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Fuzzy *n*-Jordan *-homomorphisms in induced fuzzy *C**-algebras

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Abstract

Using fixed point method, we prove the fuzzy version of the Hyers-Ulam stability of *n*-Jordan *-homomorphisms in induced fuzzy *C**-algebras associated with the following functional equation

$$f\left(\frac{x+y+z}{3}\right)+f\left(\frac{x-2y+z}{3}\right)+f\left(\frac{x+y-2z}{3}\right)=f(x).$$

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1. Introduction and preliminaries

The stability of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms in 1940. More precisely, he proposed the following problem: Given a group $\mathcal{G}$, a metric group $(\mathcal{G}', d)$ and $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $f : \mathcal{G} \to \mathcal{G}'$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in \mathcal{G}$, then there exists a homomorphism $T : \mathcal{G} \to \mathcal{G}'$ such that $d(f(x), T(x)) < \varepsilon$ for all $x \in \mathcal{G}$? Hyers [2] gave a partial solution of the Ulam's problem for the case of approximate additive mappings under the assumption that $\mathcal{G}$ and $\mathcal{G}'$ are Banach spaces. Aoki [3] generalized the Hyers' theorem for approximately additive mappings. Rassias [4] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences.

Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

We recall a fundamental result in fixed point theory.

Theorem 1.1. [5,6]*Let (X, d) be a complete generalized metric space and let $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x,\ J^{n+1} x\right) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \forall n \geq n_0$;

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of J;*

(3) y^* *is the unique fixed point of J in the set* $Y = \{y \in X | d(J^{n_0} x, y) < \infty\}$;

(4) $d(y, y^*) \leq \dfrac{1}{1 - L} d\left(y, Jy\right)$ *for all* $y \in Y$.

Isac and Rassias [7] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [8-12]).

Katsaras [13] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematics have defined fuzzy normed on a vector space from various points of view [14-20]. In particular, Bag and Samanta [21] following Cheng and Mordeson [22], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [23]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [24].

We use the definition of fuzzy normed spaces given in [16,17,21] to investigate a fuzzy version of the Hyers-Ulam stability of n-Jordan *-homomorphisms in induced fuzzy C^*- algebras associated with the following functional equation

$$f\left(\frac{x + y + z}{3}\right) + f\left(\frac{x - 2y + z}{3}\right) + f\left(\frac{x + y - 2z}{3}\right) = f(x).$$

Definition 1.2. [16-18,21] Let $\mathcal{X}$ be a complex vector space. A function $N : \mathcal{X} \times \mathbb{R} \to [0, 1]$ is called a *fuzzy norm* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathbb{R}$,

N_1: $N(x, t) = 0$ for $t \leq 0$

N_2: $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$

N_3: $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \in \mathbb{C}$-{0}

N_4: $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$

N_5: $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$

N_6: for $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair $(\mathcal{X}, N)$ is called a *fuzzy normed vector space*.

Definition 1.3. [16-18,21] Let $(\mathcal{X}, N)$ be a fuzzy normed vector space.

(1) A sequence $\{x_n\}$ in χ is said to be *convergent* if there exists an $x \in \chi$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N\text{-}\lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in χ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p}\text{-}x_n, t) > 1\text{-}\varepsilon$.

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : \mathcal{X} \to \mathcal{Y}$ between fuzzy normed vector space $\mathcal{X}$, $\mathcal{Y}$ is continuous at point $x_0 \in \mathcal{X}$ if for each sequence $\{x_n\}$ converging to x_0 in $\mathcal{X}$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : \mathcal{X} \to \mathcal{Y}$ is continuous at each $x \in \mathcal{X}$, then $f : \mathcal{X} \to \mathcal{Y}$ is said to be *continuous* on $\mathcal{X}$ (see [24]).

Definition 1.4. Let $\mathcal{X}$ be a *-algebra and $(\mathcal{X}, N)$ a fuzzy normed space.

(1) The fuzzy normed space $(\mathcal{X}, N)$ is called a fuzzy normed *-algebra if

$$N(xy, st) \geq N(x, s) \cdot N(y, t) \qquad \& \qquad N(x^*, t) = N(x, t)$$

(2) A complete fuzzy normed *-algebra is called a fuzzy Banach *-algebra.

Example 1.5. Let $(\mathcal{X}, \|\cdot\|)$ be a normed *-algebra. let

$$N(x, t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t ¿ 0, \ x \in \mathcal{X} \\ 0, & t \leq 0, x \in \mathcal{X}. \end{cases}$$

Then $N(x, t)$ is a fuzzy norm on $\mathcal{X}$ and $(\mathcal{X}, N(x, t))$ is a fuzzy normed *-algebra.

Definition 1.6. Let $(\mathcal{X}, \|\cdot\|)$ be a C^*-algebra and $N_{\mathcal{X}}$ a fuzzy norm on $\mathcal{X}$.

(1) The fuzzy normed *-algebra $(\mathcal{X}, N_{\mathcal{X}})$ is called an induced fuzzy normed *-algebra

(2) The fuzzy Banach *-algebra $(\mathcal{X}, N_{\mathcal{X}})$ is called an induced fuzzy C^*-algebra.

Definition 1.7. Let $(\mathcal{X}, N_{\mathcal{X}})$ and $(\mathcal{Y}, N)$ be induced fuzzy normed *-algebras. Then a $\mathbb{C}$-linear mapping $H : (\mathcal{X}, N_{\mathcal{X}}) \to (\mathcal{Y}, N)$ is called a *fuzzy n-Jordan *-homomorphism* if

$$H(x^n) = H(x)^n \qquad \& \qquad H(x^*) = H(x)^*$$

for all $x \in \mathcal{X}$.

Throughout this article, assume that $(\mathcal{X}, N)$ is an induced fuzzy normed *-algebra and that $(\mathcal{Y}, N)$ is an induced fuzzy C^*-algebra.

2. Main results

Lemma 2.1. *Let $(\mathcal{Z}, N)$ be a fuzzy normed vector space and let $f : \mathcal{X} \to \mathcal{Z}$ be a mapping such that*

$$N\left(f\left(\frac{x + y + z}{3}\right) + f\left(\frac{x - 2y + z}{3}\right) + f\left(\frac{x + y - 2z}{3}\right), t\right) \geq N\left(f(x), \frac{t}{2}\right) \quad (2.1)$$

for all $x, y, z \in \mathcal{X}$ and all $t > 0$. Then f is additive, i.e., $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathcal{X}$.

Proof. Letting $x = y = z = 0$ in (2.1), we get

$$N(3f(0), t) = N\left(f(0), \frac{t}{3}\right) \geq N\left(f(0), \frac{t}{2}\right)$$

for all $t > 0$. By N_5 and N_6, $N(f(0), t) = 1$ for all $t > 0$. It follows from N_2 that $f(0) = 0$.
Letting $z = -x$, $y = x$, $x = 0$ in (2.1), we get

$$N(f(0) + f(-x) + f(x), t) \geq N\left(f(0), \frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from N_2 that $f(-x) + f(x) = 0$ for all $x \in \mathcal{X}$. So

$$f(-x) = -f(x)$$

for all $x \in \mathcal{X}$.

Letting $x = 0$ and replacing y, z by $3y$, $3z$, respectively, in (2.1), we get

$$N(f(y+z)+f(-2y+z)+f(y-2z),t) \geq N\left(f(0),\frac{t}{2}\right) = 1$$

for all $t > 0$. It follows from N_2 that

$$f\left(y+z\right)+f\left(-2y+z\right)+f\left(y-2z\right) = 0 \tag{2.2}$$

for all $y, z \in \mathcal{X}$. Let $t = 2y-z$ and $s = 2z-y$ in (2.2), we obtain

$$f(t+s) = f(t) + f(s)$$

for all $t, s \in \mathcal{X}$, as desired. □

Using fixed point method, we prove the Hyers-Ulam stability of fuzzy n-Jordan *-homomorphisms in induced fuzzy C^*-algebras.

Theorem 2.2. *Let* $\varphi : \mathcal{X}^3 \to [0,\infty)$ *be a function such that there exists an* $L < \frac{3}{3^n}$ *with*

$$\varphi\left(\frac{x}{3},\frac{y}{3},\frac{z}{3}\right) \leq \frac{L}{3}\varphi(x,y,z) \tag{2.3}$$

for all $x, y, z \in \mathcal{X}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping such that*

$$\begin{aligned} N\left(f\left(\frac{\mu x+\mu y+\mu z}{3}\right)+f\left(\frac{\mu x-2\mu y+\mu z}{3}\right)+f\left(\frac{\mu x+\mu y-2\mu z}{3}\right)-\mu f(x),t\right) \\ \geq \frac{t}{t+\varphi(x,y,z)}, \end{aligned} \tag{2.4}$$

$$N\left(f(x^n)-f(x)^n,t\right) \geq \frac{t}{t+\varphi(x,0,0)}, \tag{2.5}$$

$$N(f(x^*)-f(x)^*,t) \geq \frac{t}{t+\varphi(x,0,0)} \tag{2.6}$$

for all $x, y, z \in \mathcal{X}$, *all* $t > 0$ *and all* $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$. *Then* $H(x) = N - \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right)$ *exists for each* $x \in \mathcal{X}$ *and defines a fuzzy n-Jordan *-homomorphism* $H : \mathcal{X} \to \mathcal{Y}$ *such that*

$$N\left(f(x)-H(x),t\right) \geq \frac{(1-L)t}{(1-L)\,t+\varphi(x,0,0)} \tag{2.7}$$

for all $x \in \mathcal{X}$ *and all* $t > 0$.

Proof. Letting $\mu = 1$ and $y = z = 0$ in (2.4), we get

$$N\left(3f\left(\frac{x}{3}\right)-f(x),t\right) \geq \frac{t}{t+\varphi(x,0,0)} \tag{2.8}$$

for all $x \in \mathcal{X}$.

Consider the set

$$S := \{g : \mathcal{X} \to \mathcal{Y}\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\alpha \in \mathbb{R}_+ : N\left(g(x) - h(x), \alpha t\right) \geq \frac{t}{t + \varphi(x,0,0)}, \forall x \in \mathcal{X}, \forall t > 0\right\},$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [[25], Lemma 2.1]).

Now we consider the linear mapping $J: S \to S$ such that

$$Jg(x) := 3g\left(\frac{x}{3}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \geq \frac{t}{t + \varphi(x,0,0)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Hence

$$\begin{aligned} N\left(Jg(x) - Jh(x), L\varepsilon t\right) &= N\left(3g\left(\frac{x}{3}\right) - 3h\left(\frac{x}{3}\right), L\varepsilon t\right) = N\left(g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right), \frac{L}{3}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \varphi\left(\frac{x}{3}, 0, 0\right)} \geq \frac{\frac{Lt}{3}}{\frac{Lt}{3} + \frac{L}{3}\varphi\left(x,0,0\right)} \\ &= \frac{t}{t + \varphi\left(x,0,0\right)} \end{aligned}$$

for all $x \in \mathcal{X}$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.8) that $d(f, Jf) \leq 1$.

By Theorem 1.1, there exists a mapping $H : \mathcal{X} \to \mathcal{Y}$ satisfying the following:

(1) H is a fixed point of J, i.e.,

$$H\left(\frac{x}{3}\right) = \frac{1}{3}H(x) \tag{2.9}$$

for all $x \in \mathcal{X}$. The mapping H is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.9) such that there exists a $\alpha \in (0, \infty)$ satisfying

$$N(f(x) - H(x), \alpha t) \geq \frac{t}{t + \varphi\left(x,0,0\right)}$$

for all $x \in \mathcal{X}$;

(2) $d(J^k f, H) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k\to\infty} 3^k f\left(\frac{x}{3^k}\right) = H(x)$$

for all $x \in \mathcal{X}$;
(3) $d(f,H) \le \frac{1}{1-L} d\left(f, Jf\right)$, which implies the inequality

$$d(f,H) \le \frac{1}{1-L}.$$

This implies that the inequality (2.7) holds.
It follows from (2.3) that

$$\sum_{k=0}^{\infty} 3^k \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right) < \infty$$

for all $x, y, z \in \mathcal{X}$.
By (2.4),

$$N\left(3^k f\left(\frac{\mu x + \mu y + \mu z}{3^{k+1}}\right) + 3^k f\left(\frac{\mu x - 2\mu y + \mu z}{3^{k+1}}\right) + 3^k f\left(\frac{\mu x + \mu y - 2\mu z}{3^{k+1}}\right) - \mu 3^k f\left(\frac{x}{3^k}\right), 3^k t\right) \ge \frac{t}{t + \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. So

$$N\left(3^k f\left(\frac{\mu x + \mu y + \mu z}{3^{k+1}}\right) + 3^k f\left(\frac{\mu x - 2\mu y + \mu z}{3^{k+1}}\right) + 3^k f\left(\frac{\mu x + \mu y - 2\mu z}{3^{k+1}}\right) - \mu 3^k f\left(\frac{x}{3^k}\right), t\right) \ge \frac{\frac{t}{3^k}}{\frac{t}{3^k} + \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)} = \frac{t}{t + 3^k \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)}$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Since $\lim_{k\to\infty} \frac{t}{t + 3^k \varphi\left(\frac{x}{3^k}, \frac{y}{3^k}, \frac{z}{3^k}\right)} = 1$ for all $x, y, z \in \mathcal{X}$ and all $t > 0$,

$$N\left(H\left(\frac{\mu x + \mu y + \mu z}{3}\right) + H\left(\frac{\mu x - 2\mu y + \mu z}{3}\right) + H\left(\frac{\mu x + \mu y - 2\mu z}{3}\right) - \mu H(x), t\right) = 1$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Thus

$$H\left(\frac{\mu x + \mu y + \mu z}{3}\right) + H\left(\frac{\mu x - 2\mu y + \mu z}{3}\right) + H\left(\frac{\mu x + \mu y - 2\mu z}{3}\right) = \mu H(x) \quad (2.10)$$

for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Letting $x = y = z = 0$ in (2.10), we get $H(0) = 0$. Let $\mu = 1$ and $x = 0$ in (2.10). By the same reasoning as in the proof of Lemma 2.1, one can easily show that H is additive. Letting $y = z = 0$ in (2.10), we get

$$H(\mu x) = 3H\left(\frac{\mu x}{3}\right) = \mu H(x)$$

for all $x \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. By [[26], Theorem 2.1], the mapping $H : \mathcal{X} \to \mathcal{Y}$ is $\mathbb{C}$-linear.

By (2.5),

$$N\left(3^{nk}f\left(\frac{x^k}{3^{nk}}\right) - 3^{nk}f\left(\frac{x}{3^k}\right)^n, 3^{nk}t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{3^k}, 0, 0\right)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. So

$$N\left(3^{nk}f\left(\frac{x^k}{3^{nk}}\right) - 3^{nk}f\left(\frac{x}{3^k}\right)^n, t\right) \geq \frac{\frac{t}{3^{nk}}}{\frac{t}{3^{nk}} + \varphi\left(\frac{x}{3^k}, 0, 0\right)} = \frac{t}{t + \left(3^{n-1}L\right)^k\varphi(x, 0, 0)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Since $\lim_{k\to\infty}\frac{t}{t + \left(3^{n-1}L\right)^k\varphi(x, 0, 0)} = 1$ for all $x \in \mathcal{X}$ and all $t > 0$,

$$N(H(x^n) - H(x)^n, t) = 1$$

for all $x \in \mathcal{X}$ and all $t > 0$. Thus, $H(x^n) - H(x)^n = 0$ for all $x \in \mathcal{X}$.

By (2.6),

$$N\left(3^{k}f\left(\frac{x^*}{3^{k}}\right) - 3^{k}f\left(\frac{x}{3^k}\right)^*, 3^{k}t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{3^k}, 0, 0\right)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. So

$$N\left(3^{k}f\left(\frac{x^*}{3^{k}}\right) - 3^{k}f\left(\frac{x}{3^k}\right)^*, t\right) \geq \frac{\frac{t}{3^{k}}}{\frac{t}{3^{k}} + \varphi\left(\frac{t}{3^k}, 0, 0\right)} = \frac{t}{t + 3^k\varphi\left(\frac{t}{3^k}, 0, 0\right)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. Since $\lim_{k\to\infty}\frac{t}{t + 3^k\varphi\left(\frac{t}{3^k}, 0, 0\right)} = 1$ for all $x \in \mathcal{X}$ and all $t > 0$,

$$N(H(x^*) - H(x)^*, t) = 1$$

for all $x \in \mathcal{X}$ and all $t > 0$. Thus, $H(x^*) - H(x)^* = 0$ for all $x \in \mathcal{X}$.

Therefore, the mapping $H : \mathcal{X} \to \mathcal{Y}$ is a fuzzy n-Jordan *-homomorphism. □

Corollary 2.3. *Let $\theta \geq 0$ and let p be a real number with $p > n$. Let $\mathcal{X}$ be a normed vector space with norm $\|\cdot\|$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$N\left(f\left(\frac{\mu x + \mu y + \mu z}{3}\right) + f\left(\frac{\mu x - 2\mu y + \mu z}{3}\right) + f\left(\frac{\mu x + \mu y - 2\mu z}{3}\right) - \mu f(x), t\right) \geq \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)}, \tag{2.11}$$

$$N\left(f\left(x^n\right) - f(x)^n, t\right) \geq \frac{t}{t + \theta\|x\|^p}, \tag{2.12}$$

$$N\left(f\left(x^*\right) - f(x)^*, t\right) \geq \frac{t}{t + \theta\|x\|^p} \tag{2.13}$$

*for all $x, y, z \in \mathcal{X}$, all $t > 0$ and all $\mu \in \mathbb{T}^1$. Then $H(x) = N - \lim_{n\to\infty}3^n f\left(\frac{x}{3^n}\right)$ exists for each $x \in \mathcal{X}$ and defines a fuzzy n-Jordan *-homomorphism $H : \mathcal{X} \to \mathcal{Y}$ such that*

$$N\left(f(x) - H(x), t\right) \geq \frac{\left(3^p - 3\right)t}{\left(3^p - 3\right)t + 3^p\theta\|x\|^p}$$

for all $x \in \mathcal{X}$ *and all* $t > 0$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(x, y, z) = \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

and $L = 3^{1-p}$.

Theorem 2.4. *Let* $\varphi : \mathcal{X}^3 \to [0,\infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x, y, z) \leq 3L\varphi\left(\frac{x}{3}, \frac{y}{3}, \frac{z}{3}\right)$$

for all $x, y, z \in \mathcal{X}$. *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* (2.4), (2.5), *and* (2.6). *Then* $H(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ *exists for each* $x \in \mathcal{X}$ *and defines a fuzzy n-Jordan *-homomorphism* $H : \mathcal{X} \to \mathcal{Y}$ *such that*

$$N\left(f(x) - H(x), t\right) \geq \frac{(1 - L)t}{(1 - L)t + L\varphi(x, 0, 0)} \tag{2.14}$$

for all $x \in \mathcal{X}$ *and all* $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2. Consider the linear mapping $J: S \to S$ such that

$$Jg(x) := \frac{1}{3}g(3x)$$

for all $x \in \mathcal{X}$.

It follows from (2.8) that

$$N\left(f(x) - \frac{1}{3}f(3x), \frac{1}{3}t\right) \geq \frac{t}{t + \varphi(3x, 0, 0)} \geq \frac{t}{t + 3L\varphi(x, 0, 0)}$$

for all $x \in \mathcal{X}$ and all $t > 0$. So $d(f, Jf) \leq L$. Hence

$$d\left(f, H\right) \leq \frac{L}{1 - L},$$

which implies that the inequality (2.14) holds.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let* $\theta \geq 0$ *and let p be a positive real number with* $p < 1$. *Let* $\mathcal{X}$ *be a normed vector space with norm* $||\cdot||$ *Let* $f : \mathcal{X} \to \mathcal{Y}$ *be a mapping satisfying* (2.11), (2.12), *and* (2.13). *Then* $H(x) = N - \lim_{n\to\infty} \frac{1}{3^n} f(3^n x)$ *exists for each* $x \in \mathcal{X}$ *and defines a fuzzy n-Jordan *-homomorphism* $H : \mathcal{X} \to \mathcal{Y}$ *such that*

$$N\left(f(x) - H(x), t\right) \geq \frac{\left(3 - 3^p\right)t}{\left(3 - 3^p\right)t + 3^p\theta\|x\|^p}$$

for all $x \in \mathcal{X}$ *and all* $t > 0$.

Proof. The proof follows from Theorem 2.4 by taking

$$\varphi\left(x, y, z\right) = \theta\left(\|x\|^p + \|y\|^p + \|z\|^p\right)$$

and $L = 3^{p-1}$. □

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Authors' contributions
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References

1. Ulam, SM: Problems in Modern Mathematics, Chap. VI, Science edn. Wily, New York (1940)
2. Hyers, DH: On the stability of the linear functional equation. Proc Nat Acad Sci USA. **27**, 222–224 (1941). doi:10.1073/pnas.27.4.222
3. Aoki, T: On the stability of the linear transformation in Banach spaces. J Math Soc Japan. **2**, 64–66 (1950). doi:10.2969/jmsj/00210064
4. Rassias, ThM: On the stability of the linear mapping in Banach spaces. Proc Am Math Soc. **72**, 297–300 (1978). doi:10.1090/S0002-9939-1978-0507327-1
5. Cădariu, L, Radu, V: Fixed points and the stability of Jensen's functional equation. J Inequal Pure Appl Math **4**(1):7 (2003). Art. ID 4
6. Diaz, J, Margolis, B: A fixed point theorem of the alternative for contractions on a generalized complete metric space. Bull Am Math Soc. **74**, 305–309 (1968). doi:10.1090/S0002-9904-1968-11933-0
7. Isac, G, Rassias, ThM: Stability of ψ-additive mappings: appications to nonlinear analysis. Int J Math Math Sci. **19**, 219–228 (1996). doi:10.1155/S0161171296000324
8. Cădariu, L, Radu, V: On the stability of the Cauchy functional equation: a fixed point approach. Grazer Math Ber. **346**, 43–52 (2004)
9. Cădariu, L, Radu, V: Fixed point methods for the generalized stability of functional equations in a single variable. Fixed Point Theory and Applications **2008**, 15 (2008). Art. ID 749392
10. Park, C: Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras. Fixed Point Theory and Applications **2007**, 15 (2007). Art. ID 50175
11. Park, C: Generalized Hyers-Ulam-Rassias stability of quadratic functional equations: a fixed point approach. Fixed Point Theory and Applications **2008**, 9 (2008). Art. ID 493751
12. Radu, V: The fixed point alternative and the stability of functional equations. Fixed Point Theory. **4**, 91–96 (2003)
13. Katsaras, AK: Fuzzy topological vector spaces II. Fuzzy Sets Syst. **12**, 143–154 (1984). doi:10.1016/0165-0114(84)90034-4
14. Felbin, C: Finite dimensional fuzzy normed linear spaces. Fuzzy Sets Syst. **48**, 239–248 (1992). doi:10.1016/0165-0114(92)90338-5
15. Krishna, SV, Sarma, KKM: Separation of fuzzy normed linear spaces. Fuzzy Sets Syst. **63**, 207–217 (1994). doi:10.1016/0165-0114(94)90351-4
16. Mirmostafaee, AK, Mirzavaziri, M, Moslehian, MS: Fuzzy stability of the Jensen functional equation. Fuzzy Sets Syst. **159**, 730–738 (2008). doi:10.1016/j.fss.2007.07.011
17. Mirmostafaee, AK, Moslehian, MS: Fuzzy versions of Hyers-Ulam-Rassias theorem. Fuzzy Sets Syst. **159**, 720–729 (2008). doi:10.1016/j.fss.2007.09.016
18. Mirmostafaee, AK, Moslehian, MS: Fuzzy approximately cubic mappings. Inf Sci. **178**, 3791–3798 (2008). doi:10.1016/j.ins.2008.05.032
19. Park, C: Fuzzy stability of additive functional inequalities with the fixed point alternative. J Inequal Appl **2009**, 17 (2009). Art. ID 410576
20. Xiao, JZ, Zhu, XH: Fuzzy normed spaces of opeators and its completeness. Fuzzy Sets Syst. **133**, 389–399 (2003). doi:10.1016/S0165-0114(02)00274-9
21. Bag, T, Samanta, SK: Finite dimensional fuzzy normed linear space. J Fuzzy Math. **11**, 687–705 (2003)
22. Cheng, SC, Mordeson, JM: Fuzzy linear operators and fuuzy normed linear spaces. Cacutta Math Soc. **86**, 429–436 (1994)
23. Kramosil, I, Michalek, J: Fuzzy metric and statistical metric spaces. Kybernetica. **11**, 326–334 (1975)
24. Bag, T, Samanta, SK: Fuzzy bounded linear operators. Fuzzy Set Syst. **151**, 513–547 (2005). doi:10.1016/j.fss.2004.05.004

25. Miheţ, D, Radu, V: On the stability of the additive Cauchy functional equation in random normed spaces. J Math Anal Appl. **343**, 567–572 (2008). doi:10.1016/j.jmaa.2008.01.100
26. Park, C: Homomorphisms between Poisson *JC**-algebras. Bull Braz Math Soc. **36**, 79–97 (2005). doi:10.1007/s00574-005-0029-z

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