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ORIGINAL PAPER

Solving a modified nonlinear epidemiological model of computer viruses by homotopy analysis method

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Abstract
The susceptible–infected–recovered model of computer viruses is investigated as a nonlinear system of ordinary differential equations by using the homotopy analysis method (HAM). The HAM is a flexible method which contains the auxiliary parameters and functions. This method has an important tool to adjust and control the convergence region of obtained solution. The numerical solutions are presented for various iterations, and the residual error functions are applied to show the accuracy of presented method. Several $\hbar$-curves are plotted to demonstrate the regions of convergence, and the residual errors are obtained for different values of theses regions.

Keywords Susceptible–infected–recovered model · Modified epidemiological model · Computer virus · Homotopy analysis method · $\hbar$-Curve

Introduction

In recent decades, using the computer systems has led to great changes in many fields of life. These changes are many important and wonderful where we cannot leave applying the computer systems.

Computer viruses are a malicious software program that can be written with different aims. These softwares help the user to enter the victim computer without permission. The virus should never be considered to be harmless and remain in the system. There are several types of viruses that can be categorized according to their source, technique, file type that infects, where they are hiding, the type of damage they enter, the type of operating system or the design on which they are attacking. Several computer viruses such as ILOVEYOU, Melissa, My Doom, Code Red, Sasser have been recognized. Also, in recent years, the Stuxnet virus targeted the Siemens hardware and software industry. Claims and statements allege that the virus was part of spy campaign to hit Iran's nuclear plant, Natanz. These viruses have been able to infect thousands of computers and have hurt billions dollar in computers around the world.

Thus, scientists start to work on finding methods to analyze, track, model and protect against viruses. Computer viruses are similar to biological viruses, and we can study it in two case, microscopic and macroscopic models. Therefore, many studies have proposed solutions that help us to understand how computer viruses operate. In order to control the growth and reproduction of computer viruses, many dynamical model have been presented [7, 18, 19, 29, 33–35, 40, 42]. In this study, the following modified SIR model of computer viruses

$$\begin{aligned} \frac{dS(t)}{dt} &= f_1 - \lambda S(t)I(t) - dS(t), \\ \frac{dI(t)}{dt} &= f_2 + \lambda S(t)I(t) - \varepsilon I(t) - dR(t), \\ \frac{dR(t)}{dt} &= f_3 + \varepsilon I(t) - dR(t). \end{aligned} \tag{1}$$

is illustrated where the initial conditions are in the following form

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$$S(0) = S_0(t) = S_0,$$
$$I(0) = I_0(t) = I_0, \quad (2)$$
$$R(0) = R_0(t) = R_0.$$

In recent years, many applicable methods have been applied to solve the linear and nonlinear problems [13–16, 27, 28, 31, 39, 41]. Also, many numerical and semi-analytical methods presented to solve the nonlinear epidemiological model (1) such as multi-step homotopy analysis method [17], collocation method with Chebyshev polynomials [32] and variational iteration method [29]. In this paper, the HAM [22–26] is applied to solve the nonlinear system of Eq. (1). This method has many applications to solve various problems arising in the mathematics, physics and engineering [1–6, 8–12, 14–16, 20, 21, 30, 31, 36–38]. The $\hbar$-curves are demonstrated to find the region of convergence. It is one of important abilities of HAM that the other methods do not have this ability. In order to show the efficiency and accuracy of presented approach, the residual error functions for different values of $\hbar, m$ are estimated. Several graphs of error functions are demonstrated to show the capabilities of method. List of functions, variables and initial assumptions of system (1) are presented in Table 1.

Main idea

Let L_S, L_I and L_R are the linear operators which are defined in the following form

$$L_S = \frac{\mathrm{dS}}{\mathrm{d}t}, L_I = \frac{\mathrm{dI}}{\mathrm{d}t}, L_R = \frac{\mathrm{dR}}{\mathrm{d}t} \quad (3)$$

and for constant values c_1, c_2 and c_3 we obtain

$$L_S(c_1) = 0, L_I(c_2) = 0, L_R(c_3) = 0. \quad (4)$$

Now, the following homotopy maps can be defined as

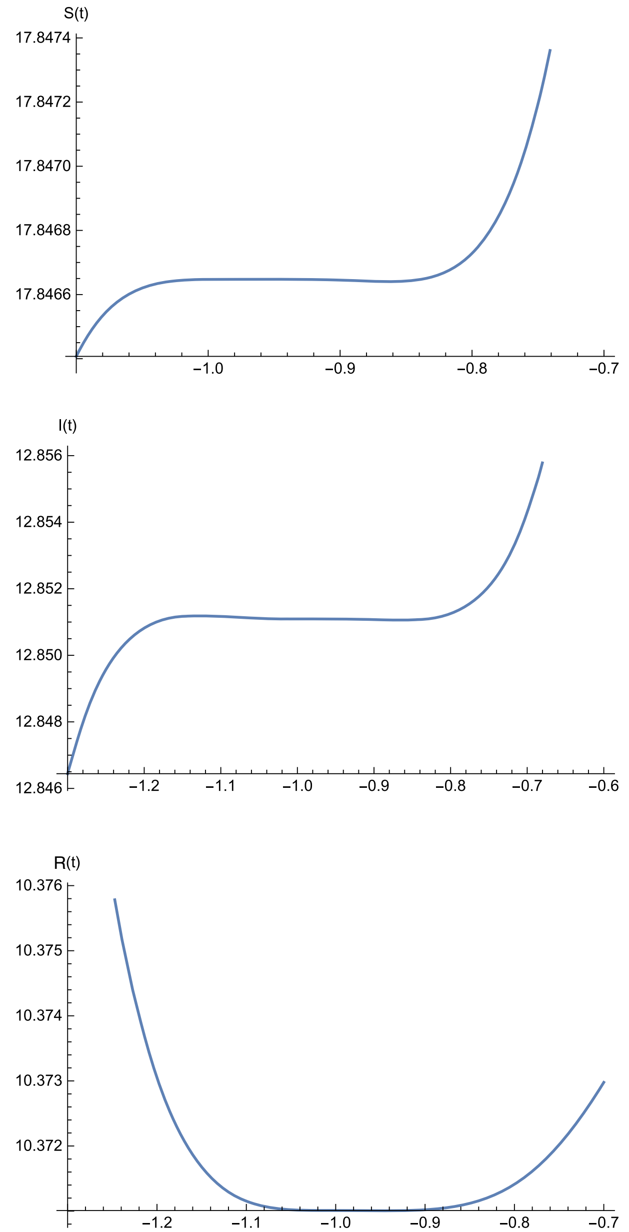

Fig. 1 $\hbar$-curves of $S(t)$, $I(t)$ and $R(t)$ for $m = 5, t = 1$

Table 1 List of parameters and functions

Parameters and functions	Meaning	Values
$S(t)$	Susceptible computers at time t	$S(0) = 20$
$I(t)$	Infected computers at time t	$I(0) = 15$
$R(t)$	Recovered computers at time t	$R(0) = 10$
f_1, f_2, f_3	Rate of external computers connected to the network	$f_1 = f_2 = f_3 = 0$
λ	Rate of infecting for susceptible computer	$\lambda = 0.001$
ε	Rate of recovery for infected computers	$\varepsilon = 0.1$
d	Rate of removing from the network	$d = 0.1$

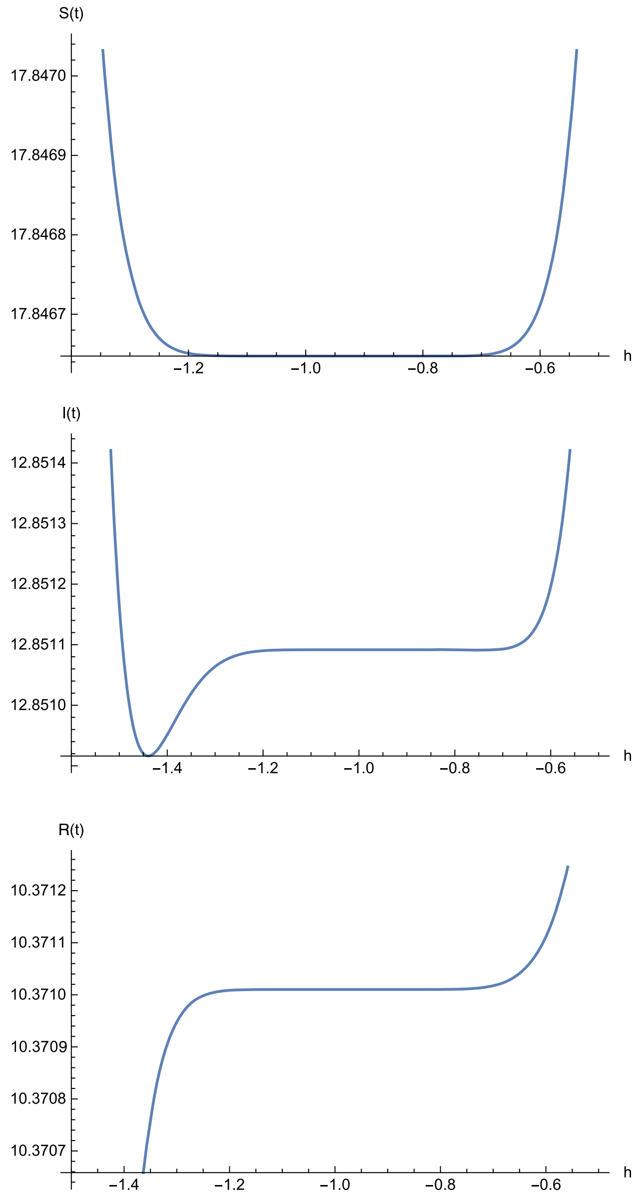

Fig. 2 $\hbar$-curves of $S(t)$, $I(t)$ and $R(t)$ for $m = 10, t = 1$

$$H_S\left(\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right) = (1-q)L_S\left[\tilde{S}(t;q) - S_0(t)\right] - q\hbar H_S(t)N_S\left[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right],$$
$$H_I\left(\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right) = (1-q)L_I\left[\tilde{I}(t;q) - I_0(t)\right] - q\hbar H_I(t)N_I\left[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right],$$
$$H_R\left(\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right) = (1-q)L_R\left[\tilde{R}(t;q) - R_0(t)\right] - q\hbar H_R(t)N_R\left[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right], \tag{5}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar \neq 0$ is a convergence control parameter, $H_S(t), H_I(t)$ and $H_R(t)$ are the auxiliary functions, L_S, L_I and L_R are the linear operators, and finally N_S, N_I and N_R are the nonlinear operators which are defined as follows

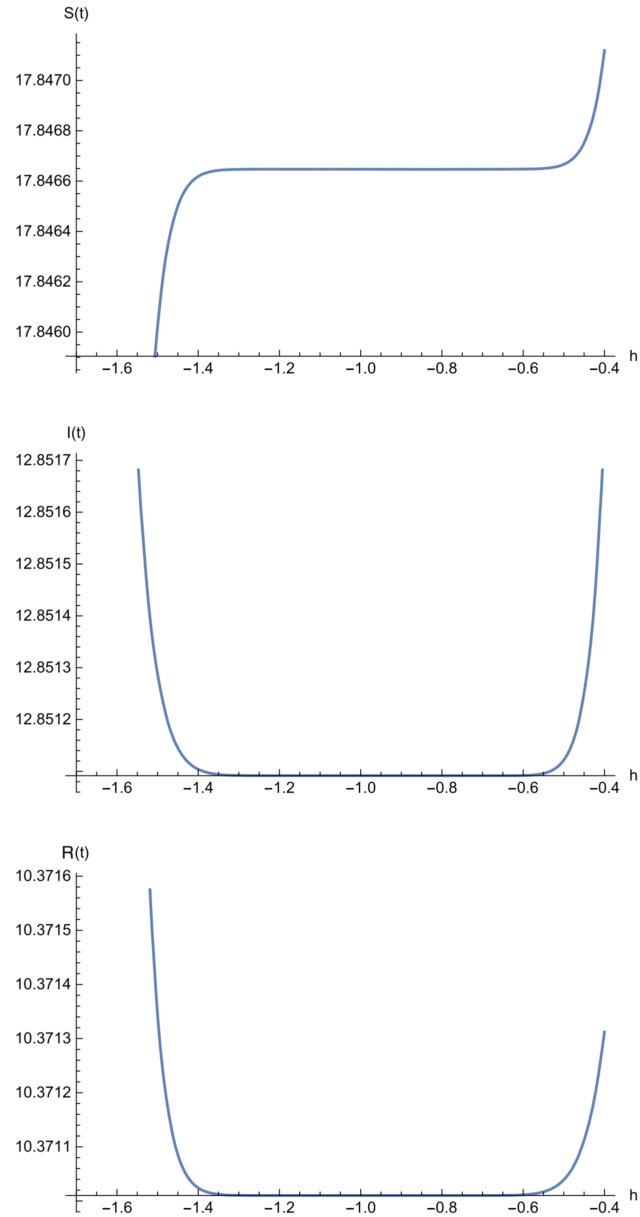

Fig. 3 $\hbar$-curves of $S(t)$, $I(t)$ and $R(t)$ for $m = 15, t = 1$

Table 2 Regions of convergence and the optimal values of $\hbar$ for $m = 5, 10, 15$ and $t = 1$

m	$\hbar_S$	$\hbar_S^*$	$\hbar_I$	$\hbar_I^*$	$\hbar_R$	$\hbar_R^*$
5	$-1.1 \le \hbar_S \le -0.8$	-0.99994	$-1.2 \le \hbar_I \le -0.8$	-1.00204	$-1.1 \le \hbar_R \le -0.9$	-0.99058
10	$-1.2 \le \hbar_S \le -0.7$	-1.0036	$-1.2 \le \hbar_I \le -0.7$	-1.0038	$-1.2 \le \hbar_R \le -0.7$	-1
15	$-1.3 \le \hbar_S \le -0.5$	-1	$-1.3 \le \hbar_I \le -0.6$	-1	$-1.3 \le \hbar_R \le -0.6$	-1

$$
\begin{aligned}
&N_S[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)] = \frac{\partial \tilde{S}(t;q)}{\mathrm{d}t} \\
&\quad - f_1 + \lambda \tilde{S}(t;q)\tilde{I}(t;q) + \mathrm{d}\tilde{S}(t;q), \\
&N_I[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)] = \frac{\partial \tilde{I}(t;q)}{\mathrm{d}t} - f_2 \\
&\quad - \lambda \tilde{S}(t;q)\tilde{I}(t;q) + \varepsilon \tilde{I}(t;q) + \mathrm{d}\tilde{R}(t;q), \\
&N_R[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)] = \frac{\partial \tilde{R}(t;q)}{\mathrm{d}t} - f_3 \\
&\quad - \varepsilon \tilde{I}(t;q) + \mathrm{d}\tilde{R}(t;q).
\end{aligned} \tag{6}
$$

According to [22–24] the zero-order deformation, equations can be defined as

$$
\begin{aligned}
&(1-q)L_S\left[\tilde{S}(t;q) - S_0(t)\right] - q\hbar H_S(t) N_S\left[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right] = 0, \\
&(1-q)L_I\left[\tilde{I}(t;q) - I_0(t)\right] - q\hbar H_I(t) N_I\left[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right] = 0, \\
&(1-q)L_R\left[\tilde{R}(t;q) - R_0(t)\right] - q\hbar H_R(t) N_R\left[\tilde{S}(t;q),\tilde{I}(t;q),\tilde{R}(t;q)\right] = 0.
\end{aligned} \tag{7}
$$

Now, we can write the Taylor series for $\tilde{S}(t;q), \tilde{I}(t;q)$ and $\tilde{R}(t;q)$ with respect to q in the following form

Table 3 Residual errors of $S_5(t), I_5(t)$ and $R_5(t)$ for different values of $\hbar$

t	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar = -1$	$\hbar_{\mathrm{opt}} = -0.99994$	$\hbar = -0.9$	$\hbar = -0.8$
0.0	0.000736	0.000023	2.66454×10^{-15}	2.66454×10^{-15}	0.000023	0.000736
0.2	0.00144214	0.0000696185	1.98321×10^{-11}	3.81375×10^{-11}	2.27788×10^{-6}	0.000388735
0.4	0.00240313	0.00014723	4.38416×10^{-10}	7.35197×10^{-10}	7.10003×10^{-6}	0.000128218
0.6	0.00366188	0.000265063	1.87637×10^{-9}	3.38906×10^{-9}	8.84562×10^{-6}	0.0000577341
0.8	0.00526424	0.000433423	1.93464×10^{-9}	6.74028×10^{-9}	6.1317×10^{-6}	0.000180563
1.0	0.00725902	0.000663706	1.1876×10^{-8}	9.02851×10^{-11}	1.58842×10^{-6}	0.000250966
$\lVert E \rVert$	0.00725902	0.000663706	1.1876×10^{-8}	6.74028×10^{-9}	0.000023	0.000736
t	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar_{\mathrm{opt}} = -1.00204$	$\hbar = -1$	$\hbar = -0.9$	$\hbar = -0.8$
0.0	0.000704	0.000022	8.43769×10^{-14}	2.66454×10^{-15}	0.000022	0.000704
0.2	0.00079364	0.0000221048	1.19964×10^{-9}	2.0886×10^{-10}	9.04759×10^{-6}	0.000553202
0.4	0.000675079	5.11674×10^{-6}	1.10223×10^{-8}	6.48732×10^{-9}	9.76643×10^{-6}	0.000345977
0.6	0.000291417	0.0000717513	3.28701×10^{-8}	4.78103×10^{-8}	0.0000295926	0.0000980449
0.8	0.000417872	0.000191176	4.72582×10^{-8}	1.955×10^{-7}	0.000046278	0.00017579
1.0	0.00151692	0.000378046	3.75195×10^{-9}	5.78838×10^{-7}	0.0000563829	0.000461654
$\lVert E \rVert$	0.00151692	0.000378046	4.72582×10^{-8}	5.78838×10^{-7}	0.0000563829	0.000704
t	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar = -1$	$\hbar_{opt} = -0.99058$	$\hbar = -0.9$	$\hbar = -0.8$
0.0	0.00016	$5. \times 10^{-6}$	0	3.70868×10^{-11}	$5. \times 10^{-6}$	0.00016
0.2	0.000761715	0.0000434312	5.69638×10^{-10}	1.31142×10^{-9}	0.000013481	0.000148655
0.4	0.00152906	0.0000988607	1.82284×10^{-8}	1.09745×10^{-8}	0.0000206331	0.000383918
0.6	0.00245803	0.000169652	1.38422×10^{-7}	6.07365×10^{-8}	0.0000172599	0.000546984
0.8	0.00354105	0.000252826	5.83309×10^{-7}	1.10979×10^{-7}	4.58752×10^{-6}	0.000639632
1.0	0.00476692	0.000343953	1.78012×10^{-6}	1.40369×10^{-9}	0.0000157769	0.000664199
$\lVert E \rVert$	0.00476692	0.000343953	1.78012×10^{-6}	1.10979×10^{-7}	0.0000206331	0.000664199

Table 4 Residual errors of $S_{10}(t), I_{10}(t)$ and $R_{10}(t)$ for different values of $\hbar$

t	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar_{\text{opt}} = -1.0036$	$\hbar = -1$	$\hbar = -0.9$	$\hbar = -0.8$	$\hbar = -0.7$
0.0	2.3552×10^{-7}	2.30233×10^{-10}	1.02141×10^{-13}	9.14824×10^{-14}	2.30053×10^{-10}	2.3552×10^{-7}	0.0000135813
0.2	7.98087×10^{-7}	1.59804×10^{-9}	1.03917×10^{-13}	1.07914×10^{-13}	5.69842×10^{-11}	4.58415×10^{-8}	6.38727×10^{-6}
0.4	1.85035×10^{-6}	5.17876×10^{-9}	9.81437×10^{-14}	1.23901×10^{-13}	5.71028×10^{-11}	4.56784×10^{-8}	1.45981×10^{-6}
0.6	3.60403×10^{-6}	1.24299×10^{-8}	1.38556×10^{-13}	1.22125×10^{-13}	7.26663×10^{-12}	7.45872×10^{-8}	1.67784×10^{-6}
0.8	6.30105×10^{-6}	2.47376×10^{-8}	5.59552×10^{-14}	1.66533×10^{-13}	1.12355×10^{-12}	6.8329×10^{-8}	3.44326×10^{-6}
1.0	0.0000101995	4.26312×10^{-8}	1.3145×10^{-13}	3.46834×10^{-13}	3.20961×10^{-11}	4.69374×10^{-8}	4.19639×10^{-6}
$\|\|E\|\|$	0.0000101995	4.26312×10^{-8}	1.38556×10^{-13}	3.46834×10^{-13}	2.30053×10^{-10}	2.3552×10^{-7}	0.0000135813
t	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar_{\text{opt}} = -1.0038$	$\hbar = -1$	$\hbar = -0.9$	$\hbar = -0.8$	$\hbar = -0.7$
0.0	2.2528×10^{-7}	2.19797×10^{-10}	1.23457×10^{-13}	9.14824×10^{-14}	2.19932×10^{-10}	2.2528×10^{-7}	0.0000129908
0.2	1.8875×10^{-7}	1.79984×10^{-10}	1.28342×10^{-13}	9.81437×10^{-14}	9.37903×10^{-11}	1.08779×10^{-7}	9.5656×10^{-6}
0.4	3.0554×10^{-7}	2.84757×10^{-9}	1.32783×10^{-13}	1.01252×10^{-13}	3.81631×10^{-10}	5.43913×10^{-8}	4.78379×10^{-6}
0.6	1.52798×10^{-6}	9.54972×10^{-9}	1.32561×10^{-13}	1.11244×10^{-13}	3.5494×10^{-10}	2.18038×10^{-7}	7.358×10^{-7}
0.8	3.77689×10^{-6}	2.17536×10^{-8}	1.03695×10^{-13}	1.51879×10^{-13}	4.57945×10^{-11}	3.47132×10^{-7}	6.45201×10^{-6}
1.0	7.35477×10^{-6}	3.95442×10^{-8}	3.39728×10^{-14}	4.14557×10^{-13}	7.11569×10^{-10}	4.17349×10^{-7}	0.0000119013
$\|\|E\|\|$	7.35477×10^{-6}	3.95442×10^{-8}	1.32783×10^{-13}	4.14557×10^{-13}	7.11569×10^{-10}	4.17349×10^{-7}	0.0000129908
t	$\hbar = -1.3$	$\hbar = -1.2$	$\hbar = -1.1$	$\hbar_{\text{opt}} = -1$	$\hbar = -0.9$	$\hbar = -0.8$	$\hbar = -0.7$
0.0	2.95245×10^{-6}	5.11999×10^{-8}	5.00062×10^{-11}	0	5.00069×10^{-11}	5.12×10^{-8}	2.95245×10^{-6}
0.2	0.0000203615	5.01628×10^{-7}	1.01934×10^{-9}	8.71525×10^{-15}	1.82436×10^{-10}	1.17196×10^{-7}	3.44612×10^{-6}
0.4	0.0000446347	1.16719×10^{-6}	2.45188×10^{-9}	1.15463×10^{-14}	1.45817×10^{-12}	1.8454×10^{-7}	7.79626×10^{-6}
0.6	0.0000746483	1.96985×10^{-6}	3.30358×10^{-9}	3.94129×10^{-14}	3.91076×10^{-10}	1.62374×10^{-7}	0.0000101862
0.8	0.000108286	2.76035×10^{-6}	1.49223×10^{-9}	1.9984×10^{-14}	7.393×10^{-10}	6.72562×10^{-8}	0.0000107469
1.0	0.000142277	3.30389×10^{-6}	6.33091×10^{-9}	1.97065×10^{-14}	8.26158×10^{-10}	8.0793×10^{-8}	9.64522×10^{-6}
$\|\|E\|\|$	0.000142277	3.30389×10^{-6}	6.33091×10^{-9}	3.94129×10^{-14}	8.26158×10^{-10}	1.8454×10^{-7}	0.0000107469

$$
\begin{aligned}
\tilde{S}(t;q) &= S_0(t) + \sum_{m=1}^{\infty} S_m(t) q^m, \\
\tilde{I}(t;q) &= I_0(t) + \sum_{m=1}^{\infty} I_m(t) q^m, \\
\tilde{R}(t;q) &= R_0(t) + \sum_{m=1}^{\infty} R_m(t) q^m,
\end{aligned} \tag{8}
$$

where

$$
S_m = \frac{1}{m!}\frac{\partial^m \tilde{S}(t;q)}{\partial q^m}\bigg|_{q=0}, I_m = \frac{1}{m!}\frac{\partial^m \tilde{I}(t;q)}{\partial q^m}\bigg|_{q=0},
$$
$$
R_m = \frac{1}{m!}\frac{\partial^m \tilde{R}(t;q)}{\partial q^m}\bigg|_{q=0}.
$$

For more analysis, the following vectors are defined

$$
\begin{aligned}
\tilde{S}_m(t) &= \left\{ S_0(t), S_1(t), \ldots, S_m(t) \right\}, \\
\tilde{I}_m(t) &= \left\{ I_0(t), I_1(t), \ldots, I_m(t) \right\}, \\
\tilde{R}_m(t) &= \left\{ R_0(t), R_1(t), \ldots, R_m(t) \right\}.
\end{aligned}
$$

Differentiating Eq. (7) m-times with respect to q, dividing by $m!$ and putting $q = 0$ the mth-order deformation equations can be obtained as follows

$$
\begin{aligned}
L_S[S_m(t) - \chi_m S_{m-1}(t)] &= \hbar H_S(t) \Re_{m,S}(\mathbf{S}_{m-1}, \mathbf{I}_{m-1}, \mathbf{R}_{m-1}), \\
L_I[I_m(t) - \chi_m I_{m-1}(t)] &= \hbar H_I(t) \Re_{m,I}(\mathbf{S}_{m-1}, \mathbf{I}_{m-1}, \mathbf{R}_{m-1}), \\
L_R[R_m(t) - \chi_m R_{m-1}(t)] &= \hbar H_R(t) \Re_{m,R}(\mathbf{S}_{m-1}, \mathbf{I}_{m-1}, \mathbf{R}_{m-1}),
\end{aligned} \tag{9}
$$

where

$$
S_m(0) = 0, I_m(0) = 0, R_m(0) = 0,
$$

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Table 5 Residual errors of $S_{15}(t), I_{15}(t)$ and $R_{15}(t)$ for different values of $\hbar$

t	$\hbar=-1.3$	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar_{opt}=-1$	$\hbar=-0.9$	$\hbar=-0.8$	$\hbar=-0.7$	$\hbar=-0.6$	$\hbar=-0.5$
0.0	3.29921×10^{-8}	7.56701×10^{-11}	2.91056×10^{-12}	1.82077×10^{-13}	2.72671×10^{-13}	7.51923×10^{-11}	3.30025×10^{-8}	2.46961×10^{-6}	0.0000701904
0.2	1.22758×10^{-7}	4.11545×10^{-10}	1.26876×10^{-12}	2.71339×10^{-13}	7.8737×10^{-13}	2.22977×10^{-12}	9.08212×10^{-9}	1.20753×10^{-6}	0.0000449263
0.4	3.02461×10^{-7}	1.20836×10^{-9}	2.64677×10^{-13}	7.74492×10^{-13}	1.38733×10^{-12}	2.22431×10^{-11}	3.6583×10^{-9}	3.39541×10^{-7}	0.0000250612
0.6	6.16321×10^{-7}	2.75659×10^{-9}	2.47047×10^{-12}	3.74367×10^{-13}	1.92779×10^{-12}	1.80065×10^{-11}	8.98214×10^{-9}	2.19959×10^{-7}	9.80585×10^{-6}
0.8	1.11237×10^{-6}	5.30973×10^{-9}	3.26672×10^{-12}	1.04139×10^{-12}	2.84572×10^{-12}	8.17524×10^{-12}	9.81097×10^{-9}	5.44873×10^{-7}	1.55886×10^{-6}
1.0	1.8332×10^{-6}	8.85967×10^{-9}	3.44436×10^{-12}	2.77822×10^{-12}	3.3924×10^{-12}	2.29727×10^{-12}	8.30763×10^{-9}	6.97877×10^{-7}	9.68337×10^{-6}
$\lVert E\rVert$	1.8332×10^{-6}	8.85967×10^{-9}	3.44436×10^{-12}	2.77822×10^{-12}	3.3924×10^{-12}	7.51923×10^{-11}	3.30025×10^{-8}	2.46961×10^{-6}	0.0000701904
t	$\hbar=-1.3$	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar_{opt}=-1$	$\hbar=-0.9$	$\hbar=-0.8$	$\hbar=-0.7$	$\hbar=-0.6$	$\hbar=-0.5$
0.0	3.15602×10^{-8}	7.7125×10^{-11}	7.27418×10^{-13}	1.82077×10^{-13}	6.36824×10^{-13}	7.24869×10^{-11}	3.15676×10^{-8}	2.36223×10^{-6}	0.0000671387
0.2	2.20155×10^{-8}	3.71703×10^{-13}	6.50147×10^{-13}	2.34035×10^{-13}	6.83453×10^{-13}	9.87121×10^{-12}	1.76896×10^{-8}	1.76249×10^{-6}	0.0000566569
0.4	7.37529×10^{-8}	5.44329×10^{-10}	2.35101×10^{-12}	2.24265×10^{-13}	6.98108×10^{-13}	6.68665×10^{-11}	1.71675×10^{-9}	9.29254×10^{-7}	0.0000426896
0.6	3.11344×10^{-7}	1.91031×10^{-9}	2.45226×10^{-12}	4.27658×10^{-13}	5.34683×10^{-13}	1.16269×10^{-10}	2.17681×10^{-8}	2.63814×10^{-8}	0.0000262555
0.8	7.47885×10^{-7}	4.34223×10^{-9}	2.14317×10^{-12}	2.54019×10^{-13}	6.49703×10^{-13}	1.16597×10^{-10}	3.87462×10^{-8}	1.00885×10^{-6}	8.28368×10^{-6}
1.0	1.4286×10^{-6}	7.77428×10^{-9}	4.34319×10^{-12}	4.33431×10^{-13}	5.98188×10^{-13}	6.4357×10^{-11}	5.00369×10^{-8}	1.93795×10^{-6}	0.0000103872
$\lVert E\rVert$	1.4286×10^{-6}	7.77428×10^{-9}	4.34319×10^{-12}	4.33431×10^{-13}	6.98108×10^{-13}	1.16597×10^{-10}	5.00369×10^{-8}	2.36223×10^{-6}	0.0000671387
t	$\hbar=-1.3$	$\hbar=-1.2$	$\hbar=-1.1$	$\hbar_{opt}=-1$	$\hbar=-0.9$	$\hbar=-0.8$	$\hbar=-0.7$	$\hbar=-0.6$	$\hbar=-0.5$
0.0	7.17409×10^{-9}	1.72804×10^{-11}	0	0	1.13687×10^{-13}	1.64562×10^{-11}	7.17444×10^{-9}	5.36871×10^{-7}	0.0000152588
0.2	7.76303×10^{-8}	2.63399×10^{-10}	6.52645×10^{-13}	3.41893×10^{-13}	3.86025×10^{-13}	5.07507×10^{-11}	1.40688×10^{-8}	5.85159×10^{-7}	7.1495×10^{-6}
0.4	1.82402×10^{-7}	6.26761×10^{-10}	7.20313×10^{-13}	4.80727×10^{-13}	8.62921×10^{-13}	4.81887×10^{-11}	2.4034×10^{-8}	1.34887×10^{-6}	0.0000249791
0.6	3.00549×10^{-7}	9.13821×10^{-10}	3.06866×10^{-13}	9.30644×10^{-13}	6.56392×10^{-13}	4.54331×10^{-12}	2.39238×10^{-8}	1.77154×10^{-6}	0.0000383372
0.8	3.91317×10^{-7}	7.33472×10^{-10}	5.31131×10^{-13}	1.75326×10^{-12}	1.82554×10^{-12}	8.23384×10^{-11}	1.54493×10^{-8}	1.87859×10^{-6}	0.0000473825
1.0	3.90141×10^{-7}	5.18874×10^{-10}	2.13696×10^{-12}	1.09068×10^{-12}	1.85918×10^{-12}	1.59379×10^{-10}	6.61942×10^{-10}	1.70211×10^{-6}	0.0000523195
$\lVert E\rVert$	3.91317×10^{-7}	9.13821×10^{-10}	2.13696×10^{-12}	1.75326×10^{-12}	1.85918×10^{-12}	1.59379×10^{-10}	2.4034×10^{-8}	1.87859×10^{-6}	0.0000523195

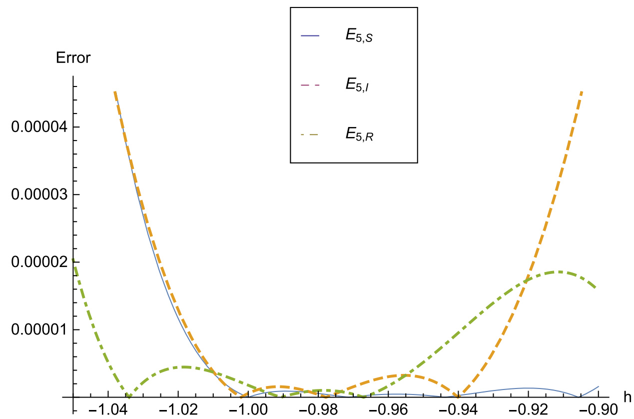

Fig. 4 Averaged residual errors $E_{5,S}, E_{5,I}, E_{5,R}$ versus $\hbar$ for $t = 1$

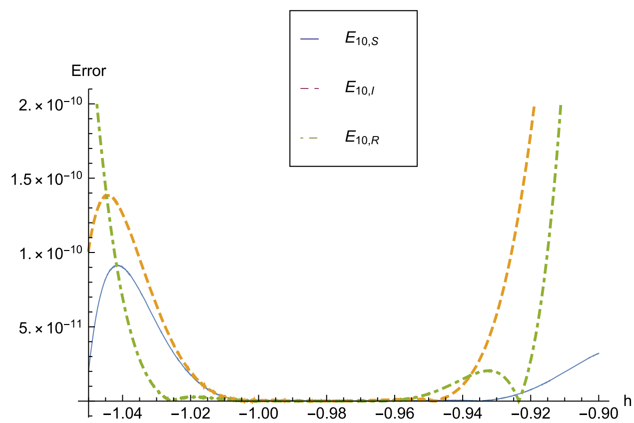

Fig. 5 Averaged residual errors $E_{10,S}, E_{10,I}, E_{10,R}$ versus $\hbar$ for $t = 1$

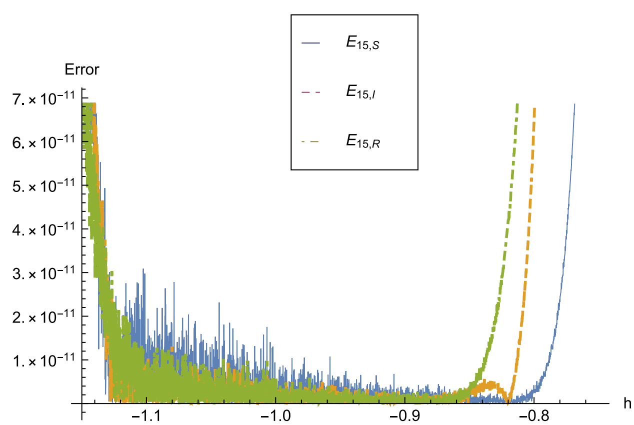

Fig. 6 Averaged residual errors $E_{15,S}, E_{15,I}, E_{15,R}$ versus $\hbar$ for $t = 1$

and

$$\begin{aligned}
\Re_{m,S}(t) &= \frac{\mathrm{d}S_{m-1}(t)}{\mathrm{d}t} - (1-\chi_m)f_1 + \lambda\sum_{i=0}^{m-1} S_i(t)I_{m-1-i}(t) \\
&\quad + \mathrm{d}S_{m-1}(t), \\
\Re_{m,I}(t) &= \frac{\mathrm{d}I_{m-1}(t)}{\mathrm{d}t} - (1-\chi_m)f_2 - \lambda\sum_{i=0}^{m-1} S_i(t)I_{m-1-i}(t) \\
&\quad + \varepsilon I_{m-1}(t) + \mathrm{d}R_{m-1}(t), \\
\Re_{m,R}(t) &= \frac{\mathrm{d}R_{m-1}(t)}{\mathrm{d}t} - (1-\chi_m)f_3 - \varepsilon I_{m-1}(t) + \mathrm{d}R_{m-1}(t),
\end{aligned} \tag{10}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1. \end{cases} \tag{11}$$

By putting $H_S(t) = H_I(t) = H_R(t) = 1$ and applying the inverse operators L_S^{-1}, L_I^{-1} and L_R^{-1}, we have

$$\begin{aligned}
S_m(t) &= \chi_m S_{m-1}(t) + \hbar\int_0^t \Re_{m,S}(t)\mathrm{d}t, \\
I_m(t) &= \chi_m I_{m-1}(t) + \hbar\int_0^t \Re_{m,I}(t)\mathrm{d}t, \\
R_m(t) &= \chi_m R_{m-1}(t) + \hbar\int_0^t \Re_{m,R}(t)\mathrm{d}t.
\end{aligned} \tag{12}$$

Finally, the mth-order approximate solution of nonlinear system (1) can be obtained as

$$S_m(t) = \sum_{j=0}^{m} S_j(t), \quad I_m(t) = \sum_{j=0}^{m} I_j(t), \quad R_m(t) = \sum_{j=0}^{m} R_j(t). \tag{13}$$

Numerical illustration

In this section, the numerical results based on the HAM are presented. The approximate solutions for $m = 5, 10, 15$ are obtained in the following form

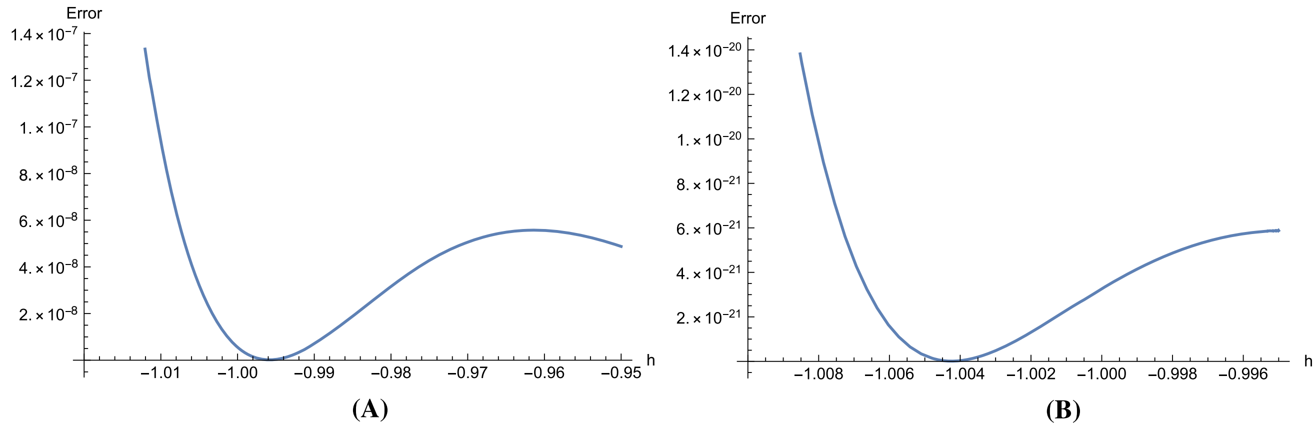

Fig. 7 Square residual errors of **a** $E'_{5,S}$ and **b** $E'_{10,S}$ versus $\hbar$ based on the OHAM

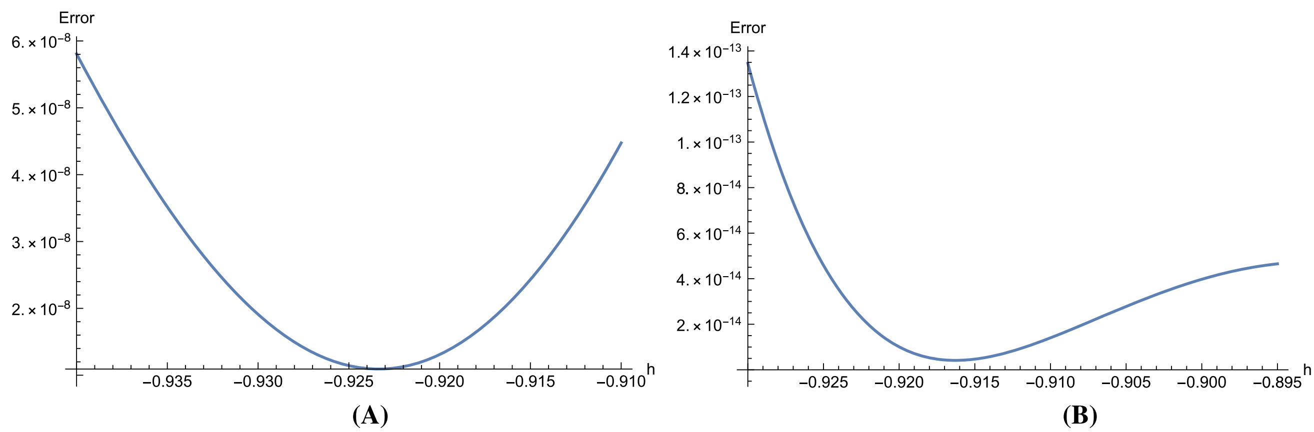

Fig. 8 Square residual errors of **a** $E'_{5,I}$ and **b** $E'_{10,I}$ versus $\hbar$ based on the OHAM

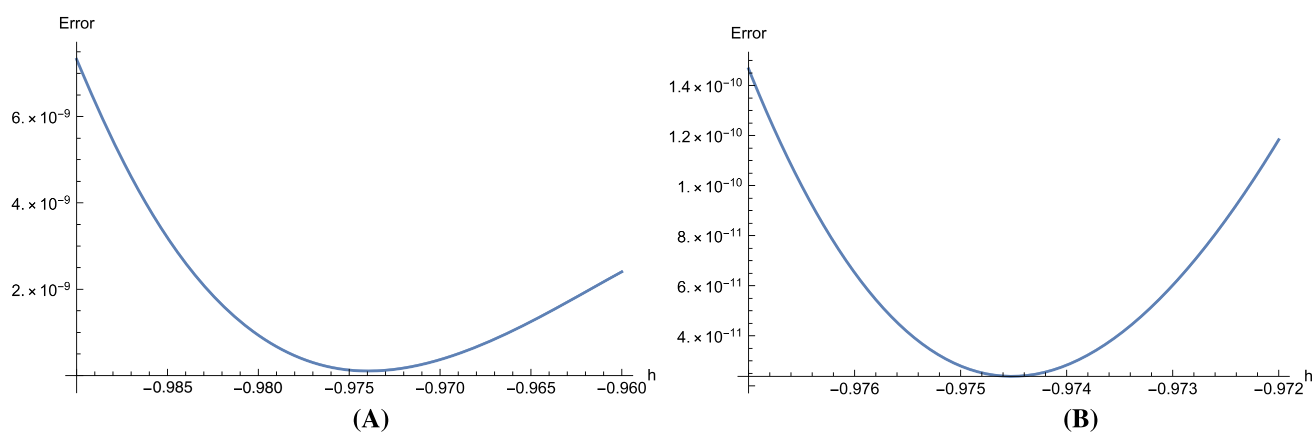

Fig. 9 Square residual errors of **a** $E'_{5,R}$ and **b** $E'_{10,R}$ versus $\hbar$ based on the OHAM

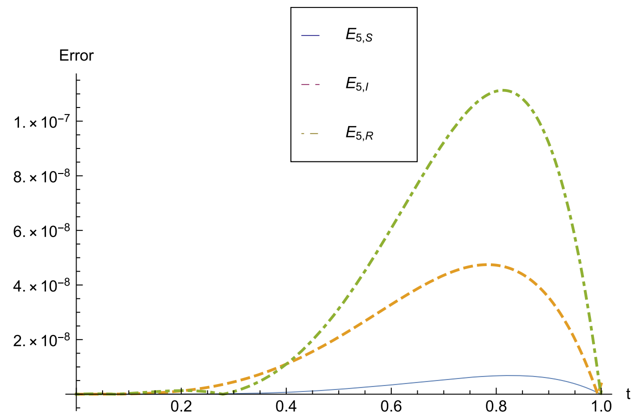

Fig. 10 Residual error functions for $E_{5,S}, E_{5,I}, E_{5,R}$ and the optimal values $\hbar^*$

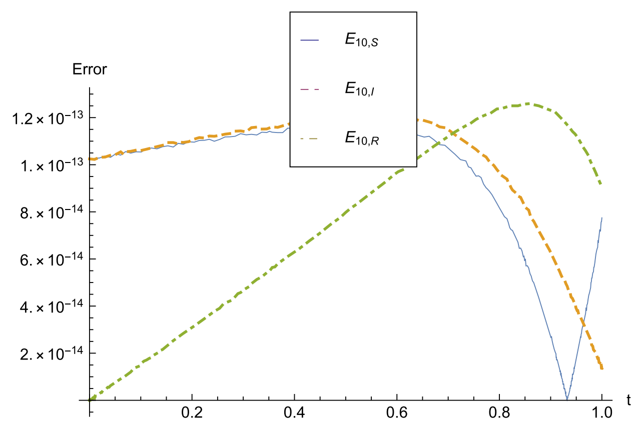

Fig. 11 Residual error functions for $E_{10,S}, E_{10,I}, E_{10,R}$ and the optimal values $\hbar^*$

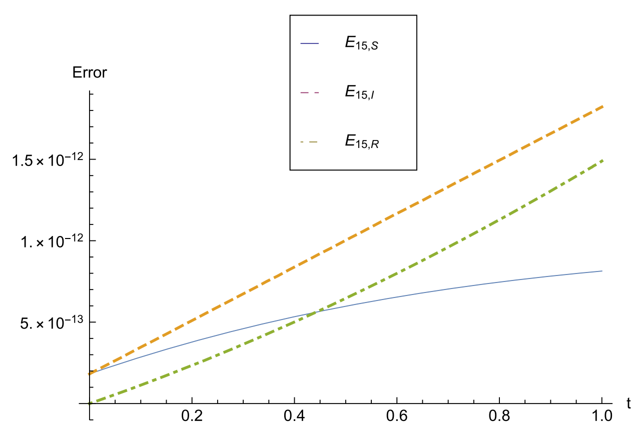

Fig. 12 Residual error functions for $E_{15,S}, E_{15,I}, E_{15,R}$ and the optimal values $\hbar^*$

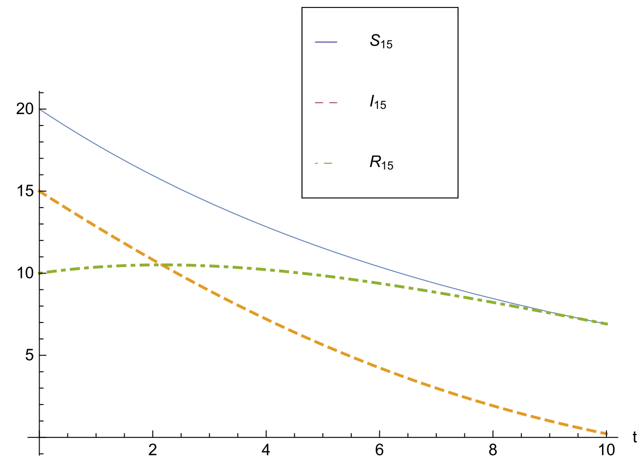

Fig. 13 Approximate solutions of $S(t)$, $I(t)$ and $R(t)$ for $m = 15, \hbar = -1$ and $t \in [0, 10]$

$$\begin{aligned}
S_5(t) = {}& 20 + 11.5ht + 23h^2t + 23h^3t + 11.5h^4t + 2.3h^5t \\
& + 1.5425h^2t^2 + 3.085h^3t^2 \\
& + 2.31375h^4t^2 + 0.617h^5t^2 + 0.0790458h^3t^3 \\
& + 0.118569h^4t^3 + 0.0474275h^5t^3 \\
& + 0.00154855h^4t^4 + 0.00123884h^5t^4 \\
& + 7.74864 \times 10^{-6}h^5t^5, \\
I_5(t) = {}& 15 + 11ht + 22h^2t + 22h^3t + 11h^4t + 2.2h^5t \\
& + 0.4575h^2t^2 + 0.915h^3t^2 \\
& + 0.68625h^4t^2 + 0.183h^5t^2 - 0.0573792h^3t^3 \\
& - 0.0860688h^4t^3 - 0.0344275h^5t^3 \\
& - 0.00203085h^4t^4 - 0.00162468h^5t^4 \\
& - 9.82135 \times 10^{-6}h^5t^5,
\end{aligned}$$

$$\begin{aligned}
R_5(t) = {}& 10 - 2.5ht - 5h^2t - 5h^3t - 2.5h^4t - 0.5h^5t \\
& - 1.35h^2t^2 - 2.7h^3t^2 \\
& - 2.025h^4t^2 - 0.54h^5t^2 - 0.06025h^3t^3 \\
& - 0.090375h^4t^3 - 0.03615h^5t^3 \\
& - 0.0000358854h^4t^4 - 0.0000287083h^5t^4 \\
& + 7.97984 \times 10^{-6}h^5t^5, \\
S_{10}(t) = {}& 20 + 23ht + 103.5h^2t + 276h^3t + 483h^4t \\
& + 579.6h^5t + 483h^6t + 276h^7t \\
& + \cdots - 3.32927 \times 10^{-10}h^9t^9 - 2.99635 \times 10^{-10}h^{10}t^9 \\
& - 5.68453 \times 10^{-13}h^{10}t^{10}, \\
I_{10}(t) = {}& 15 + 22ht + 99h^2t + 264h^3t + 462h^4t \\
& + 554.4h^5t + 462h^6t + 264h^7t \\
& + \cdots + 3.2067 \times 10^{-10}h^9t^9 + 2.88603 \\
& \times 10^{-10}h^{10}t^9 + 4.21668 \times 10^{-13}h^{10}t^{10},
\end{aligned}$$

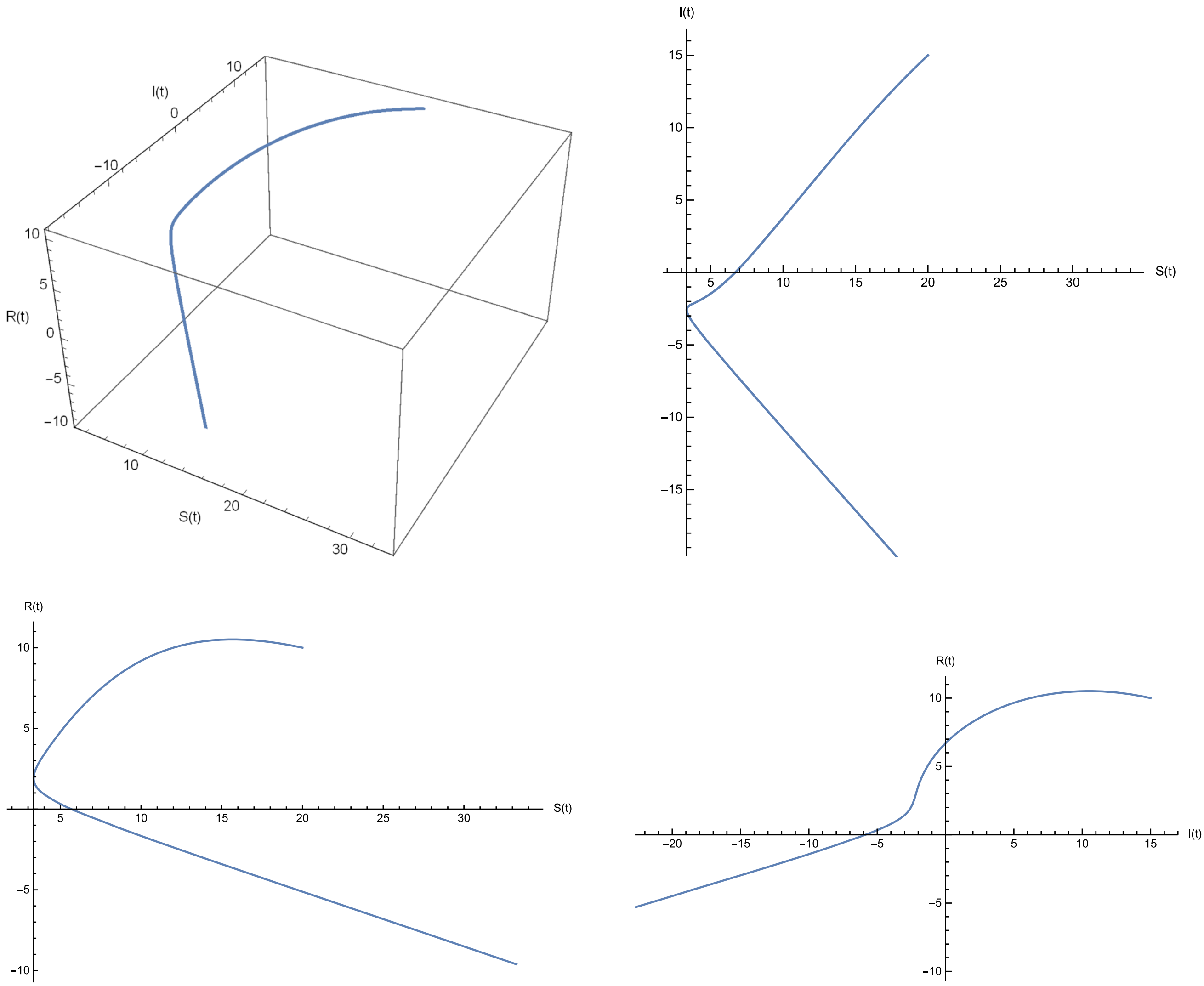

Fig. 14 Phase portraits of $S_{15}(t), I_{15}(t), R_{15}(t)$ by using the HAM

$$\begin{aligned}R_{10}(t) &= 10 - 5ht - 22.5h^2t - 60h^3t - 105h^4t \\ &\quad - 126h^5t - 105h^6t - 60h^7t \\ &\quad + \cdots - 1.34528 \times 10^{-10}h^9t^9 - 1.21075 \times 10^{-10}h^{10}t^9 \\ &\quad - 4.55198 \times 10^{-13}h^{10}t^{10}, \\ S_{15}(t) &= 20 + 34.5ht + 241.5h^2t + 1046.5h^3t \\ &\quad + 3139.5h^4t + 6906.9h^5t \\ &\quad + \cdots + 5.68303 \times 10^{-17}h^{14}t^{14} + 5.30416 \times 10^{-17}h^{15}t^{14} \\ &\quad + 1.56646 \times 10^{-20}h^{15}t^{15}, \\ I_{15}(t) &= 15 + 33ht + 231h^2t + 1001h^3t + 3003h^4t + 6606.6h^5t \\ &\quad + \cdots - 5.44064 \times 10^{-17}h^{14}t^{14} - 5.07793 \\ &\quad \times 10^{-17}h^{15}t^{14} - 7.423 \times 10^{-21}h^{15}t^{15}, \\ R_{15}(t) &= 10 - 7.5ht - 52.5h^2t - 227.5h^3t - 682.5h^4t - 1501.5h^5t \\ &\quad + \cdots + 1.61197 \times 10^{-17}h^{14}t^{14} + 1.50451 \\ &\quad \times 10^{-17}h^{15}t^{14} + 3.13449 \times 10^{-20}h^{15}t^{15},\end{aligned}$$

where $\hbar$ is the convergence control parameter of the HAM. In order to show the regions of convergence, several $\hbar$-curves are demonstrated in Figs. 1, 2 and 3. These regions are the parallel parts of $\hbar$-curves with axiom x. The regions of convergence for $m = 5, 10, 15$ and $t = 1$ are presented in Table 2.

In order to show the efficiency and accuracy of presented method, the following residual error functions are applied as follows

$$\begin{aligned}E_{m,S}(t) &= S'_m(t) - f_1 + \lambda S_m(t)I_m(t) + \mathrm{d}S_m(t), \\ E_{m,I}(t) &= I'_m(t) - f_2 - \lambda S_m(t)I_m(t) + \varepsilon I_m(t) + \mathrm{d}R_m(t), \\ E_{m,R}(t) &= R'_m(t) - f_3 - \varepsilon I_m(t) + \mathrm{d}R_m(t),\end{aligned} \tag{14}$$

and the numerical results for different values of t, m based on the presented convergence regions are obtained in

Tables 3, 4 and 5. Also, the norm of residual errors $E_{m,S}(t), E_{m,I}(t)$ and $E_{m,R}(t)$ is given in these tables. In Figs. 4, 5 and 6 the residual errors for $t = 1$ and versus $\hbar$ are demonstrated. By using these figures, the optimal values of convergence control parameter $\hbar$ can be obtained which are presented in Table 2. Also, in Figs. 7, 8, 9, 10, 11 and 12 the plots of square residual errors

$$E'_{m,S} = \int_0^{+\infty} \left\{ N_S \left[\sum_{j=0}^{m} S_j(\xi) \right] \right\}^2 \mathrm{d}\xi,$$
$$E'_{m,I} = \int_0^{+\infty} \left\{ N_I \left[\sum_{j=0}^{m} I_j(\xi) \right] \right\}^2 \mathrm{d}\xi, \tag{15}$$
$$E'_{m,R} = \int_0^{+\infty} \left\{ N_R \left[\sum_{j=0}^{m} R_j(\xi) \right] \right\}^2 \mathrm{d}\xi,$$

based on the optimal homotopy analysis method (OHAM) are shown. Figure 13 shows the approximate solution for $m = 15$ and $t \in [0, 10]$. In Fig. 14, the phase portraits of $S-I, S-R, I-R$ and $S-I-R$ for 15th-order approximation of the HAM and $\hbar = -1$ are shown.

Conclusion

In this study, the modified nonlinear SIR epidemiological model of computer viruses was illustrated and the HAM was applied to solve the presented model. It is important to note that in this method we have some auxiliary parameters and functions. One of these parameters is the convergence control parameter $\hbar$ which can be applied to adjust and control the convergence region of obtained solutions. Thus, by plotting several $\hbar$-curves and finding the regions of convergence, we showed the advantages and abilities of method. The residual errors were applied to show the efficiency and accuracy of method.

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