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CODING THEORY

# Bounds on the Cardinality of Subspace Codes with Non-maximum Code Distance

**E. M. Gabidulin†, N. I. Pilipchuk[a,*], and O.V. Trushina[a,**]**

[a] *Moscow Institute of Physics and Technology (State University), Moscow, Russia*
*e-mail*: *pilipchuk.nina@gmail.com, **oksana.trushina@gmail.com

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**Abstract**—We study subspace codes with nonmaximum code distance. As opposed to spreads, i.e., codes with the maximum subspace distance, we refer to them as nonspreads here. We consider families of nonspreads based on using the Silva–Kötter–Kschischang (SKK) subspace code construction and Gabidulin–Bossert multicomponent codes with zero prefix (MZP). We give estimates for cardinalities of nonspreads for a large number of parameters. We show that for large dimensions, the cardinalities almost attain the upper bound given by the Johnson inequality.

*Key words*: finite field, code, spreads, decoding, space, subspace, code cardinality, rank metric.

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## 1. INTRODUCTION

Interest to subspace codes has raised owing to development of ideas of random network coding [1] and distributed storage systems [2], where they have found applications.

A subspace code is a set of subspaces of a given space. Consider a finite $n$-dimensional space $W = GF(q)^n$ over a finite field $GF(q)$. Let $W(n,m)$ be the set of all $m$-dimensional subspaces of $W$, referred to as the *m-Grassmanninan*. The size of the Grassmanninan is given by the Gaussian coefficients:

$$|W(n,m)| = \begin{bmatrix} n \\ m \end{bmatrix} = \frac{(q^n-1)(q^n-q)\dots(q^n-q^{m-1})}{(q^m-1)(q^m-q)\dots(q^m-q^{m-1})}.$$

The *subspace distance* between two subspaces $U, V \in W$ can be defined as

$$\begin{aligned} d_{\rm sub}(U,V) &= \dim(U \uplus V) - \dim(U \cap V) \\ &= \dim(U) + \dim(V) - 2\dim(U \cap V), \end{aligned}$$

where $U \uplus V$ denotes the smallest subspace containing both subspaces $U$ and $V$. If $U$ and $V$ are of the same dimension $m$, the subspace distance equals

$$d_{\rm sub}(U,V) = 2(m - \dim(U \cap V)) = 2\delta,$$

where $\delta = m - \dim(U \cap V)$. This distance is also known as the *Grassmannian metric*.

If a code consists of elements of the $m$-Grassmanninan $W(n,m)$ with number of subspaces $M$, minimum distance $d_{\rm sub}$, and dimension $m$, then it is called a constant-dimension code and is denoted by $(M, n, d_{\rm sub}, m)$. A code with maximum distance $d_{\rm sub} = 2m$ is called a *spread*. We will refer to subspace codes with distances other than the maximum as *nonspreads*.

† Deceased.

Choose some spread with parameters $(n, d_{\text{sub}}, m)$. A code with parameters $(n+1, d_{\text{sub}}, m+1)$ is a nonspread. Cardinalities of these codes are related via the Johnson inequality [3]

$$M(n+1, d_{\text{sub}}, m+1) \le \frac{q^{n+1}-1}{q^{m+1}-1} M(n, d_{\text{sub}}, m). \tag{1}$$

The ratio $K_J = \dfrac{q^{n+1}-1}{q^{m+1}-1}$ will below be referred to as the *Johnson coefficient*. This coefficient shows that when passing from a spread to a nonspread, the code cardinality for the chosen parameters can at the most increase by a factor of $K_J$.

Consider a spread with length $n = mt + s$, where $t$ and $s$ are integers with $0 \le s \le m-1$. If $s = 0$, the spread is said to be *complete*, and if $s > 0$, it is said to be *partial*. It is known that there exist optimal complete spreads and for some parameters, optimal partial spreads. Thus, in [4] MZP spreads were considered, i.e., spreads constructed by the principle of multicomponent codes with zero prefix (MZP codes), which were proposed in [5]. It was shown that complete MZP spreads and partial MZP spreads attain the upper bound on the cardinality.

The cardinality of a complete MZP spread is [4]

$$M_{\text{MZP spread}} = \frac{q^n-1}{q^m-1}. \tag{2}$$

Using relations (1) and (2), one can obtain an upper bound on the cardinality of a nonspread

$$M_{\max} = K_J \frac{q^n-1}{q^m-1} = \frac{q^{n+1}-1}{q^{m+1}-1}\frac{q^n-1}{q^m-1}. \tag{3}$$

## 2. NONSPREADS FROM SKK SPREADS

In [6, 7], Silva, Kötter, and Kschischang gave a detailed description of the lifting construction of their subspace SKK code designed for transmission through a network using random linear transformations [1]. This code consists of a set of matrices of the form

$$\mathcal{M}_{\text{SKK}} = \left\{ \begin{pmatrix} \boldsymbol{I}_m & \boldsymbol{M}_{m\times(n-m)} \end{pmatrix} \right\},$$

where $\boldsymbol{I}_m$ is the identity matrix of order $m$ and $\boldsymbol{M}_{m\times(n-m)}$ is a code matrix of size $m \times (n-m)$ from a matrix rank code $\mathcal{M}_{\text{rank}}$ with rank distance $d_{\text{rank}} = \delta$ (see [8]). The subspace distance of $\mathcal{M}_{\text{SKK}}$ is twice the rank distance of the matrix code.

The cardinality of an SKK code equals the number of codewords of the rank code with rank distance $d_{\text{rank}} = \delta$ and codeword length $n - m$:

$$M_{\text{SKK}} = |\mathcal{M}_{\text{SKK}}| = |\mathcal{M}_{\text{rank}}| = q^{(n-m)k},$$

where $k = m - \delta + 1$, $\delta \le m$. For an SKK spread with parameters $(n,\ d_{\text{sub}} = 2m,\ m)$, where $\delta = m$, $n = tm$, $t \in \mathbb{N}$, we construct a nonspread by increasing the dimension and length by 1, i.e., a nonspread with parameters $(n+1, d_{\text{sub}} = 2m, m+1)$. For these parameters, $k = (m+1) - m + 1 = 2$. Let us see what is the cardinality of an SKK nonspread as a fraction of the maximum cardinality (3):

$$K_{\text{SKK}} = \frac{q^{2(n-m)}}{M_{\max}} = q^{2(n-m)} \frac{q^{m+1}-1}{q^{n+1}-1}\frac{q^m-1}{q^n-1}.$$

It grows rather fast with $m$. For $q = 2$, its values for various $m$ and $t$ are presented in Table 1.

**Table 1.** Cardinality of an SKK nonspread as a fraction of the maximum for $q = 2$.

| $m$ | $t$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 0.7225806 | 0.6719160 | 0.6601128 | 0.6572124 | 0.6564904 | 0.6563101 | 0.6562650 | 0.6562538 | 0.6562509 |
| 3 | 0.8398950 | 0.8227212 | 0.8206130 | 0.8203501 | 0.8203172 | 0.8203131 | 0.8203126 | 0.8203125 | 0.8203125 |
| 4 | 0.9135490 | 0.9085358 | 0.9082239 | 0.9082044 | 0.9082032 | 0.9082031 | 0.9082031 | 0.9082031 | 0.9082031 |
| 5 | 0.9550118 | 0.9536569 | 0.9536146 | 0.9536133 | 0.9536133 | 0.9536133 | 0.9536133 | 0.9536133 | 0.9536133 |
| 6 | 0.9770423 | 0.9766902 | 0.9766847 | 0.9766846 | 0.9766846 | 0.9766846 | 0.9766846 | 0.9766846 | 0.9766846 |
| 7 | 0.9884023 | 0.9883125 | 0.9883118 | 0.9883118 | 0.9883118 | 0.9883118 | 0.9883118 | 0.9883118 | 0.9883118 |
| 8 | 0.9941710 | 0.9941483 | 0.9941483 | 0.9941483 | 0.9941483 | 0.9941483 | 0.9941483 | 0.9941483 | 0.9941483 |
| 9 | 0.9970779 | 0.9970722 | 0.9970722 | 0.9970722 | 0.9970722 | 0.9970722 | 0.9970722 | 0.9970722 | 0.9970722 |
| 10 | 0.9985371 | 0.9985356 | 0.9985356 | 0.9985356 | 0.9985356 | 0.9985356 | 0.9985356 | 0.9985356 | 0.9985356 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 30 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 |

## 3. NONSPREADS FROM MZP SPREADS

Consider an MZP spread [4], i.e., an $(n,\, d_{\text{sub}} = 2m,\, m)$ subspace code, where $n = tm$, $t \in \mathbb{N}$, and construct a code of dimension $m+1$ with length of code matrices $n+1 = tm+1$, $t \in \mathbb{N}$, and subspace distance $d_{\text{sub}} = 2m$. The first component of this code is an SKK code, and each next component is a shift of the preceding component by a zero prefix. Thus, we have

$$
\begin{array}{ll}
\text{1st component} & \left(\boldsymbol{I}_{m+1} \quad \boldsymbol{M}^1_{(m+1)\times(n-m)}\right), \\
\text{2nd component} & \left(\boldsymbol{0}_{(m+1)\times m} \quad \boldsymbol{I}_{m+1} \quad \boldsymbol{M}^2_{(m+1)\times(n-2m)}\right), \\
(t-2)\text{nd component} & \Big(\underbrace{\boldsymbol{0}_{(m+1)\times m}\ \boldsymbol{0}_{(m+1)\times m}\ \ldots\ \boldsymbol{0}_{(m+1)\times m}}_{t-3} \quad \boldsymbol{I}_{m+1} \quad \boldsymbol{M}^{t-2}_{(m+1)\times 2m}\Big), \\
(t-1)\text{st component} & \Big(\underbrace{\boldsymbol{0}_{(m+1)\times m}\ \boldsymbol{0}_{(m+1)\times m}\ \ldots\ \boldsymbol{0}_{(m+1)\times m}}_{t-2} \quad \boldsymbol{I}_{m+1} \quad \boldsymbol{M}^{t-1}_{(m+1)\times m}\Big), \\
t\text{th component} & \Big(\underbrace{\boldsymbol{0}_{(m+1)\times m}\ \boldsymbol{0}_{(m+1)\times m}\ \ldots\ \boldsymbol{0}_{(m+1)\times m}}_{t-1} \quad \boldsymbol{I}_{m+1}\Big);
\end{array}
$$

here, $\boldsymbol{M}$ is the matrix of a rank code where the superscript indicates the number of the component and the subscript is the matrix size. For these parameters, $k = (m+1) - m + 1 = 2$.

The cardinality of this MZP nonspread is the sum of cardinalities of the components:

$$
M_{\text{MZP}} = q^{k(n-m)} + \sum_{i=1}^{t-2} q^{kmi} + 1 = \frac{q^{kn}-1}{q^{km}-1}. \tag{4}
$$

As in the case of an SKK code, let us see what is the cardinality of an MZP nonspread as a fraction of the maximum cardinality, $K_{\text{MZP}} = \dfrac{M_{\text{MZP}}}{M_{\max}}$. For $q = 2$, its values for various $m$ and $t$ are presented in Table 2.

Consider the ratio

$$
\frac{K_{\text{MZP}}}{K_{SKK}} = \frac{q^{kn}-1}{q^{km}-1}\,\frac{1}{q^{k(n-m)}} = \frac{q^{kn}-1}{q^{kn}-q^{k(n-m)}}.
$$

The ratio $\dfrac{K_{\text{MZP}}}{K_{SKK}}$ is always greater than one, so MZP nonspreads faster approach the maximum cardinality with growing dimension $m$.

**Table 2.** Cardinality of an SKK nonspread as a fraction of the maximum for $q = 2$.

| $m$ | $t$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 2 | 0.7677419 | 0.7165354 | 0.7041096 | 0.7010259 | 0.7002564 | 0.7000641 | 0.7000160 | 0.7000040 | 0.7000010 |
| 3 | 0.8530184 | 0.8357771 | 0.8336385 | 0.8333715 | 0.8333381 | 0.8333339 | 0.8333334 | 0.8333333 | 0.8333333 |
| 4 | 0.9171175 | 0.9120986 | 0.9117856 | 0.9117660 | 0.9117648 | 0.9117647 | 0.9117647 | 0.9117647 | 0.9117647 |
| 5 | 0.9559444 | 0.9545892 | 0.9545468 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 |
| 6 | 0.9772809 | 0.9769287 | 0.9769232 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 |
| 7 | 0.9884626 | 0.9883728 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 |
| 8 | 0.9941862 | 0.9941635 | 0.9941634 | 0.9941634 | 0.9941634 | 0.9941634 | 0.9941634 | 0.9941634 | 0.9941634 |
| 9 | 0.9970817 | 0.9970760 | 0.9970760 | 0.9970760 | 0.9970760 | 0.9970760 | 0.9970760 | 0.9970760 | 0.9970760 |
| 10 | 0.9985380 | 0.9985366 | 0.9985366 | 0.9985366 | 0.9985366 | 0.9985366 | 0.9985366 | 0.9985366 | 0.9985366 |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| 30 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 | 0.9999999 |

## 4. NONSPREADS FROM PARTIAL MZP SPREADS

Consider a partial MZP spread $(n,\, d_{\text{sub}} = 2m,\, m)$ [4], where $n = tm + s$, $t \in \mathbb{N}$, $1 \le s < m$, and construct a nonspread from it by increasing the length and dimension by one:

$$\begin{array}{ll}
\text{1st component} & \left(\boldsymbol{I}_{m+1} \quad \boldsymbol{M}^1_{(m+1)\times(n-m)}\right), \\
\text{2nd component} & \left(\boldsymbol{0}_{(m+1)\times m} \quad \boldsymbol{I}_{m+1} \quad \boldsymbol{M}^2_{(m+1)\times(n-2m)}\right), \\
(t-2)\text{nd component} & \left(\underbrace{\boldsymbol{0}_{(m+1)\times m} \ \boldsymbol{0}_{(m+1)\times m} \ \ldots \ \boldsymbol{0}_{(m+1)\times m}}_{t-3} \quad \boldsymbol{I}_{m+1} \quad \boldsymbol{M}^{t-2}_{(m+1)\times(2m+s)}\right), \\
(t-1)\text{st component} & \left(\underbrace{\boldsymbol{0}_{(m+1)\times m} \ \boldsymbol{0}_{(m+1)\times m} \ \ldots \ \boldsymbol{0}_{(m+1)\times m}}_{t-2} \quad \boldsymbol{I}_{m+1} \quad \boldsymbol{M}^{t-1}_{(m+1)\times(m+s)}\right), \\
t\text{th component} & \left(\underbrace{\boldsymbol{0}_{(m+1)\times m} \ \boldsymbol{0}_{(m+1)\times m} \ \ldots \ \boldsymbol{0}_{(m+1)\times m}}_{t-1} \quad \boldsymbol{I}_{m+1} \quad \boldsymbol{M}^{t}_{(m+1)\times s}\right).
\end{array}$$

Compute the cardinality of a partial MZP nonspread:

$$M_{\text{partial MZP}} = \frac{q^{kn} - q^{k(m+s)}}{q^{km} - 1} + 1. \tag{5}$$

As in the cases above, find the cardinality of a partial MZP nonspread as a fraction of the maximum. In Table 3 we give examples of some computed values.

## 5. KNOWN WORKS ON NONSPREADS

We present examples of nonspreads with the best values of the cardinality against previously known subspace codes with the same parameters. In [9], using Steiner systems, a subspace code with parameters $(n = 13,\, d_{\text{sub}} = 4,\, m = 3)$ and cardinality $M = 1\,597\,245$ was constructed. The obtained value of the cardinality coincides with the Johnson upper bound (3).

In [10, 11], subspace codes for parameters $n = 6$, $m = 3$, and $d_{\text{sub}} = 4$ with cardinality 77 and for parameters $n = 7$, $m = 3$, and $d_{\text{sub}} = 4$ with cardinality 329 were constructed.

For a code with parameters $(n = 6,\, d_{\text{sub}} = 4,\, m = 3)$ and cardinality $M = 77$ in [10], computer search was used first. Then the obtained codes were analyzed, and in a new construction, rank codes with parameters $(n = 3,\, d_{\text{rank}} = 3,\, m = 2)$ and cardinality $M = 64$ were used. Then an SKK code with parameters $(n = 6,\, d_{\text{sub}} = 4,\, m = 3)$ was constructed. Additional treatment resulted in constructing a code with cardinality $M = 77$. Since that, the cardinality value of 77 is considered to be the best for a code with parameters $(n = 6,\, d_{\text{sub}} = 4,\, m = 3)$, whereas the Johnson inequality yields an upper bound on the cardinality $M(6, 4, 3) = 81$.

**Table 3.** Cardinality of a partial MZP nonspread as a fraction of the maximum for $q = 2$.

| $m$ | $t$ | | | | | | | | | $s$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |
| 2 | 0.8024691 | 0.7291248 | 0.7074621 | 0.7018763 | 0.7004697 | 0.7001175 | 0.7000294 | 0.7000073 | 0.7000018 | 1 |
| 3 | 0.8892734 | 0.8409784 | 0.8342984 | 0.8334541 | 0.8333484 | 0.8333352 | 0.8333336 | 0.8333334 | 0.8333333 | 1 |
| | 0.9117595 | 0.8436442 | 0.8346293 | 0.8334954 | 0.8333536 | 0.8333359 | 0.8333336 | 0.8333334 | 0.8333333 | 2 |
| 4 | 0.9412305 | 0.9137060 | 0.9118864 | 0.9117723 | 0.9117652 | 0.9117647 | 0.9117647 | 0.9117647 | 0.9117647 | 1 |
| | 0.9545451 | 0.9145145 | 0.9119369 | 0.9117755 | 0.9117654 | 0.9117647 | 0.9117647 | 0.9117647 | 0.9117647 | 2 |
| | 0.9615337 | 0.9149200 | 0.9119621 | 0.9117770 | 0.9117655 | 0.9117648 | 0.9117647 | 0.9117647 | 0.9117647 | 3 |
| 5 | 0.9697041 | 0.9550330 | 0.9545607 | 0.9545459 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 1 |
| | 0.9769231 | 0.9552552 | 0.9545676 | 0.9545461 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 2 |
| | 0.9806196 | 0.9553664 | 0.9545711 | 0.9545463 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 3 |
| | 0.9824899 | 0.9554220 | 0.9545728 | 0.9545463 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 0.9545455 | 4 |
| 6 | 0.9846163 | 0.9770451 | 0.9769250 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 1 |
| | 0.9883721 | 0.9771033 | 0.9769259 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 2 |
| | 0.9902723 | 0.9771325 | 0.9769263 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 3 |
| | 0.9912280 | 0.9771470 | 0.9769266 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 4 |
| | 0.9917073 | 0.9771543 | 0.9769267 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 0.9769231 | 5 |
| 7 | 0.9922482 | 0.9884026 | 0.9883723 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 1 |
| | 0.9941634 | 0.9884175 | 0.9883724 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 2 |
| | 0.9951267 | 0.9884250 | 0.9883725 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 3 |
| | 0.9956097 | 0.9884287 | 0.9883725 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 4 |
| | 0.9958516 | 0.9884306 | 0.9883725 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 5 |
| | 0.9959727 | 0.9884315 | 0.9883726 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 0.9883721 | 6 |
| 20 | 0.9999990 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 1 |
| | 0.9999993 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 2 |
| | 0.9999994 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 3 |
| | 0.9999995 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 4 |
| | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
| | 0.9999995 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 0.9999986 | 19 |

Another code with parameters ($n = 7$, $d_{\text{sub}} = 4$, $m = 3$) and cardinality $M = 329$ was constructed by somewhat the same method in [11]. To obtain an SKK code, a rank code was preliminarily constructed and an identity matrix was prepended. Also, ideas of finite geometry were used. An upper bound on the cardinality by the Johnson inequality is 381 for this code.

In [12], an upper bound of 272 on the cardinality was presented for a code with parameters ($n = 8$, $d_{\text{sub}} = 6$, $m = 4$). The Johnson bound gives the value of 289 for this code.

Thus, except for the first paper [9], other works do not give values close to the Johnson maximum cardinality. However, to the best of our knowledge, this work was not continued to construct codes with other parameters.

Also, several works [13, 14] in the same direction are known. They use SKK codes in parallel constructions and reach a considerable increase in the number of codewords for certain parameters. Let us compare the cardinalities for the parameters considered in [13, 14].

In [13], a lower bound on the cardinality of a code with parameters $(16, 8, 8)$ was obtained: $A(16, 8, 8) = 1\,099\,562\,828\,461$. A code with parameters $(16, 8, 8)$ can be constructed as a nonspread of a complete spread. Indeed,

$$\begin{aligned} &d = 2\delta = 8 \implies \delta = 4 = m, \\ &k = m + 1 - \delta + 1 = 8 - 4 + 1 = 5, \\ &n = 3 \times 4 \implies t = 3, \end{aligned}$$

and by equation (4) with $q = 2$ we obtain $M(16, 8, 8) = \dfrac{2^{60} - 1}{2^{20} - 1}$. Then

$$\frac{M(16, 8, 8)}{A(16, 8, 8)} = 0.999954.$$

Similarly we can find

$$\frac{M(19,4,6)}{A(19,4,6)} = 0.8828625, \qquad \frac{M(18,8,9)}{A(18,8,9)} = 0.999955, \qquad \frac{M(18,6,9)}{A(18,6,9)} = 0.9948,$$
$$\frac{M(18,4,5)}{A(18,4,5)} = 0.89316, \qquad \frac{M(14,4,7)}{A(14,4,7)} = 0.884, \qquad \frac{M(12,6,6)}{A(12,6,6)} = 0.995.$$

Constructions proposed in [13, 14] are rather complicated and use nonlinear conditions, which are an obstacle for constructing codes for any values of the parameters. As is seen from the above-given relations, with the use of simple constructions proposed in the present paper one can construct codes containing more than 88% possible codewords, and for some parameter values, even more than 99%. Moreover, these constructions allow to obtain codes for any parameters, and the obtained codes can be efficiently decoded [15].

## 6. CONCLUSION

We have considered families of subspace codes of large cardinality with a nonmaximum code distance. The construction algorithm is based on the Johnson inequality: fix a spread code with subspace distance equal to twice the dimension and, keeping the distance unchanged, increase the dimension and length by one. For an initial spread, we have considered three variants: an SKK spread, MZP spread, and a partial MZP spread. We have shown that for large dimensions all the three codes almost attain the maximum cardinality bound according to the Johnson inequality.

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