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CODING THEORY

Multicomponent Codes with Maximum Code Distance[1]

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Abstract—We consider subspace codes, called multicomponent codes with zero prefix (MZP codes), whose subspace code distance is twice their dimension. We find values of parameters for which the codes are of the maximum cardinality. We construct combined codes where the last component of the multicomponent code is the code from [1] found by exhaustive search for particular parameter values. As a result, we obtain a family of subspace codes with maximum cardinality for a number of parameters. We show that the family of maximum-cardinality codes can be extended by using dual codes.

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1. INTRODUCTION

We use the following definitions and notation. Denote by $W = GF(q)^n$ an n-dimensional space over the finite field $GF(q)$. Let $W(n, m)$ be the set of m-subspaces of the ambient space W, which is referred to as a Grassmannian. The number of m-subspaces is

$$|W(n,m)| = \begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^n-1)(q^n-q)\dots(q^n-q^{m-1})}{(q^m-1)(q^m-q)\dots(q^m-q^{m-1})}.$$

Let $U, V \in W$ be two subspaces of dimensions m and r, respectively.

The intersection of two subspaces is defined as the subspace X with the *maximal* dimension such that $X \subseteq U$ and $X \subseteq V$. It is denoted by $X = U \cap V$. It is clear that $0 \le \dim X \le \min\{\dim U, \dim V\}$. If $\dim X = 0$, then subspaces U and V intersect *trivially*, i.e., have only the zero vector in common.

The *union* of two subspaces is defined as the subspace Y of the *minimal* dimension such that $U \subseteq Y$ and $V \subseteq Y$. It is denoted by $Y = U \uplus V$. It is clear that $\dim Y = \dim U + \dim V - \dim(U \cap V)$.

The *subspace distance* between subspaces U and V is defined as

$$d_{\rm sub}(U,V) = \dim Y - \dim X = \dim(U \uplus V) - \dim(U \cap V).$$

The following relations are valid:

$$d_{\rm sub}(U,V) = \dim U + \dim V - 2\dim(U \cap V) = m + r - 2\dim(U \cap V),$$
$$d_{\rm sub}(U,V) = 2\dim(U \uplus V) - m - r.$$

In particular, if the dimension of subspaces is the same, i.e., $m = r$, then the distance between subspaces is an even number:

$$d_{\rm sub}(U,V) = 2m - 2\dim(U \cap V) = 2\delta.$$

Here $\delta = m - \dim(U \cap V)$.

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A subspace code is any set of subspaces $U \in W$. If all code subspaces have equal dimensions m, i.e., belong to the same Grassmannian $W(n, m)$, such a code is said to be of dimension m.

We denote by $[n, M, d_{\mathrm{sub}} = 2\delta, m]$ a code in the subspace metric such that n is the code length, $M = M(n, d_{\mathrm{sub}} = 2\delta, m)$ is the number of code words (subspaces), d_{sub} is the code distance, and m is the dimension.

A subspace of dimension m can be specified as the space spanned by rows of a full-rank matrix of size $m \times n$ over $GF(q)$. Let a subspace U be generated by an $m \times n$ matrix $\boldsymbol{X}_1$, and let V be generated by an $m \times n$ matrix $\boldsymbol{X}_2$. Then

$$\dim(U \uplus V) = \mathrm{Rk}\left(\begin{bmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_2\end{bmatrix}\right) = \dim(U) + \dim(V) - \dim(U \cap V),$$

$$d_{\mathrm{sub}}(U, V) = 2\,\mathrm{Rk}\left(\begin{bmatrix}\boldsymbol{X}_1\\ \boldsymbol{X}_2\end{bmatrix}\right) - 2m = 2\delta.$$

The maximum possible subspace distance for a code of dimension m is $d_{\mathrm{sub}} = 2m$. In theory of finite geometries, such codes are defined as sets of *pairwise trivially intersecting* subspaces of dimension m, known as *spreads* or *partial spreads*.

The goal of the present paper is finding the parameters of MZP codes that have the cardinality coinciding with the upper bound on the cardinality of subspace codes.

2. BOUNDS ON THE CARDINALITY OF SUBSPACE CODES

The cardinality is one of the the main characteristics of a code: the greater the cardinality, the greater the transmission rate. Let us give basic presently known results on estimating the bounds on the cardinality of subspace codes.

2.1. Lower Cardinality Bounds

In the literature, much attention is given to constructing subspace codes of large cardinality. For example, such are the Silva–Koetter–Kschischang (SKK) subspace network codes [2, 3], as well as Gabidulin–Bossert multicomponent subspace network codes with zero prefix (MZP codes) [4–6].

Matrices of an SKK-code have the so-called lifting structure, consisting of the concatenation of the identity matrix $\boldsymbol{I}_m$ of order m and a matrix of a *rank* code $\boldsymbol{M}$ of size $m \times (n - m)$ with entries M_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n - m$, having *rank* distance $d_{\mathrm{r}} = \delta$:

$$\boldsymbol{X} = \begin{bmatrix}\boldsymbol{I}_m & \boldsymbol{M}\end{bmatrix} = \begin{bmatrix} 1 & 0 & \ldots & 0 & M_{11} & M_{12} & \ldots & M_{1(n-m)}\\ 0 & 1 & \ldots & 0 & M_{21} & M_{22} & \ldots & M_{2(n-m)}\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ldots & \vdots\\ 0 & 0 & \ldots & 1 & M_{m1} & M_{m2} & \ldots & M_{m(n-m)}\end{bmatrix}.$$

The code subspaces are subspaces U spanned by rows of the matrices $\boldsymbol{X}$. Let a code subspace U_1 be generated by a matrix $\boldsymbol{X}_1 = \begin{bmatrix}\boldsymbol{I}_m & \boldsymbol{M}'\end{bmatrix}$, and a code subspace U_2, by a matrix $\boldsymbol{X}_2 = \begin{bmatrix}\boldsymbol{I}_m & \boldsymbol{M}''\end{bmatrix}$. Then

$$d_{\mathrm{sub}}(U_1, U_2) = 2\,\mathrm{Rk}\left(\begin{bmatrix}\boldsymbol{I}_m & \boldsymbol{M}'\\ \boldsymbol{I}_m & \boldsymbol{M}''\end{bmatrix}\right) - 2m = 2\,\mathrm{Rk}(\boldsymbol{M}'' - \boldsymbol{M}') \geq 2\delta. \tag{1}$$

For $m \leq n - m$, the maximum cardinality of a matrix code consisting of $m \times (n - m)$ matrices and having rank distance δ is (see [7])

$$M_r = q^{(n-m)(m-\delta+1)}.$$

Hence it follows that the cardinality of an SKK code with subspace distance $d_{\text{sub}} = 2\delta$ is

$$M_{\text{skk}}(2\delta) = M(n, d_{\text{sub}} = 2\delta, m) = M_r = q^{(n-m)(m-\delta+1)}.$$

An SKK code with maximum possible code distance $d_{\text{sub}} = 2m$ has cardinality

$$M_{\text{skk}}(2m) = q^{(n-m)}.$$

A class of multicomponent codes with zero prefix (MZP codes) was proposed in 2008 in [4]. Let the length of code matrices of a code of dimension m be $n = rm + s$, where $r \geq 1$, $0 \leq s \leq m - 1$. The code consists of r components:

$$\begin{array}{ll} \text{1st component:} & [\boldsymbol{I}_m \;\; \boldsymbol{M}_1], \\ \text{2nd component:} & [\mathbf{0}_m \;\; \boldsymbol{I}_m \;\; \boldsymbol{M}_2], \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ (r-1)\text{st component:} & [\mathbf{0}_m \;\; \mathbf{0}_m \;\; \mathbf{0}_m \;\; \ldots \;\; \boldsymbol{I}_m \;\; \boldsymbol{M}_{r-1}], \\ r\text{th component:} & [\mathbf{0}_m \;\; \mathbf{0}_m \;\; \mathbf{0}_m \;\; \mathbf{0}_m \;\; \ldots \;\; \boldsymbol{I}_m \;\; \boldsymbol{M}_r]. \end{array}$$

The subspaces generated by different components trivially intersect by the constructions; therefore, pairwise subspace distances between them are maximal and are equal to $2m$.

Let us choose matrices $\boldsymbol{M}_i$ inside each fixed component so that the *rank* distance between the different matrices $\boldsymbol{M}_i'$ and $\boldsymbol{M}_i''$ is equal to m.

Let the first component be an SKK code with *subspace* distance $d_{\text{sub}} = 2m$. To this end, the matrices $\boldsymbol{M}_1$ of size $m\times(n-m) = m\times((r-1)m+s)$ are chosen from a matrix code with rank distance $d_r = m$. The cardinality of this code (the number of different matrices) is $q^{n-m} = q^{(r-1)m+s}$.

The second component is the concatenation of three matrices: the zero matrix prefix $\mathbf{0}_m$ of order m, the identity matrix $\boldsymbol{I}_m$ of order m, and a matrix $\boldsymbol{M}_2$ of size $m\times(n-2m) = m\times(r-2)m+s$ of a rank code $\mathcal{M}_2$ with rank distance $d_r = m$. The cardinality of this code (and of the whole component) is $|\mathcal{M}_2| = q^{n-2m} = q^{(r-2)m+s}$.

The $(r-1)$st component is the concatenation of the following matrices: $r-2$ zero matrix prefixes $\mathbf{0}_m$ of order m, the identity matrix $\boldsymbol{I}_m$ of order m, and a matrix $\boldsymbol{M}_{r-1}$ of size $m \times (n-(r-1)m) = m \times rm + s$ of a rank code $\mathcal{M}_{r-1}$ with rank distance $d_r = m$. The cardinality of this code (and of the whole component) is $|\mathcal{M}_{r-1}| = q^{n-(r-1)m} = q^{m+s}$.

The last, rth, component is the concatenation of the following matrices: $r-1$ zero matrix prefixes $\mathbf{0}_m$ of order m, the identity matrix $\boldsymbol{I}_m$ of order m, and a matrix $\boldsymbol{M}_r$ of size $m\times(n-rm) = m\times s$ of a rank code. This matrix cannot be of the rank m, since $s \leq m-1$. Hence, one can choose only one (arbitrary) matrix $\boldsymbol{M}_r$. The cardinality of the last component is 1.

The cardinality of the MZP code of length $n = rm + s$ with subspace distance $d_{\text{sub}} = 2m$ is

$$M_{\text{MZP}} = \sum_{i=1}^{r-1} q^{(r-i)m+s} + 1 = \frac{q^n - q^s}{q^m - 1} - (q^s - 1). \tag{2}$$

2.2. Upper Cardinality Bounds

For a code $[n, M, d_{\text{sub}} = 2\delta, m]$ of dimension m with subspace distance 2δ, an upper cardinality bound is obtained in 2003 [8]. It is of the form

$$M \leq M_{\text{Wang}} = \left\lfloor \frac{|W(n, m-\delta+1)|}{|W(m, m-\delta+1)|} \right\rfloor = \left\lfloor \frac{\begin{bmatrix} n \\ m-\delta+1 \end{bmatrix}}{\begin{bmatrix} m \\ m-\delta+1 \end{bmatrix}} \right\rfloor. \tag{3}$$

For spreads, i.e., codes of dimension m with maximum possible subspace distance $d_{\text{sub}} = 2m$, bounds were obtained in many papers, including those published long before 2003 (see, e.g., [9–12], etc.).

Represent the code length in the form

$$n = rm + s, \quad r \geq 2, \quad 0 \leq s \leq m - 1.$$

Then the upper bound (3) can be rewritten as

$$M = M(n, d_{\text{sub}} = 2m, m) \leq M_{\text{Wang}}(n, d_{\text{sub}} = 2m, m) = \frac{q^n - q^s}{q^m - 1}. \tag{4}$$

If $s = 0$, i.e., $n = rm$, then the upper bound $M_{\text{Wang}}(n, d_{\text{sub}} = 2m, m) = \dfrac{q^n - 1}{q^m - 1}$ coincides with the lower existence bound (2) for an MZP code:

$$M_{\text{MZP}} = \sum_{i=1}^{r-1} q^{(r-i)m} + 1 = \frac{q^n - 1}{q^m - 1}.$$

An algorithm for constructing subspace codes with parameters $n = rm$ was proposed in [4,13], and also in other terms, in [9–12].

For lengths $n = rm + s$, $1 \leq s \leq m - 1$, the bound $M_{\text{Wang}}(n, d_{\text{sub}} = 2m, m)$ in (4) is not tight.

For $s = 1$, an upper bound was obtained in [10] in the form

$$M = M(n, d_{\text{sub}} = 2m, m) \leq \frac{q^n - q}{q^m - 1} - (q - 1). \tag{5}$$

In this case, the cardinality (2) of an MZP code coincides with this bound.

For $s \geq 2$, results of [11] allow to modify the bound (4) as follows:

$$M = M(n, d_{\text{sub}} = 2m, m) \leq \frac{q^n - q^s}{q^m - 1} - \lfloor \theta \rfloor - 1, \tag{6}$$

where

$$2\theta = \sqrt{1 + 4q^m(q^m - q^s)} - (2q^m - 2q^s + 1). \tag{7}$$

Let us represent the parameter θ explicitly. To this end, we rewrite $\sqrt{1 + 4q^m(q^m - q^s)}$ as

$$\sqrt{1 + 4q^m(q^m - q^s)} = 2q^m \sqrt{1 - \frac{q^s}{q^m} + \frac{1}{4q^{2m}}} = 2q^m \sqrt{1 - x}, \tag{8}$$

where $x = \dfrac{q^s}{q^m} - \dfrac{1}{4q^{2m}}$. The Tailor series expansion gives

$$\begin{aligned} \sqrt{1 - x} &= 1 - \frac{1}{2}x - \sum_{k \geq 2} \frac{(2k-3)!!}{2^k \cdot k!} x^k \\ &= 1 - \frac{1}{2}\left(\frac{q^s}{q^m} - \frac{1}{4q^{2m}}\right) - \sum_{k \geq 2} \frac{(2k-3)!!}{2^k \cdot k!} \left(\frac{q^s}{q^m} - \frac{1}{4q^{2m}}\right)^k. \end{aligned} \tag{9}$$

Using (9) in (8) and (7) gives

$$\theta = \frac{q^s - 1}{2} + \frac{1}{4q^m} - \sum_{k \geq 2} \frac{(2k-3)!!}{2^k \cdot k!} \left(\frac{q^{m/k+s}}{q^m} - \frac{q^{m/k}}{4q^{2m}}\right)^k. \tag{10}$$

Next, consider the case $q = 2$. Then the first several terms of θ can be written as

$$\theta = 2^{s-1} - \frac{1}{2} - 2^{2s-m-3} - \varepsilon, \tag{11}$$

where ε is a small quantity.

Let $2s < m + 2$. Then from the integer 2^{s-1} there is subtracted a number of absolute value less than 1. Hence, in this case $\lfloor \theta \rfloor = 2^{s-1} - 1$. A similar analysis for the cases $2s = m + 2$ and $2s > m + 2$ shows that

$$\lfloor \theta \rfloor = \begin{cases} 2^{s-1} - 1 & \text{if } 2s < m + 2, \\ 2^{s-1} - 2 & \text{if } 2s = m + 2, \\ 2^{s-1} - 2^{2s-m-3} - 1 & \text{if } 2s > m + 2. \end{cases} \tag{12}$$

For instance, the upper bound (6) for $s = 2$ is of the form

$$M = M(n, d_{\text{sub}} = 2m, m) \le \frac{2^n - 2^2}{2^m - 1} - 2, \tag{13}$$

whereas the lower bound (2) is

$$M = M(n, d_{\text{sub}} = 2m, m) \ge \frac{2^n - 2^2}{2^m - 1} - 3. \tag{14}$$

For the case $s = 1$, the upper and lower bounds coincide. The question for $s \ge 2$ remains open. For one particular case, it is solved in [1]. Let $n = 8$, $m = 3$, and $d_{\text{sub}} = 6$, so that $s = 2$. In this case the existence bound (14) gives $M = \dfrac{2^n - 2^2}{2^m - 1} - 3 = 33$. On the other hand, computer search allowed the authors of [1] to find a code of cardinality 34, which coincides with the upper bound $M = \dfrac{2^n - 2^2}{2^m - 1} - 2 = 34$. We call this code a ZJSSS code (using the first letters of the authors' names); let us consider its properties in more detail.

3. ZJSSS CODE

Code subspaces of dimension $m = 3$ of the ZJSSS code are given by binary generator matrices of size 3×8. For brevity, each 8-bit row is written as a binary representation of a decimal number. For example, the number 169 corresponds to the row [1 0 1 0 1 0 0 1]. The code matrices are as follows:

$$\begin{array}{llll}
A_1 = (169, 75, 5), & A_2 = (195, 43, 6), & A_3 = (108, 29, 3), & A_4 = (130, 72, 20), \\
A_5 = (144, 68, 33), & A_6 = (65, 61, 2), & A_7 = (66, 19, 4), & A_8 = (140, 87, 1), \\
A_9 = (35, 16, 9), & A_{10} = (147, 99, 7), & A_{11} = (155, 76, 38), & A_{12} = (69, 40, 24), \\
A_{13} = (132, 103, 12), & A_{14} = (152, 88, 56), & A_{15} = (153, 94, 39), & A_{16} = (196, 34, 11), \\
A_{17} = (167, 97, 15), & A_{18} = (159, 84, 32), & A_{19} = (154, 71, 55), & A_{20} = (145, 80, 50), \\
A_{21} = (131, 54, 13), & A_{22} = (134, 74, 53), & A_{23} = (166, 18, 8), & A_{24} = (164, 64, 31), \\
A_{25} = (138, 90, 60), & A_{26} = (135, 73, 27), & A_{27} = (146, 77, 37), & A_{28} = (171, 105, 17), \\
A_{29} = (158, 79, 52), & A_{30} = (128, 89, 47), & A_{31} = (129, 22, 10), & A_{32} = (143, 83, 46), \\
A_{33} = (205, 36, 21), & A_{34} = (137, 91, 44). & &
\end{array}$$

The ZJSSS code has the maximum cardinality for length $n = 8$ and subspace distance $d_{\text{sub}} = 6$.

The construction algorithm for MZP codes allows to include the ZJSSS code as the last component to increase the cardinality. Let us take the ZJSSS code as one component:

$$\text{Last component:} \quad [\boldsymbol{X}].$$

This component consists of 3×8 matrices $\boldsymbol{X}$ of the ZJSSS code such that $d_{\mathrm{sub}}(\boldsymbol{X}'-\boldsymbol{X}'') = 2m = 6$. The cardinality of this component is 34, which is the maximum possible for this length and this code distance.

We extend this component by a zero matrix prefix $\mathbf{0}_3$ of order 3, and add the 1st component of length 11 of the MZP code:

$$\begin{array}{ll} \text{1st extra component:} & [\boldsymbol{I}_m \ \ \boldsymbol{X}_1], \\ \text{Last component:} & [\mathbf{0}_3 \ \ \boldsymbol{X}]. \end{array}$$

The 1st component consists of the identity submatrix $\boldsymbol{I}_m$ of order m and a matrix $\boldsymbol{X}_1$ of size 3×8 of a rank code: different matrices have rank distance $d_r = m = 3$. The component has subspace distance $d_{\mathrm{sub}} = 2m = 6$ and cardinality 2^8. As a result, both components form a code of length $11 = 3 \cdot 3 + 2$, subspace distance $d_{\mathrm{sub}} = 2m = 6$, and maximum possible cardinality $\dfrac{2^{11}-2^2}{2^3-1} - 2 = 290$, which corresponds to the upper cardinality bound.

If a combined code of length $3(r-1)+2$ is constructed, then the code of length $n = 3r+2$ can be constructed from it by extending all the used components with the zero matrix prefix $\mathbf{0}_3$ of order 3 and adding a new MZP-code component of length $n = 3r+2$ and cardinality $2^{3(r-1)+2}$.

4. MZP CODES OF THE MAXIMUM CARDINALITY

4.1. MZP Codes of Dimension $m = 2$

Let the code length be $n = 2r+s$, where $r \geq 2$, $s = 0, 1$. The dimension of code subspaces is $m = 2$ for all n. The cardinality of such codes

$$M_{n,\mathrm{opt}} = \frac{q^n - q^s}{q^2-1} - s$$

coincides with the upper bound.

4.2. MZP Codes of Dimension $m = 3$

Let the code length be $n = 3r+s$, where $r \geq 1$, $s = 0, 1$. The dimension of code subspaces is $m = 3$ for all n congruent with 0 or 1 modulo $m = 3$. The cardinality of the MZP-code

$$M_{n,\mathrm{opt}} = \frac{q^n - q^s}{q^3-1} - s$$

coincides with the upper bound.

For binary codes ($q = 2$) with the length $n = 3r+2$, where $r \geq 2$, the maximum cardinality is

$$M_{n,\max} = \frac{2^n - 2^2}{2^3-1} - 2.$$

These codes can be constructed based on the ZJSSS code by including it in MZP codes. In particular, the cardinality of this code $M_{8,\mathrm{opt}} = 34$ is greater by 1 than the cardinality of the MZP code.

Let us present an open problem. A conjecture: a combined code of the maximum cardinality can be constructed for other dimensions if a code of the maximum cardinality has already been constructed for small lengths. Assume that for $n = 10$ and $m = 4 = \delta$ a code of the maximum cardinality $M = 66$ is constructed; then, using it as the last component, we can construct by the above method a family of codes of the maximum cardinality for a number of parameters.

4.3. Dual Codes of the Maximum Cardinality

Let us be given an $[n, M, d_{\rm sub} = 2\delta, m]$ subspace code of dimension m. Let us construct a dual code $[n, M, d_{\rm sub} = 2\delta, n-m]$ of dimension $n-m$. For a subspace $X \in W(n,m)$ of dimension m, there exists an orthogonal complement subspace $X^\perp \in W(n, n-m)$ of dimension $n-m$.

If one of these codes has the maximum cardinality, then the other has the same cardinality [13].

Let a subspace X of dimension m be generated by an $m \times n$ matrix L of rank m. Then the orthogonal subspace $X^\perp$ of dimension $n-m$ is given by a matrix $L^\perp$ of size $(n-m) \times n$ such that

$$(L^\perp)(L^\top) = 0,$$

where $L^\top$ means the transposed matrix.

Let us construct the dual MZP code. For $j = 1, \ldots, r$, components of the MZP code of dimension m and length $n = rm + s$ are given by matrices of rank m of the form

$$L_j = \begin{bmatrix} \mathbf{0}_m & \ldots & \mathbf{0}_m & \boldsymbol{I}_m & \boldsymbol{M}_j \end{bmatrix},$$

which consist of the zero matrix prefix of size $m \times (j-1)m$, the identity submatrix $\boldsymbol{I}_m$ of order m, and a submatrix $\boldsymbol{M}_j$ of size $m \times n - jm$. The orthogonal matrix $L_j^\perp$ of rank $n-m$ is of the form

$$L_j^\perp = \begin{bmatrix} \boldsymbol{I}_{(j-1)m} & \mathbf{0}_{(j-1)m}^{m} & \mathbf{0}_{(j-1)m}^{n-jm} \\ \mathbf{0}_{n-jm}^{(j-1)m} & -\boldsymbol{M}_j^\top & \boldsymbol{I}_{n-jm} \end{bmatrix}.$$

Here, $\mathbf{0}_a^b$ means the zero matrix of size $a \times b$.

The dimension of the dual MZP code $L_j^\perp$ is $n-m$, and its subspace distance $d_{\rm sub} = 2m$. For lengths $n = rm + s$, $s = 0, 1$, these codes have the maximum cardinality. The codes for $n = 3r + 2$ with subspace distance $d_{\rm sub} = 6$ also have the maximum cardinality.

5. CONCLUSION

Let us summarize the obtained results. An MZP code has the maximum cardinality, which coincides with the upper bound, in the case where the subspace code distance $d_{\rm sub}$ is equal to twice the dimension m and the code length is $n = rm + s$, where $s = 0, 1$ and r is a positive integer.

If the subspace code distance $d_{\rm sub}$ is equal to twice the dimension $m = 2$ for any length n, then the MZP code also has the maximum cardinality. The dual code has the same length n, the same code distance $d_{\rm sub}$, and a new value of the dimension $n-m$; it also has the maximum cardinality. In this case the subspace code distance is not necessarily equal to twice the dimension.

The combination of an MZP code and the ZJSSS code (from [1]) yields a new combined MZP-ZJSSS of the maximum cardinality for dimension $m = 3$, code distance $d_{\rm sub} = 2m$, and lengths of the form $n = rm + s$, where $s = 0, 1, 2$ and r is a positive integer.

The dual code (relative to the MZP-ZJSSS code), which is of dimension $n-m$, also has the maximum cardinality.

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