ISSN 0032-9460, Problems of Information Transmission, 2008, Vol. 44, No. 1, pp. 1–11. © Pleiades Publishing, Inc., 2008.
Original Russian Text © C. Martínez, E. Stafford, R. Beivide, E.M. Gabidulin, 2008, published in Problemy Peredachi Informatsii, 2008, Vol. 44, No. 1, pp. 3–14.

CODING THEORY

Modeling Hexagonal Constellations with Eisenstein–Jacobi Graphs

C. Martínez[a1], E. Stafford[a], R. Beivide[a1], and E. M. Gabidulin[b2]

[a] *Universidad de Cantabria, Santander, Spain*
carmen.martinez@unican.es
esteban.stafford@gestion.unican.es
ramon.beivide@unican.es
[b] *Moscow Institute of Physics and Technology (State University)*
gab@mail.mipt.ru

Received October 18, 2006; in final form, November 1, 2007

Abstract—A set of signal points is called a hexagonal constellation if it is possible to define a metric so that each point has exactly six neighbors at distance 1 from it. As sets of signal points, quotient rings of the ring of Eisenstein–Jacobi integers are considered. For each quotient ring, the corresponding graph is defined. In turn, the distance between two points of a quotient ring is defined as the corresponding graph distance. Under certain restrictions, a quotient ring is a hexagonal constellation with respect to this metric. For the considered hexagonal constellations, some classes of perfect codes are known. Using graphs leads to a new way of constructing these codes based on solving a standard graph-theoretic problem of finding a perfect dominating set. Also, a relation between the proposed metric and the well-known Lee metric is considered.

DOI: 10.1134/S0032946008010018

1. INTRODUCTION

Problems of designing error-correcting codes for multi-dimensional signal spaces are intensively studied. For two-dimensional spaces, in [1] the *Mannheim metric* (an analog of the well-known *Manhattan metric*) was introduced, and error-correcting codes in this metric were proposed. In this case, sets of signal points are elements of quotient rings of the ring of Gaussian integers. In [2], this approach was extended to the case of hexagonal constellations, which were modeled by means of quotient rings of Eisenstein–Jacobi integers, and an upgraded Mannheim metric was proposed for this case.

In [3], lattices and plane tessellations were employed to derive codes based on the distance induced by the Euclidean metric over certain graphs embedded in flat tori. Both two-dimensional and hexagonal constellations were considered. It was pointed out that the metrics defined in [1, 2] are particular cases of those used to construct the codes.

In [4], the circulant graph distance has been proved to be a suitable metric for designing codes over Gaussian integers. In this metric, perfect group codes were constructed. The methodology described in [4] can also be applied to define perfect codes over hexagonal constellations. To this aim, we introduce a family of degree-six graphs whose vertices are labeled with elements of quotient rings of the ring of Eisenstein–Jacobi integers. This allows us to interpret the distance between graph vertices as a distance between elements of a quotient ring and to consider codes over the ring

[1] Supported in part by the Spanish Comision Interministerial de Ciencia y Tecnologia (CICYT), Grant TIN2004-07440-C02-01, and the Research Program of the Universidad de Cantabria, Grant 30.640.VI07.
[2] Supported in part by the Russian Foundation for Basic Research, project no. GFENa-05-01-39017.

in this metric. Next, we consider the graph-theoretic problem of finding perfect t-dominating sets for these graphs. If the problem has a solution, such a perfect set directly yields a perfect code over a quotient ring of the ring of Eisenstein–Jacobi integers.

In [4] it was also proved that the Mannheim distance defined in [1] for two-dimensional constellations is not a metric. There was given an example where the triangular inequality is not fulfilled. By the same reason, the extended Mannheim distance defined in [2] for hexagonal constellations is not a true metric either. A correct definition of a metric for two-dimensional constellations was given in [5]. In the present work, the metric is extended to the hexagonal case. In both cases, the graph-based metric constitutes a correct alternative to the Mannheim distance.

The paper is organized as follows. Section 2 presents definitions and basic facts on quotient rings of the ring of Eisenstein–Jacobi integers. In Section 3 we introduce the family of Eisenstein–Jacobi graphs and show that this family is a subset of the family of circulant graphs of degree six. In Section 4, the problem of finding perfect t-dominating sets over Eisenstein–Jacobi graphs is solved. In Section 5 we study the relationship between this new graph-based metric and the Lee metric. Results of the paper were partially announced in [6].

2. THE RING OF EISENSTEIN–JACOBI INTEGERS

The ring of *Eisenstein–Jacobi integers* (EJ-integers for short) is defined as

$$\mathbb{Z}[\rho] = \{x + y\rho \mid x, y \in \mathbb{Z}\},$$

where $\rho = (-1 + \sqrt{-3})/2$. It is easy to see that $\rho^2 + \rho + 1 = 0$. This set is closed with respect to addition, subtraction, and multiplication; i.e., it is a ring. It is known (see [7, Proposition 1.4.2]) that $\mathbb{Z}[\rho]$ is a Euclidean domain with the norm

$$\mathcal{N}\colon \mathbb{Z}[\rho] \longrightarrow \mathbb{N}, \quad x + y\rho \longmapsto x^2 + y^2 - xy.$$

Note that $\mathcal{N}(x+y\rho) = (x+y\rho)(\overline{x+y\rho})$ since $\overline{x+y\rho} = (x-y) - y\rho$. Units of $\mathbb{Z}[\rho]$ (i.e., elements with the unitary norm) are $\{\pm 1, \pm\rho, \pm\rho^2\}$.

For every nonzero element $\alpha \in \mathbb{Z}[\rho]$ we can consider the corresponding quotient ring

$$\mathbb{Z}[\rho]_\alpha = \{\beta \pmod{\alpha} \mid \beta \in \mathbb{Z}[\rho]\}.$$

Elements of the ring $\mathbb{Z}[\rho]_\alpha$ are equivalence classes of EJ-integers modulo the ideal generated by α. The ring is clearly finite, as is shown by the following result.

Theorem 1. *Let $0 \neq \alpha = a + b\rho \in \mathbb{Z}[\rho]$. Then the quotient ring $\mathbb{Z}[\rho]_\alpha$ contains $\mathcal{N}(\alpha)$ elements.*

Moreover, in some particular cases we can state the following ring isomorphism.

Theorem 2. *Let $\alpha = a + b\rho \in \mathbb{Z}[\rho]$ be such that $\gcd(a, b) = 1$. Consider $N = \mathcal{N}(\alpha)$. Then the rings $\mathbb{Z}_N$ and $\mathbb{Z}[\rho]_\alpha$ are isomorphic.*

Proof. It is sufficient to check that the mapping

$$f\colon \mathbb{Z}_N \longrightarrow \mathbb{Z}[\rho]_\alpha, \quad n \longmapsto n \pmod{\alpha},$$

is a ring isomorphism. $\triangle$

We also need the following trivial consequence of Theorem 1.

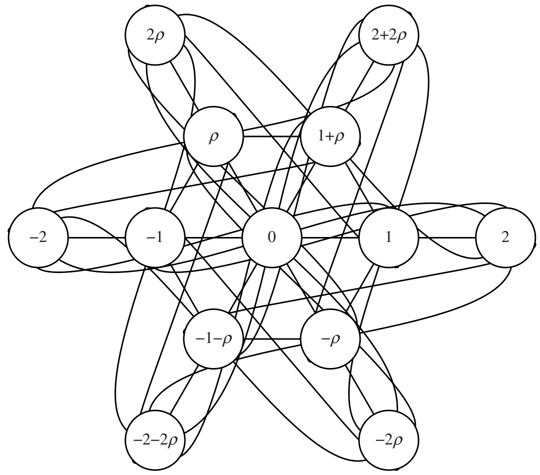

Fig. 1. The Eisenstein–Jacobi graph $EJ_{3+4\rho}$.

Corollary. *Let* $0 \neq \alpha = a + b\rho \in \mathbb{Z}[\rho]$.

1. *Let* $\beta \in \mathbb{Z}[\rho]$ *divide* α. *Then the ideal generated by* β, *i.e.*, $(\beta) \subset \mathbb{Z}[\rho]_\alpha$, *contains* $\dfrac{\mathcal{N}(\alpha)}{\mathcal{N}(\beta)}$ *elements.*
2. *Let* $\beta \in \mathbb{Z}[\rho]$ *do not divide* α, *and let* $\eta = \gcd(\beta, \alpha)$. *Then the ideal* $(\beta) \subset \mathbb{Z}[\rho]_\alpha$ *is generated by* η *and contains* $\dfrac{\mathcal{N}(\alpha)}{\mathcal{N}(\eta)}$ *elements.*

Proof. The corollary is a consequence of Theorem 1 and the Third ring isomorphism theorem (see [8, Corollary 5.9]). $\triangle$

3. EISENSTEIN–JACOBI GRAPHS

Eisenstein–Jacobi graphs are Cayley graphs over quotient rings of the ring of EJ-integers. Therefore, they are connected and vertex-symmetric. Since their adjacency pattern is determined by units of the ring $\mathbb{Z}[\rho]$, they are regular graphs of degree six. We formally define Eisenstein–Jacobi graphs as follows.

Definition 1. Let $\alpha = a + b\rho \in \mathbb{Z}[\rho]$. Consider the quotient ring $\mathbb{Z}[\rho]_\alpha$. We denote the corresponding *Eisenstein–Jacobi graph* (EJ-graph for short) by $EJ_\alpha = (V, E)$, where V is the set of vertices and E the set of edges defined as follows:

- $V = \mathbb{Z}[\rho]_\alpha$; i.e., vertices are all elements of the quotient ring;
- $E = \{(\beta, \gamma) \in V \times V \mid (\gamma - \beta) \equiv \pm 1, \pm\rho, \pm\rho^2 \pmod{\alpha}\}$; i.e., a pair of vertices is connected by an edge if and only if their difference modulo α is a unit of $\mathbb{Z}[\rho]$.

Figure 1 shows the EJ-graph generated by $\alpha = 3 + 4\rho$.

In some cases, EJ-graphs are degree-six circulant graphs. By definition, a degree six *circulant graph* with N vertices and jumps j_1, j_2, and j_3, denoted by $C_N(j_1, j_2, j_3)$, is an undirected graph in which each vertex n, $0 \le n \le N-1$, is adjacent to exactly six vertices: $n \pm j_1 \pmod{N}$, $n \pm j_2 \pmod{N}$, and $n \pm j_3 \pmod{N}$.

The following theorem characterizes EJ-graphs that are degree-six circulants.

Theorem 3. *Let* $\alpha = a + b\rho \in \mathbb{Z}[\rho]$ *be such that* $\gcd(a, b) = 1$. *Denote* $N = \mathcal{N}(\alpha)$. *Then the graphs* EJ_α *and* $C_N(a, b, a-b)$ *are isomorphic.*

Proof. Any integer n can be represented in the form $n \equiv bx - ay \pmod N$ because $\gcd(a,b) = 1$. Let us prove that the ring homomorphism defined as

$$\Phi\colon \mathbb{Z}_N \longrightarrow \mathbb{Z}[\rho]_\alpha, \quad n \equiv bx - ay \ (\mathrm{mod}\ N) \longmapsto x + y\rho \ (\mathrm{mod}\ \alpha), \tag{1}$$

is a graph isomorphism.

The mapping Φ is well defined: if $bx - ay \equiv 0 \pmod N$, then there exist $k, t \in \mathbb{Z}$ such that

$$\begin{cases} x = k(a+b) + at, \\ y = k(2b - a) + bt. \end{cases}$$

Then $\Phi(bx - ay) = x + y\rho = k((a+b) + (2b-a)\rho) + t(a + b\rho) = k(1-\rho)(a+b\rho) + t(a+b\rho) \equiv 0 \pmod \alpha$.

The mapping Φ is injective: if $\Phi(bx - ay) = x + y\rho \equiv 0 \pmod \alpha$, then there exist $c, d \in \mathbb{Z}$ such that $x + y\rho = (ac - db) + (ad + bc - db)\rho$, and

$$\begin{cases} x = ac - bd, \\ y = ad + bc - db. \end{cases}$$

Hence we obtain that $bx - ay = -d(a^2 + b^2 - ab) = -dN \equiv 0 \pmod N$.

Finally, since both sets have the same number of elements, Φ is a bijection. If two nodes, j and h, are adjacent in the circulant, then $j - h \equiv \pm a, \pm b, \pm(a-b) \pmod N$. Hence, $\Phi(j) - \Phi(h) = \Phi(j-h) \equiv \pm\Phi(a), \pm\Phi(b), \pm\Phi(a-b)$. Since $\Phi(a) = \rho$, $\Phi(b) = -1$, and $\Phi(a-b) = \Phi(a) - \Phi(b) = 1 + \rho$, the nodes $\Phi(j)$ and $\Phi(h)$ are adjacent in EJ_α. $\triangle$

There are not too much works in the literature devoted to degree-six circulant graphs. The most relevant results are presented in [9, 10].

In [9], the author focused on graphs $C_N(1, 3k+1, 3k+2)$ with the number of nodes $N = 3k^2 + 3k + 1$. The parameter k is the *graph diameter*, i.e., the maximum distance between a pair of vertices. Let us show that this family of graphs is contained in the family of EJ-graphs.

Theorem 4. *Let k be a positive integer and $N = 3k^2 + 3k + 1$. Then the graphs $EJ_{k+(2k+1)\rho}$ and $C_N(1, 3k+1, 3k+2)$ are isomorphic.*

Proof. By Theorem 3, the graphs $EJ_{k+(2k+1)\rho}$ and $C_N(k, k+1, 2k+1)$ are isomorphic. The isomorphism mapping between the graphs $C_N(1, 3k+1, 3k+2)$ and $C_N(k, k+1, 2k+1)$ is defined as the ring automorphism

$$f\colon \mathbb{Z}_N \longrightarrow \mathbb{Z}_N, \quad 1 \longmapsto k. \quad \triangle$$

In [10], the authors focused on degree-six circulants $C_N(a, b, a+b)$ defined for any number of vertices N, where $0 < a < b < \left\lfloor \frac{N}{2} \right\rfloor$. They studied some cases where this choice of parameters minimizes the diameter. Clearly, this family contains the family of EJ-graphs if $\gcd(a,b)=1$.

One might be curious about what sort of graphs are the EJ-graphs with $\gcd(a,b) \neq 1$. The problem of complete characterization of these graphs would be a future research topic. However, we can assert that an EJ-graph is circulant only if $\gcd(a,b) = 1$, since this is the only case where the quotient ring is cyclic. In addition, there are other interesting particular cases, where EJ-graphs contain square tori graphs of degree four. A *torus graph* $\mathbb{T}_b = (V, E)$ with b^2 vertices is defined by a set of vertices $V = \mathbb{Z}_b \times \mathbb{Z}_b$ where two vertices $(n, m), (n', m') \in V$ are adjacent if and only if either $n = n'$ and $m - m' \equiv \pm 1 \pmod b$ or $m = m'$ and $n - n' \equiv \pm 1 \pmod b$.

Theorem 5. *Let $0 \neq \alpha = a \in \mathbb{Z}$. Then the EJ-graphs EJ_a and $EJ_{a\rho}$ are isomorphic and contain a square torus graph $\mathbb{T}_{|a|}$ with a^2 vertices.*

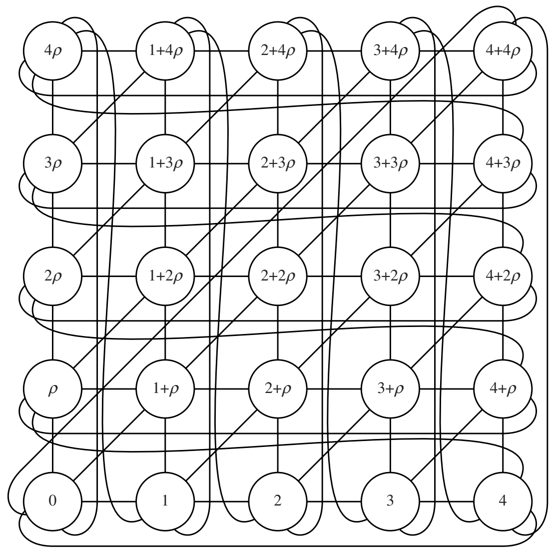

Fig. 2. The Eisenstein–Jacobi graph for $\alpha = 5\rho$.

Proof. We may assume that $0 < a \in \mathbb{Z}$. It is straightforward that $\mathbb{Z}[\rho]_a$ and $\mathbb{Z}[\rho]_{a\rho}$ are isomorphic rings. Let a mapping f be defined as

$$f\colon \mathbb{Z}_a \times \mathbb{Z}_a \longrightarrow \mathbb{Z}[\rho]_a, \quad (x, y) \longmapsto x + y\rho \pmod{a}. \quad \triangle$$

Then it can be proved that f is an embedding of the torus graph of side a in both EJ_a and $EJ_{a\rho}$.

As an example, Fig. 2 shows the EJ-graph for $\alpha = 5\rho$. Note that in this case the EJ-graph contains a square torus graph (or Lee graph) of side 5.

Finally, we review dense EJ-graphs. These are graphs with the maximum number of vertices for a given diameter t. They are generated by $t + (2t+1)\rho$ or its conjugate. Note that they have $N = 3t^2 + 3t + 1$ vertices. By Theorem 3, they are isomorphic to the corresponding circulant graph of degree six. In Fig. 3, a dense EJ-graph with diameter $t = 4$ is represented.

4. PERFECT DOMINATING SETS IN EISENSTEIN–JACOBI GRAPHS

In this section we study the existence of perfect t-dominating sets over EJ-graphs. Such sets are ideals of $\mathbb{Z}[\rho]_\alpha$. They directly lead to perfect codes over quotient rings of the ring of EJ-integers whose metric is the EJ-graph distance. Similar codes were previously considered in [3].

Let us first introduce some necessary definitions.

Definition 2. Let $\alpha = a + b\rho \in \mathbb{Z}[\rho]$. We denote by $D_\alpha(\beta, \gamma)$ the distance between vertices β and γ in EJ_α. This distance can be expressed as

$$D_\alpha(\beta, \gamma) = \min\{|x| + |y| + |z| :\, x + y\rho + z\rho^2 \equiv (\gamma - \beta) \pmod{\alpha}\}.$$

Example 1. Let $\alpha = 4 + 3\rho$. Figure 4 illustrates the distance between the points $A = -1$ and $B = 2 + 2\rho$ in the graph $EJ_{4+3\rho}$. The distance between A and B in the ring $\mathbb{Z}[\rho]$ is 3. However, in the quotient ring $\mathbb{Z}[\rho]_{4+3\rho}$ the distance is $D_{4+3\rho}(A, B) = 1$.

Given $0 \neq \alpha = a + b\rho \in \mathbb{Z}[i]$, an integer $t > 0$, and a vertex $\eta \in EJ_\alpha$, a ball of radius t centered in η embedded in EJ_α is the set

$$B_t(\eta) = \{\gamma \in EJ_\alpha \mid D_\alpha(\gamma, \eta) \le t\}.$$

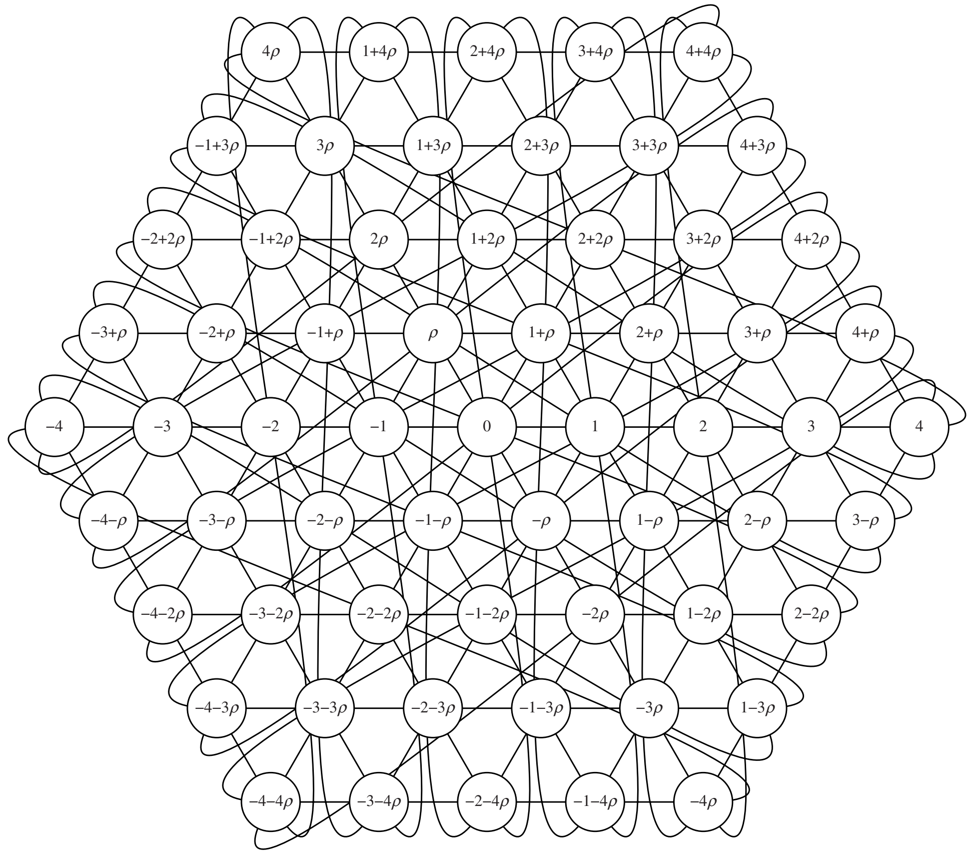

Fig. 3. Dense Eisenstein–Jacobi graph with diameter $t = 4$ (for $\alpha = 4 + 9\rho$).

For any β, the cardinality of $B_t(\beta)$ is $1 + \sum\limits_{d=1}^{t} 6d = 3t^2 + 3t + 1$.

A vertex η of EJ_α is said to *t-dominate* a vertex $\beta \in EJ_\alpha$ if $\beta \in B_t(\eta)$. A vertex subset S is called a *perfect t-dominating set* if every vertex of EJ_α is t-dominated by a unique vertex in S. Note that if S is a perfect t-dominating set, then $S + \beta \pmod{\alpha} = \{\eta + \beta \pmod{\alpha} \mid \eta \in S\}$ is also a perfect t-dominating set. Consequently, we may assume that $0 \in S$.

The following theorem states a sufficient condition for the existence of a perfect t-dominating set in an EJ-graph containing the zero vertex.

Theorem 6. *Let $0 \neq \alpha = a + b\rho \in \mathbb{Z}[\rho]$, and let t be a positive integer.*

1. *If $\beta = t + (2t+1)\rho$ divides α, then the ideal $S = (\beta) \subseteq \mathbb{Z}[\rho]_\alpha$ is a perfect t-dominating set in EJ_α.*
2. *If $-\overline{\beta} = (t+1) + (2t+1)\rho$ divides α, then the ideal $S = (-\overline{\beta}) = (\overline{\beta}) \subseteq \mathbb{Z}[\rho]_\alpha$ is a perfect t-dominating set in EJ_α.*

Proof. Let us prove the first claim of the Theorem. The second one is proved similarly.

Due to the corollary, the ideal (β) has $\dfrac{\mathcal{N}(\alpha)}{\mathcal{N}(\beta)}$ elements, where $\mathcal{N}(\beta) = 3t^2 + 3t + 1$.

First of all, we prove that $\beta = t + (2t+1)\rho$ is such that $D_\alpha(\beta, 0) = 2t + 1$. Obviously, since $\beta \equiv (t+1)\rho + t(1+\rho) \pmod{\alpha}$ and $|t+1| + |t| = 2t+1$, we have $D_\alpha(\beta, 0) \leq 2t+1$. Next, we show that this distance is exactly $2t+1$. Otherwise, the representation $\beta \equiv x + y\rho + z\rho^2 \pmod{\alpha}$ would imply $|x| + |y| + |z| \leq 2t$. Therefore, there exists $\eta \in \mathbb{Z}[\rho]$ such that $\eta\beta = x + y\rho + z\rho^2$ in $\mathbb{Z}[\rho]$. Since

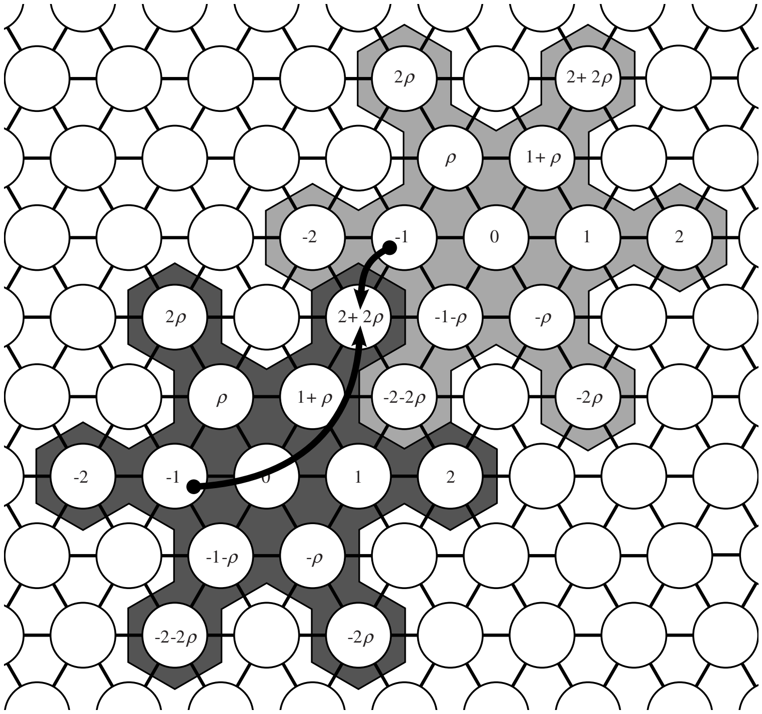

Fig. 4. Distance $D_{4+3\rho}$ between points -1 and $2+2\rho$.

$\mathcal{N}(\beta) > 3t^2$ and $\mathcal{N}(x+y\rho+z\rho^2) \le 24t^2$, we obtain $\mathcal{N}(\eta) < 8$. Now, if $\eta = c + d\rho$ with $c, d \in \mathbb{Z}$, we have $\eta\beta = (ct - d(2t+1)) + (c(2t+1) - d(t+1))\rho$ and therefore

$$\begin{cases} x - z = ct - d(2t+1), \\ y - z = c(2t+1) - d(t+1). \end{cases} \tag{2}$$

Hence we get

$$\begin{cases} |x| + |z| \ge |ct - d(2t+1)|, \\ |y| + |z| \ge |c(2t+1) - d(t+1)|. \end{cases} \tag{3}$$

Since $\mathcal{N}(\eta) \le 7$, we have three different cases to be studied separately.

- If $0 \le |c| < |d|$, then $|x| + |z| \ge |ct - d(2t+1)| \ge |d||2t+1| > 2t$.
- If $0 \le |d| < |c|$, then $|y| + |z| \ge |c(2t+1) - d(t+1)| \ge |c|(2t+1) > 2t$.
- If $0 \ne |c| = |d|$, we have two possibilities. If $c = -d$, any of the preceding cases yields $|x|+|z| > 2t$ or $|y| + |z| > 2t$. If $c = d$, from relation (2) we get $|x| + |y| \ge |c(t+1) + dt| = |2ct + c| = |c|(2t+1) > 2t$.

In any case we have $|x| + |y| + |z| > 2t$, which is a contradiction.

Finally, let us prove that all elements in the ideal (β) are at distance greater than or equal to $2t+1$ from each other. Assume the contrary. Then there exists $\eta' \in \mathbb{Z}[\rho]$ such that $D_\alpha(\eta'\beta, 0) < 2t+1$. Hence, $\eta'\beta \equiv x + y\rho + z\rho^2 \pmod{\alpha}$ with $|x| + |y| + |z| \le 2t$; i.e., there exists $\eta \in \mathbb{Z}[\rho]$ such that $\eta\beta = x + y\rho + z\rho^2$ in $\mathbb{Z}[\rho]$. We have already proved that this condition yields a contradiction, which completes the proof. $\triangle$

Hence, if $\alpha = a + b\rho \in \mathbb{Z}[\rho]$ and $\beta = t + (2t+1)\rho$ divides α, then the ideal $\mathcal{C} = (\beta)$ is a perfect code over $\mathbb{Z}[i]_\alpha$ with respect to the metric induced by the associated EJ-graph. Note that this distance differs from the Mannheim metric introduced in [2]. Our codes coincide with the codes

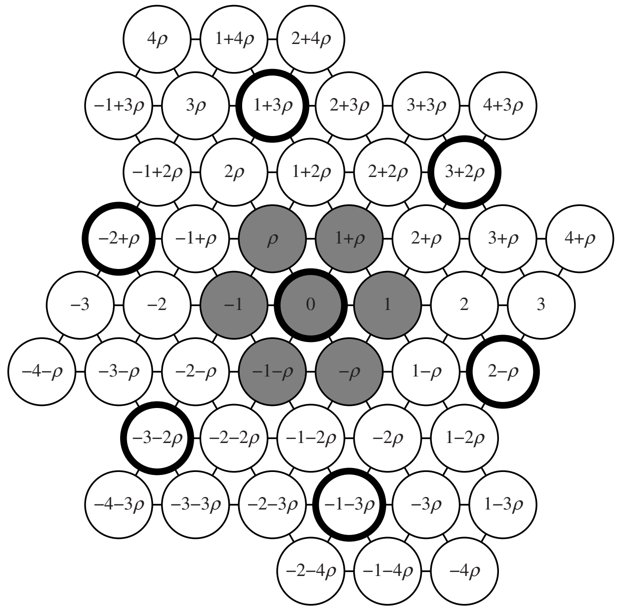

Fig. 5. Perfect 1-dominating set in $EJ_{-8-3\rho}$.

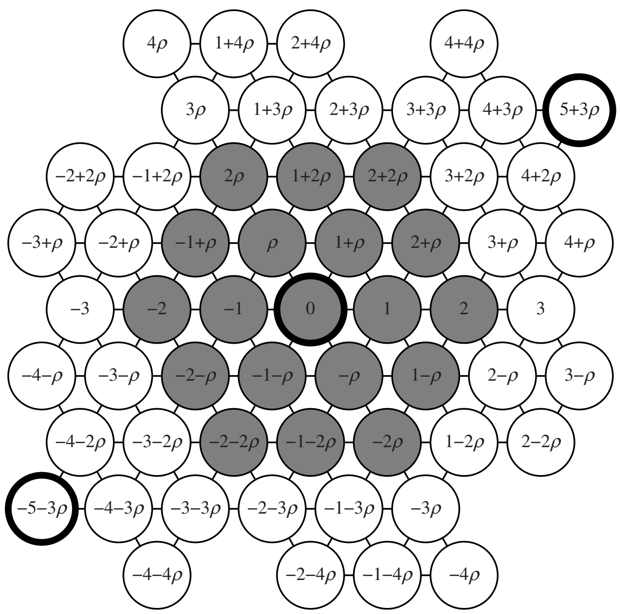

Fig. 6. Perfect 2-dominating set in $EJ_{-8-\rho}$.

proposed in [3] and obtained as lattices over flat tori. Our approach seems to be simpler and more understandable. As an example, we provide two perfect error-correcting codes constructed by our method.

Example 2. Let $\alpha_1 = (1+3\rho)^2 = -8-3\rho$. By Theorem 6, the ideal generated by $1+3\rho$ gives vertices of a perfect 1-dominating set in $EJ_{-8-3\rho}$. In Fig. 5, a plane representation of this graph and its perfect 1-dominating set are given. Now, let $\alpha_2 = (1+2\rho)(2+5\rho) = -8-\rho$. Obviously, $2+5\rho$ is a divisor of α_2. Hence, by Theorem 6, the ideal generated by $2+5\rho$ is a perfect 2-dominating set in $EJ_{-8-\rho}$. This set is represented in Fig. 6. To simplify the figures, wrap-around edges are omitted.

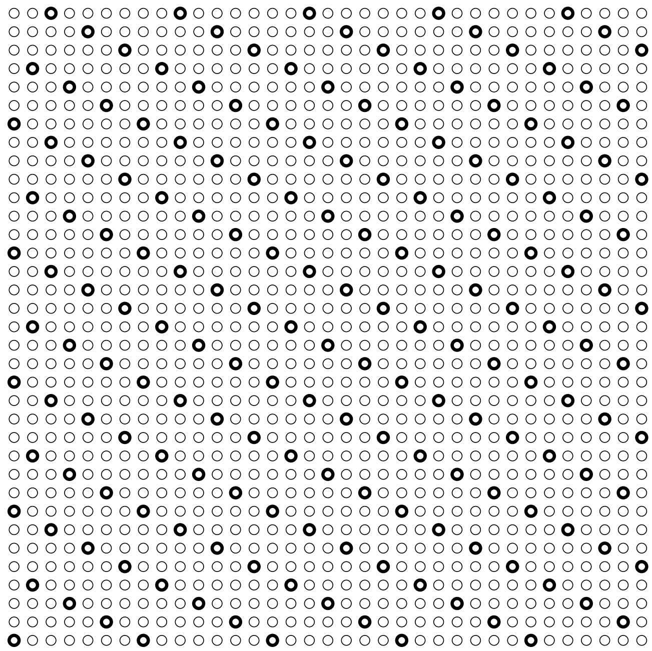

Fig. 7. Perfect 1-error-correcting code over $\mathbb{Z}_{35} \times \mathbb{Z}_{35}$ for the Eisenstein–Jacobi metric.

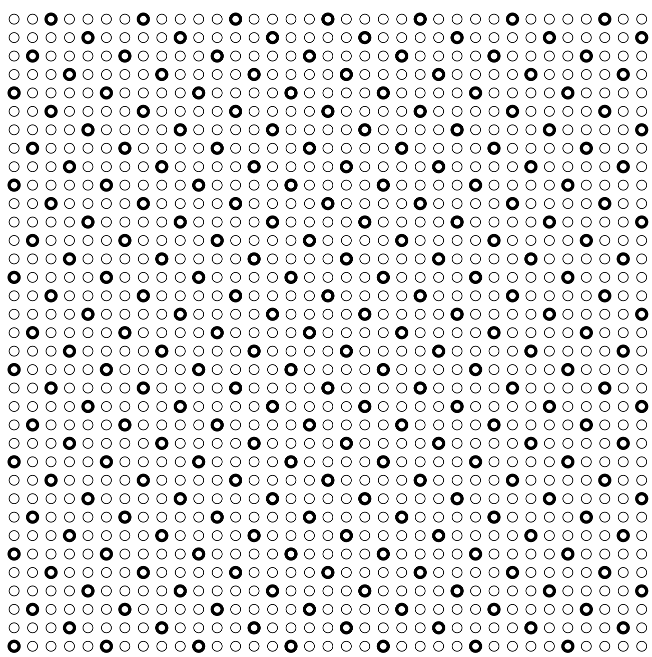

Fig. 8. Perfect 1-error-correcting code over $\mathbb{Z}_{35} \times \mathbb{Z}_{35}$ for the Lee metric.

5. EISENSTEIN–JACOBI GRAPHS AND THE LEE METRIC

In Section 4 we have introduced EJ-graphs over the quotients of the ring of EJ-integers. These graphs model hexagonal constellations. The construction of perfect dominating sets over these graphs allow us to define perfect codes for the induced graph distance in a very simple way.

In Theorem 5 we have considered the case where the EJ-graph generated by an integer $\alpha = a \in \mathbb{Z}$ (respectively, $\alpha = a\rho \in \mathbb{Z}[\rho]$) has an embedded square torus graph of side $|a|$. It is obvious that the metric induced by the torus graph is the Lee metric. As a consequence, we have that the Lee metric is an upper bound for the EJ-graph metric. Thus, we can state the following result.

Lemma. *Let t be a positive integer, and let $a = 3t^2 + 3t + 1$. Then the ideal $\mathcal{C} = (t + (2t+1)\rho)$ is a perfect code over $\mathbb{Z}_a \times \mathbb{Z}_a$ for the EJ-graph metric. Moreover, if we denote by $w_L(x + y\rho)$ the Lee weight induced by the embedded torus graph, we have*

$$d_{\max}(\mathcal{C}) := \max\{D_a(c, 0) \mid c \in \mathcal{C}\} \le \max\{w_L(c) \mid c \in \mathcal{C}\}.$$

Note that the diameter of the square torus graph of side a is either a or $a - 1$, depending on the parity of a. This is a coarse upper bound for $\max\{w_L(c) \mid c \in \mathcal{C}\}$. Next, we are going to compare the maximum distance of these EJ-codes with the maximum distance of the Lee error-correcting codes constructed by Golomb in [11].

On the one hand, it is well known [12] that given a positive integer a there exists a perfect t-Lee-error-correcting code over $\mathbb{Z}_a \times \mathbb{Z}_a$ if the integer $2t^2 + 2t + 1$ divides a. Let $a = k(2t^2 + 2t + 1)$. Then the codewords are of the form $[C_0 + r(2t^2 + 2t + 1), C_1 + s(2t^2 + 2t + 1)]$, where $0 \le r, s < k$, and $C = [C_0, C_1]$ is a codeword in a perfect t-Lee-error-correcting code over the alphabet of integers modulo $2t^2 + 2t + 1$. This code can be considered as a particular case of a perfect code over Gaussian integers presented in [13].

On the other hand, if there exists a positive integer t such that $3t^2 + 3t + 1$ divides a, then, as we have seen, there exists a perfect t-EJ-error-correcting code over $\mathbb{Z}_a \times \mathbb{Z}_a$. For example, for $a = 35$ we can construct perfect single-error-correcting EJ and Lee codes. These codes are presented in Figs. 7 and 8. The first is a perfect 1-dominating set over EJ_{35}, which is a perfect 1-error-correcting code over $\mathbb{Z}_{35} \times \mathbb{Z}_{35}$ for the EJ-graph metric. The maximum distance of this code is 22. The second is a perfect 1-error-correcting code for the Lee metric with the maximum distance equal to 33.

In these examples we have considered perfect codes over the same space $\mathbb{Z}_a \times \mathbb{Z}_a$ and with the same error-correcting capacity but in different metrics. Thus, the obtained codes have different number of codewords. If one considers codes in different metrics over the same alphabet and of the same cardinality, their error-correcting capacity will be different. For instance, taking $a = 61$, we can construct a 4-EJ-error-correcting code and a 5-Lee-error-correcting code over $\mathbb{Z}_{61} \times \mathbb{Z}_{61}$.

REFERENCES

1. Huber, K., Codes over Gaussian Integers, *IEEE Trans. Inform. Theory*, 1994, vol. 40, no. 1, pp. 207–216.

2. Huber, K., Codes over Eisenstein–Jacobi integers, in *Finite Fields: Theory, Applications, and Algorithms (Las Vegas, 1993)*, Contemp. Math., vol. 168, Providence: AMS, 1994, pp. 165–179.

3. Costa, S.I.R., Muniz, M., Agustini, E., and Palazzo, R., Graphs, Tessellations, and Perfect Codes on Flat Tori, *IEEE Trans. Inform. Theory*, 2004, vol. 50, no. 10, pp. 2363–2377.

4. Martínez, C., Beivide, R., Gutierrez, J., and Gabidulin, E., On the Perfect t-Dominating Set Problem in Circulant Graphs and Codes over Gaussian Integers, in *Proc. 2005 IEEE Int. Sympos. on Information Theory (ISIT'05), Adelaide, Australia*, 2005, pp. 254–258.

5. Martínez, C., Beivide, R., and Gabidulin, E., Perfect Codes for Metrics Induced by Circulant Graphs, *IEEE Trans. Inform. Theory*, 2007, vol. 53, no. 9, pp. 3042–3052.

6. Martínez, C., Stafford, E., Beivide, R., and Gabidulin, E., Perfect Codes over Eisenstein–Jacobi Graphs, in *Proc. 10th Int. Workshop on Algebraic and Combinatorial Coding Theory, Zvenigorod, Russia, 2006*, pp. 190–193.

7. Ireland, K. and Rosen, M., *A Classical Introduction to Modern Number Theory*, New York: Springer, 1982. Translated under the title *Klassicheskoe vvedenie v sovremennuyu teoriyu chisel*, Moscow: Mir, 1987.

8. Hungerford, T.W., *Algebra*, New York: Holt, Rinehart & Winston, 1974.

9. Shin, K.G., HARTS: A Distributed Real-Time Architecture, *Computer*, 1991, vol. 24, no. 5, pp. 25–35.

10. Aguiló, F., Fiol, M.A., and García, C., Triple Loop Networks with Small Transmission Delay, *Discrete Math.*, 1997, vol. 167/168, pp. 3–16.

11. Golomb, S.W. and Welch, L.R., Perfect Codes in the Lee Metric and the Packing of the Polyominoes, *SIAM J. Appl. Math.*, 1970, vol. 18, no. 2, pp. 302–317.

12. Berlekamp, E.R., *Algebraic Coding Theory*, New York: McGraw-Hill, 1984. Translated under the title *Algebraicheskaya teoriya kodirovaniya*, Moscow: Mir, 1971.

13. Martínez, C., Moretó, M., Beivide, R., and Gabidulin, E., A Generalization of Perfect Lee Codes over Gaussian Integers, in *Proc. 2006 IEEE Int. Sympos. on Information Theory (ISIT'06), Seattle, USA*, 2006, pp. 1070–1074.