

On Refutation Rules

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Abstract. The goal of this paper is to generalize specific techniques connected with refutation rules involving certain normal forms. In particular, a method of axiomatizing both a logic L and its complement $-L$ is introduced.

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1. Introduction

Refutation rules are rules preserving non-validity. Typical examples are the following (introduced by Łukasiewicz [2,3]).

(R_s or *reverse substitution*) $s(A)/A$ where $s(A)$ is a substitution instance of A
 (R_{mp}^L or *reverse modus ponens*) B/A where $A \rightarrow B \in L$
 (R_d or the disjunction property) $A \quad B/A \vee B$

Note that R_s, R_{mp}^L are refutation rules for every logic closed under *substitution* and *modus ponens*, while R_d is a refutation rule for Intuitionistic Logic (INT) but it is not a refutation rule for Classical Logic (CL).

A pair consisting of a set of refutation axioms (which are some non-valid formulas) and a set of refutation rules is a refutation system. It is an axiom system, just like traditional axiom systems, but it generates non-valid formulas rather than valid ones. Of course, axiom systems are notoriously bad for proof search. And so are refutation systems containing the rules: R_s and R_{mp}^L (although they have some interesting theoretical aspects (see [5,6])). However, the rule R_d is more promising for proof search (or, rather, refutation search). It has the important property that each premise is simpler than the conclusion. Also, in INT for example, we have the following.

$A \vee B \notin INT$ iff both $A \notin INT$ and $B \notin INT$.

Thus, you can reduce $A \vee B$ to A and B , when you search for a refutation of $A \vee B$.

In this paper we study refutation rules of this kind. We are interested in refutation rules of the form

$$\frac{F_1 \cdots F_k}{F}$$

where F is a certain normal form of rank $n > 0$ and each F_i is a normal form of rank $< n$. Specific rules of this kind can be extracted from [4,6]. Here we generalize these techniques obtaining a simple method of proving that a refutation system is complete for a logic L (that is, it axiomatizes the complement $-L$ of L). This also gives an axiomatization for L by some rules dual to the refutation rules. What is more, our method provides a simple refutation-search procedure that is both label-free and cycle-free. Thus, it may be simpler than standard refutation methods, in which refuting a formula is justified by failure to find a proof for it. Our concepts will be illustrated in the modal logic $S4$, where the problem of refutation search is non-trivial.

2. Refutation Rules

Let L be a propositional logic (which can be thought of as the set of propositional formulas valid in structures of some sort).

Our refutation rules involve L normal forms having the following property.

For every formula A , there are L normal forms A^1, \dots, A^l such that

$$A \in L \text{ iff } A^1 \in L, \dots, A^l \in L.$$

(Note that A^i/A is a refutation rule for L , and $A^1, \dots, A^l/A$ is a proof rule for L .)

For example, classical conjunctive normal forms are of this kind. However, normal forms based on clausal forms (see [4,6]) are more useful. They are both simple and general. We assume that a natural number $r(F)$ (called the rank of F) has been assigned to every normal form F , and some (simple) normal forms are of rank 0.

(Here for any finite sets X, Y of formulas, the symbol $X \longrightarrow Y$ stands for $\bigwedge X \rightarrow \bigvee Y$, where $\bigwedge X$ ($\bigvee X$) is a conjunction (a disjunction) of the formulas in X , and $\Box X = \{\Box A : A \in X\}$.)

Example 2.1. (For more details, see [4,6].) An $S4$ normal form F is either a \Box -free formula or a formula

$$\Box S \longrightarrow a_1$$

where $S = \Theta \cup \Delta \cup \Gamma$

$$\Theta = \{\Box a_i \equiv c_0 : 1 \leq i \leq m\}$$

$$\Delta = \{\Box b_j \equiv c_j : 1 \leq j \leq n\}$$

Γ is a finite set of \Box -free formulas, all a_i, b_j, c_0, c_j are propositional variables, and $m > 0, n \geq 0$.

The rank $r(F)$ of F is 0 if F is \Box -free, and

$r(F) = n + 1$ if $F = \Box S \longrightarrow a_1$. (Note that the number of formulas in Θ is ignored.)

For every formula A , its corresponding $S4$ normal form A' can be constructed with the property that $A \in S4$ iff $A' \in S4$.

Definition 2.2. A refutation rule for a logic L is a rule of the kind

$$X/F$$

where F is an L normal form of rank > 0 and X is a finite non-empty set of L normal forms of rank $< r(F)$ having the following property.

If $X \subseteq -L$ then $F \in -L$.

(This is usually established by some semantic (or algebraic) argument.)

Example 2.3. The following is a refutation rule for $S4$ [see [6]].

$$(R_{S4}) \quad \frac{C_1 \dots C_m F_1 \dots F_n}{F}$$

where F is an $S4$ normal form of rank > 0 , and

$$\begin{aligned} C_i &= \Gamma \longrightarrow a_i, c_0, c_1, \dots, c_n \quad (1 \leq i \leq m) \\ F_j &= \square(\square b_j \equiv c_j), \square \Delta_j^-, \square \Gamma, \square a_1, \dots, \square a_m, \square c_0 \longrightarrow b_j \\ \Delta_j^- &= \Delta - \{\square b_j \equiv c_j\} \quad (1 \leq j \leq n) \end{aligned}$$

Note that all C_i are of rank 0, and each F_j is an $S4$ normal form of rank $< r(F)$. Moreover F_j is $S4$ -equivalent to

$$F'_j = \square S, \square a_1 \longrightarrow b_j.$$

(We say that formulas A, B are $S4$ -equivalent iff $A \equiv B \in S4$.)

Definition 2.4. (i) A refutation system \mathbf{R} for a logic L is a pair (RA, RR) , where RR is a set of refutation rules for L and RA is the set of refutation axioms for L (that is, the set of all normal forms F of rank 0 such that $F \notin L$).

(ii) A normal form is \mathbf{R} -refutable iff it is derivable from RA by RR .

Note that if F is \mathbf{R} -refutable, then $F \notin L$. Thus, a refutation system generates non-valid normal forms by derivations. Of course, it need not be complete (that is, having the property that every $F \notin L$ is \mathbf{R} -refutable).

3. Completeness

When is a refutation system for L complete? We introduce a condition (called the reduction property) that proves sufficient.

Definition 3.1. Let \mathbf{R} be a refutation system for a logic L . \mathbf{R} has the reduction property iff for every normal form F of rank > 0 , the following conditions are satisfied.

- (i) The set $RR(F)$ of refutation rules of the form X/F (that is, the set $\{X/F : X/F \in RR\}$) is non-empty and finite.
- (ii) If $F \in -L$ then there is a rule $X/F \in RR(F)$ such that $X \subseteq -L$.

The reduction property also provides a proof system axiomatizing L .

Definition 3.2. Let \mathbf{R} be a refutation system for a logic L with the reduction property.

- (i) The set of proof rules for L corresponding to RR is the set $PRR = \{A_1, \dots, A_t/F : F \text{ is a normal form of rank } > 0, \text{ each } A_i \in X_i \text{ and } \{X_1/F, \dots, X_t/F\} = RR(F)\}$.
(Note that L is closed under the rules of PRR .)
- (ii) The set of proof axioms for L corresponding to RA is the set PRA of all normal forms F of rank 0 such that $F \in L$.
- (iii) The proof system for L corresponding to \mathbf{R} is the pair $\mathbf{PR} = (PRA, PRR)$.

We simply say “ F is provable” (in symbols $\vdash F$) instead of “ F is derivable from PRA by PRR ”. Observe that if $\vdash F$ then $F \in L$. And we say “ F is refutable” (in symbols $\dashv F$) instead of “ F is derivable from RA by RR ”. Of course, if $\dashv F$ then $F \notin L$.

Theorem 3.3. Let \mathbf{R} be a refutation system for a logic L with the reduction property. Then for every L normal form F we have:

- (i) Either $\vdash F$ or $\dashv F$.
- (ii) If $F \notin L$ then $\dashv F$.
- (iii) If $F \in L$ then $\vdash F$.

Proof. (i) By induction on $r(F)$.

- (1) $r(F) = 0$. Either $F \in L$ or $F \notin L$. Hence $\vdash F$ or $\dashv F$.
- (2) $r(F) > 0$ and this is true for L normal forms of rank $< r(F)$.

Since \mathbf{R} has the reduction property, $RR(F)$ is non-empty and finite. Let $X_1/F, \dots, X_t/F$

be all rules in $RR(F)$. Consider any $A \in X_i$ ($1 \leq i \leq t$). A is of rank $< r(F)$, so by the induction hypothesis, A is provable or A is refutable.

(Case 1) For some $1 \leq i \leq t$, every $A \in X_i$ is such that $\dashv A$. Then $\dashv F$ by RR .

(Case 2) For every $1 \leq i \leq t$, some $A \in X_i$ is such that $\vdash A$. Then $\vdash F$ by PRR .

Therefore either $\vdash F$ or $\dashv F$.

- (ii) Assume that $F \notin L$. Then it is not the case that $\vdash F$. (Otherwise $F \in L$.) Hence $\dashv F$ by (i).
- (iii) Assume that $F \in L$. Then it is not the case that $\dashv F$. (Otherwise $F \notin L$.) Hence $\vdash F$ by (i), as required. \square

Remark 3.4. In a specific logic, the proof system \mathbf{PR} can be replaced by some elegant proof system by showing that both PRA and PRR are derivable in that system.

Example 3.5. Let \mathbf{R} consist of the following (see [6]).

RA : All \square -free formulas that are not in CL .

RR : R_{S4} and

$$\begin{array}{l} (RG_j) \quad \frac{G_j}{F} \\ (RH_j) \quad \frac{H_j}{F} \end{array}$$

where F is an $S4$ normal form of rank > 0 and

$$\begin{aligned} G_j &= \Box\Theta, \Box\Delta_j^-, \Box\Gamma, \Box b_j, \Box c_j \longrightarrow a_1 \\ H_j &= \Box\Theta, \Box(\Box b_j \equiv c_0), \Box\Delta_j^-, \Box\Gamma, \Box(c_0 \equiv c_j) \longrightarrow a_1 \quad (1 \leq j \leq n) \end{aligned}$$

Note that G_j, H_j are $S4$ normal forms of rank $< r(F)$, and they are $S4$ -equivalent to G'_j, H'_j , respectively, where

$$\begin{aligned} G'_j &= \Box S, \Box b_j \longrightarrow a_1 \\ H'_j &= \Box S, \Box(c_0 \equiv c_j) \longrightarrow a_1 \end{aligned}$$

We remark that \mathbf{R} is a refutation system for $S4$ (see [6]), and it has the reduction property. Indeed, 3.1(i) is obvious. To show 3.1(ii), note the following fact.

If (some $C_i \in S4$ or some $F_j \in S4$) and all $G_1, \dots, G_n, H_1, \dots, H_n \in S4$, then $F \in S4$ (see [6]).

Hence if $F \notin S4$, then (all $C_j \notin S4$ and all $F_j \notin S4$) or some $G_j \notin S4$ or some $H_j \notin S4$.

Therefore (by 3.3) \mathbf{R} is complete for $S4$.

4. Refutation Search

Let \mathbf{R} be a refutation system for a logic L with the reduction property. Then for any L normal form F of rank > 0 we have:

$$F \in -L \text{ iff } X_1 \subseteq -L \text{ or } \dots \text{ or } X_t \subseteq -L$$

where $\{X_1/F, \dots, X_t/F\} = RR(F)$.

This fact provides the following simple refutation-search procedure for normal forms of rank > 0 .

Write F .

Then write X_1, \dots, X_t .

Every formula in each X_i is a normal form of rank $< r(F)$, so by repeating such reductions, the procedure terminates at some normal forms of rank 0, which are either refutation axioms or proof axioms. From them either a refutation for F or a proof for F can be obtained (see the proof of 3.3(i)). (Of course, for any formula A , its corresponding normal forms A^1, \dots, A^l can be constructed, so that this is also a refutation-search procedure for any A .)

Example 4.1. Let $F = \Box S \longrightarrow p_1$, where

$$\begin{aligned} S &= \Theta \cup \Gamma \\ \Theta &= \{\Box p_1 \equiv q, \Box p_2 \equiv q, \Box p_3 \equiv q\} \\ \Gamma &= \{p_1 \vee p_2, p_1 \vee p_3, p_2 \vee p_3\} \end{aligned}$$

and p_1, p_2, p_3, q are propositional variables.

Refutation

We have $n = 0$, so F is obtained by R_{S4} from

$$\begin{aligned} C_1 &= \Gamma \longrightarrow p_1, q \\ C_2 &= \Gamma \longrightarrow p_2, q \end{aligned}$$

$$C_3 = \Gamma \longrightarrow p_3, q$$

because each $C_i \notin CL$.

5. Conclusions

Our method is complementary to standard proof-search procedures. It is a direct refutation method, that is, refuting A is justified by the existence of a derivation, whereas standard decision methods provide indirect refutation procedures, in which refuting A is justified by failure to find a proof for A .

Our method is based on reductions and it is cycle-free, so it may be useful in logics for which tableau procedures are based on repetitions and cycles, for example in transitive modal logics (for more information, see [1]).

It may also be useful in logics for which simple contraction-free sequent systems are not easy to find (for a discussion, see [6]).

Our method is interesting because of practical applications as well as theoretical ones.

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