4.8 extend in the natural way to $k + 1$-potent compatibly involutive CIRLs ($k \in \omega$) as follows. (Item (2a) does not generalise.) If $\mathbf{A}$ is $k + 1$-potent, then the following statements hold:

1. The relation $\preceq$ defined for all $a, b \in A$ by $a \preceq b$ iff $a \to b = 1$, where $a \to b$ abbreviates $a^k \Rightarrow b$, is a quasiorder on A.
2. The relation $\varXi$ defined for all $a, b \in A$ by $a \ \varXi \ b$ iff $a \preceq b$ and $b \preceq a$ is a congruence on the term reduct $\langle A; \wedge, \vee, \to, 0, 1\rangle$. Moreover, the quotient $\langle A; \wedge, \vee, \to, 0, 1\rangle/\varXi$ is a Heyting algebra.
3. For all $a, b \in A$, it holds that ${\sim}(a \to b) \equiv a \wedge {\sim} b \pmod{\varXi}$.
4. For all $a, b \in A$, it holds that $a \wedge {\sim} a \preceq b$.

Modulo the proof of Proposition 4.8, only the proof of (2) above is non-trivial; this is an unpublished result of Busaniche and Cignoli (2008, Theorem 1.2). □

Proposition 4.8 shows that a compatibly involutive 3-potent (more generally, $k+1$-potent) CIRL is *almost* a Nelson algebra in that only (N5) may fail to hold. This naturally directs attention to the clause (N5). We have:

Proposition 4.10 *For a compatibly involutive CIRL* $\mathbf{A}$*, t.f.a.e.:*

1. $\mathbf{A} \models$ (*Nelson*).
2. $\mathbf{A} \models x^2 \leqslant y$ *and* $(\sim y)^2 \leqslant \sim x$ *implies* $x \leqslant y$.
3. *For all* $a, b \in A$*, it holds that*

   $a \preceq b$ *and* $\sim b \preceq \sim a$ *iff* $a \leq b$,

   where $\preceq$ *is the quasiorder relation of Proposition 4.8.*

Moreover, if $\mathbf{A} \models$ (*Nelson*)*, then* $\mathbf{A}$ *is distributive and 3-potent.*

Proof (1) $\iff$ (2) This is (Kozak 2014, Lemma 2.7).
(2) $\iff$ (3) This follows immediately on recalling that for all $c, d \in A$, $c \preceq d$ iff $c^2 \leq d$.

For the final statement, suppose $\mathbf{A}$ is a Nelson residuated lattice. By Corollary 4.3, $\mathbf{A}$ is 3-potent. The distributivity of $\mathbf{A}$ is established by Busaniche and Cignoli (2010, Remark 3.7). □

At this point, we are ready to use the preceding result to give an order-theoretic characterisation of the Nelson residuated lattices among the 3-potent compatibly involutive CIRLs.

Theorem 4.11 *Let* $\mathbf{A}$ *be a 3-potent compatibly involutive CIRL. The following are equivalent:*

1. $\mathbf{A} \models$ (*Nelson*).
2. *For all* $a, b \in A$*,* $a \leq b$ *iff* ($a \preceq b$ *and* $\sim b \preceq \sim a$) *iff* ($a \preceq b$ *and* $a \preceq_h^\partial b$).
3. *The relation* $\preceq \cap \preceq_h^\partial$ *is a partial order on* $\mathbf{A}$ *that coincides with* $\leq$.
4. *The relation* $\preceq \cap \preceq_h^\partial$ *is a partial order on* $\mathbf{A}$.
5. *The relation* $\preceq \cap \preceq_h^\partial$ *is anti-symmetric.*
6. *For all* $a, b \in A$*, if* $a^2 = b^2$ *and* $(\sim a)^2 = (\sim b)^2$*, then* $a = b$.

Proof Let $a, b, c \in A$.

(1) $\Rightarrow$ (2) This follows from Proposition 4.10 and Lemma 3.6.
(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) This sequence of implications is clear.
(5) $\Rightarrow$ (6) Suppose $\preceq \cap \preceq_h^\partial$ is anti-symmetric. Assume $a^2 = b^2$ and $(\sim a)^2 = (\sim b)^2$. Since for all $c, d \in A$, we have that $c \preceq d$ iff $c^2 \leq d$ iff $c^2 \leq d^2$ and $c \preceq^\partial d$ iff $(\sim d)^2 \leq \sim c$ iff $(\sim d)^2 \leq (\sim c)^2$, by our assumptions on $\mathbf{A}$ we get that $a \preceq \cap \preceq_h^\partial b$ and $b \preceq \cap \preceq_h^\partial a$. By anti-symmetry, $a = b$. Hence, for all $a, b \in A$, if $a^2 = b^2$ and $(\sim a)^2 = (\sim b)^2$, then $a = b$.
(6) $\Rightarrow$ (1) This follows by hypothesis and Proposition 4.4.

□

Remark 4.12 We speculate that Theorem 4.11 extends to a characterisation of the Nelson residuated lattices among the compatibly involutive CIRLs upon moving the assumption of 3-potency from the statement of the theorem to Item (6). □

The last condition of the preceding theorem is saying that if two elements as well as their negations generate the same filter, then they must be identical. This can be equivalently reformulated as the following:

$$\text{if } a * a = b * b \text{ and } a + a = b + b, \text{ then } a = b$$

which says that two elements are identical when they generate the same filter and the same ideal. We shall return on the connection between the identity (Nelson) and these filter/ideal-theoretic properties in Sect. 7.

Let us denote by $\mathrm{Q}(A)$ the set of all quasiorders on a given set A. The set $\mathrm{Q}(A)$ forms a complete lattice $\langle \mathrm{Q}(A); \subseteq \rangle$ under the inclusion ordering (Pöschel and Radeleczki 2008, Section 1.2). Reformulating Theorem 4.11, we have:

Corollary 4.13 *Let* $\mathbf{A}$ *be a Nelson residuated lattice. Then,* $\leq$ *is the meet of* $\preceq$ *and* $\preceq_h^\partial$ *in* $\langle \mathrm{Q}(A); \subseteq \rangle$.

Proof It suffices to observe that $\leq\; = \;\preceq \cap \preceq_h^\partial$ by Theorem 4.11. □

5 Introducing (0, 1)-congruence orderability

In their unpublished manuscript on Skolem rings (Büchi and Owens 1990), Büchi and Owens considered algebras $\mathbf{A}$ with a constant term $\mathbf{0}$ having the property that for all $a, b \in A$,

$$\text{if} \quad \Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(0, b) \quad \text{then} \quad a = b. \tag{5.1}$$

They termed such algebras 'fission free'. Büchi and Owens were mainly interested in fission free algebras for which every compact congruence is of the form $\Theta^{\mathbf{A}}(0, a)$; Skolem rings, which are term equivalent to implicative semilattices, form the main example of such algebras.

Condition (5.1) means that the mapping $a \mapsto \Theta^{\mathbf{A}}(0, a)$ on an algebra $\mathbf{A}$ with a constant term $\mathbf{0}$ is injective; it follows that the natural quasiordering on the universe of $\mathbf{A}$, defined for all $a, b \in A$ by

$$a \leq b \quad \text{iff} \quad \Theta^{\mathbf{A}}(0, a) \subseteq \Theta^{\mathbf{A}}(0, b)$$

is a partial ordering. This observation led Idziak et al. (2009) to call such algebras **0**-*congruence orderable*.

Following Idziak et al. (2009), we say an algebra $\mathbf{A}$ with a constant term $\mathbf{c}$ is $\mathbf{c}$-*congruence orderable* if, for all $a, b \in A$, it holds that

$$\Theta^{\mathbf{A}}(c, a) = \Theta^{\mathbf{A}}(c, b) \quad \text{implies} \quad a = b. \tag{5.2}$$

A variety V with a constant term $\mathbf{c}$ is $\mathbf{c}$-*congruence orderable* if every member of V is $\mathbf{c}$-congruence orderable. Boolean and Heyting algebras are **1**-congruence orderable, as are all their subreducts that include **1** in the language: implicative (semi)lattices, Hilbert algebras, Tarski algebras, and upper-bounded distributive (semi)lattices. Pseudocomplemented semilattices, equivalential algebras, and Boolean groups are also **1**-congruence orderable, while pointed sets are $\mathbf{c}$-congruence orderable for each distinguished element c.

Observe that, by the duality principle, Boolean algebras are also **0**-congruence orderable. That is, Boolean algebras are *simultaneously* $\mathbf{c}$-congruence orderable for $\mathbf{c} \in \{\mathbf{0}, \mathbf{1}\}$. Motivated by this example, and in keeping with the themes of this paper, in this section we introduce a generalisation of **0**-congruence orderability to the double-pointed case and show that Nelson residuated lattices are distinguished among the compatibly involutive CIRLs by our generalisation of congruence orderability to the case involving two constant terms. Our starting point for this exploration is the following easy characterisation of principal congruences of the form $\Theta(1, a)$ and $\Theta(0, a)$ on CIBRLs.

Lemma 5.1 *Let* $\mathbf{A}$ *be a* $k+1$*-potent CIBRL* ($k \in \omega$)*. The following statements hold for all* $a, b, c \in A$*:*

1. $a \preceq b$ *iff* $\Theta^{\mathbf{A}}(b, 1) \subseteq \Theta^{\mathbf{A}}(a, 1)$.
2. $a \preceq_h^{\partial} b$ *iff* $\sim b \preceq \sim a$ *iff* $\Theta^{\mathbf{A}}(a, 0) \subseteq \Theta^{\mathbf{A}}(b, 0)$.
3. $a^k = b^k$ *iff* $\Theta^{\mathbf{A}}(1, a) = \Theta^{\mathbf{A}}(1, b)$.
4. $(\sim a)^k = (\sim b)^k$ *iff* $\Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(b, 0)$.

Proof (1) It suffices to observe that $a \preceq b$ iff $a^k \leq b$ iff $b \in [a)$ iff $b \equiv 1 \pmod{\Theta^{\mathbf{A}}(a, 1)}$ iff $\Theta^{\mathbf{A}}(b, 1) \subseteq \Theta^{\mathbf{A}}(a, 1)$.
(2) We have $a \preceq_h^{\partial} b$ iff $\sim b \preceq \sim a$ by Lemma 3.6. Also, using (1) and Lemma 2.9, we have $\sim b \preceq \sim a$ iff $\Theta^{\mathbf{A}}(a, 0) = \Theta^{\mathbf{A}}(\sim a, 1) \subseteq \Theta^{\mathbf{A}}(\sim b, 1) = \Theta^{\mathbf{A}}(b, 0)$.
(3) This follows from (1) on recalling $a \preceq b$ iff $a^k \leq b^k$.
(4) This is the dual of (3) and holds by (3) and Lemma 2.9. □

Remark 5.2 Let $\mathbf{A}$ be a $k+1$-potent CIRL, $k \in \omega$. Then, $\mathsf{HSP}(\mathbf{A})$ satisfies $x^{k+1} \approx x^k$ and thus has EDPC. From this observation and results due to Spinks and Veroff (2007), it is easy to see that for all $a, b, c \in A$, it holds that:

1. $b \equiv c \pmod{\Theta^{\mathbf{A}}(a, 1)}$ iff $a \to b = a \to c$.
2. $b \equiv c \pmod{\Theta^{\mathbf{A}}(0, a)}$ iff $(\sim a)^k * b = (\sim a)^k * c$.

Since $a \equiv 0 \pmod{\Theta^{\mathbf{A}}(0, b)}$ iff $(\sim b)^k * a = (\sim b)^k * 0$ (by (2)) iff $(\sim b)^k * a = 0$ iff $(\sim b)^k \leq a \Rightarrow 0$ iff $(\sim b)^k \leq \sim a$ iff $\sim b \preceq \sim a$, this observation generalises (1)–(2) of Lemma 5.1. □

Corollary 5.3 *Let* $\mathbf{A}$ *be a compatibly involutive CIRL. The following are equivalent:*

1. $\mathbf{A} \models$ (*Nelson*).
2. $\mathbf{A}$ *is 3-potent and for all* $a, b \in A$*, it holds that*

$$\begin{aligned} &\Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(0, b) \text{ and} \\ &\Theta^{\mathbf{A}}(1, a) = \Theta^{\mathbf{A}}(1, b) \text{ implies } a = b. \end{aligned} \tag{5.3}$$

Proof By Lemma 5.1, we have that

$$\begin{aligned} &a^2 = b^2 \text{ iff } \Theta^{\mathbf{A}}(1, a) = \Theta^{\mathbf{A}}(1, b) \text{ and} \\ &(\sim a)^2 = (\sim b)^2 \text{ iff } \Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(0, b), \end{aligned}$$

so the result follows from Proposition 4.4. □

Recall that for an algebra $\mathbf{A}$, elements $a, b \in A$ are residually distinct if $\Theta^{\mathbf{A}}(a, b) = \nabla_A$. By analogy with the theory of $\mathbf{c}$-congruence orderable algebras, we shall say that an algebra $\mathbf{A}$ with constant terms $\mathbf{c}, \mathbf{d}$ realising residually distinct elements $c, d \in A$ is $(\mathbf{c}, \mathbf{d})$-*congruence orderable* if, for all $a, b \in A$, it holds that

$$\begin{aligned} &\Theta^{\mathbf{A}}(c, a) = \Theta^{\mathbf{A}}(c, b) \text{ and} \\ &\Theta^{\mathbf{A}}(d, a) = \Theta^{\mathbf{A}}(d, b) \quad \text{implies} \quad a = b. \end{aligned} \tag{5.4}$$

A variety V with constant terms $\mathbf{c}, \mathbf{d}$ is said to be $(\mathbf{c}, \mathbf{d})$-*congruence orderable* if every member of V is $(\mathbf{c}, \mathbf{d})$-congruence orderable.

Observe that every double-pointed variety (relative to some constant terms $\mathbf{c}, \mathbf{d}$) that is $\mathbf{c}$-congruence orderable is *a fortiori* $(\mathbf{c}, \mathbf{d})$-congruence orderable; examples include Boolean algebras, Heyting algebras, and pseudocomplemented semilattices.

Corollary 5.4 *Every Nelson residuated lattice is* $(\mathbf{0}, \mathbf{1})$*-congruence orderable.* □

Remark 5.5 With the notion of $(\mathbf{c}, \mathbf{d})$-congruence orderability to hand, it is evident that Lemma 5.1 asserts that for every $k+1$-potent CIBRL ($k \in \omega$), the following are equivalent:

1. For all $a, b \in A$, if $a^k = b^k$ and $(\sim a)^k = (\sim b)^k$, then $a = b$.
2. $\mathbf{A}$ is $(\mathbf{0}, \mathbf{1})$-congruence orderable. □

To the best of our knowledge, this is the first paper in which $(\mathbf{0}, \mathbf{1})$-congruence orderability *qua* a congruence condition has been isolated in the literature. Nonetheless, the study of $(\mathbf{0}, \mathbf{1})$-congruence orderability dates back to at least the paper (Varlet 1972) of Varlet, who essentially proves that a double p-algebra is $(\mathbf{0}, \mathbf{1})$-congruence orderable iff it is congruence regular.[7] For a brief survey of work concerning $(\mathbf{0}, \mathbf{1})$-congruence orderability in the context of double p-algebras and related structures, see Järvinen and Radeleczki (2017).

Example 5.6 Let $\mathbf{A}$ be a De Morgan algebra. By Sankappanavar (1980, Theorem 2.2), for all $a, b, c \in A$ it holds that

$$c \equiv d \pmod{\Theta^{\mathbf{A}}(0, b)} \quad \text{iff} \quad (c \vee b) \wedge \sim b = (d \vee b) \wedge \sim b$$

and

$$c \equiv d \pmod{\Theta^{\mathbf{A}}(1, b)} \quad \text{iff} \quad (c \wedge b) \vee \sim b = (d \wedge b) \vee \sim b.$$

From this characterisation of $\Theta^{\mathbf{A}}(0, b)$ and $\Theta^{\mathbf{A}}(1, b)$, it follows directly that $\mathbf{A}$ satisfies congruence condition (5.4), with $\mathbf{c} = \mathbf{0}, \mathbf{d} = \mathbf{1}$, iff for all $a, b \in A$,

$$\begin{aligned} &(a \vee b) \wedge \sim b = b \wedge \sim b \text{ and} \\ &(b \vee a) \wedge \sim a = a \wedge \sim a \text{ and} \\ &(a \wedge b) \vee \sim b = b \vee \sim b \text{ and} \\ &(b \wedge a) \vee \sim a = a \vee \sim a \quad \text{implies} \quad a = b. \end{aligned} \tag{5.5}$$

Further, it is easy to check that (5.5) holds on $\mathbf{A}$ iff $\mathbf{A} \models x \wedge \sim x \leqslant y \vee \sim y$. The variety of Kleene algebras is thus $(\mathbf{0}, \mathbf{1})$-congruence orderable; moreover, this property distinguishes the Kleene algebras among the De Morgan algebras. □

Remark 5.7 Recall that the $\langle \wedge, \vee, \sim, 0, 1\rangle$-reduct of a Nelson algebra $\mathbf{A}$ is a Kleene algebra. Conversely, Brignole and Monteiro (1967) shows that a Kleene algebra $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ that is structurally enriched with a binary operation $\rightarrow$ such that for all $a, b, c \in A$,

$$\begin{aligned} &a \wedge c \leq \sim a \vee b \text{ iff } c \leq a \rightarrow b \quad \text{and} \\ &(a \wedge b) \rightarrow c = a \rightarrow (b \rightarrow c) \end{aligned}$$

is a Nelson algebra. Example 5.6 thus suggests that Nelson algebras may be understood as Kleene algebras $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ structurally enriched with an 'implication' operation $\rightarrow$ that preserves the $(\mathbf{0}, \mathbf{1})$-congruence orderability of Kleene algebras. This is in contrast to the usual (logical) perspective of Nelson algebras, which views Nelson's constructive logic with strong negation $\mathbf{N3}$ as a conservative expansion of the intuitionistic propositional calculus by a unary logical connective $\sim$ of strong negation. □

To put Corollary 5.4 into context, we briefly consider $\mathbf{c}$-congruence orderability for CIBRLs.

Proposition 5.8 *For a CIBRL* $\mathbf{A}$*, the following statements are equivalent:*

1. $\langle A; *, \vee, \Rightarrow, 0, 1\rangle$ *is a Heyting algebra.*
2. $\mathbf{A} \models x \wedge y \approx x * y$.
3. $\mathbf{A} \models x * x \approx x$.
4. $\mathbf{A}$ *is* $\mathbf{1}$*-congruence orderable.*

Proof (1) $\iff$ (2) This is well known.

(2) $\iff$ (3) Similarly, this is well known.

(1) $\Rightarrow$ (4) This is also well known.

(4) $\Rightarrow$ (2) Suppose $\mathbf{A}$ is $\mathbf{1}$-congruence orderable. Let $a, b \in A$. From $a \wedge b \leq a, b$, we get $(a \wedge b)^2 \leq a * b$ by (Compat), whence $a * b \in [a \wedge b)$. Thus, $[a * b) \subseteq [a \wedge b)$. On the other hand, $a * b \leq a \wedge b$ by integrality. Thus, $a \wedge b \in [a * b)$ and therefore $[a \wedge b) \subseteq [a * b)$. Hence, $[a \wedge b) = [a * b)$. It follows that $\Theta^{\mathbf{A}}(a \wedge b, 1) = \Theta^{\mathbf{A}}(a * b, 1)$, whence $a \wedge b = a * b$. Hence, $\mathbf{A} \models x \wedge y \approx x * y$. □

Recall from McKinsey and Tarski (1946, Definition 1.1) that a *co-Heyting algebra* (also *Brouwerian algebra* in the literature) $\langle A; \vee, \wedge, \dot{-}, 1, 0\rangle$ is a bounded distributive lattice $\langle A; \vee, \wedge, 1, 0\rangle$ augmented with a "subtraction" operation $\dot{-}$ defined for all $a, b, c \in A$ by

$$a \dot{-} b \leq c \quad \text{iff} \quad a \leq b \vee c. \tag{5.6}$$

Proposition 5.9 *For a CIBRL* $\mathbf{A}$*, the following statements are equivalent:*

1. $\langle A; \wedge, \vee, \dot{-}, 0, 1\rangle$ *is a co-Heyting algebra.*
2. $\langle A; \wedge, \vee, \dot{-}, \neg, 0, 1\rangle$ *is a Boolean algebra.*
3. $\langle A; \wedge, \vee, \Rightarrow, \sim, 0, 1\rangle$ *is a Boolean algebra.*
4. $\mathbf{A} \models x \vee y \approx x + y$.
5. $\mathbf{A} \models x + x \approx x$.
6. $\mathbf{A}$ *is* $\mathbf{0}$*-congruence orderable.*

Proof Let $a, b \in A$.

[7] A *double p-algebra* (Katriňák 1973) is an algebra $\langle A; \wedge, \vee, {}^*, {}^+, 0, 1\rangle$ where $\langle A; \wedge, \vee, {}^*, 0, 1\rangle$ is a pseudocomplemented distributive lattice (Balbes and Dwinger 1974, Chapter VIII) and $\langle A; \wedge, \vee, {}^+, 0, 1\rangle$ is a dually pseudocomplemented distributive lattice.

(1) $\iff$ (2) It suffices to show $\neg\neg a \leq a$. For this, note $\neg\neg a \leq a$ iff $1 \dot{-} (1 \dot{-} a) \leq a$ iff $1 \leq (1 \dot{-} a) \vee a$ (by (5.6)) iff $1 \leq a \vee (1 \dot{-} a)$ iff $1 \dot{-} a \leq 1 \dot{-} a$ (by (5.6)).
(2) $\iff$ (3) Obvious, as $\neg c = 1 \dot{-} c = 1 * {\sim} c = {\sim} c$ for all $c \in A$.
(1) $\iff$ (4) This is clear.
(4) $\iff$ (5) Suppose $\mathbf{A} \models x \vee y \approx x + y$. By specialisation, $\mathbf{A} \models x + x \approx x \vee x \approx x$; thus $\mathbf{A} \models x + x \approx x$. Conversely, suppose $\mathbf{A} \models x + x \approx x$. From $a, b \leq a + b$, we get $a \vee b \leq a + b$, so it remains to show $a + b \leq a \vee b$. For this, note $a + b \leq (a \vee b) + (a \vee b)$ iff $(a \vee b) \dot{-} (a + b) \leq a \vee b$, which latter holds by integrality. Therefore, $a + b \leq (a \vee b) + (a \vee b) = a \vee b$ and thus $\mathbf{A} \models x \vee y \approx x + y$.
(2) $\Rightarrow$ (6) This is clear.
(6) $\Rightarrow$ (4) Suppose $\mathbf{A}$ is $\mathbf{0}$-congruence orderable. From $a, b \leq a + b$ we get $a \vee b \leq a + b$, whence $a \vee b \in (a \vee b]$. Thus, $(a \vee b] \subseteq (a + b]$. On the other hand, from the proof of (4) $\iff$ (5) above, we get $a + b \leq (a \vee b) + (a \vee b)$; thus $a + b \in (a \vee b]$. Therefore, $(a + b] \subseteq (a \vee b]$ and so $(a \vee b] = (a + b]$. It follows that $\Theta^{\mathbf{A}}(a \vee b, 0) = \Theta^{\mathbf{A}}(a + b, 0)$; from $\mathbf{0}$-congruence orderability, we conclude that $a \vee b = a + b$. Hence, $\mathbf{A} \models x \vee y \approx x + y$. □

Corollary 5.10 *For a compatibly involutive CIBRL* $\mathbf{A}$*, the following are equivalent:*

1. $\langle A; \wedge, \vee, \Rightarrow, {\sim}, 0, 1\rangle$ *is a Boolean algebra.*
2. $\mathbf{A}$ *is* $\mathbf{1}$*-congruence orderable.*
3. $\mathbf{A}$ *is* $\mathbf{0}$*-congruence orderable.*

Proof (1) $\Rightarrow$ (2) This is well known.
(2) $\Rightarrow$ (1) Suppose $\mathbf{A}$ is $\mathbf{1}$-congruence orderable. Then, $\langle A; \wedge, \vee, \Rightarrow, 0, 1\rangle$ is a Heyting algebra by Proposition 5.8. Since $\mathbf{A} \models x \approx (x \Rightarrow \mathbf{0}) \Rightarrow \mathbf{0}$, we conclude that $\mathbf{A}$ is a Boolean algebra.
(1) $\iff$ (3) By Proposition 5.9. □

We now turn our attention to point regularity. Following Idziak et al. (2009), a $\mathbf{c}$-regular $\mathbf{c}$-congruence orderable algebra $\mathbf{A}$ is said to be $\mathbf{c}$-*Fregean*. Compared to $\mathbf{c}$-congruence orderable algebras, $\mathbf{c}$-Fregean algebras are quite specialised and rare; Boolean and Heyting algebras, together with their subreducts having $\Rightarrow$ in the language, form the most familiar examples of $\mathbf{1}$-Fregean algebras. Interest in such algebras has nonetheless increased in recent years, owing to their connections with Fregean logics (Czelakowski and Pigozzi 2004; Font and Jansana 2009; Pigozzi 1991); the algebraisable Fregean logics include the deductive systems intermediate between the classical and intuitionistic propositional calculi (Czelakowski and Pigozzi 2004, Theorem 66, Theorem 68), together with their superimplicational fragments (Font and Jansana 2009, Section 5.2).

By analogy with the theory of Fregean varieties, we will say that an algebra $\mathbf{A}$ with constant terms $\mathbf{c}, \mathbf{d}$ realising residually distinct elements $c, d \in A$ is $(\mathbf{c}, \mathbf{d})$-*Fregean* if it is $(\mathbf{c}, \mathbf{d})$-congruence orderable and both $\mathbf{c}$-regular and $\mathbf{d}$-regular. A variety V with constant terms $\mathbf{c}, \mathbf{d}$ is $(\mathbf{c}, \mathbf{d})$-*Fregean* if every member of V is $(\mathbf{c}, \mathbf{d})$-Fregean.

Theorem 5.11 *Every Nelson residuated lattice is* $(\mathbf{0}, \mathbf{1})$*-Fregean.*

Proof This follows from Corollary 5.4, since the variety of compatibly involutive CIRLs is simultaneously $\mathbf{0}$-ideal determined and $\mathbf{1}$-ideal determined. □

In Sect. 7, we shall establish a converse to the preceding result: namely, that Nelson residuated lattices are precisely the compatibly involutive CIBRLs that are $(\mathbf{0}, \mathbf{1})$-Fregean (Corollary 7.2). For this, we shall rely on a result proved in the next section.

6 A syntactic characterisation of NRLs

In this section, we prove a result that was announced without proof in Spinks and Veroff (2018, Theorem 4.2), namely that the Nelson identity is equivalent, over compatibly involutive CIRLs, to the identity

$$x^2 \vee (x \wedge {\sim} x) \approx x. \tag{6.1}$$

This may be viewed as yet another approximation to unveiling the mystery of the Nelson identity by reducing it to an equivalent expression, which yet ended up producing the perhaps even more mysterious (but then also interesting) identity (6.1). This latter was found via a computer-based search using the automated reasoning program Prover9 (McCune 2018) with the method of proof sketches (Veroff 2001) while trying to prove the implication (2) $\Rightarrow$ (1) of Theorem 6.1. Beyond the obvious observation that satisfaction of $x^2 \vee (x \wedge {\sim} x) \approx x$ (observe that $x \leqslant x^2 \vee (x \wedge {\sim} x)$ is the only non-trivial half) says that for every element a, the least upper bound of a^2 and $a \wedge {\sim} a$ is a, we have at the moment no particular insight into this identity.

Theorem 6.1 *For a compatibly involutive CIRL* $\mathbf{A}$*, the following are equivalent:*

1. $\mathbf{A}$ *is a Nelson residuated lattice.*
2. $\mathbf{A}$ *is 3-potent and satisfies the quasi-identity:*

$$x^2 \approx y^2 \text{ and } ({\sim} x)^2 \approx ({\sim} y)^2 \text{ implies } x \approx y. \tag{4.4}$$

3. $\mathbf{A}$ *satisfies the identity:*

$$x^2 \vee (x \wedge {\sim} x) \approx x. \tag{6.1}$$

Proof (1) ⇒ (2) This has been observed previously.

(2) ⇒ (3) Let $a, b \in A$. We show

$$a^2 = \big(a^2 \vee (a \wedge \sim a)\big)^2 \quad \text{and}$$
$$(\sim a)^2 = \big(\sim(a^2 \vee (a \wedge \sim a))\big)^2$$

and then apply quasi-identity (4.4).

To see $a^2 = \big(a^2 \vee (a \wedge \sim a)\big)^2$, just observe that

$$\begin{aligned} a^2 &= (a^2)^2 \text{ (by 3-potence)} \\ &= (a^2)^2 \vee 0 \overset{(4.7)}{=} (a^2)^2 \vee (a \wedge \sim a)^2 \overset{(4.6)}{=} \big(a^2 \vee (a \wedge \sim a)\big)^2. \end{aligned}$$

It remains to show $(\sim a)^2 = \big(\sim(a^2 \vee (a \wedge \sim a))\big)^2$; we split the proof into two pieces.

We first show $(\sim a)^2 \leq \big(\sim(a^2 \vee (a \wedge \sim a))\big)^2$. From $a^2 \leq a$ and (∼-Contra), we have $\sim a \leq \sim(a^2)$, whence $(\sim a)^2 \leq \big(\sim(a^2)\big)^2$ by the doubling construction. Also, from $a \wedge \sim a \leq a$ and (∼-Contra), we have $\sim a \leq \sim(a \wedge \sim a)$, whence $(\sim a)^2 \leq \big(\sim(a \wedge \sim a)\big)^2$ by the doubling construction. Thus, $(\sim a)^2 \leq \big(\sim(a^2)\big)^2 \wedge \big(\sim(a \wedge \sim a)\big)^2$. By the doubling construction, therefore,

$$\big((\sim a)^2\big)^2 \leq \big((\sim(a^2))^2 \wedge (\sim(a \wedge \sim a))^2\big)^2 \overset{(\wedge\text{-Centrip.})}{=} \big(\sim(a^2) \wedge \sim(a \wedge \sim a)\big)^2,$$

whence $(\sim a)^2 \leq \big(\sim(a^2) \wedge \sim(a \wedge \sim a)\big)^2$ by 3-potence. But then $(\sim a)^2 \leq \big(\sim(a^2 \vee (a \wedge \sim a))\big)^2$ by De Morgan's laws and the law of double negation.

We now show $\big(\sim(a^2 \vee (a \wedge \sim a))\big)^2 \leq (\sim a)^2$. From $a \vee \sim a \leq 1$, we have $\sim(a^2) \wedge (a \vee \sim a) \leq 1 = a \Rightarrow a$, whence $\big(\sim(a^2) \wedge (a \vee \sim a)\big) * a \leq a$ by (Res). Also, $\sim(a^2) \wedge (a \vee \sim a) \leq \sim(a^2) = a \Rightarrow \sim a$, so by (Res), $\big(\sim(a^2) \wedge (a \vee \sim a)\big) * a \leq \sim a$. Therefore $\big(\sim(a^2) \wedge (a \vee \sim a)\big) * a \leq a \wedge \sim a$. But then $\big(\sim(a^2) \wedge (a \vee \sim a)\big) * a \leq a^2 \vee (a \wedge \sim a)$, whence $\sim(a^2) \wedge (a \vee \sim a) \leq a \Rightarrow \big(a^2 \vee (a \wedge \sim a)\big)$ by (Res). By (⇒-Contra), $\sim(a^2) \wedge (a \vee \sim a) \leq \sim\big(a^2 \vee (a \wedge \sim a)\big) \Rightarrow \sim a$, and by De Morgan's laws and the law of double negation, $\sim\big(a^2 \vee (a \wedge \sim a)\big) \leq \sim\big(a^2 \vee (a \wedge \sim a)\big) \Rightarrow \sim a$. By (Res), therefore, $\big(\sim(a^2 \vee (a \wedge \sim a))\big)^2 \leq \sim a$, whence $\big((\sim(a^2 \vee (a \wedge \sim a)))^2\big)^2 \leq (\sim a)^2$ by the doubling construction. By 3-potence, $\big(\sim(a^2 \vee (a \wedge \sim a))\big)^2 \leq (\sim a)^2$ as desired.

We have shown that $a^2 = \big(a^2 \vee (a \wedge \sim a)\big)^2$ and $(\sim a)^2 = \big(\sim(a^2 \vee (a \wedge \sim a))\big)^2$, so by quasi-identity (4.4), $a = a^2 \vee (a \wedge \sim a)$. Hence, $\mathbf{A} \models x^2 \vee (x \wedge \sim x) \approx x$.

(3) ⇒ (1) Let $a, b, c \in A$. Before beginning the proof proper, we verify that $\mathbf{A}$ satisfies the following identities, in the indicated order:

(i) $(x \wedge y) * \sim y \approx \mathbf{0}$
(ii) $\sim x * (x \vee \sim y) \approx x * y$
(iii) $(x \vee \sim y) * y^2 \approx x * y^2$
(iv) $(x \vee \sim y)^2 * y \approx x^2 * y$.

Proof (of (i)–(iv))

For (i), observe $a \wedge b \leq a = 1 \Rightarrow a \overset{(\Rightarrow\text{-Contra})}{=} \sim a \Rightarrow \sim 1 = \sim a \Rightarrow 0$, whence $(a \wedge b) * \sim a \leq 0$ by (Res). Hence, $\mathbf{A} \models \sim x * (x \wedge y) \approx \mathbf{0}$.
For (ii), note $(a \vee \sim b) * b \overset{(2.3)}{=} (a * b) \vee (\sim b * b) = (a * b) \vee 0 = a * b$. Hence, $\mathbf{A} \models (x \vee \sim y) * y \approx x * y$.
For (iii), notice $(a \vee \sim b) * b^2 = \big((a \vee \sim b) * b\big) * b \overset{(ii)}{=} (a * b) * b = a * b^2$. Hence, $\mathbf{A} \models (x \vee \sim y) * y^2 \approx x * y^2$.
For (iii), observe that

$$\begin{aligned} &(a \vee \sim b)^2 * b \\ &\quad = (a \vee \sim b) * \big((a \vee \sim b) * b\big) \overset{(ii)}{=} (a \vee \sim b) * (a * b) \\ &\quad = \big((a \vee \sim b) * b\big) * a \overset{(ii)}{=} (a * b) * a = a^2 * b. \end{aligned}$$

Hence, $\mathbf{A} \models (x \vee \sim y)^2 * y \approx x^2 * y$.

This completes the proof of (i)–(iv). □

Suppose now that (6.1) holds. To see $\mathbf{A}$ is a Nelson residuated lattice, we show $\mathbf{A} \models (x^2 * y) \vee (y^2 * x) \approx x * y$. To see this latter identity holds, we verify that $\mathbf{A}$ satisfies the following identities, in the indicated order:

(v) $x^2 \vee \sim x \approx x \vee \sim x$
(vi) $x * (x \wedge y) \approx x * (x * y)$
(vii) $x^2 \vee \big(x \wedge (y \vee \sim x)\big) \approx x$
(viii) $(x^2 * y) \vee (y^2 * x) \approx x * y$.

Proof *(of (v)–(viii))* For (v), it suffices to observe

$$\begin{aligned} a \vee \sim a &\overset{(\text{Hyp.})}{=} \big(a^2 \vee (a \wedge \sim a)\big) \vee \sim a \\ &= a^2 \vee \big((a \wedge \sim a) \vee \sim a\big) = a^2 \vee \sim a, \end{aligned}$$

where the final equality holds by absorption. Hence, $\mathbf{A} \models x^2 \vee \sim x \approx x \vee \sim x$.

For (vi), observe that

$$\begin{aligned} a * (a \wedge b) &= \big(a * (a \wedge b)\big) \vee 0 \\ &= \big(a * (a \wedge b)\big) \vee \big(\sim a * (a \wedge b)\big) && \text{by (i)} \\ &= (a \vee \sim a) * (a \wedge b) && \text{by (2.3)} \\ &= (a^2 \vee \sim a) * (a \wedge b) && \text{by (v)} \\ &= \big(a^2 * (a \wedge b)\big) \vee \big(\sim a * (a \wedge b)\big) && \text{by (2.3)} \\ &= \big(a^2 * (a \wedge b)\big) \vee 0 && \text{by (i)} \\ &= a^2 * (a \wedge b) \\ &\leq a^2 * b && \text{by } (\textit{Compat}),\ \text{as } a \wedge b \leq b. \end{aligned}$$

On the other hand, $a * b \leq a \wedge b$ by integrality. By (Compat), $a * (a * b) \leq a * (a \wedge b)$. Hence, $\mathbf{A} \models x * (x \wedge y) \approx x^2 * y$.

For (vii), from $\sim a \leq \sim a \vee b$ we have $a \wedge \sim a \leq a \wedge (\sim a \vee b)$, whence $a^2 \vee (a \wedge \sim a) \leq a^2 \vee \big(a \wedge \sim(a \vee b)\big)$. But $a^2 \vee (a \wedge \sim a) = a$ by hypothesis, so $a \leq a^2 \vee \big(a \wedge (\sim a \vee b)\big)$. Conversely, from $a^2 \leq a$ and $a \wedge (\sim a \vee b) \leq a$ we get $a^2 \vee \big(a \wedge (\sim a \vee b)\big) \leq a \vee a = a$. Hence, $\mathbf{A} \models x^2 \vee \big(x \wedge (\sim x \vee y)\big) \approx x$.

For (viii), put $\alpha := a \vee \sim b$ and $\beta := b$. Then:

$$\begin{aligned}
a * b &= (a \vee \sim b) * b && \text{by (ii)}\\
&= (a \vee \sim b) * \big(b^2 \vee (b \wedge (a \vee \sim b))\big) && \text{by (vii)}\\
&= \alpha * \big(\beta^2 \vee (\beta \wedge \alpha)\big) &&\\
&= (\alpha * \beta^2) \vee \big(\alpha * (\beta \wedge \alpha)\big) && \text{by (2.3)}\\
&= (\alpha * \beta^2) \vee \big(\alpha * (\alpha * \beta)\big) && \text{by (vi)}\\
&= (\alpha * \beta^2) \vee (\alpha^2 * \beta) &&\\
&= \big((a \vee \sim b) * b^2\big) \vee \big((a \vee \sim b)^2 * b\big) &&\\
&= (a * b^2) \vee (a^2 * b) && \text{by (iii), (iv).}
\end{aligned}$$

Hence, $\mathbf{A} \models (x^2 * y) \vee (y^2 * x) \approx x * y$.

This completes the proof of (v)–(viii). Since $\mathbf{A} \models$ (viii), from Proposition 4.1 we conclude $\mathbf{A}$ is a Nelson residuated lattice. □

Remark 6.2 The implication (2) ⇒ (3) in the preceding theorem can be readily deduced via the Vakarelov/Fidel twist structure semantics for Nelson algebras (see e.g. Odintsov 2003, Definition 4.1, Proposition 5.3). However, the direct proof is instructive, so we have included it here. □

7 A congruence-theoretic characterisation of NRLs

We are now finally ready to establish the announced result which rephrases the Nelson identity as a purely universal algebraic property of congruences on (compatibly involutive) residuated lattices. We shall also relate this congruence-theoretic property to an equivalent 'filter separation property' that may be instructively compared to similar filter conditions well known from lattice theory. We begin by showing that a $(\mathbf{0}, \mathbf{1})$-congruence orderable compatibly involutive CIRL must satisfy the Nelson identity.

Theorem 7.1 *Let* $\mathbf{A}$ *be a compatibly involutive CIRL. If* $\mathbf{A}$ *is* $(\mathbf{0}, \mathbf{1})$*-congruence orderable, then* $\mathbf{A}$ *is a Nelson residuated lattice.*

Proof Suppose $\mathbf{A}$ is $(\mathbf{0}, \mathbf{1})$-congruence orderable. To see $\mathbf{A}$ is a Nelson residuated lattice, it suffices to show $\mathbf{A}$ satisfies identity (6.1). To see $\mathbf{A} \models$ (6.1), we show both $\Theta^{\mathbf{A}}\big(a^2 \vee (a \wedge \sim a), 1\big) = \Theta^{\mathbf{A}}(a, 1)$ and $\Theta^{\mathbf{A}}\big(a^2 \vee (a \wedge \sim a), 0\big) = \Theta^{\mathbf{A}}(a, 0)$ and then conclude by $(\mathbf{0}, \mathbf{1})$-congruence orderability that $a^2 \vee (a \wedge \sim a) = a$.

We first show $\Theta^{\mathbf{A}}\big(a^2 \vee (a \wedge \sim a), 1\big) = \Theta^{\mathbf{A}}(a, 1)$. To begin, from $a^2, a \wedge \sim a \leq a$ we have $a^2 \vee (a \wedge \sim a) \leq a$, whence $a \in \big[a^2 \vee (a \wedge \sim a)\big)$. Thus, $[a) \subseteq \big[a^2 \vee (a \wedge \sim a)\big)$. On the other hand, we have $a^2 \leq a^2 \vee (a \wedge \sim a)$, whence $a^2 \vee (a \wedge \sim a) \in [a)$. Thus, $\big[a^2 \vee (a \wedge \sim a)\big) \subseteq [a)$. Hence, $\big[a^2 \vee (a \wedge \sim a)\big) = [a)$ and therefore

$$\Theta^{\mathbf{A}}(a^2 \vee (a \wedge \sim a), 1) = \Theta^{\mathbf{A}}(a, 1). \tag{7.1}$$

We next show $\Theta^{\mathbf{A}}\big(a^2 \vee (a \wedge \sim a), 0\big) = \Theta^{\mathbf{A}}(a, 0)$. Again, from $a^2, a \wedge \sim a \leq a$ we have $a^2 \vee (a \wedge \sim a) \leq a$, whence $a^2 \vee (a \wedge \sim a) \in (a]$. Thus, $\big(a^2 \vee (a \wedge \sim a)\big] \subseteq (a]$. On the other hand, from $a^2 \leq a^2$ we get $a \leq a \Rightarrow a^2$ by (Res), which is to say $a \leq \sim\big(a * \sim(a^2)\big)$. Therefore

$$a * \sim(a^2) \leq \sim a. \tag{7.2}$$

Also, $\sim(a^2) \leq a \Rightarrow a = 1$, so by (Res), $\sim(a^2) * a \leq a$. That is to say,

$$a * \sim(a^2) \leq a. \tag{7.3}$$

By (7.2) and (7.3), we have $a * \sim(a^2) \leq a \wedge \sim a$, whence $a * \sim(a^2) \leq a^2 \vee (a \wedge \sim a)$. That is, $\sim(a^2) * a \leq a^2 \vee (a \wedge \sim a)$, whence $\sim(a^2) \leq a \Rightarrow \big(a^2 \vee (a \wedge \sim a)\big)$ by (Res). But then $\sim(a^2) \wedge (a \vee \sim a) \leq a \Rightarrow \big(a^2 \vee (a \wedge \sim a)\big)$, which yields $a \leq \big(\sim(a^2) \wedge (a \vee \sim a)\big) \Rightarrow \big(a^2 \vee (a \wedge \sim a)\big)$. By De Morgan's laws and the law of double negation, $a \leq \sim\big(a^2 \vee (a \wedge \sim a)\big) \Rightarrow \big(a^2 \vee (a \wedge \sim a)\big)$, which is to say $a \leq \big(a^2 \vee (a \wedge \sim a)\big) + \big(a^2 \vee (a \wedge \sim a)\big)$. That is,

$$a \leq \big(a^2 \vee (a \wedge \sim a)\big)^{\bar{2}}.$$

Hence, $a \in \big(a^2 \vee (a \wedge \sim a)\big]$. Thus, $(a] \subseteq \big(a^2 \vee (a \wedge \sim a)\big]$. Hence, $\big(a^2 \vee (a \wedge \sim a)\big] = (a]$ and therefore

$$\Theta^{\mathbf{A}}\big(a^2 \vee (a \wedge \sim a), 0\big) = \Theta^{\mathbf{A}}(a, 0). \tag{7.4}$$

By (7.1), (7.4), and $(\mathbf{0}, \mathbf{1})$-congruence orderability, we conclude $a^2 \vee (a \wedge \sim a) = a$. It follows that $\mathbf{A} \models$ (6.1); thus, $\mathbf{A}$ is a Nelson residuated lattice by Theorem 6.1. □

We shall say that a CIBRL $\mathbf{A}$ satisfies the *filter separation property (FSP)* when, for all $a, b \in A$, it holds that

$$\text{if } [a) = [b) \text{ and } [\sim a) = [\sim b) \text{ then } a = b. \tag{FSP}$$

Read in the contrapositive, the (FSP) is saying that, for all $a \neq b$, there is a filter that separates either these two elements or their negations. In the context of compatibly involutive

CIRLs, the (FSP) can be equivalently restated as the condition

if $[a) = [b)$ and $(a] = (b]$ then $a = b$,

which we can read as follows: if $a \neq b$, then there is either a filter or an ideal that contains one and not the other.

One is reminded of well-known properties of lattices, e.g. that any two distinct elements can be (trivially) separated by a lattice filter; or of the more interesting theorem of Marshall Stone (1936, Theorem 64) that any two distinct elements of a Boolean algebra can be separated by a prime ideal (or, equivalently, by a prime filter). In fact, it would not be difficult to formulate our (FSP) also in terms of prime filters/ideals (for a suitable definition of primeness in the context of residuated lattices) and then use it to obtain an analogue of Stone's theorem for our algebras. Such results are at the basis of representation and duality theory for several classes of algebras having a distributive lattice term reduct.[8] In a topological context, the (FSP) may in fact be viewed as one among the useful separation of points properties, which can also sometimes be formulated in terms of homomorphisms into a generating algebra rather than filters or ideals (see e.g. the algebraic separation theorem of Clark and Davey (1998, p. 16)).

Our formulation of the (FSP) suggests that, in the context of (compatibly involutive) CIRLs, distinct elements may not be separable if we use filters alone: in particular, the elements a, a^2, and indeed a^k for all $k \geq 1$ will generate the same filter. However, if we take into account filters and ideals at the same time (either directly or through the negation), then we can recover a separation of points property. The obvious resemblance with the above-discussed notions of congruence orderability vs. $(\mathbf{0}, \mathbf{1})$-congruence orderability is not a superficial one, as the following corollary, which is the main result of the section, now shows.

Corollary 7.2 *For a compatibly involutive CIBRL* $\mathbf{A}$, *the following are equivalent:*

1. $\mathbf{A}$ *is a Nelson residuated lattice.*
2. $\mathbf{A}$ *is* $(\mathbf{0}, \mathbf{1})$*-congruence orderable.*
3. $\mathbf{A}$ *is* $(\mathbf{0}, \mathbf{1})$*-Fregean.*
4. $\mathbf{A}$ *has the (FSP).*
5. *For all* $a, b \in A$, *it holds that*

if $[a) = [b)$ *and* $(a] = (b]$ *then* $a = b$. □

[8] The topological study of Nelson algebras can be traced back at least to Cignoli (1986); Sendlewski (1990). For **N4**-lattices, a topological duality was first introduced in Odintsov (2010); see also Jansana and Rivieccio (2014). We notice in passing that the *special filters of the first kind* used in the duality of Odintsov (2010) coincide, within Nelson algebras, with our filters as defined in Sect. 2

The next corollary is an unpublished result of Busaniche and Cignoli (2008).

Corollary 7.3 *For a compatibly involutive CIRL* $\mathbf{A}$, *t.f.a.e.:*

1. $\mathbf{A}$ *is* $k+1$*-potent* ($k \in \omega$) *and satisfies the quasi-identity*

$$x^k \approx y^k \text{ and } (\sim x)^k \approx (\sim y)^k \text{ implies } x \approx y. \tag{7.5}$$

2. $\mathbf{A}$ *is a Nelson residuated lattice.*

Proof (1) ⇒ (2). Suppose $\mathbf{A}$ is $k+1$-potent. By $k+1$-potency and Lemma 5.1, we have that $\mathbf{A} \models$ (7.5) iff for all $a, b \in A$,

$\Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(0, b)$ and
$\Theta^{\mathbf{A}}(1, a) = \Theta^{\mathbf{A}}(1, b)$ implies $a = b$.

But this latter holds iff $\mathbf{A}$ is a Nelson residuated lattice.
(2) ⇒ (1). Suppose $\mathbf{A}$ is a Nelson residuated lattice. By Corollary 4.3, $\mathbf{A}$ is 3-potent. From this and Proposition 4.4, we easily conclude that $\mathbf{A} \models$ (7.5). □

We conclude the paper with an application of the main result. But first, call an algebra $\mathbf{A}$ with constant terms $\mathbf{c}$ and $\mathbf{d}$ *properly* $(\mathbf{c}, \mathbf{d})$*-congruence orderable* if $\mathbf{A}$ is $(\mathbf{c}, \mathbf{d})$-congruence orderable and neither $\mathbf{c}$-congruence orderable nor $\mathbf{d}$-congruence orderable.

Theorem 7.4 *Let* $\mathbf{A}$ *be a non-trivial subdirectly irreducible Nelson residuated lattice with monolith* μ. *Either* $\mathbf{A}$ *is simple or* $|0/\mu| = 2 = |1/\mu|$, *and all other* μ*-blocks are trivial.*

Proof We first observe that $\mathbf{A}$ is $(\mathbf{0}, \mathbf{1})$-congruence orderable by Corollary 7.2, since it is a Nelson residuated lattice. Suppose $\mathbf{A}$ is either $\mathbf{1}$-congruence orderable or $\mathbf{0}$-congruence orderable. In either case, by Corollary 5.10 we have that $\mathbf{A}$ is (term equivalent to) a Boolean algebra. As $\mathbf{A}$ is subdirectly irreducible, we conclude that $\mathbf{A}$ is simple.

Observe next from the theory of Nelson algebras that, to within isomorphism, the only simple Nelson residuated lattices are the 2-element Boolean algebra $\mathbf{2}$ (considered as a Nelson residuated lattice) and the 3-element chain $\mathbf{3}$.

Assume now that $|A| \geq 4$ and that $\mathbf{A}$ is neither $\mathbf{0}$-congruence orderable nor $\mathbf{1}$-congruence orderable. Then, $\mathbf{A}$ is properly $(\mathbf{0}, \mathbf{1})$-congruence orderable; moreover, $\mathbf{A}$ is subdirectly irreducible and not simple. Since $\mathbf{A}$ is subdirectly irreducible, from Corollary 2.2 we have that there exists an element $m_1 < 1$ such that for every $c < 1$ there exists a positive integer n for which $c^n \leq m_1$ holds. The monolith of $\mathbf{A}$ is thus $\Theta^{\mathbf{A}}(1, m_1)$.

Let $\mathbf{A}_h^\partial$ be the 'horizontal' dual of $\mathbf{A}$. Since $\mathbf{A}$ and $\mathbf{A}_h^\partial$ are term equivalent, they have the same congruences. By hypothesis, therefore, $\mathbf{A}_h^\partial$ is subdirectly irreducible, so by Corollary 3.4, there exists an element $0 < m_2$ such that for any $0 < d$ there exists a positive integer n for which $m_2 \leq d^{\bar{n}}$ holds. So the monolith of $\mathbf{A}_h^\partial$ is $\Theta^{\mathbf{A}_h^\partial}(0, m_2)$.

As the congruences on $\mathbf{A}$ and $\mathbf{A}_h^\partial$ coincide, we have that $\Theta^{\mathbf{A}}(0, m_2) = \Theta^{\mathbf{A}_h^\partial}(0, m_2)$, by properties of principal congruences. Therefore, $\Theta^{\mathbf{A}}(1, m_1) = \mu = \Theta^{\mathbf{A}}(0, m_2)$. Evidently $1 \neq m_1$ and $0 \neq m_2$. To complete the proof, we first establish the following

Claim *Let $a, b \in A$. If $a \neq 0, 1$ and $b \neq 0, 1$ and $a \equiv b \pmod{\mu}$, then $a = b$.*

Proof (of the claim) By our assumptions on a and b, for every $\theta \in \operatorname{Con} \mathbf{A}$ we have that $(0, a) \in \theta$ iff $(0, b) \in \theta$ and $(1, a) \in \theta$ iff $(1, b) \in \theta$. This implies $\Theta^{\mathbf{A}}(0, a) = \Theta^{\mathbf{A}}(0, b)$ and $\Theta^{\mathbf{A}}(1, a) = \Theta^{\mathbf{A}}(1, b)$, whence $a = b$ since $\mathbf{A}$ is properly $(\mathbf{0}, \mathbf{1})$-congruence orderable. □

To complete the proof of the theorem, we establish three cases:

- $|0/\mu| = 2$. Suppose there exist $a, b \in A$ such that $a, b \neq 0$ and $a \equiv_\mu 0$ and $b \equiv_\mu 0$. If $a = 1$, then $0 \equiv_\mu 1$ and $\mu = \Theta^{\mathbf{A}}(0, 1) = \nabla_A$. But then $\mathbf{A}$ is simple, a contradiction. Hence, $a \neq 1$ and an analogous argument establishes $b \neq 1$. By the claim, $a = b$. Hence, $0/\mu$ has at most two elements. Since $0 \neq m_1 \in \mu$, $0/\mu$ has exactly two elements.
- $|1/\mu| = 2$. This follows by an argument analogous to the case $|0/\mu| = 2$.
- $|c/\mu| = 1$, for $c \neq 0, 1$. Let $c \in A$ be such that $c \neq 0, 1$. Let $a, b \in c/\mu$. Then, $a \not\equiv 0 \pmod{\mu}$ and $a \not\equiv 1 \pmod{\mu}$, whence $a \neq 0$ and $a \neq 1$. A similar observation shows $b \neq 0$ and $b \neq 1$. By the claim, $a = b$. Hence, c/μ has exactly one element.

Conjoining the preceding three cases, we see that $|0/\mu| = 2 = |1/\mu|$, and all other μ-blocks are trivial, completing the proof. □

We close the section with the following result, which is basically Sendlewski (1984, Theorem 2.1); see also Cornejo and Viglizzo (2018, Theorem 6.12).

Corollary 7.5 *For a Nelson algebra $\mathbf{A}$, t.f.a.e.:*

1. *$\mathbf{A}$ is subdirectly irreducible.*
2. *$\mathbf{A}$ has a unique atom.*
3. *$\mathbf{A}$ has a unique co-atom.*
4. *$\mathbf{A}$ has both a unique co-atom c and a unique atom $\sim c$.* □

8 Future work

In this final section, we collate—in the form of open problems—some directions for future research that we find particularly promising.

Problem 8.1 Is there an analogue of Corollary 7.2 that holds for **N4**-lattices?

Let $x \overset{\mathbf{RM}}{\Rightarrow} y$ abbreviate $\big(x \wedge (y \Rightarrow y)\big) \Rightarrow y$ and $x \twoheadrightarrow y$ abbreviate $x \overset{\mathbf{RM}}{\Rightarrow} (x \overset{\mathbf{RM}}{\Rightarrow} y)$. Recent work due to Spinks and Veroff (2018) shows that, to within term equivalence, **N4**-lattices may be presented as certain algebras $\langle A; \wedge, \vee, *, \Rightarrow, \sim\rangle$ that: (i) generalise compatibly involutive CIRLs by dropping integrality;[9] and (ii) satisfy the following paraconsistent analogue of the Nelson identity, *viz.*

$$(x \twoheadrightarrow y) \wedge (\sim y \twoheadrightarrow \sim x) \approx x \Rightarrow y. \tag{8.1}$$

In this presentation of **N4**-lattices *qua* residuated structures, the term function induced by $\twoheadrightarrow$ corresponds to the weak implication $\to$ of **N4**-lattices. We conjecture that over every appropriate algebra $\langle A; \wedge, \vee, *, \Rightarrow, \sim\rangle$, paraconsistent analogue (8.1) of the Nelson identity is equivalent to the following congruence condition holding for all $a, b \in A$:

$$\Theta^{\mathbf{A}}\big(a, \sim(a \Rightarrow a)\big) = \Theta^{\mathbf{A}}\big(b, \sim(b \Rightarrow b)\big) \text{ and }$$
$$\Theta^{\mathbf{A}}(a, a \Rightarrow a) = \Theta^{\mathbf{A}}(b, b \Rightarrow b) \text{ implies } a = b.$$

Problem 8.2 Characterise the $(\mathbf{0}, \mathbf{1})$-congruence orderable CIBRLs.

The $(\mathbf{0}, \mathbf{1})$-congruence orderable CIBRLs must form a class of algebras K such that the subclass of K that satisfies the self-inverting identity $\sim\sim x \approx x$ coincides with the variety of Nelson residuated lattices. In other words, K is some non-involutive generalisation of the variety of Nelson residuated lattices. Observations such as our Lemma 5.1 and Remark 5.5 hint that K may be the class of all CIBRLs satisfying (a weaker version of) the identity (Nelson), where $\sim$ (as usual) abbreviates implication into $\mathbf{0}$.

Problem 8.3 Investigate the class of all CIBRLs satisfying the Nelson identity.

Problem 8.2 suggests the class K of all CIBRLs satisfying the Nelson identity (Nelson) may be of independent interest. Preliminary investigations into K lend some credence to this hypothesis. We speculate that K may be first-order definitionally equivalent to the class of all algebras $\langle A; \wedge, \vee, \to, \sim, 0, 1\rangle$ of type $\langle 2, 2, 2, 1, 0, 0\rangle$ such that the following hold:

(N1′) The reduct $\langle A; \wedge, \vee, \sim, 0, 1\rangle$ is a quasi-De Morgan algebra (with lattice ordering $\leq$) in the sense of Sankappanavar (1987, Definition 2.2).

(N2′) The relation $\preceq$ defined for all $a, b \in A$ by $a \preceq b$ iff $a \to b = 1$ is a quasiorder on A.

[9] This corresponds to the move from the integral (Nelson) to the non-integral (**N4**-lattice) case.

(N3′) The relation $\Xi := \preceq \cap (\preceq)^{-1}$ is a congruence on the reduct $\langle A; \wedge, \vee, \to, 0, 1\rangle$, and the quotient algebra $\langle A; \wedge, \vee, \to, 0, 1\rangle/\Xi$ is a Heyting algebra.
(N4′) For all $a, b \in A$, it holds that $\sim(a \to b) \equiv \sim\sim(a \wedge \sim b) \pmod{\Xi}$.
(N5′) For all $a, b \in A$, it holds that $a \leq b$ iff $a \preceq b$ and $\sim b \preceq \sim a$.

Just as items (N1)–(N5) in the definition of **N4**-lattices guarantee that they can be equivalently presented as twist structures (Odintsov 2004), we conjecture that the above items may allow one to represent this new class of algebras via a twist-structure construction of some kind. However, since all twist structures known in the literature give rise, by their very definition, to an involutive negation, the construction itself will have to be generalised in order to accommodate the failure of the double negation law. Work in this direction has been recently done in Maia et al., and one can speculate whether the *non-involutive twist structures* of [43] might be suitably modified to obtain a representation theorem for 'non-involutive Nelson algebras'.

Problem 8.4 Investigate $(\mathbf{0}, \mathbf{1})$-congruence orderability from the perspective of general algebra. Is there a characterisation theorem for congruence permutable $(\mathbf{0}, \mathbf{1})$-congruence orderable varieties similar to the characterisation theorem for congruence permutable $\mathbf{1}$-congruence orderable varieties of Idziak et al. (2009)?

The variety of equivalential algebras (Kabziński and Wroński 1991) is the paradigmatic example of a congruence permutable $\mathbf{1}$-Fregean variety inasmuch as every congruence permutable Fregean variety of algebras V possesses a binary term that turns every member of V into an equivalential algebra. This result (proved in a stronger form) is the main theorem of Idziak et al. (2009).

Problem 8.5 What is the logical counterpart, if any, of being $(\mathbf{0}, \mathbf{1})$-Fregean?

Recall from Czelakowski and Pigozzi (2004, Definition 59) that a deductive system $\mathbf{L}$ over a language type Λ is *Fregean* if, for every theory T of $\mathbf{L}$, the relativised interderivability relation $\dashv\vdash^T_{\mathbf{L}}$ defined for all Λ-formulas φ, ψ by

$$\varphi \dashv\vdash^T_{\mathbf{L}} \psi \quad \text{iff} \quad T, \varphi \vdash_{\mathbf{L}} \psi \text{ and } T, \psi \vdash_{\mathbf{L}} \varphi$$

is a congruence relation on the formula algebra $\mathbf{Fm}_\Lambda$; such logics thus enjoy a very strong property of replacement of equivalents (Font and Jansana 2009, p. 68). By Czelakowski (2001, Theorem 6.3.1), a regularly algebraisable logic is Fregean iff its equivalent quasivariety is $\mathbf{1}$-Fregean for some constant term $\mathbf{1}$; still stronger results are given in Czelakowski and Pigozzi (2004, Section 3.1). Given the relationship between Fregean logics and Fregean algebras, it is natural to enquire as to the logical counterpart, if any, of the property of being $(\mathbf{0}, \mathbf{1})$-Fregean.

Added in proof In Rivieccio and Spinks (2018), the authors generalise the well-known Vakarelov/Fidel twist structure semantics to a semantics for a non-involutive analogue of Nelson algebras; this partly addresses Problem 8.3. In ongoing work, the authors also show that the $(\mathbf{0}, \mathbf{1})$-congruence orderable CIBRLs are exactly the CIBRLs satisfying the Nelson identity, where negation $\sim$ is introduced as implication into $\mathbf{0}$; this resolves Problem 8.2.

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Compliance with ethical standards

Conflict of interest All three authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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