Soft Computing (2022) 26:8403–8426
https://doi.org/10.1007/s00500-022-07221-6

FUZZY SYSTEMS AND THEIR MATHEMATICS

On the residuation principle of *n*-dimensional R-implications

Rosana Zanotelli[1] · Bruno Moura[1] · Renata Reiser[1] · Benjamin Bedregal[2]

Accepted: 23 May 2022 / Published online: 28 June 2022

Abstract

The essential idea of the residuation principle plays a fundamental role in the residuated lattice theory based on R-implications, which are derived from left-continuous t-norms providing a general logical framework frequently applied to achieve solutions for multiple criteria and decision-making problems. The definition of n-dimensional fuzzy R-implications (n-DRI) is introduced, showing that the main properties of R-implications on the unit interval [0,1] can be preserved in the n-dimensional upper simplex $L_n([0,1])$, based on left-continuous n-dimensional t-norms. We also show a construction method for this class of implications, studying its relationships with intrinsic properties such as identity, neutrality, ordering and exchange principles. In addition, by considering the action of n-dimensional automorphisms, the conjugate of n-dimensional fuzzy R-implications is studied. The characterization of n-dimensional fuzzy R-implications and a methodology to obtain these operators from n-dimensional aggregations on $L_n([0,1])$ is discussed, as the left-continuous n-dimensional t-norms. The representable Łukasiewicz implication and minimum aggregation on $L_n([0,1])$ are considered to compare multiple alternatives in both approaches: (i) using admissible linear orders in $L_n([0,1])$ provided by a sequence of aggregations and (ii) applying the arithmetic means in n-dimensional data application, based on multiple attributes related to a selection of the best CIM (computer-integrated manufacturing) software systems obtained from decision maker evaluations. The theoretical results on n-DRI are carries out to the fuzzy module evaluation in cloud computing environments.

Keywords *n*-Dimensional fuzzy sets · *n*-Dimensional R-implications · Residuation property · Cloud computing

1 Introduction

The notion of an n-dimensional fuzzy set (n-DFS) or L_n-fuzzy set theory was introduced by Shang et al. (2010) as a special class of L-fuzzy set theory, generalizing the theories underlying many other multivalued fuzzy logics: the interval-valued fuzzy set theory (IVFS) (Sambuc 1975), the Atanassov's intuitionistic fuzzy set theory (Atanassov 1986) (A-IFS) and its interval-valued approach (Atanassov and Gargov 1998). In the L_n-fuzzy set theory, n-dimensional fuzzy set membership values are n-tuples of real numbers in [0, 1], ordered in increasing order, called n-dimensional intervals.

So, as the first question addressed in this work, does n-dimensional intervals plays a role in the development of scientific investigation? A historical and hierarchical analysis of n-DFS and other important extensions of the fuzzy set (FS) theory have been extensively studied, as highlighted by Bustince et al. (2016). As the general idea, an n-DFS considers several uncertainty levels in its membership functions, adding degrees of freedom and making it possible to directly model uncertainties in computational systems based on FS (Bedregal et al. 2012). Such uncertainties are frequently associated with many causes such as using n-ary operators modelling imprecise parameters in time-varying systems or applying distinct expert knowledge possibly obtained from questionnaires, and also including uncertain words from natural language provided by groups of experts.

Moreover, in what sense does n-dimensional fuzzy logic differ from other ones, as the hesitant fuzzy logic? As an

✉ Renata Reiser
reiser@inf.ufpel.edu.br

Rosana Zanotelli
rzanotelli@inf.ufpel.edu.br

Bruno Moura
brunomourapaz@gmail.com

Benjamin Bedregal
bedregal@dimap.ufrn.br

1 Center of Technological Development, Federal University of Pelotas, Pelotas, RS, Brazil

2 Department of Informatics and Applied Mathematics, Federal University of Rio Grande do Norte, Natal RN,, Brazil

extension of fuzzy logic (FL) as old as HFS (Torra 2010) and typical hesitant fuzzy sets (THFS) (Bedregal et al. 2014), but less exploited, the study of n-DFS motivates us to promote new investigations, consolidating this multi-valued logical approach. In particular, the possibility of modelling repeating membership degrees in an n-dimensional interval and so, preserving the frequency of such occurrence, which are based on the expert's opinions. It seems a promising area in applied researches, providing solutions for multiple criteria and decision-making problems (MCDM).

In addition, we can ask about the advantages of n-dimensional R-implications? Klement and Navara (1999), themselves predominantly started with a left-continuous t-norm and studied the corresponding class of residual implication named as R-implications. Since there, R-implications have been used as a fundamental methodology in approximate reasoning and modelling fuzzy rules in inference systems (Bedregal et al. 2009). In the broad sense, it is frequently applied to fuzzy control, analysis of vagueness in natural language and techniques of soft-computing. In the narrow sense, as the case in this article, the extension of theoretical research of R-implications contributes to a branch of many valued logic and algebraic logics enabling an investigation of deep logical questions involving the residuation property (Alcalde et al. 2005; Baczyński 2004). See other results covering operational semantics of programming languages (Espana and Estruch 2004) and including several classes of algebraic logics (Paiva et al. 2021; Santiago et al. 2019a, b). Moreover, an R-implication derived from a left-continuous t-norm is a general logical framework in mathematical morphology (González-Hidalgo and Massanet 2014), defining the morphological operators as fuzzy dilation and fuzzy erosion. Another relevant application of the residual implications in image processing Barrenechea et al. (2011); Shi et al. (2013) is concerned with subsethood fuzzy sets and similarity measures (Santos et al. 2019), performing the comparison of digital images represented by multi-valued fuzzy sets. And, the essential idea of residuation is also applied in automata theory (Farias et al. 2016).

To sum up, we are interested in the concept of n-DFS underlying the formal $L_n([0, 1])$-residuation theory related to n-dimensional R-implications (n-DRI), focusing on solutions in the MCDM applied research area. For that, we aim to expand the research from Zanotelli (2020) and completely new results have been included in this new study of n-DRI.

1.1 Main contribution

The main results of n-dimensional R-implications (n-DRI) on the class of n-dimensional simplex attempt to answer the following research questions, followed by the corresponding methodology to achieve their appropriate answers.

– In the previous literature on n-DFS, what basic concepts of fuzzy connectives need to be reported in order to consolidate the study of R-implications in the n-dimensional upper simplex $L_n([0, 1])$?

Based on the preliminaries studies, the extension of fuzzy implication from unit interval $[0, 1]$ to $L_n([0, 1])$ is revised. We leverage the main concepts on conjugate and representable n-DI, including the study of aggregating functions. In particular t-norms along with fuzzy negations are related to notions for n-dimensional intervals. Additionally, their interrelationship with the n-dimensional aggregation operator is investigated, exploring the notion of Moore-continuity and intrinsic properties as identity and exchange principles.

– How to extend the investigation of residual theory to the n-dimensional upper simplex $L_n([0, 1])$? More precisely, what are the necessary and sufficient conditions supporting that an n-DRI can be generated from a n-DT? And, in the reverse construction, what properties guarantee that an n-DT can be induced by an n-DRI?

The residuation principle study is relevant, leading to the characterization of the families of n-DRI based on the left-continuous n-DT. So, this work establishes a method of regaining (including minimal conditions) the residual operator, connecting an n-DRI together with an n-DT. In addition, a constructive methodology to obtain n-DRI is proposed based on the characterization to left-continuous n-DT, which is generated by n-DI as an residual implication on $L_n([0, 1])$. Thus, the study of conjugate-operator $\mathcal{I}^{\phi}$ on $L_n([0, 1])$ emphasizes the ϕ-conjugation on the class of n-DRI.

– How to construct n-DRI based on n-DA operators?

The interrelationship with the interval-valued aggregation and R-implications can be extended, consolidating a constructive method to obtain the n-DRI $\mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$ based on the minimum operator and left-continuous t-norms. And, analogously, the operator $\mathcal{T}_{\mathcal{I}_{I_1,\ldots,I_n}}$ can be defined, exploring the minimal conditions under which it forms an adjoint par with corresponding residuum operator $\mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$.

– And finally, what about possible applications of the constructive method to solve problems in on MCDM areas?

Using representable Łukasiewicz n-DRI and the minimum operator on $L_n([0, 1])$, we are able to compare multiple alternatives in both approaches: (i) using admissible linear orders in $L_n([0, 1])$ provided by a $\mathcal{M}$-sequence of aggregations; and (ii) applying the arithmetic means in n-dimensional data. The selected case study is based on multiple attributes related to a selection of the best CIM (computer-integrated manufacturing) software systems obtained from three decision maker evaluations. Concluding, in order to validate the theoretical results on the n-DRI, an application is executed in the Int-FLBCC environment, providing a evaluation for host selection of cloud computing (CC) allocation of virtual machines (VM).

1.2 Related papers

Extending the seminal studies of n-DFS, the main related papers exploring logical properties of $L_n([0,1])$-fuzzy connectives are summarized in Table 1, also reporting aspects related to $L_n([0,1])$-fuzzy reasoning.

They considered the following themes: (i) extending the study of fuzzy implications from $[0,1]$ to $L_n([0,1])$ and connecting main properties of n-dimensional fuzzy implications; (ii) formalizing multidimensional fuzzy sets as extension in which the membership values can have distinct dimensions; (iii) considering the Moore-continuous metric and representable Moore-continuous n-dimensional fuzzy negations, preserving equilibrium points from $[0,1]$ to $L_n([0,1])$-fuzzy approach; (iv) characterizing the t-representability n-dimensional triangular norms and conorms in terms of inclusion monotonicity property; (v) studying fuzzy grades based on n-dimensional fuzzy set; (vi) discussing on $L_n([0,1])$-fuzzy cut sets; and (vii) comparing $L_n([0,1])$-fuzzy ordered vectors via admissible orders. Many of them are related to theoretical research (TR) and decision-making problems (DMP), areas providing support to achieve solutions in decision-making problems.

Consolidating the research on the n-dimensional upper simplex $L_n([0,1])$, this paper studies the possibility of dealing with the main properties of n-DRI, exploring their application to solve a CIM-MCMD problem extended from Wen et al. (2018).

1.3 Outline of the paper

This paper is organized as follows. In preliminaries, we report the main characteristics of n-dimensional intervals in the n-dimensional upper simplex $L_n([0,1])$, including concepts as metric and left- and right-continuity w.r.t. the Moore-continuity.

We also consider the action of automorphisms and adjoint pairs on $L_n([0,1])$. In Sect. 2.5, n-dimensional fuzzy negations are briefly discussed based on results from Bedregal et al. (2012), also defining the concept of a dual constructor on $L_n([0,1])$.

In sequence, n-dimensional aggregation operators are considered joining with the definition of admissible linear orders on $L_n([0,1])$. In particular, n-dimensional t-norms are also studied including main properties in the class of representable t-norms, dual and conjugate constructions on $L_n([0,1])$.

In Sect. 3, the concepts and reasonable properties of n-dimensional fuzzy implications on $L_n([0,1])$ are also studied, as well as the properties assuring their representable expressions and conjugation constructions.

The core of the paper sits in Sects. 4 and 5, where properties of R-implications are extended to the n-dimensional interval fuzzy set approach. The residuation property and their main characterization based on left-continuity of n-dimensional t-norms are studied. In such context, the action of n-dimensional automorphisms is also discussed. And then, a method to obtain n-DRI from n-DT operators, characterizing the operators $\mathcal{I}_{T_1,\ldots,T_n}$ and reporting conditions under which main properties of implications are preserved by such operators.

An application in CIM-MCDM area is extended from hesitant fuzzy sets to n-dimensional fuzzy sets in Sect. 6, based on Łukasiewicz implication operator. And, the conclusion section highlights main results and briefly comments on further work.

2 Preliminaries

In this section, we will briefly review some basic concepts of FL, concerned with the study of n-dimensional intervals, which can be found in Bedregal et al. (2011, 2018).

2.1 *n*-Dimensional fuzzy sets

Let $X \neq \emptyset$ and $\mathbb{N}_n = \{1, 2, \ldots, n\}$. By Shang et al. (2010), an n-dimensional fuzzy set B over X is given as

$$B = \{(\mathrm{x}, \mu_{B1}(\mathrm{x}), \ldots, \mu_{Bn}(\mathrm{x})) : \mathrm{x} \in X\}, \tag{1}$$

when all membership functions $\mu_{Bi} : X \to [0,1], \forall i \in \mathbb{N}_n$ verify the condition $\mu_{B1}(\mathrm{x}) \le \cdots \le \mu_{Bn}(\mathrm{x}), \forall \mathrm{x} \in X$.

The n-dimensional upper simplex is given as Bedregal et al. (2011)

$$L_n([0,1]) = \{\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n : x_1 \le \cdots \le x_n\}, \tag{2}$$

and its elements are called n-dimensional intervals. For $i = 1, \ldots, n$, the i-th projection of $L_n([0,1])$ is the function $\pi_i : L_n([0,1]) \to [0,1]$ given by $\pi_i(x_1, \ldots, x_n) = x_i$. For a degenerate element $\mathbf{x} \in L_n([0,1])$, it holds that $\pi_i(\mathbf{x}) = \pi_j(\mathbf{x}), \forall i, j \in \mathbb{N}_n$, and it is denoted by $/x/$. So, $x = \pi_i(/x/)$, $\forall i \in \mathbb{N}_n$.

Table 1 Distribution of papers based on n-dimensional fuzzy sets

Technique	Contribution	Class
The n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite-valued fuzzy sets (Shang et al. 2010)	Definition of cut set on n-**dimensional fuzzy sets** studying the decomposition and representation theorems of n-DFS	TR
A class of fuzzy multisets with a fixed number of memberships (Bedregal et al. 2012)	Defining a generalization of Atanassov's operators for n-dimensional fuzzy values (called n-dimensional intervals)	DMP
A characterization theorem for t-Representable n-dimensional triangular norms (Bedregal et al. 2011)	Generalization of the notion of t-representability for n-**dimensional t-norms** providing a characterization theorem for this class of n-dimensional t-norms	TR
On n-dimensional strict fuzzy negations (Mezzomo et al. 2016)	Investigating properties inherent from the class of representable n-**dimensional strict fuzzy negations**	TR
Natural n-dimensional fuzzy negations for n-dimensional t-norms and t-conorms (Mezzomo et al. 2017)	Studying n-**dimensional fuzzy negations**, applying these studies mainly on natural n-dimensional fuzzy negations for n-**dimensional triangular norms and triangular conorms**	TR
An algorithm for MCDM using n-DFS, admissible orders and OWA operators (De Miguel et al. 2017)	Introducing the concept of **admissible order for** n-**DFS** presenting a construction method for those orders and studying **OWA operators for aggregating tuples**	DMP
n-Dimensional intervals and fuzzy S-implications (Zanotelli et al. 2018)	Study main properties characterizing the class of **S-implications on** $L_n([0,1])$	TR
Towards the study of main properties of n-dimensional QL-implicators (Zanotelli et al. 2018)	n-**dimensional QL-implicators** are studied considering duality and conjugation operators	TR
Equilibrium point of representable Moore-continuous n-dimensional interval fuzzy negations (Mezzomo et al. 2018)	Presentation of the main conditions guaranteeing the existence of equilibrium point in classes of **representable** (Moore-continuous) n-**dimensional interval fuzzy negations**	TR
Moore-continuous n-dimensional interval fuzzy negations (Mezzomo et al. 2018)	Characterizing the notion of (continuous) n-dimensional interval Moore-metric using the definitions of (continuous) Moore metric and n-**dimensional interval fuzzy negations**	TR
n-Dimensional fuzzy negations (Bedregal et al. 2018)	Presenting n-**representable fuzzy negations** on $L_n([0,1])$, analyzing main classes such as continuous and monotone by part	DMP
n-Dimensional Interval Uninorms (Mezzomo et al. 2019)	Presenting the notion of the neutral element, degenerate element, and monotonicity on $L_n([0,1])$ of **representable uninorms**	TR
Study on n-Dimensional R-implications (Zanotelli et al. 2019)	Study of **R-implications** on $L_n([0,1])$ considering the conditions under which main properties are preserved, and their representability from $[0,1]$ to $L_n([0,1])$	DMP
n-Dimensional intervals and fuzzy (S,N)-implications (Zanotelli et al. 2020)	Study main properties characterizing the class of(S,N)-**implications on** $L_n([0,1])$	TR
Multidimensional Fuzzy Sets (De Lima et al. 2021)	Presenting **concepts of multidimensional fuzzy sets** as a generalization of the n-dimensional fuzzy sets in which the elements can have different dimensions	TR
n-Dimensional Ordered Vectors (Milfont et al. 2021)	Aggregation functions on n-**dimensional ordered vectors** equipped with an admissible order and an application in multi-criteria group decision-making	DMP

Remark 1 The usual order $\leq$ on $[0, 1]$ is extended to higher dimensions, for $\mathbf{x}, \mathbf{y} \in L_n([0, 1])$, as follows:

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow x_i \leq y_i, \forall i \in \mathbb{N}_n. \tag{3}$$

For every non-empty set $\mathbf{A} \subseteq L_n([0, 1])$, the supremum and infimum with respect to $\leq$-order is given as:

$$\sup \mathbf{A} = (\sup\{\pi_1(\mathbf{x}) : \mathbf{x} \in \mathbf{A}\}, \ldots, \sup\{\pi_n(\mathbf{x}) : \mathbf{x} \in \mathbf{A}\}),$$
$$\inf \mathbf{A} = (\inf\{\pi_1(\mathbf{x}) : \mathbf{x} \in \mathbf{A}\}, \ldots, \inf\{\pi_n(\mathbf{x}) : \mathbf{x} \in \mathbf{A}\}).$$

In particular, when $\mathbf{A} = \{\mathbf{x}, \mathbf{y}\}$ we will use the infix notation $\mathbf{x} \vee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y}$ instead of $\sup \mathbf{A}$ and $\inf \mathbf{A}$, respectively.

According Bedregal et al. (2011), $\mathcal{L}_n([0, 1]) = (L_n([0, 1]), \vee, \wedge, /0/, /1/)$ is a distributive complete lattice, which is a continuous lattice in the sense of Gierz et al. (2003), with $/0/$ and $/1/$ being their bottom and top elements, respectively. Observe that $L_1([0, 1]) = [0, 1]$ and $L_2([0, 1])$ correspond to the usual lattice of all the closed subintervals on $[0, 1]$.

2.2 Continuity on $L_n([0, 1])$

The continuity of functions is based on the particular topology of their domain and codomain. There are topological spaces which can be derived from metrics, and in this case, the continuity of functions has an equivalent definition based on the metrics. In this section, we will consider main properties and relevant notions of continuity in metric spaces. For additional studies, see Dugundji (1966).

The function $d_M^n : L_n([0, 1]) \times L_n([0, 1]) \to \mathbb{R}^+$ given as

$$d_M^n(\mathbf{x}, \mathbf{y}) = \max(|\pi_1(\mathbf{x}) - \pi_1(\mathbf{y})|, \ldots, |\pi_n(\mathbf{x}) - \pi_n(\mathbf{y})|) \tag{4}$$

is a metric on $L_n([0, 1])$ called the n-dimensional interval Moore-metric on $L_n([0, 1])$, see details in Mezzomo et al. (2018), Proposition 3.1.

Remark 2 When the natural immersion of $L_n([0, 1])$ in $[0, 1]^n$ is considered, the n-dimensional interval Moore-metric coincides with the Chebyshev-metric (Mendelson 1990). Moreover, d_M^1 is the usual distance on real numbers restricted to $[0, 1]$ and d_M^2 is the Moore-metric (Dimuro et al. 2011).

Remark 3 Analogously what happen with the A-IFS context (which is isomorphic to IVFS), the equivalence between the Euclidean and the Hamming metrics is presented in Deschrijver et al. (2004), from a topological point of view, determining the analogous continuity notion. In particular, the (Deschrijver et al. 2004, Theorem 5.1) proves that the topologies induced by both metrics are the same. In analogous sense, the Moore and Euclidean metrics are both topologically equivalent.

Of course, the context proposed in Deschrijver and Kerre (2003) takes the complete lattice of intuitionistic fuzzy values, given as $L^* = \{(x, y) \in [0, 1]^2 : x + y \leq 1\}$. And, the (Deschrijver and Kerre 2003, Theorem 2.3) (see the third item) shows that IVFS and A-IFS are identical from a mathematical point of view. In fact, the presented bijection Ψ_3 explicits the equivalence between the Euclidean and Hamming metrics in L^* from the Euclidean and Hamming metrics on $L^2([0, 1])$, respectively.

One can observe that, since d_M^n is a metric, a continuity notion for n-dimensional unary functions is also verified. Thus, the study of the continuity of n-dimensional functions of arbitrary m-arity considers a corresponding metric on $(L_n([0, 1]))^m$.

Proposition 1 *Let $d_M^{n,m} : (L_n([0, 1]))^m \times (L_n([0, 1]))^m \to \mathbb{R}^+$ be a function such that, for $\mathbf{x} = (\mathbf{x_1}, \ldots, \mathbf{x_m}), \mathbf{y} = (\mathbf{y_1}, \ldots, \mathbf{y_m}) \in (L_n([0, 1]))^m$, it is given as follows:*

$$d_M^{n,m}(\mathbf{x}, \mathbf{y}) = \max(d_M^n(\mathbf{x_1}, \mathbf{y_1}), \ldots, d_M^n(\mathbf{x_m}, \mathbf{y_m})), \tag{5}$$

Then, $d_M^{n,m}$ is a metric on $(L_n([0, 1]))^m$.

Proof Clearly $d_M^{n,m}$ is symmetric and since d_M^n is a metric, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in (L_n([0, 1]))^m$ the next two conditions are verified:

1. $d_M^{n,m}(\mathbf{x}, \mathbf{y}) = 0$ iff $d_M^n(\mathbf{x_i}, \mathbf{y_i}) = 0, \forall i \in \mathbb{N}_n$ iff $\mathbf{x_i} = \mathbf{y_i}$, $\forall i \in \mathbb{N}_n$ iff $\mathbf{x} = \mathbf{y}$;
2. $\forall i \in \mathbb{N}_n$, $d_M^n(\mathbf{x_i}, \mathbf{z_i}) \leq d_M^n(\mathbf{x_i}, \mathbf{y_i}) + d_M^n(\mathbf{y_i}, \mathbf{z_i})$ then we obtain that

$$\begin{aligned}
&\max(d_M^n(\mathbf{x_1}, \mathbf{z_1}), \ldots, d_M^n(\mathbf{x_1}, \mathbf{z_1}))\\
&\leq \max(d_M^n(\mathbf{x}_1, \mathbf{y}_1) + d_M^n(\mathbf{y}_1, \mathbf{z}_1), \ldots\\
&\ldots, d_M^n(\mathbf{x}_m, \mathbf{y}_m) + d_M^n(\mathbf{y}_m, \mathbf{z}_m))\\
&= \max(d_M^n(\mathbf{x_1}, \mathbf{y_1}), \ldots, d_M^n(\mathbf{x_m}, \mathbf{y_m})) + \ldots\\
&\ldots + \max(d_M^n(\mathbf{y_1}, \mathbf{z_1}), \ldots, d_M^n(\mathbf{y_m}, \mathbf{z_m})).
\end{aligned}$$

So, $d_M^{n,m}(\mathbf{x}, \mathbf{z}) \leq d_M^{n,m}(\mathbf{x}, \mathbf{y}) + d_M^{n,m}(\mathbf{y}, \mathbf{z})$. Concluding, Proposition 1 is also verified. □

A function $F : (L_n([0, 1]))^m \to L_n([0, 1])$ is Moore-continuous if it is a $(d_M^{n,m}, d_M^n)$-continuous. In the following, the convergence of sequences and limits on the set of real intervals $[0, 1]$ are extended to $L_n([0, 1])$.

Definition 1 A function $f : \mathbb{N} \to L_n([0, 1])$ is an n-dimensional interval sequence (n-DS) if $f(i) = \mathbf{x}_i, \forall i \in \mathbb{N}$ usually denoted by $(\mathbf{x}_i)_{i \in \mathbb{N}}$.

Definition 2 An n-DS $(\mathbf{x}_i)_{i\in\mathbb{N}}$ converge to $\mathbf{a} \in L_n([0,1])$, denoted by $\mathbf{x}_i \to \mathbf{a}$ or $\lim_{n\to\infty} \mathbf{x}_i = \mathbf{a}$, if the following holds $\forall \varepsilon > 0, \exists n_0 \geq 0 such that d^n_M(\mathbf{x}_i, \mathbf{a}) < \varepsilon, \forall n > n_0$.

By previous definitions, $\forall k \in \mathbb{N}_{n-1}$, it holds that

(i) $\mathbf{x}_k \leq \mathbf{x}_{k+1}$, then $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is an increasing n-DS; and
(ii) $\mathbf{x}_k \geq \mathbf{x}_{k+1}$, then $(\mathbf{x}_i)_{i\in\mathbb{N}}$ is an decreasing n-DS.

A function $\mathcal{F} : (L_n([0,1]))^2 \to L_n([0,1])$ is a (right) left-continuous for the first variable if it verifies for each countable (decreasing) increasing chain $(\mathbf{x}_i)_{i\in\mathbb{N}}$ on $(L_n([0,1]), \leq)$, the following property:

$$\left(\mathcal{RC} : \lim_{i\to\infty} \mathcal{F}(\mathbf{x}_i, \mathbf{y}) = \mathcal{F}(\lim_{i\to\infty} \mathbf{x}_i, \mathbf{y})\right) \tag{6}$$

$$\mathcal{LC} : \lim_{i\to\infty} \mathcal{F}(\mathbf{x}_i, \mathbf{y}) = \mathcal{F}(\lim_{i\to\infty} \mathbf{x}_i, \mathbf{y}) \tag{7}$$

And, analogous construction can be demanding to the second variable, by taking (decreasing) increasing chain $(\mathbf{y}_i)_{i\in\mathbb{N}}$ on $(L_n([0,1]), \leq)$.

Proposition 2 *Let $(\mathbf{x}_i)_{i\in\mathbb{N}}$ be an n-DS. Then, $(\mathbf{x}_i)_i \in \mathbb{N}$ converges to $\mathbf{a} \in L_n([0,1])$ iff $(\pi_k(\mathbf{x}_i))_{i\in\mathbb{N}}$ converges to $\pi_k(\mathbf{a})$, for each $k \in \mathbb{N}_n$.*

Proof Straightforward. □

Corollary 1 *Let $(\mathbf{x}_i)_{i\in\mathbb{N}}$ be an increasing (decreasing) n-DS. Then $(\mathbf{x}_i)_{i\in\mathbb{N}}$ converges to some $\mathbf{a} \in L_n([0,1])$.*

Proof Straightforward from Proposition 2 and from the following well-known two facts:

(i) each increasing (decreasing) bounded sequence of real numbers converges;
(ii) if $(x_i)_{i\in\mathbb{N}}$ and $(y_i)_{i\in\mathbb{N}}$ are two converging increasing (decreasing) sequences of real numbers such that $x_i \leq y_i$ for each $i \in \mathbb{N}$, then $\lim_{i\to\infty} x_i \leq \lim_{i\to\infty} y_i$.

□

2.3 Automorphisms on $L_n([0,1])$

The notion of preserving n-dimensional fuzzy automorphisms (n-DFA) is discussed now. In Bedregal et al. (2012), an n-DFA is defined as a function $\varphi : L_n([0,1]) \to L_n([0,1])$ which is bijective and the following condition is satisfied: $\mathbf{x} \leq \mathbf{y}$ iff $\varphi(x) \leq \varphi(y)$. And, $Aut(L_n([0,1]))$ and $Aut([0,1])$ denote the sets of all automorphims on $L_n([0,1])$ and $[0,1]$, respectively.

From (Bedregal et al. 2012, Theorem 3.4), let $\varphi : L_n([0,1]) \to L_n([0,1])$. Then, $\varphi \in Aut(L_n([0,1]))$ iff there exists $\psi \in Aut([0,1])$ such that $\varphi(\mathbf{x}) = (\psi(\pi_1(\mathbf{x})), \ldots, \psi(\pi_n(\mathbf{x})))$. So, we will denote φ by $\widetilde{\psi}$. Thus, the following holds:

$$\widetilde{\psi}(\mathbf{x}) = (\psi(\pi_1(\mathbf{x})), \ldots, \psi(\pi_n(\mathbf{x}))). \tag{8}$$

In Bedregal et al. (2012), when $\varphi \in Aut(L_n([0,1]))$ then φ is also Moore-continuous, strictly increasing such that $\varphi(/0/) = /0/$ and $\varphi(/1/) = /1/$. Moreover, by (Bedregal et al. 2012, Proposition 3.4) when $\psi \in Aut(U)$, the following holds:

$$\widetilde{\psi^{-1}} = \widetilde{\psi}^{-1}. \tag{9}$$

And, let a function $F : (L_n([0,1]))^m \to L_n([0,1])$ and an automorphism $\varphi \in Aut(L_n([0,1]))$. The action of φ over a F is described as the function $F^{\varphi} : (L_n([0,1]))^m \to L_n([0,1])$ given as

$$F^{\varphi}(\mathbf{x}_1, \ldots, \mathbf{x}_m) = \varphi^{-1}(F(\varphi(\mathbf{x}_1), \ldots, \varphi(\mathbf{x}_m))) \tag{10}$$

Thus, F^{φ} is said the conjugate of F.

2.4 Adjoint pair on $L_n([0,1])$

Let $\mathcal{F}, \mathcal{G} : (L_n([0,1]))^2 \to L_n([0,1])$. A pair $(\mathcal{F}, \mathcal{G})$ is an adjoint pair if it satisfies the residuation principle:

$$\mathcal{RP} : \qquad \mathcal{F}(\mathbf{x}, \mathbf{y}) \leq \mathbf{z} \Leftrightarrow \mathbf{y} \leq \mathcal{G}(\mathbf{x}, \mathbf{z}). \tag{11}$$

2.5 Fuzzy negations on $L_n([0,1])$

By Bedregal et al. (2012), the notion of fuzzy negation was extended to $L_n([0,1])$ and their main concepts are reported below.

Definition 3 A function $\mathcal{N} : L_n([0,1]) \to L_n([0,1])$ is an n-dimensional fuzzy negation (n-DN) if it satisfies:

$\mathcal{N}1$: $\mathcal{N}(/0/) = /1/$ and $\mathcal{N}(/1/) = /0/$;
$\mathcal{N}2$: If $\mathbf{x} \leq \mathbf{y}$ then $\mathcal{N}(\mathbf{x}) \geq \mathcal{N}(\mathbf{y})$.

An n-DN $\mathcal{N}$ is strict if it is Moore-continuous and verifies the condition $\mathcal{N}(\mathbf{x}) < \mathcal{N}(\mathbf{y})$ when $\mathbf{y} < \mathbf{x}$. Moreover, when an n-DN $\mathcal{N}$ satisfies the involution condition:

$\mathcal{N}3$: $\mathcal{N}(\mathcal{N}(\mathbf{x})) = \mathbf{x}$ for each $\mathbf{x} \in L_n([0,1])$,

it is called a strong n-DN.

Let $\varphi \in Aut(L_n([0,1]))$. As reported in literature, see (Bedregal et al. 2018, Proposition 4.2), $\mathcal{N}$ is (strict, strong)

n DN iff $\mathcal{N}^{\varphi}$ is (strict, strong) n-DN such that, for all $\mathbf{x} \in L_n([0,1])$, it holds that:

$$\mathcal{N}^{\varphi}(\mathbf{x}) = \varphi^{-1}(\mathcal{N}(\varphi(\mathbf{x}))). \tag{12}$$

Studies on n-DN extend the preliminary results on representability of fuzzy negations (FN) (Bedregal et al. 2012, Prop. 3.1), preserving their main properties. When $N_1, \ldots, N_n$ are FN such that $N_1 \leq \cdots \leq N_n$, then the function $\widetilde{N_1 \ldots N_n} : L_n([0,1]) \to L_n([0,1])$ is the representable n-DN given as:

$$\widetilde{N_1 \ldots N_n}(\mathbf{x}) = (N_1(\pi_n(\mathbf{x})), \ldots, N_n(\pi_1(\mathbf{x}))). \tag{13}$$

When $N = N_1 = \cdots = N_n$, $\widetilde{N_1 \ldots N_n}$ is denoted as $\widetilde{N}$.

Example 1 Considering $N_{D1}, N_{D2} : [0,1] \to [0,1]$ as fuzzy negations, respectively, given as follows:

$$N_{D1}(x) = \begin{cases} 1, if\, x = 0, \\ 0, otherwise; \end{cases} \quad N_{D2}(x) = \begin{cases} 0, if\, x = 1, \\ 1, otherwise; \end{cases}$$

and, $N_S, N_K, N_R : [0,1] \to [0,1]$ given as $N_S(x) = 1 - x$, $N_K(x) = 1 - \sqrt{x}$ and $N_R(x) = 1 - x^2$. From (13), the following is verified:

(i) $\widetilde{N_{D1}, N_R, N_S, N_K, N_{D2}} : L_n([0,1]) \to L_n([0,1])$ is a representable n-DN;
(ii) $\widetilde{N_{D1}}, \widetilde{N_S}, \widetilde{N_K}, \widetilde{N_R}, \widetilde{N_{D2}} : L_n([0,1]) \to L_n([0,1])$ are the related n-dimensional interval extensions.

Reporting the strategy to obtain fuzzy negations from n-DN, based on degenerate elements (Bedregal et al. 2018):

Proposition 3 *Let $\mathcal{N}$ be an n-DN. Then, the function $N_i : U \to [0,1]$ is a fuzzy negation defined by*

$$N_i(x) = \pi_i(\mathcal{N}(/x/)), \forall i \in \mathbb{N}_n, x \in U. \tag{14}$$

Now, exploring the continuity property presented in previous work (Mezzomo et al. 2018, Theorem 3.1):

Theorem 1 (Mezzomo et al. 2018, Theorem 3.1) *Let $N_1, \ldots, N_n$ be fuzzy negations such that $N_1 \leq \cdots \leq N_n$. Then, the n-DN $\widetilde{N_1 \ldots N_n} : L_n([0,1]) \to L_n([0,1])$ given by (13) is Moore-continuous iff every N_i, $i \in \mathbb{N}_n$, is continuous.*

Corollary 2 *Each strong n-DN is Moore-continuous.*

Concluding, let $\mathcal{N}$ be a strong n-DN and $\mathcal{F} : (L_n([0,1]))^n \to L_n([0,1])$ be a function. The $\mathcal{N}$-dual of $\mathcal{F}$ is the function $\mathcal{F}_{\mathcal{N}} : (L_n([0,1]))^n \to L_n([0,1])$ given as

$$\mathcal{F}_{\mathcal{N}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathcal{N}(\mathcal{F}(\mathcal{N}(\mathbf{x}_1), \ldots, \mathcal{N}(\mathbf{x}_n))). \tag{15}$$

2.6 Aggregations and admissible orders on $L_n([0,1])$

Data analysis using aggregation functions and dealing with new trends is unable to answer both mathematical and practical concerns, motivating for study of n-dimensional aggregation functions.

By Bedregal et al. (2018), a k-ary n-dimensional aggregation function (n-DA) $M : L_n([0,1])^k \to L_n([0,1])$ is a function satisfying, for all $(\mathbf{x}_1, \ldots, \mathbf{x}_k), (\mathbf{y}_1, \ldots, \mathbf{y}_k) \in L_n([0,1])^k$, the following conditions:

$\mathcal{A}1$: $M(/0/, \ldots, /0/) = /0/$ and $M(/1/, \ldots, /1/) = /1/$;
$\mathcal{A}2$: $\mathbf{x}_i \leq \mathbf{y}_i, \forall i \in \mathbb{N}_k \Rightarrow M(\mathbf{x}_1, \ldots, \mathbf{x}_k) \leq M(\mathbf{y}_1, \ldots, \mathbf{y}_k)$.

When $n = 1$ then $L_1([0,1]) = [0,1]$ and therefore, each 1-DA is a k-ary aggregation function $M : [0,1]^k \to [0,1]$.

Example 2 For $i \in \mathbb{N}_n$, $j \in \mathbb{N}_k$ consider $\pi_i(\mathbf{x}_j) \equiv \mathbf{x}_{ij}$. The arithmetic mean and minimum operators $AM, \wedge : (L_n([0,1]))^k \to L_n([0,1])$ are given as follows:

$$AM(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \frac{1}{k}\left(\sum_{i=1}^{k} \mathbf{x}_{i1}, \ldots, \sum_{i=1}^{k} \mathbf{x}_{in}\right); \tag{16}$$

$$\bigwedge\left(\mathbf{x}_1, \ldots, \mathbf{x}_k\right) = (\min_{i=1}^{n} x_{1i}, \ldots, \min_{i=1}^{n} x_{ki}\Big). \tag{17}$$

By De Miguel et al. (2017), a linear order $\sqsubseteq$ on $L_n([0,1])$ is called admissible if $\sqsubseteq$ refines $\leq$. So, it satisfies the following condition: $\mathbf{x} \leq \mathbf{y} \Rightarrow \mathbf{x} \sqsubseteq \mathbf{y}, \forall \mathbf{x}, \mathbf{y} \in L_n([0,1])$.

Definition 4 Let $\mathcal{M} = (M_1, \ldots, M_n)$ be a sequence of aggregation functions $M_i : [0,1]^n \to [0,1]$, $\forall i \in \mathbb{N}_n$. For $\mathbf{x}, \mathbf{y} \in L_n([0,1])$, we consider the following relations:

1. $\mathbf{x} \sqsubset_{\mathcal{M}} \mathbf{y}$ iff $\exists k \in \mathbb{N}_n$, $M_j(\mathbf{x}) = M_j(\mathbf{y})$, $\forall j \in \mathbb{N}_{k-1}$ and $M_k(\mathbf{x}) < M_k(\mathbf{y})$;
2. $\mathbf{x} \sqsubseteq_{\mathcal{M}} \mathbf{y}$ iff $\mathbf{x} \sqsubset_{\mathcal{M}} \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.

The next results are shown by De Miguel et al. (2017):

Proposition 4 *Let $\mathcal{M} = (M_1 \ldots M_n)$ be a sequence of aggregations functions $M_i : [0,1]^n \to [0,1]$, for $i \in \mathbb{N}_n$. The $\sqsubseteq_{\mathcal{M}}$-order on $L_n([0,1])$ is admissible iff for each $\mathbf{x}, \mathbf{y} \in L_n([0,1])$, it holds that:*

$$M_i(\mathbf{x}) = M_i(\mathbf{y}), \forall i \in \mathbb{N}_n \Leftrightarrow \mathbf{x} = \mathbf{y}.$$

Proposition 5 *Let $\mathcal{M} = (M_1, \ldots, M_n)$ be a sequence of aggregation functions $A_i : [0,1]^n \to [0,1]$, $\forall i \in \mathbb{N}_n$ which is given as $M_i(\mathbf{x}) = \alpha_{i1}(\pi_1(\mathbf{x})) + \cdots + \alpha_{in}(\pi_n(\mathbf{x}))$, whenever $\alpha_{i1} + \cdots + \alpha_{in} = 1$ and $0 \leq \alpha_{ij} \leq 1$, $\forall i, j \in \mathbb{N}_n$. The $\sqsubseteq_{\mathcal{M}}$-order on $L_n([0,1])$ is admissible iff the corresponding matrix $[M] = (\alpha_{ij})_{n \times n}$ is regular.*

Example 3 Let $M_1, M_2, M_3, M_4 : [0,1]^4 \to [0,1]$ be aggregations defined as follows:

$$M_1(\mathbf{x}) = 0.25x_1 + 0.25x_2 + 0.25x_3 + 0.25x_4;$$
$$M_2(\mathbf{x}) = 0.5x_1 + 0.15x_2 + 0.15x_3 + 0.2x_4;$$
$$M_3(\mathbf{x}) = 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.3x_4;$$
$$M_4(\mathbf{x}) = 0.1x_1 + 0.4x_2 + 0.1x_3 + 0.4x_4$$

Under the conditions of Proposition 5, $[M] = (\alpha_{ij})_{4\times 4}$ is a regular matrix meaning that $M_i(\mathbf{x}) = M_i(\mathbf{y}), \forall i \in \mathbb{N}_4 \Leftrightarrow \mathbf{x} = \mathbf{y}$. And, therefore, the $\sqsubseteq_{\mathcal{M}}$-order on $L_4(U)$ is admissible for $M = \{M_1, M_2, M_3, M_4\}$.

2.7 Triangular norms on $L_n([0,1])$

In Mezzomo et al. (2017), the notion of t-norms on [0, 1] was extended to $L_n([0,1])$, and their main properties are reported below.

Definition 5 (Bedregal et al. 2012, Def.3.4) A function $\mathcal{T} : (L_n([0,1]))^2 \to L_n([0,1])$ is an n-dimensional t-norm (n-DT) if it is commutative, associative, monotonic w.r.t. the $\leq$-order and has $/1/$ as its neutral element. For $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n([0,1])$, it is expressed as follows:

$\mathcal{T}1$: $\mathcal{T}(\mathbf{x}, \mathbf{y}) = \mathcal{T}(\mathbf{y}, \mathbf{x})$;
$\mathcal{T}2$: $\mathcal{T}(\mathbf{x}, \mathcal{T}(\mathbf{y}, \mathbf{z})) = \mathcal{T}(\mathcal{T}(\mathbf{x}, \mathbf{y}), \mathbf{z})$;
$\mathcal{T}3$: If $\mathbf{x} \leq \mathbf{y}$, then $\mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathcal{T}(\mathbf{y}, \mathbf{z})$;
$\mathcal{T}4$. $\mathcal{T}(\mathbf{x}, /1/) = \mathbf{x}$.

Let $\mathcal{T}$ be n-DT. The natural n-DN of $\mathcal{T}$ is the function $\mathcal{N}_{\mathcal{T}} : (L_n([0,1]))^2 \to L_n([0,1])$ given as

$$\mathcal{N}_{\mathcal{T}}(\mathbf{x}) = \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}, \mathbf{z}) = /0/\}. \tag{18}$$

According to Bedregal et al. (2012), the conditions under which an n-DT on $\mathcal{L}_n([0,1])$ can be obtained from a finite subset of t-norms on [0, 1] are reported as follows.

Theorem 2 (Bedregal et al. 2011, Theorem 3.3) *If there exist t-norms $T_1, \ldots, T_n$ such that $T_1 \leq \cdots \leq T_n$ then $\widetilde{T_1, \ldots, T_n} : (L_n([0,1]))^2 \to L_n([0,1])$ is an n-DT defined by*

$$\widetilde{T_1 \ldots T_n}(\mathbf{x}, \mathbf{y}) = (T_1(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, T_n(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))).$$

Thus, the n-DT $\widetilde{T_1 \ldots T_n}$ is called of t-representable.

Example 4 Considering the t-norms on [0, 1] given as:

$$T_D(x, y) = \begin{cases} 0, & \text{if } x, y \in U, \\ \min(x, y), & \text{otherwise;} \end{cases} \quad T_P(x, y) = xy;$$
$$T_{LK}(x, y) = \max(x + y - 1, 0); \quad T_M(x, y) = \min(x, y).$$

1. The natural n-DT and its n-DN are given as $(\widetilde{T_D}, \widetilde{N_{D2}})$, $(\widetilde{T_P}, \widetilde{N_S})$, $(\widetilde{T_{LK}}, \widetilde{N_S})$, $(\widetilde{T_M}, \widetilde{N_S})$.
2. $T_D, \widetilde{T_{LK}, T_P}, T_M : L_n([0,1])^2 \to L_n([0,1])$ is an example of t-representable n-DT.

Proposition 6 (Bedregal et al. 2012, Theorem 3.6) *Let $\mathcal{T}$ be a n-DT and $\varphi \in Aut(L_n([0,1]))$. Then, the conjugate operator $\mathcal{T}^{\varphi} : (L_n([0,1]))^2 \to L_n([0,1])$ is an n-DT given as*

$$\mathcal{T}^{\varphi}(\mathbf{x}, \mathbf{y}) = \varphi^{-1}(\mathcal{T}(\varphi(\mathbf{x}), \varphi(\mathbf{y}))). \tag{19}$$

3 Fuzzy implications on $L_n([0,1])$

Studies on n-dimensional fuzzy implications (n-DI) on the lattice $(L_n([0,1]), \leq)$ were carried out, extending the preliminary studies on representability of fuzzy implications (FI) (Cornelis et al. 2004; Deschrijver et al. 2004) and also preserving their main properties. Thus, an n-DI can be seen as an extension of interval-valued fuzzy implication (Bedregal et al. 2007, 2010; Reiser et al. 2009; Zanotelli et al. 2018; Zapata et al. 2017) and of an interval-valued Atanassov' intuitionistic fuzzy implication (Reiser and Bedregal 2013; Reiser et al. 2013). So, their properties on [0, 1] can also be investigated in an n-dimensional sense on $L_n([0,1])$.

Definition 6 (Zanotelli et al. 2018, Def.7) A function $\mathcal{I} : (L_n([0,1]))^2 \to L_n([0,1])$ is a n-dimensional fuzzy implicator if $\mathcal{I}$ meets the boundary conditions:

$\mathcal{I}0(a)$: $\mathcal{I}(/1/, /1/) = \mathcal{I}(/0/, /1/) = \mathcal{I}(/0/, /0/) = /1/$;
$\mathcal{I}0(b)$: $\mathcal{I}(/1/, /0/) = /0/$.

Other properties of implicators are reported below.

$\mathcal{I}1$: $\mathbf{x} \leq \mathbf{z} \to \mathcal{I}(\mathbf{x}, \mathbf{y}) \geq \mathcal{I}(\mathbf{z}, \mathbf{y})$;
$\mathcal{I}2$: $\mathbf{y} \leq \mathbf{z} \to \mathcal{I}(\mathbf{x}, \mathbf{y}) \leq \mathcal{I}(\mathbf{x}, \mathbf{z})$;
$\mathcal{I}3$: $\mathcal{I}(/1/, \mathbf{y}) = \mathbf{y}$;
$\mathcal{I}4$: $\mathcal{I}(\mathbf{x}, \mathbf{x}) = /1/$;
$\mathcal{I}5$: $\mathcal{N}(\mathbf{x}) = \mathcal{I}(\mathbf{x}, /0/)$ is an n-DN;
$\mathcal{I}6$: $\mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{z})) = \mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x}, \mathbf{z}))$;
$\mathcal{I}7$: $\mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/ \Rightarrow \mathbf{x} \leq \mathbf{y}$;
$\mathcal{I}8$: $\mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/ \Leftrightarrow \mathbf{x} \leq \mathbf{y}$.

Definition 7 An n-dimensional fuzzy implicator $\mathcal{I}$ which also satisfies $\mathcal{I}1$ and $\mathcal{I}2$ is called an n-DI or fuzzy implication on $L_n([0,1])$.

Since the set of n-DI (denoted by $\mathcal{I}(L_n([0,1]))$) extends the set of fuzzy implications (denoted by I([0,1]), the related properties $\mathcal{I}k$ are given by Ik, $\forall k \in \mathbb{N}_8$.

See the main results from Baczyński and Jayaram research (Baczyński and Jayaram 2008a, Lemma 1.3.4), which are now extended to $L_n([0,1])$.

Proposition 7 *When $\mathcal{I} : (L_n([0,1]))^2 \to L_n([0,1])$ verifies $\mathcal{I}6$ and $\mathcal{I}8$, it satisfies $\mathcal{I}0(a)$, $\mathcal{I}0(b)$, $\mathcal{I}1$, $\mathcal{I}3$ and $\mathcal{I}4$.*

Proof Since $/0/ \le /1/$, by $\mathcal{I}8$, it holds that $\mathcal{I}0(a)$: $\mathcal{I}(/0/,/0/) = \mathcal{I}(/0/,/1/) = \mathcal{I}(/1/,/1/) = /1/$; $\mathcal{I}4$: By $\mathcal{I}8$, it is immediate that $\mathcal{I}(\mathbf{x},\mathbf{x}) = /1/$. $\mathcal{I}1$: If $\mathbf{x}_1 \le \mathbf{x}_2$, by $\mathcal{I}4$ the following is verified:

$$\begin{aligned}
&\mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathcal{I}(\mathbf{x}_2,\mathbf{y})) = /1/\\
&\Rightarrow \mathcal{I}(\mathbf{x}_2,\mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y})) = /1/ \text{ (by } \mathcal{I}6)\\
&\Rightarrow \mathbf{x}_2 \le \mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y}) \text{ (by } \mathcal{I}8)\\
&\Rightarrow \mathbf{x}_1 \le \mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y})\\
&\Rightarrow \mathcal{I}(\mathbf{x}_1,\mathcal{I})(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y}) = /1/ \text{ (by } \mathcal{I}8)\\
&\Rightarrow \mathcal{I}(\mathbf{x}_2,\mathbf{y}) \le \mathcal{I}(\mathbf{x}_1,\mathbf{y}) \text{ (by } \mathcal{I}6)
\end{aligned}$$

$\mathcal{I}3$: By $\mathcal{I}6$ and $\mathcal{I}8$, we consider both conditions:

(i) $\mathcal{I}(\mathbf{y},\mathcal{I}(/1/,\mathbf{y})) = \mathcal{I}(/1/,\mathcal{I}(\mathbf{y},\mathbf{y})) = \mathcal{I}(/1/,/1/) = /1/$. So, it holds that $\mathbf{y} \le \mathcal{I}(/1/,\mathbf{y})$;
(ii) By $\mathcal{I}4$, $\mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathcal{I}(/1/,\mathbf{y})) = /1/$. It means that $\mathcal{I}(/1/,\mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathbf{y})) = /1/$. And then, we have that $/1/ \le \mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathbf{y})$. So, $\mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathbf{y}) = /1/$. By $\mathcal{I}8$, it holds that $\mathcal{I}(/1/,\mathbf{y}) \le \mathbf{y}$. Thus, by (i) and (ii), we have that $\mathcal{I}(/1/,\mathbf{y}) = \mathbf{y}$.

$\mathcal{I}0(b)$: Straightforward from $\mathcal{I}3$, $\mathcal{I}(/1/,/0/) = /0/$. Therefore, Proposition 7 holds. □

3.1 Conjugate-operators $\mathcal{I}^{\varphi}$ on $L_n([0,1])$

This section discusses the conjugate operator acting on n-DI and preserving their main properties.

Proposition 8 *Let $\mathcal{I} : (L_n([0,1]))^2 \to L_n([0,1])$ be a function and $\varphi \in Aut(L_n([0,1]))$. Properties from $\mathcal{I}0$ to $\mathcal{I}8$ are invariant under the conjugate-operator $\mathcal{I}^{\varphi} : (L_n([0,1]))^2 \to L_n([0,1])$ given by*

$$\mathcal{I}^{\varphi}(\mathbf{x},\mathbf{y}) = \varphi^{-1}\left(\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{y}))\right). \tag{20}$$

Proof Let $\mathcal{I}$ be an n-DI verifying properties from $\mathcal{I}0$ to $\mathcal{I}8$. For $\mathbf{x},\mathbf{x}_1,\mathbf{x}_2,\mathbf{y} \in L_{n([0,1])}$ it holds that: $\mathcal{I}0$: Next boundary conditions hold straightforward once $\varphi(/0/) = \varphi^{-1}(/0/) = /0/$ and $\varphi(/1/) = \varphi^{-1}(/1/) = /1/$ and so, $\mathcal{I}$ satisfies $\mathcal{I}0$. $\mathcal{I}1$: Consider $\mathbf{x}_1 \le \mathbf{x}_2$. By the monotonicity of φ, $\varphi(/1/) = \varphi^{-1}(/1/)$ and because $\mathcal{I}$ satisfies $\mathcal{I}1$, we obtain the following expression:

$$\begin{aligned}
\mathcal{I}^{\varphi}(\mathbf{x}_1,\mathbf{y}) &= \varphi^{-1}\left(\mathcal{I}(\varphi(\mathbf{x}_1),\varphi(\mathbf{y}))\right)\\
&\ge \varphi^{-1}\left(\mathcal{I}(\varphi(\mathbf{x}_2),\varphi(\mathbf{y}))\right) = \mathcal{I}^{\varphi}(\mathbf{x}_2,\mathbf{y}).
\end{aligned}$$

$\mathcal{I}2$: Analogous to $\mathcal{I}1$.

$\mathcal{I}3 : \mathcal{I}^{\varphi}(/1/,\mathbf{y}) = \varphi^{-1}\left(\mathcal{I}(/1/,\varphi(\mathbf{y}))\right) = \varphi^{-1}(\varphi(\mathbf{y})) = \mathbf{y}.$

$\mathcal{I}4 : \mathcal{I}^{\varphi}(\mathbf{x},\mathbf{x}) = \varphi^{-1}\left(\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{x}))\right) = \varphi^{-1}(/1/) = /1/.$

$$\begin{aligned}
\mathcal{I}5 : \mathcal{I}^{\varphi}(\mathbf{x},/0/) &= \varphi^{-1}\left(\mathcal{I}(\varphi(\mathbf{x}),/0/)\right)\\
&= \varphi^{-1}(\mathcal{N}_{\mathcal{I}}(\varphi(\mathbf{x})))\\
&= \mathcal{N}_{\mathcal{I}^{\varphi}}(\mathbf{x}). \text{So, by[15, } Prop. 4.2\text{]is an-DN.}
\end{aligned}$$

$\mathcal{I}6$: If $\mathcal{I}$ satisfies the exchange principle, then

$$\begin{aligned}
\mathcal{I}^{\varphi}(\mathbf{x},\mathcal{I}^{\varphi}(\mathbf{y},\mathbf{z})) &= \mathcal{I}^{\varphi}(\mathbf{x},\varphi^{-1}(\mathcal{I}(\varphi(\mathbf{y}),\varphi(\mathbf{z}))))\\
&= \varphi^{-1}(\mathcal{I}(\varphi(\mathbf{x}),\mathcal{I}(\varphi(\mathbf{y}),\varphi(\mathbf{z}))))\\
&= \varphi^{-1}(\mathcal{I}(\varphi(\mathbf{y}),\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{z}))))\\
&= \mathcal{I}^{\varphi}(\mathbf{y},\varphi^{-1}(\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{z}))))\\
&= \mathcal{I}^{\varphi}(\mathbf{y},\mathcal{I}^{\varphi}(\mathbf{x},\mathbf{z})).
\end{aligned}$$

$\mathcal{I}7$: Since $\mathcal{I}$ verifies $\mathcal{I}7$, we obtain the next result:

$$\begin{aligned}
\mathcal{I}^{\varphi}(\mathbf{x},\mathbf{y}) = /1/ &\Leftrightarrow \varphi^{-1}\left(\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{y}))\right) = /1/\\
&\Leftrightarrow (\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{y}))) = /1/\\
&\Rightarrow \varphi(\mathbf{x}) \le \varphi(\mathbf{y}) \Leftrightarrow \mathbf{x} \le \mathbf{y}.
\end{aligned}$$

$\mathcal{I}8$: Since $\mathcal{I}$ verifies $\mathcal{I}8$, the next result is verified:

$$\begin{aligned}
\mathcal{I}^{\varphi}(\mathbf{x},\mathbf{y}) = /1/ &\Leftrightarrow \varphi^{-1}\left(\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{y}))\right) = /1/\\
&\Leftrightarrow (\mathcal{I}(\varphi(\mathbf{x}),\varphi(\mathbf{y}))) = /1/\\
&\Leftrightarrow \varphi(\mathbf{x}) \le \varphi(\mathbf{y}) \Leftrightarrow \mathbf{x} \le \mathbf{y}.
\end{aligned}$$

Concluding, Proposition 8 holds. □

3.2 Representable fuzzy implications on $L_n([0,1])$

This section studies the representability of n-DI which is invariant under their main properties (Zanotelli et al. 2018, Prop. 6).

Proposition 9 *Consider the functions $I_1,\ldots,I_n : [0,1]^2 \to [0,1]$ such that $I_1 \le \cdots \le I_n$. The function $\widetilde{I_1 \ldots I_n} : (L_n([0,1]))^2 \to L_n([0,1])$ given by*

$$\widetilde{I_1 \ldots I_n}(\mathbf{x},\mathbf{y}) = (I_1(\pi_n(\mathbf{x}),\pi_1(\mathbf{y})),\ldots,I_n(\pi_1(\mathbf{x}),\pi_n(\mathbf{y}))), \tag{21}$$

is an n-dimensional fuzzy implicator iff $I_1,\ldots,I_n$ are also fuzzy implicators.

By Proposition 9, $\mathcal{I}$ is called a representable n-dimensional fuzzy implicator if there exist fuzzy implicators $I_1 \le \cdots \le I_n$ such that $\mathcal{I} = \widetilde{I_1 \ldots I_n}$.

Remark 4 When $I_1 = \cdots = I_n = I$, the expression $\widetilde{I_1 \ldots I_n}$ in (21) is denoted by $\widetilde{I}$.

Remark 5 For $\mathbf{x}, \mathbf{y} \in L_n([0,1])$, $\widetilde{I_1 \ldots I_n} \in \mathcal{I}(L_n([0,1]))$ and $i \in \mathbb{N}_n$, the following holds:

1. $\pi_i(\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y})) = I_i(\pi_{n+1-i}(\mathbf{x}), \pi_i(\mathbf{y}))$;
2. $\pi_i(\widetilde{I_1 \ldots I_n}(/x/, /y/)) = I_i(x, y)$;
3. $\pi_i(\widetilde{I}(/x/, /y/)) = I(x, y)$.

Extending the result from Proposition 9, next proposition states that $\mathcal{I}$ is a representable n-DI if there exist fuzzy implications $I_1 \leq \cdots \leq I_n$ such that $\mathcal{I} = \widetilde{I_1 \ldots I_n}$, and its converse construction can also be verified.

Proposition 10 (Zanotelli et al. 2018, Prop. 8) *Let $I_1, \ldots, I_n : [0,1]^2 \to [0,1]$ be functions such that $I_1 \leq \cdots \leq I_n$. The function $\widetilde{I_1 \ldots I_n} : (L_n([0,1]))^2 \to L_n([0,1]) \in \mathcal{I}(L_n([0,1]))$ iff $I_i \in I(U)$ for each $i \in \mathbb{N}_n$.*

Other main properties of fuzzy implicators on [0, 1] are preserved by a representable n-DI.

Proposition 11 *Let $I_1, \ldots, I_n : [0,1]^2 \to [0,1]$ be functions such that $I_1 \leq \cdots \leq I_n$, $i \in \mathbb{N}_n$ and $k \in \{3, 5, 6, 7\}$. An n-DI $\widetilde{I_1 \ldots I_n} : (L_n([0,1]))^2 \to L_n([0,1])$ verifies the property $\mathcal{I}k$ iff each $I_i : [0,1]^2 \to [0,1]$, for $i \in \mathbb{N}_n$, verifies the corresponding property Ik.*

Proof ($\Leftarrow$) Firstly, let $I_1, \ldots, I_n \in I(U)$ such that $I_1 \leq \cdots \leq I_n$, satisfying property Ik, for $k \in \{3, 5, 6, 7, 9\}$. For $\widetilde{I_1 \ldots I_n} \in \mathcal{I}(L_n([0,1]))$ given by (21) and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n([0,1])$, the following holds:

$$\begin{aligned} \mathcal{I}3 : \widetilde{I_1 \ldots I_n}(/1/, \mathbf{y}) &= (I_1(1, y_1), \ldots, I_n(1, y_n)) \\ &= (y_1, \ldots, y_n) = \mathbf{y} \text{ (by (21); } I3) \end{aligned}$$

$$\begin{aligned} \mathcal{I}5 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, /0/) &= (I_1(x_n, 0), \ldots, I_n(x_1, 0)) \\ &= (N_{I_1}(x_n), \ldots, N_{I_n}(x_1)) \text{ (by (21))} \\ &= \widetilde{N_{I_1} \ldots N_{I_n}}(\mathbf{x}) \text{ (by (13); } I5). \end{aligned}$$

Based on Corollary 6.2 (Zanotelli et al. 2018), if $i \leq j$, $N_{I_i} \leq N_{I_j}$. So, $\widetilde{I_1 \ldots I_n}(\mathbf{x}, /0/) = \widetilde{N_{I_1} \ldots N_{I_n}}(\mathbf{x}, /0/)$ is an n-DN.

$$\begin{aligned} \mathcal{I}6 : &\widetilde{I_1 \ldots I_n}(\mathbf{x}, \widetilde{I_1 \ldots I_n}(\mathbf{y}, \mathbf{z})) \\ &= \widetilde{I_1 \ldots I_n}(\mathbf{x}, (I_1(y_n, z_1), \ldots, I_n(y_1, z_n))) \text{ (by (21))} \\ &= (I_1(x_n, I_1(y_n, z_1)), \ldots, I_n(x_1, I_n(y_1, z_n))) \\ &= (I_1(y_n, I_1(x_n, z_1)), \ldots, I_n(y_1, I_n(x_1, z_n))) \text{ (by (21), } I5) \\ &= \widetilde{I_1 \ldots I_n}(\mathbf{y}, \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{z})) \text{ (by (21)).} \end{aligned}$$

$$\begin{aligned} \mathcal{I}7 : &\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = /1/ \\ &\Leftrightarrow (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) = /1/ \text{ (by (21)).} \end{aligned}$$

So, it implies that $\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = /1/$

$$\begin{aligned} &\Leftrightarrow I_1(x_n, y_1) = 1, \ldots, I_n(x_1, y_n) = 1 \Rightarrow x_n \leq y_1 \\ &\Rightarrow x_1 \leq y_1, \ldots, x_n \leq y_n \Rightarrow \mathbf{x} \leq \mathbf{y} \text{ (by } I7). \end{aligned}$$

($\Rightarrow$) Let $\widetilde{I_1, \ldots, I_n} \in \mathcal{I}(L_n([0,1]))$, given by (21), verifying properties $\mathcal{I}k$, for $k \in \{3, 5, 6, 7, 9\}$. Based on π_i-projections, for $i \in \mathbb{N}_n$, the following holds for each $x, y, z \in U$:

$I3$: By $\mathcal{I}3$, $\widetilde{I_1 \ldots I_n}(/1/, /y/) = /y/$ implies $I_i(1, y) = y$.

$I5$: By $\mathcal{I}5$, if $\widetilde{I_1 \ldots I_n}(/x/, /0/) = \mathcal{N}(/x/)$ is a $n-$DN.

So, by Prop. 3, $I_i(x, 0) = N_i(x)$ is a fuzzy negation.

$I6$: By $\mathcal{I}6$, we have that

$$\begin{aligned} &\widetilde{I_1 \ldots I_n}(/x/, \widetilde{I_1 \ldots I_n}(/y/, /z/)) \\ &= \widetilde{I_1 \ldots I_n}(/y/, \widetilde{I_1 \ldots I_n}(/x/, /z/)). \end{aligned}$$

So, $I_i(x, I_i(y, z)) = I_i(y, I_i(x, z))$, $\forall i \in \mathbb{N}_n$.

$I7$: By $\mathcal{I}7$, $\forall i \in \mathbb{N}_n$, $I_i(x, y) = 1$

$$\Rightarrow \widetilde{I_1 \ldots I_n}(/x/, /y/) = /1/ \Rightarrow /x/ \leq /y/ \Rightarrow x \leq y.$$

Therefore, Proposition 11 is verified. □

Proposition 12 *Let $\widetilde{I_1 \ldots I_n} : (L_n([0,1]))^2 \to L_n([0,1])$ such that $I_1 \leq \cdots \leq I_n$ are fuzzy implicators on [0, 1]. However, $\widetilde{I_1 \ldots I_n}$ does not verify the properties $\mathcal{I}4$ and $\mathcal{I}8$.*

Proof It is immediate that $\widetilde{I_1, \ldots, I_n}$ does not verify $\mathcal{I}4$, by taking $\mathbf{x} = (0, \ldots, 0, 1)$, we have that

$$\widetilde{I_1, \ldots, I_n}(\mathbf{x}, \mathbf{x}) = (I_1(1, 0), I_2(0, 0), \ldots, I_n(0, 1))$$

and then, $\widetilde{I_1, \ldots, I_n}(\mathbf{x}, \mathbf{x}) = (0, 1, \ldots, 1) \neq /1/$. In addition, since $\widetilde{I_1, \ldots, I_n}$ not satisfies $\mathcal{I}4$, then it also not satisfies $\mathcal{I}8$.

□

Example 5 In the following, an example of a representable n-DI is presented. Let $I_{KD}, I_{RC}, I_{LK}, I_{WB} : [0,1]^2 \to [0,1]$ be fuzzy implications given by the expressions below:

$$I_{KD}(x, y) = \max(1 - x, y); \qquad I_{RC}(x, y) = 1 - x + xy;$$

$$I_{LK}(x, y) = \min(1, 1 - x + y); \quad I_{WB}(x, y) = \begin{cases} 1, x \leq 1, \\ y, \text{otherwise.} \end{cases}$$

Based on results stated by Baczyński and Jayaram (Baczyński and Jayaram 2008a, pp. 57, 192 and 196) the following comparisons are verified: $I_{KD} \leq I_{RC} \leq I_{LK} \leq I_{WB}$. So, the function $\widetilde{I_{KD}, I_{RC}, I_{LK}, I_{WB}}$ is a representable n-DI on $L_n([0,1])$. One can observe that, for $\mathbf{x} = (0.0, 0.1, 0.5, 0.8)$ and $\mathbf{y} = (0.2, 0.6, 0.9, 1.0)$, it holds that

$\mathcal{I}(\mathbf{x},\mathbf{y}) = \widetilde{I_{KD}, I_{RC}, I_{LK}, I_{WB}}(\mathbf{x},\mathbf{y}) = (0.2, 0.8, 1.0, 1.0)$. Moreover, based on Remark 4, it holds that:

$$\widetilde{I_{KD}}(\mathbf{x},\mathbf{y}) = (0.2, 0.6, 0.9, 1.0)\,;$$
$$\widetilde{I_{RC}}(\mathbf{x},\mathbf{y}) = (0.36, 0.8, 0.99, 1.0);$$
$$\widetilde{I_{LK}}(\mathbf{x},\mathbf{y}) = (0.4, 1.0, 1.0, 1.0);$$
$$\widetilde{I_{WB}}(\mathbf{x},\mathbf{y}) = (1.0, 1.0, 1.0, 1.0).$$

3.3 Moore-continuous functions on $L_n([0,1])$

Based on the intuitive notion and main properties of n-DI, Theorem 3 introduces the notion of Moore-continuous functions for representable n-DI.

Theorem 3 *An n-DI* $\widetilde{I_1\ldots I_n} : (L_n([0,1]))^2 \to L_n([0,1])$ *is Moore-continuous iff* I_i *is continuous, for* $i = 1,\ldots n$.

Proof ($\Rightarrow$) Let $(x_1,y_1),(x_2,y_2) \in [0,1]^2, i \in \mathbb{N}_n$ and $\varepsilon > 0$. Since $\widetilde{I_1\ldots I_n}$ be a Moore-continuous function on $L_n([0,1])$. If $(/x_1/,/y_1/),(/x_2/,/y_2/) \in (L_n([0,1]))^2$, there exists $\delta > 0$ such that $d_M^{n,2}((/x_1/,/y_1/),(/x_2/,/y_2/)) < \delta$, which implies in the following result:

$$d_M^n(\widetilde{I_1\ldots I_n}(/x_1/,/y_1/), \widetilde{I_1\ldots I_n}(/x_2/,/y_2/)) < \varepsilon.$$

Thus, if $\max(|x_1-x_2|,|y_1-y_2|) < \delta$, by Eq. (5), we have that $d_M^{n,2}((/x_1/,/y_1/),(/x_2/,/y_2/)) < \delta$. And, one can easily observe that $d_M^n(\widetilde{I_1\ldots I_n}(/x_1/,/y_1/), \widetilde{I_1\ldots I_n}(/x_2/,/y_2/)) < \varepsilon$. So, it means that the following result holds

$$d_M^n((I_1(x_1,y_1),\ldots,I_n(x_1,y_1)),\\ (I_1(x_2,y_2),\ldots,I_n(x_2,y_2))) < \varepsilon.$$

Hence, by (4), $|I_i(x_1,y_1) - I_i(x_2,y_2)| < \varepsilon$. Therefore, I_i is continuous.

($\Leftarrow$) Let $\varepsilon > 0$ and $(\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2) \in (L_n([0,1]))^2$. By continuity of I_i, for each $i \in \mathbb{N}_n$, there exists $\delta_i > 0$ such that, if $\max(|\pi_i(\mathbf{x}_1)-\pi_i(\mathbf{x}_2)|, |\pi_i(\mathbf{y}_1)-\pi_i(\mathbf{y}_2)|) < \delta_i$ then $|I_i(\pi_i(\mathbf{x}_1),\pi_i(\mathbf{y}_1)) - I_i(\pi_i(\mathbf{x}_2),\pi_i(\mathbf{y}_2))| < \varepsilon$. Thus, considering $\delta = \min\{\delta_i : i \in \mathbb{N}_n\}$, we have that

$$\begin{aligned}
&d_M^{n,2}((\mathbf{x}_1,\mathbf{y}_1),(\mathbf{x}_2,\mathbf{y}_2)) < \delta\\
&\Rightarrow \max(d_M^n(\mathbf{x}_1,\mathbf{x}_2), d_M^n(\mathbf{y}_1,\mathbf{y}_2)) < \delta\\
&\Rightarrow \max(|\pi_1(\mathbf{x}_1) - \pi_1(\mathbf{x}_2)|,\ldots,|\pi_n(\mathbf{x}_1) - \pi_n(\mathbf{x}_2)|,\\
&|\pi_1(\mathbf{y}_1) - \pi_1(\mathbf{y}_2)|,\ldots,|\pi_n(\mathbf{y}_1) - \pi_n(\mathbf{y}_2)|) < \delta\\
&\Rightarrow \max(|\pi_{n-i+1}(\mathbf{x}_1) - \pi_{n-i+1}(\mathbf{x}_2)|, |\pi_i(\mathbf{y}_1) - \pi_i(\mathbf{y}_2)|) < \delta_i\\
&\Rightarrow |I_i(\pi_{n-i+1}(\mathbf{x}_1),\pi_i(\mathbf{y}_1)) - I_i(\pi_{n-i+1}(\mathbf{x}_2),\pi_i(\mathbf{y}_2))| < \varepsilon\\
&\Rightarrow \max(|I_1(\pi_n(\mathbf{x}_1),\pi_1(\mathbf{y}_1)) - I_1(\pi_n(\mathbf{x}_2),\pi_1(\mathbf{y}_2)|,\ldots,\\
&|I_n(\pi_1(\mathbf{x}_1),\pi_n(\mathbf{y}_1)) - I_n(\pi_1(\mathbf{x}_2),\pi_n(\mathbf{y}_2)|) < \varepsilon\\
&\Rightarrow \max(|\pi_1(\widetilde{I_1,\ldots,I_n}(\mathbf{x}_1,\mathbf{y}_1)) - \pi_1(\widetilde{I_1,\ldots,I_n}(\mathbf{x}_2,\mathbf{y}_2))|,\ldots,\\
&|\pi_n(\widetilde{I_1,\ldots,I_n}(\mathbf{x}_1,\mathbf{y}_1)) - \pi_n(\widetilde{I_1,\ldots,I_n}(\mathbf{x}_2,\mathbf{y}_2))|) < \varepsilon\\
&\Rightarrow d_M^n(\widetilde{I_1,\ldots,I_n}(\mathbf{x}_1,\mathbf{y}_1), \widetilde{I_1,\ldots,I_n}(\mathbf{x}_2,\mathbf{y}_2)) < \varepsilon.
\end{aligned}$$

Therefore, $\widetilde{I_1\ldots I_n}$ is a $(d_M^{n,2}, d_M^n)$-continuous function on $L_n([0,1]) \times L_n([0,1])$ and Theorem 3 holds. □

4 R-implications on $L_n([0,1])$

The R-implication arises from the notion of residuum in intuitionistic logic or, equivalently, from the notion of residue in the theory of lattice-ordered semigroups. Observe that the R-implication is well-defined only if the t-norm is left-continuous, which justifies the name "residuum of T", since the R-implication satisfies the residuation condition when the underlying t-norm is left continuous. Moreover, a t-norm T is left-continuous if and only if it satisfies the residuation condition (Baczyński 2004).

In this section, these main characteristics of R-implications extended from $[0,1]$ to $L_n([0,1])$ are discussed.

Definition 8 A function $\mathcal{I}_{\mathcal{T}} : (L_n([0,1]))^2 \to L_n([0,1])$ is called an n-dimensional R-implication (n-DRI) if there exists n-DT $\mathcal{T} : (L_n([0,1]))^2 \to L_n([0,1])$ such that

$$\mathcal{I}_{\mathcal{T}}(\mathbf{x},\mathbf{y}) = \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(\mathbf{x},\mathbf{z}) \le \mathbf{y}\}. \qquad (22)$$

Next proposition extends results from (Baczyński and Jayaram 2008b, Theorem 5.5).

Proposition 13 *If* $\mathcal{T}$ *is an n-DT then* $\mathcal{I}_{\mathcal{T}} \in \mathcal{I}(L_n([0,1]))$. *Moreover, it verifies* $\mathcal{I}0, \mathcal{I}1, \mathcal{I}2, \mathcal{I}3$ *and* $\mathcal{I}4$. *In addition, it also verifies* $\mathcal{I}5$, *meaning that its natural negation* $\mathcal{N}_{\mathcal{I}}$ *coincides with the* $\mathcal{N}_{\mathcal{T}}$ *given in (18).*

Proof Let $\mathcal{T}$ be an n-DT and $\mathcal{I}_{\mathcal{T}}$ be the function defined by Eq. (22). Let $\mathbf{x},\mathbf{y},\mathbf{z} \in L_n([0,1])$. ($\mathcal{I}0$): The boundary conditions hold as follows:

$$\begin{aligned}
\mathcal{I}_{\mathcal{T}}(/1/,/1/) &= \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(/1/,\mathbf{z}) \le /1/\} = /1/;\\
\mathcal{I}_{\mathcal{T}}(/0/,/1/) &= \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(/0/,\mathbf{z}) \le /1/\} = /1/;\\
\mathcal{I}_{\mathcal{T}}(/0/,/0/) &= \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(/0/,\mathbf{z}) \le /0/\} = /1/;\\
\mathcal{I}_{\mathcal{T}}(/1/,/0/) &= \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(/1/,\mathbf{z}) \le /0/\} = /0/.
\end{aligned}$$

($\mathcal{I}1$): Let $\mathbf{x}_1,\mathbf{x}_2 \in L_n([0,1])$. Based on monotonicity of $\mathcal{T}$, when $\mathbf{x}_1 \le \mathbf{x}_2$, taking $\mathbf{z} \in L_n([0,1])$ such that $\mathcal{T}(\mathbf{x}_2,\mathbf{z}) \le \mathbf{y}$ we should have that $\mathcal{T}(\mathbf{x}_1,\mathbf{z}) \le \mathbf{y}$. So, the inclusion $\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}_1,\mathbf{z}) \le \mathbf{y}\} \supset \{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}_2,\mathbf{z}) \le \mathbf{y}\}$ implies that $\sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}_1,\mathbf{z}) \le \mathbf{y}\} \ge \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}_2,\mathbf{z}) \le \mathbf{y}\}$. Therefore, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_1,\mathbf{y}) \ge \mathcal{I}_{\mathcal{T}}(\mathbf{x}_2,\mathbf{y})$.
($\mathcal{I}2$): Analogous to $\mathcal{I}1$.
($\mathcal{I}3$): $\mathcal{I}_{\mathcal{T}}(/1/,\mathbf{y}) = \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(/1/,\mathbf{z}) = \mathbf{z} \le$

$\mathbf{y}\} = \mathbf{y}$.
$(\mathcal{I}4)$: $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{x}) = \sup\{\mathbf{z} \in L_n([0,1])\colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{x}\} = /1/$.
$(\mathcal{I}5)$: $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, /0/) = \sup\{\mathbf{z} \in L_n([0,1])\colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq /0/\} = \mathcal{N}_{\mathcal{T}}(\mathbf{x})$, by (18), which is an n-DN according to Bedregal et al. (2012). Concluding, Proposition 13 is verified. □

Based on results presented in Proposition 13, the natural negation $\mathcal{N}_{\mathcal{I}}$ coincides with the $\mathcal{N}_{\mathcal{T}}$ given in Eq. (18) when $\mathcal{I}$ is an n-DRI and $\mathcal{T}$ their underlying n-DT on $\mathcal{L}_n([0,1])$.

Corollary 3 *There is no representable n-DI which is an n-DRI.*

Proof Straightforward from Props. 12 and 13. □

4.1 Residuation property on $\mathcal{L}_n([0, 1])$

The following results show the necessary and sufficient conditions under which the pair of functions $(\mathcal{I}_{\mathcal{T}}, \mathcal{T})$ verifies the residuation property on $L_n([0,1])$.

Lemma 1 *Let $\mathcal{T}$ be an n-DT. If $\mathcal{T}$ is left-continuous, i.e. for each increasing n-DS $(\mathbf{y}_i)_{i\in\mathbb{N}}$ and $\mathbf{x} \in L_n([0,1])$, we have that*

$$\mathcal{LC} : \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i) = \mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i).$$

Then $\mathcal{T}$ is a join-morphism, i.e. $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n([0,1])$, we have that

$$\mathcal{T}(\mathbf{x}, \mathbf{y} \vee \mathbf{z}) = \mathcal{T}(\mathbf{x}, \mathbf{y}) \vee \mathcal{T}(\mathbf{x}, \mathbf{z}).$$

Proof $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n([0,1])$ and a increasing n-DS $(\mathbf{y}_i)_{i\in\mathbb{N}}$ and $(\mathbf{z}_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \mathbf{y}_i = \mathbf{y}$ and $\lim_{i\to\infty} \mathbf{z}_i = \mathbf{z}$. So, $\lim_{i\to\infty} \mathbf{y}_i \vee \mathbf{z}_i = \mathbf{y} \vee \mathbf{z}$ and thereby, since $\mathcal{T}$ is left-continuous,

$$\begin{aligned}\mathcal{T}(\mathbf{x}, \mathbf{y} \vee \mathbf{z}) &= \mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i \vee \mathbf{z}_i)\\ &= \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i \vee \mathbf{z}_i)\\ &= \lim_{i\to\infty}(\mathcal{T}(\mathbf{x}, \mathbf{y}_i) \vee \mathcal{T}(\mathbf{x}, \mathbf{z}_i))\\ &= \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i) \vee \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{z}_i)\\ &= \mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) \vee \mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{z}_i)\\ &= \mathcal{T}(\mathbf{x}, \mathbf{y}) \vee \mathcal{T}(\mathbf{x}, \mathbf{z}).\end{aligned}$$

□

Proposition 14 *Let $\mathcal{T}$ be an n-DT and $\mathbf{x}, \mathbf{y} \in L_n([0,1])$. If $\mathcal{T}$ is left-continuous then there exists an increasing sequence $(\mathbf{z}_i)_{i\in\mathbb{N}} \subseteq \Delta = \{\mathbf{z} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$ such that $\lim_{i\to\infty} \mathbf{z}_i = \sup \Delta$.*

Proof If $\mathbf{t}, \mathbf{s} \in \Delta$ then there exists $\mathbf{z}_1, \mathbf{z}_2 \in L_n([0,1])$ such that, $\mathbf{t} = \mathcal{T}(\mathbf{x}, \mathbf{z}_1) \leq \mathbf{y}$ and $\mathbf{s} = \mathcal{T}(\mathbf{x}, \mathbf{z}_2) \leq \mathbf{y}$. From Lemma 1, $\mathcal{T}(\mathbf{x}, \mathbf{z}_1 \vee \mathbf{z}_2) = \mathbf{t} \vee \mathbf{s} \leq \mathbf{y}$. Therefore, Δ is a directed set, i.e. for each $\mathbf{t}, \mathbf{s} \in \Delta$, $\mathbf{t} \vee \mathbf{s} \in \Delta$. Since from Bedregal et al. (2011); Zanotelli et al. (2020) $\mathcal{L}_n([0,1]) = (L_n([0,1]), \leq)$ is a complete lattice, then its dcpo (Abramski and Jung 1994, Example 2.1.14). In addition, it is a ω-continuous domain, once $L_n([0,1]) \cap \mathbb{Q}^n$ is basis of $\mathcal{L}_n([0,1])$, and therefore, from (Abramski and Jung 1994, Proposition 2.2.13), the directed subset of Δ of $\mathcal{L}_n$ contain an ω-chain $\mathbf{z}_1 \leq \mathbf{z}_2 \leq \cdots$ with the same supremum. Hence $(\mathbf{z}_i)_{i\in\mathbb{N}}$ is an increasing sequence such that $\lim_{i\to\infty} \mathbf{z}_i = \sup \Delta$. □

Theorem 4 *Let $\mathcal{T}$ be an n-DT. The following statements are equivalent:*

1. *$\mathcal{T}$ is left-continuous;*
2. *$(\mathcal{T}, \mathcal{I}_{\mathcal{T}})$ is an adjoint pair;*
3. *$\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \max\{\mathbf{z} \in L_n([0,1])\colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$.*

Proof (**1.**) $\Rightarrow$ (**2.**) Let $\mathcal{T}$ be a left-continuous n-DT and assume that $\mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}$, for some $\mathbf{x}, \mathbf{y}, \mathbf{t} \in L_n([0,1])$. So, we have that $\mathbf{t} \in \{\mathbf{z} \in L_n([0,1])\colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$. Therefore, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) \geq \mathbf{t}$. Conversely, let $\mathbf{u} = \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$. Then, by Proposition 14, there exists an increasing sequence $(\mathbf{z}_i)_{i\in\mathbb{N}}$ such that $\mathcal{T}(\mathbf{x}, \mathbf{z}_i) \leq \mathbf{y}$ and $\lim\limits_{i\to\infty} \mathbf{z}_i = \mathbf{u}$. So, if $\mathbf{t} \leq \mathbf{u}$, it holds that $\mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathcal{T}(\mathbf{x}, \mathbf{u}) = \mathcal{T}(\mathbf{x}, \lim\limits_{i\to\infty} \mathbf{z}_i) = \lim\limits_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{z}_i) \leq \mathbf{y}$.

(**2.**) $\Rightarrow$ (**3.**) Assuming that $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ verify $\mathcal{RP}$, we obtain that $\mathcal{T}(\mathbf{x}, \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})) \leq \mathbf{y}$ and then, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) \in \{\mathbf{z} : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$. Therefore, we conclude that $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \max\{\mathbf{z} : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$.

(**3.**) $\Rightarrow$ (**1.**) Let $\mathcal{T}$ be a n-DT and $(\mathbf{y}_i)_{i\in\mathbb{N}}$ be an increasing n-DS. Then by Corollary 1 it converges to a $\mathbf{z} \in L_n([0,1])$, i.e. $\mathbf{z} = \lim_{i\to\infty} \mathbf{y}_i$. Then, from the monotonicity of $\mathcal{T}$, we have the inequality $\mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) \geq \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i)$. Now, let $\mathbf{y} = \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i)$, implying that $\mathcal{T}(\mathbf{x}, \mathbf{y}_i) \leq \mathbf{y}$, $\forall i \in \mathbb{N}$. Then, $\mathbf{y}_i \in \{\mathbf{t} \in L_n([0,1])\colon \mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}\}, \forall i \in \mathbb{N}$. Based on such results we obtain that $\mathbf{y}_i \leq \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}), \forall i \in \mathbb{N}$. And so, we can deduce that $\lim_{i\to\infty} \mathbf{y}_i \leq \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$. Again, by RP, we obtain that

$$\mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) \leq \mathbf{y}$$

meaning that $\lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i) \geq \mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i)$. Therefore, $\mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) = \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i)$. Concluding, Theorem 4 is verified. □

Proposition 15 *If $\mathcal{T}$ is a left-continuous n-DT then $\mathcal{I}_{\mathcal{T}}$ is an n-DI satisfying $\mathcal{I}6$ and $\mathcal{I}8$. Moreover, $\mathcal{I}_{\mathcal{T}}$ is left-continuous w.r.t the first variable and right-continuous w.r.t. the second variable.*

Proof First observe that once $\mathcal{T}$ is left-continuous, then by Theorem 4 the pair $(\mathcal{T}, \mathcal{I}_{\mathcal{T}})$ satisfies $\mathcal{RP}$.

From Proposition 13 we know that $\mathcal{I}_{\mathcal{T}}$ is an n-DI. $(\mathcal{I}6)$: Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n([0,1])$. So, the next holds:

$$\begin{aligned}&\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathcal{I}_{\mathcal{T}}(\mathbf{y}, \mathbf{z}))\\ &= \max\{\mathbf{t} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathcal{I}_{\mathcal{T}}(\mathbf{y}, \mathbf{z})\}; \quad \text{(by (22))}\end{aligned}$$

$= \max\{\mathbf{t} \in L_n([0,1]) : \mathcal{T}(\mathbf{y}, \mathcal{T}(\mathbf{x}, \mathbf{t})) \le \mathbf{z}\};$ (by $\mathcal{RP}$)
$= \max\{\mathbf{t} \in L_n([0,1]) : \mathcal{T}(\mathbf{x}, \mathcal{T}(\mathbf{y}, \mathbf{t})) \le \mathbf{z}\};$ (by $\mathcal{T}1, \mathcal{T}3$)
$= \max\{\mathbf{t} \in L_n([0,1]) : \mathcal{T}(\mathbf{y}, \mathbf{t}) \le \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{z})\};$ (By $\mathcal{RP}$)
$= \mathcal{I}_{\mathcal{T}}(\mathbf{y}, \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{z}))$ (by (22)).

($\mathcal{I}8$): Let $\mathbf{x}, \mathbf{y} \in L_n([0,1])$. If $\mathbf{x} \le \mathbf{y}$, then $\mathcal{T}(\mathbf{x}, /1/) = \mathbf{x} \le \mathbf{y}$, by $\mathcal{RP}$ $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = /1/$. Conversely, if $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = /1/$, then because of $\mathcal{RP}$ we obtain that $\mathcal{T}(\mathbf{x}, /1/) \le \mathbf{y}$ meaning that $\mathbf{x} \le \mathbf{y}$. ($\mathcal{LC}$): Now, suppose that $\mathcal{I}_{\mathcal{T}}$ is not left-continuous w.r.t. the first variable in some point $(\mathbf{x}^*, \mathbf{y}^*) \in L_n([0,1]) \times L_n([0,1])$. So, there exists an increasing chain $(\mathbf{x}_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}^*$ and $\mathbf{a} = \lim_{i\to\infty} \mathcal{I}_{\mathcal{T}}(\mathbf{x}_i, \mathbf{y}*) \ne \mathcal{I}_{\mathcal{T}}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{b}$. Thus, because for each $i \in \mathbb{N}$, $\mathbf{x}_i \le \mathbf{x}^*$ and $\mathcal{I}_{\mathcal{T}}$ satisfy $\mathcal{I}1$, then $\mathbf{a} > \mathbf{b}$. Since, for each $i \in \mathbb{N}$, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_i, \mathbf{y}^*) \ge \mathbf{a}$ then, by Property $\mathcal{RP}$, we obtain that $\mathcal{T}(\mathbf{x}_i, \mathbf{a}) \le \mathbf{y}^*$ and therefore $\lim_{i\to\infty} \mathcal{T}(\mathbf{x}_i, \mathbf{a}) \le \mathbf{y}^*$. Hence, since $\mathcal{T}$ is left-continuous, we have $\mathcal{T}(\mathbf{x}^*, \mathbf{a}) \le \mathbf{y}^*$. Therefore, from $\mathcal{RP}$, $\mathbf{b} = \mathcal{I}_{\mathcal{T}}(\mathbf{x}^*, \mathbf{y}^*) \ge \mathbf{a}$, which is a contradiction with $\mathbf{a} > \mathbf{b}$. So, $\mathcal{I}_{\mathcal{T}}$ is a left-continuous function w.r.t. the first variable. ($\mathcal{RC}$): Now, consider $\mathcal{I}_{\mathcal{T}}$ as a binary function which does not verify the right-continuity property w.r.t. the second variable in some point $(\mathbf{x}^*, \mathbf{y}^*) \in L_n(U) \times L_n([0,1])$. Thus, there exists a decreasing chain $(\mathbf{y}_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \mathbf{y}_i = \mathbf{y}^*$ and $\mathbf{a} = \lim_{i\to\infty} \mathcal{I}_{\mathcal{T}}(\mathbf{x}^*, \mathbf{y}_i) \ne \mathcal{I}_{\mathcal{T}}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{b}$. So, because for each $i \in \mathbb{N}$, $\mathbf{y}_i \ge \mathbf{y}^*$ and $\mathcal{I}_{\mathcal{T}}$ satisfy $\mathcal{I}2$, then $\mathbf{a} > \mathbf{b}$. On the other hand, since for each $i \in \mathbb{N}$, $\mathbf{a} \le \mathcal{I}_{\mathcal{T}}(\mathbf{x}^*, \mathbf{y}_i)$ then, by $\mathcal{RP}$, we have that $\mathcal{T}(\mathbf{x}^*, \mathbf{a}) \le \mathbf{y}_i$ for each $i \in \mathbb{N}$. So, in the limit, $\mathcal{T}(\mathbf{x}^*, \mathbf{a}) \le \lim_{i\to\infty} \mathbf{y}_i = \mathbf{y}^*$ and therefore, by $\mathcal{RP}$, $\mathbf{a} \le \mathcal{I}_{\mathcal{T}}(\mathbf{x}^*, \mathbf{y}^*) = \mathbf{b}$, which is a contradiction with hypothesis $\mathbf{a} > \mathbf{b}$. Concluding, $\mathcal{I}_{\mathcal{T}}$ is a right-continuous function w.r.t. the second variable. Therefore, Proposition 15 is verified. □

4.2 Conjugation of R-implications on $L_n([0, 1])$

In the following theorem, results from (Baczyński and Jayaram 2008b, Prop.2.5.10) are extended, showing how n-dimensional automorphisms act on an n-DRI, generating a new n-DRI.

Theorem 5 *Let $\mathcal{I}_{\mathcal{T}} : (L_n([0,1]))^2 \to L_n([0,1])$ be an n-DRI and $\varphi \in Aut(L_n([0,1]))$. Then, the function $\mathcal{I}^{\varphi}_{\mathcal{T}} : (L_n([0,1]))^2 \to L_n([0,1])$ defined by Eq. (20) is an n-DRI, in fact*

$$\mathcal{I}^{\varphi}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \mathcal{I}_{\mathcal{T}^{\varphi}}(\mathbf{x}, \mathbf{y}). \tag{23}$$

Proof For $\mathbf{x}, \mathbf{y} \in L_n([0,1])$, by the continuity of bijection φ and from Eqs. (10), (19), (20) and (22), the following results are verified:

$$\mathcal{I}^{\varphi}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \varphi^{-1}(\mathcal{I}_{\mathcal{T}}(\varphi(\mathbf{x}), \varphi(\mathbf{y})))$$
$$= \varphi^{-1}(sup\{\varphi(\mathbf{z}) \in L_n([0,1]) : \mathcal{T}(\varphi(\mathbf{x}), \varphi(\mathbf{z})) \le \varphi(\mathbf{y})\})$$
$$= sup\{\varphi^{-1}(\varphi(\mathbf{z})) \in L_n([0,1]) : \mathcal{T}^{\varphi}(\mathbf{x}, \mathbf{z}) \le \varphi^{-1}(\varphi(\mathbf{y}))\}$$
$$= sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}^{\varphi}(\mathbf{x}, \mathbf{z}) \le \mathbf{y}\} = \mathcal{I}_{\mathcal{T}^{\varphi}}(\mathbf{x}, \mathbf{y})$$

From Proposition 6, $\mathcal{T}^{\varphi} : (L_n([0,1]))^2 \to L_n([0,1])$ is an n-DT implying that $\mathcal{I}_{\mathcal{T}^{\varphi}} : (L_n([0,1]))^2 \to L_n([0,1])$ is also an n-DRI. Therefore, Theorem 5 is verified. □

Corollary 4 *Let $\mathcal{I}_{\mathcal{T}} : (L_n([0,1]))^2 \to L_n([0,1])$ be an n-DRI and $\varphi \in Aut(L_n([0,1]))$. Then, Properties $\mathcal{I}3, \mathcal{I}4$ and $\mathcal{I}5$ are invariant under the conjugate φ-operator $\mathcal{I}^{\varphi}_{\mathcal{T}} : (L_n([0,1]))^2 \to L_n([0,1])$. In addition, if $\mathcal{T}$ is left-continuous then the n-DI $\mathcal{I}^{\varphi}_{\mathcal{T}}$ satisfies the following properties: $\mathcal{I}6, \mathcal{I}7$ and $\mathcal{I}9$.*

Proof It follows from Propositions 8, 13 and 15. □

4.3 Characterizing R-implications on $L_n([0, 1])$

We present a characterization of n-DRI, based on a left-continuous n-DT obtained from n-DI by a residuation principle.

Since $\mathcal{I}(L_n([0,1]))$ is a complete lattice and each $\mathcal{I} \in \mathcal{I}(L_n([0,1]))$ verifies the right boundary condition, meaning that $\mathcal{I}(\mathbf{x}, /1/) = /1/$, then the function $\mathcal{T}_{\mathcal{I}} : (L_n([0,1]))^2 \to L_n([0,1])$ given as

$$\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) = inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x}, \mathbf{t}) \ge \mathbf{y}\}, \tag{24}$$

is a well-defined function on $L_n([0,1])$.

Remark 6 Let $\widetilde{I_{RC}}$ be the n-dimensional extension of the Reichenbach fuzzy implication given in Example 5. For $\mathbf{x} > 0$ we obtain $\mathcal{T}_{\widetilde{I_{RC}}}(\mathbf{x}, /1/) = /1/$, meaning that $\mathcal{T}_{\widetilde{I_{RC}}}$ does not satisfy $\mathcal{T}4$. Concluding, Eq.(24) does not always generate an n-DT.

Remark 7 According to Corollary 3 the function $\widetilde{I_{LK}}$, reported in Example 5, does not verify the conditions defining an n-DRI. In addition, let T_{LK} be the Łukasiewicz t-norm. One can easily verify that $\mathcal{I}_{\widetilde{T_{LK}}}$ does not coincide to $\widetilde{I_{LK}}$.

Lemma 2 *Let $\mathcal{I}$ be an n-DI. If $\mathcal{I}$ is right-continuous w.r.t. the second variable, i.e. for each decreasing n-DS $(\mathbf{y}_i)_{i\in\mathbb{N}}$ and $\mathbf{x} \in L_n([0,1])$, we have that*

$$\mathcal{RC} : \lim_{i\to\infty} \mathcal{I}(\mathbf{x}, \mathbf{y}_i) = \mathcal{I}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i).$$

Then $\mathcal{I}$ is a meet-morphism, i.e. $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n([0,1])$, we have that

$$\mathcal{I}(\mathbf{x}, \mathbf{y} \wedge \mathbf{z}) = \mathcal{I}(\mathbf{x}, \mathbf{y}) \wedge \mathcal{I}(\mathbf{x}, \mathbf{z}).$$

Proof Analogous to Lemma 1. □

Proposition 16 *Let $\mathcal{I}$ be an n-DI and $\mathbf{x}, \mathbf{y} \in L_n([0,1])$. If $\mathcal{I}$ is right-continuous, then there exists an decreasing sequence $(\mathbf{z}_i)_{i\in\mathbb{N}} \subseteq \Delta = \{\mathbf{z} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{z}) \geq \mathbf{y}\}$ such that $\lim_{i\to\infty} \mathbf{z}_i = \inf \Delta$.*

Proof It follow from Proposition 14 and duality between $(L_n([0,1]), \leq)$ and $(L_n([0,1]), \geq)$, which also is a complete lattice and a ω-continuous domain. □

Proposition 17 *For $\mathcal{I} \in \mathcal{I}(L_n([0,1]))$ the following statements are equivalent:*

1. *$\mathcal{I}$ is right-continuous w.r.t. the second variable;*
2. *$(\mathcal{T}_{\mathcal{I}}, \mathcal{I})$ is an adjoint pair;*
3. *The infimum in Eq. (24) is the minimum, i.e. for $\mathbf{x}, \mathbf{y} \in L_n([0,1])$, we have that*

$$\mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}) = \min\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}\}. \tag{25}$$

Proof (**1.**) $\Rightarrow$ (**2.**) Firstly, suppose that $\mathcal{I}$ is a fuzzy implication which is right-continuous w.r.t. the second variable and also, that $\mathcal{I}(\mathbf{x},\mathbf{z}) \geq \mathbf{y}$ for some $\mathbf{x},\mathbf{y},\mathbf{z} \in L_n([0,1])$. This implies that $\mathbf{z} \in \{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}\}$ and hence $\mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}) \leq \mathbf{z}$. Conversely, if $\mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}) \leq \mathbf{z}$, for $\mathbf{x},\mathbf{y},\mathbf{z} \in L_n([0,1])$. Let $\mathbf{u} = \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y})$. From Proposition 16, there exists a decreasing sequence $(\mathbf{z}_i)_{i\in\mathbb{N}}$ such that $\mathcal{I}(\mathbf{x},\mathbf{z}_i) \geq \mathbf{y}$ and $\lim\limits_{i\to\infty} \mathbf{z}_i = \mathbf{u}$. So, if $\mathbf{u} \leq \mathbf{t}$ then, $\mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathcal{I}(\mathbf{x},\mathbf{u}) = \mathcal{I}(\mathbf{x}, \lim\limits_{i\to\infty} \mathbf{z}_i) = \lim\limits_{i\to\infty} \mathcal{I}(\mathbf{x},\mathbf{z}_i) \geq \mathbf{y}$. (**2.**) $\Rightarrow$ (**3.**) Assuming that $(\mathcal{T}_{\mathcal{I}}, \mathcal{I})$ is an adjoint pair. Since $\mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}) \leq \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y})$, one has that $\mathcal{I}(\mathbf{x},\mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y})) \geq \mathbf{y}$. So, by the definition of $\mathcal{T}_{\mathcal{I}}$ the infimum in (24) is the minimum. (**3.**) $\Rightarrow$ (**1.**) From the monotonicity of $\mathcal{I}$ w.r.t the second variable, for each decreasing sequence $(\mathbf{y}_i)_{i\in\mathbb{N}} \in L_n([0,1])$ and $\mathbf{x} \in L_n([0,1])$, the following holds:

$$\mathcal{I}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) \leq \lim_{i\to\infty} \mathcal{I}(\mathbf{x},\mathbf{y}_i). \tag{26}$$

When $\mathbf{y} = \lim\limits_{i\to\infty} \mathcal{I}(\mathbf{x},\mathbf{y}_i)$ then $\mathcal{I}(\mathbf{x},\mathbf{y}_i) \geq \mathbf{y}, \forall i \in \mathbb{N}$. Thus, $\mathbf{y}_i \in \{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}\}, \forall i \in \mathbb{N}$. So, $\mathbf{y}_i \geq \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}), \forall i \in \mathbb{N}$, meaning that $\lim\limits_{i\to\infty} \mathbf{y}_i \geq \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y})$. In addition, by (24) and $\mathcal{I}2$, it holds that

$$\mathcal{I}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) \geq \mathcal{I}(\mathbf{x}, \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y})) \geq \mathbf{y} = \lim_{i\to\infty} \mathcal{I}(\mathbf{x},\mathbf{y}_i). \tag{27}$$

From the above inequalities in Eqs. (26) and (27), we obtain the following result:

$$\mathcal{I}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) = \lim_{i\to\infty} \mathcal{I}(\mathbf{x},\mathbf{y}_i), \forall \mathbf{x}, \mathbf{y}_i \in L_n([0,1]), \forall i \in \mathbb{N}.$$

Meaning that $\mathcal{I}$ is right-continuous w.r.t. the second variable. Thus, Proposition 17 is verified. □

Theorem 6 *If a function $\mathcal{I} : (L_n([0,1]))^2 \to L_n([0,1])$ satisfies $\mathcal{I}2$, $\mathcal{I}6$, $\mathcal{I}8$ and verifies the right-continuity w.r.t. the second variable, then $\mathcal{T}_{\mathcal{I}}$ given by Eq. (25) is a left-continuous n-DT. Moreover $\mathcal{I} = \mathcal{I}_{\mathcal{T}_{\mathcal{I}}}$ and*

$$\mathcal{I}(\mathbf{x},\mathbf{y}) = \max\{\mathbf{t} \in L_n([0,1]) : \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{t}) \leq \mathbf{y}\}. \tag{28}$$

Proof First observe that once $\mathcal{I}$ is right-continuous, then by Proposition 17 the pair $(\mathcal{T}_{\mathcal{I}}, \mathcal{I})$ satisfies $\mathcal{RP}$.

Since $\mathcal{I}$ satisfies $\mathcal{I}6$ and $\mathcal{I}8$, by Proposition 7, it also satisfies $\mathcal{I}0(a)$, $\mathcal{I}0(b)$, $\mathcal{I}1$, $\mathcal{I}3$ and $\mathcal{I}4$. In particular, $\mathcal{I}$ satisfies $\mathcal{I}2$ and then it is an n-DI. For each $\mathbf{x},\mathbf{y},\mathbf{z} \in L_n([0,1])$ the following results are verified. ($\mathcal{T}1$) By $\mathcal{I}6$, for each $\mathbf{t} \in L_n([0,1])$, $\mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x},\mathbf{t})) = 1 \Leftrightarrow \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y},\mathbf{t})) = 1$. And, by $\mathcal{I}8$, it means that $\mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathbf{y} \Leftrightarrow \mathcal{I}(\mathbf{y},\mathbf{t}) \geq \mathbf{x}$. Then, the following holds:

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}) &= \inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}\}\\ &= \inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{y},\mathbf{t}) \geq \mathbf{x}\} = \mathcal{T}_{\mathcal{I}}(\mathbf{y},\mathbf{x}).\end{aligned}$$

($\mathcal{T}2$) Let $\mathbf{t} \in L_n([0,1])$. From properties $\mathcal{RP}$ and $\mathcal{I}6$:

$$\begin{aligned}\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathcal{T}_{\mathcal{I}}(\mathbf{y},\mathbf{z})) &= \inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathcal{T}_{\mathcal{I}}(\mathbf{y},\mathbf{z})\}\\ &= \inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x},\mathbf{t})) \geq \mathbf{z}\}\\ &= \inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y},\mathbf{t})) \geq \mathbf{z}\}\\ &= \inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{y},\mathbf{t}) \geq \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{z})\}\\ &= \mathcal{T}_{\mathcal{I}}(\mathbf{y}, \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{z})).\end{aligned}$$

($\mathcal{T}3$) For $\mathbf{x},\mathbf{y}_1,\mathbf{y}_2 \in L_n([0,1])$ and $\mathbf{y}_1 \leq \mathbf{y}_2$, we have that $\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}_1\} \supseteq \{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}_2\}$. Then, it implies that $\mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}_1) \leq \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{y}_2)$.

($\mathcal{T}4$) For $\mathbf{x} \in L_n([0,1])$, by $\mathcal{I}3$, we get the following: $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, /1/) = \mathcal{T}_{\mathcal{I}}(/1/, \mathbf{x}) = \inf\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(/1/, \mathbf{t}) \geq \mathbf{x}\}$. Then, $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, /1/) = \inf\{\mathbf{t} \in L_n([0,1]) : \mathbf{t} \geq \mathbf{x}\} = \mathbf{x}$. So, we conclude that $\mathcal{T}$ is a n-DT.

Now, assuming that $\mathcal{T}_{\mathcal{I}}$ is not left-continuous w.r.t. the second variable at some point $(\mathbf{x}, \mathbf{y}_0) \in (L_n([0,1]))^2$. Thus, there exists an increasing chain $(\mathbf{y}_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \mathbf{y}_i = \mathbf{y}^*$ and $\mathbf{a} = \lim_{i\to\infty} \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}_i) \neq \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}^*) = \mathbf{b}$. Since $\mathcal{T}_{\mathcal{I}}$ is increasing, then $\mathbf{a} < \mathbf{b}$ and $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}_i) \leq \mathbf{a}$, for all $i \in \mathbb{N}$. On the other hand, since for each $i \in \mathbb{N}$, $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}_i) \leq \mathbf{a}$ then, by $\mathcal{RP}$, $\mathbf{y}_i \leq \mathcal{I}(\mathbf{x}, \mathbf{a})$ for each $i \in \mathbb{N}$. So, in the limit, $\mathbf{y}^* = \lim\limits_{i\to\infty} \mathbf{y}_i \leq \mathcal{I}(\mathbf{x}, \mathbf{a})$ and therefore, by $\mathcal{RP}$, we have that $\mathbf{b} = \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}^*) \leq \mathbf{a}$ which is a contradiction to $\mathbf{a} < \mathbf{b}$. Then, $\mathcal{T}_{\mathcal{I}}$ is a left-continuous n-DT. Now, let $\mathcal{I}_{\mathcal{T}_{\mathcal{I}}}$ the n-DRI generated by $\mathcal{T}_{\mathcal{I}}$. For each $\mathbf{x},\mathbf{y} \in L_n([0,1])$, by Proposition 17, the following is verified:

$$\begin{aligned}&\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathcal{I}(\mathbf{x},\mathbf{y}))\\ &= \min\{\mathbf{t} \in L_n([0,1]) : \mathcal{I}(\mathbf{x},\mathbf{t}) \geq \mathcal{I}(\mathbf{x},\mathbf{y})\} \leq \mathbf{y}\\ &\Rightarrow \mathcal{I}(\mathbf{x},\mathbf{y}) \in \{\mathbf{t} \in L_n([0,1]) : \mathcal{T}_{\mathcal{I}}(\mathbf{x},\mathbf{t}) \leq \mathbf{y}\}.\end{aligned}$$

Then, $\mathcal{I}(\mathbf{x},\mathbf{y}) \leq \mathcal{I}_{\mathcal{T}_\mathcal{I}}(\mathbf{x},\mathbf{y})$. In addition, by $\mathcal{RP}$ and $\mathcal{I}2$, we have that:

$$\mathcal{I}_{\mathcal{T}_\mathcal{I}}(\mathbf{x},\mathbf{y}) \leq \mathcal{I}(\mathbf{x}, \mathcal{T}_\mathcal{I}(\mathbf{x}, \mathcal{I}_{\mathcal{T}_\mathcal{I}}(\mathbf{x},\mathbf{y}))) \leq \mathcal{I}(\mathbf{x},\mathbf{y}).$$

So, $\mathcal{I}_{\mathcal{T}_\mathcal{I}}(\mathbf{x},\mathbf{y}) = \mathcal{I}(\mathbf{x},\mathbf{y})$. And, Theorem 6 holds. □

We also have the following connection between left-continuous n-DT and n-DRI generated from them.

Lemma 3 *$\mathcal{T} = \mathcal{T}_{\mathcal{I}_\mathcal{T}}$ if $\mathcal{T}$ is a left-continuous n-DT.*

Proof From Proposition 15, $\mathcal{I}_\mathcal{T} \in \mathcal{I}(L_n([0,1]))$ satisfies $\mathcal{I}6$, $\mathcal{I}8$ and is right-continuous w.r.t. the second variable. By Theorem 6, the function $\mathcal{T}_{\mathcal{I}_\mathcal{T}}$ is a left-continuous n-DT. Since $\mathcal{T}$ is left-continuous and by Theorem 4 the pair $(\mathcal{T}, \mathcal{I}_\mathcal{T})$ satisfies $\mathcal{RP}$, then for $\mathbf{x},\mathbf{y} \in L_n([0,1])$ it holds that: $\mathcal{I}_\mathcal{T}(\mathbf{x}, \mathcal{T}(\mathbf{x},\mathbf{y})) = \max\{\mathbf{t} \in L_n([0,1])\colon \mathcal{T}(\mathbf{x},\mathbf{t}) \leq \mathcal{T}(\mathbf{x},\mathbf{y})\}$, i.e. $\mathcal{I}_\mathcal{T}(\mathbf{x}, \mathcal{T}(\mathbf{x},\mathbf{y})) \geq \mathbf{y}$ and therefore, we have that $\mathcal{T}(\mathbf{x},\mathbf{y}) \in \{\mathbf{t} \in L_n([0,1])\colon \mathcal{I}_\mathcal{T}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}\}$. Hence,

$$\mathcal{T}(\mathbf{x},\mathbf{y}) \geq \mathcal{T}_{\mathcal{I}_\mathcal{T}}(\mathbf{x},\mathbf{y}). \tag{29}$$

Conversely, since $\mathcal{T}$ is left-continuous, from Proposition 4, $\mathcal{T}(\mathbf{x}, \mathcal{I}_\mathcal{T}(\mathbf{x},\mathbf{z})) \leq \mathbf{z}$, for any $\mathbf{z} \in L_n([0,1])$. If $\mathbf{z} = \mathcal{T}_{\mathcal{I}_\mathcal{T}}(\mathbf{x},\mathbf{y})$, then $\mathcal{T}_{\mathcal{I}_\mathcal{T}}(\mathbf{x},\mathbf{y}) \geq \mathcal{T}(\mathbf{x}, \mathcal{I}_\mathcal{T}(\mathbf{x}, \mathcal{T}_{\mathcal{I}_\mathcal{T}}(\mathbf{x},\mathbf{y})))$. Further, since $\mathcal{I}_\mathcal{T}$ is right-continuous, from Proposition 17 we get also $\mathcal{I}_\mathcal{T}(\mathbf{x}, \mathcal{T}_{\mathcal{I}_\mathcal{T}}(\mathbf{x},\mathbf{y})) \geq \mathbf{y}$. Thereby, by $\mathcal{RP}$, we obtain the following result:

$$\mathcal{T}_{\mathcal{I}_\mathcal{T}}(\mathbf{x},\mathbf{y}) \geq \mathcal{T}(\mathbf{x},\mathbf{y}). \tag{30}$$

From Eqs. (29) and (30), we get $\mathcal{T}_{\mathcal{I}_\mathcal{T}}(\mathbf{x},\mathbf{y}) = \mathcal{T}(\mathbf{x},\mathbf{y})$, $\forall \mathbf{x},\mathbf{y} \in L_n([0,1])$. So, Lemma 3 is verified. □

Other results in the characterization of n-DRI generated from left-continuous n-DT are shown below.

Theorem 7 *For a function $\mathcal{I} : (L_n([0,1]))^2 \to L_n([0,1])$ the following statements are equivalent:*

1. *$\mathcal{I}$ is an n-DRI obtained from a left-continuous n-DT;*
2. *$\mathcal{I}$ satisfies $\mathcal{I}2, \mathcal{I}6, \mathcal{I}8$ and right-continuity w.r.t. the second variable.*

Moreover, the representation of n-DRI, up to a left-continuous n-DT, is unique in this case.

Proof Consider function $\mathcal{I} : (L_n([0,1]))^2 \to L_n([0,1])$. The following holds: (**1**.) $\Rightarrow$ (**2**.) Let $\mathcal{I}$ be an n-DRI generated from a left-continuous n-DT $\mathcal{T}$. By Lemma 3, $\mathcal{T}$ is $\mathcal{T}_\mathcal{I}$. From Proposition 15 it satisfies properties $\mathcal{I}2$, $\mathcal{I}6$ and $\mathcal{I}8$. Moreover, it is also right-continuous w.r.t. the second variable. (**2**.) $\Rightarrow$ (**1**.) Let $\mathcal{I} : (L_n([0,1]))^2 \to L_n([0,1])$ be a function satisfying $\mathcal{I}2, \mathcal{I}6, \mathcal{I}8$ and it is right-continuous w.r.t. the second variable. By Theorem 6, we get that $\mathcal{I} = \mathcal{I}_{\mathcal{T}_\mathcal{I}}$ where $\mathcal{T}_\mathcal{I}$ defined by Eq. (25) is a left-continuous n-DT. Hence $\mathcal{I}$ is an n-DRI generated from the left-continuous n-DT $\mathcal{T}_\mathcal{I}$. And, the uniqueness of the representation of an n-DRI up to a left-continuous n-DT follows from Lemma 3.

Concluding, it is shown that Theorem 7 is verified. □

The next corollary follows from Theorem 7 characterizing left-continuous t-norms on $L_n([0,1])$.

Corollary 5 *For $\mathcal{T} : (L_n([0,1]))^2 \to L_n([0,1])$, the following statements are equivalent:*

1. *$\mathcal{T}$ is a left-continuous n-DT;*
2. *There exists $\mathcal{I} \in \mathcal{FI}$, which satisfies $\mathcal{I}6$, $\mathcal{I}8$ and it is also right-continuous w.r.t. the second variable, such that $\mathcal{T}$ is given by (25).*

Proof Straightforward from Theorem 7. □

Lemma 4 *If $\mathcal{I} : L_n([0,1])^2 \to L_n([0,1])$ is continuous function except at the point $(/0/, /0/)$ satisfying $\mathcal{I}2$, $\mathcal{I}6$, $\mathcal{I}7$ and $\mathcal{I}(\mathbf{x}, /0/) = N_{D1}(\mathbf{x})$, then the function $\mathcal{T}_\mathcal{I}$ defined by (24) is a continuous n-DT.*

Proof Since $\mathcal{I}$ is right continuous w.r.t. the second variable, from Theorem 6, $\mathcal{T}_\mathcal{I}$ is a left-continuous n-DT and, by Proposition 17, $(\mathcal{T}_\mathcal{I}, \mathcal{I})$ is an adjoint pair, i.e. $\mathcal{I}$ and $\mathcal{T}_\mathcal{I}$ satisfy property $\mathcal{RP}$. Thus, in order to prove that n-DT $\mathcal{T}_\mathcal{I}$ is a continuous function, suppose the contrary, i.e. that $\mathcal{T}_\mathcal{I}$ is not right-continuous. Then there exist $\mathbf{x},\mathbf{y} \in L_n([0,1])$ and $\mathbf{z},\mathbf{z}' \in L_n([0,1])$ such that

$$\mathbf{z} = \mathcal{T}_\mathcal{I}(\mathbf{x},\mathbf{y}) < \mathbf{z}' < \lim_{\mathbf{h}\to\mathbf{0}^+} \mathcal{T}_\mathcal{I}(\mathbf{x}, \mathbf{y}+\mathbf{h}). \tag{31}$$

considering the sum of THFE.[1]

From $\mathcal{RP}$ property, we have $\mathcal{I}(\mathbf{x},\mathbf{z}) \geq \mathbf{y}$ and by $\mathcal{I}2$, we have $\mathcal{I}(\mathbf{x},\mathbf{z}') \geq \mathbf{y}$. Moreover, by $\mathcal{RP}$, we also have that $\mathbf{z}' \leq \mathcal{T}_\mathcal{I}(\mathbf{x},\mathbf{y})$, which is a contradiction. Thus, $\mathcal{T}_\mathcal{I}$ is right-continuous and hence, it is also continuous. Therefore, Lemma 4 is verified. □

5 Obtaining *n*-DRI from *n*-DA operators

This section extends main results in Liu and Wang (2006) from interval-valued fuzzy set theory to n-dimensional simplex. In particular, the class of n-DRI is constructed based on binary n-DA aggregation operators, given as the minimum operator and left-continuous t-norms.

[1] The operator $+ : L_n([0,1])^2 \to L_n([0,1])$ is given as $\mathbf{x}+\mathbf{y} = (\min(x_1+y_1, 1), \ldots, \min(x_n+y_n, 1))$, $\forall \mathbf{x} = (x_1,\ldots,x_n), \mathbf{y} = (y_1,\ldots,y_n) \in L_n([0,1])$.

Proposition 18 *Let $T_1, \ldots, T_n : [0,1]^2 \to [0,1]$ be left-continuous t-norms such that $T_1 \leq \cdots \leq T_n$. Then, for all $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in L_n([0,1])$ the function $\mathcal{T}_{T_1,\ldots,T_n} : (L_n([0,1]))^2 \to L_n([0,1])$ given as $\mathcal{T}_{T_1\ldots T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=1}^n T_1(x_{n-i+1}, y_i), \ldots, T_n(x_1, y_n)\right)$ and shortly expressed as follows:*

$$\mathcal{T}_{T_1\ldots T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*} \tag{32}$$

verifies $\mathcal{T}1$, $\mathcal{T}2$, $\mathcal{T}4$, $\mathcal{LC}$ and $\mathcal{RP}$ properties on $L_n([0,1])$.

Proof Let $T_1, \ldots, T_n : [0,1]^2 \to [0,1]$ be left-continuous t-norms such that $T_1 \leq \cdots \leq T_n$. For all $\mathbf{x}, \mathbf{y}, \mathbf{x}' \in L_n([0,1])$ the following holds:

$$(\mathcal{T}1)\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*}$$
$$= \left(\min_{i=k}^{n} T_k(y_i, x_{n-i+k})\right)_{k\in\mathbb{N}_n^*} = \left(\min_{i=k}^{n} T_k(y_{n-i+k}, x_i)\right)_{k\in N_n^*}$$
$$= \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{y}, \mathbf{x}).$$
$$(\mathcal{T}2)\mathbf{x} \leq \mathbf{x}' \Rightarrow \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*}$$
$$\leq \left(\min_{i=k}^{n} T_k(x'_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*} = \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}', \mathbf{y}).$$

$$(\mathcal{T}4)\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, /1/) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, 1)\right)_{k\in\mathbb{N}_n^*}$$
$$= \left(\min_{i=k}^{n} x_{n-i+k}\right)_{k\in\mathbb{N}_n^*} = (x_1, x_2, \ldots, x_n) = \mathbf{x}.$$

($\mathcal{LC}$) Let $\mathbf{x}^l$ be a non-decreasing sequence in $L_n([0,1])$. Then, the following holds:

$$\lim_{l\to\infty} \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}^l, \mathbf{y})$$
$$= \lim_{l\to\infty} \left(\min_{i=k}^{n} T_k(x^l_{n+k-i}, y_i)\right)_{k\in\mathbb{N}_n^*}$$
$$= \left(\min_{i=k}^{n} \lim_{l\to\infty} T_k(x^l_{n+k-i}, y_i)\right)_{k\in\mathbb{N}_n^*}$$
$$= \left(\min_{i=1}^{n} T_1(\lim_{l\to\infty} x^l_{n+1-i}, y_i), \ldots, T_n(\lim_{l\to\infty} x^l_1, y_n)\right)$$
$$= \left(\min_{i=k}^{n} T_k(\lim_{l\to\infty} x^l_{n+k-i}, y_i)\right)_{k\in\mathbb{N}_n^*} = \mathcal{T}_{T_1,\ldots,T_n}(\lim_{l\to\infty} \mathbf{x}^l, \mathbf{y}).$$

Since $\mathcal{T}_{T_1,\ldots,T_n}$ satisfies $\mathcal{LC}$ and the proof of Theorem 4 ((**1.**) $\Rightarrow$ (**2.**)) does not make use $\mathcal{T}3$, then this same proof also proves $\mathcal{RP}$ for $\mathcal{T}_{T_1,\ldots,T_n}$ and $\mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$. Concluding, Proposition 18 is verified. □

In the next proposition, it is shown that the operator $\mathcal{T}_{T_1,\ldots,T_n}$ forms an adjoint pair with its residuum operator, in spite of not being necessarily a n-dimensional t-norm.

Corollary 6 *Let $T_1, \ldots, T_n : [0,1]^2 \to [0,1]$ be left-continuous t-norms such that $T_1 \leq \cdots \leq T_n$. Then, the pair of functions $(\mathcal{T}_{T_1,\ldots,T_n}, \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}})$ is an adjoint pair.*

Proof Follows from Propositions 17 and 18. □

The characterization of an $\mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$ operator is presented in the following theorem.

Theorem 8 *Let $T_1, \ldots, T_n : [0,1]^2 \to [0,1]$ be left-continuous t-norms on $[0,1]$ such that $T_1 \leq \ldots \leq T_n$ and $I_1, \ldots, I_n$ be their corresponding residual implications. Then $\mathcal{I}_{I_1,\ldots,I_n} = \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$ if the function $\mathcal{I}_{I_1,\ldots,I_n} : (L_n([0,1]))^2 \to L_n([0,1])$, for $\forall \mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in L_n([0,1])$, is given as follows:*

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=1}^{n} I_i(x_i, y_i), \min_{i=2}^{n} I_i(x_i, y_i), \ldots, I_n(x_n, y_n)\right), \tag{33}$$

and, shortly expressed in the next equation

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^{n} I_i(x_i, y_i)\right)_{k\in\mathbb{N}_n^*}, \tag{34}$$

which is called as the $\mathcal{I}_{I_1,\ldots,I_n}$-operator.

Proof Let $T_1, \ldots, T_n : [0,1]^2 \to [0,1]$ be left-continuous t-norms on $[0,1]$ such that $T_1 \leq \cdots \leq T_n$ and $I_1, \ldots, I_n$ be their corresponding residual implications. For all $\mathbf{x}, \mathbf{y} \in L_n([0,1])$ we have the next results:

$$\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y})) \leq \mathbf{y} \Leftrightarrow$$
$$\Leftrightarrow \min_{i=k}^{n} T_k(x_{n+k-i}, \min_{j=k}^{n} I_j(x_j, y_j)) \leq y_k, \forall k \in \mathbb{N}_n$$
$$\Leftrightarrow T_k(x_k, \min_{j=k}^{n} I_j(x_j, y_j) \leq y_k, \forall k \in \mathbb{N}_n$$
$$\Leftrightarrow \min_{j=k}^{n} I_j(x_j, y_j) \leq I_k(x_k, y_k), \forall k \in \mathbb{N}_n \text{ (by } \mathcal{RP}).$$

So, $\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y})) \leq \mathbf{y}$. Then, implying that $\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) \in \{\mathbf{z} \in L_n([0,1]) : \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$. Consequently, two results need to be considered:

$$(i)\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) \leq \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$$
$$= \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}(\mathbf{x}, \mathbf{y}).$$

Now, let $\mathbf{z} \in L_n([0,1])$ such that $\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}$. Then, for each $k \in \mathbb{N}_n$, $\min_{i=k}^n T_k(x_{n+k-i}, z_i) \leq y_k$. So, there exists $i \geq k$ such that $T_k(x_{n+k-i}, z_i) \leq y_k$ and therefore,

$T_k(x_k, z_k) \leq y_k$. Hence, by $\mathcal{RP}$, it holds that $z_k \leq I_k(x_k, y_k)$ for each $k \in \{1, \ldots, n\}$ and therefore $z_k \leq \min_{i=k}^n I_i(x_i, y_i)$. And, the following holds:

$$\mathbf{z} \leq \left(\min_{i=1}^n I_i(x_i, y_i), \min_{i=2}^n I_i(x_i, y_i), \ldots, I_n(x_n, y_n) \right)$$

And, $\mathbf{z} \leq \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y})$. It holds that:

$$(ii)\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) \geq \sup\{\mathbf{z} \in L_n([0,1]) : \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\} = \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}(\mathbf{x}, \mathbf{y}).$$

Therefore, from (i) e (ii), Theorem 8 is verified. □

Corollary 7 *Let $T_1, \ldots, T_n : [0,1]^2 \rightarrow [0,1]$ be left-continuous t-norms on $[0,1]$ such that $T_1 \leq \ldots \leq T_n$ and $I_1, \ldots, I_n$ be their corresponding residual implications. Then, we have $\left(\mathcal{I}_{I_1,\ldots,I_n}, \mathcal{T}_{T_1\ldots T_n}\right)$ is an adjoint pair on $L_n([0,1])$.*

Proof It follows from Corollary 6 and Theorem 8. □

5.1 Main properties related to $\mathcal{I}_{I_1,\ldots,I_n}$-operator

Now, the conditions under which the main properties of fuzzy implications are preserved by action of the $\mathcal{I}_{I_1,\ldots,I_n}$-operator on $L_n([0,1])$, are investigated.

Proposition 19 *Let $I_1, \ldots, I_n : [0,1]^2 \rightarrow [0,1]$ be implications such that $I_n \leq \cdots \leq I_1$ satisfying Ik, for $k \in \{1, 2, 3, 4, 5, 8\}$. The $\mathcal{I}_{I_1,\ldots,I_n}$-operator expressed as (34) verifies $\mathcal{I}k$ property.*

Proof Let $I_1, \ldots, I_n : [0,1]^2 \rightarrow [0,1]$ be implications such that $I_n \leq \cdots \leq I_1$. Observe that $\min_{i=k}^{n} I_i(x_i, y_i) = I_j(x_j, y_j)$ for some $j = k, \ldots, n$, implying that: ($\mathcal{I}0$): Firstly, the boundary conditions hold:

$$\mathcal{I}_{I_1,\ldots,I_n}(/0/, /0/) = \left(\min_{i=k}^n I_i(0,0) \right)_{k \in \mathbb{N}_n} = /1/;$$

$$\mathcal{I}_{I_1,\ldots,I_n}(/0/, /1/) = \left(\min_{i=k}^n I_i(0,1) \right)_{k \in \mathbb{N}_n} = /1/;$$

$$\mathcal{I}_{I_1,\ldots,I_n}(/1/, /1/) = \left(\min_{i=k}^n I_i(1,1) \right)_{k \in \mathbb{N}_n} = /1/;$$

$$\mathcal{I}_{I_1,\ldots,I_n}(/1/, /0/) = \left(\min_{i=k}^n I_i(1,0) \right)_{k \in \mathbb{N}_n} = /0/.$$

($\mathcal{I}1$) If $\mathbf{x} \leq \mathbf{x}'$ then the following holds:

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^n I_i(x_i, y_i) \right)_{k \in \mathbb{N}_n}$$
$$= \left(\min_{i=1}^n I_i(x_i, y_i), \min_{i=2}^n I_i(x_i, y_i), \ldots, I_n(x_n, y_n) \right)$$
$$\geq \left(\min_{i=1}^n I_i(x'_i, y_i), \min_{i=2}^n I_i(x'_i, y_i), \ldots, I_n(x'_n, y_n) \right)$$
$$= \left(\min_{i=k}^n I_i(x'_i, y_i) \right)_{k \in \mathbb{N}_n} = \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}', \mathbf{y});$$

($\mathcal{I}2$) If $\mathbf{y} \leq \mathbf{y}'$ then the next results hold:

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^n I_i(x_i, y_i) \right)_{k \in \mathbb{N}_n}$$
$$= \left(\min_{i=1}^n I_i(x_i, y_i), \min_{i=2}^n I_i(x_i, y_i), \ldots, I_n(x_n, y_n) \right)$$
$$\leq \left(\min_{i=1}^n I_i(x_i, y'_i), \min_{i=2}^n I_i(x_i, y'_i), \ldots, I_n(x_n, y'_n) \right)$$
$$= \left(\min_{i=k}^n I_i(x_i, y'_i) \right)_{k \in \mathbb{N}_n} = \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}');$$

$$(\mathcal{I}3)\mathcal{I}_{I_1,\ldots,I_n}(/1/, \mathbf{y}) = \left(\min_{i=k}^n I_i(1, y_i) \right)_{k \in \mathbb{N}_n}$$
$$= \left(\min_{i=1}^n I_i(1, y_i), \min_{i=2}^n I_i(1, y_i), \ldots, I_i(1, y_n) \right)$$
$$= \left(\min_{i=1}^n y_i, \min_{i=2}^n y_i, \ldots, y_n \right) = (y_1, y_2, \ldots, y_n) = \mathbf{y};$$

$$(\mathcal{I}4)\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{x}) = \left(\min_{i=k}^n I_{I_i}(x_i, x_i) \right)_{k \in \mathbb{N}_n}$$
$$= \left(\min_{i=1}^n I_1(x_i, x_i), \min_{i=2}^n I_i(x_i, x_i), \ldots, I_n(x_n, x_n) \right) = /1/;$$

$$(\mathcal{I}5)\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, /0/) = \left(\min_{i=k}^n I_i(x_i, 0) \right)_{k \in \mathbb{N}_n^*}$$
$$= \left(\min_{i=1}^n I_i(x_i, 0), \min_{i=2}^n I_i(x_i, 0), \ldots, I_n(x_n, 0) \right)$$
$$= (\min(I_1(x_1, 0), I_2(x_2, 0), \ldots, I_n(x_n, 0)),$$
$$\min(I_2(x_2, 0), \ldots, I_n(x_n, 0)), \ldots, I_n(x_n, 0))$$
$$= (I_n(x_n, 0), I_n(x_n, 0), \ldots I_n(x_n, 0))$$
$$= (N_n(x_n), N_n(x_n) \ldots, N_n(x_n)) = \widetilde{N_n}(/x_n/)$$

$$(\mathcal{I}8)\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = /1/$$
$$\Leftrightarrow \left(\min_{i=k}^n I_i(x_i, y_i) \right)_{k \in \mathbb{N}_n} = /1/$$
$$\Leftrightarrow \min_{i=1}^n I_i(x_i, y_i) = 1 \text{ or } \ldots \text{ or } I_n(x_n, y_n) = 1$$
$$\Leftrightarrow x_i \leq y_i, \forall i \in \mathbb{N}_n \Leftrightarrow \mathbf{x} \leq \mathbf{y}.$$

Therefore, Proposition 19 is verified. □

Example 6 Let I_{LK} given in Example 5and T_{LK} given in Example 4. Consider the restriction from $L_n([0,1])$ to the interval approach on $L_2([0,1])$. By Theorem 8, we have an adjoint pair $(\mathcal{I}_{I_{LK}}, \mathcal{T}_{T_{LK}})$, such that

$$\mathcal{I}_{I_{LK}}(\mathbf{x}, \mathbf{y}) = [\min(I_{Lk}(x_1, y_2), I_{LK}(x_2, y_1)), I_{Lk}(x_2, y_2)];$$

$\mathcal{T}_{T_{LK}}(\mathbf{x},\mathbf{y}) = [T_{Lk}(x_1,y_1), \min(T_{LK}(x_1,y_2), T_{LK}(x_2,y_1))]$.

And, by Liu and Wang (2006), taking the representable n-DT $\widetilde{T_{LK}}(\mathbf{x},\mathbf{y}) = [T_{LK}(x_1,y_1), T_{LK}(x_2,y_2)]$, we also have $(\mathcal{I}_{I_{LK}}, \widetilde{T_{LK}})$ as another adjoint pair on $L_2([0,1])$.

6 Modelling a CIM-application on $L_n([0,1])$

This section extents the application described in (Wen et al. 2018, Example 1) from HFS to n-DS, in order to solve the multi-criteria decision-making (MCDM) problem considering multiple alternatives in the selection of CIM (computer-integrated manufacturing) software.

6.1 Describing the CIM-MCDM problem

In order to help the user in the selection of seven kinds of CIM software systems available in the market nowadays, a data processing company aims to clarify differences of such systems (Chen et al. 2013).

The evaluations expressed by n-DFS are shown in Table 2. In this case study, $A = \{A_1, A_2, \ldots, A_7\}$ $(n_2 = 7)$ be the set of CIM software alternatives and X be the set of 4 attributes related to functionality (x_1), usability (x_2), portability (x_3) and maturity (x_4) $(n_1 = 4)$. See Table 2, related to matrix $[D]_{7\times4} = (\mathbf{x}_{ki})_{k=1\ldots7, i=1\ldots4}$ whose elements are given as an 3-dimensional interval $\mathbf{x}_{ij}$, containing selected opinions of three decision makers and providing their evaluations with values between 0 and 1 for all alternative A_i w.r.t. each attribute.

6.2 Applying the residual implication $\mathcal{I}_{I_{LK}}$

The triangle product relation $\lhd \subseteq (L_4([0,1]))^7$ is given as $\lhd \equiv \mathcal{F}_\wedge \circ \mathcal{I}$, considering the operators:

- $\mathcal{I} \equiv \mathcal{I}_{I_{LK},I_{Lk},I_{LK}} : (L_3([0,1]))^2 \to L_3([0,1])$ introduced in Theorem 8, expressed by (34) considering the Łukasiewicz fuzzy implication I_{LK} presented in Example 5; and
- $\mathcal{F}_\wedge : L_4([0,1])^4 \to L_4([0,1])$ as the minimum operator in Eq. (17).

Taking $k, j \in \{1,\ldots,7\}$, the action of $\lhd$-operator can be given by 7×7-matrix whose elements $\mathbf{z}_{k,j} = \lhd(\mathbf{x}_{ki}, \mathbf{x}_{ji}) \in L_4([0,1])$ are given as follows:

$$\mathbf{z}_{k,j} = \begin{cases} \left(\mathcal{F}_\wedge \circ \mathcal{I}\,(\mathbf{x}_{ki}, \mathbf{x}_{li})_{(i=1\ldots4)}\right)_{(k,j=1\ldots7)}, & \text{if } k \neq j \\ (1,1,1,1), & \text{otherwise.} \end{cases} \tag{35}$$

And, the result comparisons are achieved whenever the ordered elements $\mathbf{z}_{(k,j)} \in L_4([0,1])$ are obtained based on the admissible linear order $\sqsubseteq_{[M]}$-order, described in Proposition 5.

See the action of operator in Eq. (35) resulting on the matrix $L_{7\times7} = (\mathbf{z}_{(kj)})_{k,j=1\ldots7}$ presented in Table 3.

Illustrating the action of $\lhd$-operator over the 3-dimensional intervals reported in Table 2, consider the related 1^{st}

Table 2 Information related to The n-dimensional interval components

D-matrix	x_1	x_2	x_3	x_4
A1	(0.8, 0.85, 0.95)	(0.7, 0.75, 0.8)	(0.65, 0.65, 0.80)	(0.3, 0.3, 0.35)
A2	(0.85, 0.85, 0.9)	(0.6, 0.7, 0.8)	(0.2, 0.2, 0.2)	(0.15, 0.15, 0.15)
A3	(0.2, 0.3, 0.40)	(0.4, 0.4, 0.5)	(0.9, 0.9, 1)	(0.45, 0.5, 0.65)
A4	(0.8, 0.95, 1)	(0.1, 0.15, 0.2)	(0.2, 0.2, 0.3)	(0.6, 0.7, 0.80)
A5	(0.35, 0.4, 0.5)	(0.7, 0.9,1)	(0.4, 0.4, 0.4)	(0.2, 0.3, 0.35)
A6	(0.5, 0.6, 0.7)	(0.8, 0.8, 0.9)	(0.4, 0.4, 0.6)	(0.1, 0.1, 0.2)
A7	(0.8, 0.8, 1)	(0.15, 0.2, 0.35)	(0.1, 0.1, 0.2)	(0.7, 0.7, 0.85)

Table 3 Action of $\lhd$-operator in n-dimensional intervals

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
A1	(1,1,1,1)	(0.4,0.8,0.9,0.95)	(0.4,0.65,1,1)	(0.4,0.5,1,1)	(0.55,0.6,0.9,1)	(0.7,0.75,0.8001)	(0.4,0.45,0.95,1)
A2	**(0.95,1,1,1)**	(1,1,1,1)	(0.35,0.7,1,1)	(0.4,0.95,1,1)	(0.5,1,1,1)	(0.65,0.95,1,1)	(0.5,0.9,0.95,1)
A3	(0.7,0.75,1,1)	(0.2,0.5,1,1)	(1,1,1,1)	(0.3,0.7,1,1)	(0.4,0.7,1,1)	(0.5,0.55,1,1)	(0.2,0.75,1,1)
A4	(0.55,0.9,1,1)	(0.35,0.9,0.9,1)	(0.35,0.8,1,1)	(1,1,1,1)	(0.45,0.55,1,1)	(0.4,0.65,1,1)	(0.85,0.9,1,1)
A5	(0.8,1,1,1)	(0.8,0.8,0.8,1)	(0.5,0.85,1,1)	(0.2,0.8,1,1)	(1,1,1,1)	(0.8,0.9,1,1)	(0.3,0.7,1,1)
A6	(0.9,1,1,1)	(0.6,0.8,0.95,1)	(0.6,0.7,1,1)	(0.3,0.7,1,1)	(0.8,0.8,0.9,1)	(1,1,1,1)	(0.35,0.6,1,1)
A7	(0.5,0.95,1,1)	(0.3,0.9,1,1)	(0.4,0.75,1,1)	(0.85,0.9,1,1)	(0.5,0.5,1,1)	(0.35,0.7,1,1)	(1,1,1,1)

and 2^{nd} lines (A_1 and A_2 alternatives). So, the component $\mathbf{z}_{21}$ is given as:

$$\begin{aligned}\mathbf{z}_{21} &= \mathcal{F}_{\wedge}(\mathbf{y}_i)_{i=1\ldots 4};\\ \mathbf{y}_i &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{2i}, \mathbf{x}_{1i})\\ &= (\min(I_{LK}(x_1, y_1), I_{LK}(x_2, y_2), I_{LK}(x_3, y_3)),\\ &\quad \min(I_{LK}(x_2, y_2), I_{LK}(x_3, y_3)), I(x_3, y_3)) \in L_3(U).\end{aligned}$$

It holds that:

$$\begin{aligned}\mathbf{y}_1 &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{21}, \mathbf{x}_{11}) = \mathcal{I}((0.85, 0.85, 0.9), (0.8, 0.85, 0.95))\\ &= (\min(I_{LK}(0.85, 0.8), I_{LK}(0.85, 0.85), I_{LK}(0.9, 0.95)),\\ &\quad \min(I_{LK}(0.85, 0.85), I_{LK}(0.9, 0.95)), I_{LK}(0.9, 0.95))\\ &= (\min(\min(1, 1-0.85+0.8), \min(1, 1-0.85+0.85)),\\ &\quad \min(1, 1-0.9+0.95)), \min(\min(1, 1-0.85+0.85)),\\ &\quad \min(1, 1-0.9+0.95)), \min(1, 1-0.9+0.95)) = (0.95, 1, 1);\\ \mathbf{y}_2 &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{22}, \mathbf{x}_{12}) = \mathcal{I}((0.6, 0.7, 0.8), (0.7, 0.75, 0.8))\\ &= (\min(I_{LK}(0.6, 0.7), I_{LK}(0.7, 0.75), I_{LK}(0.8, 0.8)),\\ &\quad \min(I_{LK}(0.7, 0.75), I_{LK}(0.8, 0.8)), I_{LK}(0.8, 0.8))\\ &= ((\min(1, 1-0.6+0.7), \min(1, 1-0.7+0.75),\\ &\quad \min(1, 1-0.8+0.8)), (\min(1, 1-0.7+0.75),\\ &\quad \min(1, 1-0.8+0.8)), \min(1, 1-0.8+0.8)) = (1, 1, 1);\\ \mathbf{y}_3 &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{23}, \mathbf{x}_{13}) = \mathcal{I}((0.2, 0.2, 0.2), (0.65, 0.65, 0.8))\\ &= (\min(I_{LK}(0.2, 0.65), I_{LK}(0.2, 0.65), I_{LK}(0.2, 0.8)),\\ &\quad \min(I_{LK}(0.2, 0.65), I_{LK}(0.2, 0.8)), I_{LK}(0.2, 0.8))\\ &= ((\min(1, 1-0.2+0.65), \min(1, 1-0.2+0.65),\\ &\quad \min(1, 1-0.2+0.8)), (\min(1, 1-0.2+0.65),\\ &\quad \min(1, 1-0.2+0.8)), \min(1, 1-0.2+0.8)) = (1, 1, 1);\\ \mathbf{y}_4 &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{24}, \mathbf{x}_{14}) = \mathcal{I}((0.15, 0.15, 0.15), (0.3, 0.3, 0.35))\\ &= (\min(I_{LK}(0.15, 0.3), I_{LK}(0.15, 0.3), I_{LK}(0.15, 0.35)),\\ &\quad \min(I_{LK}(0.15, 0.3), I_{LK}(0.15, 0.35)), I_{LK}(0.15, 0.35))\\ &= (\min(1, 1-0.15+0.3), \min(1, 1-0.15+0.3),\\ &\quad \min(1, 1-0.15+0.35)), \min(\min(1, 1-0.15+0.3),\\ &\quad \min(1, 1-0.15+0.35)), \min(1, 1-0.15+0.35)) = (1, 1, 1).\end{aligned}$$

And, $\mathbf{z}_{21} = \triangleleft(\mathbf{x}_{2i}, \mathbf{x}_{1i}) = \mathcal{F}_{\wedge} \circ \mathcal{I}_{I_{LK}}\,(\mathbf{x}_{2i}, \mathbf{x}_{1i})_{(i=1\ldots 4)}$, which can be expressed as follows:

$$\begin{aligned}\mathbf{z}_{21} &= (\wedge(0.95, 1, 1), \wedge(1, 1, 1), \wedge(1, 1, 1), \wedge(1, 1, 1))\\ &= (0.95, 1, 1, 1).\end{aligned}$$

It results on the ordered component $\mathbf{z}_{(21)}$, as shown in Table 3, placed in row-2 and column-1. The other components can be obtained analogously.

6.2.1 Solving the CIM-MCDM problem on $L_n([0, 1])$

Since many elements in Table 3 of 4-dimensional interval components may not be comparable by the usual partial order $\leq_{L_4(U)}$, we consider the set $\mathcal{M}$ of aggregation sequence given in Example 3, to apply the $\sqsubseteq_{[M]}$-order as the admissible linear order described in Proposition 5. Such interval data preserving strategy enables us to consider the uncertainty associated with input data modelling the possible indecision of specialist preferences. In addition, we remain capable to guarantee the comparison of all output interval data.

Thus, using this strategy, we are not restricted to the use of aggregation operators (as the arithmetic mean) performed over the elements in Table 3, presenting the ordered values related to each component $\mathbf{z}_{(kl)} \in L_4(U)$ resulting from action of $\triangleleft$-operator over data provided by evaluations in such MCDM problem.

See, the components $\mathbf{z}_{(72)} = (0.3, 0.9, 1.0, 1.0)$ and $\mathbf{z}_{(27)} = (0.5, 0.9, 0.95, 1.0)$, which are incomparable w.r.t. the partial order $\leq_{L_n([0,1])}$, meaning that $\mathbf{z}_{(72)} \not\leq_{L_n([0,1])} \mathbf{z}_{(27)}$ and $\mathbf{z}_{(27)} \not\leq_{L_n([0,1])} \mathbf{z}_{(72)}$ hold. So, the $\mathcal{M}$-aggregation ($\mathcal{M} = \{M_1, M_2, M_3, M_4\}$), as presented in Example 3, enables the comparison between $\mathbf{z}_{(27)}$ and $\mathbf{z}_{(72)}$.

In fact, $\mathbf{z}_{(27)} \sqsubset_{[M]} \mathbf{z}_{(72)}$ since we have $M_1\left(\mathbf{z}_{(54)}\right) = M_1\left(\mathbf{z}_{(45)}\right)$ but $M_2\left(\mathbf{z}_{(54)}\right) = 0.57 \leq 0.66 = M_2\left(\mathbf{z}_{(45)}\right)$. The above calculations are highlighted in the following:

$$[M]\mathbf{z}_{(72)} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25\\ 0.50 & 0.15 & 0.15 & 0.20\\ 0.20 & 0.20 & 0.30 & 0.30\\ 0.10 & 0.40 & 0.10 & 0.40 \end{bmatrix} \begin{bmatrix} 0.20\\ 0.80\\ 1.00\\ 1.00 \end{bmatrix} = \begin{bmatrix} \mathbf{0.75}\\ \mathbf{0.57}\\ 0.80\\ 0.84 \end{bmatrix};$$

$$[M]\mathbf{z}_{(27)} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25\\ 0.50 & 0.15 & 0.15 & 0.20\\ 0.20 & 0.20 & 0.30 & 0.30\\ 0.10 & 0.40 & 0.10 & 0.40 \end{bmatrix} \begin{bmatrix} 0.50\\ 0.90\\ 0.95\\ 1.00 \end{bmatrix} = \begin{bmatrix} \mathbf{0.75}\\ \mathbf{0.66}\\ 0.80\\ 0.77 \end{bmatrix}.$$

Analogously, the other component comparisons, related to $\mathbf{z}_{(i7)}$ and $\mathbf{z}_{(i7)}$, for $1 \leq i \leq 7$, can be performed. And so, we are able to conclude that $A7 > A2$. Summarizing, Table 4 presents all resulting comparisons related to 3-dimensional

Table 4 Applying admissible $\mathcal{M}$-order in the comparison degrees between alternatives in Table 5

$\mathbf{z}_{21} \sqsupset \mathbf{z}_{12}$	$\mathbf{z}_{12} \sqsubset \mathbf{z}_{21}$	$\mathbf{z}_{13} \sqsubset \mathbf{z}_{31}$	$\mathbf{z}_{14} \sqsubset \mathbf{z}_{41}$	$\mathbf{z}_{15} \sqsubset \mathbf{z}_{51}$	$\mathbf{z}_{16} \sqsubset \mathbf{z}_{61}$	$\mathbf{z}_{17} \sqsubset \mathbf{z}_{71}$
$\mathbf{z}_{31} \sqsupset \mathbf{z}_{13}$	$\mathbf{z}_{32} \sqsubset \mathbf{z}_{23}$	$\mathbf{z}_{23} \sqsupset \mathbf{z}_{32}$	$\mathbf{z}_{24} \sqsupset \mathbf{z}_{42}$	$\mathbf{z}_{25} \sqsupset \mathbf{z}_{52}$	$\mathbf{z}_{26} \sqsupset \mathbf{z}_{62}$	$\mathbf{z}_{27} \sqsupset \mathbf{z}_{72}$
$\mathbf{z}_{41} \sqsupset \mathbf{z}_{14}$	$\mathbf{z}_{42} \sqsubset \mathbf{z}_{24}$	$\mathbf{z}_{43} \sqsupset \mathbf{z}_{34}$	$\mathbf{z}_{34} \sqsubset \mathbf{z}_{43}$	$\mathbf{z}_{35} \sqsubset \mathbf{z}_{53}$	$\mathbf{z}_{36} \sqsubset \mathbf{z}_{63}$	$\mathbf{z}_{37} \sqsubset \mathbf{z}_{73}$
$\mathbf{z}_{51} \sqsupset \mathbf{z}_{15}$	$\mathbf{z}_{52} \sqsubset \mathbf{z}_{25}$	$\mathbf{z}_{53} \sqsupset \mathbf{z}_{35}$	$\mathbf{z}_{54} \sqsubset \mathbf{z}_{45}$	$\mathbf{z}_{45} \sqsupset \mathbf{z}_{54}$	$\mathbf{z}_{46} \sqsupset \mathbf{z}_{64}$	$\mathbf{z}_{47} = \mathbf{z}_{74}$
$\mathbf{z}_{61} \sqsupset \mathbf{z}_{16}$	$\mathbf{z}_{62} \sqsubset \mathbf{z}_{26}$	$\mathbf{z}_{63} \sqsupset \mathbf{z}_{36}$	$\mathbf{z}_{64} \sqsubset \mathbf{z}_{46}$	$\mathbf{z}_{65} \sqsubset \mathbf{z}_{56}$	$\mathbf{z}_{56} \sqsupset \mathbf{z}_{65}$	$\mathbf{z}_{57} \sqsubset \mathbf{z}_{75}$
$\mathbf{z}_{71} \sqsupset \mathbf{z}_{17}$	$\mathbf{z}_{72} \sqsubset \mathbf{z}_{27}$	$\mathbf{z}_{73} \sqsupset \mathbf{z}_{37}$	$\mathbf{z}_{74} = \mathbf{z}_{47}$	$\mathbf{z}_{75} \sqsupset \mathbf{z}_{57}$	$\mathbf{z}_{76} \sqsupset \mathbf{z}_{67}$	$\mathbf{z}_{67} \sqsubset \mathbf{z}_{76}$

Table 5 n-Dimensional implication degree calculation

	A1	A2	A3	A4	A5	A6	A7
A1	1.000	0.762	0.762	0.725	0.762	0.812	0.700
A2	0.987	1.000	0.762	0.837	0.875	0.900	0.837
A3	0.862	0.675	1.000	0.750	0.775	0.762	0.737
A4	0.862	0.787	0.787	1.000	0.750	0.762	0.937
A5	0.950	0.850	0.837	0.750	1.000	0.925	0.750
A6	0.975	0.837	0.825	0.750	0.875	1.000	0.737
A7	0.862	0.800	0.787	0.937	0.750	0.762	1.000

fuzzy sets reported in Table 3, where $\sqsubset_{[M]} \equiv \sqsubset$ are used by reducing notation:

Moreover, based on all resulting comparisons, one can observe that $\mathbf{z}_{(i1)} \sqsubset \mathbf{z}_{(1i)}$, for all $i \in \{1, \ldots, 7\}$. Then A_1 is the superior CIM software alternative by comparing it with other ones. The same analysis can be extended to other alternatives, resulting on next comparison: $A1 > A3 > A6 > A5 > A4 = A7 > A2$.

Now, in order to avoid the above exhaustive comparisons, we consider the data aggregating strategy based on an average operator.

6.2.2 Solving CIM-MCDM problem by applying data aggregating strategy

The arithmetic mean performed over the previous data from Table 3, given as follows

$$\mathbf{t}_{kj} = \left(\frac{1}{4} \sum_{i=1}^{4} \mathbf{z}_{kj}(i) \right), \forall k, j = 1 \ldots 7, \tag{36}$$

result in the data shown in Table 5. Thus, a new comparison can be obtained. Based on such data analysis exploited from Table 5, a new comparison is reached: $A1 > A3 > A6 > A4 = A5 = A7 > A2$.

This comparison result coincides with the results presented in Wen et al. (2018), see Example 1.

6.2.3 Discussing the results for CIM-MCDM problem

For two or more experts providing the same membership degree for an alternative w.r.t. data criterion, these degrees collapse in the hesitant decision matrix. However, when the modelling via n-dimensional sets is considered, it is possible to repeat (e.g. the smaller and/or larger) values and preserve the dimension of each vector. Thus, this process explains the frequency of repetitions related to expert opinions in addition to faithfully reflecting their evaluation for all criteria. Therefore, the rating obtained is also more reliably influenced.

In this modelling via n-dimensional sets for this CIM-MCDC proposal, even considering the same operator, which is the fuzzy Łukasiewicz implication, the comparison for $A4$ and $A5$ alternatives can be presented. However, by consider the media arithmetic as result data in the n-dimensional decision matrix, it coincides with the proposal via hesitant sets in Wen et al. (2018). And so, it is not possible to decide which is the best choice among $A4$ and $A5$ alternatives. Moreover, one can observe that such problem situation may also change the best options, just updating the input data.

Based on the second strategy considering the arithmetic mean aggregation,, only the corresponding comparison between alternatives $A4$ and $A5$ cannot be explicit ($A4 = A5$). And, the comparison $A5 > A4$ is achieved in the first strategy as suitable for these two distinct 4-dimensional intervals. However, both strategies determine that the three best alternatives are $A1$, $A3$ and $A6$. And also, the two worst options are the alternatives $A7$ and $A2$. While avoiding multivalued data type reduction and assuming an opinion of all experts, we consider admissible orders for comparison between all results in the comparison matrix.

To sum up, the use of n-dimensional intervals in $L_n([0, 1])$ in modelling MCDM problems seems more intuitively, since their inherent ordered components are relevant. Meaning that, when admissible linear order is taking, it provides detailed comparison in $L_n([0, 1])$.

7 Validating *n*-DRI in CC-environments

In order to validate the residuation study on n-DFI, this work consider the *Int-FLBCC* (***Int****erval* ***F****uzzy* ***L****oad* ***B****alancing for* ***C****loud* ***C****omputing*) model [16], improving energy consumption efficiency while maintaining good Service Level Agreement (SLA) and Quality of Service (QoS) levels. In this sense, the detection of physical machines overload to define priorities in the allocation and reallocation process of virtual machines (VM) is a great challenge to perform load balancing (LB) in cloud computing (CC-) environments .

The evaluation metrics in the dynamic consolidation approach is related to main concepts in CC-migration and performance w.r.t. SLA average violation, which have been applied under QoS restrictions. The energy consumption of CC-data centres is determined by the CPU and memory usage, disk storage, power supplies and cooling systems, described as a linear relationship between power consumption and CPU usage.

The *Int-FLBCC* modelling fuzzy system considers a Rule Base acting on three steps: Fuzzification, Inference, and Defuzzification returning as output the utilization level of each host. This system was performed using the Interval Type-2 Fuzzy Logic System Toolbox (IT2FLT) module (Castro et al. 2007) and Juzzy (Wagner 2013).

Table 6 Hosts of cloud

Vendor	Model	CPU name	CPU characteristics	Memory
1	ProLiant DL325 Gen10	AMD EPYC 7551P 2.0 GHz	32-Core, 2.0 GHz, 64MB L3 Cache	128 GB
2	PowerEdge R840	Intel Xeon Platinum 8180 2.50 GHz	28 core, 2.50 GHz, 38.5 MB L3 Cache	384 GB

(1) Hewlett Packard Enterprise (2) Dell Inc

The configuration of the *Int-FLBCC* test environment is executed in the fuzzy module evaluation for host selection of CC-allocation of VMs, considering the CloudSim (Rodrigo et al. 2011) as the toolkit for modelling and simulating CC-services. The open-source Java library Wagner (2013) is used to implement the fuzzy inference system.

For all experiments, real-world workloads provided as part of the CoMon (Park and Vivek 2006) project, a monitoring infrastructure for PlanetLab. The Infrastructure as a Service (IaaS) cloud environment represented for the tests considered 800 heterogeneous physical hosts and four types of configurations as described in Table 6. The results as well as the *Int-FLBCC* framework are available in GitHub[2] in an extended version of CloudSim 3.0.3.

The frequency of server CPU is mapped into MIPS classifications. Half of the hosts are the ProLiant DL325 Gen 10 with 4721 MIPS for each core, and the other half consists of the PowerEdge R840 server with 4520 MIPS for each core. Each server is modelled as 5 GB/s of network bandwidth. Characteristics of VM types are similar to Amazon EC2 instance types, including Medium High CPU Instance (4000 MIPS, 32 GB); Extra Large Instance (3000 MIPS, 8 GB); Small Instance (2000 MIPS, 8 GB); and Micro Instance (2000 MIPS, 16 GB). As an interval schedule, the applied measurement interval is 5 min. See main characteristics of each workload in Table 7. CPU load workload data for more than 1000 VM of servers in over 500 locations worldwide were used. And, the value of the workload data confirms that the average CPU utilization is well below 50%.

In addition, 10 workload data sets collected on different days are applied and allocated to each VM. All these experiments considered the allocation algorithms: Inter-Quartile Range (IQR), Local Regression Robust (LRR) and Median Absolute Deviation (MAD).

Table 8 describes the abbreviated name of the three fuzzy approaches, which are generated by combinations methods of implications, intersections and unions, for each of the three classical algorithms (CA): IQR, LRR and MAD.

Table 7 Characteristics of the workload data

Workload	VMs	Mean (%)	St.dev. (%)
20110303	1052	12.31	17.09
20110306	898	11.44	16.83
20110309	1061	10.7	15.57
20110322	1516	9.26	12.78
20110325	1078	10.56	14.14
20110403	1463	12.39	16.55
20110409	1358	11.12	15.09
20110411	1233	11.56	15.07
20110412	1054	11.54	15.15
20110420	1033	10.43	15.21

Table 8 Operator settings in experiments

APP	Implication	Intersection	Union
IQR2	Minimum	Maxmin	Minmax
IQR1	product	Maxmin	Minmax
IQR3	$\mathcal{I}_{I_{LK}}$	$\widetilde{T_{LK}}$	$(\widetilde{T_{LK}})_{\widetilde{N_S}}$
IQR4	$\mathcal{I}_{I_{LK}}$	$\mathcal{T}_{T_{LK}}$	$(\mathcal{T}_{T_{LK}})_{\mathcal{N}_S}$
LRR2	Minimum	Maxmin	Minmax
LRR1	product	Maxmin	Minmax
LRR3	$\mathcal{I}_{I_{LK}}$	$\widetilde{T_{LK}}$	$(\widetilde{T_{LK}})_{\widetilde{N_S}}$
LRR4	$\mathcal{I}_{I_{LK}}$	$\mathcal{T}_{T_{LK}}$	$(\mathcal{T}_{T_{LK}})_{\mathcal{N}_S}$
MAD2	Minimum	Maxmin	Minmax
MAD1	product	Maxmin	Minmax
MAD3	$\mathcal{I}_{I_{LK}}$	$\widetilde{T_{LK}}$	$(\widetilde{T_{LK}})_{\widetilde{N_S}}$
MAD4	$\mathcal{I}_{I_{LK}}$	$\mathcal{T}_{T_{LK}}$	$(\mathcal{T}_{T_{LK}})_{\mathcal{N}_S}$

(CA) Crisp Algorithm; (APP) Application Name

7.1 Discussion of results

Firstly, in Fig. 1 the evaluation based on energy consumption is presented, where it can be highlighted that the proposed aggregation and implication methods managed to achieve similar results and slightly better than the crisp algorithms.

The assessment of the SLA in cloud computing services is of fundamental importance, as it contains the metrics agreed between information technology companies and their customers, such as minimum availability of computing resources, and transfer rate of communication channels. In this sense, this evaluation seeks to analyze the average levels of SLA violation obtained through the simulations, using the implication and aggregator methods defined in this work. This assessment is shown in the graph of Fig. 2 and Table 9.

[2] https://github.com/brunomourapaz/CloudSim.

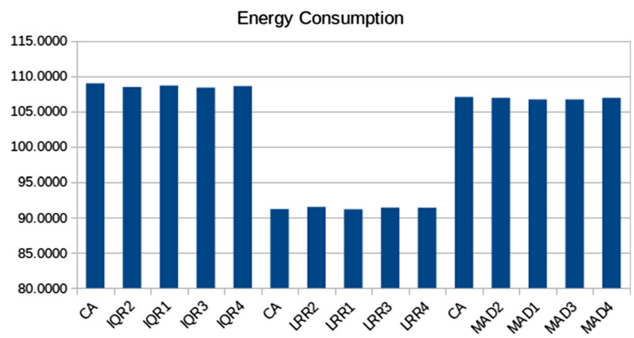

Fig. 1 Energy consumption

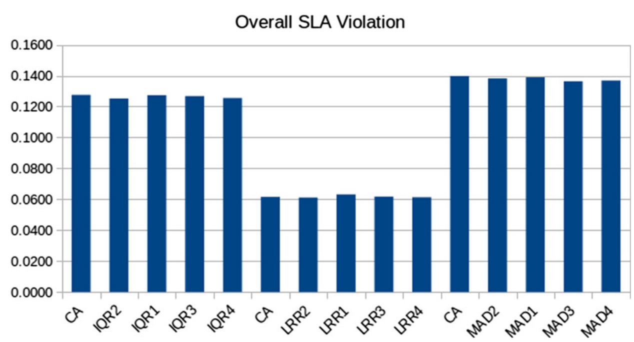

Fig. 2 Overall SLA violation

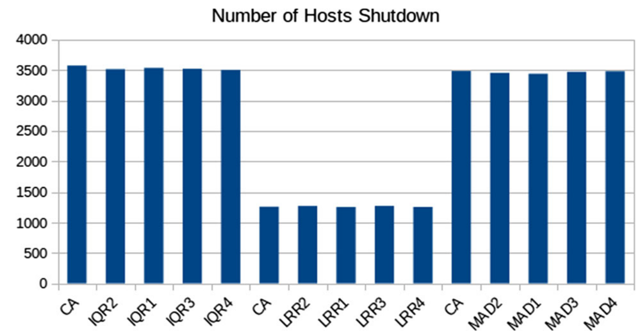

Fig. 3 Number of host shutdowns

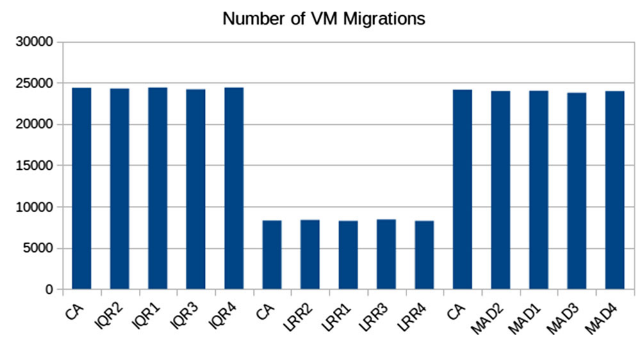

Fig. 4 Number of VM migrations

The graph in Fig. 3 shows the evaluation of the number of physical machines that were turned off in the process of dynamic consolidation of VM in the execution of the work proposal, occurring when physical machines are underloaded. The VM associated with this group of physical machines are moved to others that still have available resources, thus allowing the shutdown of underloaded physical machines and thus contributing to the reduction of energy consumption.

The number of VM migration is an important factor to be evaluated, as a high movement of VM in the data centre can cause the degradation of the performance of computational resources, and the communication channels in a CC-environment, consequently, making it possible to obtain worst levels of SLA. This evaluation is shown in the graph in Fig. 4, for each VM allocation algorithm, and applying all the proposed aggregation and implication methods.

The results obtained using the approaches in this paper, in general, reached values close to the classical algorithms used in the stages of dynamic VM consolidation. Analyzing cases IQR2, IQR1, IQR3 and IQR5 for both energy consumption and SLAV metrics the results were relatively better than the classic IQR algorithm. For MAD3, the reduction in energy consumption was slightly better than the classic case, as it was the average SLA violation. In the other cases in which the proposed approaches did not surpass the classic cases, the values reached very close levels, so validating the proposal.

Table 9 Numerical result of experiments

APP	Energy	VMM	SLAV	HS
IQR	108.9530	24363	0.1275	3574
IQR2	108.4401	24279	0.1252	3514
IQR1	108.6414	24397	0.1274	3535
IQR3	108.3517	24180	0.1267	3520
IQR4	108.5589	24395	0.1255	3501
LRR	91.1897	8286	0.0615	1260
LRR2	91.4924	8370	0.0611	1271
LRR1	91.1523	8249	0.0632	1253
LRR3	91.4060	8417	0.0617	1272
LRR4	91.3902	8247	0.0613	1255
MAD	107.0234	24134	0.1397	3484
MAD2	106.8981	23971	0.1382	3454
MAD1	106.6766	24007	0.1389	3439
MAD3	106.6782	23772	0.1363	3471
MAD4	106.8990	23963	0.1368	3481

(CA) Crisp Algorithm; (VMM) VM Migrations; (HS) Host Shutdown; (SLAV) SLA Violation

8 Conclusion

This work discusses n-dimensional R-implications, considering important properties, as residuation property and left- and right-continuity. As a main contribution, properties characterizing the class of R-implications on $L_n([0,1])$ are studied, extending ϕ-conjugation constructions under R-implications from $[0,1]$ to $L_n([0,1])$.

We also present an axiomatic characterization of n-DRI based on left-continuous n-DT, and the reverse construction, the method obtaining n-DT from n-DI considering the residuation principle. Such method provides an exhaustive study of properties of the n-DRI, as essential conditions for an n-DRI verifying the exchange principal and the ordering property are discussed.

Moreover, the class of n-DRI is constructed based on binary n-DA aggregation operators considering the minimum operator and left-continuous t-norms. Finally, an illustration on solving a CIM-MCDM application is extended from hesitant fuzzy sets to n-dimensional fuzzy sets. And, toward to the consolidation of the *Int-FLBCC* modelling fuzzy system improving performance for more than 2-dimensional approach.

Ongoing work considers the extension of other fuzzy connectives from $[0,1]$ to $L_n([0,1])$, also related to special classes of fuzzy implications as Dishkant implications and Yager implications (Reiser et al. 2009; Yager 2004), also including (T,N)-implications (Pinheiro et al. 2018) and residuated implications generated from t-subnorms (Reiser et al. 2013) and overlaps (Dimuro and Bedregal 2015).

Since inherent ordering related to n-dimensional intervals, further work considers the analysis of properties of n-DRI as continuity and monotonicity based on the admissible linear orders on $L_n([0,1])$, contributing with comparison solutions for decision making on multi-attributes based on a group of specialists.

And finally, the sequence of new results is passing through the investigation of other topics related to (left/right)-continuity on the n-dimensional upper simplex $L_n([0,1])$, but considering the extension of the Euclidean distance and/or its equivalence approach, the Hamming distance, as pointed out in literature.

Acknowledgements This work was partially supported by Brazilian Funding Agency CAPES, CNPq (311429/2020-3 and 309160/2019-7) also including PqG/FAPERGS (21/2551-0002057-1).

Funding The authors have not disclosed any funding.

Data availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest Authors declare that they have no conflicts of interest and this article does not contain any studies with human participants or animals performed by authors.

References

Abramski S, Jung A (1994) Domain theory. In: Abramsky S, Gabbay DM, Maibaum TSE (eds) Handbook of logic in computer science, vol 3. Oxford University Press, Oxford

Alcalde C, Burusco A, Fuentes-González R (2005) A constructive method for the definition of interval-valued fuzzy implication operators. Fuzzy Sets Syst 153(2):211–227

Atanassov K (1986) Intuitionistic fuzzy sets. Fuzzy Sets Syst 20(1):87–96

Atanassov K, Gargov G (1998) Elements of intuitionistic fuzzy logic. Part I. Fuzzy Sets Syst 95(1):39–52

Baczyński M (2004) Residual implications revisited. Notes on the Smets–Magrez theorem. Fuzzy Sets Syst 145(2):267–277

Baczyński M, Jayaram B (2008) Fuzzy implications. Springer, Berlin

Baczyński M, Jayaram B (2008) (S, N)-and R-implications: a state-of-the-art survey. Fuzzy Sets Syst 159(14):1836–1859

Barrenechea E, Bustince H, De Baets B, Lopez-Molina C (2011) Construction of interval-valued fuzzy relations with application to the generation of fuzzy edge images. IEEE Trans Fuzzy Syst 19(5):819–830

Bedregal B, Santiago R, Reiser R, Dimuro G (2007) The best interval representation of fuzzy S-implications and automorphisms. In: Proceedings of the 16th FUZZIEEE conference, pp 1–6

Bedregal B, Dimuro G, Reiser R, Carvalho JP, Dubois D, Kaymak U (2009) An approach to interval-valued R-implications and automorphisms. In: Proceedings of the 6th IFSA/EUSFLAT conference, pp 1–6

Bedregal B, Dimuro G, Santiago R, Reiser R (2010) On interval fuzzy S-implications. Inf Sci 180(8):1373–1389

Bedregal B, Beliakov G, Bustince H, Calvo T, Fernández J, Mesiar R (2011) A characterization theorem for t-representable n-dimensional triangular norms. In: Proceedings of 2011 EUROFUSE, pp 103–112

Bedregal B, Beliakov G, Bustince H, Calvo T, Mesiar R, Paternain D (2012) A class of fuzzy multisets with a fixed number of memberships. Inf Sci 189:1–17

Bedregal B, Reiser R, Bustince H, Lopez-Molina C, Torra V (2014) Aggregation functions for typical hesitant fuzzy elements and the action of automorphisms. Inf Sci 255:82–99

Bedregal B, Mezzomo I, Reiser R (2018) n-Dimensional fuzzy negations. IEEE Trans Fuzzy Syst 26(6):3660–3672

Moura B, Schneider G, Yamin A, Santos H, Reiser R, Bedregal B (2021) Interval-valued fuzzy logic approach for overloaded hosts in consolidation of virtual machines in cloud computing. Fuzzy Sets Syst. https://doi.org/10.1016/j.fss.2021.03.001

Bustince H, Barrenechea E, Pagola M, Fernandez J, Xu Z, Bedregal B, Montero J, Hagras H, Herrera F, De Baets B (2016) A historical account of types of fuzzy sets and their relationships. IEEE Trans Fuzzy Syst 24(1):179–194

Rodrigo C, Rajiv R, Anton B, De Rose C, Rajkumar B (2011) CloudSim: a toolkit for modeling and simulation of cloud computing environments and evaluation of resource provisioning algorithms. Soft Prac Exp 41(1):23–50

Castro J, Castillo O, Martínez L (2007) Interval type-2 fuzzy logic toolbox. Eng Lett 15(1):9–98

Chen N, Xu Z, Xia M (2013) Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. Appl Math Model 37(4):2197–2211

Park K, Vivek S (2006) CoMon: a mostly-scalable monitoring system for PlanetLab.SIGOPS. Oper Syst Rev 40(1):65–74

Cornelis C, Deschrijver G, Kerre E (2004) Implication in intuitionistic fuzzy and interval-valued fuzzy set theory: construction, classification, application. Int J Approx Reason 35(1):55–95

De Lima A, Palmeira E, Bedregal B, Bustince H (2021) Multidimensional fuzzy sets. IEEE Trans Fuzzy Syst 29(8):2195–2208

De Miguel L, Sesma-Sara M, Elkano M, Asiain M, Bustince H (2017) An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators. Inf Fusion 37:126–131

Deschrijver G, Kerre E (2003) On the relationship between some extensions of fuzzy set theory. Fuzzy Sets Syst 133(2):227–235

Deschrijver G, Cornelis C, Kerre E (2004) On the representation of intuitionistic fuzzy t-norms and t-conorms. IEEE Trans Fuzzy Syst 12(1):45–61

Dimuro G, Bedregal B, Santiago R, Reiser R (2011) Interval additive generators of interval t-norms and interval t-conorms. Inf Sci 181(18):3898–3916

Dimuro G, Bedregal B (2015) On residual implications derived from overlap functions. Inf Sci 312:78–88

Dugundji J (1966) Topology. Ally and Bacon, Boston

Espana S, Estruch V (2004) A memorizing semantics for functional logic languages. In: Proceedings of 2004 European symposium on programming, pp 109–123

Farias A, Costa V, Santiago R, Bedregal B (2016) A residuated function in a class of Mealy type L-Valued finite automaton. In: Proceedings of 2016 NAFIPS, pp 1–6

Gierz G, Hofmann K, Keimel K, Lawson J, Mislove M, Scott D (2003) Continuous lattices and domains. Cambridge University Press, Cambridge, p 93

González-Hidalgo M, Massanet S (2014) A fuzzy mathematical morphology based on discrete t-norms: fundamentals and applications to image processing. Soft Comput 18(11):2297–2311

Wagner C (2013) Juzzy—a java based toolkit for type-2 fuzzy logic. In: Proceedings of 2013 IEEE T2FUZZ, pp 45–52

Klement E, Navara M (1999) A survey on different triangular norm-based fuzzy logics. Fuzzy Sets Syst 101(2):241–251

Liu H, Wang G (2006) A note on implicators based on binary aggregation operators in interval-valued fuzzy set theory. Fuzzy Sets Syst 157(24):3231–3236

Mendelson B (1990) Introduction to topology. Courier Corporation, Chelmsford

Mezzomo I, Bedregal B, Reiser R, Bustince H, Paternain D (2016) On n-dimensional strict fuzzy negations. In: Proceedings of 2016 FUZZ-IEEE, pp 301–307

Mezzomo I, Bedregal B, Milfont T, Asmus T, Bustince B (2019) n-Dimensional interval uninorms. In: Proceedings of 2019 FUZZ-IEEE, pp 1–6

Mezzomo I, Bedregal B, Reiser R (2017) Natural n-dimensional fuzzy negations for n-dimensional t-norms and t-conorms. In: Proceedings of 2017 FUZZ-IEEE, pp 1–6

Mezzomo I, Bedregal B, Milfont T (2018) Equilibrium point of representable Moore continuous n-dimensional interval fuzzy negations. In: Proceedings of 2018 NAFIPS, pp 265–277

Mezzomo I, Bedregal B, Milfont T (2018) Moore continuous n-dimensional interval fuzzy negations. In: Proceedings 2018 FUZZ-IEEE, pp 1–6

Milfont T, Mezzomo I, Bedregal B, Mansilla E, Bustince H (2021) Aggregation functions on n-dimensional ordered vectors equipped with an admissible order and an application in multi-criteria group decision-making. Int J Approx Reason 137:34–50

Paiva R, Santiago R, Bedregal B, Rivieccio U (2021) Inflationary BL-algebras obtained from 2-dimensional general overlap functions. Fuzzy Sets Syst 418:64–83

Pinheiro J, Bedregal B, Santiago R, Santos H (2018) A study of (T, N)-implications and its use to construct a new class of fuzzy subsethood measure. Int J Approx Reason 97:1–16

Reiser R, Bedregal B, Santiago R, Dimuro G (2009) Interval valued D-implications. Trends Appl Comput Math 10(1):63–74

Reiser R, Bedregal B, Baczyński M (2013) Aggregating fuzzy implications. Inf Sci 253:126–146

Reiser R, Bedregal B (2013) Interval-valued intuitionistic fuzzy implications-construction, properties and representability. Inf Sci 248:68–88

Reiser R, Bedregal B, Visintin L (2013) Index, expressions and properties of interval-valued intuitionistic fuzzy implications. Trends Appl Comput Math 14(2):193–208

Sambuc R (1975) Fonctions and Floues: Application a l'aide au Diagnostic en Pathologie Thyroidienne. Faculté de Médecine de Marseille

Santiago R, Bedregal B, Marcos J, Caleiro C, Pinheiro J (2019) Semi-BCI algebras. J Multiple-Valued Logic Soft Comput 32(1–2):87–109

Santiago R, Bedregal B, Madeira A, Martins M (2019) On interval dynamic logic: Introducing quasi-action lattices. Sci Comput Program 175:1–16

Santos H, Couso I, Bedregal B, Takác Z, Minárová M, Asiain A, Barrenechea E, Bustince H (2019) Similarity measures, penalty functions, and fuzzy entropy from new fuzzy subsethood measures. Int J Intell Syst 34(6):1281–1302

Shang Y, Yuan X, Lee S (2010) The n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets. Commun Math Appl 60(3):442–463

Shi Y, Gasse B, Kerre E (2013) The role a fuzzy implication plays in a multi-criteria decision algorithm. Int J Gen Syst 42(1):111–120

Torra V (2010) Hesitant fuzzy sets. Int J Intell Syst 25(6):529–539

Wen M, Zhao H, Xu Z (2018) Hesitant fuzzy Lukasiewicz implication operation and its application to alternatives' sorting and clustering analysis. Soft Comput 23(2):393–405

Yager R (2004) On some new classes of implication operators and their role in approximate reasoning. Inf Sci 167(1–4):193–216

Zanotelli R, Reiser R, Bedregal B (2018) n-dimensional intervals and fuzzy S-implications. In: Proceedings of 2018 FUZZ-IEEE, pp 1–8

Zanotelli R, Reiser R, Bedregal B (2018) Towards the study of main properties of n-dimensional QL-implicators. In: Proceedings of 2018 CBSF, pp 636–646

Zanotelli R, Reiser R, Bedregal B (2019) Study on n-Dimensional R-implications. In: Proceedings of 11th EUSFLAT conference, pp 474–481

Zanotelli R, Reiser R, Bedregal B (2020) n-Dimensional (S, N)-implications. Int J Approx Reason 126:1–26

Zanotelli R (2020) n-Dimensional Fuzzy Implications: Analytical, Algebraic and Applicational Approaches. Ph.D. thesis, Universidade Federal de Pelotas, CDTEC PPGC. Pelotas. Rio Grande do Sul. Brasil

Zapata H, Bustince H, Montes S, Bedregal B, Dimuro G, Takác Z, Baczyński M, Fernandez J (2017) Interval-valued implications and interval-valued strong equality index with admissible orders. Int J Approx Reason 88:91–109